18.05 Lecture 6
February 16, 2005

Solutions to Problem Set #1

1-1 pg. 12 #9
$B_n = \bigcup_{i=n}^{\infty} A_i, C_n = \bigcap_{i=n}^{\infty} A_i$
a) $B_n \supset B_{n+1} ...$
$B_n = A_n \cup (\bigcup_{i=n+1}^{\infty} A_i) = A_n \cup B_{n+1}$
$s \in B_{n+1} \Rightarrow s \in B_{n+1} \cup A_n = B_n$
$C_n \subset C_{n+1} ...$
$C_n = A_n \cap C_{n+1}$
$s \in C_n = A_n \cap C_{n+1} \Rightarrow s \in C_{n+1}$
b) $s \in \bigcap_{n=1}^{\infty} B_n \Rightarrow s \in B_n$ for all n
$s \in \bigcup_{i=1}^{\infty} A_i$ for all n $\Rightarrow s \in$ some A_i for $i \geq n$, for all n
$\Rightarrow s \in$ infinitely many events $A_i \Rightarrow A_i$ happen infinitely often.
c) $s \in \bigcup_{n=1}^{\infty} C_n \Rightarrow s \in$ some $C_n = \bigcap_{i=n}^{\infty} A_i \Rightarrow$ for some n, s $\in$ all A_i for $i \geq n$
$\Rightarrow s \in$ all events starting at n.

1-2 pg. 18 #4
P(at least 1 fails) $= 1 - P$(neither fail) $= 1 - 0.4 = 0.6$

1-3 pg. 18 #12
$A_1, A_2, ...$
$B_1 = A_1, B_2 = A_1^c A_2, ..., B_n = A_1^c ... A_{n-1}^c A_n$
$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(B_i)$ splits the union into disjoint events, and covers the entire space.
follows from: $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$
take point (s) in $\bigcup_{i=1}^{n} A_i, \Rightarrow s \in$ at least one $\Rightarrow s \in A_1 = B_1$,
if not, $s \in A_1^c$, if $s \in A_2$, then $s \in A_1^c A_2 = B_2$, if not... etc.
at some point, the point belongs to a set.
The sequence stops when $s \in A_1^c \cap A_2^c \cap ... \cap A_{k-1}^c \cap A_k = B_k$
$\Rightarrow s \in \bigcup_{i=1}^{n} B_i . P(\bigcup_{i=1}^{n} A_i) = P(\bigcup_{i=1}^{n} B_i)$
$= \sum_{i=1}^{n} P(B_i)$ if B_i's are disjoint.
Should also prove that the point in B_i belongs in A_i. Need to prove B_i's disjoint - by construction:
$B_i, B_j \Rightarrow B_i = A_i^c \cap ... \cap A_{i-1}^c \cap A_i$
$B_j = A_1^c \cap ... \cap A_i^c \cap ... \cap A_{j-1}^c \cap A_j$
$s \in B_i \Rightarrow s \in A_i, s' \in B_j \Rightarrow s' \notin A_i$.
$\Rightarrow$ implies that $s \neq s'$

1-4 pg. 27 #5
$\#(S) = 6 \times 6 \times 6 \times 6 = 6^4$
$\#(\text{all different}) = 6 \times 5 \times 4 \times 3 = P_{6,4}$
$P(\text{all different}) = \frac{P_{6,4}}{6^4} = \frac{5}{18}$

1-5 pg. 27 #7
12 balls in 20 boxes.
P(no box receives > 1 ball, each box will have 0 or 1 balls)
also means that all balls fall into different boxes.
$\#(S) = 20^{12}$
$\#(\text{all different}) = 20 \times 19 ... \times 9 = P_{20,12}$

$P(...) = \frac{P_{20,12}}{20^{12}}$

1-6 pg. 27 #10
100 balls, r red balls.
$A_i = \{\text{draw red at step i}\}$
think of arranging the balls in 100 spots in a row.
a) $P(A_1) = \frac{r}{100}$
b) $P(A_{50})$
sample space = sequences of length 50.
$\#(S) = 100 \times 99 \times ... \times 50 = P_{100,50}$
$\#(A_{50}) = r \times P_{99,49}$ red on 50. There are 99 balls left, r choices to put red on 50.
$P(A_{50}) = \frac{r}{100}$, same as part a.
c) As shown in part b, the particular draw doesn't matter, probability is the same.
$P(A_{100}) = \frac{r}{100}$

1-7 pg. 34 #6
Seat n people in n spots.
$\#(S) = n!$
#(AB sit together) =?
visualize n seats, you have n-1 choices for the pair.
2(n-1) ways to seat the pair, because you can switch the two people.
but, need to account for the (n-2) people remaining!
$\#(AB) = 2(n-1)(n-2)!$
therefore, $P = \frac{2(n-1)!}{n!} = \frac{2}{n}$
or, think of the pair as 1 entity. There are (n-1) entities, permute them, multiply by 2 to swap the pair.

1-8 pg. 34 #11
Out of 100, choose 12. $\#(S) = \binom{100}{12}$
#(AB are on committee) = $\binom{98}{10}$, choose 10 from the 98 remaining.
$\mathbb{P} = \frac{\binom{98}{10}}{\binom{100}{12}}$

1-9 pg. 34 #16
50 states × 2 senators each.
a) Select 8 , $\#(S) = \binom{100}{8}$
#(state 1 or state 2) = $\binom{98}{6}\binom{2}{2} + \binom{2}{1}\binom{98}{7}$
or, calculate: $1 - \mathbb{P}(\text{neither chosen}) = 1 - \frac{\binom{98}{8}}{\binom{100}{8}}$
b) #(one senator from each state) = 2^{50}
select group of 50 = $\binom{100}{50}$

1-10 pg. 34 #17
In the sample space, only consider the positions of the aces in the hands.
$\#(S) = \binom{52}{4}$, #(all go to 1 player) = $4 \times \binom{13}{4}$
$\mathbb{P} = 4 \times \frac{\binom{13}{4}}{\binom{52}{4}}$

1-11
r balls, n boxes, no box is empty.
first of all, put 1 ball in each box from the beginning.
r-n balls remain to be distributed in n boxes.

$$\binom{n+(r-n)-1}{r-n} = \binom{r-1}{r-n}$$

1-12
30 people, 12 months.
$\mathbb{P}$(6 months with 3 birthdays, 6 months with 2 birthdays)
$\#(S) = 12^{30}$
Need to choose the 6 months with 3 or 2 birthdays, then the multinomial coefficient:

$$\#(\text{possibilities}) = \binom{12}{6}\binom{30}{3,3,3,3,3,3,2,2,2,2,2,2}$$

** End of Lecture 6