Exam 2

November 04, 2004

You have 1 hour 20 min to solve the following problems. The problems worth 10 points each. You can use your notes, books, calculators, etc. Show your reasoning.

1. Recall that a *simple graph* is an indirected graph without loops and multiple edges. The number of simple graphs on the n vertices labelled $1, \ldots, n$ equals $2^{\binom{n}{2}}$ because, for each of the $\binom{n}{2}$ pairs $\{i, j\} \subset \{1, \ldots, n\}$, a graph either contains the edge (i, j) or not. A vertex of a graph is called *isolated* if there are no edges adjacent to it.

   (a) Find the number of simple graph on n labelled vertices as above such either the vertex 1 or the vertex 2 (or both) is isolated.

   (b) Find an expression for the number of simple graphs on n labelled vertices with no isolated vertex. (Your answer may involve a summation.)

2. (a) Solve the recurrence relation $a_{n+2} = a_{n+1} + 6a_n$, $n \geq 0$, with the initial conditions $a_0 = 2$ and $a_1 = 1$.

   (b) Solve the recurrence relation $a_{n+2} = a_{n+1} + 6a_n + 6$, $n \geq 0$, with the initial conditions $a_0 = 1$ and $a_1 = 0$.

3. Let $f(n)$ be the number of ways to partition the set $\{1, \ldots, n\}$ into nonempty blocks and then linearly order elements in each block. For example, $f(3) = 13$, corresponding to the following set partitions with ordered elements in each block 123, 132, 213, 231, 312, 321, 1|23, 1|32, 2|13, 2|31, 3|12, 3|21, 1|2|3. Assume that $f(0) = 1$. Find the exponential generating function $\sum_{n \geq 0} f(n)\, x^n/n!$. (You do not need to give an explicit formula for the number $f(n)$.)

4. Let $g(n)$ be the number of lattice paths from $(0, 0)$ to $(2n, 0)$ with steps $(1, 1)$, $(1, -1)$, $(2, 0)$ that never go below the x-axis. For example $g(0) = 1$, $g(1) = 2$, $g(2) = 6$.

   (a) Calculate the number $g(3)$.

   (b) Find the generating function $\sum_{n \geq 0} g(n)\, x^n$.