18.05 Lecture 3
February 7, 2005

$P_{n,k} = \frac{n!}{(n-k)!}$ - choose k out of n, order counts, without replacement.
n^k - choose k out of n, order counts, with replacement.
$C_{n,k} = \frac{n!}{k!(n-k)!}$ - choose k out of n, order doesn't count, without replacement.

§1.9 Multinomial Coefficients
These values are used to split objects into groups of various sizes.
$s_1, s_2, ..., s_n$ - n elements such that n_1 in group 1, n_2 in group 2, ..., n_k in group k.
$n_1 + ... + n_k = n$

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3} \times ... \times \binom{n-n_1-...-n_{k-2}}{n_{k-1}}\binom{n_k}{n_k}$$

$$= \frac{n!}{n_1!(n-n_1)!} \times \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \times \frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!} \times ... \times \frac{(n-n_1-...-n_{k-2})!}{n_{k-1}!(n-n_1-...-n_{k-1})!} \times 1$$

$$= \frac{n!}{n_1!n_2!...n_{k-1}!n_k!} = \binom{n}{n_1, n_2, ..., n_k}$$

These combinations are called multinomial coefficients.

Further explanation: You have n "spots" in which you have n! ways to place your elements.
However, you can permute the elements within a particular group and the splitting is still the same.
You must therefore divide out these internal permutations.
This is a "distinguishable permutations" situation.

Example #1 - 20 members of a club need to be split into 3 committees (A, B, C) of 8, 8, and 4 people, respectively. How many ways are there to split the club into these committees?

$$\text{ways to split} = \binom{20}{8,8,4} = \frac{20!}{8!8!4!}$$

Example #2 - When rolling 12 dice, what is the probability that 6 pairs are thrown?
This can be thought of as "each number appears twice"
There are 6^{12} possibilities for the dice throws, as each of the 12 dice has 6 possible values.
In pairs, the only freedom is **where** the dice show up.

$$\binom{12}{2,2,2,2,2,2} = \frac{12!}{(2!)^6} \rightsquigarrow \mathbb{P} = \frac{12!}{(2!)^6 6^{12}} = 0.0034$$

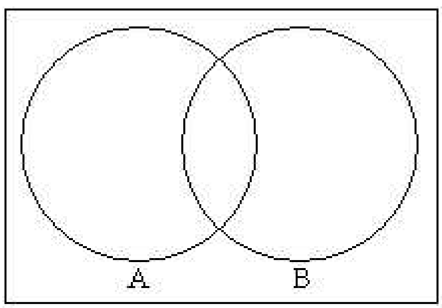
Example #3 - Playing Bridge
Players A, B, C, and D each get 13 cards.
$\mathbb{P}(A - 6\heartsuit s, B - 4\heartsuit s, C - 2\heartsuit s, D - 1\heartsuit) = ?$

$$\mathbb{P} = \frac{\binom{13}{6,4,2,1}\binom{39}{7,9,11,12}}{\binom{52}{13,13,13,13}} = \frac{(\text{choose } \heartsuit s)(\text{choose other cards})}{(\text{ways to arrange all cards})} = 0.00196$$

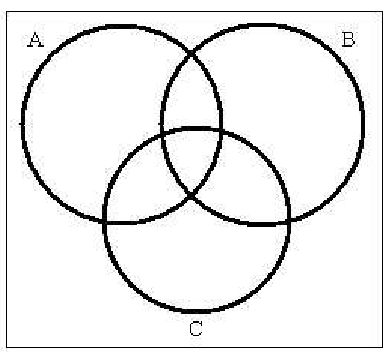
Note - If it didn't matter who got the cards, multiply by 4! to arrange people around the hands.
Alternate way to solve - just track the locations of the $\heartsuit$ s

$$\mathbb{P} = \frac{\binom{13}{6}\binom{13}{4}\binom{13}{2}\binom{13}{1}}{\binom{52}{13}}$$

Probabilities of Unions of Events:



$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(AB)$



$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(AB) - \mathbb{P}(BC) - \mathbb{P}(AC) + \mathbb{P}(ABC)$

§1.10 - Calculating a Union of Events - $\mathbb{P}$(union of events)
$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(AB)$ (Figure 1)
$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(AB) - \mathbb{P}(BC) - \mathbb{P}(AC) + \mathbb{P}(ABC)$ (Figure 2)

Theorem:

$$\mathbb{P}(\bigcup_{i=1}^{n} A_i) = \sum_{i \leq n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i A_j) + \sum_{i<j<k} \mathbb{P}(A_i A_j A_k) - ... + (-1)^{n+1} \mathbb{P}(A_i ... A_n)$$

Express each disjoint piece, then add them up according to what sets each piece belongs or doesn't belong to.
$A_1 \cup ... \cup A_n$ can be split into a disjoint partition of sets:

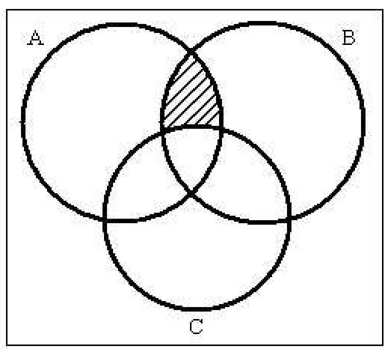
$$A_{i1} \cap A_{i2} \cap ... \cap A_{ik} \cap A^c_{i(k+1)} \cap ... \cap A^c_{in}$$

where k = last set the piece is a part of.

$$\mathbb{P}(\bigcup_{i=1}^{n} A_i) = \sum \mathbb{P}(\text{disjoint partition})$$

To check if the theorem is correct, see how many times each partition is counted.
$\mathbb{P}(A_1), \mathbb{P}(A_2), ..., \mathbb{P}(A_k)$ - k times
$\sum_{i<j} \mathbb{P}(A_i A_j) - \binom{k}{2}$ times
(needs to contain A_i and A_j in k different intersections.)

Example: Consider the piece $A \cap B \cap C^c$, as shown:



This piece is counted: $\mathbb{P}(A \cup B \cup C)$ = once.
$\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C)$ = counted twice.
$-\mathbb{P}(AB) - \mathbb{P}(AC) - \mathbb{P}(BC)$ = subtracted once.
$+\mathbb{P}(ABC)$ = counted zero times.
The sum: 2 - 1 + 0 = 1, piece only counted once.

Example: Consider the piece $A_1 \cap A_2 \cap A_3 \cap A_4^c$
k = 3, n = 4.
$\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \mathbb{P}(A_4)$ = counted k times (3 times).
$-\mathbb{P}(A_1A_2) - \mathbb{P}(A_1A_3) - \mathbb{P}(A_1A_4) - \mathbb{P}(A_2A_3) - \mathbb{P}(A_2A_4) - \mathbb{P}(A_3A_4)$ = counted $\binom{k}{2}$ times (3 times).
as follows: $\sum_{i<j<k}$ = counted $\binom{k}{3}$ times (1 time).
total in general: $k - \binom{k}{2} + \binom{k}{3} - \binom{k}{4} + ... + (-1)^{k+1}\binom{k}{k}$ = sum of times counted.

To simplify, this is a binomial situation.

$$0 = (1-1)^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^i (1)^{(k-i)} = \binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} ...$$

$$0 = 1 - \text{sum of times counted}$$

therefore, all disjoint pieces are counted once.

** End of Lecture 3