18.05 Lecture 27
April 15, 2005

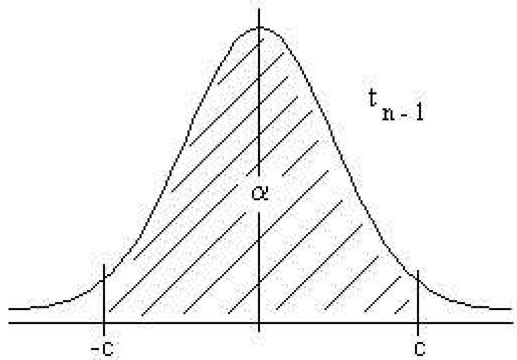
Take sample $X_1, ..., X_n \sim N(0,1)$

$$A = \frac{\sqrt{n}(\overline{x} - \mu)}{\sigma} \sim N(0,1), B = \frac{n(\overline{x^2} - (\overline{x})^2}{\sigma^2} \sim \chi^2_{n-1}$$

A, B - independent.
To determine the confidence interval for μ, must eliminate σ from A:

$$\frac{A}{\sqrt{\frac{1}{n-1}B}} = \frac{Z^0}{\sqrt{\frac{1}{n-1}(z_1^2 + ... + z_{n-1}^2)}} \sim t_{n-1}$$

Where $Z_0, Z_1, .., Z_{n-1} \sim N(0,1)$
The standard normal is a symmetric distribution, and $\frac{1}{n-1}(Z_1^2 + ... + Z_{n-1}^2) \to \mathbb{E}Z_1^2 = 1$



So t_n-distribution still looks like a normal distribution (especially for large n), and it is symmetric about zero.

Given $\alpha \in (0,1)$ find c, $t_{n-1}(-c, c) = \alpha$

$$-c \leq \frac{A}{\sqrt{\frac{1}{n-1}B}} \leq c$$

with probability = confidence (α)

$$-c \leq \frac{\sqrt{n}(\overline{x} - \mu)}{\sigma} / \sqrt{\frac{1}{n-1}\frac{n(\overline{x^2} - (\overline{x})^2)}{\sigma^2}} \leq c$$

$$-c \leq \frac{\overline{x} - \mu}{\sqrt{\frac{1}{n-1}(\overline{x^2} - (\overline{x})^2)}} \leq c$$

$$\overline{x} - c\sqrt{\frac{1}{n-1}(\overline{x^2} - (\overline{x})^2)} \leq \mu \leq \overline{x} + c\sqrt{\frac{1}{n-1}(\overline{x^2} - (\overline{x})^2)}$$

By the law of large numbers, $\overline{x} \to \mathbb{E}X = \mu$
The center of the interval is a typical estimator (for example, MLE).
error $\propto$ estimate of variance $\approx \sqrt{\frac{\sigma^2}{n}}$ for large n.
$\hat{\sigma}^2 = \overline{x^2} - (\overline{x})^2$ is a sample variance and it converges to the true variance,

by LLN $\hat{\sigma}^2 \to \sigma^2$

$$\mathbb{E}\hat{\sigma}^2 = \mathbb{E}\frac{1}{n}(x_1^2 + ... + x_n^2) - \mathbb{E}(\frac{1}{n}(x_1 + ... + x_n))^2 =$$

$$= \mathbb{E}X_1^2 - \frac{1}{n^2}\sum_{i,j}\mathbb{E}X_iX_j = \mathbb{E}X_1^2 - \frac{1}{n^2}(n\mathbb{E}X_1^2 + n(n+1)(\mathbb{E}X_1)^2)$$

Note that for $i \neq j, \mathbb{E}X_iX_j = \mathbb{E}X_i\mathbb{E}X_j = (\mathbb{E}X_1)^2 = \mu^2$, n(n - 1) terms with different indices.

$$\mathbb{E}\hat{\sigma}^2 = \frac{n-1}{n}\mathbb{E}X_1^2 - \frac{n-1}{n}(\mathbb{E}X_1)^2 =$$

$$= \frac{n-1}{n}(\mathbb{E}X_1^2 - (\mathbb{E}X_1)^2) = \frac{n-1}{n}\mathrm{Var}(X_1) = \frac{n-1}{n}\sigma^2$$

Therefore:

$$\mathbb{E}\hat{\sigma}^2 = \frac{n-1}{n}\sigma^2 < \sigma^2$$

Good estimator, but more often than not, less than actual.
So, to compensate for the lower error:

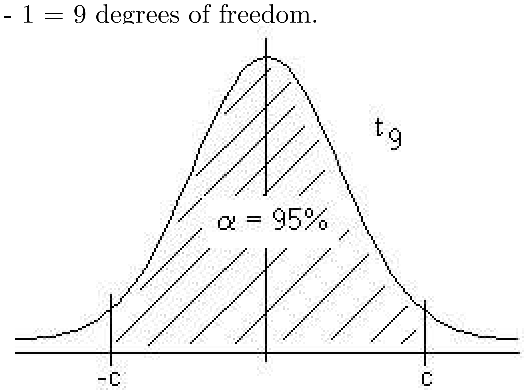
$$\mathbb{E}\frac{n}{n-1}\hat{\sigma}^2 = \sigma^2$$

Consider $(\sigma')^2 = \frac{n}{n-1}\hat{\sigma}^2$, unbiased sample variance.

$$\pm c\sqrt{\frac{1}{n-1}(\overline{x^2} - (\overline{x})^2)} = \pm c\sqrt{\frac{1}{n-1}\hat{\sigma}^2} = \pm c\sqrt{\frac{1}{n}(\sigma')^2}$$

$$\overline{x} - c\sqrt{\frac{(\sigma')^2}{n}} \leq \mu \leq \overline{x} + c\sqrt{\frac{(\sigma')^2}{n}}$$

§7.5 pg. 140 Example: Lactic Acid in Cheese
0.86, 1.53, 1.57, ..., 1.58, n = 10
$\sim N(\mu, \sigma^2), \overline{x} = 1.379, \hat{\sigma}^2 = \overline{x^2} - (\overline{x})^2 = 0.0966$
Predict parameters with confidence $\alpha = 95\%$
Use a t-distribution with n - 1 = 9 degrees of freedom.



See table: $(-\infty, c) = 0.975$ gives $c = 2.262$

$$\overline{x} - 2.262\sqrt{\frac{1}{9}\hat{\sigma}^2} \leq \mu \leq \overline{x} + 2.262\sqrt{\frac{1}{9}\hat{\sigma}^2}$$

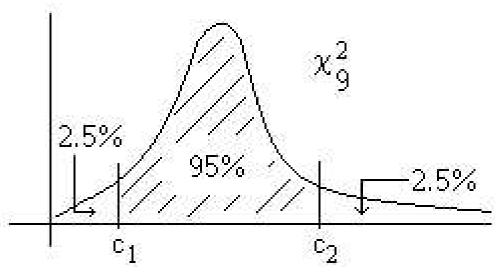
$$0.6377 \leq \mu \leq 2.1203$$

Large interval due to a high guarantee and a small number of samples.
If we change α to 90% c = 1.833, interval: $1.189 \leq \mu \leq 1.569$
Much better sized interval.

Confidence interval for variance:

$$c_1 \leq \frac{n\hat{\sigma}^2}{\sigma^2} \leq c_2$$

where the c values come from the χ^2 distribution



Not symmetric, all positive points given for χ^2 distribution.
$c_1 = 2.7, c_2 = 19.02 \rightarrow 0.0508 \leq \sigma^2 \leq 0.3579$
again, wide interval as result of small n and high confidence.

Sketch of Fisher's theorem.
$z_1, ..., z_n \sim N(0,1)$
$\sqrt{n}\overline{z} = \frac{1}{\sqrt{n}}(z_1 + ... + z_n) \sim N(0,1)$

$$n(\overline{z^2} - (\overline{z})^2) = n(\frac{1}{n}\sum z_i^2 - (\frac{1}{n}\sum z_i)^2) = \sum z_i^2 - (\frac{1}{\sqrt{n}}(z_1 + ... + z_n))^2 \sim \chi^2_{n-1}$$

$$f(z_1, ..., z_n) = (\frac{1}{\sqrt{2\pi}})^n e^{-1/2\sum z_i^2} = (\frac{1}{\sqrt{2\pi}})^n e^{-1/2r^2}$$

$$f(y_1, ..., y_n) = (\frac{1}{\sqrt{2\pi}})^n e^{-1/2r^2} = (\frac{1}{\sqrt{2\pi}})^n e^{-1/2\sum y_i^2}$$

The graph is symmetric with respect to rotation, so rotating the coordinates gives again i.i.d. standard normal sequence.

$$\prod_i \frac{1}{\sqrt{2\pi}} e^{-1/2y_i^2} \rightarrow y_1, ..., y_n - i.i.d. N(0,1)$$

Choose coordinate system such that:

$$y_1 = \frac{1}{\sqrt{n}}(z_1 + ... + z_n), \text{ i.e. } \vec{v}_1 = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}) \text{ - new first axis.}$$

Choose all other vectors however you want to make a new orthogonal basis:

$$y_1^2 + ... + y_n^2 = z_1^2 + .. + z_n^2$$

since the length does not change after rotation!

$$\sqrt{n}\overline{z} = y_1 \sim N(0,1)$$

$$n(\overline{z^2} - (\overline{z})^2) = \sum y_i^2 - y_1^2 = y_2^2 + ... + y_n^2 \sim \chi^2_{n-1}$$

** End of Lecture 27