18.05 Lecture 21
April 1, 2005

Normal Distribution

Standard Normal Distribution, N(0, 1)
p.d.f.:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

m.g.f.:

$$\phi(t) = \mathbb{E}(e^{tX}) = e^{t^2/2}$$

Proof - Simplify integral by completing the square:

$$\phi(t) = \int e^{tx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int e^{tx - x^2/2} dx =$$

$$\frac{1}{\sqrt{2\pi}} \int e^{t^2/2 - t^2/2 + tx - x^2/2} dx = \frac{1}{\sqrt{2\pi}} e^{t^2/2} \int e^{-\frac{1}{2}(t-x)^2} dx$$

Then, perform the change of variables y = x - t:

$$= \frac{1}{\sqrt{2\pi}} e^{t^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy = e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy = e^{t^2/2} \int f(x) dx = e^{t^2/2}$$

Use the m.g.f. to find expectation of X and X^2 and therefore $\text{Var}(X)$:

$$\mathbb{E}(X) = \phi'(0) = te^{t^2/2}|_{t=0} = 0; \mathbb{E}(X^2) = \phi''(0) = e^{t^2/2}t^2 + e^{t^2/2}|_{t=0} = 1; \text{Var}(X) = 1$$

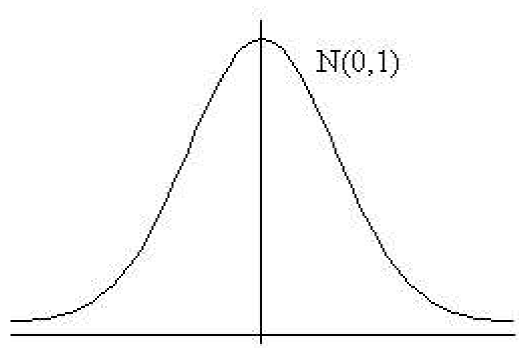
Consider $X \sim N(0, 1), Y = \sigma X + \mu$, find the distribution of Y:

$$\mathbb{P}(Y \leq y) = \mathbb{P}(\sigma X + \mu \leq y) = \mathbb{P}(X \leq \frac{y - \mu}{\sigma}) = \int_{-\infty}^{\frac{y-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

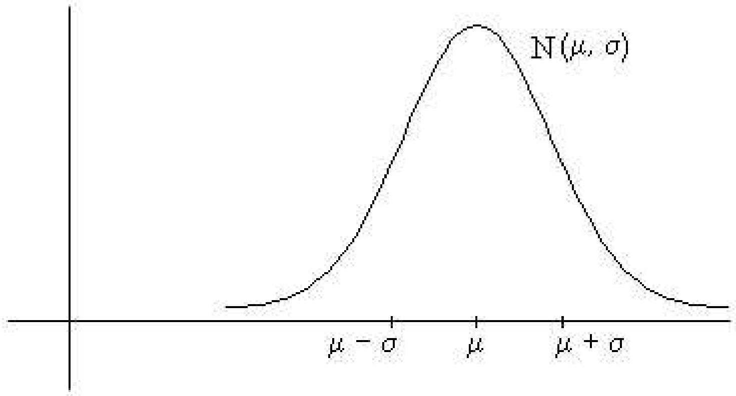
p.d.f. of Y:

$$f(y) = \frac{\partial \mathbb{P}(Y \leq y)}{\partial y} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \frac{1}{\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \rightarrow N(\mu, \sigma)$$

$\mathbb{E}Y = \mathbb{E}(\sigma X + \mu) = \sigma(0) + \mu(1) = \mu$
$\mathbb{E}(Y - \mu)^2 = \mathbb{E}(\sigma X + \mu - \mu)^2 = \sigma^2 \mathbb{E}(X^2) = \sigma^2$ - variance of $N(\mu, \sigma)$
$\sigma = \sqrt{\text{Var}(X)}$ - standard deviation



To describe an altered standard normal distribution N(0, 1) to a normal distribution $N(\mu, \sigma)$,
The peak is located at the new mean μ, and the point of inflection occurs σ away from μ



Moment Generating Function of $N(\mu, \sigma)$;
$Y = \sigma X + \mu$

$$\phi(t) = \mathbb{E}e^{tY} = \mathbb{E}e^{t(\sigma X+\mu)} = \mathbb{E}e^{(t\sigma)X}e^{t\mu} = e^{t\mu}\mathbb{E}e^{(t\sigma)X} = e^{t\mu}e^{(t\sigma)^2/2} = e^{t\mu+t^2(\sigma)^2/2}$$

Note: $X_1 \sim N(\mu_1, \sigma_1), ..., X_n \sim N(\mu_n, \sigma_n)$ - independent.
$Y = X_1 + ... + X_n$, distribution of Y:
Use moment generating function:

$$\mathbb{E}e^{tY} = \mathbb{E}e^{t(X_1+...+X_n)} = \mathbb{E}e^{tX_1}...e^{tX_n} = \mathbb{E}e^{tX_1}...\mathbb{E}e^{tX_n} = e^{\mu_1 t+\sigma_1^2 t^2/2} \times ... \times e^{\mu_n t+\sigma_n^2 t^2/2}$$

$$= e^{\sum \mu_i t + \sum \sigma_i^2 t^2/2} \sim N(\sum \mu_i, \sqrt{\sum \sigma_i^2})$$

The sum of different normal distributions is still normal!
This is not always true for other distributions (such as exponential)

Example:
$X \sim N(\mu, \sigma), Y = cX$, find that the distribution is still normal:
$Y = c(\sigma N(0,1) + \mu) = (c\sigma)N(0,1) + (\mu c)$
$Y \sim cN(\mu, \sigma) = N(c\mu, c\sigma)$

Example:
$Y \sim N(\mu, \sigma)$
$\mathbb{P}(a \leq Y \leq b) = \mathbb{P}(a \leq \sigma x + \mu \leq b) = \mathbb{P}(\frac{a-\mu}{\sigma} \leq X \leq \frac{b-\mu}{\sigma})$
This indicates the new limits for the standard normal.

Example:
Suppose that the heights of women: $X \sim N(65, 1)$ and men: $Y \sim N(68, 2)$
$\mathbb{P}$(randomly chosen woman taller than randomly chosen man)
$\mathbb{P}(X > Y) = \mathbb{P}(X - Y > 0)$
$Z = X - Y \sim N(65 - 68, \sqrt{1^2 + 2^2}) = N(-3, \sqrt(5))$
$\mathbb{P}(Z > 0) = \mathbb{P}(\frac{Z-(-3)}{\sqrt{5}} > \frac{-(-3)}{\sqrt{5}}) = \mathbb{P}(\text{standard normal } > \frac{3}{\sqrt{5}} = 1.342) = 0.09$
Probability values tabulated in the back of the textbook.

Central Limit Theorem
Flip 100 coins, expect 50 tails, somewhere 45-50 is considered typical.

Flip 10,000 coins, expect 5,000 tails, and the deviation can be larger, perhaps 4,950-5,050 is typical.

$X_i = \{1(tail); 0(head)\}$

$$\frac{\text{number of tails}}{n} = \frac{X_1 + ... + X_n}{n} \to \mathbb{E}(X_1) = \frac{1}{2} \text{ by LLN } \mathrm{Var}(X_1) = \frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}$$

But, how do you describe the deviations?
$X_1, X_2, ..., X_n$ are independent with some distribution $\mathbb{P}$

$$\mu = \mathbb{E}X_1, \sigma^2 = \mathrm{Var}(X_1); \overline{x} = \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathbb{E}X_1 = \mu$$

$\overline{x} - \mu$ on the order of $\sqrt{n} \to \frac{\sqrt{n}(\overline{x}-\mu)}{\sigma}$ behaves like standard normal.

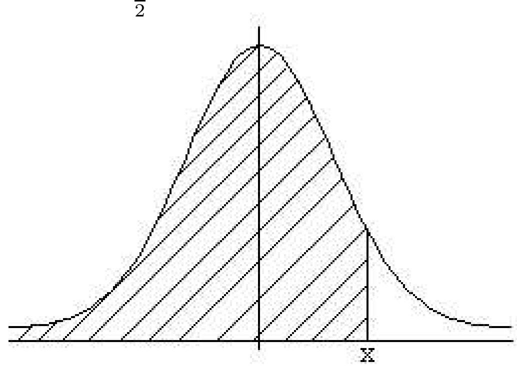
$$\frac{\sqrt{n}(\overline{x} - \mu)}{\sigma} \text{ is approximately standard normal } N(0,1) \text{ for large n}$$

$$\mathbb{P}(\frac{\sqrt{n}(\overline{x} - \mu)}{\sigma} \leq x) \overrightarrow{n \to \infty} \mathbb{P}(\text{standard normal } \leq x) = N(0,1)(-\infty, x)$$

This is useful in terms of statistics to describe outcomes as likely or unlikely in an experiment.

$$\mathbb{P}(\text{number of tails} \leq 4900) = \mathbb{P}(X_1 + ... + X_{10,000} \leq 4,900) = \mathbb{P}(\overline{x} \leq 0.49) =$$

$$= \mathbb{P}(\frac{\sqrt{10,000}(\overline{x} - \frac{1}{2})}{\frac{1}{2}} \leq \frac{\sqrt{10,000}(0.49 - 0.5)}{\frac{1}{2}}) \approx N(0,1)(-\infty, -\frac{100(0.01)}{\frac{1}{2}} = -2) = 0.0267$$



Tabulated values always give for positive X, area to the left.

In the table, look up -2 by finding the value for 2 and taking the complement.

** End of Lecture 21