18.05 Lecture 20
March 30, 2005

## §5.4 **Poisson Distribution**

$\Pi(\lambda)$, parameter $\lambda > 0$, random variable takes values: $\{0, 1, 2, ...\}$
p.f.:

$$f(x) = \mathbb{P}(X = x) = \frac{\lambda^x}{x!}e^{-\lambda}; e^{-\lambda}\sum_{x\geq 0}\frac{\lambda^x}{x!} = e^{-\lambda} \times e^{\lambda} = 1$$

Moment generating function:

$$\Phi(t) = \mathbb{E}e^{-tX} = \sum_{x\geq 0} e^{tX} \times \frac{\lambda^x}{x!}e^{-\lambda} = \sum_{x\geq 0}\frac{(e^t\lambda)^x}{x!}e^{-\lambda} = e^{-\lambda}\sum_{x\geq 0}\frac{(e^t\lambda)^x}{x!} = e^{-\lambda}e^{e^t\lambda} = e^{\lambda(e^t-1)}$$

$\mathbb{E}X^k = \Phi^k(0)$
$\mathbb{E}X = \Phi'(0) = e^{\lambda(e^t-1)} \times \lambda e^t|_{t=0} = \lambda$
$\mathbb{E}X^2 = \Phi''(0) = (\lambda e^{\lambda(e^t-1)+t})'|_{t=0} = \lambda e^{\lambda(e^t-1)+t}(\lambda e^t + 1)|_{t=0} = \lambda(\lambda+1)$
$\text{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$
If $X_1 \sim \Pi(\lambda_1), X_2 \sim \Pi(\lambda_2), ...X_n \sim \Pi(\lambda_n)$, all independent:
$Y = X_1 + ... + X_n$, find moment generating function of Y,

$$\Phi(t) = \mathbb{E}e^{tY} = \mathbb{E}e^{t(X_1+...+X_n)} = \mathbb{E}e^{tX_1} \times ... \times e^{tX_n}$$

By independence:

$$\mathbb{E}e^{tX_1}\mathbb{E}e^{tX_2} \times ... \times \mathbb{E}e^{tX_n} = e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)}...e^{\lambda_n(e^t-1)}$$

Moment generating function of $\Pi(\lambda_1 + ... + \lambda_n)$:

$$\Phi(t) = e^{(\lambda_1+\lambda_2+...+\lambda_n)(e^t-1)}$$

If dependent, for example:
$X_1, X_1 - 2X_1 \in \{0, 2, 4, ...\}$ - skips odd numbers, so not Poisson.

**Approximation of Binomial:**
$X_1, ..., X_n \sim B(p), \mathbb{P}(X_i = 1) = p, \mathbb{P}(X_1 = 0) = 1 - p$
$Y = X_1 + .. + X_n \sim B(n, p), \mathbb{P}(Y = k) = \binom{n}{k}p^k(1-p)^{n-k}$
If p is very small, n is large; $np = \lambda$
$p = 1/100, n = 100; np = 1$

$$\binom{n}{k}p^k(1-p)^{n-k} = \binom{n}{k}(\frac{\lambda}{n})^k(1-\frac{\lambda}{n})^{n-k} = \lambda^k(1-\frac{\lambda}{n})^{-k}(1-\frac{\lambda}{n!})^n\binom{n}{k}\frac{1}{n^k}$$

Many factors can be eliminated when n is large $\rightarrow$

$$\lim_{n\rightarrow\infty}(1+\frac{x}{n})^n = e^x$$

$$\binom{n}{k}\frac{1}{n^k} = \frac{n!}{k!(n-k)!}\frac{1}{n^k} = \frac{(n-k+1)(n-k+2)...n}{n \times n \times ... \times n}\frac{1}{k!}$$

Simplify the left fraction:

$$(1-\frac{k-1}{n})(1-\frac{k-2}{n})...(1-\frac{1}{n}) \rightarrow 1$$

$$\to \frac{1}{k!}$$

So, in the end:

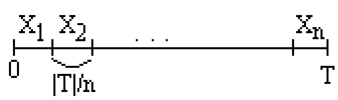
$$\binom{n}{k} p^k (1-p)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}$$

Poisson distribution with parameter $\lambda$ results.

Example:
$B(100, 1/100) \approx \Pi(1); \mathbb{P}(2) \approx \frac{1}{2}e^{-1} = \frac{e^{-1}}{2}$ very close to actual.

Counting Processes: Wrong connections to a phone number, number of typos in a book on a page, number of bacteria on a part of a plate.

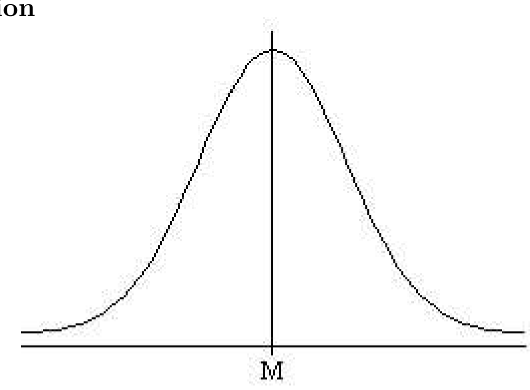
Properties:
1) Count(S) - a count of random objects in a region $S \subseteq T$
$\mathbb{E}(count(S)) = \lambda \times |S|$, where $|S|$ - size of S
(property of proportionality)
2) Counts on disjoint regions are independent.
3) $\mathbb{P}(count(S) \geq 2)$ is very small if the size of the region is small.
1, 2, and 3 lead to $count(S) \sim \Pi(\lambda |S|), \lambda$ - intensity parameter.




A region from [0, T] is split into n sections, each section has size $|T|/n$
The count on each region is $X_1, ..., X_n$
By 2), $X_1, ..., X_n$ are independent. $\mathbb{P}(X_i \geq 2)$ is small if n is large.
By 1), $\mathbb{E}X_i = \lambda\frac{|T|}{n} = 0(\mathbb{P}(X_1 = 0)) + 1(\mathbb{P}(X_1 = 1) + 2(\mathbb{P}(X_1 = 2)) + ...$
But, over 1 the value is very small.
$\mathbb{P}(X_i = 1) \approx \frac{\lambda|T|}{n}$
$\mathbb{P}(X_1 = 0) \approx 1 - \frac{\lambda|T|}{n}$

$$\mathbb{P}(count(T) = k) = \mathbb{P}(X_1 + ... + X_n = k) \approx B(n, \frac{\lambda|T|}{n}) \approx \Pi(\lambda|T|) \approx \frac{(\lambda|T|)^k}{k!} e^{-\lambda|T|}$$

§5.6 - **Normal Distribution**

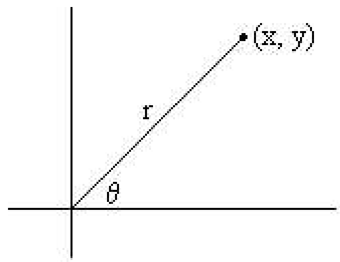


$$(\int e^{\frac{-x^2}{2}} dx)^2$$

Change variables to facilitate integration:

$$= \int e^{\frac{-x^2}{2}} dx \times \int e^{\frac{-y^2}{2}} dy = \int \int e^{\frac{-(x^2+y^2)}{2}} dxdy$$

Convert to polar:




$$= \int_0^{2\pi} \int_0^{\infty} e^{-\frac{1}{2}r^2} r dr d\theta = 2\pi \int_0^{\infty} e^{-\frac{1}{2}r^2} r dr = 2\pi \int_0^{\infty} e^{-\frac{1}{2}r^2} r d(\frac{r^2}{2}) = 2\pi \int_0^{\infty} e^{-t} dt = 2\pi$$

So, original integral area = $\sqrt{2\pi}$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = 1$$

p.d.f.:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

Standard normal distribution, N(0, 1)

** End of Lecture 20