INFINITY AND EXPERIENCE

by

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B.S., University of California at Los Angeles, 1991

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Submitted to the Department of
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ABSTRACT

This dissertation explores the relationship between our experience and knowledge of the infinite. The first chapter is a critical examination of Shaughan Lavine's project in *Understanding the Infinite*. Lavine argues that although we do not experience the infinite, we can explain how we acquire knowledge of the infinite by appeal to our experience of the indefinitely large. I argue that Lavine's proposal fails.

In the second chapter I argue that, contrary to what Lavine and others have claimed, we can have "experiences of the infinite." In particular, I argue that we can have a perceptual illusion of an infinite sequence when we see certain pictures.

In the third chapter, I argue that the experiences of the infinite discussed in the second chapter help us defend a central tenet of modal structuralism, namely the claim that there could exist infinitely many objects. In order to show that we have evidence for this modal claim, I explain how, in general, we can use pictures to establish that the depicted object could exist. I argue that upon seeing a picture, we can obtain evidence that a picture represents a "coherent" rather than an "incoherent" spatial configuration, and furthermore, that if we obtain evidence that a picture represents a coherent spatial configuration, we thereby obtain evidence that the depicted object could exist. I then use this "picture method" to show that by seeing a picture of an infinite sequence, we can obtain evidence for the modal claim that an infinite sequence could exist.

Thesis Supervisor: Robert Stalnaker
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Contents

Introduction

We have a tremendous amount of knowledge about the infinite. We know that there are infinitely many primes, that the set of natural numbers is smaller than the set of reals, that every bounded infinite set of reals has a least upper bound. Or so it seems. For, some philosophers argue that it seems impossible to know propositions about the infinite because we do not experience the infinite.

John Locke, for example, wonders how we could come to the idea of the infinite and so know propositions about the infinite if "the objects we converse with are so much short of any approach or proportion to that largeness."[1] David Hume in *A Treatise of Human Nature* concludes that we cannot form concepts of the infinite and consequently obtain knowledge of the infinite. As he writes,

> 'Tis universally allow'd, that the capacity of the mind is limited, and can never attain a full and adequate conception of infinity: And tho' it were not allow'd, 'twou'd be sufficiently evident from plainest observation and experience.[2]

More recently, Shaughan Lavine, in his book *Understanding the Infinite*,[3] presents an epistemological puzzle about the infinite by arguing that knowledge of the infinite seems impossible because the infinite is "remote from our experience," in the sense that we do

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[1] John Locke, *An Essay Concerning Human Understanding*, vol. 1, Ed., Alexander Fraser, Dover Publication, inc., New York, 1959. p. 277.

[2] David Hume, *A Treatise of Human Nature*, 2nd edition. Ed. L.A. Selby-Bigge. Oxford University Press, Oxford, 1978. p. 26.

[3] Shaughan Lavine, *Understanding the Infinite*. Harvard University Press, Cambridge, Massachusetts and London, England, 1994.

not experience anything suitably like "infinite mathematical objects." In this dissertation I will investigate responses to this epistemological puzzle about the infinite. In so doing, I will investigate the relationship between experience and knowledge of the infinite.

The first chapter is a critical examination of Lavine's project in *Understanding the Infinite*. Lavine provides a response to the epistemological puzzle by arguing that our knowledge of the infinite is grounded in our experience of the indefinitely large. More specifically, he proposes that we can justifiably extrapolate from a finite version of Zermelo-Frankel set theory with Choice—which I will call Fin(ZFC)—to full-fledged ZFC, and furthermore he argues that our experience of the indefinitely large provides us with grounds for believing the axioms of Fin(ZFC). I argue that Lavine's project fails at its second step because it seems impossible to show that both the finite version of Extensionality and the "appropriate" finite version of the Axiom of Infinity are true principles about finite sets.

Since I argue that Lavine's program fails, we are left with a puzzle about the infinite. The rest of the thesis takes steps to resolve this puzzle. In the second chapter I argue that we can have a *perceptual illusion of an infinite sequence*. In particular, I argue that we can see certain sequences as having a property that only an infinite sequence has.

One benefit of showing that we can have a perceptual illusion of the infinite is that it helps clarify the relationship between experience and beliefs formed on the basis of experience. For, to sustain the claim that we can have an illusion of the infinite, I argue that even though an object appears to us in a certain way, we do not always believe nor are we disposed to believe that the object is that way. Furthermore, in some cases—in particular, in cases where we see pictures—we are disposed to believe the opposite of what we see.

A further benefit of showing that we can have illusions of the infinite is that these illusions ultimately provide us with *modal* knowledge of the infinite. In the third chapter, I argue that the experiences of the infinite discussed in the second chapter provide evidence for the modal claim that there could exist infinitely many objects.

To show that these experiences provide evidence for this modal claim about the infinite, I introduce machinery for establishing modal claims. I show how, in general, we can use pictures to support the contention that the depicted object could exist. I argue that upon seeing a picture, we can provide evidence for the claim that a picture represents a "coherent" rather than an "incoherent" spatial configuration. I then argue that if we have evidence that a picture represents a coherent spatial configuration, we thereby obtain evidence that the depicted object could exist. To support these claims, I provide an account of coherence and explain why coherence provides grounds for possibility. I also present a test of coherence which enables us to determine whether a picture represents a coherent or an incoherent spatial configuration.

After presenting the "picture method," I use it to show that by seeing pictures of an infinite sequence we obtain evidence for the modal claim that an infinite sequence could exist. To make my case, I appeal to the fact that we have a perceptual illusion of the infinite when we look at certain pictures. Showing that we have evidence that an infinite sequence could exist provides a partial solution to the puzzle about the infinite.

In addition to providing a partial solution to the epistemological puzzle, showing that we have evidence for this modal claim is critically important to modal structuralism, a position in the philosophy of mathematics. Indeed, the modal structuralist must hold that an infinite sequence of concrete objects could exist, and, at present, the leading proponent of modal structuralism does not provide a convincing defense of this claim. Furthermore, in a recent article,[4] Bob Hale argues that the modal structuralist faces a serious epistemological puzzle because it does not appear that the modal structuralist can defend the claim that infinitely many concrete objects could exist. My defense of the claim that infinitely many concrete objects could exist not only supplies modal structuralism with a defense of one of its central tenets but also helps point out certain difficulties with Hale's arguments.

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4 Bob Hale, "Structuralism's Unpaid Epistemological Debts." *Philosophia Mathematica* 3 Vol. 4 (1996) pp. 124-147.

Chapter 1

Lavine on Understanding the Infinite

We know all sorts of trivial and not so trivial claims about infinite sets, functions with infinite domains, and geometrical objects consisting of infinitely many points. Or so it seems. For, Shaughan Lavine in his book *Understanding the Infinite* poses a puzzle about our knowledge of "infinite mathematical objects":

> In sharp contrast to the situation about '2+2=4', many of those who are skeptical about the existence of infinite combinatorial collections would want to doubt or deny the Axiom of Choice—not only its truth, but its acceptability in any form whatever. General facts about the infinite are not robust in the same way that the facts of counting, computing, and bookkeeping are. Moreover, it is not at all clear what we can fall back on as a source of mathematical knowledge concerning the infinite—what can play the role that bunches and sequences of moments, objects, or words seems so well suited to play for small finite mathematical objects. It is that lack that raises the problem posed by the remoteness of the infinite: it seems that we cannot have grounds to know what we find we actually do know about the infinite.[1]

According to Lavine, then, we have experiences that could possibly lead to an account of our knowledge of "finite mathematical objects." We see finite patterns and finite groups of objects; we see pairs and triples and so forth. In the case of the infinite, however, no experiences appear to provide a basis for our extensive knowledge of infinite mathematical objects. We do not see infinitely many things. We do not see patterns of infinite

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[1] Shaughan Lavine, *Understanding the Infinite*. Harvard University Press, Cambridge, Massachusetts and London, England, 1994. p. 164.

complexity. In short, we do not appear to have any kind of experience of the infinite. So, although it appears that we can appeal to experience to ground our knowledge of the finite, we have no such luxury in the case of the infinite. We thus wonder how we can know what we appear to know about the infinite.

This puzzle about the infinite provides a focal point for this thesis. In the rest of this thesis I will examine and propose responses to this puzzle. In the first chapter, I will criticize Lavine's response to the puzzle. In the second and third chapters I will work towards a partial response to the puzzle. In particular I will argue that we can have a *perceptual illusion of an infinite sequence* when we look at certain pictures, and furthermore, we can appeal to this illusion to show that we have modal knowledge about an infinite sequence.

Lavine responds to the puzzle about the infinite by arguing that our knowledge of the infinite is grounded in our experience of the indefinitely large. More specifically, he introduces the theory he calls "The Theory of Zillions." The Theory of Zillions is a finite version of Zermelo-Fraenkel Set Theory with Choice (ZFC). The Theory of Zillions is a *version* of ZFC because we obtain the central axioms of the Theory of Zillions by suitably relativizing the quantifiers in the axioms of ZFC. It is a *finite* version of ZFC because every finite subset of the axioms of Fin(ZFC) has a model whose domain contains only a finite number of finite sets. I will call the Theory of Zillions 'Finite Zermelo-Fraenkel Set Theory' or, for short, 'Fin(ZFC)'. By appeal to Fin(ZFC), Lavine argues that our knowledge of ZFC originates from our experience of the indefinitely large.

His argument proceeds in two steps. He contends that our knowledge of the axioms of ZFC is derived—through a process of "extrapolation"—from our knowledge of the axioms of Fin(ZFC). He then argues that our knowledge of the axioms of Fin(ZFC) is in turn derived from our experience of the indefinitely large. This two-step procedure supplies an explanation of how our knowledge of the infinite, in particular our knowledge that the axioms of ZFC are true, is derived from our experience of the indefinitely large.

In this first chapter, I will criticize Lavine's positive proposal that our experience of the indefinitely large provides intuitive evidence for the axioms of Fin(ZFC). In particular, I will concentrate on two axioms of Fin(ZFC): Zillion, which is the finite version of the Axiom of Infinity, and Relativized Extensionality, which is the finite version of the Axiom of Extensionality. I will argue that Lavine fails to show that Zillion is a self-evident principle about finite sets and furthermore that we have reason to prefer a finite set theory that contains a generalized version of Zillion over one that contains Zillion. I will then argue that "Generalized Zillion" and Relativized Extensionality undermine one another. In particular, there appears to be no way to sustain the claim that both Relativized Extensionality and Generalized Zillion are self-evident principles about finite sets.

Establishing these claims requires familiarizing ourselves with some of the machinery on which Lavine relies and also with some of the special notions he introduces. The first part of this chapter, then, will be spent laying some of the necessary groundwork.

I. Fin(ZFC): An Overview

First, I need to be more explicit about Fin(ZFC).[2] We obtain the language of Fin(ZFC) by adding to the language of first-order set theory[3] the constant '∅', the

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[2] I will follow Lavine's presentation of Fin(ZFC) as given in *Understanding the Infinite.* Lavine bases his mathematical work on Jan Mycielski's. Mycielski introduces finite versions of theories in his paper "Locally Finite Theories," *Journal of Symbolic Logic* 51 (1986). pp. 59-62.

[3] The only non-logical symbol in the basic language of first-order set theory is '∈'.

two-place function-symbol 'A', ('$A(x,y)$' has the intuitive meaning of '$x \cup \{y\}$'), and infinitely-many one-place predicates of the form 'Ω_p' where 'p' is replaced by a numeral.[4,5] Although the Ω's are predicates, I will abbreviate '$\Omega_p(x)$' as '$x \in \Omega_p$'.

We can give a straightforward inductive specification of the terms of Fin(ZFC): any variable is a term; the constant '$\varnothing$' is a term; and whenever t_1 and t_2 are replaced by terms, '$A(t_1,t_2)$' is a term. As for the formulas of Fin(ZFC), every sentence of the form '$\Omega_p(x)$', '$t_1 \in t_2$', or '$t_1 = t_2$', where t_1 and t_2 are replaced by terms of Fin(ZFC), is a formula. If ϕ and ψ are formulas, then $\neg\phi$ and $\phi \wedge \psi$ are formulas. As for the existential quantifier, if ϕ is a formula, then $\exists x \in \Omega_p\phi$ and $\exists x\phi$ are formulas.

Before stating the axioms of Fin(ZFC), I want to indicate briefly how Lavine wants us to understand these axioms. In giving this summary, I will rely on the notion of availability and the notion of an object's being available in virtue of the availability of another object. I will discuss these notions later. For now, we can suppose that an object is available if it is somehow given to the mind.

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4 Lavine relies on Fin(ZFC) to explain how we come to know about the infinite. Accordingly, we should be able to come to know the axioms of Fin(ZFC) and understand the language of Fin(ZFC) without a prior understanding of or knowledge of the infinite. However, given the above description of the language of Fin(ZFC), one might question whether Lavine can accomplish this. In particular, since that the subscripts can be replaced by any numeral, the language appears to contain infinitely many symbols. As a result, one might worry that to understand the language of Fin(ZFC), we must have a prior understanding of the infinite because we cannot describe the language of Fin(ZFC) without reference to an infinite or a potentially infinite number of objects. Lavine thinks he can avoid this problem by providing a schematic specification of the language of Fin(ZFC). As he writes, "I have not said anything characterizing all symbols of the language; I gave a test for whether an antecedently given symbol is a symbol of the language. That is, the specification of the predicate symbols is itself schematic." (Footnote 20, *Understanding the Infinite*. p. 269) Because the specification of the language is schematic, Lavine believes that not only can we avoid commitment to an infinite number of symbols but also we can understand the language of Fin(ZFC) without a prior understanding of or knowledge of the infinite. For more information on these issue see §VI.4 and the rest of Footnote 20 in §VIII.3.1.

5 Officially, Lavine contends that the subscripts are replaced by signs for rational numbers. This allows him to insert a new Ω between any other two. We can replace the subscripts by numerals because, according to Lavine, we employ, in any given context, only finitely many formulas of Fin(ZFC) and so employ only finitely many Ω's. We can thus renumber the Ω's so that the largest Ω has a subscript less than n, for some natural number n. See see §VI.4 and footnote 20 in *Understanding the Infinite*.

As I mentioned, Lavine claims that the axioms of Fin(ZFC) codify some of our intuitions about indefinitely large sets. In particular, the Ω's bounding the quantifiers in the axioms are supposed to represent indefinitely large finite sets. We should think of Ω_0 as an indefinitely large set that contains objects that are "available to the first degree." From Ω_0 we form a larger indefinitely large set Ω_1 which not only contains all objects in Ω_0 but also contains every object that is "available in virtue of the availability of an object in Ω_0." Similarly, from Ω_1 we form a larger indefinitely large set Ω_2 which not only contains all objects in Ω_1 but also contains every object that is "available in virtue of the availability of an object in Ω_1." And so forth. The Ω's thus form a hierarchy of indefinitely large sets.

All the quantifiers in the axioms of Fin(ZFC) are bounded by Ω's, and these bounds increase as quantifiers appear deeper inside the axiom. As a result, many axioms tell us what objects are available in virtue of the availability of other objects. In reading the axioms of Fin(ZFC), the reader should keep in mind the intuitive meanings of the Ω's.

In stating the axioms, we will have occasion to refer to a special class of the well-formed formulas, namely the regular relativizations of ZFC formulas. If ϕ is a formula of ZFC, ϕ' is a *regular relativization* of ϕ if ϕ' is obtained from ϕ by bounding *all* the quantifiers in ϕ with Ω's in accord with the following constraint: whenever a quantifier bound by Ω_p occurs within the scope of a quantifier bound by Ω_q, then $p>q$.[6] For example, the formula $\forall w \forall x \exists y \forall z (z \in y \leftrightarrow (z=w) \vee (z=x))$ has regular relativizations

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[6] Note that most of the axioms of Fin(ZFC) are *semiregular relativizations* of formulas of ZFC. A semiregular relativization is like a regular relativization except that bounds on alike quantifiers can be the same, but bounds must be increased when the quantifiers are different. For example, the formula $\forall w \forall x \exists y \forall z (z \in y \leftrightarrow (z=w) \vee (z=x))$ has regular relativizations $\forall w \in \Omega_1 \forall x \in \Omega_1 \exists y \in \Omega_2 \forall z \in \Omega_3 (z \in y \leftrightarrow (z=w) \vee (z=x))$ and $\forall w \in \Omega_2 \forall x \in \Omega_2 \exists y \in \Omega_5 \forall z \in \Omega_7 (z \in y \leftrightarrow (z=w) \vee (z=x))$.

$\forall w \in \Omega_1 \forall x \in \Omega_2 \exists y \in \Omega_3 \forall z \in \Omega_4 (z \in y \leftrightarrow (z=w) \vee (z=x))$ and

$\forall w \in \Omega_2 \forall x \in \Omega_3 \exists y \in \Omega_5 \forall z \in \Omega_7 (z \in y \leftrightarrow (z=w) \vee (z=x))$.

The axioms of Fin(ZFC) fall into four categories:

- Relativizations of the Axioms of Equality
- Axioms Governing '$\varnothing$' and 'A'
- Axioms of Indefinitely Large Size
- Relativizations of the Axioms of ZFC

The axiom schemata in the first category are as follows:

(1) $$\forall x \in \Omega_p (x=x)$$

where 'p' is replaced by a numeral.[7]

(2) $$\forall xy \in \Omega_p (x=y \rightarrow y=x)$$

(3) $$\forall xyz \in \Omega_p (x=y \wedge y=z \rightarrow x=z)$$

(4) $$(\forall x,y,z,u \in \Omega_p)(x=y \wedge z=u \rightarrow A(x,z)=A(y,u))$$

(5) $$(\forall x,y,z,u \in \Omega_p)(x=y \wedge z=u \wedge x \in z \rightarrow y \in u)$$

Next come the axioms governing '$\varnothing$' and 'A'.

(6) $$\forall x \in \Omega_0 (x \notin \varnothing)$$

(7) $$\forall xyz \in \Omega_0 (z \in A(x,y) \leftrightarrow z \in x \vee z=y).$$[8]

The third category of axioms, the Axioms of Indefinitely Large Size, tells us a bit about the Ω's. As I mentioned, the Ω's are supposed to be indefinitely large sets. The following axiom schemata give us some information about how they are built up:

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[7] From now on I will assume that the subscripts 'p', 'q', 'r', and 't' are replaced by numerals in all instances of schemata.

[8] Note that Lavine restricts his attention to models where the range of $A(x,y)$ is in Ω_1 and not in Ω_0. In these models Ω_0 need not be infinite for (7) to hold. We should also note that these axioms are not schemata. I believe Lavine does not use the schema $\ulcorner \forall xyz \in \Omega_p (z \in A(x,y) \leftrightarrow z \in x \vee z=y) \urcorner$ in place of axiom (7) because every instance of this schema besides axiom (7) is derivable from axiom (11).

(8) $$\varnothing \in \Omega_p$$

(9) $$\forall x,y(x,y \in \Omega_p \rightarrow A(x,y) \in \Omega_q)$$

where $p<q$.

Axiom (9) tells us how the Ω's grow. For $p<q$, the adjunction of any two sets in Ω_p is in Ω_q. Or similarly, we can understand this axiom as saying that the adjunction of two sets x and y is available in virtue of the availability of x and y.

(10) $$\forall x(x \in \Omega_p \rightarrow x \in \Omega_q)$$

where $p<q$.

Axiom (10) tells us that the Ω's are such that $\Omega_p \subseteq \Omega_q \subseteq \Omega_r, \ldots$, where $p<q<r<\ldots$

(11) $$(\forall x_1,\ldots,x_n \in \Omega_p)((\forall x \in \Omega_q)\phi \leftrightarrow (\forall x \in \Omega_r)\phi)$$

where $p<q$ and $p<r$ and $(\forall x \in \Omega_q)\phi$ and $(\forall x \in \Omega_r)\phi$ are regular relativizations of formulas of ZFC.

Axiom (11) tells us that the Ω's are in some sense indiscernible. In particular, the Ω's are so large that we cannot distinguish between them using regular relativizations of formulas of ZFC.

The final category of axioms consists of relativizations of the axioms of ZFC. For simplicity I will state the axioms using numerical subscripts for the Ω's. The subscripts can be replaced by any numerals that respect the ordering of the original subscripts. For example, if 'Ω_1', 'Ω_2', 'Ω_3', and 'Ω_4' are the only Ω-symbols occurring in the axiom, then the subscripts '1', '2', '3', and '4' can respectively be replaced by '1', '7', '9', and '10', by '2', '5', '6', and '7', and so forth. To aid in the understanding of these axioms, I will say that x is a Ω_p-member to indicate that x is in Ω_p. The axioms are as follows:

Relativized Extensionality

(12) $$(\forall xy \in \Omega_0)(\forall z \in \Omega_1(z \in x \leftrightarrow z \in y) \rightarrow x=y)$$ [9]

In words, for all Ω_0-members x and y, if every Ω_1-member z is in x just in case it is in y, then $x = y$. Equivalently, if x and y are distinct members of Ω_0, then there is a Ω_1-member z that witnesses that x and y are different, i.e., there is a Ω_1-member z that is either in x but not in y or in y but not in x.

Intuitively, we can understand this axiom as stating that, whenever distinct sets x and y are available, there is a witness set which is available in virtue of the availability of both x and y. So, notice that the existential quantifier signals that some object is available in virtue of the availability of other objects.

Relativized Foundation

(13) $$(\forall x \in \Omega_0)(x \neq \varnothing \rightarrow (\exists y \in \Omega_1)(y \in x \wedge (\forall z \in \Omega_2)(z \in x \rightarrow z \notin y)))$$

Every non-empty Ω_0-member x has, as a member, a Ω_1-member y that is, as far as Ω_2 knows, an $\in$-least element of x, i.e., no element in Ω_2 is both in x and in y. Note that the set that witnesses the existential quantifier here might change when different Ω's are substituted. For instance, if we replace 'Ω_2' by 'Ω_3' then the axiom reads that every non-empty Ω_0-member x has as a member a Ω_1-member y that is, as far as Ω_3 knows, an $\in$-least element of x.

Relativized Weak Union

(14) $$(\forall x \in \Omega_0)(\exists y \in \Omega_1)(\forall zu \in \Omega_2)(z \in u \wedge u \in x \rightarrow z \in y)$$

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[9] The '$\rightarrow$' in the axiom can be replaced by '$\leftrightarrow$'.

For every Ω_0-member x there is a Ω_1-member y that contains all Ω_2-members z that satisfy the following: z is a member of some Ω_2-member u of x. So, y contains every member of a member of x only if Ω_2 contains every member of x and every member of a member of x.

Relativized Choice

(15) $(\forall x \in \Omega_0)(\forall yzu \in \Omega_1)[(y \in x \wedge z \in x \wedge y \neq z \rightarrow \neg(u \in y \wedge u \in z))] \rightarrow$
$(\exists y \in \Omega_1)(\forall z \in \Omega_2)(z \in x \wedge z \neq \varnothing \rightarrow$
$(\exists u \in \Omega_3)(u \in z \wedge u \in z) \wedge (\forall uv \in \Omega_2)(u \in z \wedge u \in y \wedge v \in z \wedge v \in y \rightarrow u=v)))$

If Ω_1 thinks that all members of a Ω_0-member x have no members in common, then there is some Ω_1-member "choice" set y of x. This choice set y contains, for each Ω_2-member z of x, exactly one Ω_3-member u of z.

Relativized Power Set

(16) $(\forall x \in \Omega_0)(\exists y \in \Omega_1)(\forall z \in \Omega_2)(z \in y \leftrightarrow (\forall u \in \Omega_3)(u \in z \rightarrow u \in x))$

For every Ω_0-member x there is a Ω_1-member y that contains all and only Ω_2-members z that Ω_3 thinks are subsets of x.

Relativized Replacement

(17) $(\forall x_0,\ldots,x_n \in \Omega_0)((\forall xyz \in \Omega_1)(f(x,y,x_0,\ldots,x_n) \wedge f(x,z,x_0,\ldots,x_n) \rightarrow y=z) \rightarrow$
$(\forall x \in \Omega_1)(\exists y \in \Omega_2)(\forall z \in \Omega_3)(z \in y \leftrightarrow (\exists u \in \Omega_4)(u \in x \wedge f(u,z,x_0,\ldots,x_n))))$

where ' $\phi(x,y,x_0,\ldots,x_n)$ ' is replaced by any regular formula[10] with the following properties: any 'Ω_p' that appears in the formula has a subscripts greater than 3; only the variables $x,y,x_0,\ldots,x_n$ occur free in the formula; the variables u and v do not occur in the formula.

[10] Recall that a regular formula is any formula whose quantifiers are bound according to the following constraint: whenever a quantifier bound by Ω_p occurs within the scope of a quantifier bound by Ω_q, then $p>q$

Unrelativized, the antecedent of the axiom says that y in the formula $\phi(x,y,x_0,\ldots,x_n)$ is a function of $x,x_0,\ldots,x_n$, i.e., for every $x,x_0,\ldots,x_n$ there is exactly one output $g(x,x_0,\ldots,x_n)$ such that $\phi(x,g(x,x_0,\ldots,x_n),x_0,\ldots,x_n)$. Relativized, the antecedent says that Ω_1 thinks that y in the formula $\phi(x,y,x_0,\ldots,x_n)$ is a function of $x,x_0,\ldots,x_n$. The consequent of the axiom says that for all Ω_1-members x there is a Ω_2-member y that contains all and only those Ω_3-members z that satisfy the following condition: $z = g(u,x_0,\ldots,x_n)$ for some Ω_4-member u of x. So, y is the image of the set x under g unless Ω_3 or Ω_4 is missing members of x or certain members of the range of g.

Relativized Infinity (Zillion)

(18) $$(\exists x \in \Omega_0)(\varnothing \in x \wedge (\forall y \in \Omega_1)(y \in x \rightarrow y \cup \{y\} \in x))$$

Zillion says that there is a Ω_0-member x that contains $\varnothing$ and contains the "successor" of every Ω_1-member y of x. Intuitively, this axiom says that there is a set that is available to the first degree which contains the "successors" of certain members of x. As I will explain in detail later, a finite set can witness the truth of Zillion. Off hand, this might seem plausible since Zillion does not require that the successor of *every* member of x be in x. Rather it requires only that the successor of every Ω_1-member of x be in x.

II. "Extrapolation"

Now that we have reviewed the formal machinery, we can get a better idea of how Lavine attempts to solve the problem of the infinite. As I noted, his project has two steps. In the first step, he tries to provide a link between ZFC and Fin(ZFC) by claiming that ZFC, in some sense, arose from the Fin(ZFC). In particular, he claims that ZFC arose for

Fin(ZFC) by a process he calls "extrapolation."[11] In this section I will briefly explain this first step of Lavine's response to the problem of the infinite.

Clearly, some straightforward manipulations transform Fin(ZFC) into ZFC. We eliminate the axioms governing the Ω's, i.e., axioms (8)–(11), and we omit all predicates of the form 'Ω_p' in the rest of the axioms. From this we can obtain ZFC, the standard axioms of equality, and the axioms governing '∅' and 'A'.[12] Lavine, however, has a more elaborate story to tell about how ZFC arose from Fin(ZFC). He contends that ZFC and all its infinitary commitments arose by extrapolation from Fin(ZFC).

To understand Lavine's notion of extrapolation, we first must explain the notion of the indefinitely large. According to Lavine, a set is indefinitely large if we think that it is too large to count. Off hand, this characterization appears to suggest that only infinite sets are indefinitely large, as no finite set seems too large to count: given enough time, space, and memory, we can count any finite number of things. Lavine concurs. However, he emphasizes that, in certain circumstances, we deem some finite sets too large to count. When I look at my friend's thick mane of hair, I realize that counting how many hairs she has is out of the question. There are just too many. So, relative to certain contexts we deem certain finite sets too large to count. These sets are not too large to count in all contexts. Change our interests and resources, and we can count a previously "uncountable" set. So,

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[11] Lavine believes that "the notion of infinite size that results from extrapolating from the notion of indefinitely large size is the one that actually was operative in the development of modern axiomatic set theory." And he believes that "the picture of the infinite—of the use of ellipsis—with which the founders of set theory started was one that developed out of their experience of indefinitely large size, even though the source was largely unconscious." Lavine, *Understanding the Infinite*, p 251.

[12] I say that we *can* obtain ZFC plus the standard axioms of equality because when we drop the Ω's, we do not obtain full Leibniz Law. We obtain only the sentence '$\forall xyzu(x=y \wedge z=u \rightarrow A(x,y)=A(y,u))$'. However, all instances of Leibniz Law are provable by induction of the complexity of formulas.

if we deem a set too large to count in a particular context, then this set is indefinitely large in that context.

Extrapolation, put simply, is just "dropping the dependence on context."[13] When we extrapolate, we contemplate sets that are too large to count, simpliciter, i.e., sets that are too large to count *independent of any context.* In this way, we contemplate infinite sets. For, "Infinite is nothing more than too large to count—too large to count in a context-independent sense."[14] So, we come to the infinite via the finite by extrapolating.

So far, I have indicated how extrapolation supposedly leads us from the concept of the indefinitely large to the concept of the infinite. But how does extrapolation lead us from Fin(ZFC) to ZFC? According to Lavine, the Ω's that show up in the axioms of Fin(ZFC) represent indefinitely large sets. Indeed, he argues that the axioms governing the *Ω's, axioms (8)–(11), simply codify some of our intuitions that arise from our* experience of the indefinitely large. Because the Ω's are indefinitely large, contextual features play a role in our understanding of axioms that contain symbols for them. For, it is only relative to some context that we view a set as indefinitely large. When we extrapolate (or equivalently, drop the dependence on context), our interpretation of the axioms of Fin(ZFC) changes.

As I noted, dropping the dependence on context appears to have a straightforward formal interpretation: we simply drop the bounds on the Ω's. Lavine suggests, however, that we should interpret the dropping of the bounds as a way of setting the bounds equal. As he writes,

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[13] Lavine, *Understanding the Infinite*, p. 249.
[14] Ibid., p. 248.

> The process of extrapolation—of dropping the Ω's that serve as bounds—is my proposed formal analysis of what we actually do: we fail to attend to the differences between the bounds, the indefinitely large quantities. There is a trivial sense in which dropping the bounds is nothing more than a matter of convenience, since they can be resupplied in a simple and automatic way, but how should the effect of dropping the bounds be understood when we take the theory without them seriously, not as a mere abbreviation. I think the effect of dropping the bounds is best understood as the result of setting the bounds equal.[15]

So, we extrapolate from Fin(ZFC) to ZFC by setting the bounds equal, or equivalently accepting the schema $\Omega_p = \Omega_q$. Accepting this schema appears to have the same effect as dropping the bounds on the quantifiers. For, in setting the bounds equal we assume that the quantifiers all have the same domain, and so the bounds become superfluous. Overall, then, extrapolation allows us to eliminate the Ω's in Fin(ZFC); we thus obtain ZFC.[16]

III. The Axioms of Indefinitely Large Size

To show that our knowledge of ZFC is grounded in our experience of the indefinitely large, Lavine must explain not only how ZFC arises from Fin(ZFC) but also how our knowledge of the principles of Fin(ZFC) are grounded in our experience of the indefinitely large. In the rest of the chapter I will be concerned with this second step of Lavine's project. I will concentrate mainly on Lavine's discussion of Zillion and Relativized Extensionality.

Before examining these axioms, however, it is helpful to discuss Lavine's arguments in support of the Axioms of Indefinitely Large Size, i.e., axioms (8)–(11). For, in these arguments Lavine relies on three crucial notions: a notion of the indefinitely large,

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[15] Ibid., p. 257.

[16] Note that when we eliminate the Ω's in the axioms of Fin(ZFC), we obtain the axioms of ZFC, the axioms of equality (without full Leibniz Law), and we obtain six additional axioms. Two of these six additional axioms govern '∅' and '*A*'. The rest are the unrelativized axioms of indefinitely large size. None of the six additional axioms enables us to prove theorems that we could not already prove in ZFC.

a notion of the "availability" of a mathematical object, and a notion of availability function. These notions play a central role in Lavine's justification of Zillion and Relativized Extensionality. The discussion of axioms (8)–(11) will provide an introduction these notions.

Given that Lavine thinks axioms (8)–(11) systematize intuitions arising from our experience of the indefinitely large, we should think of the Ω's that show up in the axioms as indefinitely large sets.

Axiom (8) states that $\varnothing \in \Omega_p$. That is, it states that if Ω_p is an indefinitely large set of sets, then $\varnothing$ is in Ω_p. Lavine appears to justify this axiom with the following two claims.

(19) An indefinitely large set contains whatever is available.[17]

(20) Denoting an object by a closed term, i.e., a term with no free variables, makes that object available.[18]

Axiom (8) follows from these two claims. We have used the closed term '$\varnothing$' to denote $\varnothing$, so $\varnothing$ is available. And since indefinitely large sets contain whatever is available, $\varnothing$ is in Ω_p.

As for (19), it is difficult to see why Lavine holds it. He first mentions the notion of availability about one-hundred pages before setting out his theory of the indefinitely large. He discusses Charles Parsons's examination of a central tenet of the iterative conception of set, namely that sets are formed at stages from certain *available* sets. Parsons ultimately adopts a modal notion of availability to explain this tenet of the iterative conception. As Parsons writes,

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[17] "All that follows from our conception of Ω_0 as indefinitely large is that if something is available, then it is in Ω_0, ..." Lavine, *Understanding the Infinite*, p. 262.

[18] "We make the plausible assumption that objects we actually do denote by closed terms in our basic notation are available." Lavine, *Understanding the Infinite*, p. 262.

> The idea that any available objects can be formed into a set is, I believe, correct, provided that it is expressed abstractly enough, so that 'availability' has neither the force of existence at a particular *time* nor of giveness to the human mind, and formation is not thought of as an action or Husserlian *Akt.* What we need to do is to replace the language of time and activity by the more bloodless language of potentiality and actuality.[19]

Given that Lavine initially cites Parsons, one might think that Lavine's notion of availability is connected to Parsons's. This thought, however, is incorrect. Parsons explains what it is for a mathematical object to be available without requiring that availability be an epistemic notion. Lavine, however, is interested in "epistemically availability." As he writes, "The main sort of availability we shall consider for motivational purposes is epistemic availability with respect to a particular purpose, discourse, state of knowledge, or the like."[20]

But the mere restriction to epistemic availability does not support the connection between availability and the indefinitely large stated in (19). For, many indefinitely large sets do *not* contain whatever is epistemically available. For instance, the set of even numbers is indefinitely large, but 2, which I assume is an epistemically available number, is not in this set.

I believe it is best to assume that Lavine holds (19) because he has restricted his attention to "closed indefinitely large sets," where a closed indefinitely large set is an indefinitely large set that contains whatever is available.

Lavine provides an example of a closed indefinitely large set. He imagines a child who has a huge bucket of beans which she relies on to display numbers and perform

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19 Charles Parsons, "What is the Iterative Conception of Set." In *Philosophy of Mathematics: Selected Readings*, Second Edition. Edited by Paul Benacerraf and Hilary Putnam. Cambridge University Press. Cambridge, 1983. p. 526.

20 Lavine, *Understanding the Infinite*, p. 261.

various operations on numbers. The child's counting out, say, nine beans or seeing nine beans makes the number nine available to the child. Furthermore, in the child's present circumstance, the child considers the total number of beans in the bucket as far too many to count. Here, then, we have an example of a closed indefinitely large set. The set of beans is an indefinitely large set because the child thinks it is far too large to count. This indefinitely large set of beans contains every available number because the only numbers that the child has available in these circumstances are numbers she can form using beans in the buckets. So, the only numbers available to the child are numbers that are less than the number of beans in the bucket. In what follows, when I speak of indefinitely large sets, I will have in mind closed indefinitely large sets.

As for (20), Lavine provides further information about epistemic availability that helps us see why he believes that it holds. As he writes,

> We make the plausible assumption that objects [i.e., mathematical objects] we actually do denote by closed terms in our basic notation are available. In our prime example of availability, epistemic availability, that assumption is clearly reasonable. Thus, for example, schema (6) [$X_0<\omega_0$] brings with it the assumption that any number whose notation we do employ is available, that is, in *A* [an indefinitely large set].[21]

Here Lavine provides conditions under which it is permissible to say that a mathematical object is epistemically availability. He contends that if we write down the name of a mathematical object using standard notation, then the object is made epistemically available.

The above passage suggests that Lavine believes that we can see that (20) is true by reflecting on the notion of epistemic availability. However, this defense of (20) is

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[21] Lavine, *Understanding the Infinite*, p. 262.

questionable. I take it that an object is epistemically available as long as we have some sort of mental access to the object or it is somehow given to our mind. If this is the case, why should writing down a sign for an object ensure that the object is available to us? For the most part, signs do not resemble what they signify. And, certainly someone could accidently write down a '1' followed by an '2' and yet have no mental access to 12.

I believe, however, that we can provide some motivation for a slightly revised version of (20). In particular, consider the following:

Denotation Condition: A mathematical object is epistemically available in a context, if we, in that context, employ as well as understand sentences containing a closed term in standard notation that denotes that object.

We might also say that a mathematical object is epistemically available in a context, if we, in that context, employ a closed term in standard notation that denotes that object and also know how to *use* this term. The denotation condition differs from (20) in that it requires us to understand the sentence containing the term or similarly know how to use the term in question.

The denotation condition has some plausibility in the case of natural numbers. Arabic numerals are the canonical notation for small natural numbers. Because this notation is standardly taught, we expect any competent English speaker to know which number we are talking about when we use an Arabic numeral. And if the speaker knows which number is under discussion, it seem plausible to say that this number is available to her.

Notice that in making the denotation account plausible, the use of Arabic numerals is key. We do not believe that we make a number available when we use definite descriptions, such as 'the number of boys in the yard', or when we write out complicated expressions involving function symbols, such as '$2 + 324 + 7 - 6 + 132 + 16{,}867 - \sqrt{9}$'. For, if we use either of these expressions, a competent speaker might legitimately wonder which number we are referring to. However, something seems to be amiss if an English speaker wonders which number 5 is.[22] So, if we suppose that a mathematical object is epistemically available so long as we know which object is under discussion, then small numbers are made available when we employ an Arabic numeral to refer to them.

Lavine's discussion of the rest of the axioms governing the Ω's bring out further features of the notion of availability. The form of axioms (9), (10) and (11) is different from that of axiom (8): in particular, axioms (9), (10) and (11) have more than one occurrence of 'Ω' in them.

Lavine extends the bean scenario to provide support for these axioms. So, we again imagine a child who has a huge bucket of beans that she considers, in her present circumstances, as much too large to count. And again we imagine that the child makes a number available to herself by seeing or by counting out that number of beans. In such a situation, it is clear that if the child starts out with ω_0 beans in her bucket, she may at some point run out of beans. In particular, she may perform a bean-operation whose value is larger than ω_0. For example, she may try to add numbers whose sum is bigger than ω_0. The motivated child responds by getting more beans and thus increasing the number of available beans. The child can always get enough beans, say ω_1 beans, to calculate the sum of any two numbers less than ω_0. Similarly, for many other simple operations,[23] the child is able to get enough beans to perform the desired calculation. In general, the following

[22] Someone might wonder whether 5 is Lucy's favorite number or Cathy's. This case can be distinguished from the case where we wonder which number 5 is, simpliciter.

[23] We must restrict what sorts of bean-operations are permissible. If the operation is too complex, it is implausible to say that the child will not be able to get enough beans to perform it.

"bean-principle" appears to be true: from our original pile of beans, we can create a new pile of beans that allows us to calculate the result of relatively simple operations performed with the original pile.

If we suppose that Ω_0 contains numbers less than ω_0 and Ω_1 contains numbers less than ω_1, then we can see how the story is relevant to axiom (9). Axiom (9) states that $\forall x,y(x,y \in \Omega_0 \rightarrow A(x,y) \in \Omega_1)$. That is, it states that Ω_1 contains all objects that can be obtained by applying A to objects in Ω_0. Because adjunction is a simple operation, axiom (9) appears to be supported by the bean-principle.

The bean-principle also provides support for axiom (10). Axiom (10) states that $\forall x(x \in \Omega_0 \rightarrow x \in \Omega_1)$. That is, it states that Ω_0 is a subset of Ω_1. Since the identity function is an elementary operation, axiom (10) also appears to be supported by the bean-principle.

Axioms (9) and (10) bring out further features of the notion of availability. The axioms illustrate two related notions that Lavine relies on, the notion of degrees of availability and, in connection with it, the notion of an "availability function." As for the notion of degrees of availability, Lavine contends that objects are not just available simpliciter. Rather objects are available to different degrees. In particular, Ω_0 contains all objects that are "available to the first degree." Ω_1 contains all things that are "available to the second degree," i.e., all objects that are available in virtue of the fact that certain objects in Ω_0 are available. Ω_2 contains all things that are "available to the third degree," i.e., all objects that are available in virtue of the fact that certain objects in Ω_1 are available. And so forth.

I take it that an object y is available in virtue of another object x, if x's availability somehow facilitates y's availability. For example, suppose we make an object available by seeing it. Then, y is available in virtue of the availability of x if it is in virtue of seeing x that we are able to see y. Lavine provides an example that suggests that this is how he wants us to understand the notion of "availability to the n^{th} degree."[24] He supposes that stars are made available by seeing them and that stars that are directly pointed out to us are available to the first degree. Other stars are then made available in virtue of these initially selected stars. For example, suppose we are told about the stars that are directly beneath stars that are available to the first degree. Then, the stars beneath the stars available to the first degree are available to the second degree. Indeed, seeing stars that are available to the first degree puts us in the position to see the stars that are beneath them.

The notion of an availability function is the formal counterpart to the notion of degrees of availability. Availability functions are functions that map objects in Ω_i to objects in Ω_{i+1}. The notion of an availability function is meant to capture the idea that the things in Ω_1 are available because the things in Ω_0 are. As Lavine writes, "The things available in virtue of the availability of the members of Ω_0 are just those things that can be obtained from members of Ω_0 via the availability functions."[25] In the star example above, the relevant availability function is the function that takes a star to a star directly beneath it. Later we will examine the notion of an availability function in more detail as it is important in assessing Lavine's argument for both Zillion and Relativized Extensionality.

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[24] Lavine, *Understanding the Infinite*, p. 263. The example I present is a simplified version of Lavine's.

[25] Ibid., p. 261.

The notion of degrees of availability and that of an availability function are relevant to axioms (9) and (10) because these axioms require that certain functions are availability functions. In particular, (9) tells us that A is an availability function and (10) tells us that the identity function is an availability function.

Axioms (9) and (10) also bring out features of the models of Fin(ZFC). In models of Fin(ZFC) the Ω's form a hierarchy. That is, the Ω's are such that $\Omega_0 \subseteq \Omega_1 \subseteq \Omega_2 \ldots$, and for any set of availability functions $\mathcal{F}$, the range of $f_i \in \mathcal{F}$, when f_i is restricted to Ω_p, is contained in Ω_q for $p<q$, i.e., rang($f_i \restriction \Omega_p$) $\subseteq \Omega_q$. So, we start out with an initial set of things, Ω_0, and we build up a larger set by applying functions to the things in Ω_0. We continue this process by applying functions to the things in Ω_1. We thereby generate a larger set Ω_2. By continuing this process, we obtain a hierarchy of Ω's.

Others have proposed models of mathematical theories that are similar to this. In "Mathematics Without Foundations"[26] Hilary Putnam supplies a modal interpretation of set theoretic sentences that has properties similar to those of Lavine's interpretation of Fin(ZFC). Putnam start with models of set theory that are well-founded physical graphs. The points of the graph are physical points, each point representing a set. The points are related by arrows that indicate membership. Putnam suggests that we interpret sentences of set theory by starting with a physical graph and then considering possible extension of it. For example, if '$\forall x \exists y Pxy$' is a sentence of set theory and 'P' has no quantifiers, then '$\forall x \exists y Pxy$' is interpreted as follows. If G is a physical graph and b is a point on G, then it is possible that there is an extension, G', of G and a point c in G' such that $P(b,c)$.

[26] Hilary Putnam, "Mathematics Without Foundations." In *Philosophy of Mathematics: Selected Readings*, Second Edition. Edited by Paul Benacerraf and Hilary Putnam. Cambridge University Press. Cambridge. 1983. pp. 295-311.

Putnam's G's are similar to Lavine's Ω's. The deeper the quantifiers are imbedded the more G's we need to find such that $G \subseteq G' \subseteq G'' \subseteq \ldots$. Also the notion of an accessibility function is present. We get from G to a larger graph G' by an accessibility relation that holds between graphs and their extensions.

We can also view Charles Parsons work in "Ontology and Mathematics" as a precursor to Lavine's theory.[27] Parsons gives a modal interpretation of arithmetic. He introduces a model of arithmetic which consists of finite possible worlds. The worlds are related by an accessibility relation that ensures that, for each world, there is a world accessible from it which contains one more thing. Again, there are similarities with Fin(ZFC). The worlds are progressively larger, and we get from one possible world to a larger one by an accessibility relation that holds between possible worlds.

We now turn to the last axiom governing the Ω's, axiom (11): $(\forall x_1,\ldots,x_n \in \Omega_p)((\forall x \in \Omega_q)\phi \leftrightarrow (\forall x \in \Omega_r)\phi)$. Axiom (11) states that for $p < q < r$, we can replace an occurrence of 'Ω_q' with the larger 'Ω_r' without a change in truth value. Another observation about the child's indefinitely large pile of beans helps provide grounds for axiom (11). Even though we have stipulated that there are ω_0 beans in the original pile, the child does not know how many beans are in the pile. The child knows only that there are a lot of beans. As far as she is concerned, there could be ω_1 beans in the original pile. As she sees it, ω_0 and ω_1 are indiscernible. We can characterize this situation more formally as follows. If we replace every occurrence of 'Ω_0' in '$\phi(\Omega_0)$' by 'Ω_1', then '$\phi(\Omega_0)$' and '$\phi(\Omega_1)$' do not differ in meaning in any way that matters. Axiom (11) captures the idea that the Ω's are indiscernible in the manner suggested.

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[27] Charles Parsons, "Ontology and Mathematics." In *Mathematics in Philosophy: Selected Essays.* Cornell University Press. Ithaca, New York. 1983. pp.37-62.

IV. Zillion: Some Preliminaries

Now that I have introduced the special notions on which Lavine relies, I will begin my critique of his project. I will examine his support for certain relativizations of the axioms of ZFC, in particular axiom (18), Zillion, and axiom (12), Relativized Extensionality. In the rest of the chapter, I will point out weaknesses in Lavine's argument that Zillion is a self-evident principle about finite sets and argue that he ultimately needs to support a more general version of Zillion. I will then argue that "Generalized Zillion" and Relativized Extensionality undermine each other. In particular, I will argue that no plausible interpretation of the notion of availability or availability function enables us to show that both Relativized Extensionality and Generalized Zillion are true principles about finite sets.

As we have seen, Zillion says that there is a set x in Ω_0 that contains $\varnothing$ and contains the "successor" of every Ω_1-member of x, or in symbols,

$$(\exists x \in \Omega_0)(\varnothing \in x \wedge (\forall y \in \Omega_1)(y \in x \rightarrow y \cup \{y\} \in x)) \tag{18}$$

Axiom (18) is a principle about sets. Indeed, Fin(ZFC) is a theory about indefinitely large finite sets, so we assume that the members of the Ω's are sets and that the quantifiers range over sets. But even though (18) is about sets, we can plausibly view (18) as a principle about numbers. We can do this because we can identify the set $\varnothing$ and sets formed from $\varnothing$ by adjunction with numbers. Furthermore, if we make this identification, we can plausibly assume that members of the Ω's are numbers and so assume that the quantifiers in (18) range over numbers.

More specifically, we can identify the natural numbers with "the von Neumann numbers," i.e., the sets $\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ldots$ and so forth. We identify 0 with $\varnothing$ and 1 with $\{\varnothing\}$, 2 with $\{\varnothing,\{\varnothing\}\}$, 3 with $\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}$, etc., and the successor function with the function that takes a set x to the set $x \cup \{x\}$. Once such an identification is made, it is plausible to assume that the Ω's contain only numbers and so assume that the quantifiers in (18) range over numbers. So, we can read (18) as follows: there is a number x in Ω_0 that contains 0 and contains the successor of every number that is less than x and that is in Ω_1. In what follows, I will often rely on this identification of the von Neuman numbers with the natural numbers in order to explain under what conditions (18) is true.

Axiom (18) is the relativized version of the following Axiom of Infinity:

(21) $$\exists x(\varnothing \in x \wedge \forall y(y \in x \rightarrow y \cup \{y\} \in x))$$

In some respects, (18) and (21) are similar. An infinite set witnesses the truth of (21), and if we choose the Ω's appropriately, an infinite set also witnesses the truth of (18). In particular, the infinite set $\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\},\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\},\ldots\}$, which is identified with the first infinite number and standardly called ω, witnesses the truth of (21), and it is the smallest set that does. Furthermore, if $\Omega_0 = \Omega_1 = \{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\},\ldots,\omega\}$, then ω also witnesses the truth of (18).[28]

Although an infinite set can witness the truth of both (18) and (21), these axioms are critically different. (21) is true only if an infinite set exists.[29] On the other hand, we can choose the Ω's so that a finite set witnesses the truth of (18). For example, let $\Omega_0 = \{0,1,3\}$ and $\Omega_1 = \{0,1,3\}$. (Assume that a numeral here is an abbreviation for the appropriate set term that denotes the corresponding von Neumann number.) 3 witnesses the truth of

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[28] Note that the axioms governing the Ω's do not preclude ω from being in the Ω's.

[29] (21) does not entail that ω exists. We also need the Axiom of Separation.

Zillion. For $0 \in 3$, and since $1 \in 3$ and $2 \in 3$, the successor of every number that is in both 3 and Ω_1 is in 3. Observe that 3 witnesses the truth of (18) here because 2 is not in Ω_1, and since 2 is not in Ω_1, the successor of 2 need not be in the number that witnesses the truth of Zillion.

In general, the following principle is true:

(22) A finite von Neumann number witnesses the truth of Zillion only if the predecessor of that number is not in Ω_1.

We can see that this principle is true as follows. Suppose x is finite von Neumann number in Ω_0. And suppose that the predecessor of x is in Ω_1. We know that since x witnesses the truth of (18), it must contain the successor of every number that is both in x and in Ω_1. But since the predecessor of x is both in x and in Ω_1, then x must be in the number that witnesses the truth of (18). So, x cannot witness the truth of (18).

An even more general way of stating this condition is as follows:

(23) A finite von Neumann number witnesses the truth of Zillion only if the predecessor function is not an availability function.

To see this, suppose that the predecessor function is an availability function. In this case, the predecessor of every member of Ω_0 would be in Ω_1. In particular, for all x, if x is a finite von Neumann number in Ω_0, then the predecessor of x is in Ω_1. By (22), then, no finite von Neumann number witnesses the truth of (18).

So far, we have seen that a finite set *can* witness the truth of (18). But Lavine needs more than this because he wants to establish that Zillion is self-evident principle about indefinitely large finite sets. To make his case, he relies on intuitions that supposedly arise from our experiences of indefinitely large numbers. As he writes,

> It seems self-evident that in any fixed context there will be a number whose predecessor is not available to any degree employed in that context, and that is all the Axiom of Zillion says.[30]

Lavine provides an example to bolster his claim. He contends that in certain contexts the indefinitely large number $10^{10^{10^{10}}}$ is available, but not all of its predecessors are available. As he writes,

> It seems clear, for example, that $10^{10^{10^{10}}}$ is epistemically available in many actual contexts even though there is no actual context in which all of the natural numbers below it are epistemically available.[31]

So to show that Zillion is self-evident principle about finite sets, Lavine appeals to intuitions that arise from our experiences of indefinitely-large, finite numbers.[32]

V. Two Accounts of Availability

We have seen that Lavine claims that it is self-evident that a number may be available even though not all of its predecessors are available. But why are certain numbers available and others not? To determine the merit of Lavine's contention, we need an account of availability. I will offer two accounts of availability and argue that, with certain caveats, both accounts appear to make it plausible that a number is available although not all its predecessors are available.

In connection with axiom (8), I discussed what I called the denotation condition for availability: a number is available in a context, if, in that context, we understand and write down a sentence containing a term, in standard notation, that denotes it.

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[30] Lavine, *Understanding the Infinite*, p. 299.

[31] Ibid., p. 294. For further relevant comments, see p. 298.

[32] We should note that in these passages Lavine claims that an indefinitely large number is in an indefinitely large set. So, not only are the Ωs indefinitely large but the Ωs contain objects that are indefinitely large.

The denotation account supplies one way of filling in Lavine's argument for Zillion. In almost any given context, we will write down a relatively small number of symbols and always fewer than $10^{10^{10^{10}}}$. So, if '$10^{10^{10^{10}}}$' is one of the numerals we write down, then, because $10^{10^{10^{10}}}$ is so large, we will not write down all of its predecessors.

However, there appear to be difficulties with relying on the denotation account to argue for Zillion. First, in Section III, I suggested that the denotation account seems plausible when our notation is restricted to Arabic numerals. For, when we use an Arabic numeral to denote a number, we appear to know which number is under discussion, and so it appears that the number is made available. Lavine, however, employs terms that contain function symbols as well as Arabic numerals. This is problematic because we do not always know which number is under discussion when we use terms containing function symbols. Indeed, a competent speaker might not know which number is denoted by '$8+6-10+346+12{,}345-\sqrt{16}$'. So, Lavine appears to rely on a less plausible version of the denotation account because he assume that we make numbers available when we employ terms containing function symbols.

However, even though employing some terms containing function symbols does not make a number available, employing others appears to. Because it is practically impossible to use Arabic numerals to denote extremely large numbers, we have devised standard ways of representing large numbers. One technique is the use of scientific notation. Another is the use of a symbol for exponentiation. Because such notation is standardly employed and because we cannot employ Arabic numerals to denote truly huge numbers, perhaps numbers are made available when we use such notation. Afterall, we cannot give an informative answer to the question 'Which number is $10^{10^{10^{10}}}$?'.

But this observation brings us to another worry about employing the denotation account in support of (18). Lavine claims that he can provide motivation for Fin(ZFC) that is acceptable to finitists as well as non-finitists. However, many finitists and even some non-finitists would reject a denotation account that allows for terms containing function symbols.

One problem is that compact notation, which is standardly used, does not always deliver numbers that we think are intuitively available. For example, suppose we add a symbol for the Ackermann function.[33] The Ackermann function K is defined as follows:

$$K(0,y) = n + 1$$

$$K(x + 1,0) = K(x,1)$$

$$K(x + 1,y + 1) = K(x,K(x + 1,y))$$

Some of the values on the diagonal are as follows:

$$K(0,0) = 1$$

$$K(1,1) = 3$$

$$K(2,2) = 7$$

$$K(3,3) = 61$$

$$K(4,4) = \left.2^{2^{\cdot^{\cdot^{2}}}}\right\}n \text{, where } n = 7 \text{, i.e., } K(4,4) = \text{a stack of 2's that is 7 high.}$$

As for $K(5,5)$, it is quite large. To get an idea of how large, consider that

$$K(5,3) = \left.2^{2^{\cdot^{\cdot^{2}}}}\right\}n \text{, where } n = \left.2^{2^{\cdot^{\cdot^{2}}}}\right\}m \text{ where } m = 25{,}536.$$

Richard Crandall gives us a sense of just how big this is:

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[33] The Ackermann function is the standard example of a computable function that is not primitive recursive.

> The fifth [Ackermann number $K(5,5)$] is so large that it could not be written on a universe-sized piece of paper, even using exponential notation! Compared with the fifth Ackermann number, the mighty googolplex [$=10^{10^{10^{10}}}$] is but a spit in the proverbial bucket.[34]

By adding a symbol for the Ackermann function, then, we can denote truly huge numbers using fewer than five symbols.

Many might doubt that such huge numbers are made available when we write down and understand the relevant terms. For, these numbers are so incomprehensibly large that it does not seem plausible to say that these huge numbers are made available just because we write down and understanding the relevant terms. Furthermore, at least some finitists would deny that the Ackermann function delivers available numbers. W. W. Tait, for instance, explicitly defends the thesis that finitistic methods are encapsulated in primitive recursive arithmetic.[35] The Ackermann function, however, is not primitive recursive.

But even if Lavine admits only primitive recursive functions, some finitists will still find his claims controversial. In his paper "Finitism and Intuitive Knowledge,"[36] Charles Parson argues that exponentiation may not be an acceptable finitist function. He contends that even though we can intuit that some primitive recursive functions are well-defined, in particular addition and multiplication, we cannot do the same for exponentiation.

So, the difficulty with using the denotation account to defend (18) is that compact notation does not always deliver what many would think of as available numbers. In

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[34] Richard E. Crandall, "The Challenge of Large Numbers." *Scientific American*, February 1997. p. 78.

[35] W. W. Tait, "Finitism", *The Journal of Philosophy* 78, 1981. p. 524-546.

[36] Charles Parsons, "Finitism and Intuitive Knowledge," Forthcoming in Matthias Schirn (Ed.), *Philosophy of Mathematics Today*, Oxford University Press.

response, Lavine can simply restrict his notation accordingly. In particular, he can restrict himself to Arabic numerals. For, in almost any given context, we will write down a relatively small number of symbols and always fewer than 1,346,231. So, if '1,346,231' is one of the numerals we write down, then, because 1,346,231 is so large, we will not, in any one context, write down every one of its predecessors.

We can provide a different way of understanding Lavine's argument for Zillion by supplying a different account of availability, an account which comes from Lavine's bean example. In the bean example, 7, for example, is available to the child because she can see a pile of seven beans. So, let us say a number n is available relative to a context, if in that context a person sees n objects. If we rely on this account, it seems correct to suppose that in certain contexts an indefinitely large number is available but not all of its predecessors are. For in certain contexts we see n objects, for sufficiently large n, but we do not see m objects for every m less than n. For example, perhaps looking into the sky ensures that the number of stars in the sky is available to us. But in this context not all numbers less than the number of stars are available. For, in this context, although we see a differentiable group of n stars, we do not see a group of m stars for all m less than n.

More generally, we have reason to believe that given *any* context, an indefinitely large number and its predecessor are never both available in that context. Axiom (11) tells us that the indefinitely large Ω's that bound the quantifiers are indiscernible. It is also reasonable to claim that indefinitely large numbers that are in these indefinitely large Ω's are also indiscernible. The reason is that experiences of indefinitely large quantities appear phenomenologically indiscernible to us. For example, seeing 10,067 pebbles does not appear to be phenomenologically different from seeing 10,068 or 10,066 pebbles. When I look up into the sky I experience some number of stars, but it makes no difference whether I experience 10,068 or 10,066 stars as I can detect no difference between these experiences.

This indescirnibility has implications about what numbers are available in any one context. Since my experience of an indefinitely large number and its predecessor are indiscernible, in any *one* context it is not the case that both an indefinitely large number and its predecessor are available in that context. To see this, consider the following example. Suppose that from my window I see John running around the Charles, and I see no other person running around the Charles. In such a circumstance, it seems plausible to say that John is made available to me, even though my view of him would not enable me to conclude that it is John that I see rather than his twin. However, if I not only see John but also see his twin running around the Charles, one begins to doubt that, in this context, either man is made available to me. For even though I need not be able to distinguish an object from all others in order for it to be available to me, I should be able to distinguish it from other objects in the immediate context. Accordingly, since my experience of 12,983 objects appears the same to me as my experience of 12,982 objects, in one context both numbers cannot be made available.

One might, however, have certain reservations about the contention that a large number *n* is made available by seeing *n* things. If there are three apples before me, it seems reasonable to say that seeing these apples makes the number three available to me. For, I can see the apples as three in number. My experience makes me aware of the number of apples before me. But how can looking at 109,843 pebbles make the number 109,843 available? My experience does not appear to make we aware of the number of pebbles.

Indeed, when I see a large quantity of things, I am not aware of how many things are before me.

To respond to this criticism, we need to distinguish between being aware of a number because we form beliefs about it on the basis of experience and being aware of a number because we are somehow conscious of that many things. If we cash out awareness in terms of belief, then I am not aware of 109,843 pebbles when I see them because I do not form the belief that there are 109,843 pebbles before me. However, if we cash out awareness in terms of consciousness, then arguably I am aware of 109,843 pebbles because seeing them is a way of making me conscious of them. If we appeal to this second sense of awareness, then seeing 109,843 pebbles does make me aware of them, and so perhaps is a way of making the number 109,843 available.

VI. A Problem with Lavine's Support of Zillion

So far we have tried to substantiate Lavine's argument for (18) by presenting two accounts of availability. However, even if one believes that these accounts are plausible, we have reason to question Lavine's argument in support of (18).

To see this, recall principle (23) which says a finite von Neumann number witnesses the truth of (18) only if the predecessor function is not an availability function. The critical question is whether the intuition Lavine cites in support of Zillion—i.e., the intuition that an indefinitely large number is available but not all of its predecessors are available—ensures that the predecessor function is not an availability function. That is, is it the case that if an indefinitely large number is available but not all of its predecessors are available, then the predecessor function is not an availability function?

The answer is no.[37] For, even if the predecessor is an availability function, there can be an available indefinitely-large number whose predecessors are not all available. For example, let $\Omega_0 = \{0, 1, 2, 10^{10}\text{-}1, 10^{10}\}$ and $\Omega_1 \subseteq \{0, 1, 2, 10^{10}\text{-}2, 10^{10}\text{-}1, 10^{10}\}$. In this case, the predecessor is an availability function because the predecessor of every member of Ω_0 is in Ω_1. If 10^{10} is an available number, then it is clear that in every context, i.e., in every Ω, not all of its predecessors are available. So, it appears that this choice of the Ω's is consistent with Lavine's intuition about indefinitely large numbers. For, there is a available number not all of whose predecessors are available. But since the predecessor function is an availability function, by (23), we know that a finite number cannot witness the truth of (18). So, in this example, no finite number witnesses the truth of (18), even though there is an available indefinitely-large number whose predecessors are not all available.

Lavine has a response to this problem. As he writes,

> Thus, sentence (25) $[\forall x \in \Omega_0(x \neq \varnothing \rightarrow \exists y \in \Omega_1(y \in x \wedge y \cup \{y\} \notin x))]$ has the effect of adding the predecessor function to the availability functions.[38] Sentence (25) is not plausible in a theory that allows indefinitely large sets into Ω_0, since it imposes restrictions on what may be found to be a member of any [hereditarily finite set] in Ω_0; for any x, it asserts that there is a y such that $[y \cup \{y\}]$ is not in x, and that exclusion is contrary to the possibility that x is indefinitely large.[39]

So, Lavine contends that our intuitions about indefinitely large sets support the claim that the predecessor function is not an availability function. In particular, he relies on the

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[37] Vann McGee pointed this problem out to me.

[38] To see how (25) adds the predecessor function to the availability functions, suppose some number, say 4, is in Ω_0, where the numeral '4' here denote the corresponding von Neumann number. Since 4 is in Ω_0, (25) tells us that there must be a y in Ω_1 such that y is in 4 but $y \cup \{y\} \notin 4$. Since the only member of 4 that satisfies this requirement is 3, (25) requires that $3 \in \Omega_1$.

[39] Lavine, *Understanding the Infinite*, p. 299.

intuition that if x is an indefinitely large set, then we should not add an axiom that restricts sets from being in x.

The problems with relying on this intuition is that we can just as plausibly use it to argue for the opposite conclusion, namely that the predecessor function *is* an availability function. By excluding the predecessor function from the set of availability function, we bar sets from being in the indefinitely large Ω's. In particular, not every predecessor of a member of Ω_0 can be in Ω_1. According to the above passage, barring these sets from the indefinitely large set Ω_1 would be contrary to the fact that Ω_1 is indefinitely large.

Although Lavine fails to show that the predecessor function is not an availability function, he at least realizes that to defend the claim that a finite set witnesses the truth of Zillion he must give reasons for believing that the predecessor is not an availability function. So, as it stands, Lavine has not succeeded in showing that we have reason to believe that a finite set witnesses the truth of Zillion. However, the option appears to be open to him to defend this claim by arguing that predecessor is not an availability function. Indeed, a promising argument along these lines is to appeal to the indescirnibility of indefinitely large numbers. As I argued in the last section, we have reason to believe that in any context, it is not the case that both an indefinitely large number and its predecessor is available. We can use this fact to argue that predecessor is not an availability function.

However, even if such a defense of Zillion is possible, it is ultimately of little use. For, in the next section, I will argue that there is little point in defending Zillion because Lavine needs to defend a more general claim than Zillion.

VII. Why Zillion?

As we have seen, Lavine argues for Zillion, i.e., (18) by appealing to intuitions we have about indefinitely large numbers. This argument in support of (18) is plausible because we can find a finite von Neumann number that witnesses the truth of (18). It is natural to wonder whether Lavine's argument still works if we identify numbers with other ω-sequences of sets.

For example, suppose we identify numbers with Zermelo numbers, i.e., suppose we identify 0 with $\varnothing$, 1 with $\{\varnothing\}$, 2 with $\{\{\varnothing\}\}$, 3 with $\{\{\{\varnothing\}\}\}$, and so forth. The Axiom of Infinity associated with this identification is

(24) $$\exists x(\varnothing \in x \wedge \forall y(y \in x \rightarrow \{y\} \in x))$$

This axiom coupled with the Axiom of Separation guarantees the existence of the set $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}, \ldots\}$, which we identify with the first infinite number and, for present purposes, dub 'ϖ'. The relativized version of (24) is

(25) $$(\exists x \in \Omega_0)(\varnothing \in x \wedge (\forall y \in \Omega_1)(y \in x \rightarrow \{y\} \in x))$$

As in the case of (18) and (21), an infinite set witnesses the truth of both (24) and (25), as long as the Ω's are chosen appropriately. In particular, ϖ witnesses the truth of both (24) and (25).

However, (25) is critically different from (18). In particular, even though we can choose the Ω's so that a finite set witnesses the truth of (25), the witness set will *not* be a Zermelo number. No finite Zermelo number witnesses the truth of (25). Indeed, any set satisfying (25) must contain $\varnothing$ and $\{\varnothing\}$, but no finite Zermelo number contains both these sets.

This observation has implications for Lavine's argument in support of (18). If we use (25) in Fin(ZFC) rather than (18), Lavine's intuitions about numbers will not lend support to (25). Since the set that (25) postulates is not a Zermelo number, we cannot translate (25) into a statement about numbers. For, it is not plausible to read the quantifiers in (25) as ranging over numbers. Accordingly, intuitions about numbers have no bearing on the truth of (25).

One might respond that this discovery does not pose a problem for Lavine. One might argue that as long as he can provide justification for (18), it does not matter that he cannot provide justification for (25). It is enough to provide justification for *one* finite set theory. He need not provide a justification for all the competing finite set theories.

This response works as long as (25) or some other relativization of an axiom of infinity does not provide a "less arbitrary" finite set theory than the one obtained using (18). For, if there is a finite set theory that is less arbitrary than Fin(ZFC) and Lavine cannot defend this less arbitrary theory, then I believe Lavine's choice of Fin(ZFC) is questionable.

I have in mind here a certain way one set theory is less arbitrary than another. I believe that one set theory is less arbitrary than another if the first posits all sets of a "similar" sort whereas the other posits the existence of only some sets of this particular kind. For example, a set theory that posits the existence of the sets $\{\varnothing\}$ and $\{\varnothing,\{\varnothing\}\}$, but does not posit $\{\{\varnothing\}\}$ is more arbitrary than a set theory that posits the existence of all three of these sets. For, these sets are all similar in that all have less than two members, all are hereditarily finite, and all are of low rank. In fact, there seems to be no important difference between these sets that would justify positing some but not all of them. I also take it that one theory is less arbitrary than another theory if a fragment of the one theory is less arbitrary than the corresponding fragment of the other.[40] So, if T-$\{A\}$ is the theory T without axiom A and if T-$\{A\}$ is less arbitrary than T'-$\{A'\}$ where A' corresponds to A, then T is less arbitrary than T'.

To show that there is a finite set theory that is "less arbitrary" than Fin(ZFC), which contains (18), I will argue that there is set theory less arbitrary than the set theory containing (21), the unrelativized version of (18). To see this, let us recall the different Axioms of Infinity from which the relativizations are obtained. As we have seen, (18) is the relativization of,

(21) $$\exists x(\varnothing \in x \wedge \forall y(y \in x \rightarrow y \cup \{y\} \in x))$$

whereas,

(25) $$(\exists x \in \Omega_0)(\varnothing \in x \wedge (\forall y \in \Omega_1)(y \in x \rightarrow \{y\} \in x))$$

is a relativization of a different Axiom of Infinity:

(24) $$\exists x(\varnothing \in x \wedge \forall y(y \in x \rightarrow \{y\} \in x))$$

There is yet another Axiom of Infinity,

(26) $$\exists x(\varnothing \in x \wedge \forall y \forall z(y \in x \wedge z \in x \rightarrow y \cup \{z\} \in x))$$

which has as its relativization,

(27) $$(\exists x \in \Omega_0)(\varnothing \in x \wedge (\forall y \in \Omega_1)(\forall z \in \Omega_1)(y \in x \wedge z \in x \rightarrow y \cup \{z\} \in x))$$

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[40] Theories have corresponding fragments, if a version of each axiom of the one theory is contained in the other theory.

I believe that we have reason to prefer the finite theory containing (27) over the finite theory containing (18) or the finite set theory containing (24). The reason is that we have some reason to prefer the unrelativized theory containing (26) over both the theory containing (21) or that containing (24). For, let Z^* be the theory containing all of the axioms of ZFC except Replacement, Choice and Infinity. I contend that $Z^* \cup \{(26)\}$ is less arbitrary and therefore a better theory than both $Z^* \cup \{(24)\}$ and $Z^* \cup \{(21)\}$.

To see this, observe that $Z^* \cup \{(24)\}$ and $Z^* \cup \{(21)\}$ fail to postulate the existence of the most basic infinite sets. Neither ensures the existence of the set containing all the finite well-founded sets. Furthermore, $Z^* \cup \{(24)\}$ fails to guarantee the existence of a set containing all the finite von Neumann numbers, and similarly, $Z^* \cup \{(21)\}$ fails to guarantee the existence of a set containing all the finite Zermelo numbers.[41] $Z^* \cup \{(26)\}$, on the other hand, guarantees the existence of all of these sets.

If we adopt $Z^* \cup \{(24)\}$, then we seem to state a preference for some infinite sets over others, for we end up with a theory that postulates the existence of the Zermelo set, but not the Von Neumann set. But why should we think that the Zermelo set exists, but the von Neumann set does not? Similarly why should we think that the Zermelo set exists, but a set containing all hereditarily finite sets does not? If we postulate one of these infinite sets, why not the other? In what way are these sets importantly different? In adding (26) to Z^* rather than (21) or (24), we avoid postulating some of these infinite sets but not others. Accordingly, $Z^* \cup \{(26)\}$ seems to be the least arbitrary of the lot.[42]

41 Gabriel Uzquiano discusses these claims in his MIT thesis. He also discusses the merit of each of these theories.

42 One might also claim that $Z^* \cup \{(26)\}$ is preferable because its natural model is $V_{2\omega}$ whereas the models of $Z^* \cup \{(24)\}$ and $Z^* \cup \{(21)\}$ are not identifiable with a V_α. Lavine, however, would argue against such a contention because he does not believe that the iterative hierarchy provides the basis for our conception of sets. Accordingly, he would not take this as reason to prefer $Z^* \cup \{(26)\}$. My argument in the text avoid appealing to the iterative hierarchy. I simply point out that a theory that posits some infinite sets but not other similar infinite sets seems more arbitrary than a set theory that posits all similar infinite sets.

If $Z^* \cup \{(26)\}$ is a less arbitrary theory than $Z^* \cup \{(24)\}$ or $Z^* \cup \{(21)\}$, then $ZFC^* \cup \{(26)\}$ is a less arbitrary theory than $ZFC^* \cup \{(24)\}$ or $ZFC^* \cup \{(21)\}$, where ZFC^* is the theory containing all of the axioms of ZFC except Infinity. Since the relativized version of $ZFC^* \cup \{(21)\}$ is simply Fin(ZFC), it follows that a finite set theory containing (27) is less arbitrary than Fin(ZFC).

This observation appears to pose a problem for Lavine. For, he cannot rely on intuitions about indefinitely large numbers to support (27). For, just as we cannot read the quantifiers in (24) as ranging over numbers, we cannot read the quantifiers in (27) as ranging over numbers.

However, even though intuitions about numbers are no help with respect to (27), we might be able to appeal to intuitions about sets, In particular, the following holds:

(28) A finite set witnesses the truth of (27) only if membership is not an availability function.[43]

That is, a finite set witnesses the truth of (27) only if it is not the case that for any set x in Ω_0 all the members of x are available in Ω_1. To show that (28) holds, suppose that x is finite set in Ω_0 that witnesses the truth of (27). If membership is an availability function, then x must contain every von Neumann number and so, contrary to our hypothesis, must be infinite. For, $\varnothing$ is in x, so x contains the successor of $\varnothing$. And if a von Neumann number n is in x, then since membership is an availability function, n is in Ω_1, and by (27) the successor of n is in x. So, x contains all finite von Neumann numbers.

Overall, then, we have some reason to prefer a finite set theory with (27) over one with Zillion. But if this is the case, Lavine cannot appeal to intuitions about numbers to

[43] I have followed Lavine and used the term 'availability function' to talk about relations as well as functions.

defend this preferred finite set theory. Rather, to defend (27), or what I will call "Generalized Zillion," he must establish the more general claim about the membership relation, that membership is not an availability function.

In what follows I will argue that the requirement that membership is not an availability function conflicts with Lavine's support of Relativized Extensionality. I will argue that no plausible interpretation of the notion of availability or availability function enables us to show that Generalized Zillion and Relativized Extensionality are true principles about finite sets.

As a final note, I will indicate how the results of this section highlight the difference between Lavine's notion of availability and the notion of availability employed iterative account.[44] The iterative conception provides a story about the set formation process. At the first stage, all non-collection are available. From these available object all collections are formed. At the second stage, then, all non-collections as well as all collections of non-collections are available. From all available objects at the second stage, all collections are formed. At the third stage, then all collections of available objects from the second stage are available. And so the story goes. So, on the iterative conception a set is available at a stage, only if all its members are available at the proceeding stage.

We can now see that the notion of availability at work in the iterative conception is quite different from the notion of availability at work in Lavine's conception. Accordingly to Lavine, certain sets must be available before all of their members are. Indeed, as (28) indicates, some sets must be available at an earlier "stage" than any of their members.

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[44] See George Boolos, "The Iterative Conception of Set," *Journal of Philosophy* 68 (1971). pp. 215-231. And also his "Iteration Again," *Philosophical Topics* 26 (1989). pp. 5-21.

VIII. Relativized Extensionality: Some Preliminaries

To establish that there appears to be no way to support both Generalized Zillion and Relativized Extensionality, we need to examine Relativized Extensionality. Let us recall that Relativized Extensionality reads as follows.

(12) $$(\forall xy \in \Omega_0)(\forall z \in \Omega_1(z \in x \leftrightarrow z \in y) \rightarrow x=y)$$

We can understand this axiom as stating that if x and y are distinct members of Ω_0, then there is a z in Ω_1 that witnesses that x and y are different, i.e., there is a z in Ω_1 that is either in x but not in y or in y but not in x. In the rest of the chapter I will examine Lavine's argument for this principle and argue that it does not seem possible to show that both Relativized Extensionality and Generalized Zillion are true principles about finite sets.

Lavine claims that Relativized Extensionality is a self-evident principle about sets. No doubt that *unrelativized* Extensionality is self-evident. But saying why it is self-evident is delicate matter. We can first observe that the justification of Extensionality is different from that of other set theoretic axioms. We may try to justify other axioms by arguing that if we think of sets in such and such a manner, then the axioms are easily seen to be true. In this way, we argue that the axioms, in some sense, follow from a certain conception of set, and so the axioms are self-evident with respect to this conception. However, we do not justify the Axiom of Extensionality by showing that it follows from some conception of set. Rather it is a constraint on any conception of set that Extensionality should be true of it. If a conception of set fails to ensure the truth of other axioms besides Extensionality, many will be dismayed, but it is arguable that we are not required to discard the conception. However, if someone offers a conception of set for which Extensionality fails, we suspect that she is not offering a conception of set. As George Boolos writes, "But a theory that did not affirm that the objects with which it dealt were identical if they had the same members would only by charity be called a theory of *sets* alone."[45] So, a conception of set is unacceptable if Extensionality is not true of it.

What, then, is an acceptable justification for the Extensionality? Boolos gives a plausible characterization in the following passage:

> It seems probable, nevertheless, that whatever justification for accepting the axiom of Extensionality there may be, it is more likely to resemble the justification for adopting most of the classical examples of analytic sentences, such as 'bachelors are unmarried' or 'siblings have siblings' than is the justification for accepting any other axioms of set theory. That the concepts of *set* and *being a member of* obey the axiom of Extensionality is a far more central feature of our use of them than is the fact that they obey any other axiom.[46]

So, Extensionality is self-evident because it is true in virtue of facts about how we use certain words.

Lavine's justification of Relativized Extensionality is very different from that given for Extensionality. He claims that Relativized Extensionality "reflects the intuition that a set that distinguishes two sets is available in virtue of the fact that the two sets are."[47] However, although there is something incoherent about denying Extensionality, off hand, it appears that we do not do a disservice to the concepts of *set* or *availability* by holding that two sets are available even though no witness set is available.

But what, then, is the justification of this axiom? The justification for Relativized Extensionality employs the notion of degrees of availability, a notion we first discussed in connection with axioms (9) and (10). In discussing those axioms, I noted that when Lavine

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[45] Boolos, "The Iterative Conception of Set," p. 229.
[46] Boolos, "The Iterative Conception of Set," p. 229.
[47] Lavine, *Understanding the Infinite*, p. 295.

says that the set c is available *in virtue of the fact* that sets a and b are, he takes it that c is the value of an availability function on arguments a and b, where the relevant availability function is a function that takes members of Ω_i to Ω_{i+1}. If we rephrase Lavine's justification of Relativized Extensionality using the notion of availability functions, we get the following: Relativized Extensionality reflects the intuition that there is an availability function that takes two sets in Ω_i to a set in Ω_{i+1} that witnesses their difference.

We again face the notion of availability. This time, however, it does not seem that we have to give an account of the availability of a mathematical object. Rather we need to give an account of the notion of an availability function. In the next section I will examine the notion of an availability function. I will argue that no plausible account of availability function ensures that both Relativized Extensionality and Generalized Zillion are true principles about finite sets.

Let me end this section by commenting on why one might begin to sense a tension between Relativized Extensionality and Generalized Zillion. We have seen that a finite set x witnesses the truth of Generalized Zillion only if membership is not an availability function, i.e., only if certain members of a set in Ω_0 are barred from being in Ω_1. In this way, Generalized Zillion requires that some members of some sets are *not* available in Ω_1. Relativized Extensionality, on the other hand, requires that some members of some sets *are* available in Ω_1. It tells us that whenever we have two sets, some member of one of them must be available. So, Generalized Zillion tells us that certain members of sets are not available while Relativized Extensionality tells us that certain members of sets are available. Of course, these two requirements are not inconsistent. However, it looks as if generating intuitions that will support both these claims may be a lot to ask. How can we ensure that certain special members of a set are available but not all of its members are available? The challenge in the next section is to find an account of the notion of an availability function that has the right hit and miss feature.

IX. Availability Functions

In what follows I will examine two accounts of the notion of availability function and argue that each either classifies membership as an availability function or fails to classify a witness function as an availability function. The first account comes from Lavine's discussion of the Axioms of Indefinitely Large Size. In this discussion, Lavine presents an example that provides us with one interpretation of the notion of an availability function. In particular, Lavine observes that a child who has only seventy beans but wants to know what 10×10 is can always get more beans and then calculate the answer. In general, a child can always enlarge her original pile of beans in order to calculate simple operations performed with the original pile. In the example, the larger and larger piles of beans correspond to the larger and larger Ω's, and the operations the child perform can be taken to represent availability functions.

If these operations represent availability function, then, this example suggests a way to flesh out the notion of availability function. Since the child is performing operations with a pile of beans, the sorts of operations she performs can be represented physically. Perhaps availability functions are functions that have some sort of physical interpretation.

The star example fosters this interpretation of the notion of an availability function. In this example, Lavine supposes that stars that are directly pointed out to us are available to the first degree. Other stars are then made available in virtue of these initially selected stars. For example, perhaps we are told about the stars that are directly beneath stars that are available to the first degree. These stars are then available to the second degree. The relevant availability function here is the function that takes every star to the star directly below it. Again Lavine employs a physical model, and the functions he discusses have a physical representation.

It is a bit of a stretch, but we might be able to provide a physical interpretation for a witness function. Suppose that red marbles are tokens of set *a*, blue marbles are tokens of the set *b*, and so forth. When presented with two sets we simply need to get rid of matching elements on the sets. Any leftover marble will be an appropriate witness to the difference of the two sets.

But even though this account of availability function might lend some plausibility to Relativized Extensionality, it appears to undermine the claim that a finite set witnesses the truth of Generalized Zillion. As I stated in (28), a finite set witnesses the truth of Generalized Zillion only if the membership relation is not an availability function. Since membership has a simply physical interpretation—any bean in the pile is a member—this account suggests that membership is an availability function.

This brings me to the second account of the notion of availability function. Perhaps, we can flesh out the notion of an availability function by adopting a version of the denotation account for availability functions. Perhaps, a function is an availability function if we write down a name for it using "standard notation."

On this account, it appears that a finite set does not witness the truth of Generalized Zillion. Indeed, the membership relation is standard notation, so membership appears to be an availability function.

We can alter the denotation account to avoid this conclusion. Perhaps, a function is available if using standard notation, we can write down the name of its output, or in the case of a relation, its outputs. In this case, membership is not an availability function because if we are just given the name of a set, we often cannot write down the names of all the set's members.

But even though this new condition might lend some plausibility to Zillion, this new condition does not appear to make it plausible that a witness function is an availability function. For, to make that claim plausible, it appears that we need a general way of specifying the output of a witness function. The following comment of Yannis Moschovakis helps us see why this approach is fruitless:

> *The Axiom of Choice is the only Zermelo axiom other than Extensionality which is not a special case of the General Comprehension Principle.* This is misstated on occasion, to make the claim that the axiom of Choice is the only one which *demands the existence* of objects for which it does not supply a definition, which is not true: the Extensionality and Power Set Axioms do the same, in a more fundamental if indirect manner.[48]

Moschovakis's comments tell us that, in general, for any two sets Extensionality does *not* provide us with a formula in the language of first-order set theory that gives the entrance condition for a unique set that distinguishes them. In the case of Extensionality we cannot provide a formula which is such that given any two sets, this formula defines a witness set.[49]

One might think that we can get around this feature of Extensionality. If we appeal to the Axiom of Choice, then for any two sets we can posit the existence of a function that

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[48] Yiannis Moschovakis, *Notes on Set Theory*. Springer-Verlag, New York, 1994. p. 120.

[49] On the constructible universe, we can define a well-ordering on the universe. For any two constructible sets, we can write down a formula that defines the least member of their symmetric deference. So, for any two constructible sets, we have a formula that is true of a set witnessing their difference.

takes these sets to a unique member of their symmetric difference. However, since we appeal to the Axiom of Choice, we do not generate a name for this witness set. Indeed, all we know is that for every two sets there is some function for which a witness set is a value. In general, we can provide a name for neither the function nor its output.

Overall then, neither the physical or the denotation account of an availability function enables us to argue that both Relativized Extensionality and Generalized Zillion are true principles about finite sets. In general, it seems doubtful that any account of availability functions will have the needed features. The account must ensure that membership is not an availability function and simultaneously ensure that a restriction of membership is an availability function. Indeed, the account must ensure that not all the members of the symmetric difference of the available sets x and $\varnothing$ are available. However, it must also ensure that a member of the symmetric difference of x and $\varnothing$ is available. But what principled reason could we have for thinking that the symmetric difference is not an availability function, yet believe that a restriction of it is an availability function? On the face of it, the symmetric difference is more familiar and far simpler than the needed restriction. Furthermore, the symmetric difference is part of standard set-theoretic notation whereas the needed restriction is not. All these facts suggest that we have more reason to believe that the symmetric difference is an availability function than we have to believe that a restriction of it is.

X. The Final Conflict

I believe that there is a further conflict between Generalized Zillion and Relativized Extensionality. As I noted previously, Lavine claims that Relativized Extensionality "reflects the intuition that a set that distinguishes two sets is available in virtue of the fact that the two sets are."[50] He then fleshes out the notion of 'in virtue of' phrase here in terms of availability functions. In the previous section, we saw that there appears to be no account of the notion of availability function that provides us with a way of showing that both Relativized Extensionality and Generalized Zillion are true principles about finite sets.

One might think, however, that to understand Lavine's defense, we must employ the intuitive notion on which the notion of availability function rests. As I noted, the notion of availability function is the formal counterpart to the notion of degrees of availability.

In this section I will argue that relying on the intuitive reading of the notion of availability function also does not provides us with a way of showing that both Relativized Extensionality and Generalized Zillion are true principles about finite sets. According to the intuitive reading, Generalized Zillion is a true principles about finite sets only if it is not the case that all the members of a set x are available in virtue of the availability of x. And Relativized Extensionality holds only if in virtue of the availability of two sets, a set witnessing their difference is available. I will argue that no account of set availability appears to support the contention that these axioms are true principles about finite sets and, furthermore, that certain accounts serve to undermine these axioms.

So far we have examined two ways of spelling out availability, an experiential account and a denotation account. Thinking about set availability in some experiential way is not promising. On such an account, a set is available if we see a group of tokens of its members. Relativized Extensionality appears to hold on this account because in virtue of making two set available, we make all their members available, for we see each member of

[50] Lavine, *Understanding the Infinite*, p. 295.

the group. However, this account of set availability clearly undermines Generalized Zillion. For, in making a set available, we thereby make all of its members available.

As for the denotation account, the denotation account says that a set is available if we write down and understand a sentence that contains a standard name for the set.

But what is standard notation for sets? We encountered this question in connection with Zillion. At that point we assumed that we could employ Arabic numerals. Now we need names for a variety of sets, not just the von Neumann numbers. I assume, then, that our standard notation comes from the language of ZFC.[51] The language of ZFC contains the language of first-order logic plus the two place predicate '$\in$'.

With only these basic resources, however, we have no names. We can follow Lavine's lead here. When discussing Fin(ZFC), he adds the constant '$\varnothing$' and the function symbol 'A' where '$A(x,y)$' means the same as '$x \cup \{y\}$'. In this way, we can form the names '$\varnothing$', '$A(A(\varnothing,\varnothing),\varnothing)$', '$A(\varnothing,A(A(\varnothing,\varnothing),\varnothing))$', and so forth.

But if we only have these names, then it is not the case that both Relativized Extensionality and Generalized Zillion are true principles about finite sets. For, suppose I have written down and understood the symbols '$A(A(A(\varnothing,\varnothing),A(\varnothing,\varnothing)),A(A(\varnothing,\varnothing),A(\varnothing,\varnothing)))$' and '$\varnothing$'. The sets $A(\varnothing,\varnothing)$ and $A(A(\varnothing,\varnothing),A(\varnothing,\varnothing))$ are the only sets that witness the difference between these two sets. Are either of these witnesses available in virtue of the availability of $A(A(A(\varnothing,\varnothing),A(\varnothing,\varnothing)),A(A(\varnothing,\varnothing),A(\varnothing,\varnothing)))$ and $\varnothing$? To answer this question we must consider two options: either '$A(\varnothing,\varnothing)$' and '$A(A(\varnothing,\varnothing),A(\varnothing,\varnothing))$' do *not* occur syncategormatically within '$A(A(A(\varnothing,\varnothing),A(\varnothing,\varnothing)),A(A(\varnothing,\varnothing),A(\varnothing,\varnothing)))$' or they do.

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[51] We can also assume that the notation comes from Fin(ZFC). The Appendix shows how to introduce new symbols to the language of Fin(ZFC)

If '$A(\varnothing,\varnothing)$' and '$A(A(\varnothing,\varnothing),A(\varnothing,\varnothing))$' do *not* occur syncategormatically within '$A(A(A(\varnothing,\varnothing),A(\varnothing,\varnothing)),A(A(\varnothing,\varnothing),A(\varnothing,\varnothing)))$' but rather have independent significance, then in writing down and understanding sentences containing '$A(A(A(\varnothing,\varnothing),A(\varnothing,\varnothing)),A(A(\varnothing,\varnothing),A(\varnothing,\varnothing)))$' we also write down and understand a sentence containing '$A(\varnothing,\varnothing)$' and '$A(A(\varnothing,\varnothing),A(\varnothing,\varnothing))$'. However, if we assume that the parts of these terms have independent significance, then a finite set cannot witness the truth of Generalized Zillion. For, when we write down any set using this notation, we write down the names of all of its members. So, it appears that in virtue of making a set available we make all its members available. On the other hand, if '$A(\varnothing,\varnothing)$' and '$A(A(\varnothing,\varnothing),A(\varnothing,\varnothing))$' occur syncategormatically within '$A(A(A(\varnothing,\varnothing),A(\varnothing,\varnothing)),A(A(\varnothing,\varnothing),A(\varnothing,\varnothing)))$', then in writing down these two sets we do not succeed in making a witness set available. So, in this case, Relativized Extensionality turns out to be a false principle about finite sets.

The situation is even worse than this example suggests. For, there is no principled reason to limit the language in this way, i.e., to include only one constant and one function symbol.[52] This constant and function symbol are no different than other standard constants and function symbols that can be or have been added to the language of ZFC. Indeed, we can and do expand the language of ZFC by adding standard names for functions, relations, and constants. Adding these symbols does not increase the expressive power of the

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[52] There is a slight problem with axiom (8). Lavine originally states axiom (8) as follows: $c \in \Omega_p$ where 'c' is replaced by any constant in the language of set theory. I stated axiom (8) as $\varnothing \in \Omega_p$ since '$\varnothing$' is the only constant that Lavine adds to the language of Fin(ZFC). However, if we are allowed to expand the language of Fin(ZFC), it appears that we cannot assume that the original form of axiom (8) holds for all new constants. For, since we can introduce infinitely many constants, there would be no finite models of Fin(ZFC). I suppose we can just use axiom (8) as I stated it. For all Lavine needs from axiom (8) is to ensure that the Ω's are non-empty. Indeed, in his paper "Locally Finite Theories," Mycielski does not include axiom (8). Instead he simply requires that the Ω's be non-empty.

language or the deductive power of the theory. Once we expand the language of Fin(ZFC), however, it is completely implausible to say that for any two available sets x and y, there is a witnesses set that is available in virtue of the availability of x and y.

To see this, first I will describe how to introduce a function or a constant to ZFC. To make such introductions, we must find some formula $\phi(x_1,\ldots,x_n,y)$ in the basic language of set theory and with no free variables other than $x_1,\ldots,x_n,y$. Additionally, $\phi(x_1,\ldots,x_n,y)$ must be such that ZFC entails $\forall x_1,\ldots,x_n\exists!y\phi(x_1,\ldots,x_n,y)$. If these criteria are satisfied, we can let $F(x_1,\ldots,x_n)$ be the unique y such that $\phi(x_1,\ldots,x_n,y)$. For example, we introduce '$\varnothing$' with the formula $\phi(y)$: $\forall z(z \notin y)$. Using the Extensionality and Separation Axioms, we can prove that there is a unique y such that $\phi(y)$. We then let $\varnothing$ be the y such that $\phi(y)$.

In certain respects it does not matter what new symbols we introduce by this method. For, we have introduced the functions and constants in such a way that any formula containing a function symbol or a constant is equivalent to a formula in the basic language of set theory. However, since the denotation account requires standard notation, we rely on standard function symbols and constants to name sets. The following names are standard: $2^{\aleph_0}$, $\langle S(\varnothing),SS(\varnothing)\rangle$, $\omega + 1$.

Class terms of the form '$\{x|\phi(x_1,\ldots,x_n,x)\}$', where $\phi(x_1,\ldots,x_n,x)$ is replaced by a formula of the language of ZFC, are another standard device to name sets. The class term '$\{y|\phi(x_1,\ldots,x_n,x)\}$' refers to a set as long as ZFC proves that $\forall x_1,\ldots,x_n\exists y\forall x(x \in y \leftrightarrow \phi(x_1,\ldots,x_n,x))$. We can use this device to name sets only if we add class terms to the language of ZFC. As in the case of defined function symbols and constants, adding class terms to the language does not increase the expressive power of the language or the deductive power of the theory.[53, 54]

Once we expand the language of Fin(ZFC), it is completely implausible to say that for any two available sets x and y, there is a witness set that is available in virtue of the availability of x and y. Terms of the form '$\{x|\phi(x)\}$' provide a nice example. No ϕ's need be available for us to understand a sentence containing the term '$\{x|\phi(x)\}$'. Indeed this is one of the benefits of class abstracts. When a set has lots of members, it is impossible to think about it by somehow running through its members. Class abstracts allow us to think about a set without thinking about its members, as the set is presented to us via an open formula. If we employ class abstracts, then it looks as if a witness set for two sets is not made available in virtue of the availability of those two sets.

As for other symbols, it is not always clear whether they denote the same set. For example, without a computation, we do not know whether V_4 is identical to the set $\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\},\{\{\varnothing\}\},\{\varnothing,\{\{\varnothing\}\},\{\varnothing,\{\varnothing\{\varnothing\}\}\},\{\{\varnothing\},\{\{\varnothing\}\}\}\}$, where V_4 is the fourth level of the iterative hierarchy.

In sum, then, if we add only the constant '$\varnothing$' and the function symbol 'A' to the basic language of ZFC, we cannot ensure both that Relativized Extensionality and Generalized Zillion are true principles about finite sets. And once we expand the language, it is no longer obvious when two terms denote the same or different sets. So, it just seems patently false to say that in virtue of two sets being available a set that witnesses their difference is available.

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53 See Azriel Levy, *Basic Set Theory*. Springer-Verlag, Berlin Heidelberg New York, 1979. p. 13 Theorems 4.5 and 4.6.

54 This similarity between class terms and functions is not surprising. We can view class terms as simply a convenient way of introducing functions. If for all $x_1,\ldots,x_n$ there is a set y such that $y=\{x|\phi(x_1,\ldots,x_n,x)\}$, then by appeal to Extensionality we know that for all $x_1,\ldots,x_n$, $\{x|\phi(x_1,\ldots,x_n,x)\}$ is the unique y such that $\psi(y)$, where $\psi(y)$ is $\forall x(x\in y\leftrightarrow\phi(x_1,\ldots,x_n,x))$.

XI. Summary

The course of the argument has been as follows. I argued that Lavine fails to establish that it is self-evident that a finite set witnesses the truth of Zillion. I then pointed out that even if Lavine could establish this, the maneuver is insufficient to motivate the "appropriate" finite version of ZFC. For, the appropriate version of finite set theory contains Generalized Zillion rather than Zillion. To motivate a finite set theory containing Generalized Extensionality, he must make it plausible that membership is not an availability function. I then turned to Relativized Extensionality. My overall point in connection with Relativized Extensionality was that it does not seem possible to show that both Relativized Extensionality and Generalized Zillion are true principles about finite sets. No account of availability functions enabled us to make the case. Furthermore, when we understand the notion of availability function more intuitively, Lavine's defense of Relativized Extensionality turns out to be highly implausible.

XII. Appendix

We have examined how to add constants and function symbols to the language of ZFC. Using this as a guide, we can see how we add them to Fin(ZFC). Since we are making such introductions in Fin(ZFC), we must rely on the axioms of Fin(ZFC) to ensure that the open formulas in our definitions are satisfied by a unique set. We add such notation to Fin(ZFC) as follows. Suppose we have shown in ZFC that $F(x_1,\ldots,x_n)$ is the unique y such that $\phi(x_1,\ldots,x_n,y)$ where $\phi(x_1,\ldots,x_n,y)$ is a formula in the basic language with no free variables other than $x_1,\ldots,x_n,y$. That is we have introduced 'F' to the language of ZFC by proving from ZFC that $\forall x_1,\ldots,x_n \exists! y \phi(x_1,\ldots,x_n,y)$.

We can introduce a corresponding 'F' in Fin(ZFC) as follows. First, we suppose that 'Ω_1','Ω_2',...,'Ω_n' are the only Ωs that we have written down. We then let $F_1(x_1,\ldots,x_n)$ be the unique y such that $\phi_1(x_1,\ldots,x_n,y)$ where $\phi_1(x_1,\ldots,x_n,y)$ is a regular relativization of ϕ obtained by bounding the first quantifier that occurs in ϕ with Ω_2.[55] We can define $F_1(x_1, \ldots, x_n)$ in this way because we can prove $\forall x_1,\ldots,x_n \in \Omega_0 \exists! y \in \Omega_1 \phi_1(x_1,\ldots,x_n,y)$ from the axioms of Fin(ZFC). We can carry out this proof since $\forall x_1,\ldots,x_n \exists! y \phi_1(x_1,\ldots,x_n,y)$ follows from ZFC just in case $\forall x_1,\ldots,x_n \in \Omega_0 \exists! y \in \Omega_1 \phi_1(x_1,\ldots,x_n,y)$ follows from Fin(ZFC).[56] We have thus defined a function that corresponds to 'F', but this function is restricted to members of Ω_0 and its values are members of Ω_1. We can continue this process, however, and define $F_2(x_1,\ldots,x_n)$ as $\phi_2(x_1,\ldots,x_n,y)$ where $\phi_2(x_1,\ldots,x_n,y)$ is the regular relativization of ϕ obtained by increasing the subscripts of the Ω's that show up in ϕ_1 by 1. So, we now have a function corresponding to 'F' except this function is restricted to members of Ω_1, and it outputs members of Ω_2. We can consolidate the above information by adding an 'F' to the language such that for $x_1,\ldots,x_n$ in Ω_p, $F(x_1,\ldots,x_n)$ is the unique y such that $\phi_{p+1}(x_1,\ldots,x_n,y)$.[57]

Through arguments resembling those in the previous paragraph, we can also show that if ϕ is a formula of the language of ZFC and '$\{x|\phi(x)\}$' names a set, then '$\{x|\phi^*(x)\}$' names a set in the language of Fin(ZFC) where ϕ^* is a relativization of ϕ. So, if we add class terms to the language of Fin(ZFC), we can name sets in Fin(ZFC) using them.

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[55] We could single out a unique relativization by requiring that we use the regular relativization that contains the Ω's with the smallest subscripts. We can do this because at any one time we have written down only finitely many Ω's.

[56] For a proof of this, see Theorem 3.3 in *Understanding the Infinite* (p. 273).

[57] Note that each function in ZFC can be associated with more than one function in Fin(ZFC). If we start out with different regular relativizations of ϕ we end up with different functions in Fin(ZFC). So, for each function in ZFC, there is a family of functions in Fin(ZFC) that we can associate with it. Also note that if more Ω's are written down, our definition of F must be updated.

Chapter 2

Experience of the Infinite

It is often taken as a datum that we do not experience the infinite. For example, in his *An Essay Concerning Human Understanding* John Locke claims that we cannot form a conception of the infinite because we lack experience of the infinite. As he writes,

> As for the idea of finite, there is no great difficulty. The obvious portions of extension that affect our senses, carry with them into the mind the idea of the finite:... The difficulty is, how we come by those *boundless ideas* of eternity and immensity; since the objects we converse with come so much short of any approach or proportion to that largeness.[1]

In *A Treatise of Human Nature,* David Hume suggests a similar worry:

> 'Tis universally allow'd, that the capacity of the mind is limited, and can never attain a full and adequate conception of infinity: And tho' it were not allow'd, 'twou'd be sufficiently evident from plainest observation and experience.[2]

And as we have seen in the first chapter, Shaughan Lavine argues that it is puzzling how we have knowledge of the infinite because "we have absolutely no experience of any kind of the infinite."[3]

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1 John Locke, *An Essay Concerning Human Understanding*, vol. 1, Ed., Alexander Fraser, Dover Publication, inc., New York, 1959. p. 277.

2 David Hume, *A Treatise of Human Nature*, second edition. Ed. L.A. Selby-Bigge. Oxford University Press, Oxford, 1978. p. 26.

3 Shaughan Lavine, *Understanding the Infinite*. Harvard University Press, Cambridge, Massachusetts and London, England, 1994. p. 8.

Contrary to these philosophers, I will argue that when we view pictures such as Figure 1, we have an "experience of the infinite," or more specifically, we have a *perceptual illusion of an infinite sequence.*

Figure 1

Showing that we can experience an infinite sequence is important to two areas of inquiry. First, the discussion of these experiences sheds light on the relationship between experience and beliefs formed on the basis of experience. For, to sustain the claim that these experiences are experiences of the infinite, I will use the example of pictures to argue that even though an object appears to us in a certain way, we do not always believe nor are we always disposed to believe that the object is that way. Furthermore, I will use the example of pictures to argue that we can see an object as having some property, but come to believe, on the basis of seeing the object, that the object does not have that property. So, the particularly tricky case of the infinite will hopefully provide general information about the relationship between our experience and the beliefs formed on the basis of that experience.

Second, these experiences provide a partial solution to the epistemological puzzle about the infinite introduced in Chapter 1. In Chapter 3, I will argue that the illusions of the infinite I will describe supply us with modal knowledge of the infinite. Accordingly, the experiences of the infinite discussed in this chapter provide the foundation for a partial response to the puzzle about the infinite.

I should note that I will *not* argue that the experiences of the infinite discussed in this chapter provide a solution to Locke's puzzle about the infinite. Locke wonders how we can form a *concept* of the infinite if we do not experience of the infinite. But even though I will not provide a response to this empiricist puzzle, as I explain in the final section, we should not summarily dismiss these experiences as a way to provide a solution to the empiricist puzzle.

In this chapter, then, I will argue that we can have experiences of the infinite and discus how these experiences shed light on the relationship between experience and beliefs formed on the basis of experience. I will postpone the argument that these experience supply us with modal knowledge of the infinite until the next chapter.

I. Some Distinctions

First, I will make a few distinctions in order to facilitate my discussion of these experiences of the infinite.

The first distinction is between two uses of the expression 'experience'. In one use we apply the expression to sense experiences. As usually characterized, sense experiences have at least two distinctive characteristics. Sense experiences are, in a certain sense, passive. As long as we are in the right place at the right time, we need not do anything to have such experiences. Second, sense experiences have phenomenological qualities.[4] There is something it is like to have these experiences.[5]

We can contrast this use of 'experience' with another. We sometimes use the phrase 'experience' to refer to an action we perform as opposed to a sense experience we undergo. For example, if a farmer asks a potential apprentice whether she has farming experience, he is not asking whether she has seen a freshly planted field or heard the crow of roosters in the morning. Rather, the farmer is asking the apprentice whether she has performed certain actions that have culminated in her attaining farming skills. He is asking her about what I will call her "act experiences."

Act experiences, then, are actions we perform. When discussing act experiences, the term 'experience' is used to refer to the act itself, for example, the act of driving a car or flying a kite. Act experiences have two distinguishing characteristics. First, they are active, unlike sense experiences which are passive. For an act experience, it is not enough to be in the right place at the right time. Rather, to have such experiences, we need to do something. Indeed, the experience is the act which the individual performs. Furthermore,

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[4] A phenomenological quality of an experience is how things seem to someone who has the experience. For example, the way a red patch looks to someone is a phenomenological quality of the experience of the red patch, and the way rain sounds to someone is a phenomenological quality of the experience of rain. Phenomenological qualities of experiences are often called qualia.

[5] One might also want to characterize sense experiences as having intentional features. However, whether all sense experience have intentional features is debatable. On the one hand, it seems plausible to say that seeing a red house is an experience that is directed toward a certain object. But it seems less plausible to hold that tickles, itches, and pains have intentional features.

although an individual might have certain sense experiences when he performs an action—for example he might experience hunger pangs or he might feel the ground beneath his feet—an act experience is not identified by its associated phenomenological qualities.

The distinction between sense experience and act experience is important because it helps clarify what I mean by the claim that we can have experiences of the infinite. There are at least two ways one might understand this claim. We can understand it as the claim that we can have an "act experience of the infinite," where we have an act experience of the infinite if we "construct" an infinite sequence of objects.[6] In this case, when I say that I will show that we can have experiences of the infinite, one expects an argument that we can construct an infinite sequence of objects. On the other hand, we can understand the claim that we can have experiences of the infinite as the claim that we can have sense experiences of the infinite. In what follows, I will argue that we can have sense experiences of an infinite sequence.

Besides noting the distinction between sense experience and act experience, we should also note another important distinction, the distinction between the potential infinite and the actual infinite. This distinction is usually made by relying on the notion of

[6] We can think of a construction as either a mental or a physical act. For instance, the intuitionists and empiricists, such as Locke, seem to think that constructing a natural number is a kind of mental act. But others have described the construction of a "number" as the formation of a physical string that is composed of strokes and so seem to think of a construction in terms of a physical act, such as the physical act of drawing one stroke on the left end of a string of strokes. Although it is plausible to refer to physical constructions as experiences, it is debatable whether we should refer to mental constructions as experiences. Off hand, it sounds awkward to say that someone has had an experience of a mental act such as learning or believing. However, Locke, for instance, appears to think that we are conscious of certain mental acts and that this consciousness is in part responsible for the formation of number concepts including the concept of infinity. So, although the construction is a mental process, some philosophers believe that our awareness or experience of this mental process is important in forming certain concepts.

construction.[7] The F's are potentially infinite if, given some F's, it is always possible to construct an F that is different from any previously constructed F. The F's are actually infinite if there are infinitely many Fs. If we have a primitive symbol for a function such as the successor function, we can replace the constructivist notions in this characterization with modal notions. In particular, if s is a symbol for "a successor function" on the F's, then the F's are potentially infinite if the following holds: necessarily for all x, it is possible that there exists a y such that $s(x) = y$. The F's are actually infinite if it is possible that for every x there is a y such that $s(x) = y$.[8] In this paper, I will focus on experiences of completed infinite sequences rather than potentially infinite sequences.

One last distinction I need to make is between different notions of *seeing*. In one sense of 'sees', if we see an object and that object is made of infinitely many parts, then it follows that we see all the parts of that object and so see infinitely many objects. I am not interested in this sense of *seeing*. For, ultimately I want to show that we can acquire knowledge about the infinite from our experience of the infinite. To acquire knowledge when we see an object, we need to be "aware" of what we see. If we are able to "see" all the parts of an object only because we see the whole object, we are not necessarily aware of all these parts of the object.

So, the sense of seeing in which I am interested is the sense in which we see an object because we are aware of that object. But what is it to be aware of an object? I

7 Michael Dummett seems to introduce a notion that is akin to the notion of the potential infinite. See, for example, his discussion of the notion of an indefinitely extensible concept in his *Frege: Philosophy of Mathematics* (Harvard University Press, Cambridge, MA, 1991), pp. 316-319, and also in his article "The Philosophical Significance of Gödel's Theorem" in *Truth and Other Enigmas* (Harvard University Press, Cambridge, MA 1978), pp.195-199.

8 Notice that in *S5* we can affix a necessity operator to the claim that it is possible that for every x there is a y such that $s(x) = y$.

suggest that to be aware of an object is to be able to differentiate that object from its surroundings on the basis of how that object appears.[9] In this sense of 'sees', it is correct to say that John sees an apple on the brown table because he is able to differentiate the apple from its environment on the basis of how the apple appears to him. However, it is incorrect to say that John sees a red sticker on the apple when the red sticker blends in perfectly with the color of the apple. For, although John is casually connected to the red sticker, he does not see the red sticker because he does not differentiate the red sticker from its environment on the basis of how the sticker appears to him.

In describing cases involving this sense of 'sees', it is often useful to describe the situation in question as one in which a person sees an object as an F. For, this form of expression indicates not only what object a person is casually connected to but also how that object phenomenologically appears to the person, or similarly, how his senses *represent* the object. For example, we might say that a person sees an apple as round and red to indicate that the person sees an apple that appears round and red to him, or, similarly, to indicate that the this person's senses represent the apple as round and red.

It is important to note that there are many ways to understand the claim that a person S sees an object as an F. So far, I have described a phenomenological use. In this use, when we say that S sees an object as an F, we characterize how that object phenomenologically appears to S. However, in a different use, we use this phrase to characterize what beliefs we form based on our sense experience. In this doxastic use, when we say that S sees an object o as an F, we indicate that on the basis of a perception, S believes, of o, that it is F. In what follows, when I say that S sees an object as an F, I

[9] This characterization is based roughly on Fred Dretske's characterization of non-epistemic seeing in his *Seeing and Knowing*. University of Chicago Press, Chicago, 1969.

intend to characterize how an object phenomenologically appears to S. When I want to use the doxastic sense of 'sees', I will simply claim that a person forms certain beliefs on the basis of what he perceives.[10]

II. The Indefinitely Large and the Infinite

Now that I have made certain important distinctions, I will begin my investigation as to whether we can have experiences of the infinite.

There are many ways one might argue that we can have sense experiences of a completed infinite sequence. Let me first briefly indicate some of the ways I will *not* argue. I will not argue that we see an infinite object as infinite. Neither will I argue that we see a finite object as infinite. Rather, I will argue that when we see certain objects, namely certain sequences, we see them as having the property: *containing, as a proper part, a shrunken duplicate*. Seeing a sequence as containing a shrunken duplicate is an experience of an infinite sequence because, as I will argue, if the sequence did contain a shrunken duplicate, it would have infinitely many parts. So, the experience is a perceptual illusion of an infinite sequence.

We do not see all sequences as containing a shrunken duplicate. The sequences that we see as containing a shrunken duplicate are certain indefinitely large sequences. The idea that we encounter the infinite in our experiences of the indefinitely large is not new.

Jorge Luis Borges[11] nicely describes the motivation for thinking that our experiences of the indefinitely large might count as experiences of the infinite. Borges

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[10] These distinctions are based roughly on distinctions made by Fred Dretske in his *Seeing and Knowing*. University of Chicago Press, Chicago, 1969. Also see Fred Dretske, *Naturalizing the Mind*. MIT Press, Cambridge, 1995. esp. pp. 67-68.

[11] Jorge Luis Borges, "Funes, the Memorious." *Ficciones*. Edited and Translated by Anthony Kerrigan. Grove Weidenfeld, New York, 1962. p. 107-115.

recounts an encounter with a young man, Funes, who had unmatched powers of perception and memory.

> We, in a glance, perceive three wine glasses on the table; Funes saw all the shoots, clusters, and grapes of the vine. He remembered the shapes of the clouds in the south at dawn on the 30th of April of 1882, and he could compare them in recollection with the marbled grain in the design of a leather-bound book which he had seen only once, and with the lines in the spray which an oar raised in the Rio negro on the eve of the battle of the Quebracho.[12]

At some point, Funes endeavored to name every aspect and every detail of every scene he had ever seen. However, "two considerations dissuaded him; the thought that the task was interminable and the thought that it was useless."[13] When we are presented with such a large number of things, we sometimes think that we are, in a certain way, dealing with the infinite.

We have seen in the first chapter that Shaughan Lavine also suggests that we, in some sense, encounter the infinite in our experience of the indefinitely large. As he writes,

> ...the thought that the number of grains of sand on the beach might as well be infinite for many practical purposes is what leads to the concept of the infinite...[14]

> So where does the concept of the infinite come from? Training to be sure, but how did it begin? My proposal is that it began with an extrapolation from experience of indefinitely large size.[15]

So, Lavine suggests that our concept of the infinite is rooted in our experiences of the indefinitely large.

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[12] Borges, "Funes, the Memorious," p. 112.
[13] Ibid., p. 114.
[14] Lavine, *Understanding the Infinite*, p. 256.
[15] Ibid., p. 248.

The connections Borges and Lavine suggest between our experience of the indefinitely large and the infinite are insufficient to show that we have experiences of the infinite. In fact, in most cases, although experiences of the indefinitely large might, in some way, seem like encounters with the infinite, these experiences are not plausibly called experiences of the infinite.

For example, consider cases where we see an indefinitely large pile of similarly sized objects, for example, when we see a large pile of pebbles or when we see a truck filled with sand. Seeing indefinitely large piles and heaps is not plausibly called an experience of the infinite. For, when we see a large pile of pebbles, it does not appear that we see the pile as infinite or as having a property that only an infinite object could have. Furthermore, we see the pile as bounded both in its extent and in the dimension of its parts. We should note, however, that even though we see the pile as finite, it is not the case that we see the pile as containing, say, 100,324 pebbles. Indeed, we refer to these experience as experiences of the indefinitely large precisely because we do not see the pile as containing 100,324 pebbles.

A quite different example of an experience of the indefinitely large is a case where we are in the midst of a randomly scattered group of similarly sized objects, for example when we are in the midst of a forest.[16] Although we do not see the forest as finite—for the boundaries of the forest are not apparent—seeing the forest, nevertheless, is not an experience of the infinite. Although our perception of the forest leaves open the possibility of their being infinitely many trees, we do not see the forest as infinite nor do we appear to see the forest as having a property that only an infinite object could have.

[16] Notice that although the trees are approximately the same size, the trees that are farther away appear smaller than those that are closer.

III. Illusions of the Infinite

There are, however, other experiences of indefinitely large quantities that do count as experiences of the infinite. In particular, we can see certain indefinitely large sequences as containing, as a proper part, a shrunken duplicate. These are experiences of the infinite because, as I will argue, if the sequences in question did contain a shrunken duplicates, then they would contain infinitely many members. In this way, we can have an illusion of an infinite sequence. I believe we have such a case when we see Figure 1, which appears in the beginning of this paper.

The first thing we should note is that Figure 1 is a picture. When discussing visual experiences of pictures, we must be aware that there are two layers of representations at work. First, the picture depicts or represents some object. Second, the picture looks some way to us, and we represent the picture as looking that way.

In Chapter 3, I will be concerned with the first level of representation. In particular, I will argue that Figure 1 represents an infinite sequence. For now, however, I am not interested in what this picture represents, but rather, in what we see when we see the picture as an object in itself. That is, I am interested in how our experience represents this picture. As I have indicated, I believe that we can see Figure 1 as having the property *containing a shrunken duplicate as a proper part.* I also believe that if this picture did have this property, the sequence of rectangles in the picture would be infinite. To establish these claims, I will first indicate why we see Figure 1 and other similar figures as containing a shrunken duplicate. I will then argue that if Figure 1 did contain a shrunken duplicate, it would contain infinitely many rectangles.

To show that we see Figure 1 as containing a shrunken duplicate of itself, consider Figure 2. The top picture in Figure 2 is simply Figure 1. For reference, I have labeled this top picture "sequence A." Below sequence A, I have drawn another sequence of rectangles which I have labeled "sequence B."

I contend that we see two relationships between sequence A and sequence B.

(1) We see an exact duplicate of sequence B as a proper part of sequence A.

(2) We see sequence B as a shrunken duplicate of all of sequence A.

Both of these claims are clear from inspection of Figure 2. We can verify (1) by noticing that sequence B appears to be an exact duplicate of the sequence we would obtain if we delete the first rectangle-woman pair in sequence A. We verify (2) by recognizing that sequence A and sequence B appear to be similar in all respects except for size.

If both (1) and (2) are true, then we see Figure 1 as containing, as a proper part, a shrunken duplicate. For, we recognize that sequence B is simply a duplicate of the sequence we obtain by deleting the first rectangle in sequence A, and, what holds for sequence B holds for its exact duplicate.

We can create a similar kind of case by using a picture that consists of simply a sequence of rectangles. For instance, consider Figure 3, which is a simplification of Figure 2. The top picture in Figure 3 is a sequence of rectangles. The bottom picture in Figure 3 is the sequence obtained when the first rectangle in the top sequence is deleted. As in the case of Figure 2, the bottom picture appears to be an exact duplicate of a proper part of the top picture. In addition, the bottom picture appears to us as a duplicate, in all respects except for size, of the top picture in Figure 3. Since we see the two pictures in

Sequence *A*

Sequence *B*

Figure 2

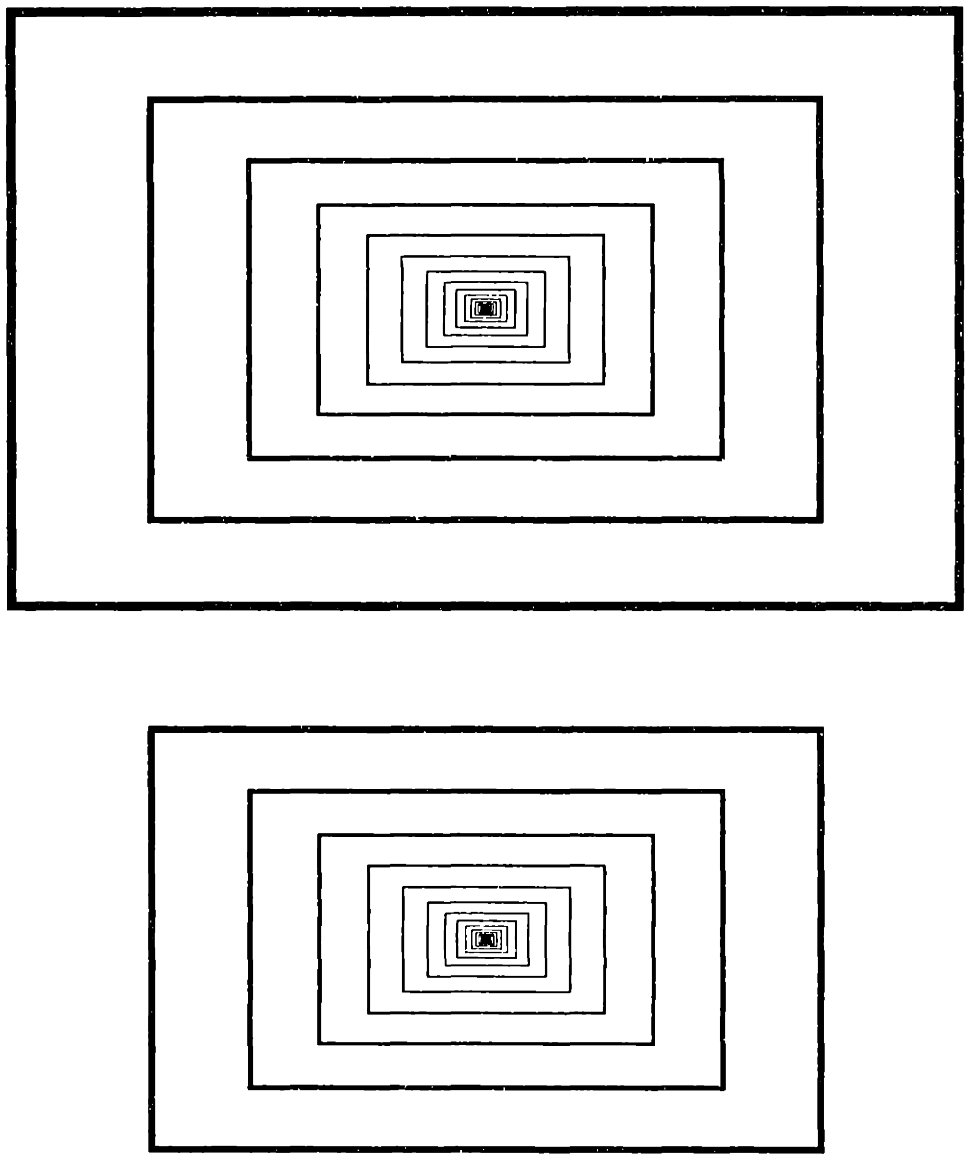

Figure 3

Figure 3 as related in these two ways, it follows that we see the top picture as containing a shrunken duplicate.

Notice that the rectangles need not be embedded in one another to produce the desired effect. For instance, consider Figure 4. The bottom picture in this figure is the sequence obtained by deleting the first rectangle in the top sequence. As with the previous examples, we see the top picture in Figure 4 as containing a miniaturized duplicate of

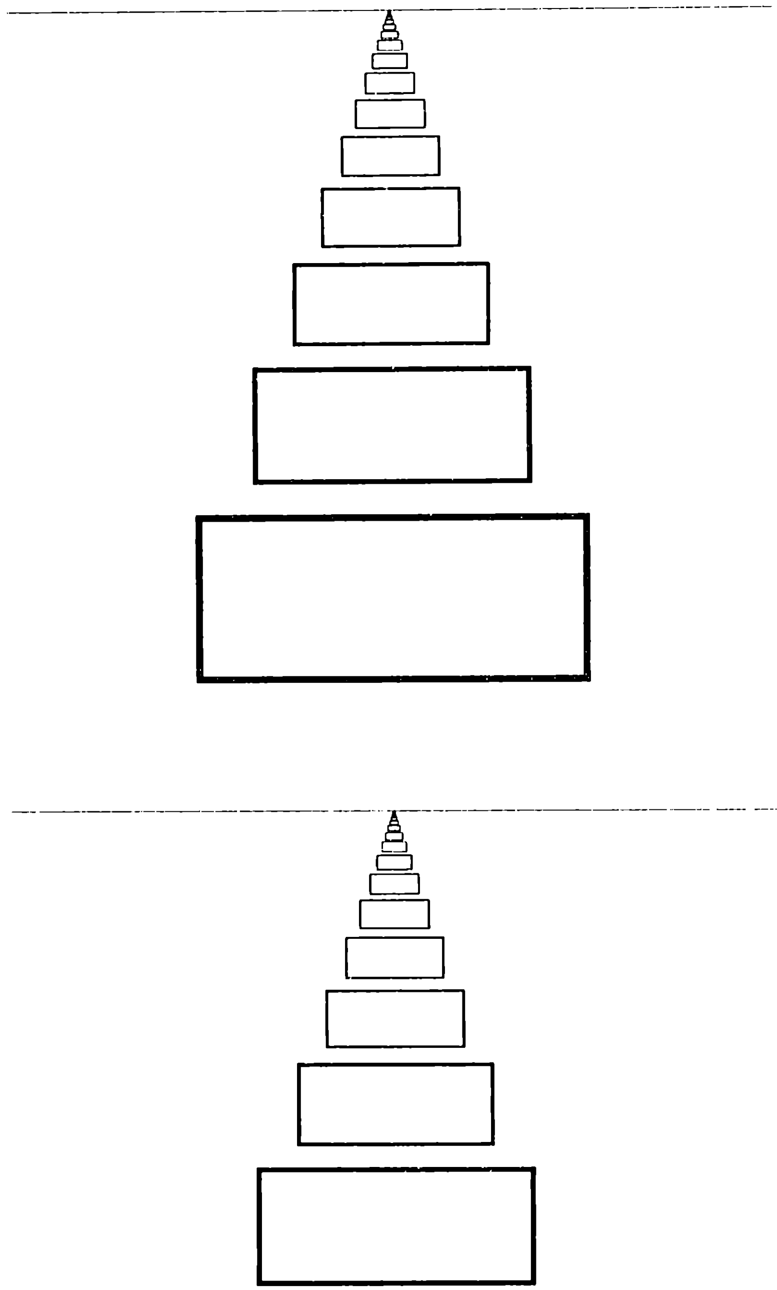

Figure 4

itself. Indeed, not only does an exact duplicate of the bottom picture appear to be a proper part of the top sequence but the bottom sequence appears to be a duplicate, except for size, of a proper part of the top sequence.

We now come to a crucial point in the argument. I contend that in seeing these sequences as containing a shrunken duplicate, we have a perceptual illusion of an infinite sequence. To show this, consider Figure 1. I contend that if Figure 1 did contain a shrunken duplicate, then it would contain infinitely many rectangles. To show this, I will argue that if Figure 1 contains finitely many rectangles, Figure 1 does not contain a shrunken duplicate. So, suppose that the number of rectangles in Figure 1 is finite. Let us call the sequence that is obtained by deleting the first rectangle in Figure 1, MiniFigure 1. Since Figure 1 contains only a finite number of rectangles, say n, MiniFigure 1 contains one less rectangle and so contains n-1 rectangles. MiniFigure 1, then, is not a shrunken duplicate of Figure 1. For, MiniFigure 1 contains one less rectangle than Figure 1. Accordingly, if Figure 1 contains finitely many rectangles, Figure 1 does not contain a shrunken duplicate. It follows that if Figure 1 contains a shrunken duplicate as a proper part, then Figure 1 contains infinitely many rectangles. So, in seeing Figure 1 as containing a shrunken duplicate as a proper part, we have an illusion of an infinite sequence.

Before considering some objections to this argument, I need to make a couple of clarifications. At the beginning of the section, I claimed that I would *not* show that we can see a sequence or any other object *as infinite*. Rather, I would show that we can see a sequence as containing a shrunken duplicate. At this point, some might think that I have not kept my word. For, in showing that we see Figure 1 as containing a shrunken duplicate, some might think that I have also shown that we see Figure 1 as infinite. The reasoning for this conclusion might go as follows. I have shown that we see Figure 1 as having a property which is such that if Figure 1 did have this property, it would contain infinitely many parts. Furthermore, the following principle appears to hold:

(3) If a person sees an object o as an F and if the claim that x is an F entails that x is a G, then he sees o as a G.

Therefore, it appears that I have shown that we see Figure 1 as infinite.

The problem with this line of reasoning is that (3) is false. Consider the following example. The claim that x is a square entails that x is a diamond. But we can see a square as a square, and yet not see a square as a diamond. Indeed, to see a square as a diamond, we need to see the square forty-five degrees off the x and y axis. Accordingly, we must differentiate between seeing a sequence as containing a shrunken duplicate and seeing the sequence as infinite. If my argument is successful, I have established that we can see a sequence as containing a shrunken duplicate, but this argument is insufficient to show that we can see a sequence as infinite.

IV. An Objection

I will now turn to some objections to the argument in the preceding section. It is possible to pose a serious objection to the argument that if our experience of Figure 1 were veridical, then the figure would contain infinitely many rectangles. To bring out the worry, consider the following example. Suppose that there are two closed boxes that look identical. They appear to be the same size, color, and so forth. It thus seems correct to say that we see the boxes as duplicates. But even if we see these boxes as duplicates, surely it would be fallacious to argue that if the boxes were duplicates, then they would contain the same things. For, we do not see the boxes as duplicates with respect to what they contain. But isn't this same kind of mistake made in the above argument? To show that we see Figure 1 as containing a shrunken duplicate, we need to see Figure 1 and MiniFigure 1 as duplicates with respect to the number of rectangles they contain. But since we see these figures as containing an *indefinite* number of rectangles, we cannot see the two figures as containing the same number of rectangles.

To respond to this criticism, first notice that to see Figure 1 and MiniFigure 1 as duplicates with respect to the number of rectangles they contain, we need not see Figure 1 or MiniFigure 1 as containing a particular number of rectangles. To see Figure 1 as containing the same number of rectangles as MiniFigure 1, we need only see the rectangles of Figure 1 as one-one correlated with those of MiniFigure 1. Indeed, to see the number of knives as the same as the number of forks, we need not see the knives and forks as both seven in number. It is enough to see the forks and knives as one-one paired.

But do we see the rectangles of Figure 1 as one-one correlated with those of MiniFigure 1? I believe that we do. We see these sequences as so correlated because we see Figure 1 and MiniFigure 1 as having the same structure. In particular, we see these sequences as having the same shape, and we see these sequences as having the same kinds of parts arranged in the same way. Furthermore, we see the corresponding parts of these sequences as having the same shape and as having the same kinds of parts arranged in the same way. We see this correlation even in the "fuzzy" parts of the image. As the images get smaller, it becomes more and more difficult to differentiate between the different rectangles. Nevertheless, we see the fuzzy regions of both sequences as similar in structure. The sequences are indefinite in the same ways. Because we see these sequence as similar in these respects, we see the sequence as one-one correlated.

So, the indefiniteness of the sequences does not preclude us from seeing the sequences as one-one correlated. We can see the sequences as containing an indefinite number of rectangles, yet still see the sequences as containing the same number of rectangles. We are able to do this because, even though the sequence gets fuzzy as the images get smaller, we still see the sequences as one-one correlated.

V. Sense Experience and Beliefs Formed on the Basis of Sense Experience

Given certain assumptions about the relationship between sense experience and beliefs, one can make further objections to the claim that we can have an illusion of an infinite sequence. In this section I will examine these objections and challenge the underlying assumptions about the relationship between what we see and what we believe.

Some philosophers believe that perception should be analyzed in terms of belief and so accept that

(4) If a person sees an object *o* as an ***F***, then he believes that *o* is ***F***,

Those who accept (4) might deny that we see a sequence as containing a shrunken duplicate, for, we are never fooled into believing that the sequence actually contains a shrunken duplicate. Indeed, we know that the image is finite.

However, as many others have noted,[17] there are many counter examples to (4). For example, consider the Muller-Lyer figure (Figure 5). I see the top line in the figure as

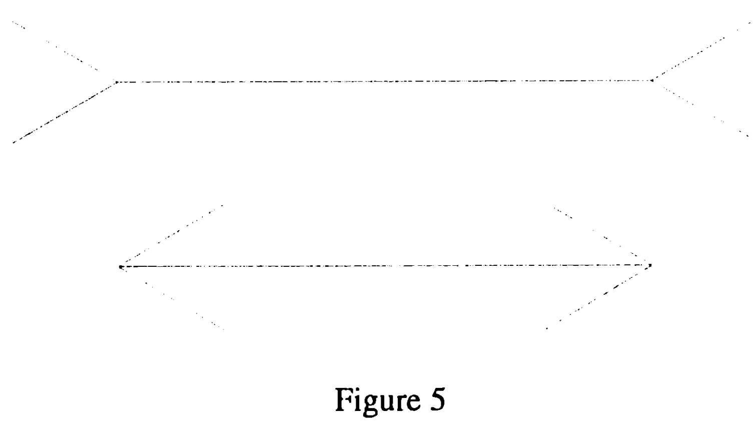

Figure 5

[17] Frank Jackson, *Perception*. Cambridge University Press, Cambridge, 1977. pp. 37-38.

longer than the bottom line. Nevertheless, I know that the two lines are the same length because I have read about this illusion. In this case, then, having certain background beliefs affects what beliefs I form on the basis of my sense experience. Accordingly, we can see an object as an *F* and yet do not believe that the object is an *F*. The example of the infinite is similar to the Muller-Lyer example. We know that Figure 1 does not contain infinitely many rectangles, and so, we do not believe that the sequence contains a shrunken duplicate. Nevertheless, we can see the sequence as containing a shrunken duplicate.

A similar objection to my argument relies on a slightly weaker version of (4). Some philosophers hold that perceptions should be analyzed as inclinations or dispositions to believe and so hold that

(5) If a person sees an object *o* as an *F*, then he is disposed to believe that *o* is *F*.

If (5) holds, then it appears that we cannot see a sequence as containing a shrunken duplicate because we are never disposed to believe that the sequence contains a shrunken duplicate.

Principle (5) has also been challenged. Gareth Evans, for example, presents several arguments against this view in *The Varieties of Reference*.[18] Instead of reviewing his arguments here, I will present a new counter example to (5) in which we see an object as an *F*, but are not disposed to believe that the object is an *F*. The counter example is given in Figure 6. Figure 6 is a two-dimensional picture of a three-dimensional cube. Although the picture is two-dimensional, it appears to have certain features that only a three-dimensional object could have. In particular, when we look at this figure, we see line *a* as

[18] Gareth Evans, *The Varieties of Reference*.Edited by John McDowell. Oxford University Press, Oxford, 1982. pp.124, 229.

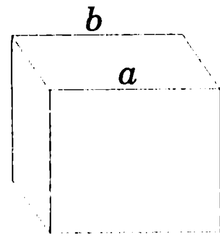

Figure 6

in front of line b. But even though we see the picture as having this three-dimensional feature, we are not disposed to believe that line a is in front of line b. Indeed, we are in no way taken in by the artist's representation of a three-dimensional object. We do not believe that the picture is a three-dimensional cube.

It is a bit difficult to determine what is going on in this case. This case is not an example of aspect shifting, i.e., it is not the case that at one moment, we see line a as in front of line b, and then at a later moment, we see line a as on the same plane as line b. Nor is this an example in which our experience has contradictory content, i.e., it is not the case that we see line a as in front of line b and simultaneously see line a as not in front of line b. Rather, what seems to be going on in this case is that even though we can see line a as in front of line b, this is only one aspect of how the picture appears to us, and this one aspect of how the picture appears is insufficient to produce the disposition to believe that the picture is three-dimensional. Indeed, what beliefs we are disposed to form on the basis of perception are caused not simply by one of the ways an object appears to us but rather by the combination of all the ways an object appears to us. When we take into account the entire appearance of the picture, we are not disposed, on the basis of perception, to believe that line a is in front of line b.

A similar story can be told in the infinite sequence case. Even though we see the sequence as containing a shrunken duplicate, perhaps other aspects of how the sequence looks temper our disposition to believe that the sequence contains a shrunken duplicate. So, in the end we are not disposed to believe that the sequence contains a shrunken duplicate even though we see the sequence as containing a shrunken duplicate.

Figure 6 has certain further interesting features. I have claimed that we see line *a* as in front of line *b*, but we are not disposed to believe that line *a* is in front of line *b*. However, something further is true in the case of Figure 6. Because we recognize immediately that the picture is a two-dimensional representation of a three-dimensional object, we are disposed to believe that line *a* is *not* in front of line *b*. So, we see line *a* as in front of line *b*, but are disposed to believe the opposite of what we see. Accordingly, Figure 6 illustrates that even the following principle, which is weaker than (4) and (5) also fails:

(6) If a person sees an object *o* as an *F*, then it is not the case that he is disposed to believe that *o* is *not F*.

That (6) fails is relevant to the case of the infinite. Certain features of pictures of sequences might dispose someone to believe that the sequences in the pictures are finite. For instance, Figure 3 contains a black dot in its center, and so, some might contend that when we see Figure 3, we are disposed to believe that the sequence in the picture is finite. In this case, then, we see the sequence in Figure 3 as containing a shrunken duplicate, but we are disposed to believe that the sequence is finite and so disposed to believe that it does not contain a shrunken duplicate. My discussion of Figure 6 shows that the case of the infinite is not an anomaly.

What seems to be going on in these cases is that even though we see, for example, line *a* as in front of line *b*, other aspects of the way the picture looks dispose us to believe that the picture is two-dimensional. As I noted, what beliefs we are disposed to form on the basis of perception are caused not simply by one of the ways the object appears to us, but rather by the combination of all the ways the object appears to us. Accordingly, we must also factor in that when we look at the picture, we see the picture as drawn on a two-dimensional surface. When we take into account the entire appearance of the object, in some cases, we are disposed to believe that the object is not an F even though the object appears to be an F. In the case of Figure 6, line a appears to be in front of line b, but when we factor in all the aspects of how the picture appears, we are disposed to believe that line a is not in front of line b.

Overall, then, we should not discount the example of the infinite because it goes against certain accounts that analyze perception in terms of belief. For, as I have argued, such accounts are implausible.

VI. The "Content" of Our Experience of the Infinite

In the proceeding reflections about the relationship between experience and belief, certain interesting features of the illusion of the infinite have been brought out. In particular, I have indicated that we see Figure 3 as containing a black dot in its center. One thus might conclude that we see the sequence as finite.[19] Accordingly, part of the content of such an experience of the infinite seems to be that the sequence is finite. We have seen previously that part of the content of the experience of the infinite is that the sequence

[19] I should note that I am not convinced that we see any of these sequences as finite. For one thing, I believe that these pictures are similar to what we see when we are in the midst of a forest. Just as we do not see the boundaries of the forest, we do not see the boundaries of these sequences. Furthermore, the inference from the claim that we see the sequence as containing a black dot to the claim that we see the sequences as finite is questionable. For the inference appears to rely on (3). Although I have these reservations, I will accept the claim that we see the sequences as finite because some readers might hold that this accurately represents the content of the experience.

contains a shrunken duplicate of itself. The question I ask in this section is whether it is possible for an experience to have a content consisting of both the claim that the sequence is finite and that the sequence contains a shrunken duplicate of itself.

Initially, one might think this is problematic because if the sequence did contain a shrunken duplicate it would be infinite. However, we cannot see the sequence as infinite and simultaneously see the sequence as finite. That is, we cannot have an experience with contradictory content. This objection, however, does not succeed. As I discussed earlier, the claim that we see the sequence as infinite does *not* follow from the claim that we see the sequence as containing a shrunken duplicate. So, it is incorrect to conclude that we see the sequence as simultaneously finite and infinite.[20]

But even though the experience does not have a contradictory content, one might think that we cannot simultaneously see the sequence as both finite and as containing a shrunken duplicate. For, the properties that lead us to see the sequence as finite preclude us from seeing the sequence as containing a shrunken duplicate.

There are two ways to respond to this objection. First, one can claim that the experience of the infinite is a case of aspect shifting. That is, we do not simultaneously see the sequence as finite and as containing a shrunken duplicate. Rather, at one point we see the sequence as finite, and then the figure "flips" on us and we see the sequence as containing a shrunken duplicate. The problem with this response is that it does not respect the phenomenology of the case at hand. The figure does not appear to "flip" on us.

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[20] Even if one could show that the experience has contradictory content, it is debatable whether this undermines the example of the infinite. In his article "The Waterfall Illusion" Tim Crane argues that in the waterfall illusion we see an object as moving and simultaneously we see the same object as not moving. Crane argues that such examples are not problematic unless the contents of perceptual experiences are conceptual. Tim Crane, "The Waterfall Illusion." *Analysis* 48 (1988). pp. 142-47. also see his "Concepts in Perception." *Analysis* 48 (1988). pp. 150-53.

A more promising response to the objection is to challenge the contention that the properties that lead us to see the sequence as finite preclude us from seeing the sequence as containing a shrunken duplicate. I contend that seeing a black dot in the center of the top picture in Figure 3 does not preclude us from seeing the top sequence in this figure as containing a shrunken duplicate. For, to see this sequence as containing a shrunken duplicate, we need to see the top sequence in Figure 3 and the bottom sequence in Figure 3 as similar in structure, and we can see the two sequences as similar in structure even though we see a black dot in the center.

Notice that the case of the infinite is thus similar to the case of the box in Figure 6. In the case of Figure 6, we see line *a* as in front of line *b*, but we also see the figure as drawn on a two-dimensional surface. Just as in the case of the infinite, this case is not problematic because seeing the figure as drawn on a two-dimensional surface does not preclude us from seeing line *a* as in front of line *b*.

VII. But It's an Illusion

So far, I have argued that, contrary to what many philosophers have argued, we can have an "experience of the infinite." In particular, we can have a perceptual illusion of an infinite sequence. In the introduction I claimed that showing that we can have such experiences is interesting because it ultimately helps us respond to the epistemological puzzle about the infinite that Lavine describes. In particular, in Chapter 3 I will argue that we can appeal to these illusions to show that we have modal knowledge of the infinite.

However, at this point, one might be skeptical that the experiences I have described can serve the purpose I say they can. In particular, one might believe that because these experiences are *illusions*, I cannot appeal to them to respond to the epistemological puzzle about the infinite.

One concern along this line is that because an illusion is a misrepresentation of the way the world is, it cannot provide us with information that results in knowledge about the world. To respond to this concern, we need to notice that although illusions do not appear to tell us how the world *is*, it does not follow that they do not tell us how the world *could be*. For, even though illusions misrepresents our immediate environment, they still might supply us with modal knowledge. Of course, at this point, it is unclear how an illusion can supply us with modal knowledge. In Chapter 3, I will explain how it is that certain illusory experiences can provide us with knowledge that certain objects could exist.

Another concern about the possibility of illusions providing us with any kind of knowledge is that when we undergo an illusion, our mind appears to manufactures the experience, creating something that is not really there. As a result, this experience does not appear to provide us with reliable information.

To respond, we must notice that the illusion of the infinite is not the kind of illusion where our mind manufactures the experience. Indeed, the illusion of the infinite is not a hallucination. When a person hallucinates, his experience is not caused by external stimuli. Rather, something internal to the person causes him to undergo the experience. His mind manufactures the experience. One might legitimately question whether a hallucination can provide information that results in knowledge. Our experience of the infinite, however, is not a hallucination. The arrangement of lines in Figure 5 causes the picture to look a certain way to us. We do not make the picture look that way. We do not mentally add properties to the picture so that it appears a certain way to us. So, because the illusion of the infinite is caused by certain features of the picture, we cannot rule out, off hand, the possibility that this experience can supply us with reliable information and thus knowledge.

So far, I have simply defended the weak claim that, off hand, we should not conclude that these illusion cannot serve as a basis for modal knowledge. I now want to indicate why these experiences might be particularly well suited for the task. That is, I want to indicate why the illusion of the infinite is particularly useful in providing information about the infinite. Most illusions we undergo do not appear to be informative. The reason is that we can just as well imagine what we appear to see. For example, we need not have an illusion of a golden mountain to know what a golden mountain would look like. For, we can perfectly well imagine what a golden mountain would look like. The case of the infinite is different. If someone asks us to imagine what we would see if we were to see an entire infinite sequence, we would have difficulty imagining what we are asked to imagine. The illusion of the infinite is special because it provides a way for us to extend our powers of imagination. The illusion of the infinite supplies our imagination with something that it would have difficulty producing on its own.

Before closing, I want to note that as with the epistemological puzzle, one should not summarily conclude that we cannot use these experiences of the infinite to provide a response the empiricist puzzle simply because these experiences are illusions. In particular, even though illusions misrepresents our immediate environment, they still might supply us with concepts of objects that do not actually exist. For example, some might hold that to have a concept one must know what it is like to see an object that falls under that concept. In this case, an illusion might supply the needed experience.

Furthermore, since the experience of the infinite is not a hallucination, we cannot reject the use of this illusion on the grounds that our mind manufactures our experience and so supplies inherently questionable information.

VIII. Summary

Overall, then, I have argued that we can have an illusion of an infinite sequence because we can see certain indefinitely long sequences as containing, as a proper part, a shrunken duplicate. I have argued that we should not discount the example of the infinite on the basis that it goes against certain accounts that analyze perception in terms of belief. For, such accounts are implausible. Furthermore, I have argued that there is no problem in holding that the content of this experience consists both of the claim that the sequence contains a shrunken duplicate and the claim that the sequence terminates. Lastly, I have argued that we should not summarily dismiss this experience of an infinite sequence as a way to obtain modal knowledge on the grounds that the experience is an illusion. This last result leads into the last chapter, where I will argue that the experiences I have described provide us with modal knowledge of the infinite.

Chapter 3

Pictures as a Guide to Possibility: The Case of the Infinite

The feasibility of philosophical positions often rests on possibility claims. The case of skepticism about the external world is a prime example. The skeptic relies on the premise that there could be a world that lacks physical objects, but appears just as our world appears to us. The standard practice for establishing that a proposition is possible is to argue that the proposition is conceivable, where the proposition's conceivability is presumed to provide evidence for its possibility. What I propose to do here is to describe a way of establishing possibility claims. I will explain how pictures can be an invaluable resource in establishing that the depicted object could exist in three-dimensional space.

When drawing modal conclusions from pictures, we must proceed with care. Some pictures are deceptive. In particular, some pictures appear to depict an object that could exist in three-dimensional space, but they do not depict such an object. So, to sustain the claim that a picture provides evidence that the depicted object could exist in three-dimensional space, I need to explain how we determine that a picture depicts an object that could exist in three-dimensional space rather than an object that could not. I will argue that upon seeing a picture, we can obtain evidence that a picture represents a "coherent spatial configuration" rather than an "incoherent spatial configuration," and furthermore, that if we have evidence that a picture represents a coherent spatial configuration, we thereby obtain evidence that the depicted object could exist in three-dimensional space. Part of my argument will involve explicating the notion of coherence at work here. I will provide not only a characterization of coherence but also a test for coherence. I will call this method of establishing possibility claims, "the picture method."

Part of my interest in the picture method is that it has implications for modal structuralism, a position in the philosophy of mathematics.[1] In short, we can use this method to defend modal structuralism.

As its name suggests, modal structuralism is a form of structuralism. In particular, a modal structuralist accepts the platitude that mathematics is the study of structures. Modal structuralism differs from other varieties of structuralism in that it avoids commitment to mathematical objects. It avoids such commitment by translating mathematical statements into second-order modal statements that contain n´ terms that purport to refer to, or quantifiers that purport to range over, mathematical objects.

Because modal structuralism dispenses with mathematical objects, it appears to avoid a well-known epistemological puzzle facing a platonist philosophy.[2] The platonist must provide an account of knowledge of and reference to mathematical objects, but since mathematical objects are abstract, it appears that an account will not be forthcoming. The modal structuralist circumvents this requirement because the modal structuralist explains how to avoid reference to these problematic entities. As a result, the modal structuralist need not provide an account of knowledge of and reference to mathematical entities.

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1 Hilary Putnam originally presented this view in his paper "Mathematics without Foundations." *Philosophy of Mathematics, Selected Readings*, Second Edition. Eds. Paul Benacerraf and Hilary Putnam. Cambridge University Press, Cambridge, 1983. pp. 295-311. More recently, Geoffrey Hellman has provided a detailed defense of the view. See his *Mathematics Without Numbers*. Oxford University Press, Oxford, 1989.

2 Paul Benacerraf, "Mathematical Truth." *The Journal of Philosophy* 70 (1973). pp. 661-679.

Modal structuralism, however, does not appear entirely free of epistemological difficulties. As others have pointed out,[3] modal structuralism is viable only if an ω-sequence could exist.[4] But at present, modal structuralism appears to lack a defense of this claim, and so it faces an epistemological challenge.

In what follows, I will explain why the modal structuralist must hold that an ω-sequence could exist and argue that, at present, the leading proponent of modal structuralism does not provide a convincing defense of this claim. I will then provide the needed defense. In particular, I will introduce the picture method for establishing possibility claims. I will then use the picture method to argue that there could be an ω-sequence of concrete objects. I will argue that pictures such as Figure 1 provide the necessary evidence. By showing that we have grounds for this modal claim, I provide the modal structuralist with a defense of one of the his central tenets.

I will close the chapter by discussing an objection to my proposal that a modal structuralist can provide grounds for the claim that there could be an ω-sequence of concrete objects. In a recent paper[5] Bob Hale argues that the modal structuralist cannot provide evidence for this claim. Hale thus concludes that modal structuralism faces an epistemological difficulty just as significant as the one that platonism faces. Clearly, my defense of the claim that an ω-sequence could exist is contrary to Hale's conclusion. In the

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[3] Charles Parsons, "Structuralist View of Mathematical Objects." *Synthese* 84 (1990). p. 319. Bob Hale, "Structuralism's Unpaid Epistemological Debts." *Philosophia Mathematica* 3 Vol. 4 (1996). pp. 124-147. Hellman, *Mathematics Without Numbers.*

[4] An ω-sequence is an infinite set X that has a structure similar to that of the natural numbers. That is, the sequence has a first member, "0"; every member in the sequence has a "successor"; and there are no members in the sequence that follow infinitely many others. More formally, an ω-sequence is a set X such that there is an object 0 which is a "distinguished member" of X, there is a one-one and total function f from X onto X minus 0, and every member of X can be reached from 0 by finitely many iterations of f.

[5] Bob Hale, "Structuralism's Unpaid Epistemological Debts." *Philosophia Mathematica* 3 Vol. 4 (1996). pp. 124-147.

Figure 1

last two sections I will lay out Hale's argument and explain why it does not undermine my defense of this claim.

I. The Evidentiary Burdens of Modal Structuralism

First off, I will explain why the modal structuralist must support the claim that an ω-sequence could exist. I will then argue that the modal structuralist seems to face an epistemological challenge because, at present, he lacks a convincing defense of this central tenet.

To understand why a modal structuralist must provide grounds for the claim that an ω-sequence could exist, we need a characterization of the modal structuralist's suggested translation of mathematical statements. As I have noted, the modal structuralist proposes a way to eliminate reference to mathematical objects and thereby eliminate such objects. He eliminates reference to mathematical objects by showing how to translate each mathematical sentence into a sentence that contains neither quantifiers that purport to range over, nor singular terms that purport to refer to, mathematical objects.

In the case of arithmetic, it is easiest to see this translation as proceeding in three stages. First, an arithmetical statement is translated into a statement that is about ω-sequences. This enables the modal structuralist to eliminate reference to natural numbers. This statement about ω-sequences is then rendered in second-order logic so as to avoid reference to all types of mathematical objects, including ω-sequences, sets, and so forth. Finally, in what appears to be an attempt to avoid epistemological tangles, the modal structuralist replaces the second-order translation with a corresponding statement in *modal* second-order logic. In the final analysis, then, the modal structuralist translates arithmetical statements into statements of modal second-order logic.

Off hand, the first stage of the translation seems plausible, as there is a predicable way of mating each member of and operation on the natural numbers to a corresponding member of and operation on any ω-sequence. For instance, 0 corresponds to the first member in an ω-sequence; 1, to the second member; 2, to the third; and so forth. Also, the successor function corresponds to the function f that orders the ω-sequence. Using this correspondence as a rough guide, the modal structuralist can translate arithmetical statements into statements that contain no terms that purport to refer to, or quantifiers that purport to range over, natural numbers. For example, the statement that every number has a successor can be translated into the statement that, for all x, if x is an ω-sequence, whose "successor" function is f, then for every member a of x there is a member b of x such that $b = f(a)$. In general, in the first stage, an arithmetical statement A is translated as follows:

(1) For all x if x is an ω-sequence, then $A^*(x)$

where A^* is obtained from A by rewriting A so that the resulting formula speaks of operations on, and objects in, x. If all arithmetical statements are translated as the modal structuralist suggests, reference to natural numbers appears to be eliminated.

However, as it stands, this translation does not eliminate reference to all mathematical objects, as (1) contains what appear to be a quantifier over ω-sequences. To avoid reference to ω-sequences and other mathematical objects, the modal structuralist carries the translation a step further. Using second-order logic, we can rewrite (1) in an attempt to eliminate all terms that purport to refer to, and quantifiers that purport to range over, mathematical objects. (Those interested in the translation should see footnote 6.)[6] Whether the second-order translation is successful in eliminating reference to mathematical objects depends on one's views about the ontological commitments of second-order modal logic. Although the translation uses second-order logic, in what follows I will appeal to (1) because it is easier to work with than the corresponding statement in second-order logic.

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[6] I will show how we can translate an arithmetical statement A directly into a statement of second-order modal logic. For an arithmetical statement A whose only nonlogical constants are 0 and a symbol S for the successor function, we obtain a translation as follows. First, we form the conditional whose antecedent is the conjunction PA of the second-order Peano Axioms and whose consequent is A. (Note that the conjunction PA of these second-order axioms is finite.) PA specifies the conditions under which a sequence formed from 0 and S is an ω-sequence. Second, we obtain the open sentences PA' and A' from PA and A, respectively, by replacing all occurrences of 0 with the variable z, replacing all occurrences of S with the second-order variable R, and relativizing all the quantifiers in A and PA with the second-order variable X. We then bind the newly introduced variables with universal quantifiers to form $\forall X \forall x \forall R(PA' \rightarrow A')$, which is the desired second-order translation of A.

A modal structuralist finds the translation, as it stands, unacceptable. The problem is that if there are no ω-sequences, then for any arithmetical statement A, the translation of A is vacuously true. So, to avoid wholesale vacuous truth on the current translation, one must defend the claim that an ω-sequence of objects *actually* exists.

For the modal structuralist, defending such a claim is unappealing. Part of his impetus for providing a translation of mathematical statements is to eliminate mathematical objects and thus to avoid the corresponding epistemological difficulties. But if, after giving the translation, he must defend the claim that an ω-sequence exists, he again faces a web of epistemological difficulties. The ω-sequences in question consist either of abstract objects or of concrete ones. Supposing that it consists of abstract objects is unattractive. For, if a modal structuralist tries to defend the claim that an ω-sequence of abstract objects exists, he appears to confront the epistemological difficulties he originally tried to avoid.[7] Supposing that the ω-sequence consists of concrete objects is also unattractive because the modal structuralist would most likely defend such a claim by appeal to the fact that our current scientific theories postulate the existence of infinitely many space-time points. But as Parsons queries "Can we rule out the possibility that physics will abandon infinitely divisible space-time and replace it with some 'quantized' conception?"[8] If we cannot, then the modal structuralist is left with the unsettling possibility that future physics will undermine the justification of our mathematical theories.

[7] It may be that abstract objects other than mathematical objects pose less epistemological difficulties. For instance, if the abstract objects are "quasi-concrete" objects, i.e., if the abstract objects "are determined by intrinsic relations to concrete objects," (Charles Parsons, *Mathematics in Philosophy*. Cornell University Press, New York, 1983. p. 25.), then we might be able to provide an explanation of knowledge of, and reference to, such objects. In particular, perhaps we come to know about quasi-concrete objects by examining the concrete objects to which they are intrinsically related.

[8] Parsons, "The Structuralist View of Mathematical Objects," p. 315.

In what appears to be an attempt to avoid such epistemological tangles, the modal structuralist modifies the suggested translation of arithmetical statements. He proposes that instead of translating an arithmetical statement A as (1), we should translate A as follows:

(2) $\Box$(For all x if x is an ω-sequence, then $A^*(x)$).[9]

So, the modal structuralist proposes that we simply affix a necessity operator to the front of (1). Because the statement is prefixed with a modal operator, to avoid wholesale vacuous truth, the modal structuralist need not defend the claim that an ω-sequence exists.

However, the modal structuralist is not entirely free of epistemological burdens. If he accepts (2) as the translation of arithmetical statements, then to avoid wholesale vacuous truth, he must defend the claim that

(3) $\Diamond$(There is an ω-sequence)

Indeed, if it is necessarily false that an ω-sequence exists, then all statements of the form (2) are true. For, if there could be no ω-sequences, then of course it is necessary that for all x either x is not an ω-sequence or $A^*(x)$. Accordingly, if there could be no ω-sequences, for any arithmetical statement A, the translation of A is true.

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[9] Exactly what the relevant notion of necessity is here is a difficulty question. In his "Structuralist View of Mathematical Objects" Charles Parsons offers four possible interpretations of the modal operator in this context. The four interpretations are as follows: "strictly logical, in a sense connected with formal logic;" "logical in the sense more usual in discussions of modality, which takes account of the constraints of non-logical concepts, in my opinion best called 'metaphysical';" "mathematical;" and "physical." (Charles Parsons, "Structuralist View of Mathematical Objects," p. 319.) Parsons maintains that the necessity operator should be interpreted as mathematical necessity.

Off hand, it seems that defending (3) is preferable to defending the claim that an ω-sequences actually exists. At a minimum, we have more options in defending (3). For instance, we can try to argue that it is conceivable or imaginable that an ω-sequence exists.

However, one should not be overly optimistic about the ease at which a modal structuralist can provide a defense of (3). To see why, let us consider a defense of (3) given by Geoffrey Hellman, one of the leading defenders of modal structuralism. He appears to believe that (3) is true because we can argue for the claim that

(4) $\Diamond$(There is an ω-sequence of concrete objects)

Hellman believes that an infinite sequence of concrete objects could exist because "even arch-opponents of "completed infinities" concede the coherence"[10] of a models of "Euclidean time or space."[11] Here, Hellman appears to contend that since current physical theories postulate the existence of infinite space, models of infinite space are coherent and thus possible. He then appeals to second-order logic to concludes that models consisting of an ω-sequence of objects are possible.

Is Hellman's defense plausible? It is if one believes that the current physical theories cannot be challenged on philosophical grounds. That is, it is if one believes that by dint of a claim's being part of our current scientific theory, this claim expresses a coherent possibility.

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[10] Hellman, *Mathematics without Numbers*, p. 30.

[11] Charles Parsons also believe that the claim that infinitely many concrete objects exists follows from the claims about coherence, but he appears to be more cautious in endorsing the claim that our current physical theories are coherent: "There may be epistemological problems about how we know a statement like [(4)] to be true. But its truth seems to follow from the supposition that theories in physics describe coherent possibilities, and perhaps it can be seen in more direct and intuitive ways." Charles Parsons, "Structuralist View of Mathematical Objects," p. 320.

Of course, many will not be convinced by an appeal to the authority of science. And those that are not will find it less than evident that there could exist infinitely many concrete objects. Although some claims about what is possible strike us as evident—for example, the claim that I could have worn a black shirt yesterday seems obviously true because I can readily imagine that yesterday I put on a black shirt rather than a white one—the claim that it is possible that there are infinitely many concrete objects does not appear to have this character. In particular, it is not at all clear that we can imagine a situation in which there are infinitely many concrete objects.[12]

In his article "Structuralism's Unpaid Epistemological Debts"[13] Bob Hale presents a related criticism of Hellman's defense. Hale contends that Hellman mistakenly assumes that because the claim about ω-sequence is only about what is possible, it is a weak claim and so needs no defense. To show that such a stance is incorrect, Hale points out that it would be incorrect to think that just because the supposition that my mind could exist without my body is a claim about the merely possible, this supposition needs no support, and our default position is acceptance. Indeed, we do not accept this claim without supporting arguments. In fact, many philosophers argue that this claim is false. According

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[12] To bring out the full difficulty of the task facing the modal structuralist, we would need to consider different interpretations of the modal operator. As I noted in footnote 9, Charles Parsons offers four possible interpretations of the modal operator in this context: physical, logical, mathematical, metaphysical. I believe none of these interpretations ensures that the claim that infinitely many concrete objects could exist is evident. As for physical possibility, unless one believes that by dint of being a scientific theory, a scientific theory guarantees the coherence of its claims, it does not appear to be evident that it is physically possible that infinitely many concrete objects could exist. As for logical possibility, I agree with Parsons that logical possibility is not clearly distinct form mathematical possibility. And, since it is not clear how one goes about establishing mathematical possibility, I believe it is a stretch to say that it is evident that it is mathematically possible that infinitely many objects could exist. This leaves metaphysical possibility, and establishing the metaphysical possibility of this claim about the infinite requires an argument from conceivability.

[13] See footnote 5.

to Hale, just as we need evidence that my mind could exist without my body, we need evidence that there could be infinitely many concrete objects.

So, we have seen that the leading proponent of modal structuralism fails to provide an adequate defense of (3). The modal structuralist thus appears to face the epistemological challenge of explaining how we know that (3) holds. Of course, this is not to say that no defense of (3) is possible. For all I have said, nothing bars a modal structuralist from supplying a defense of (3). However, I will not survey the possible defenses that a modal structuralist might wage. Rather, I will offer a defense of my own.

In particular, in what follows, I will lay the groundwork for a defense of (3). I will explain how to appeal to pictures in order to establish that the depicted object could exist. I will then apply this general method to show that by seeing pictures of an infinite sequence, we obtain evidence for (4) and thus we obtain evidence for (3).

II. The Use of 'Depicts' and its Cognates

Before providing a sketch of the picture method, however, I will explain how I intend to use 'depicts' and its cognates. In giving this account I am not explicating the notion of depiction. Indeed, a key feature of this account is that the claim that a picture depicts an object does not entail that the depicted object could exist in three-dimensional space.

We have all had the experience of seeing certain pictures as having properties similar to a three-dimensional object. When we look at a two-dimensional picture of a cube, for example, we see some of the lines in the picture as being in front of others. Pictures designed to look like three-dimensional objects are special because our visual system produces an experience whose content contains claims that a e purportedly about a three-dimensional object. Our visual system is able to do this because it draws on mechanisms that are used to perceive three-dimensional objects. When we look at three-dimensional objects, our perceptual system produces, from the two-dimensional pattern encoded in our rods and cones, an experience whose content contains claims that are purportedly about a three-dimensional object. This process is automatic and unconscious, but once it is accomplished we end up with an experience that purports to be of a three-dimensional object. We can then extract from the content of this experience a list of statements that are purportedly about a three-dimensional configuration. By looking at "perspective pictures," i.e., pictures that look like three-dimensional objects, these automatic and unconscious mechanisms also produce an experience whose content contains claims that are purportedly about a three-dimensional object, and from the content of this experience, we can extract a description that is purportedly about a three-dimensional object.

An artist employs a variety of techniques to produce perspective pictures. For instance, an artist can use rules of linear perspective. These rules specify how to draw images in two-dimensions so that the resulting picture appears to look like a three-dimensional spatial configuration. One rule tells the artist how to draw lines so that the our experience of these lines will be similar to our experience of parallel lines that recede in the distance. Besides using rules of linear perspective, the artist can also employ certain shading techniques to mimic in two-dimensions what happens when light hits a solid object.

It is important to recognize that the artist can apply these techniques of perspective *locally* and still produce a picture that appears to look *globally* like a three-dimensional object. Figure 2 provides an example. In this picture, the rules of perspective are applied locally. Each corner is constructed so that our visual system is provided locally with the needed cues to produce an experience whose content contains claims that are purportedly about a three-dimensional object. I will discuss this picture in more detail latter. In particular, I will explain how we can distinguish this picture from pictures "of" three-dimensional objects.

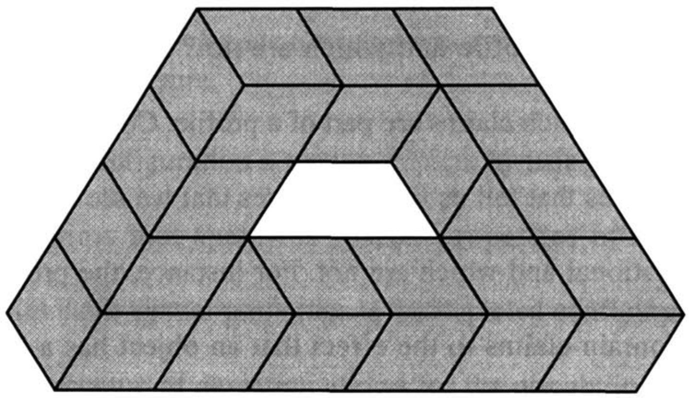

Figure 2

To characterize the notion of depiction, let us restrict our attention to perspective pictures. We have seen that when we look at a picture, we can generate, from the content of our experience, a list of statements that are purportedly about a three-dimensional object. Let us call the list of statements generated from such a picture a "profile" generated from this picture. For example, a profile generated from Figure 2 contains the claim that there is an object that is three-dimensional; that this object consists of four connected bars that meet at ninety degree angles; that the bar at the top of this object contains three-square blocks; that the top bar is orientated so that its front surface makes an acute angle with the ground; and so forth.

It is important to notice that a profile generated from a picture does not contain, for every feature we see the picture as having, a claim that describes this feature. For example, no profile generated from Figure 2 contains the claim that a two-dimensional object is made of ink nor does it contain the claim that this two-dimensional object has 'Figure 2' written beneath it. Although these claims are part of the content of our experience of this picture, they are not part of a profile generated from it.

This observation, however, brings up an important question: how do we determine which claims are part of a profile and which are not? I believe that conventions of representation determine which claims are part of a profile. Conventions of representation are, in short, general rules that tell us which features that we see a perspective picture as having are representational and which are not. For instance, the profile generated from Figure 2 does not contain claims to the effect that an object has a feature only a two-dimensional object has. The reason is that a convention of representation tells us that we should ignore aspects of the picture that destroy the illusion of three-dimensionality. As a result, we do not include in the profile claims that correspond to features that undermine this illusion. So, even though we see the picture as having certain properties that only a two-dimensional object has, we ultimately do not include in the profile the claim that the object has such and such two-dimensional feature. Conventions of representation also tell us to ignore certain features of the picture that are unique to pictures. For instance, we ignore that the picture has a frame, and we ignore that the picture is drawn on a canvas. So, the claim that 'Figure 2' is written beneath Figure 2 is thus not part of the profile generated from Figure 2. Conventions of representation help us generate a profile by helping us distinguish representational from nonrepresentational features of a picture.

My characterization of the notion of a profile allows for the possibility that more than one profile can be generated from a picture. For, we can extract more claims from the content of our experience of a picture on one occasion than on another. To define 'depicts', I need to introduce the notion of a maximal profile. The "maximal profile" generated from a picture is the list that contains, for every representational feature of a picture, a corresponding claim about that feature. Using the notion of maximal profile, we can define 'depicts' as follows:

Depiction Condition: A picture depicts a configuration that fits the maximal profile generated from this picture.

So, we determine what configuration a picture depicts by using rules of interpretation to generate a list of claims that specifies spatial properties of a configuration. The configuration that has these spatial properties is the depicted configuration.

The proposed account of depiction allows for the possibility that a picture depicts an object that could not exist in three-dimensional space. For, an object that could not exist in three-dimensional space can fit a profile generated by a picture and so be depicted by a picture. Notice, then, in saying that a picture depicts an object, we are not thereby committed to the claim that the depicted object could exist in three-dimensional space.

III. Pictures as a Guide to Possibility: A Sketch

So far, we have seen that the modal structuralist owes us a defense of (3), i.e. the claim that an ω-sequence could exist. In this and the next section, I will lay the groundwork for this defense of (3). I will introduce a method for establishing possibility claims. I will describe how to appeal to pictures in order to establish that the depicted object could exist. I will then apply this general method to show that by seeing pictures of an infinite sequence, we obtain evidence for (4), i.e., the claim that an ω-sequence of concrete objects could exist, and thus we obtain evidence for (3).

I should inform the reader that in this section I will help myself to an admittedly controversial notion, namely the notion of a "coherent" spatial configuration. I will discuss this notion in detail in the next section. In this section I simply want to provide an outline of the picture method.

The picture method is well suited to the task of establishing that the depicted object could exist. In fact, the picture method often fares better than methods where we have *only* a description of the object in question and, on the basis of our understanding of this description, attempt to show that something satisfying that description could exist. For, it is often difficult to determine, on the basis of our grasp of a description, whether an object satisfying that description could exist. But when we couple the description with a picture of the object, we sometimes obtain the needed evidence. Pictures sometimes succeed where mere descriptions fail because experiences have a force that narratives sometimes lack. To see this, consider the following story which, I presume, the reader will find almost impossible to understand.

A man with suction cups attached to his feet takes a journey around four connected bars. Initially, he walks up the inside of the first bar which is twelve feet long and which lies perpendicular to his body. When he reaches the top of the bar, he faces the end of a fifteen foot bar which is parallel to his body. The man steps onto this second bar and walks along it until he confronts a third bar which is parallel to his body. He steps onto this third bar, walks twelve feet, and encounters a fifteen-foot bar which is again parallel to his body.

He steps onto this last bar, walks along it, and eventually finds himself back at his initial position.

The difficulty in trying to determine whether it is possible that a man could make such a journey in three-dimensional space is *not* that this story is nonsense. Indeed, each sentence is clear and comprehensible. Rather, the difficulty is that it is unclear how the whole story fits together. As a result, we are uncertain as to whether this story describes a case of possible motion in three-dimensional space.

Pictures provide us with the needed evidence. In particular, Figure 3 provides us

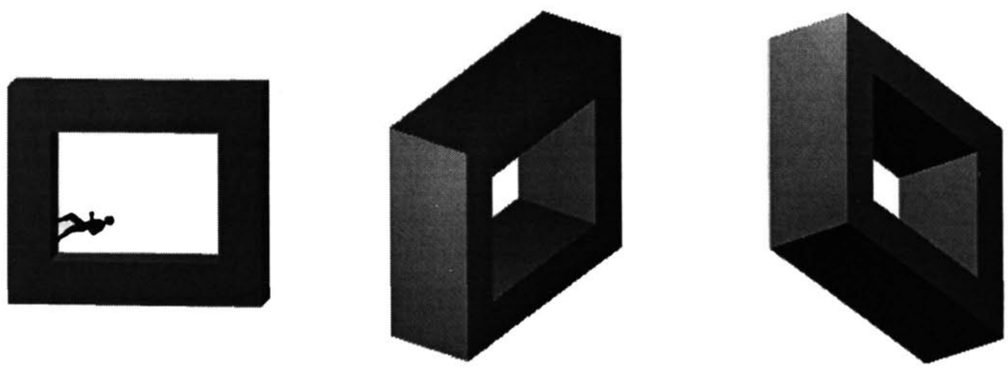

Figure 3

with grounds for believing that the story describes a case of possible motion in three-dimensional space. Figure 3 contains three pictures of the object on which the man walks and depicts his initial position. The pictures show us how the object would look from different vantage points. I believe that by seeing Figure 3, we have adequate grounds for believing that the depicted object could exist in three-dimensional space, and consequently, for believing that the man's journey is a case of possible motion in three-dimensional space. The reason is that seeing these pictures gives us evidence that the pictures represent a coherent spatial arrangement and from this observation of coherence, we can infer that the depicted object could exist in three-dimensional space. These pictures thus have a force that the description of the man journey lacks.

However, when drawing modal conclusions from our experience of pictures, we must proceed with care, for, pictures can be deceptive. Fortunately, however, pictures are deceptive in a way that need not fool us. To see this, recall Figure 2 from the previous

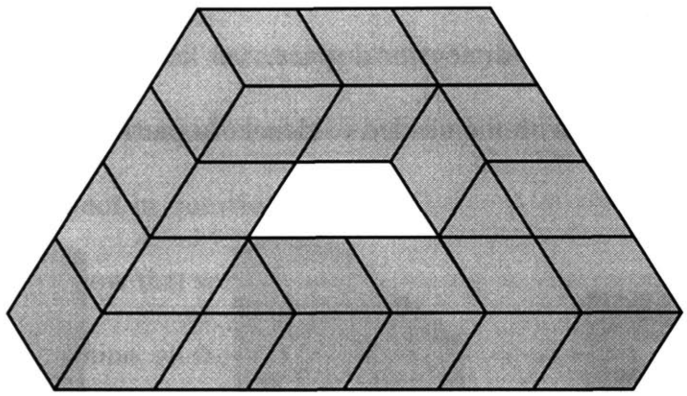

Figure 2

section. Just like Figure 3, Figure 2 appears to depict a three-dimensional object. However, on closer inspection, we begin to see that this picture is deceiving. In particular, if we look at the bottom left corner while covering up the bottom right corner, the bottom bar appears to be in the background. If we do the opposite, i.e., look at the bottom right corner while covering up the bottom left corner, the bottom bar appears to be in the foreground. Looking at these two corners separately, thus, makes us expect that the bottom bar extending from the right corner does not meet the bottom bar extending from the left corner. However, when we look at the figure as a whole, it appears to depict an object consisting of four connected bars. Because we form a contradictory interpretation of the object, we have reason to believe that the picture does *not* depict a coherent spatial configuration. We thus have evidence that the picture does *not* depict an object that could exist in three-dimensional space. So, even though some pictures are deceptive, we need not be duped.

So far, then, I contend that if we see certain kinds of pictures and if we determine that these pictures represent a coherent spatial configuration, then seeing these pictures provides evidence that the depicted object could exist in three-dimensional space. Accordingly, I contend that not only can we detect that a picture represents a coherent spatial configuration, but also we can use the claim that a picture represents a coherent spatial configuration as grounds for concluding that the depicted object could exist in three-dimensional space.

Before closing this section, let me briefly indicate how the picture method is related to the standard method of establishing modal claims. The standard way to establish that a proposition is possible is to argue that it is conceivable, where the proposition's conceivability is presumed to provide evidence for its possibility. Does the picture method provide a way of conceiving that an object exists? The answer to this question depends on one's conception of conceivability. Some conceptions of conceivability make room for the impact of visual images on our ability to conceive that an object could exist. Other conceptions do not. The discussion in this section demonstrates that conceptions of conceivability that make room for the impact of visual images must provide a method for filtering out deceptive images such as Figure 2.

IV. Pictures as a Guide to Possibility: A Detailed Argument

My argument, as it stands, requires supplementation. In particular, I need to provide a characterization of the notion of coherence. In the rest of this section, I will provide an account of coherence as well as a test for coherence.

I suggest the following characterization of coherence:

Coherence Condition: A picture represents a coherent spatial configuration if and only if it can be "naturally" mapped into three-dimensional space and the resulting object can be "filled in" so as to produce a three-dimensional object that fits the maximal profile generated from the picture.

I need to make two comments about this condition. First, I need to indicate what I mean by a "natural" map. We know that a two-dimensional image can be mapped to infinitely many three-dimensional objects. Raphael's *School of Athens*, for example, can be mapped to my left foot, the fly on the window sill, the cup on the table, and so forth. Such maps, however, are not natural maps. A natural map respects the geometrical information contained in the picture. For example, we might see one point in the picture as in front of another. A natural map maps these two points in the picture to two points in three-dimensional space, one of which is in front of the other. To ensure that a map is natural, we can require that the map preserves linear distances up to scale and relative positions. That is, if point a_1 on the picture is mapped to the point b_1 in three-dimensional space and point a_2 on the picture is mapped to point b_2 in three-dimensional space, then the distance and the relative position between a_1 and a_2 as represented in the picture must be proportionally preserved when these points are mapped to b_1 and b_2.

My second comment is about the filling-in clause of the coherence condition. It is necessary to include this clause because a natural map from a two-dimensional surface will often supply only some of the outside surfaces of the depicted object. A natural map supplies only some of the surfaces because pictures often depict an object that has surfaces that are not "seen" in the picture, but nevertheless are "implied" by the picture. It is thus necessary to fill in the rest of the object after mapping. This "filling in" should be done in a way that meets our expectations as to what the depicted object is like. For example, in looking at Figure 3, we expect that the depicted object is composed of four rectangular bars that meet at right angles. Since the map from the two-dimensional image to three-dimensional space does not generate the backside of the rectangle, it is necessary to "fill in" the resulting three-dimensional object. To avoid cumbersome sentences in what follows, I will drop the filing-in clause when discussing the coherence condition and simply say that the resulting object fits the maximal profile generated from the picture.

Now that we have a characterization of coherence, we can see why meeting the coherence condition entails possibility. It entail possibility because if the two-dimensional picture can be mapped into three-dimensional space in such a way that the resulting object fits the maximal profile generated from the picture, then a three-dimensional object can have all the features that the depicted object has. Accordingly, the depicted object could exist in three-dimensional space.

We can similarly argue that falling to meet the coherence condition entails impossibility in three-dimensional space. That is, if a two-dimensional picture *cannot* be naturally mapped into three-dimensional space in such a way that the resulting object fits the maximal profile generated from the picture, then a three-dimensional object cannot have all the features that the depicted object has. Accordingly, the depicted object could not exist in three-dimensional space.

Initially, the argument from incoherence to impossibility might seem problematic. For, certain pictures do not appear to meet the coherence condition, but they appear to depict an object that could exist in three-dimensional space. I have in mind pictures that "flip" on us. That is, at one moment we see the picture in one way and then the next moment the picture looks completely different. The classic example is given in Figure 4. Figure 4 does not appear to meet the coherence condition because if we wait for the figure to invert, we can see point *A* as in front of point *B*, and we can also see point *B* as in front of point *A*. (Figure 5 makes the two views more apparent.) As a result, the geometrical

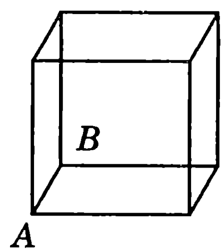

Figure 4

Figure 5

information contained in the picture appears to be contradictory. So, it appears that there is no map from this picture to three-dimensional space that preserves the geometrical information in the picture. But Figure 4 appears to depict a cube and so appears to depict an object that could exist in three-dimensional space.

I believe that the difficulty here is only apparent. The problem with this "counter example" is that it is incorrect to say that Figure 4 depicts, in my sense, a single object. We determine what object a picture depicts on the basis of how the picture appears to us. Because the picture looks some way to us, we can obtain a profile from the picture. In the case of Figure 4, however, we cannot generate the needed profile because the picture looks one way at one moment and then looks an entirely different way at another moment. We have two entirely different experiences of the picture and all its part. The best we can do, then, is set up two profiles for the picture. It thus is incorrect to say that the picture depicts a single three-dimensional object.

This example illustrates that there are a variety of pictures that depict neither a coherent spatial configuration nor an incoherent one. This occurs because in some cases it is either indeterminate what object a picture depicts. We thus cannot sort pictures into two piles: pictures that represents coherent spatial configurations and pictures that represent incoherent spatial configurations. We need a third pile for undecided cases and cases where the picture does not depict a single object.

So far, then, I have explained what coherence consists in. But my characterization of coherence is still incomplete. My arguments in the previous section suggest that we can detect whether a picture depicts a coherent spatial configuration simply by looking at the picture, but the coherence condition does not make it clear how we determine, by inspecting the picture, that a picture represents a coherent spatial configuration. For example, the coherence condition says that Figure 2 represents a coherent spatial arrangement just in case the two-dimensional picture can be naturally mapped into three-dimensional space in such a way that the resulting object consists of four connected bars, one of which contains four blocks, and so forth. But how do we determine whether Figure 2 satisfies this condition? Must we resort to some fancy proof or computer aided mapping program? Must we have sophisticated mathematical knowledge to determine whether the coherence condition is satisfied?[14] Or is it possible to detect whether a picture depicts a coherent spatial configuration simply by looking at the picture?

I believe that visual inspection of a picture is sufficient and suggest the following test for coherence:

Coherence Test: A picture passes the coherence test if and only if we cannot generate contradictory profile when looking at different parts of the picture as well as the whole picture.

I have already tacitly appealed to this test when discussing Figure 2. There, I argued that this figure does not represent a coherent spatial configuration because we form a contradictory profile when looking at the different parts of the picture as well as the whole picture. When we look at the right corner, the bottom beam looks as if it is in the background. When we look at the left corner, the bottom beam looks as if it is in the foreground. We thus expect that these two sides cannot meet. However, when looking at the whole picture, the bottom beam appears as a single bar.

A few comments are in order about the suggested coherence test. Recall that a profile generated from a picture does not contain, for every feature we see the picture as having, a claim that describes this feature. For example, consider Figure 6. If we cover up

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[14] The stakes in answering this question are high. Ultimately I want to use the picture method to defend modal structuralism, a position that attempts to eliminate mathematical objects. If defending modal structuralism requires knowledge of mathematical claims, then the argument for it would be circular. However, if determining whether a picture represents a coherence spatial configuration does not require mathematical knowledge, then the defense of modal structuralism is not circular. For, even though the coherence condition appeals to mathematical concepts, the defense of modal structuralism will not be circular as long determining whether something passes the coherence test does not require mathematical knowledge.

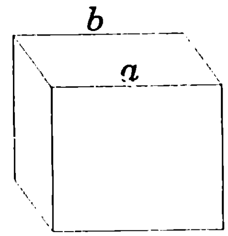

Figure 6

every part of the picture except the middle sections of line *a* and line *b*, then we see line *a* and line *b* as on the same plane. One might worry that this leads us to construct a contradictory profile for the picture. However, since focusing on only a small portion of the picture destroys the illusion of depth, conventions of representation tell us that we should not include, in a profile, the claim that line *a* is *not* in front of line *b*. When constructing a profile from a picture, we consider only the representational features of the picture, i.e. the features that conventions of representation deem representational.

My next comment about the coherence test concerns how the coherence condition and the coherence test are related. In particular, I need to show that the coherence test is a test for coherence.

To show that the coherence test is a test for coherence, I first need to establish that if a picture represents a coherent spatial configuration, then our test indicates this and, second, that if our test indicates that the picture depicts a coherent spatial configuration, then the picture depicts such a configuration. To establish the first claim, notice that if the picture can be naturally mapped into three dimensional space in such a way that the resulting object fits the maximal profile generated from the picture, then the picture and the depicted object are geometrically similar. In particular, how the picture appears geometrically is how the depicted object is. Because coherence guarantees that the depicted object could exist in three-dimensional space, the depicted object has non-contradictory properties. Since how the picture appears geometrically is how the depicted object is, it follows that we cannot generate a contradictory profile when we look at the picture. Accordingly, the picture passes the coherence test.

We now need to show that if a picture passes the coherence test, then the picture represents a coherent spatial configuration. We know that if a picture passes the coherence test, we cannot generate a contradictory profile from the picture. So, the maximal profile generated from the picture must be non-contradictory. Because it is non-contradictory, we can reproduce the features listed in the maximal profile on a three-dimensional graph. Accordingly, the depicted object could exist in three-dimensional space.

But what guarantees that there is a natural map from the picture to the depicted object? To answer this question, notice that I have characterized the notion of depiction so that the geometrical properties we see the picture as having are similar to the geometrical properties that the depicted object has. In particular, if we see two points in the picture as a certain distance apart and at a certain position relative to each other, then the depicted object contains corresponding points which are at a proportional distance apart and similar position relative to one another. Accordingly, there is a natural map from the picture to the depicted object. So, the picture represents a coherence spatial configuration.

Now that I have discussed both the coherence condition and the coherence test, I can clearly indicate the kind of support that coherence provides for possibility. If a picture represents a coherent spatial configuration, then we have indisputable grounds for concluding that the depicted object could exist in three-dimensional space. Similarly, if a picture passes the coherence test, then we have indisputable grounds for claiming that the picture represents a coherent spatial configuration, and so we have indisputable grounds for concluding that the depicted object could exist in three-dimensional space.

However, it is important to notice that our route for establishing possibility requires us to *provide evidence* that a picture passes the coherence test. Ultimately, then, our reason for holding that a picture depicts an object that could exist in three-dimensional space comes from evidence that this picture passes the coherence test.

But, what kind of evidence do we have for the claim that a picture passes the coherence test? Well, we need to look at the whole picture as well as its parts and then try to determine whether we can form a contradictory profile from the picture. The key point to notice, here, is that in making this determination, we are fallible. As I stated earlier, the content of an experience of a picture contains claims that are purportedly about a three-dimensional object and claims that are about a two-dimensional object. Using rules of representation, we extract from the content of our experience claims that purport to be about a three-dimensional object. In performing this extraction, our inattention might lead us to form a non-contradictory profile from the picture even though it is possible to form a contradictory one. Because we are fallible in assessing whether a picture passes the coherence test, the picture method allows us to provide defeasible justification of possibility.

Overall, then, I have suggested a way to use experiences of pictures to support possibility claims. If we do not form a contradictory profile when looking at a picture, then we have reason to believe that the picture represents a coherent spatial arrangement. As a result, we have reason to believe that the depicted object could exist in three-dimensional space.

V. Pictorial Evidence that an ω-Sequence Could Exist

In the rest of this essay, I will apply the machin ry I have developed to the case of an infinite sequence. I will argue that by using this machinery, we can provide grounds for (4), i.e., the claim that an ω-sequence of concrete objects could exist. By providing grounds for this claim, I offer a defense of modal structuralism.

To show (4), I will argue that a particular kind of infinite sequence could exist, in particular a *Dedekind infinite* sequence, where a set of objects X forms a Dedekind infinite sequence if there is a one-one function f that maps X onto all, except for one member, of X. Arguing that a Dedekind infinite sequence could exist provides a route for establishing (4), even though not all Dedekind infinite sequences are ω-sequences.[15] The reason is that we can apply a theorem of Richard Dedekind to extract an ω-sequence from a Dedekind infinite sequence. Dedekind appeals to second-order logic to show that if a Dedekind infinite sequence exists then it contains an ω-sequence as a proper part.[16] The injective function on the Dedekind infinite sequence supplies us with the "successor" function for the ω-sequence. In particular, the ancestral of this injective function is the needed "successor" function for the ω-sequence. So, if we know that a Dedekind infinite sequence of concrete object could exist, by using second order logic we can show that an ω-sequence of those concrete objects could exist. In what follows, then, my task will be to argue that we have grounds for believing that a particular Dedekind infinite sequence could exist.

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[15] A Dedekind infinite sequence may contain members that follow infinitely many others

[16] Richard Dedekind, *Essays on the Theory of Numbers*. Dover Publications, NY, 1963. p. 68. (Theorem 72)

The particular Dedekind infinite sequence I am interested in is a Dedekind infinite sequence of rectangles, the description of which is as follows. The sequence of rectangles contains rectangles that are of the same size. In this sequence, there is a first rectangle, and two feet after this rectangle, there is another rectangle which is parallel to the first; and, two feet after this second rectangle is another rectangle, and so forth. In general, for any rectangle r in this sequence there is another rectangle r' that follows two feet after r and that is different from any rectangles lying behind r.[17]

Have we described a possible situation? As with the story about the man's journey around four connected bars, it is difficult to determine whether the description forms a coherent whole. To make this determination, I suggest that we apply the picture method to obtain the needed evidence. To use this method, we need to produce a picture and argue that not only is this picture a picture of a Dedekind infinite sequence but also this picture passes the coherence test. If we can do this, we obtain evidence that the sequence could exist.

Off hand, however, this suggestion does not seem promising. The problem is that it does not appear that we can produce a picture of a Dedekind infinite sequence. For, a picture depicts a Dedekind infinite sequence only if a Dedekind infinite sequence satisfies the maximal profile generated from the picture and no finite sequence satisfies the maximal profile. But how can we produce a picture that generates such a profile? Any picture we draw has only finitely many rectangles. So, it appears that the maximal profile will contain the claim that the sequence is finite, and so only finite sequences will satisfy it. Accordingly, it appears that we can produce only a picture that depicts a finite sequence.

[17] Although it may not be obvious from the description, this sequence is a Dedekind infinite sequence. The function that plays the role of the one-one function is the function that takes each rectangle to the rectangle directly following it.

I believe that we can produce a picture of an infinite sequence as opposed to a finite sequence. I will argue that we can produce a picture of an infinite sequence because we can produce a picture that is such that, when we look at it, we have an *illusion of an infinite sequence*, and consequently, we must include in the profile generated from this picture a claim that only a Dedekind infinite sequence can satisfy.

To make my case, first I will remind the reader how it is that we have an illusion of the infinite when we look at certain pictures. I will then explain why this ensures that only a Dedekind infinite sequence satisfies the maximal profile generated from this picture.

The picture that provides us with this illusionary effect is Figure 7. As I stated in the Chapter 2, in saying that we have an illusion of an infinite sequence when we look at Figure 7, I do not mean that we actually see the figure as containing infinitely many rectangles. Rather, we have an illusion of the infinite in that we see the figure as having the

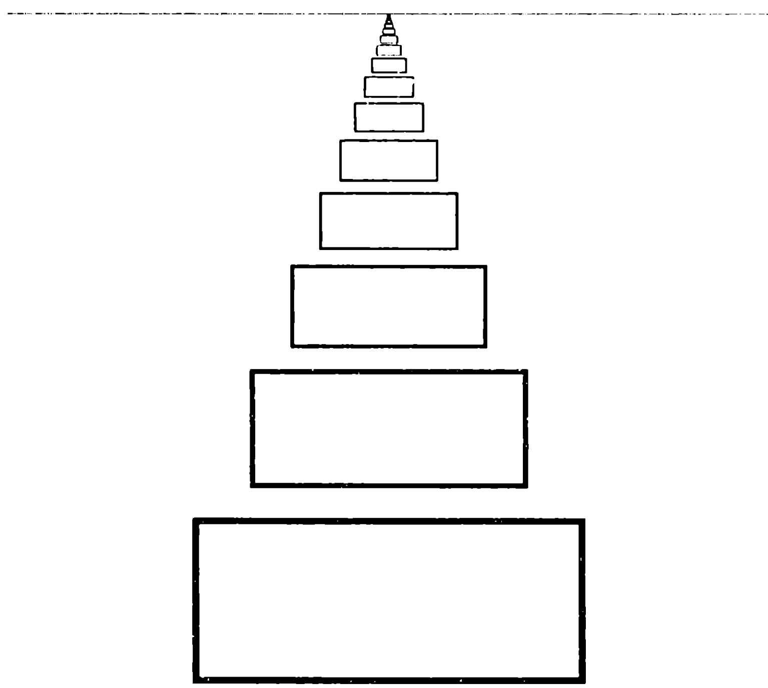

Figure 7

property: *containing a shrunken duplicate of itself*. This is an illusion of the infinite because if Figure 7 did have that property, it would contain infinitely many rectangles.

To see that we have an illusion of the infinite, I have placed Figure 7 above another figure, Figure 8. Figure 8 contains an exact duplicate of the sequence obtained from

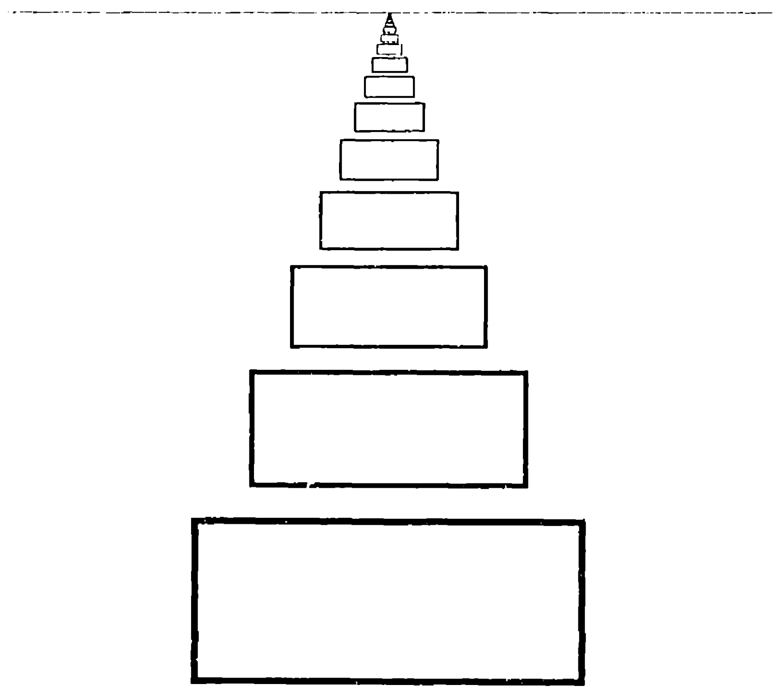

Figure 7

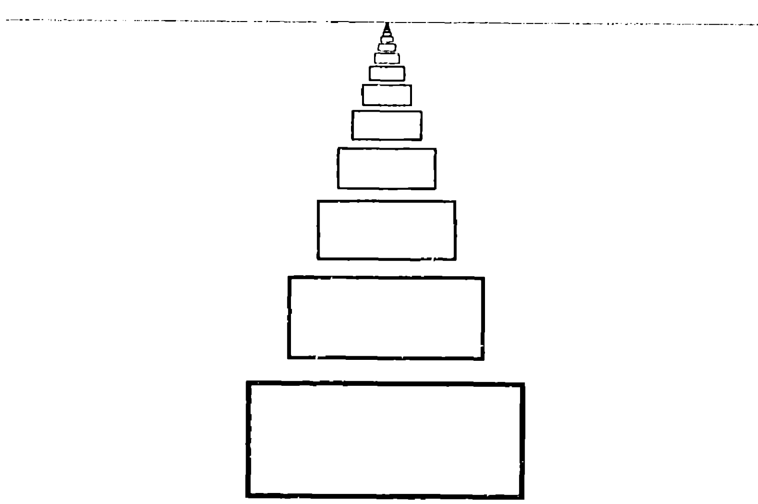

Figure 8

Figure 7 when the first rectangle in Figure 7 is deleted. As I discussed in the Chapter 2, I contend that by inspecting Figure 7 and Figure 8, we can verify that the following hold:

(5) We see an exact duplicate of Figure 8 as a proper part of Figure 7.

(6) We see Figure 8 as a shrunken duplicate of all of Figure 7.

And if both (5) and (6) are true, we represent Figure 7 as having the property, *containing a shrunken duplicate of itself.*

I contend that in seeing Figure 7 as containing a shrunken duplicate, we have a perceptual illusion of an infinite sequence. For, if Figure 7 did contain a shrunken duplicate, then it would contain infinitely many rectangles. To show this, I will argue that if Figure 7 contains finitely many rectangles, Figure 7 does not contain a shrunken duplicate. So, suppose that the number of rectangles in Figure 7 is finite. Let us call the sequence that is obtained by deleting the first rectangle in Figure 7, MiniFigure 7. Since Figure 7 contains only a finite number of rectangles, say n, MiniFigure 7 contains one less rectangle and so contains n-1 rectangles. MiniFigure 7, then, is not a shrunken duplicate of Figure 7. Accordingly, if Figure 7 contains a shrunken duplicate as a proper part, then Figure 7 contains infinitely many rectangles. So, in seeing Figure 7 as containing a shrunken duplicate as a proper part, we have an illusion of an infinite sequence.

So far, I have argued that we have an illusion of an infinite sequence when we see Figure 7. Establishing this claim enables us to show that Figure 7 is a picture of a Dedekind infinite sequence of rectangles. As I said previously, Figure 7 depicts a Dedekind infinite sequence only if a Dedekind infinite sequence satisfies the maximal profile generated from Figure 7 and no finite sequence satisfies this profile. I believe that the maximal profile generated for Figure 7 includes the claim that the sequence contains a shrunken duplicate. Indeed, I have argued that we see Figure 7 as containing a shrunken duplicate. Furthermore, I believe that conventions of representation tell us that the claim that an object contains a shrunken duplicate is part of the maximal profile generated from this picture. Indeed, the property *containing a shrunken duplicate* is not a property that is characteristic of pictures, such as the properties of *having a frame*, *drawn on a two-dimensional canvas*, and *having the label* 'Figure 7'. Furthermore, in seeing the picture as containing a shrunken duplicate, we preserve the illusion of depth in the picture. So, conventions appear to indicate that the maximal profile generated from Figure 7 includes the claim that the sequence contains a shrunken duplicate. As a result, the sequence that satisfies the maximal profile must be a Dedekind infinite sequence, and so Figure 7 depicts such a sequence.

Now that we have produced a picture of an infinite sequence, we are close to the goal of this section, namely showing that a Dedekind infinite sequence of rectangles could exist. The final step is to show that this picture passes the coherence test. That is, we must show we do not form a contradictory profile when looking at different parts of Figure 7 as well as the whole picture.

Off hand, one might think that such a task is destined to fail. One might think that when we look at Figure 7 we form a *contradictory* profile. In particular, on the basis of seeing the picture, we include in the profile the claim that the sequence contains a shrunken duplicate. But also, since we see the ink blot near the horizon line, it appears that we also include in the profile the claim that the sequence contains only finitely many rectangles. So, it appears we form a contradictory profile.

To see what is wrong with this argument, we must remember that not every feature of the picture corresponds to a claim in the profile. Indeed, conventions of representation tell us that some features of Figure 7 are not representational. In particular, conventions tell us that we should not include in the profile the claim that the sequence contains some rectangles followed by a large blotch. For one thing, when we look at the part of the picture that just contains the ink blot, we destroy the illusion of depth in the picture. So, conventions of representation tell us that we should not include in a profile the claim that there is a blot on the horizon. Furthermore, the feature of the ink's coalescing at one point is like the feature of having a frame or of being drawn on paper. It is unique to the picture. The picture has this feature by dint of being a picture, and so conventional of representation dictate that the feature is not representational.

The observations made in the above paragraph are similar to observations we made about Figure 6, the picture of the cube. We noted that this figure has certain nonrepresentational features. The feature *drawn on paper* is such a nonrepresentational feature of Figure 6. On the other hand, the feature *line a being in front of line b* is a representational feature. Conventions of representation tell us which features are representational and which are not representational. Once we apply conventions of representation, we can see that it is incorrect to include in the maximal profile generated from Figure 7 the claim that the depicted sequence is finite.

With these clarification, we can now explain why Figure 7 passes the coherence test. When we look at Figure 7 and its different parts, we form a non-contradictory profile. When we look at the whole picture, we include in the profile the claim that sequence contains a shrunken duplicate. And, looking at the parts of the picture does not require us to include a claim in the profile that contradicts this claim about the shrunken duplicate. When we look at the last few rectangles, we do not add to the profile a claim contradicting the claim that the sequence contains a shrunken duplicate. When we look at the first few rectangles, we include in the profile claims about the shape and position of the first few rectangles, claims that do not contradict the claim that the sequence contains a shrunken duplicate. By producing a non-contradictory profile, we obtain evidence that this picture passes the coherence test.

We have thus shown that we have ground for the claim that a Dedekind infinite sequence could exist. We have reason to believe that Figure 7 is a picture of a Dedekind infinite sequence. Furthermore, we have reason to believe that this picture passes the coherence test. Accordingly, our experience of Figure 7 give us grounds for the claim that Dedekind infinitely many concrete objects could exist in three-dimensional space.

As I noted at the beginning of this section, showing that we have evidence that Dedekind infinitely many concrete objects could exist is one short step from showing that we have evidence that an ω-sequence of concrete objects could exist. By appealing to second-order logic, we can extract an ω-sequence from a Dedekind infinite sequence.

VI. But It's an Illusion

At this point the reader might have certain concerns about the fact that I have used an illusion to provide grounds for a modal claim. In Chapter 2 I already addressed certain worries about using an illusion as a basis for knowledge. In particular, I argued that even though an illusion is a misrepresentation of the way the world is, this does not obviously bar it from providing us with modal knowledge. I also pointed out that we cannot discount the possibility that this illusion provides us with modal knowledge on the basis that our mind manufactures the illusion, creating something that is not really there. For, as I argued, the arrangement of lines in Figure 7 causes the picture to look a certain way to us. We do not add properties to the picture.

Now that I have described the picture method, however, one might have specific concerns about my use of the illusion of the infinite. For instance, one might worry that because the illusion of the infinite is in our head and does not correspond to some real feature of the picture, showing that the picture passes the coherence test does not appear to provide grounds for the claim that an infinite sequence could exist. The problem is that the coherence test is supposed to give us evidence that we can map Figure 7 into three-dimensional space so as to preserve the geometrical features of Figure 7. But it appears that we can preserve these features of Figure 7 by mapping Figure 7 to a finite sequence in three-dimensional space. For, our experience of a finite sequence is similar to our experience of Figure 7.

This argument misstates the coherence condition. The coherence condition requires that the mapping from Figure 7 to a three-dimensional configuration is such that the resulting configuration fits the maximal profile generated from Figure 7. But what claims are in the maximal profile from Figure 7? I have argued that the claim that a sequence contains a shrunken duplicate is part of the maximal profile generated from Figure 7. So, the required mapping cannot map Figure 7 to a finite sequence. For, a finite sequence does not satisfy the maximal profile generated from Figure 7.

That the coherence condition accommodates illusory features of a picture should be no surprise. All perspective pictures have illusory features. These pictures appear to have three-dimensional properties, but they are two-dimensional. The coherence condition requires that such *illusory* features of the picture are *real* features of the depicted object. Indeed, the mapping from a two-dimensional picture to a three-dimensional configuration is such that the resulting object not only appears to fit but actually fits the maximal profile generated from the picture.

We should note, however, that the illusion of the infinite differs from the illusion of depth that is produced in the picture of a cube, for instance. In particular, certain three-dimensional finite sequences appear to contain a shrunken duplicate but, for the most part, no other object besides a cube appears as a cube. This difference between the illusion of the infinite and other illusory aspects of a picture, however, turns out to be unimportant. For, as I have argued, conventions dictate that the feature *containing a shrunken duplicate* is a representational feature of the picture. So, just as a picture "of" a cube depicts an object that has six sides which meet at ninety degree angles, Figure 6 depicts an object that contains a shrunken duplicate. So, Figure 6 depicts a sequence that not only appears to contain a shrunken duplicate but actually contains one.

VII. An Epistemological Puzzle for Modal Structuralism

So far, I have argued that the modal structuralist can provide a defense of (4), i.e., the claim that there could exist an ω-sequence of concrete objects. My argument, however, is contrary to Bob Hale's conclusion in his recent article "Structuralism's Unpaid Epistemological Debts." There, Hale argues that it appears to be impossible for the modal structuralist to provide credible support for (3), i.e., the claim that an ω-sequence of objects could exist. The problem is that to show (3), the modal structuralist must establish

(4).[18] But since Hale believes there appears to be no way for a modal structuralist to establish (4), he concludes that modal structuralism generates a special epistemological puzzle that seems just as tough as the epistemological puzzle that platonism generates. In this section, I will discuss why Hale believes that it is impossible to provide grounds for (4). In the next section I will explain why his argument does not undermine my argument for (4). I also will point out specific difficulties with his argument against modal structuralism.

Hale claims that the only way a modal structuralist can argue for (4) is by using conceivability as a grounds for possibility, where the notion of conceivability is explicated in terms of imaginability.

An initial thought in providing such an explication is simply to say that *P* is conceivable if *P* is imaginable. This characterization seems to work in some cases. For, example, I take it that I can imagine that ten of my books are stacked in a pile on my living room floor, and so this proposition is conceivable. But others propositions are not amenable to this treatment. For example, can we imagine that someone's mind is distinct

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[18] Hale contends that it is not plausible for a modal structuralist to defend the claim that an ω-sequence of abstract objects could exist. For to defend this claim, the modal structuralist must show either that an actual ω-sequence of abstract objects exists or that if there is no actual ω-sequence of abstract objects, then there could be such a sequence. None of these options, however, is acceptable to the modal structuralist. According to Hale, the first option is unacceptable because it requires the modal structuralist to reintroduce abstract objects and thus the corresponding epistemological difficulties. (Note that whether there is an epistemological difficulty depends on the type of abstract object introduced. For instance, if the abstract objects are "quasi-concrete" objects, i.e., if the abstract objects "are determined by intrinsic relations to concrete objects," (Charles Parsons, *Mathematics in Philosophy*. Cornell University Press, New York, 1983. p. 25.), then we might be able to provide an explanation of our knowledge of and reference to such objects. In particular, perhaps we come to know about quasi-concrete objects by examining the concrete objects to which they are intrinsically related.) The second option also appears to be unacceptable because, to pursue it, the modal structuralist must provide an explanation as to why it is a *contingent* fact that an ω-sequence of abstract objects does not exist, but Hale contends that there appears to be no explanation of this *contingency*. (See Bob Hale and Crispin Wright, "*A reductio ad surdum*—Field on the Contingency of Mathematical Objects." *Mind* 103 (1994). pp 169-84.) So, Hale concludes that the modal structuralist must defend (4).

from his body? It is hard to know. For, how do we determine whether we have succeeded in imagining such a thing? How can we settle any kind of dispute as to whether this claim is conceivable? So, although this characterization of conceivability enables us to verify that certain claims are possible, it is no help in settling disputed cases.

One method of settling disputes is to attempt to establish the conceivability of *P* by appeal to the imaginability of some other situation which we characterize using a description that is not simply a rewording of a sentence expressing the proposition *P*. The idea, then, is to convince someone of *P*'s possibility by describing a situation without employing a sentence that expresses *P*. For example, we might try to convince someone that John could be in pain by describing a situation in which John falls off a building and subsequently cries and screams. The problem with this suggestion, however, is that when we produce such a description, it is unclear why we should take the imaginability of the situation described as evidence for *P*'s conceivability.

Clearly, the situation we describe must be related to the proposition that *P* in some way. But exactly what counts as the appropriate way?

Hale ultimately offers the following answer: "...to conceive of *P*'s being the case is to imagine a situation which distinctively favours the hypothesis *P*...",[19] or similarly to conceive of *P*'s being the case is to imagine a situation "of which it would be unreasonable, in the absence of further data, not to believe that, were it to obtain, it would be the case that *P*."[20,21] This account emphasize the fact that to conceive of *P*, we need to

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[19] Hale, "Structuralism's Unpaid Epistemological Debts," p. 139.

[20] Ibid., p. 139.

[21] Hale relies heavily on Stephen Yablo's discussion in "Is Conceivability a Guide to Possibility?". Stephen Yablo, "Is Conceivability a Guide to Possibility?" *Philosophy and Phenomenological Research* 53 (1993). pp 1-42. Cf. Yablo who writes, "conceiving of *P* is a way of imagining that *P*; it is imagining a world of which *P* is held to be a true description," or similarly, imagining "a world that I take to verify that *P*." Yablo, "Is Conceivability a Guide to Possibility?" p. 27.

not only describe a situation without reference to *P* but also convince ourselves that the situation described is one *of which* we justifiably believe that *P* is true.[22] This account enables us to see why describing John's falling and screaming is a way of convincing someone that John could be in pain. The situation in which John falls is described without employing the word 'pain' or a cognate, and it is a situation of which we justifiably believe that John is in pain. So, as long as one agrees that we can indeed imagine a situation in which John falls and screams, then we have reason to believe that we can conceive that John is in pain.

To show that the modal structuralist cannot defend (4), Hale "conveniently" divides his argument into two parts.[23] He argues for the following two claims:

(7) We cannot conceive that an infinite sequence of concrete objects is constructed.

(8) We cannot conceive that such a sequence exists independently of construction.[24]

Hale argues for (7) by relying on the following key premise:[25]

(9) If it is imaginable that someone constructs an infinite sequence of concrete objects, then it should be possible to imagine a situation in which that person has evidence that this construction has been performed.

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[22] One virtue of this treatment is that it leaves room for undecided cases, cases where we cannot determine whether *P* is conceivable or inconceivable. For, we might not be able to imagine either a situation that distinctively favors *P* or a situation that distinctively favors not *P*.

[23] Although Hale says that is *convenient* to divide the argument into these two parts, he never explains why he proceeds as he does.

[24] As I noted in footnote 22, one virtue of Hale's explication of conceivability is that it leaves room for undecided cases, cases where we cannot determine whether *P* is conceivable or inconceivable. Hale takes advantage of this aspect of his account. He does not argue that it is *inconceivable* that an infinite sequence exists. Rather he argues that it is *not conceivable* that such a sequence exists.

[25] For discussions about whether constructing an infinite sequence is possible see Paul Benacerraf, "Tasks, Super-tasks and Modern Eleatics." *The Journal of Philosophy* 59 (1962). pp. 765-784. Also see James Thomson, "Tasks and Super-tasks." *Analysis* 15 (1954). pp. 1-13.

As Hale writes,

> So, if it is possible that Hercules has performed a supertask [i.e., a task consisting of infinitely many sub tasks], it should be possible to envisage a situation in which he would have adequate grounds for believing himself to have succeeded.[26]

He accepts this principle because the construction of an infinite sequence of concrete objects is a *physical* task, and he thinks that we should, in principle, be able to come up with empirical evidence that a physical task has been accomplished.

Because I will challenge (9), it is not necessary to go into extensive detail about the rest of Hale's argument for (7). Suffice it to say that he argues that to imagine a situation in which we would have empirical evidence that an infinite sequence has been constructed, we would need to imagine a situation in which we would have evidence that an infinite sequence exists *independently* of any construction. So, he contends that to argue that an infinite sequence could exist, the modal structuralist must ultimately argue that we can conceive that an infinite sequence exists independently of construction.

According to Hale, though, it is also not promising to argue that we can conceive that an infinite sequence of concrete objects exists independently of construction. He claims that to conceive that an infinite sequence exists independently of any construction, we "must supply a description—necessarily finite—of a possible situation, no empirically adequate theoretical account of which could avoid postulating the existence of a completed ω-sequence."[27] Since the description of the situation in question will be of only finitely many observable pieces of evidence, he contends that "It beggars belief that any such

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[26] Hale, "Structuralism's Unpaid Epistemological Debts," p. 142.

[27] Ibid., p. 145. Note that the description cannot be simply a rewording of the claim that an ω-sequence exists. Rather, the description is of certain observable phenomena that might support this claim.

description could rule out all but theories which postulate the existence of infinitely many concrete objects..."[28] and so concludes that (8) holds, i.e., that we cannot conceive that an ω-sequence exists independently of any construction.

Overall, then, Hale claims that although modal structuralism appears to resolve one of the platonist's most perplexing epistemological puzzle, it seems to generate a quite difficult epistemological puzzle of its own.

VIII. Why Hale's Argument Fails

If my argument in support of modal structuralism is correct, then something must be wrong with Hale's argument. For, I have argued that we have grounds for the claim that infinitely many concrete objects could exist. But Hale concludes that the modal structuralist cannot establish this modal claim.

To determine how Hale's argument fails, we must determine whether he believes that the picture method supplies us with a way of conceiving that certain objects could exists.

Given that Hale assumes that to conceive that an infinite sequence exists, we must "must supply a description—necessarily finite—of a possible situation, no empirically adequate theoretical account of which could avoid postulating the existence of a completed ω-sequence," it appears that he does not consider the picture method as a way of conceiving that an infinite sequence exists. For, if we use the picture method, we need not supply a description of observable evidence and show that no empirically adequate theory could avoid postulating the existence of a completed ω-sequence. Rather, we simply argue that we can produce a picture of an infinite sequence and that this picture passes the

[28] Hale, "Structuralism's Unpaid Epistemological Debts," p. 145.

coherence test. It thus appears that Hale does not classify the picture method as a way of conceiving that an object exists. But if this is the case, then none of his arguments undermines what I have said in support of the claim that an infinite sequence could exist. So, Hale fails to show that a modal structuralist cannot establish that an infinite sequence of concrete objects could exist. For, Hale has not considered all the ways the modal structuralist might provide evidence for this modal claim. The notion of conceivability he discusses provides one route to possibility. But there are others.

Hale's argument faces further difficulties besides this one. As we have seen, he thinks that modal structuralism faces a special epistemological difficulty because he believes that both (7) and (8) hold. I contend that even if we do not consider the picture method as a way of establishing possibility claims, Hale fails to establish either of these claims.

As we have seen, the crucial assumption in arguing for (7) is as follows:

(9) If it is imaginable that someone constructs an infinite sequence of concrete objects, then it should be possible to imagine a situation in which that person has evidence that this construction has been performed.

The general principle upon which (9) seems to rely is as follows:

(10) If it is imaginable that someone performs a physical task, then it should be possible to imagine a situation in which that person has evidence that this task has been performed.

This general principle, however, is false. For, it is possible to perform a task that is such that no one has evidence that the task has been performed. For example, suppose there is a bucket full of marbles, and suppose that John, who is alone and wears gloves, closes his eyes, grabs a marble, returns the marble to the bucket, and then mixes up the marbles. John has thereby succeeded in performing the task T, where T is the task of choosing a *particular* marble m and making sure that no one has evidence that m has been chosen. Although we can imagine a situation in which John performs this task, we cannot imagine a situation in which John performs this task and someone has evidence that he has performed this task. For, if John performs T, then no one has evidence that T is performed. So, since performing some physical task requires one to destroy all evidence that the task is performed, (10) is false.

As for the second part of Hale's argument, we have seen that he believes that, with respect to a description of observable phenomena that might serve as evidence that infinitely many concrete objects exist,

> It beggars belief that any such description could rule out all but theories which postulate the existence of completed concrete ω-sequences—that the facts it would record, were it true, would in principle defy explanation in terms of a theory which confined itself resolutely to the finite, as far as its concrete ontology goes.[29]

And from this, he concludes that we cannot conceive that infinitely many objects exist independently of any construction. But this restriction on what is required to conceive that an infinite sequence exists is too strong. Indeed, even Hale believes that we can conceive that John, say, is in pain.[30] But surely no matter what pieces of observable evidence we may cite, they do not "rule out all but theories which postulate" that John is in pain nor do they "defy explanation" in terms of theory that says that John is not in pain.

In sum, Hale appears to rely on verificationist premises that many will find implausible. Indeed, a nonverificationist believes that some tasks might be intrinsically or,

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[29] Hale, "Structuralism's Unpaid Epistemological Debts," p. 145.
[30] Ibid., pp. 138-9.

in principle, unverifiable. So, to a nonverificationist, it seems plausible to think that we can imagine that a physical task is performed even though we might never be able to image a situation in which we have evidence that this task is performed. In addition, the default stance of a nonverificationist is that whether P is possibly true should not depend on our ability to verify something. As a result a nonverificationist may take exception to the contention that to establish that P is possible, we need to verify that a situation is such that, if it obtained, P would hold in it.[31]

IX. Summary

Overall, then, I have offered a way to establish possibility claims. I have argued that if we can show that a picture depicts a certain object and if we form a non-contradictory profile of this picture, then we thereby obtain evidence that the depicted object could exist in three-dimensional space. This technique for establishing modal claims is important because the tenability of philosophical positions often rests on modal claims. In particular, the tenability of modal structuralism rests on the claim that an ω-sequence could exist. I have argued that the picture method helps us defend modal structuralism because we can use it to provide evidence for the claim that a Dedekind infinite sequence could exist. I have argued that we can produce a picture of a Dedekind infinite sequence and furthermore that this picture passes the coherence test. In so doing, I have provided evidence for the claim that a Dedekind infinite sequence could exist. To complete the argument in support of modal structuralism, I appealed to a theorem of

[31] Hale points out that the conceivability condition does not equate P's possibility with P's verifiability and so contends that the account does not import an objectionable verificationist element. See Hale, "Structuralism's Unpaid Epistemological Debts," p. 139. However, notice that the account does require that to determine whether P is possible, we must convince ourselves that the situation described is one of which we justifiably believe that P is true. My point is that some nonverificationists might object to this condition as well.

Dedekind which enabled me to construct an ω-sequence from a Dedekind infinite sequence.

Conclusion

I began this thesis by presenting an epistemological puzzle about the infinite: how is it that we have knowledge of infinite mathematical objects if we do not experience anything suitably like "infinite mathematical objects?" In this dissertation I have investigated a response to this puzzle as well as proposed a partial solution to it.

In the first chapter, I argued that Lavine's response to this puzzle does not work. As we have seen, Lavine tries to respond to the puzzle by arguing that the axioms of Fin(ZFC) are true principles about finite sets. I argued that he fails to show this. In particular, I argued that it does not seem possible to show that both Relativized Extensionality and Generalized Zillion are true principles about finite sets. No account of availability or availability functions enabled us to support both principles. Furthermore, when we try to understand the notion of availability function more intuitively, Relativized Extensionality became highly implausible.

In the second and the third chapters I took steps to respond to the epistemological puzzle. In the second chapter I argued that, contrary to what many have assumed, we can have "experiences of the infinite" when we see certain pictures. In particular, we can see sequences in these pictures as containing, as a proper part, a shrunken duplicate, and so when we look at these pictures, we have a *perceptual illusion of an infinite sequence*. In this chapter, I also explained how experiences of pictures help undermine certain assumptions about the relationship between experience and beliefs formed on the basis of experience.

In the third chapter, I set out to show that the experiences discussed in the second chapter provide us with *modal* knowledge of the infinite. Specifically, I set out to show that these experiences provide us with evidence that there could exist infinitely many concrete objects.

To show that we have evidence for this modal claim, I introduced machinery for establishing modal claims. I showed how, in general, we can use pictures to support the contention that the depicted object could exist. I argued that upon seeing a picture, we can provide evidence that a picture represents a "coherent" rather than an "incoherent" spatial configuration. I then argue that if we have evidence that a picture represents a coherent spatial configuration, we thereby obtain evidence that the depicted object could exist. To support these claims, I provided an account of coherence and explained why coherence provides grounds for possibility. I also presented a test of coherence.

I then used the "picture method" to show that by seeing pictures of an infinite sequence we obtain evidence for the modal claim that an infinite sequence could exist. To make my case, I argued that since we a perceptual illusion of the infinite when we look at certain pictures, these pictures represent an infinite sequence. In addition I argued that we have evidence that these pictures represent a coherent spatial configuration. I thus concluded that we have evidence that an infinite sequence could exist.

An added benefit of showing that we have evidence that an infinite sequence could exist is that it helped us provide a defense of a central tenet of modal structuralism. The modal structuralist must defend the claim that an infinite sequence of concrete objects could exist, and, at present, he appears to lack a defense of this central tenet. So, in addition to providing a response to the epistemological puzzle, my argument in the third chapter shores up the modal structuralist position.

Overall, I hope that the arguments in this thesis serve to show that not only is the infinite not as remote from our experience as we initially might believe but also our knowledge of the infinite is not as mysterious as it first seems.

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