18.05 Lecture 24
April 8, 2005

Bayes Estimator.
Prior Distribution $f(\theta) \rightarrow$ compute posterior $f(\theta|X_1, ..., X_n)$
Bayes's Estimator = expectation of the posterior.
$\mathbb{E}(X-a)^2 \rightarrow$ minimize a $\rightarrow a = \mathbb{E}X$

Example: $B(p), f(p) = \text{Beta}(\alpha, \beta) \rightarrow f(p|x_1, ..., x_n) = \text{Beta}(\alpha + \sum x_i, \beta + n - \sum x_i)$

$$\theta(x_1, ..., x_n) = \frac{\alpha + \sum x_i}{\alpha + \beta + n}$$

Example: Poisson Distribution

$$\Pi(\lambda), f(x|\lambda) = \frac{\lambda^x}{x! e^{-\lambda}}$$

Joint p.f.:

$$f(x_1, ..., x_n|\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i}}{x_i!} e^{-\lambda} = \frac{\lambda^{\sum x_i}}{\Pi x_i!} e^{-n\lambda}$$

If $f(\lambda)$ is the prior distribution, posterior:

$$f(\lambda|x_1, ..., x_n) = \frac{\frac{\lambda^{\sum x_i}}{\Pi x_i!} e^{-n\lambda} f(\lambda)}{g(x_1 ... x_n) = \int \frac{\lambda^{\sum x_i}}{\Pi x_i!} e^{-n\lambda} f(\lambda) d\lambda}$$

Note that g does not depend on λ:

$$f(\lambda|x_1, ..., x_n) \sim \lambda^{\sum x_i} e^{-n\lambda} f(\lambda)$$

Need to choose the appropriate prior distribution, Gamma distribution works for Poisson.

Take $f(\lambda)$ - p.d.f. of $\Gamma(\alpha, \beta)$,

$$f(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$$

$f(\lambda|x_1, ..., x_n) \sim \lambda^{\sum x_i + \alpha - 1} e^{-(n+\beta)\lambda} \rightarrow \Gamma(\alpha + \sum x_i, \beta + n)$
Bayes Estimator:

$$\lambda(x_1, ..., x_n) = \mathbb{E}X = \frac{\alpha + \sum x_i}{n + \beta}$$

Once again, balances both prior intuition and data, by law of large numbers:

$$\lambda(x_1, ..., x_n) = \frac{\alpha/n + \sum x_i/n}{1 + \beta/n} \xrightarrow{n \rightarrow \infty} \overline{x} \rightarrow \mathbb{E}(X_1) \rightarrow \lambda$$

The estimator approaches what you're looking for, with large n.

Exponential $E(\alpha), f(x|\alpha) = \alpha e^{-\alpha x}, x \geq 0$
$f(x_1, ..., x_n|\alpha) = \Pi_{i=1}^{n} \alpha e^{-\alpha x_i} = \alpha^n e^{-(\sum x_i)\alpha}$
If $f(\alpha)$ - prior, the posterior:

$$f(\alpha|x_1,...,x_n) \sim \alpha^n e^{-(\sum x_i)\alpha} f(\alpha)$$

Once again, a Gamma distribution is implied.
Choose $f(\alpha) - \Gamma(u,v)$

$$f(\alpha) = \frac{v^u}{\Gamma(u)} \alpha^{u-1} e^{-v\alpha}$$

New posterior:

$$f(\alpha|x_1,...,x_n) \sim \alpha^{n+u-1} e^{-(\sum x_i + v)u} \rightarrow \Gamma(u+n, v+\sum x_i)$$

Bayes Estimator:

$$\alpha(x_1,...,x_n) = \frac{u+n}{v+\sum x_i} = \frac{u/n+1}{v/n+\sum x_i/n} \xrightarrow{n \rightarrow \infty} \frac{1}{\overline{x}} \rightarrow \frac{1}{\mathbb{E}X} = \alpha$$

Normal Distribution:

$$N(\mu,\sigma), f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$f(x_1,...,x_n|\mu,\sigma) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2}$$

It is difficult to find simple prior when both μ, σ are unknown.
Say that σ is given, and μ is the only parameter:

$$\text{Prior: } f(\mu) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2b^2}(\mu-a)^2} = N(a,b)$$

Posterior:

$$f(\mu|X_1,...,X_n) \sim e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2 - \frac{1}{2b^2}(\mu-a)^2}$$

Simplify the exponent:

$$= \frac{1}{2\sigma^2}\sum(x_i^2 - 2\mu x_i + \mu^2) + \frac{1}{2b^2}(\mu^2 - 2a\mu + a^2) = \mu^2(\frac{n}{2\sigma^2} + \frac{1}{2b^2}) - 2\mu(\frac{\sum x_i}{2\sigma^2} + \frac{a}{2b^2}) + ...$$

$$= \mu^2 A - 2\mu B + ... = A(\mu^2 - 2\mu\frac{B}{A} + (\frac{B}{A})^2) - \frac{B^2}{A} + ... = A(\mu - \frac{B}{A})^2 + ...$$

$$f(\mu|X_1,...,X_n) \sim e^{-A(\mu-\frac{B}{A})^2} = e^{-\frac{1}{2(1/\sqrt{2A})^2}}(\mu - \frac{B}{A})^2 = N(\frac{B}{A}, \frac{1}{\sqrt{2A}}) = N(\frac{\sigma^2 A + nb^2\overline{x}}{\sigma^2+nb^2}, \frac{\sigma^2 b^2}{\sigma^2+nb^2})$$

Normal Bayes Estimator:

$$\mu(X_1,...,X_n) = \frac{\sigma^2 a + nb^2\overline{x}}{\sigma^2+nb^2} = \frac{\sigma^2 a/n + b^2\overline{x}}{\sigma^2/n + b^2} \xrightarrow{n \rightarrow \infty} \overline{x} \rightarrow \mathbb{E}(X_1) = \mu$$

** End of Lecture 24