18.05 Lecture 30
April 27, 2005

Bayes Decision Rule
$\xi(1)\alpha_1(\delta) + \xi(2)\alpha_2(\delta) \rightarrow$ minimize.

$$\delta = \{H_1 : \frac{f_1(\mathbf{x})}{f_2(\mathbf{x})} > \frac{\xi(2)}{\xi(1)}; H_2 : \text{ if } <; H_1 \text{ or } H_2 : \text{ if } =\}$$

Example: see pg. 469, Problem 3
$H_0 : f_1(x) = 1$ for $0 \leq x \leq 1$
$H_1 : f_2(x) = 2x$ for $0 \leq x \leq 1$
Sample 1 point x_1
Minimize $3\alpha_0(\delta) + 1\alpha_1(\delta)$

$$\delta = \{H_0 : \frac{1}{2x_1} > \frac{1}{3}; H_1 : \frac{1}{2x_1} < \frac{1}{3}; \text{either if equal}\}$$

Simplify the expression:

$$\delta = \{H_0 : x_1 \leq \frac{3}{2}; H_1 : x_1 > \frac{3}{2}\}$$

Since x_1 is always between 0 and 1, H_0 is always chosen. $\delta = H_0$ always.

Errors:
$\alpha_0(\delta) = \mathbb{P}_0(\delta \neq H_0) = 0$
$\alpha_1(\delta) = \mathbb{P}_1(\delta \neq H_1) = 1$
We made the α_0 very important in the weighting, so it ended up being 0.

Most powerful test for two simple hypotheses.
Consider a class $K_\alpha = \{\delta \text{ such that } \alpha_1(\delta) \leq \alpha \in [0, 1]\}$
Take the following decision rule:

$$\delta = \{H_1 : \frac{f_1(\mathbf{x})}{f_2(\mathbf{x})} \geq c; H_2 : \frac{f_1(\mathbf{x})}{f_2(\mathbf{x})} < c\}$$

Calculate the constant from the confidence level α:

$$\alpha_1(\delta) = \mathbb{P}_1(\delta \neq H_1) = \mathbb{P}_1(\frac{f_1(\mathbf{x})}{f_2(\mathbf{x})} < c) = \alpha$$

Sometimes it is difficult to find c, if discrete, but consider the simplest continuous case first:
Find $\xi(1), \xi(2)$ such that $\xi(1) + \xi(2) = 1, \frac{\xi(2)}{\xi(1)} = c$
Then, δ is a Bayes decision rule.
$\xi(1)\alpha_1(\delta) + \xi(2)\alpha_2(\delta) \leq \xi(1)\alpha_1(\delta') + \xi(2)\alpha_2(\delta')$
for any decision rule δ'
If $\delta' \in K_\alpha$ then $\alpha_1(\delta') \leq \alpha$.
Note: $\alpha_1(\delta) = \alpha$, so: $\xi(1)\alpha + \xi(2)\alpha_2(\delta) \leq \xi(1)\alpha + \xi(2)\alpha_2(\delta')$
Therefore: $\alpha_2(\delta) \leq \alpha_2(\delta'), \delta$ is the best (mosst powerful) decision rule in K_α

Example:
$H_1 : N(0, 1), H_2 : N(1, 1), \alpha_1(\delta) = 0.05$

$$\frac{f_1(\mathbf{x})}{f_2(\mathbf{x})} = e^{-\frac{1}{2}\sum x_i^2 + \frac{1}{2}\sum (x_i-1)^2} = e^{\frac{n}{2} - \sum x_i} \geq c$$

Always simplify first:

$$\frac{n}{2} - \sum x_i \geq \log(c), \sum x_i \leq \frac{n}{2} + \log(c), \sum x_i \leq c'$$

The decision rule becomes:

$$\delta = \{H_1 : \sum x_i \leq c'; H_2 : \sum x_i > c'\}$$

Now, find c'
$\alpha_1(\delta) = \mathbb{P}_1(\sum x_i > c')$
recall, subscript on $\mathbb{P}$ indicates that $x_1, ..., x_n \sim N(0,1)$
Make into standard normal:

$$\mathbb{P}_1(\frac{\sum x_i}{\sqrt{n}} > \frac{c'}{\sqrt{n}}) = 0.05$$

Check the table for $\mathbb{P}(z > c'') = 0.05, c'' = 1.64, c' = \sqrt{n}(1.64)$

Note: a very common error with the central limit theorem:

$$\sum x_i \rightarrow \sqrt{n}(\frac{\frac{1}{n}\sum x_i - \mu}{\sigma}) \rightarrow \frac{\sum x_i - n\mu}{\sqrt{n}\sigma}$$

These two conversions are the same! Don't combine techniques from both.

The Bayes decision rule now becomes:

$$\delta = \{H_1 : \sum x_i \leq 1.64\sqrt{n}; H_2 : \sum x_i > 1.64\sqrt{n}\}$$

Error of Type 2:
$\alpha_2(\delta) = \mathbb{P}_2(\sum x_i \leq c = 1.64\sqrt{n})$
Note: subscript indicates that $X_1, ..., X_n \sim N(1,1)$

$$= \mathbb{P}_2(\frac{\sum x_i - n(1)}{\sqrt{n}} \leq \frac{1.64\sqrt{n} - n}{\sqrt{n}}) = \mathbb{P}_2(z \leq 1.64 - \sqrt{n})$$

Use tables for standard normal to get the probability.
If n = 9 $\rightarrow \mathbb{P}_2(z \leq 1.64 - \sqrt{9}) = \mathbb{P}_2(z \leq -1.355) = 0.0877$

Example:
$H_1 : N(0,2), H_2 : N(0,3), \alpha_1(\delta) = 0.05$

$$\frac{f_1(\mathbf{x})}{f_2(\mathbf{x})} = \frac{(\frac{1}{2\sqrt{2\pi}})^n e^{-\sum \frac{1}{2(2)}x_i^2}}{(\frac{1}{3\sqrt{2\pi}})^n e^{-\sum \frac{1}{2(3)}x_i^2}} = (\frac{3}{2})^{n/2} e^{-\frac{1}{12}\sum x_i^2} \geq c$$

$$\delta = \{H_1 : \sum x_i^2 \leq c'; H_2 : \sum x_i^2 > c'\}$$

This is intuitive, as the sum of squares $\sim$ sample variance.
If small $\rightarrow \sigma = 2$
If large $\rightarrow \sigma = 3$

$\alpha_1(\delta) = \mathbb{P}_1(\sum x_i^2 > c') = \mathbb{P}_1(\sum \frac{x_i^2}{2} > \frac{c'}{2}) = \mathbb{P}_1(\chi_n^2 > c'') = 0.05$
If n = 10, $\mathbb{P}_1(\chi_{10}^2 > c'') = 0.05; c'' = 18.31, c' = 36.62$

Can find error of type 2 in the same way as earlier:
$\mathbb{P}(\chi_n^2 > \frac{c'}{3}) \rightarrow \mathbb{P}(\chi_{10}^2 > 12.1) \approx 0.7$
A difference of 1 in variance is a huge deal!
Large type 2 error results, small n.

** End of Lecture 30