Section 14

Optional stopping. Inequalities for martingales.

Consider a sequence of $\sigma$-algebras $(\mathcal{B}_n)_{n \geq 0}$ such that $\mathcal{B}_n \subseteq \mathcal{B}_{n+1}$. Integer valued r.v. $\tau \in \{1, 2, \ldots\}$ is called a stopping time if $\{\tau \leq n\} \in \mathcal{B}_n$. Let us denote by $\mathcal{B}_\tau$ a $\sigma$-algebra of the events $B$ such that

$$\{\tau \leq n\} \cap B \in \mathcal{B}_n, \quad \forall n \geq 1.$$ 

If $(X_n)$ is adapted to $(\mathcal{B}_n)$ then random variables such as $X_\tau$ or $\sum_{k=1}^\tau X_k$ are measurable on $\mathcal{B}_\tau$. For example, $\{X_\tau \in A\} = \bigcup_{n \geq 1} \{\tau = n\} \cap \{X_n \in A\} = \bigcup_{n \geq 1} \left(\{\tau \leq n\} \setminus \{\tau \leq n-1\}\right) \cap \{X_n \in A\} \in \mathcal{B}_\tau$.

Theorem 31 (Optional stopping) Let $(X_n, \mathcal{B}_n)$ be a martingale and $\tau_1, \tau_2 < \infty$ be stopping times such that $\mathbb{E}|X_{\tau_2}| < \infty$, $\lim_{n \to \infty} \mathbb{E}|X_n|1(\tau_1 \leq \tau_2) = 0$. (14.0.1)

Then on the event $\{\tau_1 \leq \tau_2\}$

$$\mathbb{E}(X_{\tau_2}|\mathcal{B}_{\tau_1}) = X_{\tau_1}.$$

More precisely, for any set $A \in \mathcal{B}_{\tau_1}$,

$$\mathbb{E}(X_{\tau_2}1_A|1(\tau_1 \leq \tau_2)) = \mathbb{E}(X_{\tau_1}1_A|1(\tau_1 \leq \tau_2)).$$

If $(X_n, \mathcal{B}_n)$ is a submartingale then equality is replaced by $\geq$.

Remark. If stopping times $\tau_1, \tau_2$ are bounded then (14.0.1) is satisfied. As the next example shows, without some control of the stopping times the statement is not true.

Example. Consider an i.i.d. sequence $(X_n)$ such that

$$\mathbb{P}(X_n = \pm 2^n) = \frac{1}{2}.$$ 

If $\mathcal{B}_n = \sigma(X_1, \ldots, X_n)$ then $(S_n, \mathcal{B}_n)$ is a martingale. Let $\tau_1 = 1$ and $\tau_2 = \min\{k \geq 1, S_k > 0\}$. Clearly, $S_{\tau_2} = 2$ because if $\tau_2 = k$ then

$$S_{\tau_2} = S_k = -2 - 2^2 - \ldots - 2^{k-1} + 2^k = 2.$$ 

However, $2 = \mathbb{E}(S_{\tau_2}|\mathcal{B}_1) \neq S_{\tau_1} = X_1$. 

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The second condition in (14.0.1) is violated since $\mathbb{P}(\tau_2 = n) = 2^{-n}$ and

$$
\mathbb{E}[S_n | I(n \leq \tau_2)] = 2\mathbb{P}(\tau_2 = n) + (2^{n+1} - 2)\mathbb{P}(n + 1 \leq \tau_2) = 2 \neq 0.
$$

**Proof of Theorem 31.** Consider a set $A \in \mathcal{B}_n$. We have,

$$
\mathbb{E}X_{\tau_2}I_2(\tau_1 \leq \tau_2) = \sum_{n \geq 1} \mathbb{E}X_{\tau_2}I(\tau_1 = n)I(\tau_1 \leq \tau_2)
$$

\[= \sum_{n \geq 1} \mathbb{E}X_nI(\tau_1 = n)I(\tau_1 \leq \tau_2) \quad \text{(by definition)}
\]

To prove (*), it is enough to prove that for $A_n = A \cap \{\tau_1 = n\} \in \mathcal{B}_n$,

$$
\mathbb{E}X_{\tau_2}I_n(n \leq \tau_2) = \mathbb{E}X_nI_n(n \leq \tau_2).
$$

We can write

$$
\mathbb{E}X_nI_n(n \leq \tau_2) = \mathbb{E}X_nI_2(\tau_2 = n) + \mathbb{E}X_nI_n(n + 1 \leq \tau_2)
$$

\[= \mathbb{E}X_{\tau_2}I_{n+1}(\tau_2 = n) + \mathbb{E}X_nI_n(n + 1 \leq \tau_2) \quad \text{(by martingale property)}
\]

\[\{ \text{since } \{n + 1 \leq \tau_2\} = \{\tau_2 \leq n\} \cap \mathcal{B}_n, \text{ by martingale property}\}
\]

\[= \mathbb{E}X_{\tau_2}I_{n+1}(\tau_2 = n) + \mathbb{E}X_nI_n(n + 1 \leq \tau_2)
\]

\[\{ \text{by induction}\}
\]

\[= \sum_{n \leq k < m} \mathbb{E}X_{\tau_2}I_n(\tau_2 = k) + \mathbb{E}X_nI_n(m \leq \tau_2)
\]

By (14.0.1), the last term

$$
\mathbb{E}X_nI_n(m \leq \tau_2) \leq \mathbb{E}X_nI(m \leq \tau_2) \to 0 \text{ as } m \to \infty.
$$

Since

$$
X_{\tau_2}I_n(n \leq \tau_2 \leq m) \to X_{\tau_2}I_n(n \leq \tau_2) \text{ as } m \to \infty
$$

and $\mathbb{E}[X_{\tau_2}] < \infty$, by dominated convergence theorem,

$$
\mathbb{E}X_{\tau_2}I_n(n \leq \tau_2 < m) \to \mathbb{E}X_{\tau_2}I_n(n \leq \tau_2).
$$

This proves (14.0.2).

**Theorem 32 (Doob’s inequality) If $(X_n, \mathcal{B}_n)$ is a submartingale then for $Y_n = \max_{1 \leq k \leq n} X_k$ and $M > 0$

$$
\mathbb{P}(Y_n \geq M) \leq \frac{1}{M} \mathbb{E}X_nI(Y_n \geq M) \leq \frac{1}{M} \mathbb{E}X_n^+.
$$

**Proof.** Define a stopping time

$$
\tau_1 = \begin{cases} 
\min\{k : X_k \geq M, k \leq n\} & \text{if such } k \text{ exists,} \\
n & \text{otherwise.}
\end{cases}
$$

Let $\tau_2 = n$ so that $\tau_1 \leq \tau_2$. By Theorem 31,

$$
\mathbb{E}(X_n | \mathcal{B}_{\tau_1}) = \mathbb{E}(X_{\tau_2} | \mathcal{B}_{\tau_1}) \geq X_{\tau_1}.
$$

Let us apply this to the set $A = \{Y_n = \max_{1 \leq k \leq n} X_k \geq M\}$ which belongs to $\mathcal{B}_{\tau_1}$ because

$$
A \cap \{\tau_1 \leq k\} = \left\{\max_{1 \leq i \leq k} X_i \geq M\right\} \in \mathcal{B}_k.
$$

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On the event $A$, $X_{\tau_1} \geq M$ and, therefore,

$$\mathbb{E}X_n I_A = \mathbb{E}X_{\tau_1} I_A \geq \mathbb{E}X_{\tau_1} I_A \geq M \mathbb{E}I_A = M \mathbb{P}(A).$$

On the other hand, $\mathbb{E}X_n I_A \leq \mathbb{E}X_n^+$ and this finishes the proof.

As a corollary we obtain the second Kolmogorov’s inequality. If $(X_i)$ are independent and $\mathbb{E}X_i = 0$ then $S_n = \sum_{1 \leq i \leq n} X_i$ is a martingale and $S_n^2$ is a submartingale. Therefore,

$$\mathbb{P}\left( \max_{1 \leq k \leq n} |S_k| \geq M \right) = \mathbb{P}\left( \max_{1 \leq k \leq n} S_k^2 \geq M^2 \right) \leq \frac{1}{M^2} \mathbb{E}S_n^2 = \frac{1}{M^2} \sum_{1 \leq k \leq n} \text{Var}(X_k).$$

**Exercises.**

1. Show that for any random variable $Y$, $\mathbb{E}|Y|^p = \int_0^\infty p t^{p-1} \mathbb{P}(|Y| \geq t) dt$.
2. Let $X, Y$ be two non-negative random variables such that for every $t > 0$, $\mathbb{P}(Y \geq t) \leq t^{-1} \int X I(Y \geq t) d\mathbb{P}$. For any $p > 1$, $\|f\|_p = (\int |f|^p d\mathbb{P})^{1/p}$ and $1/p + 1/q = 1$, show that $\|Y\|_p \leq q \|X\|_p$.
3. Given a non-negative submartingale $(X_n, B_n)$, let $X_n^\ast := \max_{0 \leq j \leq n} X_j$ and $X_n := \max_{j \geq 1} X_j$. Prove that for any $p > 1$ and $1/p + 1/q = 1$, $\|X_n^\ast\|_p \leq q \sup_n \|X_n\|_p$. *Hint:* use exercise 2 and Doob’s maximal inequality.

**Doob’s upcrossing inequality.** Let $(X_n, B_n)_{n \geq 1}$ be a submartingale. Given two real numbers $a < b$ we will define a sequence of stopping times $(\tau_n)$ when $X_n$ is crossing $a$ downward and $b$ upward as in figure 14.1. Namely, we define

$$\tau_1 = \min\{n \geq 1, X_n \leq a\}, \quad \tau_2 = \min\{n > \tau_2 : X_n \geq b\}$$

and, by induction, for $k \geq 2$

$$\tau_{2k-1} = \min\{n > \tau_{2k-2}, X_n \leq a\}, \quad \tau_{2k} = \min\{n > 2k - 1, X_n \geq b\}.$$

Define

$$\nu(a, b, n) = \max\{k : \tau_{2k} \leq n\}$$

- the number of upward crossings of $[a, b]$ before time $n$.

**Theorem 33 (Doob’s upcrossing inequality)** We have,

$$\mathbb{E}\nu(a, b, n) \leq \frac{\mathbb{E}(X_n - a)^+}{b - a}. \quad (14.0.4)$$

**Proof.** Since $x \to (x - a)^+$ is increasing convex function, $Z_n = (X_n - a)^+$ is also a submartingale. Clearly,

$$\mu_X(a, b, n) = \nu_Z(0, b - a, n)$$

which means that it is enough to prove (14.0.4) for nonnegative submartingales. From now on we can assume that $0 \leq X_n$ and we would like to show that

$$\mathbb{E}\nu(0, b, n) \leq \frac{\mathbb{E}X_n}{b}.$$
Let us define a sequence of r.v.s
\[ \eta_j = \begin{cases} 
1, & \tau_{2k-1} < j \leq \tau_{2k} \text{ for some } k \\
0, & \text{otherwise}, 
\end{cases} \]
i.e. \( \eta_j \) is the indicator of the event that at time \( j \) the process is crossing \([0, b]\) upward. Define \( X_0 = 0 \). Then
\[ b\nu(0, b, n) \leq \sum_{j=1}^{n} \eta_j (X_j - X_{j-1}) = \sum_{j=1}^{n} I(\eta_j = 1)(X_j - X_{j-1}). \]
The event
\[ \{ \eta_j = 1 \} = \bigcup_{k} \{ \tau_{2k-1} < j \leq \tau_{2k} \} = \bigcup_{k} \left\{ \tau_{2k-1} \leq j - 1 \right\} \setminus \left\{ \tau_{2k} \leq j - 1 \right\} \in \mathcal{B}_{j-1} \]
i.e. the fact that at time \( j \) we are crossing upward is determined completely by the sequence up to time \( j - 1 \). Then
\[ b\nu(0, b, n) \leq \sum_{j=1}^{n} E\nu \left( I(\eta_j = 1)(X_j - X_{j-1}) \right| \mathcal{B}_{j-1} \right) = \sum_{j=1}^{n} E\nu(\eta_j = 1)E(X_j - X_{j-1} \left| \mathcal{B}_{j-1} \right) \]
\[ = \sum_{j=1}^{n} E\nu(\eta_j = 1)(E(X_j \left| \mathcal{B}_{j-1} \right) - X_{j-1}) \leq \sum_{j=1}^{n} E(X_j - X_{j-1}) = EX_n, \]
where in the last inequality we used that \( (X_j, \mathcal{B}_j) \) is a submartingale, \( E(X_j \left| \mathcal{B}_{j-1} \right) \geq X_{j-1} \), which implies that
\[ I(\eta_j = 1)(E(X_j \left| \mathcal{B}_{j-1} \right) - X_{j-1}) \leq E(X_j \left| \mathcal{B}_{j-1} \right) - X_{j-1}. \]
This finishes the proof.