18.175 Theory of Probability
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Section 26

Laws of Brownian motion at stopping times. Skorohod’s imbedding.

Let $W_t$ be the Brownian motion.

**Theorem 63** If $\tau$ is a stopping time such that $E\tau < \infty$ then $E W_\tau = 0$ and $E W^2_\tau = E\tau$.

**Proof.** Let us start with the case when a stopping time $\tau$ takes finite number of values

$$\tau \in \{t_1, \ldots, t_n\}.$$  

If $F_{t_j} = \sigma\{W_t; t \leq t_j\}$ then $(W_{t_j}, F_{t_j})$ is a martingale since

$$E(W_{t_j} | F_{t_{j-1}}) = E(W_{t_j} - W_{t_{j-1}} + W_{t_{j-1}} | F_{t_{j-1}}) = W_{t_{j-1}}.$$  

By optional stopping theorem for martingales, $E W_\tau = E W_{t_1} = 0$. Next, let us prove that $E W^2_\tau = E\tau$ by induction on $n$. If $n = 1$ then $\tau = t_1$ and

$$E W^2_\tau = E W^2_{t_1} = t_1 = E\tau.$$  

To make an induction step from $n - 1$ to $n$, define a stopping time $\alpha = \tau \wedge t_{n-1}$ and write

$$E W^2_\tau = E(W_\alpha + W_\tau - W_\alpha)^2 = E W^2_\alpha + E(W_\tau - W_\alpha)^2 + 2E W_\alpha(W_\tau - W_\alpha).$$  

First of all, by induction assumption, $E W^2_\alpha = E\alpha$. Moreover, $\tau \neq \alpha$ only if $\tau = t_n$ in which case $\alpha = t_{n-1}$. The event

$$\{\tau = t_n\} = \{\tau \leq t_{n-1}\}^c \in F_{t_{n-1}}$$  

and, therefore,

$$E W_\alpha(W_\tau - W_\alpha) = E W_{t_{n-1}}(W_{t_n} - W_{t_{n-1}})I(\tau = t_n) = 0.$$  

Similarly,

$$E(W_\tau - W_\alpha)^2 = E E(I(\tau = t_n)(W_{t_n} - W_{t_{n-1}})^2 | F_{t_{n-1}}) = (t_n - t_{n-1})P(\tau = t_n).$$  

Therefore,

$$E W^2_\tau = E\alpha + (t_n - t_{n-1})P(\tau = t_n) = E\tau$$  

and this finishes the proof of the induction step. Next, let us consider the case of a uniformly bounded stopping time $\tau \leq M < \infty$. In the previous lecture we defined a dyadic approximation

$$\tau_n = \frac{[2^n \tau] + 1}{2^n}$$
which is also a stopping time, $\tau_n \downarrow \tau$, and by sample continuity $W_{\tau_n} \to W_\tau$ a.s. Since $(\tau_n)$ are uniformly bounded, $\mathbb{E}\tau_n \to \mathbb{E}\tau$. To prove that $\mathbb{E}W^2_{\tau_n} \to \mathbb{E}W^2_\tau$ we need to show that the sequence $(W^2_{\tau_n})$ is uniformly integrable. Notice that $\tau_n < 2M$ and, therefore, $\tau_n$ takes possible values of the type $k/2^n$ for $k \leq k_0 = [2^n(2M)]$. Since the sequence

$$(W_{1/2^n}, \ldots, W_{k_0/2^n}, W_{2M})$$

is a martingale, adapted to a corresponding sequence of $\mathcal{F}_t$, and $\tau_n$ and $2M$ are two stopping times such that $\tau_n < 2M$, by Optional Stopping Theorem 31, $W_{\tau_n} = \mathbb{E}(W_{2M}|\mathcal{F}_{\tau_n})$. By Jensen’s inequality,

$$W^4_{\tau_n} \leq \mathbb{E}(W^4_{2M}|\mathcal{F}_{\tau_n}), \quad \mathbb{E}W^4_{\tau_n} \leq \mathbb{E}W^4_{2M} = 6M.$$

and uniform integrability follows by Hölder’s and Chebyshev’s inequalities,

$$\mathbb{E}W^2_{\tau_n}1(|W_{\tau_n}| > N) \leq (\mathbb{E}W^4_{\tau_n})^{1/2}(\mathbb{P}(|W_{\tau_n}| > N))^{1/2} \leq \frac{6M}{N^2} \to 0$$

as $N \to \infty$, uniformly over $n$. This proves that $\mathbb{E}W^2_{\tau_n} \to \mathbb{E}W^2_\tau$. Since $\tau_n$ takes finite number of values, by the previous case, $\mathbb{E}W^2_{\tau_n} = \mathbb{E}\tau_n$ and letting $n \to \infty$ proves

$$\mathbb{E}W^2_\tau = \mathbb{E}\tau. \tag{26.0.1}$$

Before we consider the general case, let us notice that for two bounded stopping times $\tau \leq \rho \leq M$ one can similarly show that

$$\mathbb{E}(W_\rho - W_\tau)W_\tau = 0. \tag{26.0.2}$$

Namely, one can approximate the stopping times by dyadic stopping times and using that by the optional stopping theorem $(W_{\tau_n}, \mathcal{F}_{\tau_n}), (W_\rho, \mathcal{F}_\rho)$ is a martingale,

$$\mathbb{E}(W_\rho - W_\tau)W_\tau = \mathbb{E}W_\tau(\mathbb{E}(W_\rho|\mathcal{F}_{\tau_n}) - W_\tau_n) = 0.$$

Finally, we consider the general case. Let us define $\tau(n) = \min(\tau, n)$. For $m \leq n$, $\tau(m) \leq \tau(n)$ and

$$\mathbb{E}(W_{\tau(n)} - W_{\tau(m)})^2 = \mathbb{E}W^2_{\tau(n)} - \mathbb{E}W^2_{\tau(m)} - 2\mathbb{E}W_{\tau(m)}(W_{\tau(n)} - W_{\tau(m)}) = \mathbb{E}\tau(n) - \mathbb{E}\tau(m)$$

using (26.0.1), (26.0.2) and the fact that $\tau(n), \tau(m)$ are bounded stopping times. Since $\tau(n) \uparrow \tau$, Fatou’s lemma and the monotone convergence theorem imply

$$\mathbb{E}(W_\tau - W_{\tau(m)})^2 \leq \liminf_{n \to \infty}(\mathbb{E}\tau(n) - \mathbb{E}\tau(m)) = \mathbb{E}\tau - \mathbb{E}\tau(m).$$

Letting $m \to \infty$ shows that

$$\lim_{m \to \infty} \mathbb{E}(W_\tau - W_{\tau(m)})^2 = 0$$

which means that $\mathbb{E}W^2_{\tau(m)} \to \mathbb{E}W^2_\tau$. Since $\mathbb{E}W^2_{\tau(m)} = \mathbb{E}\tau(m)$ by the previous case and $\mathbb{E}\tau(m) \to \mathbb{E}\tau$ by the monotone convergence theorem, this implies that $\mathbb{E}W^2_\tau = \mathbb{E}\tau$. \hfill \square

**Theorem 64 (Skorohod’s imbedding)** Let $Y$ be a random variable such that $\mathbb{E}Y = 0$ and $\mathbb{E}Y^2 < \infty$. There exists a stopping time $\tau < \infty$ such that $\mathcal{L}(W_\tau) = \mathcal{L}(Y)$.

**Proof.** Let us start with the simplest case when $Y$ takes only two values, $Y \in \{-a, b\}$ for $a, b > 0$. The condition $\mathbb{E}Y = 0$ determines the distribution of $Y$,

$$pb + (1 - p)(-a) = 0 \quad \text{and} \quad p = \frac{a}{a + b}. \tag{26.0.3}$$

Let $\tau = \inf\{t > 0, W_t = -a \text{ or } b\}$ be a hitting time of the two-sided boundary $-a, b$. The tail probability of $\tau$ can be bounded by

$$\mathbb{P}(\tau > n) \leq \mathbb{P}(|W_{\tau+1} - W_j| < a + b, 0 \leq j \leq n - 1) = \mathbb{P}(|W_1| < a + b)^n = \gamma^n.$$
Therefore, \( \mathbb{E} \tau < \infty \) and by the previous theorem, \( \mathbb{E} W_\tau = 0 \). Since \( W_\tau \in \{-a, b\} \) we must have

\[
\mathcal{L}(W_\tau) = \mathcal{L}(Y).
\]

Let us now consider the general case. If \( \mu \) is the law of \( Y \), let us define \( Y \) by the identity \( Y = Y(x) = x \) on its sample probability space \((\mathbb{R}, \mathcal{B}, \mu)\). Let us construct a sequence of \( \sigma \)-algebras

\[
\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \ldots \subseteq \mathcal{B}
\]

as follows. Let \( \mathcal{B}_1 \) be generated by the set \((-\infty, 0)\), i.e.

\[
\mathcal{B}_1 = \left\{ \emptyset, \mathbb{R}, (-\infty, 0), [0, +\infty) \right\}.
\]

Given \( \mathcal{B}_j \), let us define \( \mathcal{B}_{j+1} \) by splitting each finite interval \([c, d) \subseteq \mathcal{B}_j \) into two intervals \([c, (c + d)/2) \) and \(((c + d)/2, d) \) and splitting infinite interval \((-\infty, -j) \) into \((-\infty, -(j + 1)) \) and \(-(j + 1), -j) \) and similarly splitting \([j, +\infty) \) into \([j, j + 1) \) and \([j + 1, \infty) \). Consider a right-closed martingale

\[
Y_j = \mathbb{E}(Y \mid \mathcal{B}_j).
\]

It is almost obvious that \( \mathcal{B} = \sigma(\bigcup_{j} \mathcal{B}_j) \), which we leave as an exercise. Then, by the Levy martingale convergence, Lemma 35, \( Y_j \rightarrow \mathbb{E}(Y \mid \mathcal{B}) = Y \) a.s. Since \( Y_j \) is measurable on \( \mathcal{B}_j \), it must be constant on each simple set \([c, d) \subseteq \mathcal{B}_j \). If \( Y_j(x) = y \) for \( x \in [c, d) \) then, since \( Y_j = \mathbb{E}(Y \mid \mathcal{B}_j) \),

\[
y\mu([c, d)) = \mathbb{E}Y_j I_{[c,d)} = \mathbb{E}Y I_{[c,d)} = \int_{[c,d)} x d\mu(x)
\]

and

\[
y = \frac{1}{\mu((c,d)]} \int_{(c,d]} x d\mu(x).
\]

Since in the \( \sigma \)-algebra \( \mathcal{B}_{j+1} \) the interval \([c, d) \) is split into two intervals, the random variable \( Y_{j+1} \) can take only two values, say \( y_1 < y_2 \), on the interval \([c, d) \) and, since \( (Y_j, \mathcal{B}_j) \) is a martingale,

\[
\mathbb{E}(Y_{j+1} \mid \mathcal{B}_j) = Y_j = 0.
\]

We will define stopping times \( \tau_n \) such that \( \mathcal{L}(W_{\tau_n}) = \mathcal{L}(Y_n) \) iteratively as follows. Since \( Y_1 \) takes only two values \(-a \) and \( b \), let \( \tau_1 = \inf\{t > 0, W_t = -a \text{ or } b\} \) and we proved above that \( \mathcal{L}(W_{\tau_1}) = \mathcal{L}(Y_1) \). Given \( \tau_j \) define \( \tau_{j+1} \) as follows:

if \( W_{\tau_j} = y \) for \( y \) in (26.0.4) then \( \tau_{j+1} = \inf\{t > \tau_j, W_t = y_1 \text{ or } y_2\} \).

Let us explain why \( \mathcal{L}(W_{\tau_j}) = \mathcal{L}(Y_j) \). First of all, by construction, \( W_{\tau_j} \) takes the same values as \( Y_j \). If \( C_j \) is the \( \sigma \)-algebra generated by the disjoint sets \( \{W_{\tau_j} = y\} \) for \( y \) as in (26.0.4), i.e. for possible values of \( Y_j \), then \( W_{\tau_j} \) is \( C_j \) measurable, \( C_j \subseteq C_{j+1}, C_j \subseteq \mathcal{F}_{\tau_j} \) and at each step simple sets in \( C_j \) are split in two,

\[
\{W_{\tau_j} = y\} = \{W_{\tau_{j+1}} = y_1\} \cup \{W_{\tau_{j+1}} = y_2\}.
\]

By Markov’s property of the Brownian motion and Theorem 63, \( \mathbb{E}(W_{\tau_{j+1}} - W_{\tau_j} \mid \mathcal{F}_{\tau_j}) = 0 \) and, therefore,

\[
\mathbb{E}(W_{\tau_{j+1}} \mid \mathcal{C}_j) - W_{\tau_j} = 0.
\]

Since on each simple set \( \{W_{\tau_j} = y\} \) in \( C_j \), the random variable \( W_{\tau_{j+1}} \) takes only two values \( y_1 \) and \( y_2 \), this equation allows us to compute the probabilities of these simple sets recursively as in (26.0.3),

\[
P(W_{\tau_{j+1}} = y_2) = \frac{y_2 - y}{y_2 - y_1} P(W_{\tau_j} = y).
\]

By (26.0.5), \( Y_j \)'s satisfy the same recursive equations and this proves that \( \mathcal{L}(W_{\tau_n}) = \mathcal{L}(Y_n) \). The sequence
$\tau_n$ is monotone, so it converges $\tau_n \uparrow \tau$ to some stopping time $\tau$. Since

$$E\tau_n = EW_{\tau_n}^2 = EY_n^2 \leq EY^2 < \infty,$$

we have $E\tau = \lim E\tau_n \leq EY^2 < \infty$ and, therefore, $\tau < \infty$ a.s. Then $W_{\tau_n} \Rightarrow W_{\tau}$ a.s. by sample continuity and since $\mathcal{L}(W_{\tau_n}) = \mathcal{L}(Y_n) \rightarrow \mathcal{L}(Y)$, this proves that $\mathcal{L}(W_{\tau}) = \mathcal{L}(Y)$. 

$\square$