Section 27

Laws of the Iterated Logarithm.

For convenience of notations let us denote $\ell(t) = \log \log t$.

**Theorem 65** (LIL) Let $W_t$ be the Brownian motion and $u(t) = \sqrt{2t \ell(t)}$. Then

$$\limsup_{t \to \infty} \frac{W_t}{u(t)} = 1.$$ 

Let us briefly describe the main idea that gives origin to the function $u(t)$. For $a > 1$, consider a geometric sequence $t = a^k$ and take a look at the probabilities of the following events

$$P(W_{a^k} \geq Lu(a^k)) = P\left(\frac{W_{a^k}}{\sqrt{a^k}} \geq \frac{Lu(a^k)}{\sqrt{a^k}}\right) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\ell(a^k)}} \exp\left(-\frac{1}{2} \frac{L^2 2a^k \ell(a^k)}{a^k}\right) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\ell(a^k)}} \left(\frac{1}{k \log a}\right)^{L^2}. \quad (27.0.1)$$

This series will converge or diverge depending on whether $L > 1$ or $L < 1$. Even though these events are not independent in some sense they are "almost independent" and the Borel-Cantelli lemma would imply that the upper limit of $W_{a^k}$ behaves like $u(a^k)$. Some technical work will complete this main idea. Let us start with the following.

**Lemma 49** For any $\varepsilon > 0$,

$$\limsup_{s \to \infty} \sup_{s \leq t \leq (1+\varepsilon)s} \{ \frac{|W_t - W_s|}{u(s)} : s \leq t \leq (1+\varepsilon)s \} \leq 4\sqrt{\varepsilon} \quad a.s.$$ 

**Proof.** Let $\varepsilon, \alpha > 0$, $t_k = (1+\varepsilon)^k$ and $M_k = \alpha u(t_k)$. By symmetry, (25.0.1) and the Gaussian tail estimate in Lemma 46

$$P\left(\sup_{t_k \leq t \leq t_{k+1}} |W_t - W_{t_k}| \geq M_k\right) \leq 2P\left(\sup_{0 \leq t \leq t_{k+1} - t_k} W_t \geq M_k\right) \leq \mathcal{N}(0, t_{k+1} - t_k)(M_k, \infty) \leq 4 \exp\left(-\frac{1}{2} \frac{M_k^2}{(t_{k+1} - t_k)}\right) = 4 \exp\left(-\frac{\alpha^2 2t_k \ell(t_k)}{2\varepsilon t_k}\right) \leq 4 \exp\left(-\frac{\alpha^2 2t_k \ell(t_k)}{2\varepsilon t_k}\right) = 4 \left(\frac{1}{\varepsilon t_k}\right)^{\alpha^2}.$$ 

If $\alpha^2 > \varepsilon$, the sum of these probabilities converges and by the Borel-Cantelli lemma, for large enough $k$,

$$\sup_{t_k \leq t \leq t_{k+1}} |W_t - W_{t_k}| \leq \alpha u(t_k).$$
It is easy to see that for small enough \( \varepsilon \), \( u(t_{k+1})/u(t_k) < 1 + \varepsilon \leq 2 \). If \( k \) is such that \( t_k \leq s \leq t_{k+1} \) then, clearly, \( t_k \leq s \leq t \leq t_{k+2} \) and, therefore, for large enough \( k \),

\[
|W_t - W_s| \leq 2a\varepsilon(t_k) + 2\varepsilon(t_{k+1}) \leq (2\alpha + \alpha(1 + \varepsilon))u(s) \leq 4\alpha u(s).
\]

Letting \( \alpha \to \sqrt{\varepsilon} \) over some sequence finishes the proof.

\[\Box\]

**Proof of Theorem 65.** For \( L = 1 + \gamma > 1 \), (27.0.1) and the Borel-Cantelli lemma imply that

\[ W_{t_k} \leq (1 + \gamma)u(t_k) \]

for large enough \( k \). If \( t_k = (1 + \varepsilon)^k \) then Lemma 49 implies that with probability one for large enough \( t \) (if \( t_k = t < t_{k+1} \))

\[
\frac{W_t}{u(t)} = \frac{W_{t_k}}{u(t_k)} + \frac{W_t - W_{t_k}}{u(t_k)} \leq (1 + \gamma) + 4\varepsilon.
\]

Letting \( \varepsilon, \gamma \to 0 \) over some sequences proves that with probability one

\[
\limsup_{t \to \infty} \frac{W_t}{u(t)} \leq 1.
\]

To prove that upper limit is equal to one we will use the Borel-Cantelli lemma for independent increments \( W_{a^k} - W_{a^{k-1}} \) for large values of the parameter \( a > 1 \). If \( 0 < \gamma < 1 \) then, similarly to (27.0.1),

\[
\Pr \left( W_{a^k} - W_{a^{k-1}} \geq (1 - \gamma)u(a^k - a^{k-1}) \right) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{(1 - \gamma)\sqrt{2\ell(a^k - a^{k-1})}} \left( \frac{1}{\log(a^k - a^{k-1})} \right)^{(1 - \gamma)^2}.
\]

The series diverges and, since these events are independent, they occur infinitely often with probability one. We already proved (by (27.0.1)) that for \( \varepsilon > 0 \) for large enough \( k \), \( W_{a^k}/u(a^k) \leq 1 + \varepsilon \) and, therefore, by symmetry \( W_{a^k}/u(a^k) \geq -(1 + \varepsilon) \). This gives

\[
\frac{W_{a^k}}{u(a^k)} \geq (1 - \gamma)\frac{u(a^k - a^{k-1})}{u(a^k)} + \frac{W_{a^{k-1}}}{u(a^{k-1})} \\
\geq (1 - \gamma)\frac{u(a^k - a^{k-1})}{u(a^k)} - (1 + \varepsilon)\frac{u(a^{k-1})}{u(a^k)} \\
= (1 - \gamma)\sqrt{\frac{(a^k - a^{k-1})\ell(a^k - a^{k-1})}{a^k\ell(a^k)}} - (1 + \varepsilon)\sqrt{\frac{a^{k-1}\ell(a^{k-1})}{a^k\ell(a^k)}}
\]

and

\[
\limsup_{t \to \infty} \frac{W_t}{u(t)} \geq \limsup_{k \to \infty} \frac{W_{a^k}}{u(a^k)} \geq (1 - \gamma)\sqrt{\left( 1 - \frac{1}{a} \right)} - (1 + \varepsilon)\sqrt{\frac{1}{a}}.
\]

Letting \( \gamma \to 0 \) and \( a \to \infty \) over some sequences proves that the upper limit is equal to one.

\[\Box\]

The LIL for Brownian motion will imply the LIL for sums of independent random variables via Skorohod’s imbedding.

**Theorem 66** Suppose that \( Y_1, \ldots, Y_n \) are i.i.d. and \( \mathbb{E}Y_i = 0, \mathbb{E}Y_i^2 = 1 \). If \( S_n = Y_1 + \ldots + Y_n \) then

\[
\limsup_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1 \text{ a.s.}
\]

**Proof.** Let us define a stopping time \( \tau(1) \) such that \( W_{\tau(1)} \overset{\text{d}}{=} Y_1 \). By Markov property, the increment of the process after stopping time is independent of the process before stopping time and has the law of the Brownian motion. Therefore, we can define \( \tau(2) \) such that \( W_{\tau(1)+\tau(2)} - W_{\tau(1)} \overset{\text{d}}{=} Y_2 \) and, by independence,
$W_{\tau(1)+\tau(2)} \overset{d}{=} Y_1 + Y_2$ and $\tau(1), \tau(2)$ are i.i.d. By induction, we can define i.i.d. $\tau(1), \ldots, \tau(n)$ such that $S_n \overset{d}{=} W_{T(n)}$ where $T(n) = \tau(1) + \ldots + \tau(n)$. We have

$$\frac{S_n}{u(n)} \overset{d}{=} \frac{W_{T(n)}}{u(n)} = \frac{W_n}{u(n)} + \frac{W_{T(n)} - W_n}{u(n)}.$$  

By the LIL for the Brownian motion,

$$\limsup_{n \to \infty} \frac{W_n}{u(n)} = 1.$$  

By the strong law of large numbers, $T(n)/n \to \mathbb{E}\tau(1) = \mathbb{E}Y_1^2 = 1$ a.s. For any $\varepsilon > 0$, Lemma 49 implies that for large $n$

$$\frac{|W_{T(n)} - W_n|}{u(n)} \leq 4\sqrt{\varepsilon}$$  

and letting $\varepsilon \to 0$ finishes the proof.

LIL for Brownian motion also implies a local LIL:

$$\limsup_{t \to 0} \frac{W_t}{\sqrt{2t\log(1/t)}} = 1.$$  

It is easy to check that if $W_t$ is a Brownian motion then $tW_{1/t}$ is also the Brownian motion and the result follows by a change of variable $t \to 1/t$. To check that $tW_{1/t}$ is a Brownian motion notice that for $t < s$,

$$\mathbb{E}tW_{1/t} \left( sW_{1/s} - tW_{1/t} \right) = st - \frac{t^2}{s} - \frac{t}{s} - t = t - t = 0$$  

and

$$\mathbb{E} \left( tW_{1/t} - sW_{1/s} \right)^2 = t + s - 2t = s - t.$$  

\[\square\]