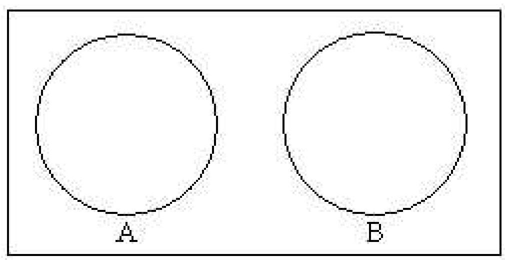
18.05 Lecture 5
February 14, 2005

§2.2 Independence of events.
$\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}$;
Definition - A and B are independent if $\mathbb{P}(A|B) = \mathbb{P}(A)$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} = \mathbb{P}(A) \rightsquigarrow \mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B)$$

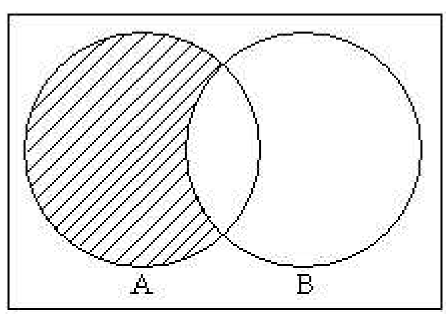
Experiments can be physically independent (roll 1 die, then roll another die), or seem physically related and still be independent.
Example: A = {odd}, B = {1, 2, 3, 4}. Related events, but independent.
$\mathbb{P}(A) = \frac{1}{2}.\mathbb{P}(B) = \frac{2}{3}.AB = \{1, 3\}$
$\mathbb{P}(AB) = \frac{1}{2} \times \frac{2}{3} = \mathbb{P}(AB) = \frac{1}{3}$, therefore independent.

Independence does not imply that the sets do not intersect.



Disjoint $\neq$ Independent.

If A, B are independent, find $\mathbb{P}(AB^c)$
$\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B)$
$AB^c = A \setminus AB$, as shown:



so, $\mathbb{P}(AB^c) = \mathbb{P}(A) - \mathbb{P}(AB)$
$= \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B)$
$= \mathbb{P}(A)(1 - \mathbb{P}(B))$
$= \mathbb{P}(A)\mathbb{P}(B^c)$
therefore, A and B^c are independent as well.
similarly, A^c and B^c are independent. See Pset 3 for proof.

Independence allows you to find $\mathbb{P}$(intersection) through simple multiplication.

Example: Toss an unfair coin twice, these are independent events.
$\mathbb{P}(H) = p, 0 \leq p \leq 1,$ find $\mathbb{P}(``TH'')$ = tails first, heads second
$\mathbb{P}(``TH'') = \mathbb{P}(T)\mathbb{P}(H) = (1-p)p$
Since this is an unfair coin, the probability is **not** just $\frac{1}{4}$
If fair, $\frac{TH}{HH+HT+TH+TT} = \frac{1}{4}$

If you have several events: $A_1, A_2, ...A_n$ that you need to prove independent:
It is necessary to show that **any** subset is independent.
Total subsets: $A_{i1}, A_{i2}, ..., A_{ik}, 2 \leq k \leq n$
Prove: $\mathbb{P}(A_{i1}A_{i2}...A_{ik}) = \mathbb{P}(A_{i1})\mathbb{P}(A_{i2})...\mathbb{P}(A_{ik})$
You could prove that any 2 events are independent, which is called "pairwise" independence, but this is not sufficient to prove that all events are independent.

Example of pairwise independence:
Consider a tetrahedral die, equally weighted.
Three of the faces are each colored red, blue, and green,
but the last face is multicolored, containing red, blue and green.
$\mathbb{P}(\text{red}) = 2/4 = 1/2 = \mathbb{P}(\text{blue}) = \mathbb{P}(\text{green})$
$\mathbb{P}(\text{red and blue}) = 1/4 = 1/2 \times 1/2 = \mathbb{P}(\text{red})\mathbb{P}(\text{blue})$
Therefore, the pair {red, blue} is independent.
The same can be proven for {red, green} and {blue, green}.
but, what about all three together?
$\mathbb{P}(\text{red, blue, and green}) = 1/4 \neq \mathbb{P}(\text{red})\mathbb{P}(\text{blue})\mathbb{P}(\text{green}) = 1/8$, not fully independent.

Example: $\mathbb{P}(H) = p, \mathbb{P}(T) = 1-p$ for unfair coin
Toss the coin 5 times $\rightsquigarrow \mathbb{P}(\text{``HTHTT''})$
$= \mathbb{P}(H)\mathbb{P}(T)\mathbb{P}(H)\mathbb{P}(T)\mathbb{P}(T)$
$= p(1-p)p(1-p)(1-p) = p^2(1-p)^3$

Example: Find $\mathbb{P}$(get 2H and 3T, in any order)
= sum of probabilities for ordering
$= \mathbb{P}(HHTTT) + \mathbb{P}(HTHTT) = ...$
$= p^2(1-p)^3 + p^2(1-p)^3 + ...$
$= \binom{5}{2}p^2(1-p)^3$

General Example: Throw a coin n times, $\mathbb{P}$(k heads out of n throws)

$$= \binom{n}{k}p^k(1-p)^{n-k}$$

Example: Toss a coin until the result is "heads;" there are n tosses before H results.
$\mathbb{P}$(number of tosses = n) =?
needs to result as "TTT....TH," number of T's = (n - 1)

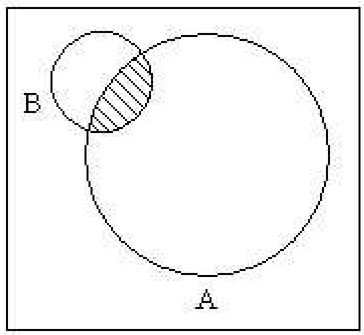
$$\mathbb{P}(\text{tosses = n}) = \mathbb{P}(TT...H) = (1-p)^{n-1}p$$

Example: In a criminal case, witnesses give a specific description of the couple seen fleeing the scene.
$\mathbb{P}(\text{random couple meets description}) = 8.3 \times 10^{-8} = p$
We know at the beginning that 1 couple exists. Perhaps a better question to be asked is:
Given a couple exists, what is the probability that another couple fits the same description?
$\mathbb{P}$(2 couples exists)
$A = \mathbb{P}(\text{at least 1 couple}), B = \mathbb{P}(\text{at least 2 couples}),$ find $\mathbb{P}(B|A)$
$\mathbb{P}(B|A) = \frac{\mathbb{P}(BA)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B)}{\mathbb{P}(A)}$

Out of n couples, $\mathbb{P}(A) = \mathbb{P}(\text{at least 1 couple}) = 1 - \mathbb{P}(\text{no couples}) = 1 - \prod_{i=1}^{n}(1-p)$
Each couple doesn't satisfy the description, if no couples exist.
Use independence property, and multiply.
$\mathbb{P}(A) = 1 - (1-p)^n$
$\mathbb{P}(B) = \mathbb{P}(\text{at least two}) = 1 - \mathbb{P}(\text{0 couples}) - \mathbb{P}(\text{exactly 1 couple})$
$= 1 - (1-p)^n - n \times p(1-p)^{n-1}$, keep in mind that $\mathbb{P}$(exactly 1) falls into $\mathbb{P}$(k out of n)

$$\mathbb{P}(B|A) = \frac{1 - (1-p)^n - np(1-p)^{n-1}}{1 - (1-p)^n}$$

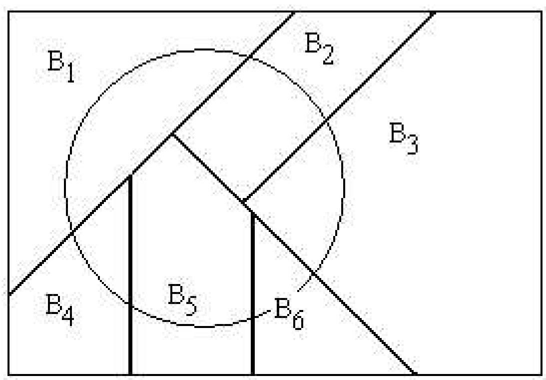
If n = 8 million people, $\mathbb{P}(B|A) = 0.2966$, which is within reasonable doubt!
$\mathbb{P}(\text{2 couples}) < \mathbb{P}(\text{1 couple})$, but given that 1 couple exists, the probability that 2 exist is not insignificant.



In the large sample space, the probability that B occurs when we know that A occured is significant!

§2.3 **Bayes's Theorem**

It is sometimes useful to separate a sample space S into a set of disjoint partitions:



$B_1, ..., B_k$ - a partition of sample space S.

$B_i \cap B_j = \emptyset$, for $i \neq j, S = \bigcup_{i=1}^{k} B_i$ (disjoint)
Total probability: $\mathbb{P}(A) = \sum_{i=1}^{k} \mathbb{P}(AB_i) = \sum_{i=1}^{k} \mathbb{P}(A|B_i)\mathbb{P}(B_i)$
(all AB_i are disjoint, $\bigcup_{i=1}^{k} AB_i = A$)

** End of Lecture 5