Conformal Loop Ensembles and the Gaussian free field

by

Samuel Stewart Watson

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Abstract

The study of two-dimensional statistical physics models leads naturally to the analysis of various conformally invariant mathematical objects, such as the Gaussian free field, the Schramm-Loewner evolution, and the conformal loop ensemble. Just as Brownian motion is a scaling limit of discrete random walks, these objects serve as universal scaling limits of functions or paths associated with the underlying discrete models. We establish a new convergence result for percolation, a well-studied discrete model. We also study random sets of points surrounded by exceptional numbers of conformal loop ensemble loops and establish the existence of a random generalized function describing the nesting of the conformal loop ensemble. Using this framework, we study the relationship between Gaussian free field extrema and nesting extrema of the ensemble of Gaussian free field level loops. Finally, we describe a coupling between the the set of all Gaussian free field level loops and a conformal loop ensemble growth process introduced by Werner and Wu. We prove that the dynamics are determined by the conformal loop ensemble in this coupling, and we use this result to construct a conformally invariant metric space.

Thesis Supervisor: Scott R. Sheffield
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Acknowledgments

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Chapter 1

Percolation

This chapter presents joint work with Dana Mendelson and Asaf Nachmias. It appears verbatim in [38].

1.1 Introduction

Let $\Omega \subset \mathbb{C}$ be a nonempty Jordan domain, and let A, B, C, D be four points on $\partial\Omega$ ordered counter-clockwise. Let P^δ denote the critical site percolation measure on the triangular lattice with mesh size $\delta > 0$, that is, each site in the lattice is independently declared *open* or *closed* with probability $1/2$ each. The Cardy-Smirnov formula [72] states that as $\delta \to 0$, the probability $P^\delta(AB \leftrightarrow CD)$ that there exists a path of open sites in Ω starting at the arc AB and ending at the arc CD converges to a limit that is a conformal invariant of the four-pointed domain (see Figure 1-1). Our main theorem establishes a power law rate for this convergence under mild regularity hypotheses.

Theorem 1.1.1. Let (Ω, A, B, C, D) be a four-pointed Jordan domain bounded by finitely many analytic arcs meeting at positive interior angles. There exists $c > 0$ such that

$$P^\delta(AB \leftrightarrow CD) - \lim_{\delta \to 0} P^\delta(AB \leftrightarrow CD) = O(\delta^c),$$

where the implied constants depend only on (Ω, A, B, C, D).

We prove Theorem 1.1.1 for all $c < 1/6$, with better exponents for certain domains (see Remark 1.2.2).

Schramm posed the problem of improving estimates on percolation arm events (see Problem 3.1 in [71]). In Section 1.6, we obtain the following improvement of the estimate found in [75] for the probability that the origin is connected to $\{z : |z| = R\}$ in the upper half-plane.

Theorem 1.1.2. Let $\{0 \leftrightarrow S_R\}$ denote the event that there exists an open path from the origin to the semicircle S_R of radius R in critical site percolation on the

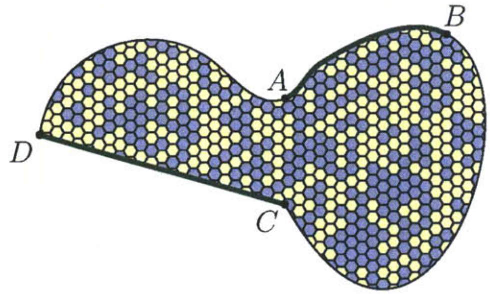

Figure 1-1: We picture triangular site percolation by coloring the faces of the dual hexagonal lattice. Smirnov's theorem states that the probability of a yellow crossing from boundary arc AB to boundary arc CD converges, as the mesh size tends to 0, to a limit which is a conformal invariant of the four-pointed domain (Ω, A, B, C, D). In the sample shown, the yellow crossing event $\{AB \leftrightarrow CD\}$ occurs.

triangular lattice in the half-plane. Then

$$\mathbb{P}(0 \leftrightarrow S_R) = e^{O(\sqrt{\log\log R})} R^{-1/3} = (\log R)^{O(1/\sqrt{\log\log R})} R^{-1/3}.$$

Our methods also yield the estimate $e^{O(\sqrt{\log\log R})} R^{-1/6\beta}$ for the probability that the origin is connected to $\{z : |z| = R\}$ in the sector centered at the origin of angle $2\pi\beta$. We remark that our methods are insufficient to give better estimates for the probability that the origin is connected to $\{z : |z| = R\}$ in the full plane (the so-called *one-arm* exponent, which takes the value $5/48$, [31]) and multiple arm events either in the full or half plane.

In his proof of Cardy's formula, Smirnov constructs a discrete observable $G_\delta : \Omega^\delta \to \mathbb{C}$, defined as a complex linear combination of crossing probabilities, and shows that G_δ converges as $\delta \to 0$ to a conformal map. The crossing probabilities and their limits can be then read off G_δ and its limit. A similar high-level strategy was also used by Smirnov [74] and Chelkak and Smirnov [9] to show that the interfaces of the critical Ising and FK-Ising model converge to SLE curves. See [15] for a comprehensive survey of this subject.

We note that the power law rate of convergence is obtained for the FK-Ising model ([74, 22]) more directly than for percolation, because the combinatorial relations in the Ising model establish that "discrete Cauchy-Riemann" equations hold precisely. In particular, in the case of the Ising model one can work with discrete second derivatives and obtain discrete harmonic functions. By contrast, for percolation the observable G_δ is only known to be approximately analytic. Thus it is necessary to control the global effects of these local deviations from exact analyticity. To accomplish this, we use a Cauchy integral formula with an elliptic function kernel in place of the usual $z \mapsto 1/z$.

The half-plane arm exponent, as well as the validity of Smirnov's theorem is widely believed to be universal in the sense that it should hold for any reasonable two-dimensional lattice. Nevertheless, so far it is an open problem to prove Smirnov's theorem even for the case of bond percolation on the square lattice. The value of the exponent does, however, depend on the dimension. For example, in high dimensions (that is, dimension at least 19 in the usual nearest-neighbor lattice, or dimension at least 6 on lattices which are spread-out enough) its value is -3 [27]. To the best of our knowledge, there are no predictions in dimensions $3, 4, 5$. As for the error terms, in dimension 2 it is believed that the correct bound for $\mathbb{P}(0 \leftrightarrow S_R)$ of Theorem 1.1.2 is $\Theta(R^{-1/3})$ (we are unable to prove this here). In general, it is believed that the polynomial decay should have no logarithmic corrections except for at dimension 6, the upper critical dimension (see [76]).

Finally, we remark that Theorem 1.1.1 has been independently proved by Binder, Chayes, and Lei [6] using different methods. Their approach applies to arbitrary simply connected domains, while our proof achieves explicit exponents for the subclass of piecewise analytic domains (see Remark 1.2.2).

Acknowledgements

We thank Vincent Beffara, Gady Kozma and Steffen Rohde, and Scott Sheffield for helpful discussions. We specifically thank Scott for suggesting the idea to use elliptic functions in the proof of Theorem 1.2.1 and for his help with the proof of Proposition 1.3.6.

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1.2 Set-up and notation

Throughout the paper, we consider piecewise analytic Jordan domains Ω with positive interior angles. That is, $\partial\Omega$ is a Jordan curve which can be written as the concatenation of finitely many analytic arcs $\gamma_1, \ldots, \gamma_N$. Recall that an arc is said to be analytic if it can be realized as the image of a closed subinterval $I \subset \mathbb{R}$ under a real-analytic function from I to $\mathbb{C}$. We will call the point at which two such arcs meet a corner, and we will denote the collection of corners by $\{x_j\}_{j=1,\ldots,N}$. Our hypotheses imply that there is a well-defined interior angle at each corner, and we impose the condition that each such angle lies in $(0, 2\pi]$. We define $\tau := \exp(2\pi i/3)$ and let Ω have three marked boundary points, labeled $x(1)$, $x(\tau)$, and $x(\tau^2)$ in counterclockwise order. We denote the angles at marked points by $2\pi\alpha_j$ and those at unmarked points by $2\pi\beta_i$.

Denote by Ω^δ the sites of the triangular lattice with mesh size δ which are contained in Ω or have a neighbor contained in Ω and consider critical site percolation on Ω^δ. Let $(\Omega^\delta)^*$ be the sites of the hexagonal lattice dual to Ω^δ (that is, $(\Omega^\delta)^*$ are the centers of the triangles of Ω^δ). We depict open and closed sites by coloring the corresponding hexagonal faces yellow and blue, respectively. For $z, z' \in \partial\Omega$, let $[z, z']$ denote the counter-clockwise boundary arc from z to z'. As in [72], the following events play a central role (see Figure 1-2):

$$E^{\delta}_{\tau^k}(z) = \left\{ \begin{array}{c} \exists \text{ a simple open path from } [x(\tau^{k+2}), x(\tau^k)] \text{ to } [x(\tau^k), x(\tau^{k+1})] \\ \text{separating } z \text{ from } [x(\tau^{k+1}), x(\tau^{k+2})] \end{array} \right\},$$

for $k \in \{0, 1, 2\}$. Let $H^{\delta}_{\tau^k} = \mathbb{P}(E^{\delta}_{\tau^k})$ and for z and $z + \eta$ neighbors in $(\Omega^{\delta})^*$, define $P^{\delta}_{\tau^k}(z, \eta) = \mathbb{P}(E^{\delta}_{\tau^k}(z + \eta) \setminus E^{\delta}_{\tau^k}(z))$. Following [3], we define

$$G^{\delta} := H^{\delta}_1 + \tau H^{\delta}_{\tau} + \tau^2 H^{\delta}_{\tau^2}, \qquad S^{\delta} := H^{\delta}_1 + H^{\delta}_{\tau} + H^{\delta}_{\tau^2}.$$

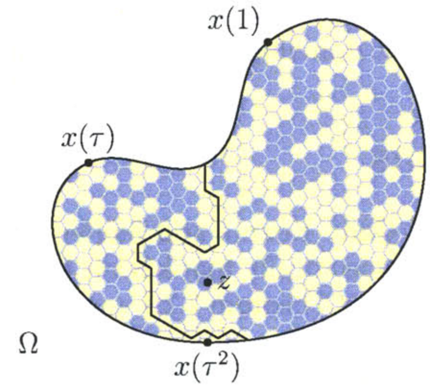


Figure 1-2: The event $E^{\delta}_1(z)$ occurs when there exists a simple open path separating z from $[x(\tau), x(\tau^2)]$.

We extend the domain of G^{δ} from the lattice $(\Omega^{\delta})^*$ to all of Ω by triangulating each hexagonal face and linearly interpolating in each resulting triangle. The possible triangulations for each face are  and  and rotations thereof. We will see that the choice of triangulation is immaterial. We obtain Theorem 1.1.1 as a corollary of the following theorem.

Theorem 1.2.1. Let $(\Omega, x(1), x(\tau), x(\tau^2))$ be a three-pointed, simply connected Jordan domain bounded by finitely many analytic arcs meeting at positive interior angles, and let T be the triangular domain with vertices $1, \tau$, and τ^2. Then there exists $c > 0$ so that $|G^{\delta}(z) - \phi(z)| = O(\delta^c)$, where ϕ is the conformal map from $(\Omega, x(1), x(\tau), x(\tau^2))$ to $(T, 1, \tau, \tau^2)$, and where the implied constants depend only on the three-pointed domain.

Remark 1.2.2. Our methods establish Theorem 1.2.1 (and thus Theorem 1.1.1) for any exponent

$$c < \min_{i,j} \left(\frac{2}{3}, \frac{1}{6\alpha_i}, \frac{1}{2\beta_j} \right). \tag{1.2.1}$$

These exponents are essentially the best possible given our approach, because no piecewise-linear interpolant of a function on a lattice of mesh δ can approximate the conformal map to T with error better than $\delta^{\min_{i,j}(1/6\alpha_i, 1/2\beta_j)}$ due to behavior near the boundary.

Remark 1.2.3. Our proof of Theorem 1.1.1 uses results whose proofs require SLE tools, but only for two purposes: (1) to handle the case where the domain contains reflex angles (that is, some interior angle formed at the intersection of two of the bounding analytic arcs is greater than π), and (2) to obtain the sharp exponent discussed in Remark 1.2.2. Without SLE machinery, we obtain Theorem 1.1.1 for domains without reflex angles and for exponents $c < \min_{i,j}(c_3, 1/6\alpha_i, 1/6\beta_j)$, where c_3 is the three-arm whole-plane exponent (which is known to be 2/3, but only by using an SLE convergence result). See Remark 1.5.3 for further discussion of this point.

Remark 1.2.4. In [75], a bound of $R^{-1/3+o(1)}$ for the half-plane arm exponent was proved using SLE calculations and the fact that the percolation exploration path converges to SLE_6 as proved by Smirnov [72] and Camia-Newman [8]. By contrast, our proof follows from Proposition 1.5.6, which is a variation of Theorem 1.1.1 proved by similar methods. The only SLE result on which our proof of Theorem 1.1.2 depends is the statement $c_3 > 1/3$, where c_3 is the three-arm whole-plane exponent.

For two quantities $f(\delta)$ and $g(\delta)$, we use the usual asymptotic notation $f = O(g)$ to mean that there exist constants C and $\delta_0 > 0$ so that $|f(\delta)| \le C|g(\delta)|$ for all $0 < \delta < \delta_0$. We use the notation $f \lesssim g$ to mean $f = O(g)$ as $\delta \to 0$, and we write $f \asymp g$ to mean $f = O(g)$ and $g = O(f)$. We sometimes use C to denote an arbitrary constant.

1.3 Preliminaries

First we recall some results from [72]. The first is a Hölder norm estimate of H_{τ^k} and is obtained via Russo-Seymour-Welsh estimates.

Lemma 1.3.1 (Lemma 2.2 in [72]). There exist $C, c > 0$ depending only on Ω such that for all $\delta > 0$, the c-Hölder norm of $H^\delta_{\tau^k}$ is bounded above by C. That is,

$$|H^\delta_{\tau^k}(z) - H^\delta_{\tau^k}(z')| \le C|z - z'|^c, \tag{1.3.1}$$

for $\tau^k \in \{1, \tau, \tau^2\}$.

Our second estimate is Smirnov's "color switching" lemma.

Proposition 1.3.2 (Lemma 2.1 in [72]). For every vertex $z \in (\Omega^\delta)^*$ and $k \in \{0, 1, 2\}$, we have

$$P^\delta_{\tau^k}(z, \eta) = P^\delta_{\tau^{k+1}}(z, \tau\eta).$$

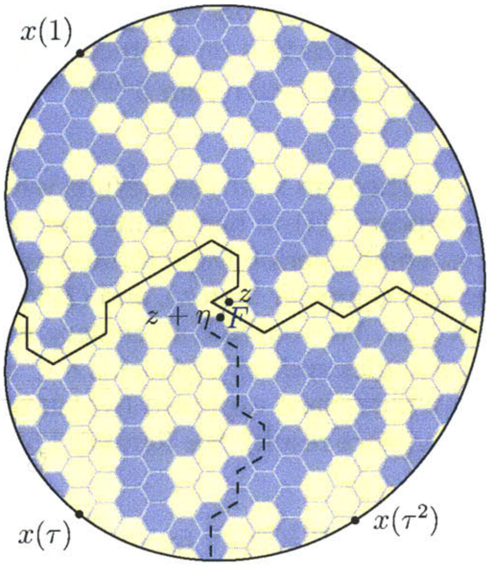


Figure 1-3: The event $E_1(z) \setminus E_1(z+\eta)$ occurs if and only if there are disjoint yellow arms from z to $[x(\tau^2), x(1)]$ and from z to $[x(1), x(\tau)]$ forming a simple path separating z from $[x(\tau), x(\tau^2)]$, as well as a blue arm from $z+\eta$ to $[x(\tau), x(\tau^2)]$ which prevents a yellow path from separating $z+\eta$ as well.

We will sometimes drop the superscript δ from the notation when it's clear from context. If F is a hexagonal face in $(\Omega^\delta)^*$, let $V(F)$ denote the set of vertices of F and define for each $z \in V(F)$ the vector η pointing to the adjacent vertex counterclockwise from z. Define the difference (see Figure 1-4(a))

$$R_k(z) := |P_{\tau^k}(z + \tau^k\eta, -\tau^k\eta) - P_{\tau^k}(z + \tau^{k+1}\eta, -\tau^k\eta)|.$$

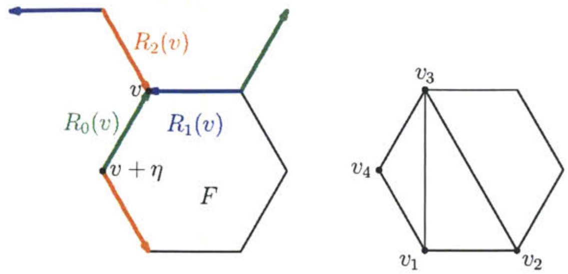
Define $z' = z + \tau\eta - \eta$ and rewrite $P_{\tau^k}(z', \eta)$ as $P^{\Omega'}_{\tau^k}(z, \eta)$, where Ω' is obtained by translating Ω by $z - z'$ (and $P^{\Omega'}$ refers to probability with respect to Ω'). Define the events $E'_{\tau^k}(z)$ with respect to Ω', and define $x'(\tau^k)$ to be $x(\tau^k)$ translated by $z - z'$.

Given $k, l \in \{0,1,2\}$, $\sigma \in \{-1,1\}$, and $z \in (\Omega^\delta)^*$, we say that the event $E^{\text{five arm}}_{\tau^k,\tau^l,\sigma}(z)$ occurs if

- $\sigma = 1$, and $E_{\tau^k}(z) \setminus E_{\tau^k}(z + \tau^k\eta)$ occurs, and the arm from z to $[x(\tau^{l+1}), x(\tau^{l+2})]$ fails to connect in Ω', or
- $\sigma = -1$, and $E'_{\tau^k}(z) \setminus E'_{\tau^k}(z + \tau^k\eta)$ occurs, and the arm from z to $[x'(\tau^{l+1}), x'(\tau^{l+2})]$ fails to connect in Ω.

For $z_0 \in \Omega$, we define $E^{\text{five arm}}_{\tau^k,\tau^l,\sigma}(z)$ to be the union of $E^{\text{five arm}}_{\tau^k,\tau^l,\sigma}(z)$ as z ranges over the vertices of the hexagonal face containing z_0.

Note that these are indeed five-arm events because two additional arms are required to prevent the failed arm from connecting elsewhere on $[x(\tau^l), x(\tau^{l+1})]$



(a) (b)

Figure 1-4: (a) Each arrow represents the probability of a three-arm event as shown in Figure 1-3. The quantity $R_0(z)$ is defined to be the difference between the probabilities represented by the two green arrows. Similarly, $R_1(z)$ is shown in blue and $R_2(z)$ is shown in orange. (b) Suppose that the triangle $z_1z_2z_3$ is in the triangulation of the face F. For z in the interior of this triangle, we bound $\bar{\partial}G^\delta(z)$ by applying (1.3.4) to triangles $z_2z_1z_4$ and $z_1z_4z_3$.

(see Figure 1-5).

Proposition 1.3.3. If F is a hexagonal face in $(\Omega^\delta)^*$, then for z_0 in the interior of F we have

$$\delta|\bar{\partial}G^\delta(z_0)| \leq 3\sqrt{3} \max_{z\in V(F),k\in\{0,1,2\}} R_k(z) \tag{1.3.2}$$

$$\leq 54\sqrt{3} \max_{k,l\in\{0,1,2\},\sigma\in\{-1,1\}} \mathbb{P}(E^{\text{five arm}}_{\tau^k,\tau^l,\sigma}(z_0)). \tag{1.3.3}$$

Proof. The main idea in the following proof is suggested in [72]. For (1.3.2), we first observe that for $z \in V(F)$, we have

$$\begin{aligned}\delta\left[\frac{\partial}{\partial\eta}H_{\tau^k}(z) - \frac{\partial}{\partial(\tau\eta)}H_{\tau^{k+1}}(z)\right] &= P_{\tau^k}(z,\eta) - P_{\tau^k}(z+\eta,-\eta)\\ &\quad - P_{\tau^{k+1}}(z,\tau\eta) + P_{\tau^{k+1}}(z+\tau\eta,-\tau\eta)\\ &= P_{\tau^{k+1}}(z+\tau\eta,-\tau\eta) - P_{\tau^k}(z+\eta,-\eta)\\ &= P_{\tau^k}(z+\tau\eta,-\eta) - P_{\tau^k}(z+\eta,-\eta),\end{aligned}$$

by Proposition 1.3.2. Suppose that the triangle T with vertices z, $z+\eta$, and $z+\tau\eta$ is in the triangulation of F. Then for z in the interior of T, we may write $\delta\bar{\partial}$ as

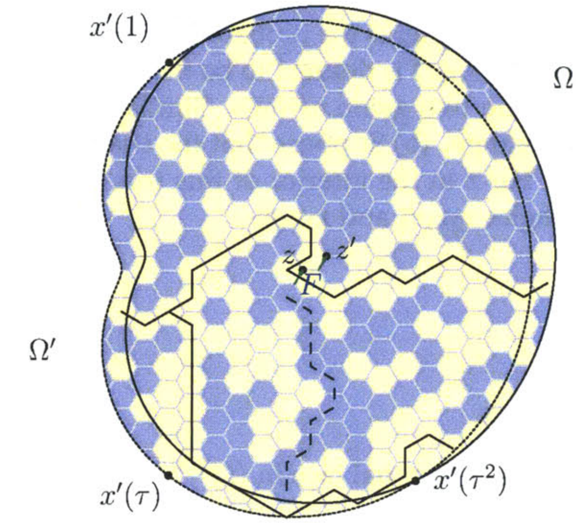


Figure 1-5: The symmetric difference of the events $E_1(z) \setminus E_1(z+\eta)$ and $E'_1(z') \setminus E'_1(z'+\eta)$ can occur in six ways. One way for the event to occur is shown above: the three requisite arms are present in Ω, so the event $E_1(z) \setminus E_1(z+\eta)$ occurs. However, the blue arm fails to connect to $[x'(\tau), x'(\tau^2)]$ in Ω'. This requires two additional yellow arms to prevent the blue arm from connecting elsewhere on $[x'(\tau), x'(\tau^2)]$. This event is denoted $E^{\text{five arm}}_{1,1,1}(z)$. The first subscript τ^k specifies that the three-arm event under consideration involves the blue arm touching down on $[x(\tau^{k+1}), x(\tau^{k+2})]$. The second subscript τ^l indicates that the boundary arc $[x(\tau^{l+1}), x(\tau^{l+2})]$ is involved in a failed connection. The third subscript σ describes whether the failed connection occurs in Ω but not Ω' (in which case we say $\sigma = 1$), or vice versa ($\sigma = -1$).

$\delta\lambda\left(\frac{\partial}{\partial\eta} - \frac{1}{\tau}\frac{\partial}{\partial(\tau\eta)}\right)$, where $\lambda = 1/2 + i/(2\sqrt{3})$. We obtain

$$\begin{aligned}
&\delta\left|\lambda\left(\frac{\partial}{\partial\eta} - \frac{1}{\tau}\ \frac{\partial}{\partial(\tau\eta)}\right)(H_1 + \tau H_\tau + \tau^2 H_{\tau^2})\right| \\
&\quad = |\lambda|\left|\left(\frac{\partial H_1}{\partial\eta} - \frac{\partial H_\tau}{\partial\tau\eta}\right) + \tau\left(\frac{\partial H_\tau}{\partial\tau\eta} - \frac{\partial H_{\tau^2}}{\partial\tau^2\eta}\right) + \tau^2\left(\frac{\partial H_\tau^2}{\partial\tau^2\eta} - \frac{\partial H_1}{\partial\eta}\right)\right| \\
&\quad \le \sqrt{3}\max_{k\in\{0,1,2\}}\left|P_{\tau^k}(z+\tau^k\eta, -\tau^k\eta) - P_{\tau^k}(z+\tau^{k+1}\eta, -\tau^k\eta)\right|. \quad (1.3.4)
\end{aligned}$$

For triangles whose vertices are not consecutive vertices of the hexagon, we obtain a similar bound by applying (1.3.4) two or three times (see Figure 1-4(b)).

For the bound in (1.3.3), we let $A = E_{\tau^k}(z) \setminus E_{\tau^k}(z+\tau^k\eta)$ and $B = E'_{\tau^k}(z) \setminus E'_{\tau^k}(z+\tau^k\eta)$ and apply $|\mathbb{P}(A) - \mathbb{P}(B)| \le \mathbb{P}(A \triangle B)$, where $A \triangle B$ denotes the symmetric difference of A and B. Note that $A \triangle B \subset \bigcup_{k,l,\sigma} E^{\text{five arm}}_{\tau^k,\tau^l,\sigma}(z_0)$, since some arm in Ω must fail to connect in Ω', or vice versa. Applying a union bound as k and l range over $\{0,1,2\}$ and σ ranges over $\{-1,1\}$ yields the result. □

Finally, we need the following a priori estimates for $H_{\tau^k}(z)$ when z is near $\partial\Omega$.

Proposition 1.3.4. Let $(\Omega, x(1), x(\tau), x(\tau^2))$ be a three-pointed Jordan domain. There exists $c > 0$ such that for every $z \in (\Omega_\delta)^*$ which is closer to $[x(\tau^{k+1}), x(\tau^{k+2})]$ than to $\partial\Omega \setminus [x(\tau^{k+1}), x(\tau^{k+2})]$, the following statements hold.
(i) $H_{\tau^k}(z) \lesssim \operatorname{dist}(z, \partial\Omega)^c$.
(ii) $|S(z) - 1| \lesssim \operatorname{dist}(z, \partial\Omega)^c$.
(iii) $\operatorname{dist}(G^\delta(z), [x(\tau^{k+1}), x(\tau^{k+2})]) \lesssim \operatorname{dist}(z, [\tau^{k+1}, \tau^{k+2}])^c$,
with implied constants depending only on $(\Omega, x(1), x(\tau), x(\tau^2))$.

Proof. (i) For $w \in [x(\tau^{k+1}), x(\tau^{k+2})]$, define $D_1(w)$ and $D_2(w)$ to be the distances from w to the boundary arcs $[x(\tau^{k+2}), x(\tau^k)]$ and $[x(\tau^k), x(\tau^{k+1})]$, respectively. Let $D = \inf_{w\in[x(\tau^{k+1}),x(\tau^{k+2})]}\max(D_1(w), D_2(w)) > 0$. Let $z' \in [x(\tau^{k+1}), x(\tau^{k+2})]$ be a closest point to z, and consider the annulus centered at z' with inner radius $|z - z'|$ and outer radius R. Then $E_{\tau^k}(z)$ entails a crossing of this annulus, which has probability $O(|z-z'|^c)$ by Russo-Seymour-Welsh.

(ii) Again let $z' \in [x(\tau^{k+1}), x(\tau^{k+2})]$ be a point nearest to z. Consider the event that there is a yellow crossing from $[x(\tau^{k+2}), x(\tau^k)]$ to $[x(\tau^{k+1}), z']$ and the event that there is a blue crossing from $[x(\tau^k), x(\tau^{k+1})]$ to $[z', x(\tau^{k+2})]$. These events are mutually exclusive, and their union has probability 1. Since these two events have probability $H_{\tau^{k+1}}(z)$ and $H_{\tau^{k+2}}(z)$, we see that

$$H_{\tau^k}(z) + (H_{\tau^{k+1}}(z) + H_{\tau^{k+2}}(z)) = O((\operatorname{dist}(z, \partial\Omega)^c) + 1.$$

(iii) This statement says that G maps points near each boundary arc to the corresponding image segment in the triangle, and it follows directly from (i). □

1.3.1 Percolation Estimates

In this subsection we present several percolation-related estimates in preparation for the proof of Theorem 1.2.1. We think of these lattices as embedded in $\mathbb{R}^2$ with mesh size δ, and distances are measured in the Euclidean metric.

Define $\mathcal{C}_\theta^k(r,R)$ to be the event that there exist k disjoint crossings of alternating colors from the inner to the outer boundary of an annular section $A_\theta(r,R)$ of angle θ and inner radius r and outer radius R. The following is a well-known result on the half-annulus two-arm and three-arm exponents. We refer the reader to [31, Appendix A] for a proof.

Proposition 1.3.5. We have

$$P^\delta(\mathcal{C}_\pi^2(r,R)) \asymp \frac{r}{R}, \text{ and}$$
$$P^\delta(\mathcal{C}_\pi^3(r,R)) \asymp \left(\frac{r}{R}\right)^2.$$

In the next proposition, we show that the exponents in the estimates above are continuous in the angle θ.

Proposition 1.3.6. For all $\varepsilon > 0$, there exists $\alpha = \alpha(\varepsilon) > 0$ so that

$$P^\delta(\mathcal{C}_{\pi+\alpha}^2(r,R)) \lesssim \left(\frac{r}{R}\right)^{1-\varepsilon}, \text{ and} \tag{1.3.5}$$
$$P^\delta(\mathcal{C}_{\pi+\alpha}^3(r,R)) \lesssim \left(\frac{r}{R}\right)^{2-\varepsilon}, \tag{1.3.6}$$

with implied constants depending only on ε.

Proof. We only prove (1.3.5) since the proof of (1.3.6) is essentially the same. We begin by showing that there exists $C > 0$ so that for all $r > 0$ and $R > 0$, there exists $\delta_0 = \delta_0(r,R,\varepsilon) > 0$ for which $P^\delta(\mathcal{C}_{\pi+\alpha}^2(r,R)) \le C\left(\frac{r}{R}\right)^{1-\varepsilon}$ holds when $0 < \delta < \delta_0$. For this statement, we may assume without loss of generality that $R = 1$.

Consider the sector of angle $\pi + \alpha$ as a union of a sector of angle π with a sector of angle α. Divide the sector of angle α into $\lceil (1-r)\alpha^{-1} \rceil$ curvilinear quadrilaterals of radial dimension α, as shown in Figure 1-6. Let $s \in \{1, \ldots, \lceil (1-r)\alpha^{-1} \rceil\}$ and note that the event $\mathcal{C}_{\pi+\alpha}^2(r,1) \setminus \mathcal{C}_\pi^2(r,1)$ entails the existence of a quadrilateral of distance $s\alpha$ from the inner circle of radius r such that there is a three-arm crossing of alternating colors of the half-annulus with inner radius α and outer radius $s\alpha \wedge (1 - r - (s+1)\alpha)$.

In the case $s\alpha \le (1-r)/2$, there is also a two-arm crossing from the annulus of inner radius $s\alpha$ and outer radius $s\alpha + r$ (see Figure 1-6(a)). If $s \in [2^k, 2^{k+1}]$ and $s\alpha \le (1-r)/2$, then the probability that both of these events occur is $O\left(\left(\frac{\alpha}{s\alpha}\right)^2\left(\frac{\alpha}{\alpha s + r}\right)\right) = O(\frac{\alpha}{2^k})$ by Proposition 1.3.5. Applying a union bound over s we obtain

$$P^\delta(\mathcal{C}_{\pi+\alpha}^2(r,1) \setminus \mathcal{C}_\pi^2(r,1)) \le c\alpha \log^{-1} \alpha. \tag{1.3.7}$$

Since $\mathcal{C}^2_\pi(r,1) \subset \mathcal{C}^2_{\pi+\alpha}(r,1)$, (1.3.7) implies

$$\begin{aligned} P^\delta(\mathcal{C}^2_{\pi+\alpha}(r,1)) &\le P^\delta(\mathcal{C}^2_\pi(r,1) + c\alpha\log\alpha^{-1} \\ &\le c(r + \alpha\log\alpha^{-1}). \end{aligned}$$

In the case $s\alpha > (1-r)/2$, the event $\mathcal{C}^2_{\pi+\alpha}(r,1) \setminus \mathcal{C}^2_\pi(r,1)$ implies the existence of a two-arm crossing of alternating colors from the annulus of inner radius $s\alpha$ and outer radius $s\alpha - r$ and a similar computation yields $P^\delta(\mathcal{C}^2_{\pi+\alpha}(r,1)) \le c(r + \alpha\log\alpha^{-1})$ in this case as well.

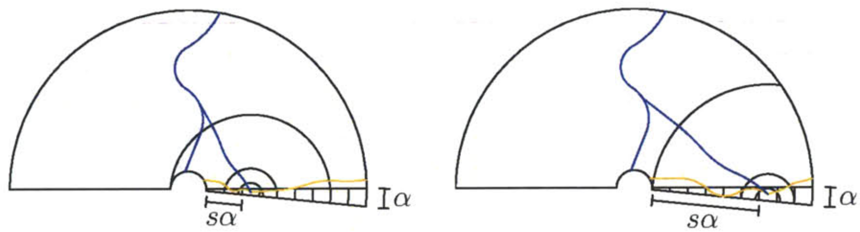


Figure 1-6: The cases (a) $s\alpha \le (1-r)/2$ and (b) $s\alpha > (1-r)/2$ for the event $\mathcal{C}^2_{\pi+\alpha}(r,R) \setminus \mathcal{C}^2_\pi(r,R)$ in Proposition 1.3.6.

Finally, to show that δ_0 may be taken to be independent of r and R, we apply a multiplicative argument. Let $K > 0$ be large enough and δ_0 small enough that $P^\delta(\mathcal{C}^2_\pi(r,R)) < (1/K)^{1-\varepsilon}$ for all $0 < \delta < \delta_0$. Insert concentric arcs of radii $r, rK, rK^2, \ldots, rK^{\lfloor \log_K(R/r) \rfloor}$ between the arcs of radii r and R, and consider the regions between successive pairs of these arcs. Since a crossing from the arcs of radius r to the arc of radius R implies that each of these regions is crossed, we have

$$\begin{aligned} P^\delta(\mathcal{C}^2_{\pi+\alpha}(r,R)) &\le \prod_{k=1}^{\lfloor \log_K(R/r) \rfloor} P^\delta\left(\mathcal{C}^2_{\pi+\alpha}\left(rK^k, rK^{k+1}\right)\right) \\ &\le C\left(\frac{r}{R}\right)^{1-\varepsilon}. \end{aligned}$$

□

Remark 1.3.7. In particular, by taking $r = \delta$, the previous results yields bounds for half-disk crossing probabilities for $z \in \partial\Omega$.

Using Smirnov's theorem, we can generalize one-arm estimates to annulus sectors of any angle.

Proposition 1.3.8. For every $\varepsilon > 0$,

$$P^\delta(\mathcal{C}^1_\theta(r,R)) \lesssim \left(\frac{r}{R}\right)^{\frac{1}{3\theta}-\varepsilon}. \tag{1.3.8}$$

Proof. Smirnov's theorem implies that for all r and R there exists $\delta_0 = \delta_0(r, R, \varepsilon) > 0$ so that for all $0 < \delta < \delta_0$, we have $P^\delta(\mathcal{C}^1_\theta(r,R)) \le (r/R)^{1/3\theta - \varepsilon}$. As in the previous proposition, we can remove the dependence on r and R with a multiplicative argument. □

We can generalize the previous results for annular regions to a neighborhood of a meeting point of two analytic arcs. We let $\mathcal{C}^k_{\Omega,z}(r,R)$ denote the event that there exist k disjoint crossings of alternating color contained in Ω and connecting the circles of radius r and R centered at z. We have the following corollary of Propositions 1.3.6 and 1.3.8.

Corollary 1.3.9. Let $\varepsilon > 0$, let $\alpha = \alpha(\varepsilon)$ be an angle satisfying the conclusion in Proposition 1.3.6. Let Ω be a piecewise analytic Jordan domain in $\mathbb{R}^2$. Fix $z \in \partial\Omega$ and suppose that z is not a corner of Ω. Let $R_0 = R_0(z, \varepsilon) > 0$ be sufficiently small that $B_{R_0}(z) \cap \Omega$ is contained in a sector centered at z and having angle $\pi + \alpha$ and radius R_0. Then for all $k \in \{1,2,3\}$ and for all $0 < r < R \le R_0$,

$$P^\delta(\mathcal{C}^k_{\Omega,z}(r,R)) \lesssim \left(\frac{r}{R}\right)^{k(k+1)/6-\varepsilon}, \tag{1.3.9}$$

with implied constants depending only on ε.

Proof. Since the event $\mathcal{C}^k_\Omega(r,R)$ implies a crossing of a sector of angle $\pi + \alpha$ with inner and outer radii of r and R,

$$P^\delta(\mathcal{C}^k_\Omega(r,R)) \le P^\delta(\mathcal{C}^k_{\pi+\alpha}(r,R))$$

and we can estimate the probability on the right by Proposition 1.3.6 for $k \in \{2,3\}$ or Proposition 1.3.8 for $k = 1$. □

We conclude this section by recording a generalization of the previous corollary for corners $z \in \partial\Omega$. The proof of this proposition uses convergence of the exploration path to SLE_6. We know how to remove this dependence on SLE results only when $k = 1$, where Smirnov's theorem suffices. We use (1.3.10) when $k \in \{2,3\}$ only to handle the case where Ω has reflex angles and to obtain the sharp exponent discussed in Remark 1.2.2.

Proposition 1.3.10. Suppose that $z \in \partial\Omega$ is a corner of Ω, but otherwise the hypotheses and variable definitions are the same as in Corollary 1.3.9. Then the conclusion holds, with (1.3.9) replaced by

$$P^\delta(\mathcal{C}^k_{\Omega,z}(r,R)) \lesssim \left(\frac{r}{R}\right)^{k(k+1)/12\theta-\varepsilon}, \tag{1.3.10}$$

where $2\pi\theta$ is the angle formed by $\partial\Omega$ at z.

Proof. Define $a^\theta_{k,\delta}(r,R)$ to be the probability of k disjoint crossings of alternating color from inner to outer radius in $\{z : \arg z \in (0, 2\pi\theta) \text{ and } r < |z| < R\}$. In [75], it is shown that

$$\lim_{\delta\to 0} a^{1/2}_{k,\delta}(1,R) = R^{-k(k+1)/6+o(1)}, \tag{1.3.11}$$

using the convergence of the percolation exploration path to SLE_6. By the invariance of the law of SLE_6 under the conformal map $z \mapsto z^{2\theta}$, we conclude that (1.3.11) generalizes to

$$\lim_{\delta\to 0} a^{\theta}_{k,\delta}(1,R) = R^{-k(k+1)/12\theta+o(1)}.$$

The following multiplicative property is also used in [75]: for all $k < r \le r' \le r''$, we have

$$a^{1/2}_{k,\delta}(r,r'') \le a^{1/2}_{k,\delta}(r,r')a^{1/2}_{k,\delta}(r',r''). \tag{1.3.12}$$

This inequality still holds with $1/2$ replaced by θ. The proof in [75] for the case $\theta = 1/2$ relies only on these two facts and therefore generalizes to (1.3.10) for the sector domain $\{z : \arg z \in (0, 2\pi\theta)\}$. The extension of this result to piecewise real-analytic Jordan domains with positive interior angles is obtained by following the same argument carried out in Corollary 1.3.9 for $\theta = 1/2$. □

1.4 Proof of Main Theorem

1.4.1 Background and set-up

We begin by recalling few definitions and facts from complex analysis and differential geometry. See [1], [71], and [34] for more details. If $a, b \in \mathbb{C}$ are linearly independent over $\mathbb{R}$ and P is a parallelogram with vertex set $\{0, a, b, a+b\}$, then a function $f : P \to \mathbb{C} \cup \{\infty\}$ is said to be doubly-periodic if $f(z+a) = f(z)$ for z on the segment from 0 to b and $f(z+b) = f(z)$ for all z on the segment from 0 to a. If f is continuous, then such a function may be extended by periodicity to a continuous function defined on $\mathbb{C}$. An elliptic function is a doubly-periodic function whose extension to $\mathbb{C}$ is analytic outside of a set of isolated poles. Given distinct points $p_1, p_2 \in P$, there exists an elliptic function f with simple poles at p_1, p_2 (and no other poles) [71, Proposition 3.4]. One way to obtain such a function is to define the Weierstrass product

$$\sigma(\zeta) = \zeta \prod_{\substack{(j,k)\in\mathbb{Z}^2 \\ (j,k)\neq(0,0)}} \left(1 - \frac{\zeta}{aj+bk}\right) \exp\left(\frac{\zeta}{aj+bk} + \frac{\zeta^2}{2(aj+bk)^2}\right)$$

and set

$$f(\zeta) = \frac{\sigma((\zeta - (p_1+p_2)/2))^2}{\sigma(\zeta - p_1)\sigma(\zeta - p_2)}. \tag{1.4.1}$$

We recall the definitions of the differential forms $d\zeta = dx + i\,dy$ and $d\overline{\zeta} = dx - i\,dy$. Note that $d\overline{\zeta} \wedge d\zeta = 2i dA$, where dA is the two-dimensional area measure and $\wedge$ is the usual wedge product. Recall that the exterior derivative d maps k-forms to

$(k+1)$-forms and satisfies

$$df = \partial f \, d\zeta + \bar{\partial} f d\bar{\zeta}, \text{ and } d(f d\zeta) = df \wedge d\zeta \tag{1.4.2}$$

for all smooth functions f.

Let $\phi : (\Omega, x(1), x(\tau), x(\tau^2)) \to (T, 1, \tau, \tau^2)$ be the unique conformal map from Ω to the equilateral triangle T with vertices $1, \tau$, and τ^2 which maps $x(\tau^k)$ to τ^k for $k \in \{0,1,2\}$. Let $\delta > 0$ be small and define Ω_{edge} to be such that $\Omega \setminus \Omega_{\text{edge}}$ is the set of all hexagonal faces of $(\Omega_\delta)^*$ completely contained in Ω. Let T_{edge} be the image of Ω_{edge} under ϕ.

We modify G^δ to obtain a function $\tilde{G}^\delta$ for which the lattice points on the boundary of $\Omega \setminus \Omega_{\text{edge}}$ are mapped to the boundary of T. Specifically, we set

$$\tilde{G}^\delta(z) = \begin{cases} \tau^k & z \text{ is adjacent to } x(\tau^k) \\ \operatorname{proj}\left(G^\delta(z), [\tau^k, \tau^{k+1}]\right) & \text{if } z \text{ is not adjacent to } x(\tau^k) \\ & \qquad \text{but is adjacent to } [x(\tau^{k+1}), x(\tau^{k+2})] \\ G^\delta(z) & \text{otherwise,} \end{cases}$$

where we are using the notation $\operatorname{proj}(z, L)$ for the projection of a complex number z onto the line $L \subset \mathbb{C}$. Now linearly interpolate to extend $\tilde{G}^\delta$ to a function on Ω, and define $J : T \to T$ by $J(w) = \tilde{G}^\delta(\phi^{-1}(w))$.

Schwarz-reflect 17 times to extend J to the parallelogram P in Figure 1-8. For example, if r is the reflection across the line through 1 and τ, then for w in the triangle $r(T)$, we define $J(w) = r \circ J \circ r(w)$. Define an elliptic function g_w via (1.4.1) with period parallelogram P and poles at $p_1 = w_0 := (1 + \tau + \tau^2)/3$ and $p_2 = w$ varying over the grey triangle K in Figure 1-8.

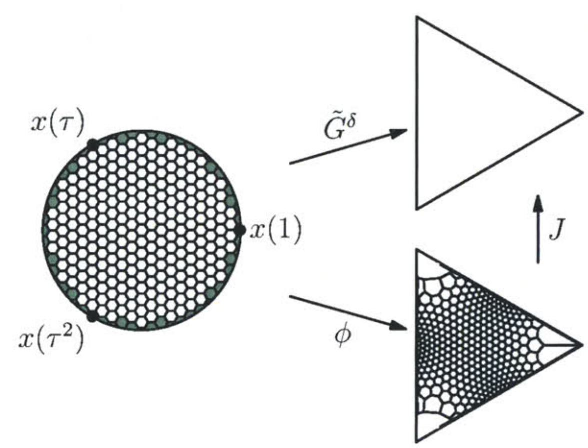


Figure 1-7: The function J is defined as the composition of $\tilde{G}^\delta$ with the inverse of the Riemann map from Ω to the triangle. The region T_{edge} is the image under the conformal map ϕ from Ω to T of the region Ω_{edge}, shown in green.

We will also need a result from the theory of Sobolev spaces. If $U \subset \mathbb{R}^2$ is a bounded domain, and $1 \le p < \infty$, we define the Sobolev space $W^{1,p}(U)$ to be the set of all functions $u : U \to \mathbb{R}$ such that the weak partial derivatives of u, $\frac{\partial u}{\partial x}$, and $\frac{\partial u}{\partial y}$ are in $L^p(U)$; see [16] for more details. We equip $W^{1,p}(U)$ with the norm

$$\|u\|_{W^{1,p}} := \|u\|_{L^p(U)} + \left\|\frac{\partial u}{\partial x}\right\|_{L^p(U)} + \left\|\frac{\partial u}{\partial y}\right\|_{L^p(U)}.$$

Denote by id the identity function from P to P, and define $C^\infty(P)$ to be the set of smooth, real-valued functions from P. Since J is piecewise-affine on P, the real and imaginary parts of J are in $W^{1,1}(P)$. Since J is defined so that $J : T \to T$ takes vertices to vertices and boundary segments to boundary segments, $J - \text{id}$ is continuous and doubly-periodic. Since smooth functions are dense in $W^{1,1}(P)$ and $L^\infty(P)$ [16], for each $\varepsilon > 0$ we obtain a pair of smooth functions $Q_1, Q_2 \in C^\infty(P)$ such that

$$\begin{aligned} &|Q(w) - (J(w) - w)| < \varepsilon \text{ for all } w \in P, \\ &\|Q_1 - \text{Re}(J - \text{id})\|_{W^{1,1}} < \varepsilon, \text{ and} \\ &\|Q_2 - \text{Im}(J - \text{id})\|_{W^{1,1}} < \varepsilon, \end{aligned} \tag{1.4.3}$$

where $Q = Q_1 + iQ_2$ (for see [16] §5.3.3 and §C.5, for example). Defining Q_1 and Q_2 to be bump function convolutions, we arrange for Q_1 and Q_2 to inherit periodicity from $J - \text{id}$. We note that by choosing ε sufficiently small in (1.4.3), we can for every $\varepsilon' > 0$ choose Q so that

$$\int_P |\partial Q - \partial(J - \text{id})|\,|g_w|\,dA < \varepsilon', \tag{1.4.4}$$

where dA refers to two-dimensional Lebesgue measure. One way to see this is to define $f(z) = \partial Q(z) - \partial(J(z) - z)$ and note that for $R > 0$, we have

$$\|fg\|_{L^1} \le R\|f\|_{L^1} + \|f\|_{L^\infty}\|\mathbf{1}_{\{|g|>R\}}g\|_{L^1}. \tag{1.4.5}$$

By the dominated convergence theorem, we may choose R sufficiently large that the second term on the right-hand side is less than $\varepsilon'/2$. Once R is chosen, we may choose Q so that $\|f\|_{L_1} \le \varepsilon'/(2R)$, by (1.4.3). Then (1.4.4) follows from (1.4.5).

1.4.2 Proof of main theorems

Proof of Theorem 1.2.1. The following calculation is similar to the proof of the Cauchy integral formula, but with two key changes: we keep track of the $\bar\partial$ term, and we use the elliptic function g_w in place of the usual kernel $\zeta \mapsto 1/\zeta$. Choose $r > 0$ sufficiently small that the balls B_1 and B_2 of radius r around w_0 and w are disjoint, and apply Stokes' theorem to the region $P \setminus (B_1 \cup B_2)$ to obtain that for smooth,

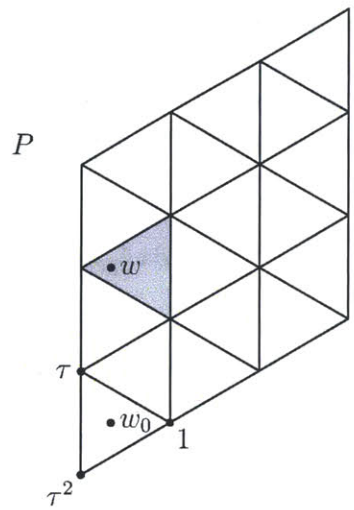


Figure 1-8: We extend $J(w)$ to a function on the parallelogram P, which is a union of 18 small triangles. The elliptic function $g_w : P \to \hat{\mathbb{C}}$ has poles at w_0 fixed and w varying in the gray region.

complex-valued, periodic functions Q on P, we have

$$-\int_{|\zeta-w|=r} Q(\zeta)g_w(\zeta)\,d\zeta - \int_{|\zeta-w_0|=r} Q(\zeta)g_w(\zeta)\,d\zeta = \int_{P\setminus(B_1\cup B_2)} d(Qg_w d\zeta).$$

Note that the integral around $\partial\Omega$ vanishes by periodicity. Applying (1.4.2) and the product rule, we obtain

$$\begin{aligned}\int_{P\setminus(B_1\cup B_2)} d(Qg_w d\zeta) &= \int_{P\setminus(B_1\cup B_2)} \left[(\partial Q d\zeta + \bar\partial Q d\bar\zeta)g_w + (\partial g_w d\zeta + \bar\partial g_w d\bar\zeta)Q\right] \wedge d\zeta \\ &= \int_{P\setminus(B_1\cup B_2)} \bar\partial Q(\zeta)g_w(\zeta)\,d\bar\zeta \wedge d\zeta.\end{aligned}$$

Let Q be a smooth, complex-valued, periodic function on P such that (1.4.3) and (1.4.4) are satisfied with $\varepsilon = \varepsilon' = \delta^{100}$, say. Since Q is bounded and g has an integrable pole at ζ, we can take $r \to 0$ and apply the dominated convergence theorem. We obtain

$$2\pi i Q(w)\operatorname{Res}(g_w, w) + 2\pi i Q(w_0)\operatorname{Res}(g_w, w_0) = 2i\int_P \bar\partial Q(\zeta)g_w(\zeta)\,dA(\zeta), \quad (1.4.6)$$

where $dA(\zeta) = dx\,dy$ is notation for the area differential. The key step of the proof is to bound the right-hand side of (1.4.6) by $O(\delta^c)$. To do this, we first consider J in place of Q, and we estimate the integral over the regions $T \setminus T_{\text{edge}}$ and T_{edge} separately. We postpone the details of these calculations to the following section, along with stronger lemma statements (Lemmas 1.5.2 and 1.5.4).

Lemma 1.4.1. There exists $c > 0$ so that

$$\int_{T_{\text{edge}}} \bar{\partial} J(\zeta) g_w(\zeta) dA(\zeta) = O(\delta^c), \tag{1.4.7}$$

where the implied constants depend only on the three-pointed domain.

Lemma 1.4.2. There exists $c > 0$ so that

$$\int_{T \setminus T_{\text{edge}}} \bar{\partial} J(\zeta) g_w(\zeta) dA(\zeta) = O(\delta^c), \tag{1.4.8}$$

where the implied constants depend only on the three-pointed domain.

Since $\text{Res}(g_w, w)$ is a continuous function of w with no zeros in K, there exists $C > 0$ such that

$$0 < C^{-1} < \text{Res}(g_w, w) < C < \infty, \qquad \forall w \in K,$$

and similarly for the residue at w_0. Therefore, (1.4.8) implies that $Q(w)$ is within $O(\delta^c)$ of a constant function, as w ranges over the gray triangle shown in Figure 1-8. By considering w to be one of the vertices of the gray triangle (so that $J(w) - w = 0$), we see that this constant function is $O(\delta^c)$. We conclude that $Q(w) = O(\delta^c)$. By (1.4.3), this implies $J(w) - w = O(\delta^c)$. By definition, this is equivalent to $\tilde{G}^\delta(z) - \phi(z) = O(\delta^c)$. The theorem follows, since $\tilde{G}^\delta$ agrees with G^δ except on the outermost layer of lattice points. □

We combine the rate of convergence for $H_1 + \tau H_\tau + \tau^2 H_{\tau^2}$ with the rate of convergence for $H_1 + H_\tau + H_{\tau^2}$ near $\partial\Omega$ to prove the rate of convergence of the crossing probabilities.

Proof of Theorem 1.1.1. Let $z \in [x(1), x(\tau)]$. First we note that $H^\delta_{\tau^2}(z) = O(\delta^c)$ by Proposition 1.3.4. Hence, by Theorem 1.2.1,

$$H^\delta_1(z) + \tau H^\delta_\tau(z) = \phi(z) + O(\delta^c)\,.$$

We also have that

$$H^\delta_1(z) + H^\delta_\tau(z) = 1 + O(\delta^c)\,,$$

since $S^\delta(z) = 1 + O(\delta^c)$ by Proposition 1.3.4 (ii). Since the vectors $(1,1), (1,\tau) \in \mathbb{C}^2$ are linearly independent, this concludes the proof. □

1.5 Bounding the error integral

1.5.1 Piecewise analytic Jordan domains

In this section, we prove the two lemmas used in the proof of the main theorem. We often treat the conformal map $\phi(z)$ like a power of z when z is near a corner of the domain Ω. To make this precise, we use the following theorem from the conformal map literature [35].

Theorem 1.5.1. If Ω is a Jordan domain part of whose boundary consists of two analytic arcs meeting at a positive angle $2\pi\alpha$ at the origin, and if $\phi : \Omega \to \mathbb{H}$ is a Riemann map sending 0 to 0, then there exists a neighborhood B of the origin and continuous functions $\rho_1, \rho_2 : B \cap \overline{\Omega} \to \mathbb{C}$ and $\rho_3, \rho_4 : \phi(B \cap \overline{\Omega}) \to \mathbb{C}$ for which

$$\phi(z) = z^{1/(2\alpha)}\rho_1(z), \qquad \phi'(z) = z^{1/(2\alpha)-1}\rho_2(z),$$
$$\phi^{-1}(z) = z^{2\alpha}\rho_3(z), \text{ and } \qquad (\phi^{-1})'(z) = z^{2\alpha-1}\rho_4(z)$$

and $\rho_i(0) \neq 0$ for $i \in \{1,2,3,4\}$.

We choose a collection $\mathcal{B}$ of disks covering the boundary of Ω as follows (see Figure 1-9). For each $z \in \partial\Omega$, choose a disk $B(z)$ centered at z and small enough that the boundary arc (or arcs) containing z admits a Taylor expansion in $B(z)$. If necessary, shrink $B(z)$ so that $\partial\Omega$ is well-approximated by its tangent (or tangents, if z is a corner point) in $B(z)$, in the sense of Propositions 1.3.6 and 1.3.10. If necessary, shrink $B(z)$ once more to ensure that $\Omega \cap B(z)$ has one component. From this collection of open disks, extract a finite subcover $\mathcal{B} = (B_j)_{j=1}^{p}$ of $\partial\Omega$ containing $\mathcal{B}_{\text{corners}} = \{B(z) : z \text{ is a corner point}\}$. Then $\mathcal{B}$ is an annular region whose interior has positive distance from $\partial\Omega$. Thus, for all sufficiently small δ, $\mathcal{B}$ covers Ω_{edge}. Note that this cover has been chosen in a manner which depends only on Ω and ε, and in particular is independent of δ.

Throughout our discussion, we permit the constants in statements involving asymptotic notation to depend only on the three-pointed domain. We also use C to represent an arbitrary constant which depends only on the three-pointed domain. When working with the variable ε, we will frequently relabel small constant multiples of ε as ε from one line to the next.

Lemma 1.5.2. Let J, g_w be as in Section 1.4, and suppose that the angle measures at marked points are $2\pi\alpha_i$ for $i = 1,2,3$, and remaining angles are $2\pi\beta_j$ for $j = 1,2,\ldots,n$. For every $\varepsilon > 0$,

$$\int_{T_{\text{edge}}} \bar{\partial} J(\zeta) g_w(\zeta) dA(\zeta) \lesssim \delta^{\min_{i,j}\left(1,\frac{1}{6\alpha_i},\frac{1}{2\beta_j}\right)-\varepsilon}, \tag{1.5.1}$$

where the implied constants depend only on ε and the three-pointed domain.

Proof of Lemma. Let $\mathcal{B}$ be as described above. Since the number of disks in $\mathcal{B}$ is bounded independently of δ, it suffices to demonstrate that (1.5.1) holds for each one. Let $B \in \mathcal{B}$, and let $\pi\beta$ be the angle formed by $\partial\Omega$ center of B.

To bound $\left|\int_{T_{\text{edge}} \cap \phi(\Omega \cap B)} \bar{\partial} J(\zeta) g_w(\zeta) dA(\zeta)\right|$, we index all the faces $\{F_k\}$ intersecting $\partial\Omega$ in such a way that the distance from F_k to the center of B is $\asymp k\delta$ for all k; this is possible since $\partial\Omega$ is piecewise smooth. We will bound the integral over each

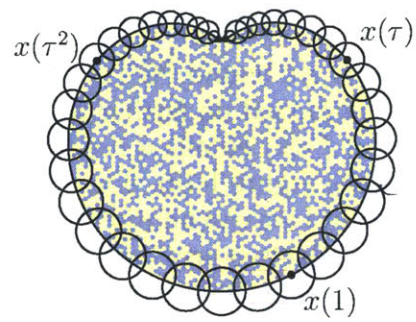


Figure 1-9: We cover the boundary with finitely many small disks, so that the boundary is approximately straight in each disk. Moreover, we ensure that every corner and every marked point is centered at one of these disks. More disks are required in regions of high curvature, as illustrated here for a domain bounded by a limaçon.

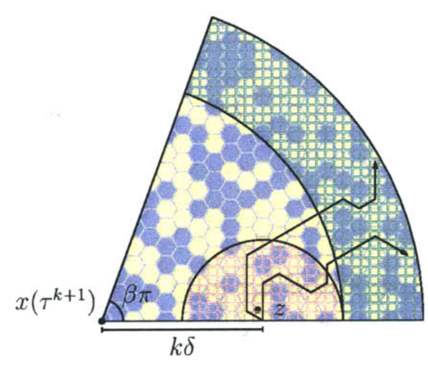


Figure 1-10: If z is adjacent to the side $[x(\tau^{k+1}), x(\tau^{k+2})]$, then the distance from $G^{\delta}(z)$ to ∂T is equal to the probability $H_{\tau^k}(z)$. This probability is bounded by that of a two-arm half-plane event with radius $k\delta$ and the two-arm β-annulus event with inner radius $2k\delta$ and constant-order outer radius.

F_k and then sum over k (see Figure 1-10). Let $\zeta \in T_{\text{edge}} \cap \phi(F_k)$ and suppose that $[\tau, \tau^2]$ is the closest boundary arc. We rewrite

$$\bar{\partial} J(\zeta) = \bar{\partial} \tilde{G}^\delta(\phi^{-1}(\zeta))(\phi^{-1})'(\zeta), \tag{1.5.2}$$

and we define $z = \phi^{-1}(\zeta)$. First we bound $\bar{\partial}\tilde{G}^\delta(z)$. In modifying $G^\delta(z)$ to obtain $\tilde{G}^\delta(z)$, the image of z has to be moved no farther than $H_1(z) = \mathbb{P}(E_1(z))$, by the definition of $\tilde{G}^\delta(z)$. The event $E_1(z)$ entails a two-arm half-disk crossing and a two-arm β-annulus crossing (see Figure 1-10). Since these events occur in disjoint regions, they are independent and we can bound $\mathbb{P}(E_1(z))$ by the product of their probabilities. By Corollary 1.3.9, the two-arm half-plane exponent in Ω, is 1 and by Proposition 1.3.10 the two-arm β-annulus exponent is $1/2\beta$. Thus the probability of $E_1(z)$ is at most $(k\delta)^{1/2\beta-\varepsilon}(1/k)^{1-\varepsilon}$. Hence for $z+\eta$ in the outermost layer and z a neighbor of $z+\eta$, we have

$$\begin{aligned}
&\frac{1}{\delta}\left(\tilde{G}^\delta(z+\eta) - \tilde{G}^\delta(z)\right) \\
&\quad = \frac{1}{\delta}\left(\tilde{G}^\delta(z+\eta) - G^\delta(z+\eta) + G^\delta(z+\eta) - G^\delta(z) + G^\delta(z) - \tilde{G}^\delta(z)\right) \\
&\quad \le \frac{1}{\delta}\left((k\delta)^{1/2\beta-\varepsilon}(1/k)^{1-\varepsilon} + G^\delta(z+\eta) - G^\delta(z)\right). \\
&\quad \asymp \delta^{-1}(k\delta)^{1/2\beta-\varepsilon}(1/k)^{1-\varepsilon}.
\end{aligned} \tag{1.5.3}$$

In the last step we use a shifted domain trick (see the proof of the second inequality in Proposition 1.3.3 and Figure 1-5) and apply the trivial inequality $\mathbb{P}(A \setminus B) \le \mathbb{P}(A)$. Using (1.5.3) to bound each term of the expression $\bar{\partial}\tilde{G}^\delta(z) = \left(\frac{\partial}{\partial\eta} - \frac{1}{\tau}\frac{\partial}{\partial(\tau\eta)}\right)\tilde{G}^\delta(z)$, we get $\bar{\partial}\tilde{G}^\delta(z) \asymp \delta^{-1}(k\delta)^{1/2\beta-\varepsilon}(1/k)^{1-\varepsilon}$.

We assume that the location z of the pole is in the face nearest to the center of B (since that is the worst case) and also that the image of the center of B is not a vertex of the equilateral triangle T. We obtain

$$\begin{aligned}
&\left| \int_{T_{\text{edge}} \cap \phi(\Omega \cap B_j)} \bar{\partial} J(\zeta) g_w(\zeta) dA(\zeta) \right| \\
&\qquad \le \sum_{k=1}^{C/\delta} \sup_{z \in F_k} |\bar{\partial}\tilde{G}^\delta(z)(\phi^{-1})'(\phi(z)) g_w(\phi^{-1}(z))| \text{area}(\phi(F_k))
\end{aligned} \tag{1.5.4}$$

by replacing the integrand with its supremum on each F_k and summing over k. We use the estimate $\text{area}(\phi(F_k)) \lesssim \sup_{z\in F_k} |\phi'(z)|^2\delta^2$ and use Theorem 1.5.1 to estimate the factors involving ϕ. We bound the right-hand side of (1.5.4) by

$$\begin{aligned}
&\lesssim \sum_{k=1}^{C/\delta} \overbrace{\delta^{-1}(k\delta)^{1/2\beta-\varepsilon}(1/k)^{1-\varepsilon}}^{\bar{\partial}\tilde{G}^{\delta}} \overbrace{(k\delta)^{1-1/2\beta}}^{(\phi^{-1})'(\phi(z))} \overbrace{(k\delta)^{-1/2\beta}}^{g_w} \overbrace{\delta^2(k\delta)^{2/2\beta-2}}^{\text{area}(\phi(F_k))} \\
&\asymp \delta^{1-\varepsilon}\left(\sum_{k=1}^{C/\delta}(k\delta)^{1/2\beta-2}\delta\right) \\
&\asymp \begin{cases} \delta^{1-\varepsilon} & \text{if } 2\beta \leq 1 \\ \delta^{1/2\beta-\varepsilon} & \text{if } 2\beta > 1. \end{cases}
\end{aligned}$$

We have evaluated the sum by noting that the factor in parentheses is a convergent Riemann sum when the exponent is at least -1. When the exponent is less than -1, the summation over k gives a constant factor, leaving the contributions of the powers of δ.

If the center of B_j is a marked point, the proof is essentially the same and the net effect is to replace $1/2\beta$ with $1/6\alpha$ throughout the calculation. These replacements are justified either by fewer percolation arms (when the exponent appears in an arm event estimate), or by the angle of $\pi/3$ at the vertices of the triangle T (when the exponent appears because of the conformal map ϕ). □

Remark 1.5.3. We can remove the dependence on SLE by using Smirnov's theorem to estimate one-arm β-annulus probabilities (instead of using Proposition 1.3.10). The result is that we obtain (1.5.1) with the right-hand side replaced by

$$\delta^{\min_{i,j}\left(1,\frac{1}{6\alpha_i},\frac{1}{6\beta_j}\right)-\varepsilon}.$$

Lemma 1.5.4. Let $J, g_w, \{\alpha_i\}, \{\beta_j\}$ be as in the statement of Lemma 1.5.2. Let $c_3 = 2/3$ be the 3-arm whole-plane exponent. Then

$$\int_{T\setminus T_{\text{edge}}} \bar{\partial}J(\zeta)g_w(\zeta)dA(\zeta) \lesssim \delta^{\min_{i,j}\left(c_3,\frac{1}{6\alpha_i},\frac{1}{2\beta_j}\right)-\varepsilon}, \tag{1.5.5}$$

where the implied constants depend only on ε and the three-pointed domain.

Proof of Lemma. We will use Proposition 1.3.3 to bound $\bar{\partial}G$. Let $\mathcal{B}$ be as above and note that $\text{dist}(\partial\Omega, \Omega \setminus \bigcup\mathcal{B}) > 0$ by the discussion preceding Lemma 1.5.2.

We first handle $\Omega \setminus \bigcup\mathcal{B}$. Suppose that one of the five-arm events of Figure 1-5 occurs, say $E_{1,1,1}^{\text{five arm}}(z)$. Let b be the point nearest $x(\tau^2)$ where a blue arm touches down in the shifted domain, and let s be the number of lattice units along the boundary from b to $x(\tau^2)$. When $z \notin \bigcup\mathcal{B}$ (see Figure 1-11), z is well away from the boundary thus we note that such a five arm event entails the existence of:

1. a 3-arm whole-plane event in alternating colors at z, in a ball of radius $\Theta(1)$,
2. a 3-arm half-annulus event of alternating colors originating at b, in a semicircle of radius $s\delta/2$, and

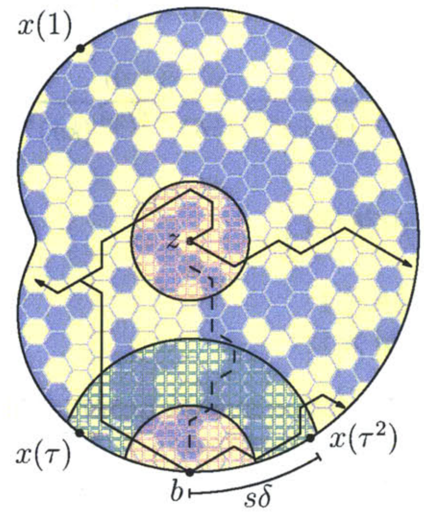


Figure 1-11: To bound the probability of the five-arm difference event described in Proposition 1.3.3, we consider three regions which contain two-arm or three-arm crossing events (these regions are shown in green and red, respectively).

3. a 2-arm half-annulus event in an annulus of inner radius $s\delta/2$ and outer radius $\Theta(1)$.

Since the derivative of the conformal map is bounded above and below for z away from the boundary, we can ignore the contribution of $\phi'(\phi^{-1}(z))$ in (1.5.2) and calculate

$$\begin{aligned} |\bar{\partial}J(z)| &\lesssim \delta^{-1}\sum_{s=1}^{C/\delta} \overbrace{(\delta^{c_3-\varepsilon})}^{\text{3-arm whole-plane}} \times \overbrace{(1/s)^2}^{\text{3-arm half-plane}} \times \overbrace{(s\delta)}^{\text{2-arm half-ann.}} \\ &\lesssim \delta^{c_3-\varepsilon}. \end{aligned}$$

Hence we have

$$\left|\int_{\phi(\Omega\setminus\mathcal{B})} \bar{\partial}J(\zeta)g_w(\zeta)dA(\zeta)\right| \lesssim \delta^{c_3-\varepsilon}\int_{\phi(\Omega\setminus\mathcal{B})}|g_w(\zeta)|\,dA(\zeta) \lesssim \delta^{c_3-\varepsilon},$$

since a simple pole is integrable with respect to area measure.

To bound the integral of the union of the balls in $\mathcal{B}$, we handle each $B \in \mathcal{B}$ separately. We first consider a ball centered at a marked corner, say $x(\tau)$. Once again, for each z and each percolation configuration, we define $b \in \partial\Omega$ to be the point nearest $x(\tau^2)$ at which a blue arm from z touches down in the shifted domain. This time we let s be the graph distance from b to the boundary point z_{foot} nearest to z (see Figure 1-14) and index the faces $F_{n,k}$ in such a way that if $z \in F_{n,k}$, $|x(\tau) - z| \asymp k\delta$ and $\operatorname{dist}(\partial\Omega, z) \asymp n\delta$. As above, we bound $|\bar{\partial}\tilde{G}^\delta(z)|$ using percolation arm estimates in each hexagonal face and sum over all the faces in $\phi(\Omega \cap B)$. By symmetry, it suffices to sum over only the faces which are closer to the boundary

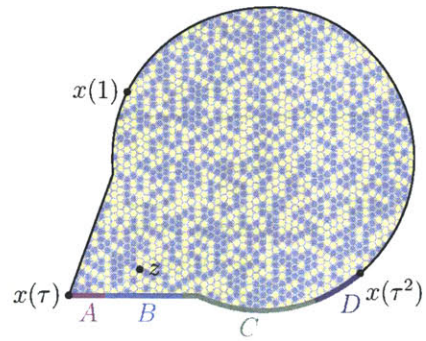


Figure 1-12: We sum over the possible locations for b, considering the cases $b \in A$, $b \in B$, $b \in C$, and $b \in D$ separately.

arc $[x(\tau), x(\tau^2)]$ than to the boundary arc $[x(1), x(\tau)]$.

Suppose that the corner at B is one of the three marked points and has interior angle $\alpha\pi$. We bound $|\bar{\partial}\tilde{G}^\delta|$ by summing over all possible locations for b. We consider four cases:

- Case A: b is closest to the corner at $x(\tau)$ (Figure 1-13(a)),
- Case B: b is within $k/2$ units of z_{foot} (Figure 1-13(b)),
- Case C: b is more than $k/2$ units to the right of z_{foot} but closer to z_{foot} than to $x(\tau^2)$ (Figure 1-13(c)), and
- Case D: b is closest to $x(\tau^2)$ (Figure 1-13(d)).

For simplicity, we assume that $[x(\tau), x(\tau^2)]$ is a real analytic arc (that is, that there are no corners between $x(\tau)$ and $x(\tau^2)$). It will be apparent that similar estimates hold when additional corners are accounted for.

Denote by $P(z,b)$ the contribution to $\bar{\partial}G^\delta$ of the five-arm event with missed connection at b (see Figure 1-5). As in (1.5.4), we bound the sum for Case A by a constant times

$$\sum_{k=1}^{C/\delta}\sum_{n=1}^{Ck}\sum_{r=1}^{k/2}\overbrace{\underbrace{\delta^{-1}}_{\bar{\partial}}\;\underbrace{n^{-c_3-\varepsilon}}_{\text{3-arm disk.}}\;\underbrace{r^{-2}}_{\text{3-arm half-disk.}}\;\underbrace{\left(\frac{r}{k}\right)^{1/2\alpha-\varepsilon}}_{\text{2-arm }\alpha\text{-ann.}}\;\underbrace{(k\delta)^{1/6\alpha-\varepsilon}}_{\text{1-arm }\alpha\text{-ann.}}}^{P(z,b)}$$
$$\times\;\overbrace{(k\delta)^{1-1/6\alpha}}^{(\phi^{-1})'(\phi(F_{n,k}))}\;\overbrace{\delta^2(k\delta)^{1/3\alpha-2}}^{\text{area}(\phi(F_{n,k})))}\;\overbrace{(k\delta)^{-1/6\alpha}}^{g_w}$$
$$\lesssim \delta^{\min(c_3,1/6\alpha)-\varepsilon}.$$

We upper bound the contribution of Case B by a constant times

$$\sum_{k=1}^{C/\delta}\sum_{n=1}^{Ck}\sum_{s=1}^{k/2}\overbrace{\underbrace{\delta^{-1}}_{\tilde{\partial}}\underbrace{n^{-c_3-\varepsilon}}_{\text{3-arm disk}}\underbrace{\frac{1}{s^2+n^2}}_{\text{3-arm half disk}}\underbrace{\frac{\sqrt{s^2+n^2}}{k}}_{\text{2-arm half-ann.}}\underbrace{(k\delta)^{1/6\alpha-\varepsilon}}_{\text{1-arm }\alpha\text{-ann.}}}^{P(z,b)}$$

$$\times\overbrace{(k\delta)^{1-1/6\alpha}}^{(\phi^{-1})'(\phi(F_{n,k}))}\overbrace{\delta^2(k\delta)^{1/3\alpha-2}}^{\text{area}(\phi(F_{n,k}))}\overbrace{(k\delta)^{-1/6\alpha}}^{g_w}$$

$$\lesssim \delta^{\min(c_3,1/6\alpha)-\varepsilon}.$$

For Case C, we get

$$\sum_{k=1}^{C/\delta}\sum_{n=1}^{Ck}\sum_{r=k/2}^{C/\delta}\overbrace{\underbrace{\delta^{-1}}_{\tilde{\partial}}\underbrace{n^{-c_3-\varepsilon}}_{\text{3-arm disk.}}\underbrace{k^{-2}}_{\text{3-arm half-disk.}}}^{P(z,b)}(k\delta)^{1-1/6\alpha}\delta^2(k\delta)^{1/3\alpha-2}(k\delta)^{-1/6\alpha}$$

$$\lesssim \delta^{1/6\alpha-\varepsilon}.$$

For Case D, we denote by $2\pi\gamma$ the angle at $x(\tau^2)$ and by t the number of lattice units from $x(\tau^2)$ to b. We obtain

$$\sum_{k=1}^{C/\delta}\sum_{n=1}^{Ck}\sum_{t=1}^{C/\delta}\overbrace{\underbrace{\delta^{-1}}_{\tilde{\partial}}\underbrace{n^{-c_3-\varepsilon}}_{\text{3-arm disk.}}\underbrace{t^{-2}}_{\text{3-arm half-disk.}}\underbrace{(t\delta)^{1/\gamma}}_{\text{2-arm }\gamma\text{ ann}}}^{P(z,b)}(k\delta)^{1-1/6\gamma}\delta^2(k\delta)^{1/3\gamma-2}(k\delta)^{-1/6\gamma}$$

$$\lesssim \delta^{1/2\gamma-\varepsilon}.$$

The proofs for the bounds in a disk whose center is not marked are essentially the same as these. As in the proof of Lemma 1.5.2, the net effect is to replace $1/6\alpha$ with $1/2\beta$. □

Remark 1.5.5. As in Remark 1.5.3, we can remove the dependence on SLE by using Smirnov's theorem instead of Proposition 1.3.10, under the additional assumption that $\partial\Omega$ has no reflex angles (that is, $\max_{i,j}(\alpha_i,\beta_j)\le 1/2$). By using the weaker one-arm β-annulus bound in place of the two-arm and three-arm bounds, we obtain (1.5.5) with the right-hand side replaced by

$$\delta^{\min_{i,j}\left(c_3,\frac{1}{6\alpha_i},\frac{1}{6\beta_j}\right)-\varepsilon}.$$

Without the help of SLE, our techniques break down in the presence of reflex angles.

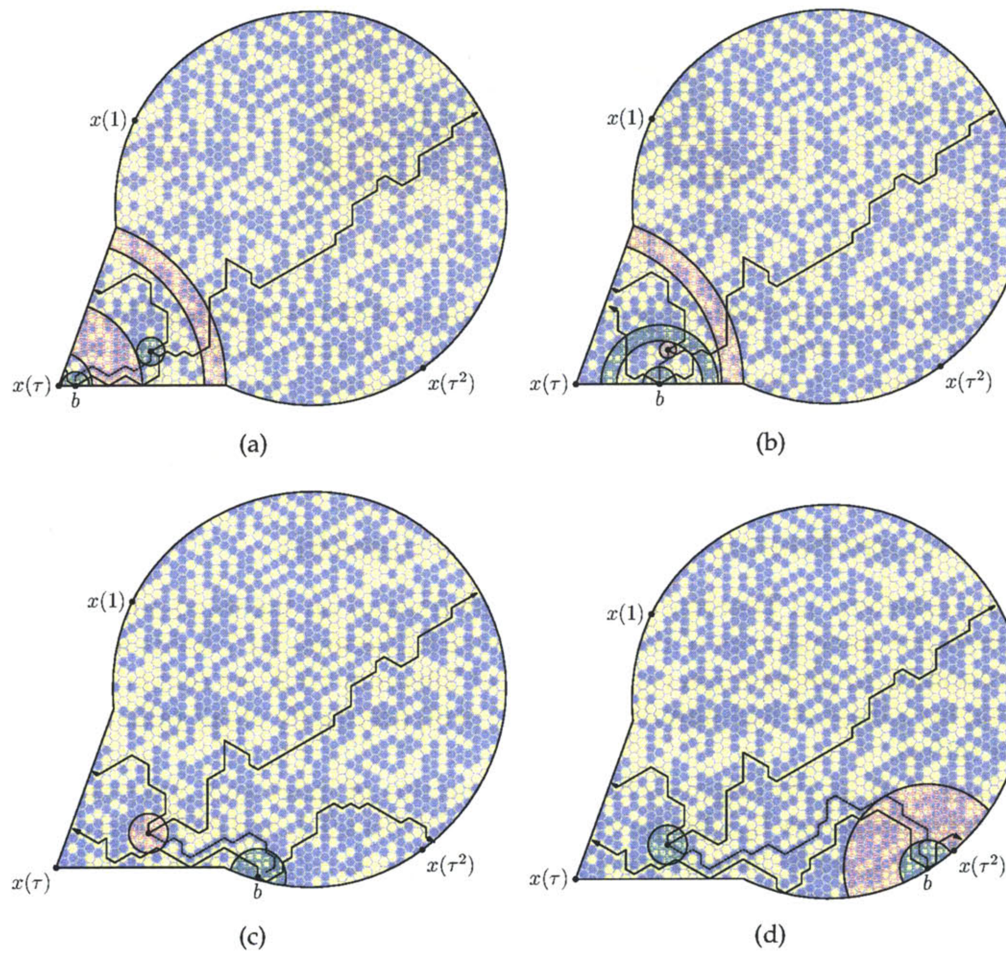


Figure 1-13: Assuming that z is near a marked corner, we have four cases to consider: (a) b is close to $x(\tau)$, (b) b is close to z, (c) b is between z and $x(\tau^2)$ but far from both, and (d) b is close to $x(\tau^2)$. For a closer view of the corner with additional labels, see Figure 1-14.

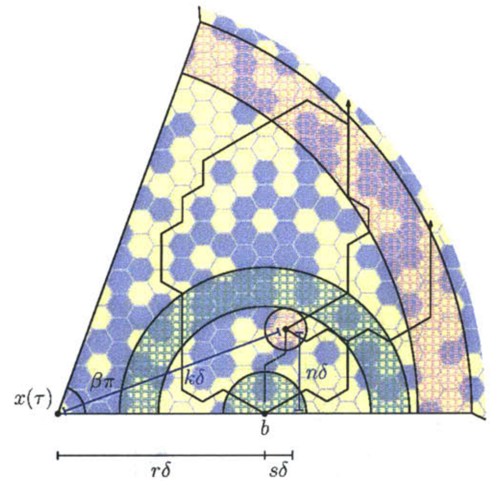


Figure 1-14: A close-up view of the corner of Figure 1-13(b), with labels illustrating the roles of k, n, r, and s. The faces are indexeed by k and n in such a way that the distance from z to the corner is $\asymp k\delta$ and the distance from z to $\partial\Omega$ is $\asymp n\delta$. Similarly, the faces intersecting the boundary are indexed so that the distance along the boundary from the corner to b is $\asymp r\delta$ and the distance from b to z_{foot} is $\asymp s\delta$.

1.5.2 Uniform bounds for half-annulus domains

While the constants in Theorem 1.1.1 generally depend on the three-pointed domain, there are some classes of domains for which Theorem 1.1.1 holds with uniform constants. In preparation for the proof of Theorem 1.1.2, we obtain uniform constants for a class of half-annulus domains with arbitrarily small ratio of inner to outer radius.

Let $\Omega_{r,R} \subset \mathbb{H}$ be the origin-centered half-annulus of inner and outer radius r and R, respectively. Let T_{unit} be the triangle with vertices $0, 1$, and $e^{i\pi/3}$, and define $\phi_{r,R} : \Omega_{r,R} \to T_{\text{unit}}$ to be the conformal map sending $-R$, $-r$, and R to $e^{i\pi/3}, 0$, and 1, respectively. For $r \geq 0$, define $S_r = \{re^{i\theta} : 0 \leq \theta \leq \pi\}$.

Proposition 1.5.6. For all $0 < c < c_3 = 2/3$ and $0 < \delta \leq r \leq 1/2$, we have

$$P^\delta(S_r \leftrightarrow S_1) - \phi_{r,1}(r) = O(r^{-1/3}\delta^c) = O(\delta^{c-1/3}), \tag{1.5.6}$$

where the implied constants depend only on c and, in particular, are uniform over $r \in (0, 1/2]$.

Remark 1.5.7. To ensure that the interval $(0, c_3 - 1/3)$ of possible exponents c is nonempty, we need the SLE result that the three-arm whole-plane exponent c_3 is greater than $1/3$.

Proof. We proceed by modifying Lemmas 1.5.2 and 1.5.4 to prove (1.5.1) and (1.5.5) with constants uniform over the domains $\Omega_{r,1}$. For $z \in \mathbb{C}$ and $\rho \geq 0$, let $B(z, \rho)$ be the disk of radius ρ centered at z. For the integral over $\Omega_{r,1} \setminus B(0,1/2)$ we obtain a bound of $O(\delta^{2/3-\varepsilon})$ by Lemmas 1.5.2 and 1.5.4, so it suffices to consider the integral over $\Omega_{r,1} \cap B(0,1/2)$.

Fix $\varepsilon > 0$, and determine $\alpha(\varepsilon)$ from Proposition 1.3.6. Choose $\eta(\varepsilon)$ small enough that $B(i,\eta) \setminus B(0,1)$ is contained in a sector of angle $\pi + \alpha$ centered at i. Cover S_r with finitely many balls of radius $2r\eta$ in such a way that $\bigcup_{w \in S_r} B(w, r\eta)$ is contained in the union U of the balls. By Lemmas 1.5.2 and 1.5.4 and rescaling (1.5.1) and (1.5.5) by a factor of r, we find that $\int_U |\bar{\partial} J g_w| \, dA = O(r^{-1/3}\delta^{c_3-\varepsilon})$. So it remains to consider the integral over the annulus $A' := \{z : r(1+\eta) < |z| < 1/2\}$. We reduce further to considering the integral over the left half $\{z \in A' : \pi/2 < \arg(z) < \pi\}$ of A', since the contribution from the right half of A' is smaller. We compute this integral similarly to those in Lemmas 1.5.2 and 1.5.4 (see Figure 1-15): we index the faces $F_{n,k}$ in such a way that $|F_{n,k}| - r \asymp k\delta$ and $\operatorname{dist}(F_{n,k}, \mathbb{R}) \asymp n\delta$ and, for $z \in F_{n,k}$ we bound

$$P(z,b) \lesssim$$
$$\underbrace{\delta^{-1}}_{\delta} \underbrace{(n \wedge k)^{-c_3+\varepsilon}}_{\text{3-arm disk.}} \underbrace{\left(\frac{\delta}{s\delta \wedge \eta r}\right)^{2-\varepsilon}}_{\text{3-arm half-disk.}} \underbrace{\left(\frac{s\delta \wedge \eta r}{\eta r}\right)^{1-\varepsilon}}_{\text{2-arm half ann.}} \underbrace{\left(\frac{r}{k\delta + r}\right)^{1-\varepsilon}}_{\text{2-arm half-ann.}} \underbrace{(k\delta + r)^{1/3-\varepsilon}}_{\text{1-arm half-ann}'}.$$

Figure 1-16 shows how to write $\phi_{r,1}$ as a composition of simpler conformal maps. Using this composition, we compute

$$\begin{aligned} \phi_{r,1}(z) &\asymp \frac{(z+r)^{2/3}}{z^{1/3}}, \\ \phi'_{r,1}(z) &\asymp \frac{z-r}{z^{4/3}(z+r)^{1/3}}, \text{ and} \\ \phi_{r,1}^{-1}(\phi_{r,1}(z)) &\asymp \frac{z^{4/3}(z+r)^{1/3}}{z-r}. \end{aligned}$$

Using these estimates, we can upper bound $\int |\bar{\partial} J g_w| \, dA$ by summing over the faces $F_{n,k}$. We obtain

$$\sum_{k=\eta r/\delta}^{C/\delta} \sum_{n=1}^{Ck} \sum_{s=1}^{Cr/\delta} P(z,b) \overbrace{(k\delta + r)^{1/3} (k\delta)^{1/3}}^{(\phi^{-1})'(\phi(F_{n,k}))} \overbrace{\delta^2 (k\delta+r)^{-2/3} (k\delta)^{-2/3}}^{\text{area}(\phi(F_{n,k}))} \overbrace{\frac{(k\delta + r)^{1/3}}{(k\delta)^{2/3}}}^{g_w}$$
$$\lesssim r^{-1/3}\delta^{c_3-\varepsilon}.$$

□

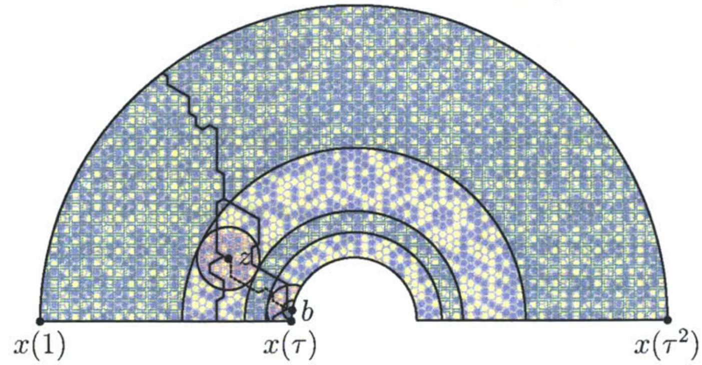


Figure 1-15: We use crossing events for the five regions shown to bound the probability of a five-arm event for which $b \in S_r$.

1.6 Half-plane exponent

We begin with a lemma about the conformal maps $\phi_{r,R} : \Omega_{r,R} \to T_{\text{unit}}$; see Subsection 1.5.2 for notation.

Lemma 1.6.1. There exist $a_1, a_2 > 0$ so that for all $r, R > 0$ such that $r/R < 1/2$, we have

$$a_1 \le \frac{\phi_{r,R}(r)}{(r/R)^{1/3}} \le a_2. \tag{1.6.1}$$

Proof. By scaling, we may assume $R = 1$. Consider the sequence of conformal maps illustrated in Figure 1-16. Let us call these maps f_n for $n = 1, 2, \ldots, 5$, so that $f_n : D_n \to D_{n+1}$. Since the domains are Jordan, we may regard f_n as a continuous map defined on the closure of each domain. Define the compositions $\tilde{f}_n = f_n \circ f_{n-1} \cdots \circ f_1$.

For $n \ge 2$, let $K_n \subset D_n$ denote the image of

$$K_1 := \{z : |z| = r \text{ and } \arg z \in [0, \pi/2]\} \cup [r, 1/2]$$

under $\tilde{f}_{n-1}$. For $n \in \{2, 3, 5\}$, regard f_n as having been analytically continued in a neighborhood of every straight boundary (by Schwarz reflection), and define m_n and M_n to be the infimum and supremum of $f_n'(z)$ as z ranges over K_n and r ranges over $[0, 1/2]$.

We claim that $0 < m_n < M_n < \infty$ for all $n \in \{2, 3, 5\}$. For $n = 5$, this follows from the continuity of f_n' and the fact that the derivative of a conformal map cannot vanish. For $n = 3$, this follows from the joint continuity of the Möbius map $(z - w)/(1 - \overline{w}z)$ in w and z.

The case $n = 2$ requires more care, since the eccentricity of D_2 depends on r. We introduce the notation $D_{2,r}$ and $f_{2,r}$ to indicate this dependence. Let $I \subset (0, 1/2)$ be an interval. We claim that for every fixed $z \in \bigcap_{r \in I} D_{2,r}$, the quantity $f_{2,r}'(z)$ is continuous in r. We first recall some definitions from complex analysis: given a simply connected domain $U \subset \mathbb{C}$ and a point $z \in U$, we will say that a Riemann map $\varphi : \mathbb{D} \to U$ is *normalized* if $\varphi(0) = z$ and $\varphi'(0) > 0$. Recall that a sequence of open sets $U_n \subset \mathbb{C}$ converges to an open set $U \subset \mathbb{C}$ in the Carathéodory sense with respect to $z \in U$ if (a) for all compact $K \subset U$ containing z, we have $K \subset U_n$ for all n sufficiently large, and (b) U contains every open set satisfying condition (a). If $U_n \to U$ in the Carathéodory sense, then the normalized Riemann maps $\varphi_n : \mathbb{D} \to U_n$ converge uniformly on compact subsets to the normalized Riemann map $\varphi : \mathbb{D} \to U$ [82]. Observe that if $r_n \to r$, D_{2,r_n} converges to $D_{2,r}$ with respect to 0 in the Carathéodory sense. Hence $f_{2,r_n} \to f_{2,r}$ uniformly on compact sets, which in turn implies that $f'_{2,r_n} \to f'_{2,r}$ uniformly on compact sets. In particular, we obtain joint continuity of $f'_{2,r}(z)$ in z and r. It follows that the infimum and supremum of $|f'_r(z)|$ over $(z,r) \in K_n \times [0,1/2]$ are achieved, which implies $0 < m_2 < M_2 < \infty$.

Since $f_1(r) = 2r/(1+r^2)$, we have

$$r \leq f_1(r) \leq 2r.$$

We note that each f_n is monotone on the real line, and apply $f_5 \circ f_4 \circ f_3 \circ f_2$ to the inequality above. Using our derivative bounds, we obtain

$$m_5 \left(m_2 m_3 r\right)^{1/3} \leq \tilde{f}_4(r) \leq M_5 \left(2M_2 M_3 r\right)^{1/3},$$

thus the result holds with $a_1 = m_5(m_2 m_3)^{1/3}$ and $a_2 = M_5(2M_2M_3)^{1/3}$. □

Remark 1.6.2. Numerical evidence suggests that Lemma 1.6.1 holds with $a_1 = 1$ and $a_2 \approx 1.426$.

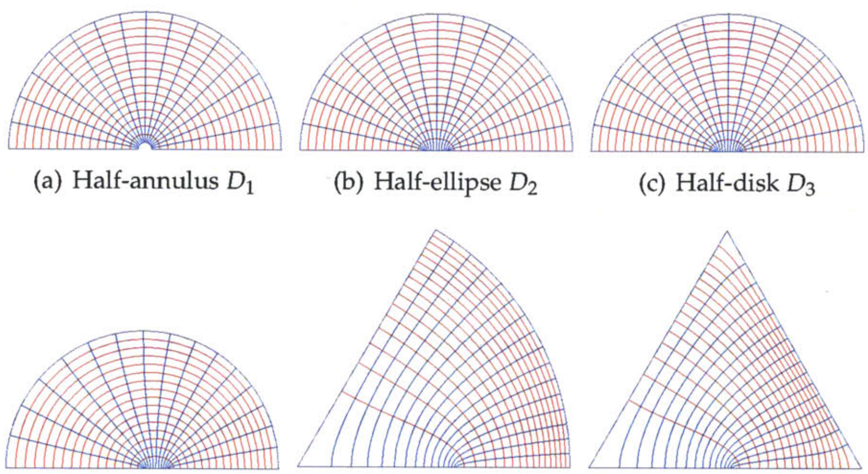
We denote by P the measure $P^{\delta=1}$ corresponding to site percolation on the triangular lattice with unit mesh size.

Lemma 1.6.3. For all $0 < c < c_3 - 1/3 = 1/3$ there exists $R_0 > 1$ such that for all $R \geq R_0$ and for all $r \leq \frac{1}{2}R$,

$$\left|P^{\delta=1}(S_r \leftrightarrow S_R) - \phi_{r,R}(r)\right| \leq \frac{a_1}{10} R^{-c}. \tag{1.6.2}$$

Proof. This follows immediately from Proposition 1.5.6, by rescaling by a factor of R. Note that we have used the openness of interval $(0, c_3 - 1/3)$ to deal with the multiplicative constant in the bound given by Proposition 1.5.6. □

Proof of Theorem 1.1.2. Let $\varepsilon > 0$, and define $R_0 = e^{\sqrt{\log\log R}}$. We assume that R is sufficiently large that R_0 satisfies the statement of Lemma 1.6.3. Define $\alpha = 1/(1 - 3c)$ and $n = \lfloor \log_{R_0} \log_\alpha R \rfloor$. Let $R_k = R_0^{\alpha^k}$ for $1 \leq k \leq n-1$, and let $R_n = R$. We first prove the upper bound. Since an open path from 0 to S_R includes a crossing from S_{R_k} to $S_{R_{k+1}}$ for all $0 \leq k < n$, we may use Lemma 1.6.1, Lemma 1.6.3, and



(a) Half-annulus D_1 (b) Half-ellipse D_2 (c) Half-disk D_3

(d) Half-disk D_4 (e) Sector D_5 (f) Equilateral triangle D_6

Figure 1-16: Panels (b) through (f) show the images of the half-annulus in panel (a) under successive conformal maps. Composing these maps gives the conformal map $\phi_{r,R}$ from the half-annulus to the equilateral triangle which sends $-R$, $-r$, and R to $e^{i\pi/3}$, 0, and 1. The ratio of outer radius to inner radius is 20 for the half-annulus shown. The map from D_1 to D_2 is a suitable scaling of $z \mapsto z + 1/z$. The map from D_2 to D_3 is the restriction of the conformal map from an ellipse to the disk. From D_3 to D_4, a Möbius map moves the image of $-r$ to the origin. The map from D_4 to D_5 is the cube root, and the map from D_5 to D_6 is the restriction to a sector of the Schwarz-Christoffel map from the disk to the regular hexagon.

independence to compute

$$\begin{aligned} P(0 \leftrightarrow S_R) &\le \prod_{k=0}^{n-1} P(S_{R_k} \leftrightarrow S_{R_{k+1}}) \\ &\le \prod_{k=0}^{n-1} \left[a_2 \left(\frac{R_{k+1}}{R_k} \right)^{-1/3} + \frac{a_1}{10} R_{k+1}^{-c} \right], \end{aligned}$$

by (1.6.2). Factoring out the first term in brackets and splitting the product, we obtain

$$\begin{aligned} P(0 \leftrightarrow S_R) &\le \prod_{k=0}^{n-1} a_2 \prod_{k=0}^{n-1} \left(\frac{R_{k+1}}{R_k} \right)^{-1/3} \prod_{k=0}^{n-1} \left[1 + a_1 (10a_2)^{-1} R_{k+1}^{1/3-c} R_k^{-1/3} \right] \\ &\le (a_1/10 + a_2)^{n-1} (R/R_0)^{-1/3}, \end{aligned}$$

because the second term in brackets simplifies to $a_1/(10a_2)$ by our choice of R_k. Substituting the value of n gives

$$\begin{aligned} P(0 \leftrightarrow S_R) &\le R_0^{1/3} (\log \alpha)^{-\log(a_1/10+a_2)/\log R_0} (\log R)^{\log(a_1/10+a_2)/\log R_0} R^{-1/3} \\ &\le e^{C\sqrt{\log\log R}} R^{-1/3}, \end{aligned}$$

for some constant C and for sufficiently large R, which gives the upper bound.

For the lower bound (see Figure 1-17), we define $R_k' = 2R_k$. Define E_k to be the event that there is an open crossing of $\Omega_{R_k, R_k'}$ from $[R_k, R_k']$ to $[-R_k', -R_k]$. By the Russo-Seymour-Welsh inequality, this probability is bounded below by a constant p which does not depend on k. Note that there is a path from the origin to S_R if the following events occur:

1. there is an open path from the origin to $S_{R_0'}$,
2. there is an open path from S_{R_k} to $S_{R_{k+1}'}$ for all $0 \le k < n$, and
3. E_k occurs for all $0 \le k < n$.

Since these events are increasing, we can use the FKG inequality to lower bound the probability of their intersection by the product of their probabilities. We obtain

$$\begin{aligned} P(0 \leftrightarrow S_R) &\ge P(0 \leftrightarrow S_{R_0}) \prod_{k=0}^{n-1} P(S_{R_k} \leftrightarrow S_{R_{k+1}'}) \prod_{k=0}^{n-1} P(E_k) \\ &\ge R_0^{-1/2} p^{n-1} \prod_{k=0}^{n-1} \left[a_1 \left(\frac{R_{k+1}'}{R_k} \right)^{-1/3} - \frac{a_1}{10} (R_{k+1}')^{-c} \right], \end{aligned}$$

since $P(0 \leftrightarrow S_{R_0}) = R_0^{-1/3+o(1)} \gtrsim R_0^{-1/2}$, by the Cardy-Smirnov theorem. Factor-

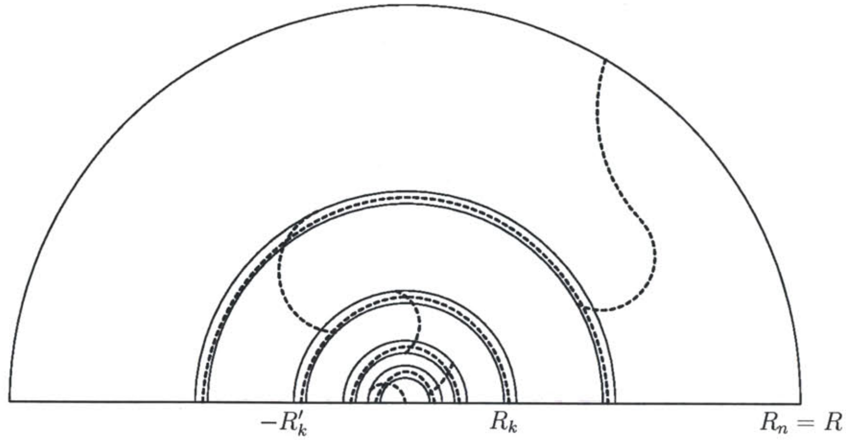


Figure 1-17: If there are segment-to-segment crossings of each narrow half-annulus, crossings from S_{R_k} to $S_{R'_k}$ for each $0 \le k \le n$, and an open path from the origin to S_{R_0}, then there is an open path from the origin to S_R. The figure shown is an image under radial logarithmic scaling $(r,\theta) \mapsto (\log r, \theta)$.

ing as before and simplifying, we obtain

$$\begin{aligned}
P(0 \leftrightarrow S_R) &\ge R_0^{-1/2}(a_1 p)^{n-1} \prod_{k=0}^{n-1} \left(\frac{R'_{k+1}}{R_k}\right)^{-1/3} \prod_{k=0}^{n-1} \left[1 - \frac{1}{10}(R'_{k+1})^{1/3-c} R_k^{-1/3}\right] \\
&\ge R_0^{-1/2} 2^{-n/3} (a_1 p)^{n-1} \prod_{k=0}^{n-1} \left(\frac{R_{k+1}}{R_k}\right)^{-1/3} \prod_{k=0}^{n-1} \left[1 - \frac{2^{1/3-c}}{10} R_{k+1}^{1/3-c} R_k^{-1/3}\right] \\
&\ge R_0^{-1/2} 2^{-n/3} \left[a_1 p (1 - 2^{1/3-c}/10)\right]^{n-1} (R/R_0)^{-1/3} \\
&\ge e^{-C\sqrt{\log\log R}} R^{-1/3},
\end{aligned}$$

for some constant $C > 0$ and sufficiently large R. □

Chapter 2

Nesting in the conformal loop ensemble

This chapter presents two joint works with Jason Miller and David Wilson. It appears almost verbatim in [49] and [47].

2.1 Introduction

The conformal loop ensemble CLE_κ for $\kappa \in (8/3, 8)$ is the canonical conformally invariant measure on countably infinite collections of non-crossing loops in a simply connected domain $D \subsetneq \mathbb{C}$ [65, 69]. It is the loop analogue of SLE_κ, the canonical conformally invariant measure on non-crossing paths. Just as SLE_κ arises as the scaling limit of a single interface in many two-dimensional discrete models, CLE_κ is a limiting law for the joint distribution of all of the interfaces. Figures 2-1 and 2-2 show two discrete loop models believed or known to have CLE_κ as a scaling limit. Figure 2-3 illustrates these scaling limits CLE_κ for several values of κ.

2.1.1 Overview of main results

Fix a simply connected domain $D \subsetneq \mathbb{C}$ and let Γ be a CLE_κ in D. For each point $z \in D$ and $\varepsilon > 0$, we let $\mathcal{N}_z(\varepsilon)$ be the number of loops of Γ which surround $B(z, \varepsilon)$, the ball of radius ε centered at z. We study the behavior of the *extremes* of $\mathcal{N}_z(\varepsilon)$ as $\varepsilon \to 0$, that is, points where $\mathcal{N}_z(\varepsilon)$ grows unusually quickly or slowly (Theorem 3.8.7). We also analyze a more general setting in which each of the loops is assigned an i.i.d. weight sampled from a given law μ. This in turn is connected with the extremes of the continuum Gaussian free field (GFF) [23] when $\kappa = 4$ and $\mu(\{-\sigma\}) = \mu(\{\sigma\}) = \frac{1}{2}$ for a particular value of $\sigma > 0$ (Theorem 2.1.2 and Theorem 2.1.3).

2.1.2 Extremes

Fix $\alpha \geq 0$. Recall that the Hausdorff α-measure $\mathcal{H}_\alpha$ of a set $E \subset \mathbb{C}$ is given by

$$\mathcal{H}_\alpha(E) = \lim_{\delta \to 0} \left(\inf \left\{ \sum_i \left(\operatorname{diam}(F_i) \right)^\alpha \, : \, \bigcup_i F_i \supseteq E, \; \operatorname{diam}(F_i) < \delta \right\} \right),$$

where the infimum is over all countable collections $\{F_i\}$ of sets. The Hausdorff dimension of E is defined to be

$$\dim_{\mathcal{H}}(E) := \inf\{\alpha \geq 0 \, : \, \mathcal{H}_\alpha(E) = 0\}.$$

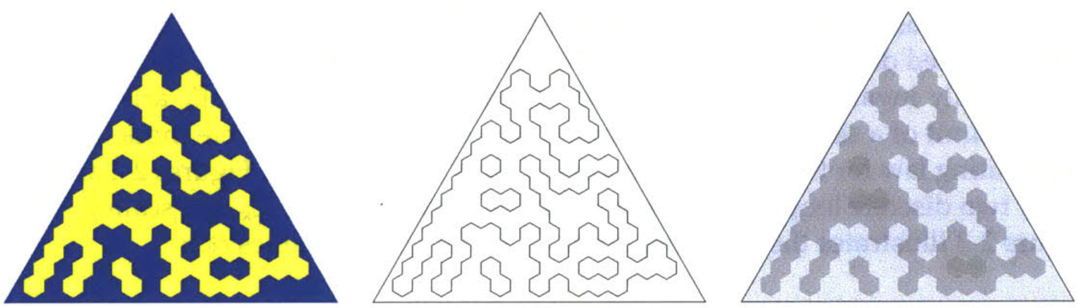
For each $z \in D$ and $\varepsilon > 0$, let

$$\widetilde{\mathcal{N}_z}(\varepsilon) := \frac{\mathcal{N}_z(\varepsilon)}{\log(1/\varepsilon)}. \tag{2.1.1}$$

For $\nu \geq 0$, we define

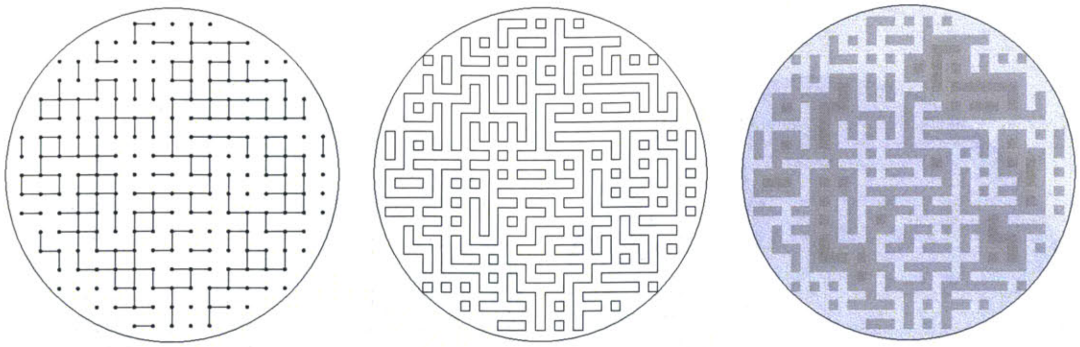
$$\Phi_\nu(\mathrm{CLE}_\kappa) := \Phi_\nu(\Gamma) := \left\{ z \in D \, : \, \lim_{\varepsilon \to 0} \widetilde{\mathcal{N}_z}(\varepsilon) = \nu \right\}. \tag{2.1.2}$$

Our first result gives the almost-sure Hausdorff dimension of $\Phi_\nu(\mathrm{CLE}_\kappa)$. The dimension is given in terms of the distribution of the conformal radius of the con-



(a) Site percolation. (b) $O(n)$ loop model. Percolation corresponds to $n = 1$ and $x = 1$, which is in the dense phase. (c) Area shaded by nesting of loops.

Figure 2-1: Nesting of loops in the $O(n)$ loop model. Each $O(n)$ loop configuration has probability proportional to $x^{\text{total length of loops}} \times n^{\#\text{ loops}}$. For a certain critical value of x, the $O(n)$ model for $0 \leq n \leq 2$ has a "dilute phase", which is believed to converge CLE_κ for $8/3 < \kappa \leq 4$ with $n = -2\cos(4\pi/\kappa)$. For x above this critical value, the $O(n)$ loop model is in a "dense phase", which is believed to converge to CLE_κ for $4 \leq \kappa \leq 8$, again with $n = -2\cos(4\pi/\kappa)$. See [24] for further background.



(a) Critical FK bond configuration. Here $q = 2$. (b) Loops separating FK clusters from dual clusters. (c) Area shaded by nesting of loops.

Figure 2-2: Nesting of loops separating critical Fortuin-Kasteleyn (FK) clusters from dual clusters. Each FK bond configuration has probability proportional to $(p/(1-p))^{\#\,\text{edges}} \times q^{\#\,\text{clusters}}$ [18], where there is believed to be a critical point at $p = 1/(1+1/\sqrt{q})$ (proved for $q \geq 1$ [5]). For $0 \leq q \leq 4$, these loops are believed to have the same large-scale behavior as the $O(n)$ model loops for $n = \sqrt{q}$ in the dense phase, that is, to converge to CLE_κ for $4 \leq \kappa \leq 8$ (see [56, 24]).

nected component of the outermost loop surrounding the origin in a CLE_κ in the unit disk. More precisely, recall that the *conformal radius* $\text{CR}(z, U)$ of a simply connected proper domain $U \subset \mathbb{C}$ with respect to a point $z \in U$ is defined to be $|\varphi'(0)|$ where $\varphi\colon \mathbb{D} \to U$ is a conformal map which sends 0 to z. For each $z \in D$, let $\mathcal{L}_z^k$ be the kth largest loop of Γ which surrounds z, and let U_z^k be the connected component of the open set $D \setminus \mathcal{L}_z^k$ which contains z. Take $D = \mathbb{D}$ and let $T = -\log(\text{CR}(0, U_0^1))$. The log moment generating function of T was computed in [61] and is given by

$$\Lambda_\kappa(\lambda) := \log \mathbb{E}\left[e^{\lambda T}\right] = \log\left(\frac{-\cos(4\pi/\kappa)}{\cos\left(\pi\sqrt{(1-\frac{4}{\kappa})^2 + \frac{8\lambda}{\kappa}}\right)}\right), \tag{2.1.3}$$

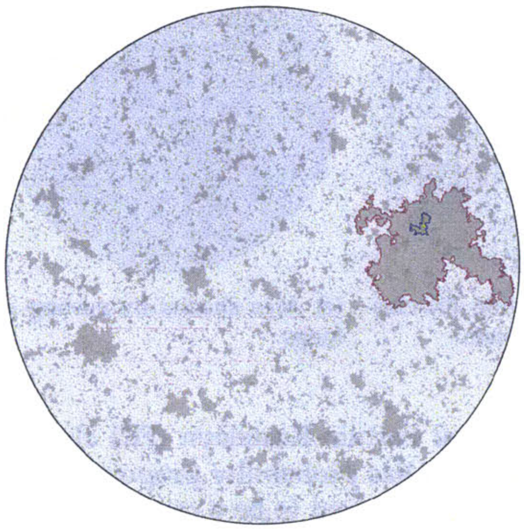
for $-\infty < \lambda < 1 - \frac{2}{\kappa} - \frac{3\kappa}{32}$. The almost-sure value of $\dim_{\mathcal{H}} \Phi_\nu(\Gamma)$ is given in terms of the *Fenchel-Legendre transform* $\Lambda_\kappa^\star : \mathbb{R} \to [0, \infty]$ of Λ_κ, which is defined by

$$\Lambda_\kappa^\star(x) := \sup_{\lambda \in \mathbb{R}} (\lambda x - \Lambda_\kappa(\lambda)) .$$

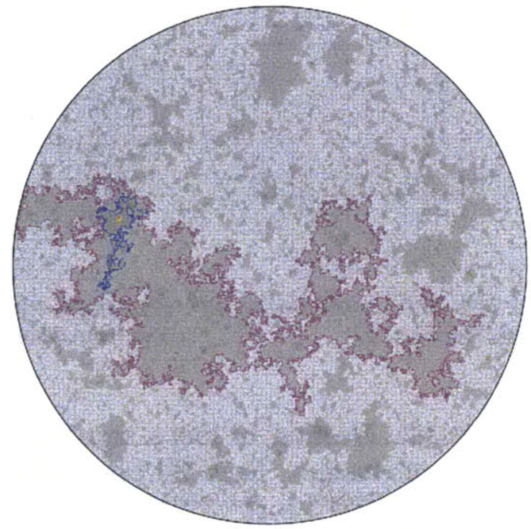
We also define

$$\gamma_\kappa(\nu) = \begin{cases} \nu \Lambda_\kappa^\star(1/\nu) & \text{if } \nu > 0 \\ 1 - \frac{2}{\kappa} - \frac{3\kappa}{32} & \text{if } \nu = 0 . \end{cases} \tag{2.1.4}$$

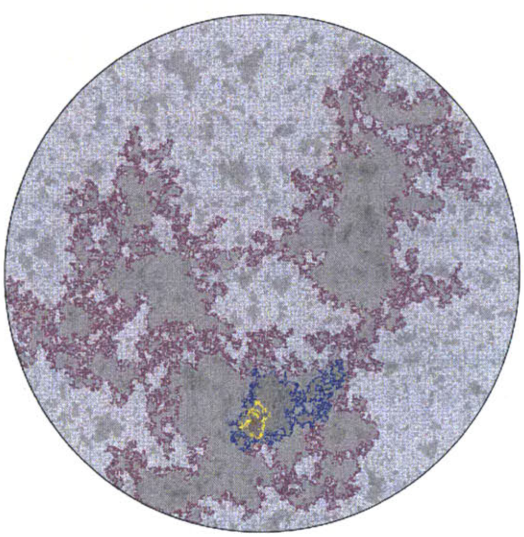
For each $\kappa \in (8/3, 8)$, let $\nu_{\max}$ be the unique value of $\nu \geq 0$ such that $\gamma_\kappa(\nu) = 2$.



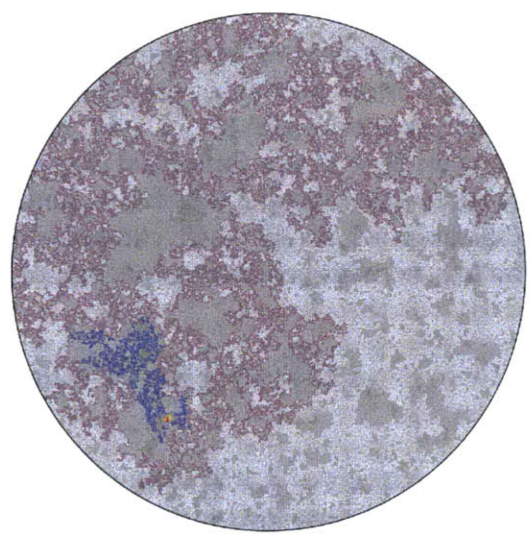
(a) CLE_3 (from critical Ising model)



(b) CLE_4 (from the FK model with $q = 4$) $\star$



(c) $CLE_{16/3}$ (from the FK model with $q = 2$)



(d) CLE_6 (from critical bond percolation) $\star$

Figure 2-3: Simulations of discrete loop models which converge to (or are believed to converge to, indicated with $\star$) CLE_κ in the fine mesh limit. For each of the CLE_κ's, one particular nested sequence of loops is outlined. For CLE_κ, almost all of the points in the domain are surrounded by an infinite nested sequence of loops, though the discrete samples shown here display only a few orders of nesting.

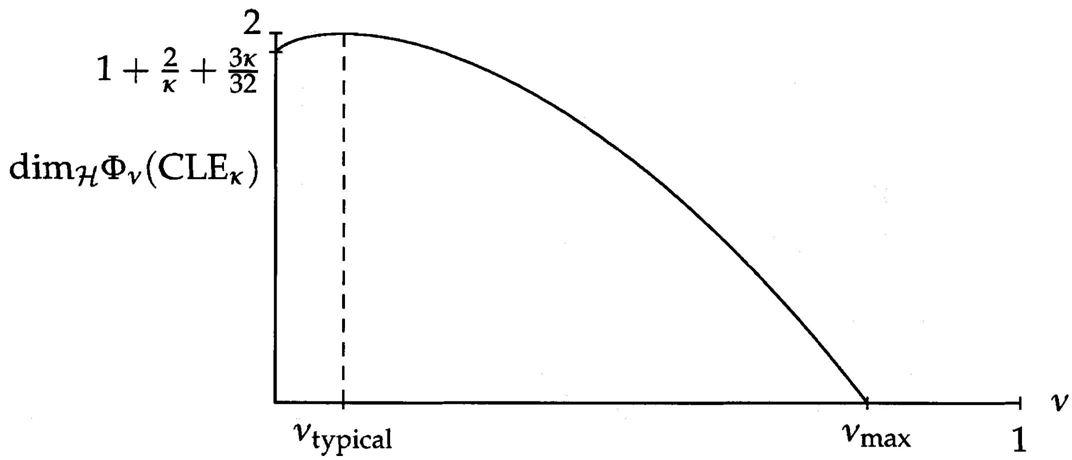


Figure 2-4: Suppose that $D \subsetneq \mathbb{C}$ is a simply connected domain and let Γ be a CLE_κ in D. For $\kappa \in (8/3, 8)$ and $\nu \geq 0$, we let $\Phi_\nu(\Gamma)$ be the set of points z for which the number of loops $\mathcal{N}_z(\varepsilon)$ of Γ surrounding $B(z,\varepsilon)$ is $(\nu + o(1))\log(1/\varepsilon)$ as $\varepsilon \to 0$. The plot above shows how the the almost-sure Hausdorff dimension of $\Phi_\nu(\mathrm{CLE}_\kappa)$ established in Theorem 3.8.7 depends on ν (the figure is for $\kappa = 6$, but the behavior is similar for other values of κ). The value $1 + \frac{2}{\kappa} + \frac{3\kappa}{32} = \dim_{\mathcal{H}} \Phi_\nu(\mathrm{CLE}_\kappa)$ is the almost-sure Hausdorff dimension of the CLE_κ gasket [61, 51, 45], which is the set of points in D which are not surrounded by any loop of Γ.

Theorem 2.1.1. If $0 \leq \nu \leq \nu_{\max}$, then almost surely

$$\dim_{\mathcal{H}} \Phi_\nu(\mathrm{CLE}_\kappa) = 2 - \gamma_\kappa(\nu) \tag{2.1.5}$$

and $\Phi_\nu(\mathrm{CLE}_\kappa)$ is dense in D. If $\nu_{\max} < \nu$, then $\Phi_\nu(\mathrm{CLE}_\kappa)$ is almost surely empty.

Moreover, if Γ is a CLE_κ in D, $\varphi \colon D \to \acute{D}$ is a conformal transformation, $\acute{\Gamma} := \varphi(\Gamma)$, and $\Phi_\nu(\acute{\Gamma})$ is defined to be the corresponding set of extremes of $\acute{\Gamma}$, then $\Phi_\nu(\acute{\Gamma}) = \varphi(\Phi_\nu(\Gamma))$ almost surely.

We also show in Theorem 2.4.9 that $\Phi_{\nu_{\max}}(\Gamma)$ is almost surely uncountably infinite for all $\kappa \in (8/3, 8)$. This contrasts with the critical case for thick points of the Gaussian free field: it has only been proved that the set of critical thick points is infinite (not necessarily uncountably infinite); see Theorem 1.1 of [23].

See Figure 2-4 for a plot of the Hausdorff dimension of $\Phi_\nu(\mathrm{CLE}_6)$ as a function of ν. The discrete analog of Theorem 3.8.7 would be to give the growth exponent of the set of points which are surrounded by unusually few or many loops for a given model as the size of the mesh tends to zero. Theorem 3.8.7 gives predictions for these exponents. Since CLE_6 is the scaling limit of the interfaces of critical percolation on the triangular lattice [73, 7, 8], Theorem 3.8.7 predicts that the typical point in critical percolation is surrounded by $(0.09189\ldots + o(1))\log(1/\varepsilon)$ loops as $\varepsilon \to 0$, where $\varepsilon > 0$ is the lattice spacing.

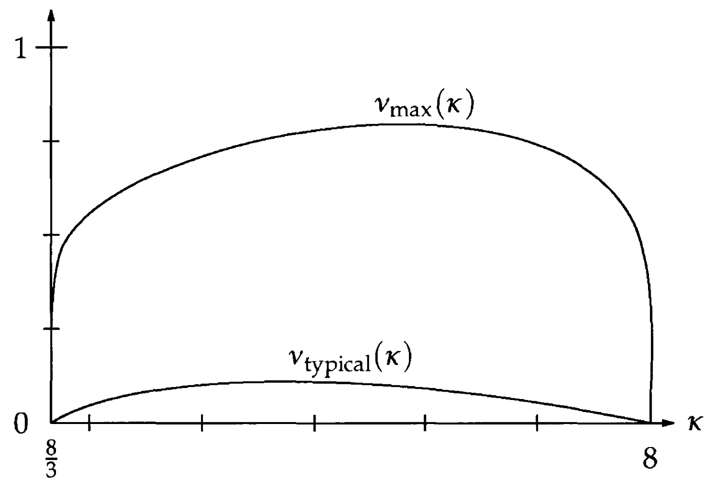


Figure 2-5: Plotted as a function of κ are the typical nesting and maximal nesting constants (ν_{typical} and $\nu_{\max}$, respectively). For example, when $\kappa = 6$, Lebesgue almost all points are surrounded by $(0.091888149\ldots + o(1))\log(1/\varepsilon)$ loops with inradius at least ε, while some points are surrounded by as many as $(0.79577041\ldots + o(1))\log(1/\varepsilon)$ loops.

We give a brief explanation of the proof for the case $\nu = 1/\mathbb{E}T$: by the renewal property of CLE_κ (Proposition 2.2.3), the random variables $\log \text{CR}(z, U_z^k) - \log \text{CR}(z, U_z^{k+1})$ are i.i.d. and equal in distribution to T. It follows from the law of large numbers (and basic distortion estimates for conformal maps) that, for $z \in D$ fixed, $\widetilde{\mathcal{N}}_z(\varepsilon) \to 1/\mathbb{E}T$ as $\varepsilon \to 0$, almost surely. By the Fubini-Tonelli theorem, we conclude that the expected Lebesgue measure of the set of points for which $\widetilde{\mathcal{N}}_z(\varepsilon) \not\to 1/\mathbb{E}T$ is 0. It follows that almost surely, there is a full-measure set of points z for which $\widetilde{\mathcal{N}}_z(\varepsilon) \to 1/\mathbb{E}T$. In other words, $\nu = \nu_{\text{typical}} := 1/\mathbb{E}T$ corresponds to typical behavior, while points in $\Phi_\nu(\text{CLE}_\kappa)$ for $\nu \neq 1/\mathbb{E}T$ have exceptional loop-count growth.

The idea to prove Theorem 3.8.7 for other values of ν is to use a multi-scale refinement of the second moment method [23, 11]. The main challenge in applying the second moment method to obtain the lower bound of the dimension of the set $\Phi_\nu(\text{CLE}_\kappa)$ in Theorem 3.8.7 is to deal with the complicated geometry of CLE loops. In particular, for any pair of points $z, w \in D$ and $\varepsilon > 0$, there is a positive probability that single loop will come within distance ε of both z and w. To circumvent this difficulty, we restrict our attention to a special class of points $z \in \Phi_\nu(\text{CLE}_\kappa)$ in which we have precise control of the geometry of the loops which surround z at every length scale.

Recall that the CLE gasket is the set of points $z \in D$ which are not surrounded by any loop of Γ. Equivalently, the gasket is the closure of the union of the set of outermost loops of Γ. Its expectation dimension, the growth exponent of the expected minimum number of balls of radius $\varepsilon > 0$ necessary to cover the gasket as

$\varepsilon \to 0$, is given by $1 + \frac{2}{\kappa} + \frac{3\kappa}{32}$ [61]. It is proved in [51] using Brownian loop soups that the almost-sure Hausdorff dimension of the gasket when $\kappa \in (8/3, 4]$ and it is shown in [45] that this result holds for $\kappa \in (4, 8)$ as well. We show in Proposition 2.2.17 that the limit as $\nu \to 0$ of $\dim_{\mathcal{H}} \Phi_\nu(\Gamma)$ is $1 + \frac{2}{\kappa} + \frac{3\kappa}{32}$ (equivalently, γ_κ is right continuous at 0). Consequently, from the perspective of Hausdorff dimension, there is no non-trivial intermediate scale of loop count growth which lies between logarithmic growth and the gasket.

Theorem 3.8.7 is a special case of a more general result, stated as Theorem 2.5.3 in Section 2.5, in which we associate with each loop $\mathcal{L}$ of Γ an i.i.d. weight $\xi_{\mathcal{L}}$ distributed according to some probability measure μ. For each $\alpha > 0$, we give the almost-sure Hausdorff dimension of the set

$$\Phi_\alpha^\mu(\Gamma) := \left\{ z \in D : \lim_{\varepsilon \to 0^+} \widetilde{\mathcal{S}}_\varepsilon(z) = \alpha \right\}$$

of extremes of the normalized weighted loop counts

$$\widetilde{\mathcal{S}}_z(\varepsilon) = \frac{1}{\log(1/\varepsilon)} \mathcal{S}_z(\varepsilon) \qquad \text{where} \qquad \mathcal{S}_z(\varepsilon) = \sum_{\mathcal{L} \in \Gamma_z(\varepsilon)} \xi_{\mathcal{L}}, \tag{2.1.6}$$

and $\Gamma_z(\varepsilon)$ is the set of loops of Γ which surround $B(z, \varepsilon)$. This dimension is given in terms of $\Lambda_\kappa^\star$ and the Fenchel-Legendre transform $\Lambda_\mu^\star$ of μ. Although the dimension for general weight measures μ and $\kappa \in (8/3, 8)$ is given by a complicated optimization problem, when $\kappa = 4$ and μ is a signed Bernoulli distribution, this dimension takes a particularly nice form. We state this result as our second theorem.

Theorem 2.1.2. Fix $\sigma > 0$, and define $\mu_{\mathrm{B}}(\{\sigma\}) = \mu_{\mathrm{B}}(\{-\sigma\}) = \frac{1}{2}$. In the special case $\kappa = 4$ and $\mu = \mu_{\mathrm{B}}$, we have

$$\dim_{\mathcal{H}} \Phi_\alpha^{\mu_B}(\Gamma) = \max\left(0, 2 - \frac{\pi^2}{2\sigma^2}\alpha^2\right) \tag{2.1.7}$$

almost surely.

This case has a special interpretation which explains the formula (2.1.7) for the dimension. It is proved in [44] that the random height field $\mathcal{S}_z(\varepsilon)$ converges in the space of distributions as $\varepsilon \to 0$ to a two-dimensional Gaussian free field h, and the loops Γ can be thought of as the level sets of h (since h is distribution-valued, h does not have level sets strictly speaking, but there is a way to make this precise). This suggests a correspondence between the extremes of $\mathcal{S}_z(\varepsilon)$ and the extremes of h. Although h is a distribution-valued random variable and does not have a well-defined value at any given point, extreme values of h (also called *thick points*) can be defined by considering the average $h_\varepsilon(z)$ of h on $\partial B(z, \varepsilon)$ and defining $T(\alpha)$ to be the set of points z for which $h_\varepsilon(z)$ grows like $\alpha \log(1/\varepsilon)$ as $\varepsilon \to 0$. It is shown in [23] that $\dim_{\mathcal{H}} T(\alpha) = 2 - \pi\alpha^2$, which equals $\dim_{\mathcal{H}} \Phi^{\mu_{\mathrm{B}}}$ when $\sigma = \sqrt{\pi/2}$ and $\kappa = 4$. The following theorem relates exceptional loop count growth with the extremes of the GFF. Loosely speaking, it says that for each α there is a unique value of ν for which "most" of the α-thick points have loop counts $\widetilde{\mathcal{N}} \approx \nu$.

Theorem 2.1.3. Let $\kappa = 4$ and $\mu_B(\{\sigma\}) = \mu_B(\{-\sigma\}) = \frac{1}{2}$ for some $\sigma > 0$. For every $\alpha \in \mathbb{R}$, there exists a unique $\nu = \nu(\alpha) \geq 0$ such that the Hausdorff dimension of the set of points with $\widetilde{\mathcal{S}}_z(\varepsilon) \to \alpha$ as $\varepsilon \to 0$ is equal to the Hausdorff dimension of the set of points with $\widetilde{\mathcal{S}}_z(\varepsilon) \to \alpha$ and $\widetilde{\mathcal{N}}_z(\varepsilon) \to \nu$ as $\varepsilon \to 0$. Moreover, we have

$$\nu(\alpha) = \frac{\alpha}{\sigma} \coth\left(\frac{\pi^2 \alpha}{\sigma}\right).$$

The usual multiple of the GFF is obtained by taking $\sigma = \sqrt{\pi/2}$. Theorems 2.1.2 and 2.1.3 are proved in Section 2.5.

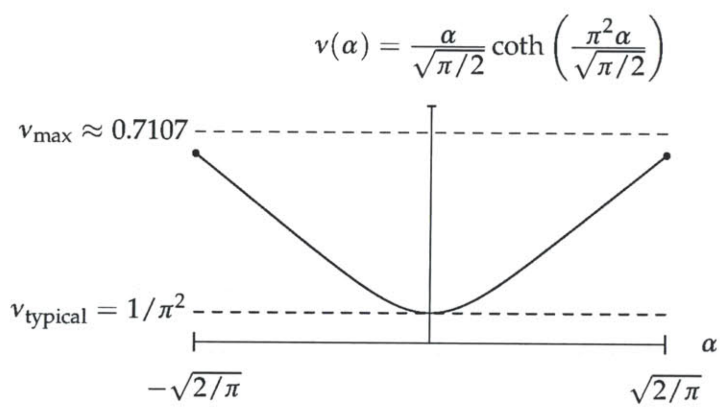


Figure 2-6: A graph of $\nu(\alpha)$ versus α, which gives the typical loop growth $\nu \log(1/\varepsilon)$ corresponding to each point with signed loop growth $\alpha \log(1/\varepsilon)$, for $\alpha \in \left[-\sqrt{2/\pi}, \sqrt{2/\pi}\right]$. Also shown is the value $\nu_{\max}$ beyond which there are no points having growth $\nu \log(1/\varepsilon)$. The graph does not reach the dashed line because it is not optimal to use the maximum number of loops: the advantage of having many loops (and thus many terms in the sum $\mathcal{S}_z(\varepsilon)$) is offset by the disadvantage of having fewer points which are surrounded by many loops. This optimization problem is the one described in the proof of Theorem 2.1.3.

2.2 Preliminaries

We first give a brief overview of the exploration tree based construction of CLE_κ in Sections 2.2.1 and 2.2.2. In Section 2.2.3, we review some facts from large deviations, and then in Section 2.2.4 we collect several estimates for random walks. Finally, in Section 2.2.5 we apply these estimates to establish asymptotics for the probability that $\widetilde{\mathcal{N}}_z(\varepsilon) \approx \nu$ for $\nu \geq 0$.

2.2.1 The continuum exploration tree

In this section, we review the exploration tree construction of CLE_κ for $\kappa \in (8/3, 8)$ given in [65]. We begin by briefly recalling the definition of the SLE_κ and $\text{SLE}_\kappa(\rho)$ processes. There are many surveys on the subject (for example, [83, 30]) to which we refer the reader for a more detailed introduction. The *radial Loewner transform* of a continuous process $W : [0, \infty) \to \partial\mathbb{D}$ is defined as follows. For $z \in \mathbb{C}$, define $g_t(z)$ to be the solution of the ordinary differential equation

$$\frac{\partial g_t(z)}{\partial t} = -g_t(z)\frac{g_t(z) + W_t}{g_t(z) - W_t}, \quad g_0(z) = z. \tag{2.2.1}$$

This ODE is well-defined until $g_t(z) = W_t$, and the swallowing time T^z is defined to be the first time at which this occurs, or ∞ if it never occurs. For $t \geq 0$ the *hull* K_t is defined to be the set of points $K_t := \{z \in \mathbb{D} : T^z \leq t\}$ of $\mathbb{D}$ swallowed by time t. For each $t \geq 0$, g_t is the unique conformal transformation $\mathbb{D} \setminus K_t \to \mathbb{D}$ with $g_t(0) = 0$ and $g_t'(0) > 0$. We refer to W as the *driving function* of the Loewner evolution, and we refer to the random growth process $(K_t)_{t \geq 0}$ as the radial Loewner transform of W.

Radial SLE_κ, introduced by Schramm [57], is defined to be the Loewner transform $(K_t)_{t\geq 0}$ of the driving function $W_t = \exp(i\sqrt{\kappa}B_t)$, where B_t is a standard Brownian motion. Time is parametrized by log-conformal radius, which means that $g_t'(0) = e^t$ for all $t \geq 0$. It was proved by Rohde and Schramm [56] ($\kappa \neq 8$) and Lawler, Schramm, and Werner [32] ($\kappa = 8$) that there is a curve η from 1 to 0 in $\overline{\mathbb{D}}$ such that $\mathbb{D} \setminus K_t$ is the unique connected component of $\mathbb{D} \setminus \eta[0, t]$ containing 0. We say that η *generates* the process K_t and call η the radial SLE_κ *trace*. The driving function can be recovered from η by the relation $W_t = \lim_{z \to \eta(t)} g_t(z)$, where the limit is taken with $z \in \mathbb{D} \setminus K_t$.

Let $D \subsetneq \mathbb{C}$ be a simply connected domain. For any conformal transformation $\varphi : \mathbb{D} \to D$, we take the image of radial SLE_κ in $\mathbb{D}$ under φ to be the definition of radial SLE_κ in D from $\varphi(1)$ to $\varphi(0)$ (with $\varphi(1)$ interpreted as a prime end). If φ extends continuously to $\overline{\mathbb{D}}$ (equivalently if ∂D is given by a closed curve, see [55, Theorem 2.1]) then radial SLE_κ in D is almost surely a continuous curve. It was proved by Garban, Rohde, and Schramm [20] that for $\kappa < 8$, radial SLE_κ in an arbitrary proper simply connected domain is almost surely continuous except possibly at its starting point.

We now describe the radial $\text{SLE}_\kappa(\rho)$ processes, a natural generalization of radial SLE_κ first introduced in [29, Section 8.3]. For $w, v \in \partial\mathbb{D}$, radial $\text{SLE}_\kappa(\rho)$ with starting configuration (w, v) is the Loewner transform of W, where the pair (W, V) solves the system of SDEs

$$\begin{cases} dV_t = -V_t \dfrac{V_t + W_t}{V_t - W_t}\, dt \\ dW_t = i\sqrt{\kappa} W_t\, dB_t - \left(\dfrac{\kappa}{2} W_t + \dfrac{\rho}{2} W_t \dfrac{W_t + V_t}{W_t - V_t}\right) dt, \end{cases} \tag{2.2.2}$$

with $W_0 = w$ and $V_0 = v$. The *force point* V_t satisfies $V_t = g_t(v)$ for all $t \geq 0$. The system (2.2.2) has a unique solution up to the collision time $\tau_{\text{col}} := \inf\{t \geq 0 : W_t = V_t\}$. The weight $\rho = \kappa - 6$ is special because $\text{SLE}_\kappa(\kappa - 6)$ is *target invariant* [63]: radial $\text{SLE}_\kappa(\kappa - 6)$ in $\mathbb{D}$ with starting configuration (w, v) and target $a \in \mathbb{D}$ has the same law (modulo time change) as an ordinary chordal SLE_κ in $\mathbb{D}$ from w to v, up to the first time the curve disconnects a and v [63].

We now explain how to construct a solution to (2.2.2) which is defined even after the collision time τ_{col}. A more detailed treatment is provided in [65, Section 3]; we give here a brief summary following [61]. We first consider $\rho > -2$. For $t \leq \tau_{\text{col}}$, we define the process $\theta_t = \arg W_t - \arg V_t$ taking values in $[0, 2\pi]$ and find using Itô's formula and (2.2.2) that θ_t satisfies the SDE

$$d\theta_t = \sqrt{\kappa}\, dB_t + \frac{\rho + 2}{2} \cot(\theta_t/2)\, dt. \tag{2.2.3}$$

Moreover, a similar calculation shows that W_t and V_t may be recovered from θ_t via

$$\begin{aligned} \arg W_t &= \arg w + \sqrt{\kappa} B_t + \frac{\rho}{2} \int_0^t \cot(\theta_s/2)\, ds. \\ \arg V_t &= \arg W_t - \theta_t. \end{aligned} \tag{2.2.4}$$

Motivated by (2.2.3) and (2.2.4), we define a process θ_t for $t \geq 0$ by evolving θ_t according to (2.2.3) on each interval of time for which $\theta_t \notin \{0, 2\pi\}$ and instantaneously reflecting at the endpoints (so that the set $\{t : \theta_t \in \{0, 2\pi\}\}$ has Lebesgue measure zero). There is a unique process θ_t with these properties [65, Proposition 4.2]. We define W_t and V_t for $t \geq 0$ using (2.2.4), and we define radial $\text{SLE}_\kappa(\rho)$ in $\mathbb{D}$ with starting configuration (w, v) to be the radial Loewner transform of W.

When $-\kappa/2 - 2 < \rho \leq -2$, the integral in (2.2.4) is infinite, and a different approach is required. One possibility is to lift θ_t to a continuous real-valued process $\tilde{\theta}_t$ as follows. Instead of reflecting the process θ to the interior of $[0, 2\pi]$, we flip a fair coin at every hitting time T of $2\pi\mathbb{Z}$ to determine whether to subsequently evolve θ in $(\tilde{\theta}_T, \tilde{\theta}_T + 2\pi)$ or $(\tilde{\theta}_T - 2\pi, \tilde{\theta}_T)$. Cancellation introduced by the coin tosses is sufficient to make the integral in (2.2.4) finite (see [84] for more discussion of this point). We define W_t and V_t according to (2.2.4) with $\tilde{\theta}$ in place of θ, and we define $\text{SLE}_\kappa(\rho)$ to be the radial Loewner transform of W. We now drop the $\tilde{\theta}$ notation and will write θ for the lifted process whenever $-\kappa/2 - 2 < \rho \leq -2$.

For $\rho \geq \kappa/2 - 2$, the laws of radial $\text{SLE}_\kappa(\rho)$ and ordinary radial SLE_κ are mutually absolutely continuous up to any fixed positive time, so $\text{SLE}_\kappa(\rho)$ is a.s. generated by a curve by the result of [56]. In [39] it is established that $\text{SLE}_\kappa(\rho)$ is a.s. generated by a curve for all $\rho > -2$ (see Remark 2.2.2). The continuity of radial $\text{SLE}_\kappa(\rho)$ has not yet been established for $\rho \in (-\kappa/2 - 2, -2]$. Radial $\text{SLE}_\kappa(\rho)$ in a general proper simply connected domain is defined again by conformal transformation, but the analogue of the continuity result of [20] is not known for $\rho \neq 0$.

The target invariance of radial $\text{SLE}_\kappa(\kappa - 6)$ processes continues to hold after time τ_{col}, and using this property we can construct a coupling of radial $\text{SLE}_\kappa(\kappa - 6)$

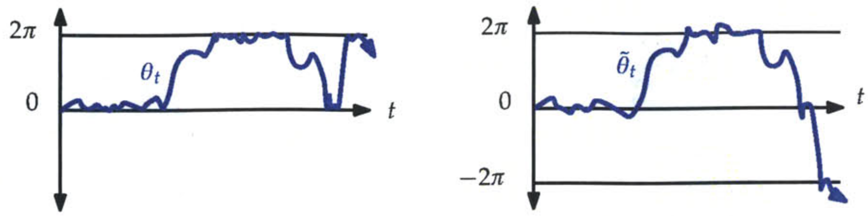


Figure 2-7: For $-\kappa/2 - 2 < \rho \leq -2$, we modify the $[0, 2\pi]$-valued process θ to obtain a process $\tilde{\theta}$ taking values in $\mathbb{R}$: after each hitting time T of $2\pi\mathbb{Z}$, an independent coin toss determines whether the next excursion of $\tilde{\theta}$ is in the interval above $\tilde{\theta}_T$ or the interval below $\tilde{\theta}_T$.

processes targeted at a countable dense subset of $\mathbb{D}$.

Proposition 2.2.1 ([65, Section 4.2]). Let $\{a_k\}_{k\in\mathbb{N}}$ be a countable dense sequence in $\mathbb{D}$. There exists a coupling of radial $\mathrm{SLE}_\kappa(\kappa-6)$ processes η^{a_k} in $\mathbb{D}$ from 1 to a_k started from $(w, v) = (1, 1e^{i0^-})$ such that for any $k, \ell \in \mathbb{N}$, η^{a_k} and η^{a_ℓ} agree a.s. (modulo time change) up to the first time that the curves separate a_k and a_ℓ and evolve independently thereafter.

From the coupling $\{\eta^{a_k}\}_{k\in\mathbb{N}}$ defined in Proposition 2.2.1, we can almost surely uniquely define (modulo time change) for each $a \in \overline{\mathbb{D}}$ a process η^a targeted at a, by considering a subsequence $\{a_{k_n}\}_{n\in\mathbb{N}}$ converging to a. Then η^a is a radial $\mathrm{SLE}_\kappa(\kappa-6)$, and we write θ_t^a, W_t^a, V_t^a for the corresponding processes of (2.2.3) and (2.2.4). The complete collection of curves $\{\eta^a\}_{a\in\overline{\mathbb{D}}}$ is the *branching* $\mathrm{SLE}_\kappa(\kappa-6)$ or *continuum exploration tree* of [65].

2.2.2 Loops from exploration trees

The CLE_κ loops $\{\mathcal{L}_a^k\}$ surrounding a point $a \in \mathbb{D}$ are defined in terms of the branch η^a of the exploration tree as follows [65] (see Figure 2-8):

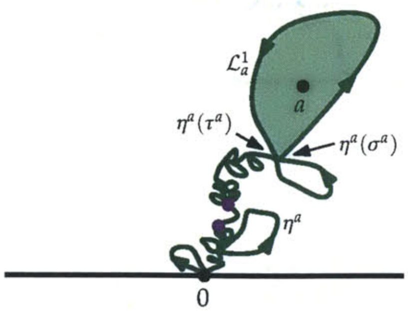
Suppose $8/3 < \kappa \leq 4$.

- Let $\tau^a = \inf\{t \geq 0 : \theta_t^a \in \{\pm 2\pi\}\}$ be the first time that η^a forms a loop around a.
- If $\tau^a = \infty$, then there are no loops surrounding a. If $\tau^a < \infty$, let $\sigma^a = \sup\{s < \tau^a : \theta_s^a = 0\}$. Then $\mathcal{L}_a^1$ is defined to be $\eta^a[\sigma^a, \tau^a]$. If $\theta_{\tau^a}^a = 2\pi$ (resp. $\theta_{\tau^a}^a = -2\pi$) then $\mathcal{L}_a^1$ has a counterclockwise (resp. clockwise) orientation.

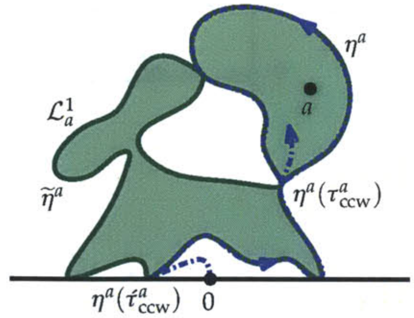
The next loop $\mathcal{L}_a^2$ is defined similarly, with the interior of $\mathcal{L}_a^1$ playing the role of $\mathbb{D}$. Continuing this process yields the full sequence $\{\mathcal{L}_a^k\}$ of loops.

Suppose $4 < \kappa < 8$.

- Let τ_{ccw}^a be the first time at which θ_t^a completes an upcrossing of an interval of the form $[2\pi k, 2\pi(k+1)]$ for $k \in \mathbb{Z}$. This is the first time η^a forms a counterclockwise loop surrounding a.



(a) $8/3 < \kappa \leq 4$



(b) $4 < \kappa < 8$

Figure 2-8: Branching $\text{SLE}_\kappa(\kappa-6)$ construction of CLE_κ process Γ in $\mathbb{H}$. (a) When $8/3 < \kappa \leq 4$, the branch η^a targeted at a traces out a countable sequence of disjoint simple loops until the first time that the process θ_t^a hits $\pm 2\pi$. If we define $\sigma^a = \sup\{s < \tau^a : \theta_s^a = 0\}$, then the outermost CLE_κ loop surrounding a is $\eta^a[\sigma^a, \tau^a]$. The schematic figure above disguises an important feature of the $\text{SLE}_\kappa(\kappa-6)$ process: η^a is tracing out a CLE loop at Lebesgue almost all times. So, for example, there are countably many small loops between the two points marked with a purple dot. (b) For each $a \in \mathbb{H}$, η^a (dashed blue line) is the branch of the exploration tree targeted at a. Marginally it evolves as a radial $\text{SLE}_\kappa(\kappa-6)$ which, whenever it hits the domain boundary or its past hull, continues in the complementary connected component containing a. Let τ^a_{ccw} be the first time t that η^a completes a counterclockwise loop surrounding a; the location of the force point at time τ^a_{ccw} is $v^a := \eta^a(\acute{\tau}^a_{\text{ccw}})$ for some $\acute{\tau}^a_{\text{ccw}} < \tau^a_{\text{ccw}}$. The outermost loop $\mathcal{L}_a^1$ of Γ surrounding a is $\eta^{v^a}|_{[\acute{\tau}^a_{\text{ccw}},\infty]}$. Successive loops are defined in analogous fashion. $\mathcal{L}_a^1$ is necessarily counterclockwise and pinned at $\eta^a(\acute{\tau}^a_{\text{ccw}})$. It is disjoint from the domain boundary if and only if a is first surrounded by a clockwise loop.

- If $\tau^a_{\text{ccw}} = \infty$ then there are no loops surrounding a and we set $\mathcal{L}_a$ to be the empty sequence. If $\tau^a_{\text{ccw}} < \infty$ let $\acute{\tau}^a_{\text{ccw}} := \sup\{t < \tau^a_{\text{ccw}} : \theta_t^a = 0\}$, let $v^a := \eta^a(\acute{\tau}^a_{\text{ccw}})$, and let $\widetilde{\eta}^a$ be the branch η^{v^a}, reparametrized so that $\widetilde{\eta}^a|_{[0,\tau^a_{\text{ccw}}]} = \eta^a|_{[0,\tau^a_{\text{ccw}}]}$. The outermost loop $\mathcal{L}_a^1$ surrounding a is defined to be $\widetilde{\eta}^a|_{[\acute{\tau}^a_{\text{ccw}},\infty]}$.

If $\mathcal{L}_a^1$ is defined, it is necessarily counterclockwise and pinned at $\eta^a(\acute{\tau}^a_{\text{ccw}})$; moreover, $\eta^a(\acute{\tau}^a_{\text{ccw}})$ is in the boundary if and only if η^a has not previously made a clockwise loop around a [65, Lemma 5.2]. The next loop $\mathcal{L}_a^2$ surrounding a is then defined in analogous fashion, and continuing in this way gives the full CLE_κ process Γ in $\mathbb{D}$. See Figs. 2-8 and 2-9.

Remark 2.2.2. For $4 < \kappa < 8$, assuming that chordal $\text{SLE}_\kappa(\kappa-6)$ processes are generated by continuous curves with reversible law [65, Conjecture 3.11], it was shown [65, Proposition 5.1 and Theorem 5.4] that CLE_κ loops are continuous, and that the law of the full ensemble is independent of the choice of root for the exploration tree. This conjecture was proved in [39, Theorem 1.3] and [41, Theorem 1.1

Figure 2-9: When $4 < \kappa < 8$, clockwise loops of η^a (dashed blue line) are not CLE loops, but correspond either to complementary connected components of CLE loops (left panel) or complementary connected components of chains of CLE loops (right panel). The CLE process is renewed within each clockwise loop (Proposition 2.2.3).

and Theorem 1.2], so the properties hold. (They are immediate for $\kappa \in (8/3, 4]$ by the equivalence of CLE_κ and the outer boundaries of loop soups [70].)

The CLE_κ process in a general proper simply connected domain is defined by conformal transformation, so the law of CLE_κ is conformally invariant. Moreover, conditional on the collection of all of the outermost loops, the law of the loops contained in the connected component D^a of $\mathbb{D} \setminus \mathcal{L}_a^1$ containing a is equal to that of a CLE_κ in D^a independently of the loops of Γ which are not contained in D^a.

Proposition 2.2.3. For any fixed countable set $\{a_j\}_{j\in\mathbb{N}} \subseteq \mathbb{D}$, conditioned on $\{\mathcal{L}_{a_j}^1\}_{j\in\mathbb{N}}$ the collections of loops within the connected components of $\mathbb{D} \setminus \bigcup_j \mathcal{L}_{a_j}^1$ are distributed as independent CLE_κ processes.

Note that when $\kappa \in (4, 8)$ a complementary component need not be surrounded by any $\mathcal{L}_a^1$; see Figure 2-9.

2.2.3 Large deviations

We review some basic results from the theory of large deviations, including the Fenchel-Legendre transform and Cramér's theorem. Let μ be a probability measure on $\mathbb{R}$. The logarithmic moment generating function, also known as the cumulant generating function, of μ is defined by

$$\Lambda(\lambda) = \Lambda_\mu(\lambda) = \log \mathbb{E}\left[e^{\lambda X}\right],$$

where X is a random variable with law μ. The Fenchel-Legendre transform $\Lambda^\star : \mathbb{R} \to [0,\infty]$ of Λ is given by [12, Section 2.2]

$$\Lambda^\star(x) := \sup_{\lambda\in\mathbb{R}}(\lambda x - \Lambda(\lambda)) .$$

We now recall Cramér's theorem in $\mathbb{R}$, as stated in [12, Theorem 2.2.3]:

Theorem 2.2.4 (Cramér's theorem). Let X be a real-valued random variable and let Λ be the logarithmic moment generating function of the distribution of X. Let $S_n = \sum_{i=1}^n X_i$ be a sum of i.i.d. copies of X. For every closed set $F \subset \mathbb{R}$ and open set $G \subset \mathbb{R}$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left[\frac{1}{n}S_n \in F\right] \leq -\inf_{y\in F} \Lambda^\star(y), \text{ and}$$
$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left[\frac{1}{n}S_n \in G\right] \geq -\inf_{y\in G} \Lambda^\star(y) .$$

Moreover,

$$\mathbb{P}\left[\frac{1}{n}S_n \in F\right] \leq 2\exp\left(-n \inf_{y\in F} \Lambda^\star(y)\right) . \tag{2.2.5}$$

Following [12, Section 2.2.1], we let $\mathcal{D}_\Lambda := \{\lambda : \Lambda(\lambda) < \infty\}$ and $\mathcal{D}_{\Lambda^\star} = \{x : \Lambda^\star(x) < \infty\}$ be the sets where Λ and $\Lambda^\star$ are finite, respectively, and let $\mathcal{F}_\Lambda = \{\Lambda'(\lambda) : \lambda \in \mathcal{D}_\Lambda^\circ\}$, where A° denotes the interior of a set $A \subset \mathbb{R}$. The following proposition summarizes some basic properties of Λ and $\Lambda^\star$.

Proposition 2.2.5. Suppose that μ is a probability measure on $\mathbb{R}$, let Λ be its log moment generating function, and assume that $\mathcal{D}_\Lambda \neq \{0\}$. Let a and b denote the essential infimum and supremum of a μ-distributed random variable X (with $a = -\infty$ and/or $b = \infty$ allowed). Then Λ and its Fenchel-Legendre transform $\Lambda^\star$ have the following properties:

(i) Λ and $\Lambda^\star$ are convex
(ii) $\Lambda^\star$ is nonnegative
(iii) $\mathcal{F}_\Lambda \subset \mathcal{D}_{\Lambda^\star}$
(iv) Λ is smooth on $\mathcal{D}_\Lambda^\circ$ and $\Lambda^\star$ is smooth on $\mathcal{F}_\Lambda^\circ$
(v) If $\mathcal{D}_\Lambda = \mathbb{R}$, then $\mathcal{F}_\Lambda^\circ = (a,b)$
(vi) If $(-\infty,0] \subset \mathcal{D}_\Lambda$, then $(a, a+\delta) \subset \mathcal{F}_\Lambda^\circ$ for some $\delta > 0$
(vii) If $[0,\infty) \subset \mathcal{D}_\Lambda$, then $(b-\delta, b) \subset \mathcal{F}_\Lambda^\circ$ for some $\delta > 0$
(viii) $\Lambda^\star$ is continuously differentiable on (a,b)
(ix) If $-\infty < a$, then $(\Lambda^\star)'(x) \to -\infty$ as $x \downarrow a$
(x) If $b < \infty$, then $(\Lambda^\star)'(x) \to +\infty$ as $x \uparrow b$

Proof. For ((i))–((iv)), we refer the reader to [12, Section 2.2.1].

To prove ((v)), note that

$$\Lambda'(\lambda) = \frac{\mathbb{E}[Xe^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]}.$$

Therefore,

$$a = \frac{\mathbb{E}[ae^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]} \le \Lambda'(\lambda) \le \frac{\mathbb{E}[be^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]} = b\,.$$

Thus $\mathcal{F}_\Lambda \subseteq [a,b]$, which gives $\mathcal{F}_\Lambda^\circ \subseteq (a,b)$.

This leaves us to prove the reverse inclusion. Suppose $c \in (a,b)$, and let $Y = X - c$.

Then

$$\Lambda'(\lambda) = \frac{\mathbb{E}[Xe^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]} = c + \frac{\mathbb{E}[Ye^{\lambda Y}]}{\mathbb{E}[e^{\lambda Y}]} = c + \frac{\mathbb{E}[\mathbf{1}_{\{Y\ge 0\}}Ye^{\lambda Y}] + \mathbb{E}[\mathbf{1}_{\{Y<0\}}Ye^{\lambda Y}]}{\mathbb{E}[e^{\lambda Y}]}\,.$$

Since $\mathcal{D}_\Lambda = \mathbb{R}$, the tails of X and Y decay rapidly enough for each of the above expected values to be finite. Since $\mathbb{P}[Y > 0] > 0$, the first term in the numerator diverges as $\lambda \to \infty$, while the second term decreases monotonically in absolute value. So for sufficiently large λ, we have $\Lambda'(\lambda) > c$. Similarly, for sufficiently large negative λ we have $\Lambda'(\lambda) \le c$. Since Λ is smooth, Λ' is continuous, so $c \in \mathcal{F}_\Lambda$. The proofs of ((vi)) and ((vii)) are analogous.

To prove ((viii)), note that $\mathcal{F}_\Lambda^\circ = (\tilde{a},\tilde{b})$ for some $a \le \tilde{a} < \tilde{b} \le b$. By ((iv)), $\Lambda^\star$ is smooth on $(\tilde{a},\tilde{b})$. Therefore, it suffices to consider the possibility that $a < \tilde{a}$ or $\tilde{b} < b$. Suppose first that $\tilde{b} < b$. By the proof of ((v)), $\tilde{b} < b$ implies that $\mathcal{D}_\Lambda = (\lambda_1,\lambda_2)$ for some $\lambda_2 < \infty$. Furthermore, observe that $\Lambda'(\lambda) \to \tilde{b}$ as $\lambda \nearrow \lambda_2$. It follows that $\Lambda(\lambda_2) < \infty$, and by convexity of Λ we have for all $\tilde{b} \le x < b$,

$$\Lambda^\star(x) = \sup_\lambda[x\lambda - \Lambda(\lambda)] = x\lambda_2 - \Lambda(\lambda_2)\,.$$

In other words, $\Lambda^\star$ is smooth on $(\tilde{a},\tilde{b})$ and is affine on $(\tilde{b},b)$ with slope matching the left-hand derivative at $\tilde{b}$. Similarly, if $a < \tilde{a}$, then $\Lambda^\star$ is affine on $(a,\tilde{a})$ with slope matching the right-hand derivative of $\Lambda^\star$ at $\tilde{a}$. Therefore, $\Lambda^\star$ is continuously differentiable on (a,b).

To prove ((ix)), we note that since X is bounded below, $D_\Lambda^\circ = (-\infty,\xi)$ for some $0 \le \xi \le +\infty$. Moreover, there exists $\varepsilon > 0$ so that $(a,a+\varepsilon) \subset \mathcal{F}_\Lambda^\circ$, by essentially the same argument we used to prove ((v)) above. Let $\widehat{\mathcal{D}} = \{\lambda : \Lambda'(\lambda) \in (a,a+\varepsilon)\}$, and note that the left endpoint of $\widehat{\mathcal{D}}$ is $-\infty$. Since Λ' is smooth and strictly increasing on $\widehat{\mathcal{D}}$ (see [12, Exercise 2.2.24]), there exists a monotone bijective function $\lambda : (a,a+\varepsilon) \to \widehat{\mathcal{D}}$ for which $\Lambda'(\lambda(x)) = x$. In the definition of $\Lambda^\star(x)$, the supremum is achieved at $\lambda = \lambda(x)$. Differentiating, we obtain

$$\begin{aligned}(\Lambda^\star)'(x) &= \frac{d}{dx}[x\lambda(x) - \Lambda(\lambda(x))] \\ &= \lambda(x) + x\lambda'(x) - \lambda'(x)\Lambda'(\lambda(x)) \\ &= \lambda(x)\,.\end{aligned} \tag{2.2.6}$$

Since the monotonicity of λ implies that $\lambda(x) \to -\infty$ as $x \to a$, this concludes the proof.

The proof of ((x)) is similar. ☐

The following is a simple corollary of Cramér's theorem which is applicable to a sequence of bounded, closed intervals contained in $\mathcal{F}_\Lambda^\circ$. The proof is routine and is omitted.

Corollary 2.2.6. Suppose $c, d \in \mathbb{R}$, $c < d$, and $[c, d] \subset \mathcal{F}_\Lambda^\circ$. If $c_n \to c$ and $d_n \to d$, then

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left[\frac{1}{n} S_n \in [c_n, d_n]\right] = -\inf_{y\in[c,d]} \Lambda^\star(y) .$$

We also have the following adaptation of Cramér's theorem for which the number of i.i.d. summands is not fixed.

Corollary 2.2.7. Let X be a positive real-valued random variable with exponential tails (that is, $\mathbb{E}[e^{\lambda_0 X}] < \infty$ for some $\lambda_0 > 0$), and let $\Lambda(\lambda) = \log \mathbb{E}[e^{\lambda X}]$. Let $S_n = \sum_{i=1}^n X_i$ be a sum of i.i.d. copies of X, and let $N_r = \min\{n : S_n \geq r\}$. If $0 < v_1 < v_2$, then

$$\lim_{r\to\infty} \frac{1}{r} \log \mathbb{P}[v_1 r \leq N_r \leq v_2 r] = -\inf_{v\in[v_1,v_2]} v\Lambda^\star(1/v) . \tag{2.2.7}$$

This is the origin of the expression $v\Lambda^\star(1/v)$ in (2.1.4).

Proof. Recall that

$$\Lambda^\star(x) = \sup_\lambda [\lambda x - \Lambda(\lambda)] ,$$

where the bracketed expression is 0 when $\lambda = 0$. Because X has exponential tails, $\Lambda'(0) = \mathbb{E}[X]$ exists. If $x < \mathbb{E}[X]$, then for some sufficiently small negative λ, the bracketed expression is positive, so $\Lambda^\star(x) > 0$. Likewise, if $x > \mathbb{E}[X]$ then $\Lambda^\star(x) > 0$. Recall also that $\Lambda^\star(\mathbb{E}[X]) = 0$.

Let $a = \operatorname{ess\,inf} X \in [0, \infty)$ be the essential infimum of X and $b = \operatorname{ess\,sup} X \in (0, \infty]$ be the essential supremum of X.

Because $\Lambda^\star$ is convex on $[a, b]$, by Lemma 2.2.8 proved below, $v\Lambda^\star(1/v)$ is convex on $[1/b, 1/a]$. The expression $v\Lambda^\star(1/v)$ is 0 when $v = 1/\mathbb{E}[X]$ and is positive elsewhere on $[1/b, 1/a]$, so it is strictly decreasing for $v \leq 1/\mathbb{E}[X]$ and strictly increasing for $v \geq 1/\mathbb{E}[X]$.

There are three possible cases for the relative order of $1/\mathbb{E}[X]$, v_1, and v_2. For example, suppose $v_1 < v_2 < 1/\mathbb{E}[X]$. We write

$$\{v_1 r \leq N_r \leq v_2 r\} = \left\{ \sum_{1\leq i\leq \lceil v_1 r\rceil - 1} X_i < r \right\} \cap \left\{ \sum_{1\leq i\leq \lfloor v_2 r\rfloor} X_i \geq r \right\} =: E_r \cap F_r .$$

Since $v\Lambda^\star(1/v)$ is continuous on $(1/b, 1/a)$, by Cramér's theorem,

$$\mathbb{P}[E_r^c] = e^{-v_1 r\Lambda^\star(1/v_1)(1+o(1))} \quad \text{and} \quad \mathbb{P}[F_r] = e^{-v_2 r\Lambda^\star(1/v_2)(1+o(1))} ,$$

except when $1/\nu_1 = 1/b$, in which case the expression for $\mathbb{P}[E_r^c]$ becomes an upper bound. Therefore,

$$\mathbb{P}[E_r \cap F_r] = P[F_r] - \mathbb{P}[F_r \cap E_r^c] = e^{-\nu_2 r \Lambda^\star(1/\nu_2)(1+o(1))},$$

which gives (2.2.7). The proof for the case $1/\mathbb{E}[X] < \nu_1 < \nu_2$ is analogous, and in the case $\nu_1 < 1/\mathbb{E}[X] < \nu_2$, both sides of (2.2.7) are 0. □

Lemma 2.2.8. Suppose that f is a convex function on $[a,b] \subseteq [0,\infty]$. Then $x \mapsto xf(1/x)$ is a convex function on $[1/b, 1/a]$.

Proof. Since f is convex, it can be expressed as $f(x) = \sup_i(\alpha_i + \beta_i x)$ for some pair of sequences of reals $\{\alpha_i\}_{i\in\mathbb{N}}$ and $\{\beta_i\}_{i\in\mathbb{N}}$. For $x \in [0,\infty]$ we can write $xf(1/x) = \sup_i(\alpha_i x + \beta_i)$, so it too is convex. □

Proposition 2.2.9. Let X be a nonnegative real-valued random variable, and let $\Lambda(\lambda) = \log \mathbb{E}[e^{\lambda X}]$. Then

$$\lim_{\nu\downarrow 0} \nu\Lambda^\star(1/\nu) = \sup\{\lambda : \Lambda(\lambda) < \infty\}. \tag{2.2.8}$$

Proof. Let $\lambda_0 = \sup\{\lambda : \Lambda(\lambda) < \infty\}$, and note that $0 \le \lambda_0 \le \infty$. Recall that $\Lambda^\star(x) := \sup_\lambda(\lambda x - \Lambda(\lambda))$, so

$$\nu\Lambda^\star(1/\nu) = \sup_\lambda(\lambda - \nu\Lambda(\lambda)).$$

The supremum is not achieved for any $\lambda > \lambda_0$. If $\lambda_0 > 0$, then $\mathbb{E}[X] < \infty$ and for $\nu \le 1/\mathbb{E}[X]$ the supremum is achieved over the set $\lambda \ge 0$ [12, Lem. 2.2.5(b)]. For any $\lambda \ge 0$ we have $\Lambda(\lambda) \ge 0$, so $\nu\Lambda^\star(1/\nu) \le \lambda_0$ for $0 < \nu \le 1/\mathbb{E}[X]$. On the other hand, for any $\lambda < \lambda_0$ we have $\Lambda(\lambda) < \infty$, so $\liminf_{\nu\downarrow 0} \nu\Lambda^\star(1/\nu) \ge \lambda$. Thus $\lim_{\nu\downarrow 0} \nu\Lambda^\star(1/\nu) = \lambda_0$ when $\lambda_0 > 0$.

Next suppose $\lambda_0 = 0$. Then the supremum is achieved over the set $\lambda \le 0$, for which $\Lambda(\lambda) \le 0$. For any $\varepsilon > 0$, there is a $\delta > 0$ for which $-\varepsilon \le \Lambda(\lambda) \le 0$ whenever $-\delta \le \lambda \le 0$. Since $\lambda_0 = 0$, $\Pr[X = 0] < 1$, so $\Lambda(-\delta) < 0$. Let $\nu_0 = -\delta/\Lambda(-\delta)$. By the convexity of Λ, for $0 \le \nu \le \nu_0$, the supremum is achieved for $\lambda \in [-\delta, 0]$. For λ in this range, $\lambda - \nu\Lambda(\lambda) \le \varepsilon\nu$, so $0 \le \nu\Lambda^\star(1/\nu) \le \varepsilon\nu$ when $0 < \nu \le \nu_0$. Hence $\lim_{\nu\downarrow 0} \nu\Lambda^\star(1/\nu) = 0$ when $\lambda_0 = 0$. □

We conclude by giving a parametrization of the graph of the function γ_κ over the interval $(0,\infty)$.

Proposition 2.2.10. Recall the definition of Λ_κ in (2.1.3). The graph of γ_κ over the interval $(0,\infty)$ is equal to the set

$$\left\{\left(\frac{1}{\Lambda_\kappa'(\lambda)}, \lambda - \frac{\Lambda_\kappa(\lambda)}{\Lambda_\kappa'(\lambda)}\right) : -\infty < \lambda < 1 - \frac{2}{\kappa} - \frac{3\kappa}{32}\right\}. \tag{2.2.9}$$

Proof. Recall that $\Lambda_\kappa^*(x) = \sup_{\lambda\in\mathbb{R}}[\lambda x - \Lambda_\kappa(\lambda)]$. Since Λ_κ' is continuous and strictly increasing, the maximizing value of λ for a given value of x is the unique $\lambda \in \mathbb{R}$ such that $\Lambda_\kappa'(\lambda) = x$. If we let λ be this maximizing value, then we have

$$\Lambda_\kappa^*(x) = \lambda x - \Lambda_\kappa(\lambda). \tag{2.2.10}$$

Differentiating (2.1.3) shows that as λ ranges from $-\infty$ to $1 - 2/\kappa - 3\kappa/32$, $\Lambda_\kappa'(\lambda)$ ranges from 0 to ∞. Using (2.2.10) and writing $\nu = 1/x$, we obtain

$$\nu = \frac{1}{x} = \frac{1}{\Lambda_\kappa'(\lambda)},$$

and

$$\nu\Lambda_\kappa^*(1/\nu) = \frac{1}{\Lambda_\kappa'(\lambda)}\left(\lambda\Lambda_\kappa'(\lambda) - \Lambda_\kappa(\lambda)\right) = \lambda - \frac{\Lambda_\kappa(\lambda)}{\Lambda_\kappa'(\lambda)}.$$

Therefore, $\{(\nu, \nu\Lambda_\kappa^*(1/\nu)) : 0 < \nu < \infty\}$ is equal to (2.2.9). □

Proposition 2.2.11. The function γ_κ is strictly convex over $[0, \infty)$.

Proof. Define $x(\lambda) = 1/\Lambda_\kappa'(\lambda)$ and $y(\lambda) = \lambda - \Lambda_\kappa(\lambda)/\Lambda_\kappa'(\lambda)$. By Proposition 2.2.10, the second derivative of γ_κ is given by

$$\frac{\frac{d}{d\lambda}\left[\frac{y'(\lambda)}{x'(\lambda)}\right]}{x'(\lambda)} = \frac{8\pi^2\sin^2\left(\frac{\pi}{\kappa}\sqrt{8\kappa\lambda + (\kappa-4)^2}\right)\tan\left(\frac{\pi}{\kappa}\sqrt{8\kappa\lambda + (\kappa-4)^2}\right)}{2\pi\sqrt{8\kappa\lambda + (\kappa-4)^2} - \kappa\sin\left(\frac{2\pi}{\kappa}\sqrt{8\kappa\lambda + (\kappa-4)^2}\right)}. \tag{2.2.11}$$

It is straightforward to confirm that $\sin^2 t\tan t/(2t - \sin(2t)) > 0$ for all $t \in [0, \pi/2)$ (where we extend the definition to $t = 0$ by taking the limit of the expression as $t \searrow 0$). Similarly, $\sinh^2(2t)\tanh(t)/(\sinh(2t) - 2t) > 0$ for all $t \le 0$ (again extending to $t = 0$ by taking a limit). Setting $t = \frac{\pi}{\kappa}\sqrt{8\kappa\lambda + (\kappa-4)^2}$, these observations imply that the second derivative of γ_κ is positive for all λ less than $1 - 2/\kappa - 3\kappa/32$. □

2.2.4 Overshoot estimates

Let $\{S_n\}_{n\in\mathbb{N}}$ be a random walk in $\mathbb{R}$ whose increments are nonnegative and have exponential moments. In this section, we will bound the tails of S_n stopped at

(i) the first time that it exceeds a given threshold (Lemma 2.2.12) and at
(ii) a random time which is stochastically dominated by a geometric random variable (Lemma 2.2.13).

Lemma 2.2.12. Suppose $\{X_j\}_{j\in\mathbb{N}}$ are nonnegative i.i.d. random variables for which $\mathbb{E}[X_1] > 0$ and $\mathbb{E}[e^{\lambda_0 X_1}] < \infty$ for some $\lambda_0 > 0$. Let $S_n = \sum_{j=1}^n X_j$ and $\tau_x = \inf\{n \ge 0 : S_n \ge x\}$. Then there exists $C > 0$ (depending on the law of X_1 and λ_0) such that $\mathbb{P}[S_{\tau_x} - x \ge \alpha] \le C\exp(-\lambda_0\alpha)$ for all $x \ge 0$ and $\alpha > 0$.

Proof. Since $\mathbb{E}[X_1] > 0$, we may choose $v > 0$ so that $\mathbb{P}[X_1 \geq v] \geq \frac{1}{2}$. We partition $(-\infty, x)$ into intervals of length v:

$$(-\infty, x) = \bigcup_{k=0}^{\infty} I_k \qquad \text{where} \qquad I_k = [x-(k+1)v, x-kv)\,.$$

Then we partition the event $S_{\tau_x} - x \geq \alpha$ into subevents

$$\{S_{\tau_x} - x \geq \alpha\} = \bigcup_{n=0}^{\infty}\bigcup_{k=0}^{\infty} E_{n,k} \qquad \text{where} \qquad E_{n,k} = \{S_n \in I_k, S_{n+1} \geq x+\alpha\}\,.$$

The event $E_{n,k}$ implies $X_{n+1} \geq kv + \alpha$, and since X_{n+1} is independent of S_n, we have

$$\mathbb{P}[E_{n,k}] \leq \mathbb{P}[S_n \in I_k] \times \frac{\mathbb{E}[e^{\lambda_0 X}]}{e^{\lambda_0(kv+\alpha)}}\,.$$

On the event $S_n \in I_k$, since X_{n+1} is independent of what occurred earlier and is larger than v with probability at least $\frac{1}{2}$, we have $\mathbb{P}[S_{n+1} \in I_k | S_n \in I_k, S_{n-1}, \dots, S_1] \leq \frac{1}{2}$. Thus

$$\sum_{n=0}^{\infty} \mathbb{P}[S_n \in I_k] = \mathbb{E}[|\{n : S_n \in I_k\}|] \leq 2\,.$$

Thus

$$\mathbb{P}[S_{\tau_x} - x \geq \alpha] \leq \sum_{k=0}^{\infty} \frac{2\,\mathbb{E}[e^{\lambda_0 X}]}{e^{\lambda_0(kv+\alpha)}} \leq \frac{2\,\mathbb{E}[e^{\lambda_0 X}]}{1-e^{-\lambda_0 v}} \times e^{-\lambda_0 \alpha}\,.$$

□

Lemma 2.2.13. Let $\{X_j\}_{j\in\mathbb{N}}$ be an i.i.d. sequence of random variables and let $S_n = \sum_{i=1}^{n} X_i$. Let N be a positive integer-valued random variable, which need not be independent of the X_j's. Suppose that there exists $\lambda_0 > 0$ for which $\mathbb{E}[e^{\lambda_0 X_1}] < \infty$ and $q \in (0,1)$ for which $\mathbb{P}[N \geq k] \leq q^{k-1}$ for every $k \in \mathbb{N}$. Then there exist constants $C, c > 0$ (depending on q and the law of X_1) for which $\mathbb{P}[S_N > \alpha] \leq C\exp(-c\alpha)$ for every $\alpha > 0$.

Proof. Since $q\mathbb{E}[e^{\lambda X}]$ is a continuous function of λ which is finite for $\lambda = \lambda_0 > 0$ and less than 1 for $\lambda = 0$, there is some $c > 0$ for which $q\mathbb{E}[e^{2cX}] < 1$. The Cauchy-Schwarz inequality gives

$$\begin{aligned}
\mathbb{E}[e^{cS_N}] &= \mathbb{E}\left[\sum_{k=1}^{\infty} e^{cS_k}\mathbf{1}_{\{N=k\}}\right] \\
&\leq \sum_{k=1}^{\infty} \sqrt{\mathbb{E}[e^{2cS_k}]\mathbb{P}[N=k]} \\
&= q^{-1/2}\sum_{k=1}^{\infty}\left(\sqrt{\mathbb{E}[e^{2cX_1}]q}\right)^k < \infty.
\end{aligned}$$

We conclude using the Markov inequality $\mathbb{P}[S_N > \alpha] \leq e^{-c\alpha}\mathbb{E}[e^{cS_N}]$. □

2.2.5 CLE estimates

We establish two technical estimates in this section. Lemma 2.2.18 uses Cramér's theorem to compute the asymptotics of the probability that the number $\mathcal{N}_z(\varepsilon)$ of CLE loops surrounding $B(z,\varepsilon)$ has a certain rate of growth as $\varepsilon \to 0$. We begin by reminding the reader of the Koebe distortion theorem and the Koebe quarter theorem.

Theorem 2.2.14. (Koebe distortion theorem) If $f : \mathbb{D} \to \mathbb{C}$ is an injective analytic function and $f(0) = 0$, then

$$\frac{r}{(1+r)^2}|f'(0)| \le |f(re^{i\theta})| \le \frac{r}{(1-r)^2}|f'(0)|, \quad \text{for } \theta \in \mathbb{R} \text{ and } 0 \le r < 1\,.$$

The lower bound implies the Koebe quarter theorem, which says that $B(0,|f'(0)|/4) \subset f(\mathbb{D})$ [30, Theorem 3.17]. Combining the quarter theorem with the Schwarz lemma [30, Lemma 2.1], we obtain the following corollary.

Corollary 2.2.15. If $D \subsetneq \mathbb{C}$ is a simply connected domain, $z \in D$, and $f : \mathbb{D} \to D$ is a conformal map sending 0 to z, then the inradius $\operatorname{inrad}(z;D) := \inf_{w\in\mathbb{C}\setminus D} |z-w|$ and the conformal radius $\operatorname{CR}(z;D) := |f'(0)|$ satisfy

$$\operatorname{inrad}(z;D) \le \operatorname{CR}(z;D) \le 4 \operatorname{inrad}(z;D)\,.$$

We also record the following corollary of the distortion theorem, see [30, Proposition 3.26] for a proof.

Corollary 2.2.16. There exists a constant C_r such that for all conformal maps f from the unit disk to $\mathbb{C}$ with $f(0) = 0$ and $f'(0) = 1$ and for all $|z| \le r$,

$$|f(z) - z| \le C_r|z|^2.$$

Furthermore, by de Branges's theorem this statement holds with $C_r = (2-r)/(1-r)^2$.

In preparation for the proof of Lemma 2.2.18 below, we establish the continuity of the function γ_κ defined in (2.1.4). Throughout, we let $\nu_{\max}$ be the unique solution to $\gamma_\kappa(\nu) = 2$.

Proposition 2.2.17. The function γ_κ is continuous. In particular,

$$\lim_{\nu\downarrow 0} \gamma_\kappa(\nu) = 1 - \frac{2}{\kappa} - \frac{3\kappa}{32}\,. \tag{2.2.12}$$

Recall that two minus the quantity on the right side of (2.2.12) is the almost-sure Hausdorff dimension of the CLE_κ gasket [61, 51, 45].

Proof of Proposition 2.2.17. The continuity of γ_κ on $(0,\infty)$ follows from Proposition 2.2.5 ((iv)), and the continuity at 0 follows from Proposition 2.2.9 and the fact that (2.1.3) blows up at $\lambda_0 = 1 - \frac{2}{\kappa} - \frac{3\kappa}{32}$ but not for $\lambda < \lambda_0$. $\square$

Lemma 2.2.18. Let $\kappa \in (8/3,8)$, $0 \leq \nu \leq \nu_{\max}$, and $0 < a \leq b$. Then for all functions $\varepsilon \mapsto \delta(\varepsilon)$ decreasing to 0 sufficiently slowly as $\varepsilon \to 0$ and for all proper simply connected domains D and points $z \in D$ satisfying $a \leq \mathrm{CR}(z;D) \leq b$, we have

$$\begin{cases} \lim_{\varepsilon\to 0} \dfrac{\log \mathbb{P}[\nu \leq \widetilde{\mathcal{N}}_z(\varepsilon) \leq \nu + \delta(\varepsilon)]}{\log \varepsilon} = \gamma_\kappa(\nu) \quad \text{for } \nu > 0 \\ \lim_{\varepsilon\to 0} \dfrac{\log \mathbb{P}[\frac{1}{2}\delta(\varepsilon) \leq \widetilde{\mathcal{N}}_z(\varepsilon) \leq \delta(\varepsilon)]}{\log \varepsilon} = \gamma_\kappa(0) \quad \text{for } \nu = 0\,, \end{cases} \tag{2.2.13}$$

where γ_κ is defined in (2.1.4), and the convergence is uniform in the domain D.

Proof. Let $\{T_i\}_{i\in\mathbb{N}}$ be the sequence of log conformal radius increments associated with z. That is, defining U_z^i to be the connected component of $D \setminus \mathcal{L}_z^i$ which contains z, we have $T_i := \log \mathrm{CR}(z;U_z^{i-1}) - \log \mathrm{CR}(z;U_z^i)$. Let

$$S_n := \sum_{i=1}^n T_i = \log \mathrm{CR}(z;D) - \log \mathrm{CR}(z;U_z^n) \quad \text{for } n \in \mathbb{N}\,.$$

As in Corollary 2.2.7, we let $N_r = \min\{n : S_n \geq r\}$.

By Corollary 2.2.15 and the hypotheses of the lemma, we have

$$\log(a/4) - \log \mathrm{inrad}(z;U_z^n) \leq S_n \leq \log b - \log \mathrm{inrad}(z;U_z^n)\,. \tag{2.2.14}$$

Suppose first that $\nu > 0$. Let

$$E := \{(\nu+\eta)\log(1/\varepsilon) \leq N_{\log(a/4)+\log(1/\varepsilon)}\}, \text{ and}$$
$$F := \{N_{\log(b)+\log(1/\varepsilon)} \leq (\nu+\delta_0-\eta)\log(1/\varepsilon)\}.$$

It follows from (2.2.14) that for all fixed $\delta_0 > 0$ and $0 < \eta < \delta_0/2$ and for all $\varepsilon = \varepsilon(\eta) > 0$ sufficiently small, we have

$$\{\nu \leq \widetilde{\mathcal{N}}_z(\varepsilon) \leq \nu + \delta_0\} \supset E \cap F.$$

By Corollary 2.2.7, $\log \mathbb{P}[E]/\log\varepsilon \to \inf_{\xi\in[\nu+\eta,\nu+\delta_0-\eta]} \gamma_\kappa(\xi)$. Furthermore, Cramér's theorem implies that $\mathbb{P}[F \mid E] = \varepsilon^{o(1)}$. It follows that

$$\liminf_{\varepsilon\to 0} \frac{\log \mathbb{P}[\nu \leq \widetilde{\mathcal{N}}_z(\varepsilon) \leq \nu+\delta_0]}{\log\varepsilon} \geq \inf_{\xi\in[\nu+\eta,\nu+\delta_0-\eta]} \gamma_\kappa(\xi)\,.$$

Letting $\eta \to 0$ and using an analogous argument to upper bound the limit supremum of the quotient on the left-hand side, we find that

$$\lim_{\varepsilon\to 0} \frac{\log \mathbb{P}[\nu \leq \widetilde{\mathcal{N}}_z(\varepsilon) \leq \nu+\delta_0]}{\log\varepsilon} = \inf_{\xi\in[\nu,\nu+\delta_0]} \gamma_\kappa(\xi)\,.$$

By the continuity of γ_κ on $[0,\infty)$, we may choose $\delta(\varepsilon) \downarrow 0$ so that (2.2.13) holds. The proof for $\nu = 0$ is similar. As above, we show that for $\delta_0 > 0$ fixed, we have

$$\lim_{\varepsilon \to 0} \frac{\log \mathbb{P}[\delta_0/2 \le \widetilde{\mathcal{N}}_z(\varepsilon) \le \delta_0]}{\log \varepsilon} = \inf_{\xi \in [\delta_0/2, \delta_0]} \gamma_\kappa(\xi) .$$

Again, choose $\delta(\varepsilon) \downarrow 0$ so that (2.2.13) holds. □

2.3 Full-plane CLE

Let $D \subsetneq \mathbb{C}$ be a proper simply connected domain and let Γ be a CLE_κ in D. For each $z \in D$, let $\mathcal{L}_z^j$ be the jth largest loop of Γ which surrounds z. In Section 2.3.1 we estimate the tail behavior of the number of such loops $\mathcal{L}_z^j$ which intersect the boundary of a ball $B(z,r)$ in D. Using these estimates, in Section 2.3.2 we show rapid convergence of CLE_κ on large domains D as $D \to \mathbb{C}$.

2.3.1 Regularity of CLE

Lemma 2.3.1. For each $\kappa \in (8/3, 8)$ there exists $p_1 = p_1(\kappa) > 0$ such that for any proper simply connected domain D and $z \in D$,

$$\mathbb{P}[\mathcal{L}_z^1 \cap \partial D = \emptyset] \ge p_1 > 0.$$

Proof. If $\kappa \in (8/3, 4]$, we can take $p_1 = 1$ since the loops of such CLEs almost surely do not intersect the boundary of D. Assume $\kappa \in (4, 8)$. By the conformal invariance of CLE, the boundary avoidance probability is independent of the domain D and the point z, so we take $D = \mathbb{D}$. Let $\eta = \eta^0$ be the branch of the exploration process of Γ targeted at 0 and let (W, V) be the driving pair for η. Let τ be an almost surely positive and finite stopping time such that $\eta|_{[0,\tau]}$ almost surely does not surround 0 and $\eta(\tau) \ne V_\tau$ almost surely. Then $\eta|_{[\tau,\infty)}$ evolves as an ordinary chordal SLE_κ process in the connected component of $\mathbb{D} \setminus \eta([0,\tau])$ containing 0 targeted at V_τ, up until disconnecting V_τ from 0. In particular, $\eta|_{(\tau,\infty)}$ almost surely intersects the right side of $\eta|_{[0,\tau]}$ before surrounding 0. Since η is almost surely not space filling [56] and cannot trace itself, this implies that, almost surely, there exists $z \in \mathbb{Q}^2 \cap \mathbb{D}$ such that the probability that η makes a clockwise loop around z before surrounding 0 is positive. This in turn implies that with positive probability, the branch η^z of the exploration tree targeted at z makes a clockwise loop around z before making a counterclockwise loop around z. By [65, Lemma 5.2], this implies $\mathbb{P}[\mathcal{L}_z^1 \cap \partial \mathbb{D} = \emptyset] > 0$. □

Suppose that $D = \mathbb{D}$. By the continuity of CLE_κ loops, Lemma 2.3.1 implies there exists $r_0 = r_0(\kappa) < 1$ such that

$$\mathbb{P}[\mathcal{L}_0^1 \subset B(0, r_0)] \ge \frac{p_1}{2}. \tag{2.3.1}$$

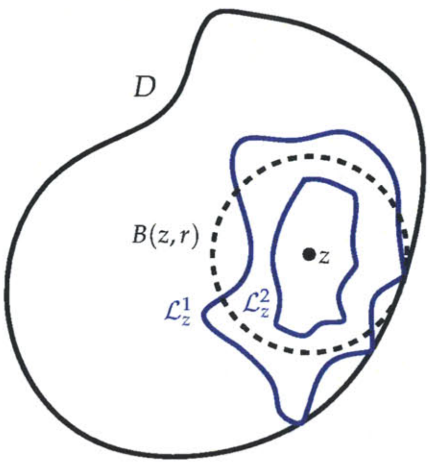


Figure 2-10: Illustration of Lemma 2.3.2. With uniformly positive probability $p = p(\kappa) > 0$, the second outermost CLE_κ loop surrounding a point $z \in D$ is contained within the largest ball $B(z,r)$ centered at z and contained within the domain.

Lemma 2.3.2. For each $\kappa \in (8/3, 8)$ there exists $p = p(\kappa) > 0$ such that for any proper simply connected domain D and $z \in D$,

$$\mathbb{P}[\mathcal{L}_z^2 \subseteq B(z, \text{dist}(z, \partial D))] \geq p.$$

(See Figure 2-10.)

Proof. Let D_1 be the connected component of $D \setminus \mathcal{L}_z^1$ which contains z and let $X = \text{CR}(z; D_1) / \text{CR}(z; D) \leq 1$. Let $\varphi\colon \mathbb{D} \to D_1$ be a conformal map with $\varphi(0) = z$, and let $r = \text{dist}(z, \partial D)$. By Corollary 2.2.15, we have $\text{CR}(z; D) \leq 4r$, hence

$$|\varphi'(0)| = \text{CR}(z; D_1) = \text{CR}(z; D) \cdot \frac{\text{CR}(z; D_1)}{\text{CR}(z; D)} \leq 4Xr.$$

Theorem 2.2.14 then implies for $|w| < r_0 < 1$, we have

$$|\varphi(w) - z| \leq 4Xr \frac{r_0}{(1 - r_0)^2}. \tag{2.3.2}$$

Since the distribution of $-\log X$ has a positive density on $(0, \infty)$ [61], the probability of the event $E = \{X \leq (1 - r_0)^2 / (4r_0)\}$ is bounded below by $p_2 = p_2(\kappa) > 0$ depending only κ. On E, the right hand side of (2.3.2) is bounded above by r, i.e., $\varphi(r_0 \mathbb{D}) \subset B(z, r)$. By the conformal invariance and renewal property of CLE, the loop $\mathcal{L}_z^2$ in D is distributed as the image under φ of the loop $\mathcal{L}_0^1$ in $\mathbb{D}$, which is independent of X. Thus, by (2.3.1), $\mathbb{P}[\mathcal{L}_z^2 \subseteq B(z,r)] \geq \mathbb{P}[E] \mathbb{P}[\mathcal{L}_z^2 \subseteq B(z,r) \mid E] \geq (p_2)(p_1/2) =: p > 0$. $\square$

For the CLE$_\kappa$ Γ in D, $z \in D$ and $r > 0$ we define

$$J^{\cap}_{z,r} := \min\{j \geq 1 : \mathcal{L}^j_z \cap B(z,r) \neq \varnothing\} \tag{2.3.3a}$$

$$J^{\subset}_{z,r} := \min\{j \geq 1 : \mathcal{L}^j_z \subset B(z,r)\}. \tag{2.3.3b}$$

Corollary 2.3.3. $J^{\subset}_{z,r} - J^{\cap}_{z,r}$ is stochastically dominated by $2\widetilde{N}$ where $\widetilde{N}$ is a geometric random variable with parameter $p = p(\kappa) > 0$ which depends only on $\kappa \in (8/3, 8)$.

Proof. Immediate from Lemma 2.3.2 and the renewal property of CLE$_\kappa$. □

Lemma 2.3.4. For each $\kappa \in (8/3, 8)$ there exist $c_1 > 0$ and $c_2 > 0$ such that for any proper simply connected domain D and point $z \in D$, for any positive numbers r and R for which $r < R$ and $B(z,R) \subset D$, a CLE$_\kappa$ in D contains a loop $\mathcal{L}$ surrounding z for which $\mathcal{L} \subset B(z,R)$ and $\mathcal{L} \cap B(z,r) = \varnothing$ with probability at least $1 - (c_1 r/R)^{c_2}$.

Proof. For convenience we let $x = \log(R/r)$ and rescale so that $R = 1$. For the CLE$_\kappa$ Γ, let $\lambda_j = -\log \mathrm{CR}(\mathcal{L}^j_z(\Gamma))$. By the renewal property of CLE$_\kappa$, $\{\lambda_{j+1} - \lambda_j\}$ form an i.i.d. sequence, and their distribution has exponential tails [61]. Now $\min(\{\lambda_j\} \cap (0,\infty)) = \lambda_{J^{\cap}_{z,1}}$, which by Lemma 2.2.12 is dominated by a distribution which has exponential tails and depends only on κ. By Cramér's theorem, there is a constant $c > 0$ so that $\lambda_{J^{\cap}_{z,1}+cx} \leq x - \log 4$ except with probability exponentially small in x. By Corollary 2.3.3, $J^{\subset}_{z,1} - J^{\cap}_{z,1}$ is stochastically dominated by twice a geometric random variable, and so $J^{\subset}_{z,1} \leq J^{\cap}_{z,1} + cx$ except with probability exponentially small in x. If both of these high probability events occur, then $\mathcal{L}_{J^{\subset}_{z,1}} \cap B(z, e^{-x}) = \varnothing$. □

2.3.2 Rapid convergence to full-plane CLE

Recall that the total variation distance between two probability measures μ and ν (on a common probability space) is defined by

$$\|\mu - \nu\|_{\mathrm{TV}} := \sup\{|\mu(A) - \nu(A)| \,:\, A \text{ measurable}\}.$$

Lemma 2.3.5. Let $\{X_j\}_{j\in\mathbb{N}}$ be non-negative i.i.d. random variables whose law has a positive density with respect to Lebesgue measure on $(0,\infty)$ and for which there exists $\lambda_0 > 0$ such that $\mathbb{E}[e^{\lambda_0 X_1}] < \infty$. Let $S_n = \sum_{j=1}^n X_j$. For $M > 0$, let $\tau_M = \min\{n \geq 0 : S_n \geq M\}$ and let ρ_M be the law of the "overshoot" $S_{\tau_M} - M$. There exists a probability measure ρ supported on $(0,\infty)$ with exponential tails and a constant $C > 1$ such that

$$\|\rho_M - \rho\|_{\mathrm{TV}} \leq Ce^{-M/C}.$$

It is known that there is a limiting measure ρ (for example, see [21, Chapt. 3.10]). If $f(x)$ is the density function for the law of X_1, then the density function of $S_{\tau_M} -$

S_{τ_M-1} converges as $M \to \infty$ to

$$\frac{xf(x)}{\int_0^\infty t\,f(t)\,dt}\,dx\,,$$

and the law of the overshoot $S_{\tau_M} - M$ converges as $M \to \infty$ to

$$\frac{\int_x^\infty f(t)\,dt}{\int_0^\infty t\,f(t)\,dt}\,dx\,.$$

For our results we need this convergence to be exponentially fast, for which we did not find a proof, so we provide one.

Proof of Lemma 2.3.5. For $M > N > 0$, we construct a coupling between ρ_N and ρ_M as follows. We take $S_0 = 0$ and $\widehat{S}_0 = N - M$, and then take $\{X_j\}_{j\in\mathbb{N}}$ and $\{\widehat{X}_j\}_{j\in\mathbb{N}}$ to be two i.i.d. sequences with law as in the statement of the lemma, with the two sequences coupled with one another in a manner that we shall describe momentarily. We let $S_n = \sum_{i=1}^n X_i$ and $\widehat{S}_n = \widehat{S}_0 + \sum_{i=1}^n \widehat{X}_i$. Define stopping times

$$\tau_N = \min\{n \geq 0 : S_n \geq N\} \qquad \text{and} \qquad \widehat{\tau}_N = \min\{n \geq 0 : \widehat{S}_n \geq N\}.$$

Then $S_{\tau_N} - N \sim \rho_N$ and $\widehat{S}_{\widehat{\tau}_N} - N \sim \rho_M$. We will couple the X_j's and $\widehat{X}_j$'s so that with high probability $S_{\tau_N} = \widehat{S}_{\widehat{\tau}_N}$.

Lemma 2.2.12 implies that there exists a law $\widetilde{\rho}$ on $(0,\infty)$ with exponential tails such that $\widetilde{\rho}$ stochastically dominates ρ_M for all $M > 0$. We choose θ to be big enough so that $\widetilde{\rho}([0,2\theta]) \geq 1/2$.

We inductively define a sequence of pairs of integers (i_k, j_k) for $k \in \{0,1,2,\dots\}$ starting with $(i_0, j_0) = (0,0)$. If $S_{i_k} + \theta \leq \widehat{S}_{j_k}$ then we set $(i_{k+1}, j_{k+1}) := (i_k + 1, j_k)$ and sample $X_{i_{k+1}}$ independently of the previous random variables. If $\widehat{S}_{j_k} + \theta \leq S_{i_k}$, then we set $(i_{k+1}, j_{k+1}) := (i_k, j_k + 1)$ and sample $\widehat{X}_{j_{k+1}}$ independently of the previous random variables. Otherwise, $\left|S_{i_k} - \widehat{S}_{j_k}\right| \leq \theta$. In that case, we set $(i_{k+1}, j_{k+1}) := (i_k + 1, j_k + 1)$ and sample $(X_{i_{k+1}}, \widehat{X}_{j_{k+1}})$ independently of the previous random variables and coupled so as to maximize the probability that $S_{i_{k+1}} = \widehat{S}_{j_{k+1}}$. Note that once the walks coalesce, they never separate.

We partition the set of steps into epochs. We adopt the convention that the kth step is from time $k-1$ to time k. The first epoch starts at time $k = 0$. For the epoch starting at time k (whose first step is $k+1$), we let

$$\ell(k) = \min\left\{k' \geq k : \min(S_{i_{k'}}, \widehat{S}_{j_{k'}}) \geq \max(S_{i_k}, \widehat{S}_{j_k}) - \theta\right\}.$$

Let E_k be the event

$$E_k = \{|S_{i_{\ell(k)}} - \widehat{S}_{j_{\ell(k)}}| \leq \theta\}.$$

By our choice of θ, $\mathbb{P}[E_k] \geq 1/2$. If event E_k occurs, then we let $\ell(k) + 1$ be the last step of the epoch, and the next epoch starts at time $\ell(k) + 1$. Otherwise, we let $\ell(k)$

be the last step of the epoch, and the next epoch starts at time $\ell(k)$.

Let $D(t)$ denote the total variation distance between the law of X_1 and the law of $t + X_1$. Since X_1 has a density with respect to Lebesgue measure which is positive in $(0, \infty)$, it follows that

$$q := \sup_{0 \le t \le \theta} D(t) < 1.$$

In particular, if the event E occurs, i.e., $\left|S_{i_{\ell(k)}} - \widehat{S}_{j_{\ell(k)}}\right| \le \theta$, and the walks have not already coalesced, then $\mathbb{P}[S_{i_{\ell(k)+1}} \ne \widehat{S}_{j_{\ell(k)+1}}] \le q$.

Let $Y_k = \max(S_{i_k}, \widehat{S}_{j_k})$. For the epoch starting at time k, the difference $Y_{\ell(k)} - Y_k$ is dominated by a random variable with exponential tails, since $\widetilde{\rho}$ has exponential tails. On the event E_k there is one more step of size $Y_{\ell(k)+1} - Y_{\ell(k)}$ in the epoch. This step size is dominated by the maximum of two independent copies of the random variable X_1 and therefore has exponential tails. Thus if k' is the start of the next epoch, then $Y_{k'} - Y_k$ is dominated by a fixed distribution (depending only on the law of X_1) which has exponential tails. It follows from Cramér's theorem that for some $c > 0$, it is exponentially unlikely that the number of epochs (before the walks overshoot N) is less than cN.

For each epoch, the walks have a $(1-q)\mathbb{P}[E_k] > 0$ chance of coalescing if they have not done so already. After cN epochs, the walkers have coalesced except with probability exponentially small in N, and except with exponentially small probability, these epochs all occur before the walkers overshoot N. □

Lemma 2.3.6. Let X be a random variables whose law is the difference in log conformal radii of successive CLE$_\kappa$ loops. Let f_M denote the density function of $X - M$ conditional on $X \ge M$. For some constant C_κ depending only on κ,

$$\sup_M f_M \le C_\kappa \times \exp\left[-(1 - 2/\kappa - 3\kappa/32)x\right].$$

For all M and all $x > 1$, the actual density is within a constant factor of this upper bound.

Proof. The density function for the law of X is [61, eqn. 4]

$$-\frac{\kappa\cos(4\pi/\kappa)}{4\pi}\sum_{j=0}^{\infty}(-1)^j(j+\tfrac{1}{2})\exp\left[-\frac{(j+\frac{1}{2})^2 - (1-\frac{4}{\kappa})^2}{8/\kappa}x\right].$$

For large enough x, the first term dominates the sum of the other terms. For small x, a different formula [61, Thm. 1] implies that the density is bounded by a constant. Integrating, we obtain $\mathbb{P}[X \ge M]$ to within constants, and then obtain the conditional probability to within constants. □

For a collection Γ of nested noncrossing loops in $\mathbb{C}$, let $\Gamma|_{B(z,r)^+}$ denote the collection of loops in Γ which are in the connected component of $\mathbb{C} \setminus \{\mathcal{L} \in \Gamma :$

$\mathcal{L}$ surrounds $B(z,r)\}$ containing z. If Γ is a CLE_κ in a proper simply connected domain containing $B(z,r)$, then $\Gamma|_{B(z,r)^+} = \Gamma|_{U_z^{J^\cap_{z,r}-1}}$.

Theorem 2.3.7. For $\kappa \in (8/3,8)$ there is a unique measure on nested noncrossing loops in $\mathbb{C}$, "full-plane CLE_κ", to which CLE_κ's on large domains D rapidly converge in the following sense. There are constants $C > 0$ and $\alpha > 0$ (depending on κ) such that for any $z \in \mathbb{C}$, $r > 0$, and simply connected proper domain D containing $B(z,r)$, a full-plane CLE_κ $\Gamma_\mathbb{C}$ and a CLE_κ Γ_D on D can be coupled so that with probability at least $1 - C(r/\operatorname{dist}(z,\partial D))^\alpha$, there is a conformal map φ from $\Gamma_\mathbb{C}|_{B(z,r)^+}$ to $\Gamma_D|_{B(z,r)^+}$ which has low distortion in the sense that $|\varphi'(z) - 1| < C(r/\operatorname{dist}(z,\partial D))^\alpha$ on $\Gamma_\mathbb{C}|_{B(z,r)^+}$. Full-plane CLE_κ is invariant under scalings, translations, and rotations.

Proof. We first prove for that $x > 0$, the stated estimates hold for $z = 0$ and $r = 1$, with $\mathbb{C}$ and D replaced by any two proper simply connected domains D_1 and D_2 which both contain the ball $B(0,e^x)$.

For $i \in \{1,2\}$, let Γ_i denote the CLE_κ on D_i. Let $\lambda_j^{(i)} = -\log \operatorname{CR}(0, \mathcal{L}_0^j(\Gamma_i))$. Note that $\lambda_0^{(i)} \le -x$ for $i \in \{1,2\}$. Furthermore, $\{\lambda_{j+1}^{(i)} - \lambda_j^{(i)}\}_{j\in\mathbb{N}}$ is an i.i.d. positive sequence whose terms have a continuous distribution with exponential tails [61]. Therefore, by Lemma 2.3.5, there is a stationary point process $\lambda^{(0)}$ on $\mathbb{R}$ with i.i.d. increments from this same distribution, and the sequences $\lambda^{(1)}$ and $\lambda^{(2)}$ can be coupled to $\lambda^{(0)}$ so that $\lambda^{(i)} \cap (-\frac{3}{4}x, \infty) = \lambda^{(0)} \cap (-\frac{3}{4}x,\infty)$ for $i \in \{1,2\}$, except with probability exponentially small in x.

Let

$$a := \min\left(\lambda^{(0)} \cap \left(-\frac{3}{4}x, \infty\right)\right). \tag{2.3.4}$$

By Lemma 2.2.12 or Lemma 2.3.6, $a \in (-\frac{3}{4}x, -\frac{1}{2}x)$ except with probability exponentially small in x.

We shall couple the two CLE_κ processes Γ_1 and Γ_2 as follows. First we generate the random point process $\lambda^{(0)}$. Then we sample the negative log conformal radii of the loops of Γ_1 and Γ_2 surrounding 0, so as to maximize the probability that these coincide with $\lambda^{(0)}$ on $(-\frac{3}{4}x, \infty)$. If either $\lambda^{(1)}$ or $\lambda^{(2)}$ does not coincide with $\lambda^{(0)}$ on $(-\frac{3}{4}x,\infty)$, then we may complete the construction of Γ_1 and Γ_2 independently. Otherwise, we construct Γ_1 and Γ_2 up to and including the loop with conformal radius e^{-a}, where a is defined in (2.3.4). Let $\mathcal{L}_a^{(i)}$ denote the loop of Γ_i whose negative log conformal radius (seen from 0) is a, and let $U_a^{(i)}$ denote the connected component of $\mathbb{C} \setminus \mathcal{L}_a^{(i)}$ containing 0. Then we sample a random CLE_κ $\Gamma_\mathbb{D}$ of the unit disk $\mathbb{D}$ which is independent of a and the portions of Γ_1 and Γ_2 constructed thus far. (We can either take $\Gamma_\mathbb{D}$ to be independent of $\lambda^{(0)}$, or so that the negative log conformal radii of $\Gamma_\mathbb{D}$'s loops surrounding 0 coincide with $(\lambda^{(0)} - a) \cap (0,\infty)$.) Then we let $\psi^{(i)}$ be the conformal map from $\mathbb{D}$ to $U_a^{(i)}$ with $\psi^{(i)}(0) = 0$ and $(\psi^{(i)})'(0) > 0$, and set the restriction of Γ_i to $U_a^{(i)}$ to be $\psi^{(i)}(\Gamma_\mathbb{D})$. If there are any bounded connected components of $\mathbb{C} \setminus \mathcal{L}_a^{(i)}$ other than $U_a^{(i)}$, then we generate the restriction of Γ_i to these components independently of everything else generated thus far. The resulting loop processes Γ_i are distributed according to the conformal loop ensemble on D_i, and have been coupled to be similar near 0.

Let $\psi = \psi^{(2)} \circ (\psi^{(1)})^{-1}$ be the conformal map from $U_a^{(1)}$ to $U_a^{(2)}$ for which $\psi(0) = 0$ and $\psi'(0) = 1$. Assuming $a \in (-\frac{3}{4}x, -\frac{1}{2}x)$ and $a \in \lambda^{(1)}$ and $a \in \lambda^{(2)}$, the Koebe distortion theorem implies that on $B(0, e^{x/4})$, $|\psi' - 1|$ is exponentially small in x.

By Lemma 2.3.4, except with probability exponentially small in x, both Γ_1 and Γ_2 contain a loop surrounding $B(0,1)$ which is contained in $B(0, e^{x/4})$. It is possible that ψ maps a loop of Γ_1 surrounding $\mathbb{D}$ to a loop of Γ_2 intersecting $\mathbb{D}$ or vice versa. But since ψ has exponentially low distortion, any such loop would have to have inradius exponentially close to 1. The expected number of loops of Γ_1 with negative log conformal radius between $-\log 4$ and 1 is bounded by a constant, so by the Koebe quarter theorem, the expected number of loops of Γ_1 with inradius between $1/e$ and 1 is bounded by a constant. Let $D_3 = e^u D_1$ be a third domain, where u is independent of everything else and uniformly distributed on $(0,1)$. It is evident that $e^u \Gamma_1$ has no loop with inradius exponentially (in x) close to 1, except with probability exponentially small in x. On the other hand, we can couple a CLE_κ on D_3 to Γ_1 in the same manner that we did for domain D_2, and deduce that Γ_1 must also have no loop with inradius exponentially close to 1, except with probability exponentially small in x. We conclude that it is exponentially unlikely for there to be a loop of Γ_1 surrounding $B(0,1)$ which ψ maps to a loop of Γ_2 intersecting $B(0,1)$ or vice versa. Thus $\psi(\Gamma_1|_{B(z,1)^+}) = \Gamma_2|_{B(z,1)^+}$, except with probability exponentially small in x.

The corresponding estimates for general z and r and domains D_1 and D_2 containing $B(z,r)$ follows from the conformal invariance of CLE_κ.

Given the above estimates for any two proper simply connected domains which contain a sufficiently large ball around the origin, it is not difficult to take a limit. For some sufficiently large constant k_0 (which depends on κ), we let Γ_k be a CLE_κ in the domain $B(0, e^k)$, where $k \geq k_0$ is an integer. For each k, we couple Γ_{k+1} and Γ_k as described above. With probability 1 all but finitely many of the couplings have that $\Gamma_{k+1}|_{B(0,e^{k/2})^+} = \psi_k(\Gamma_k|_{B(0,e^{k/2})^+})$ for a low-distortion conformal map ψ_k, so suppose that this is the case for all $k \geq \ell$. The conformal maps ψ_k (for $k \geq \ell$) approach the identity map sufficiently rapidly that for each $m \geq \ell$, the infinite composition $\cdots \circ \psi_{m+1} \circ \psi_m$ is well defined and converges uniformly on compact subsets to a limiting conformal map with distortion exponentially small in m. We define $\Gamma_{\mathbb{C}}|_{B(0,e^{m/2})^+}$ to be the image of $\Gamma_m|_{B(0,e^{m/2})^+}$ under this infinite composition. These satisfy the consistency condition $\Gamma_{\mathbb{C}}|_{B(0,\exp(m_1/2))^+} \subseteq \Gamma_{\mathbb{C}}|_{B(0,\exp(m_2/2))^+}$ for $m_2 \geq m_1 \geq \ell$, so then we define $\Gamma_{\mathbb{C}} = \bigcup_{m \geq \ell} \Gamma_{\mathbb{C}}|_{B(0,e^{m/2})^+}$. For any other proper simply connected domain D containing a sufficiently large ball around the origin, we couple Γ_D to $\Gamma_{\lfloor \log \mathrm{dist}(0,\partial D) \rfloor}$ as described above, and with high probability it will be close to $\Gamma_{\mathbb{C}}$ in the sense described in the theorem.

It is evident from this construction of $\Gamma_{\mathbb{C}}$ that it is rotationally invariant around

0. Next we check that $\Gamma_\mathbb{C}$ is invariant under transformations of the form $z \mapsto Az + C$ where $A, C \in \mathbb{C}$ and $A \neq 0$. Note that $A\Gamma_\mathbb{C} + C$ restricted to a ball $B(0,r)$ is arbitrarily well approximated by CLE on $B(C, A2^k)$ for sufficiently large k. But by the coupling for simply connected proper domains, the CLEs on $B(C, A2^k)$ and $B(0, 2^k)$ restricted to $B(0,r)$ approximate each other arbitrarily well for sufficiently large k, and by construction, $\Gamma_\mathbb{C}$ restricted to $B(0,r)$ is arbitrarily well approximated by CLE on $B(0,2^k)$ restricted to $B(0,r)$ when k is sufficiently large. Thus full-plane CLE is invariant under affine transformations.

Finally, if there were more than one loop measure that approximates CLE on simply connected proper domains in the sense of the theorem, then for a sufficiently large ball the measures would be different within the ball. Since for some sufficiently large proper simply connected domain D, CLE on D restricted to the ball would be well-approximated by both measures, we conclude that full-plane CLE is unique. □

2.4 Nesting dimension

In this section we prove Theorem 3.8.7, which gives the Hausdorff dimension of the set $\Phi_\nu(\Gamma)$ for a CLE_κ Γ in a simply connected proper domain $D \subsetneq \mathbb{C}$. Define

$$\Phi_\nu^+(\Gamma) := \left\{ z \in D : \liminf_{r\to 0} \widetilde{\mathcal{N}}_z(r;\Gamma) \geq \nu \right\}$$
$$\Phi_\nu^-(\Gamma) := \left\{ z \in D : \limsup_{r\to 0} \widetilde{\mathcal{N}}_z(r;\Gamma) \leq \nu \right\}.$$

Then the sets $\Phi_\nu^\pm(\Gamma)$ are monotone in ν, and $\Phi_\nu(\Gamma) = \Phi_\nu^+(\Gamma) \cap \Phi_\nu^-(\Gamma)$. (We suppress Γ from the notation when it is clear from context.)

Proposition 2.4.1. $\Phi_\nu^+(\Gamma)$ and $\Phi_\nu^-(\Gamma)$ are invariant under conformal maps.

Proof. Let $\varphi\colon D \to D'$ be a conformal map, and let Γ be a CLE_κ in D; $\varphi(\Gamma)$ is a CLE_κ in D'. By Theorem 2.2.14, for all $\varepsilon > 0$ small enough

$$\mathcal{N}_z\left(16\varepsilon|\varphi'(z)|^{-1};\Gamma\right) \leq \mathcal{N}_{\varphi(z)}(\varepsilon;\varphi(\Gamma)) \leq \mathcal{N}_z\left(\tfrac{1}{16}\varepsilon|\varphi'(z)|^{-1};\Gamma\right).$$

But

$$\begin{aligned}\liminf_{\varepsilon\to 0^+} \frac{\mathcal{N}_z(16^{\pm 1}\varepsilon|\varphi'(z)|^{-1};\Gamma)}{\log(1/\varepsilon)} &= \liminf_{\varepsilon\to 0^+} \frac{\mathcal{N}_z(\varepsilon;\Gamma)}{\log(1/(16^{\mp 1}\varepsilon|\varphi'(z)|)} \\ &= \liminf_{\varepsilon\to 0^+} \frac{\mathcal{N}_z(\varepsilon;\Gamma)}{\log(1/\varepsilon)}.\end{aligned}$$

Thus

$$\liminf_{\varepsilon\to 0^+} \widetilde{\mathcal{N}}_z(\varepsilon;\Gamma) = \liminf_{\varepsilon\to 0^+} \widetilde{\mathcal{N}}_{\varphi(z)}(\varepsilon;\varphi(\Gamma)),$$

so $\varphi(\Phi_\nu^+(\Gamma)) = \Phi_\nu^+(\varphi(\Gamma))$. Similarly, $\varphi(\Phi_\nu^-(\Gamma)) = \Phi_\nu^-(\varphi(\Gamma))$. □

Observe that conformal maps preserve Hausdorff dimension: away from the boundary, conformal maps are bi-Lipschitz, and the Hausdorff dimension of a countable union of sets is the maximum of the Hausdorff dimensions. So we may restrict our attention to the case where the domain D is the unit disk $\mathbb{D}$.

2.4.1 Upper bound

Let Γ be a CLE_κ in $\mathbb{D}$. Here we upper bound the Hausdorff dimension of $\Phi_\nu^\pm(\Gamma)$. Recall that γ_κ is defined in (2.1.4) and that $\nu_{\max}$ is the unique value of $\nu \geq 0$ such that $\gamma_\kappa(\nu) = 2$. Moreover, $\gamma_\kappa(\nu) \in [0,2)$ for $0 \leq \nu < \nu_{\max}$.

Proposition 2.4.2. If $0 \leq \nu \leq \nu_{\text{typical}}$, then $\dim_{\mathcal{H}} \Phi_\nu^-(\mathrm{CLE}_\kappa) \leq 2 - \gamma_\kappa(\nu)$ almost surely. If $\nu_{\text{typical}} \leq \nu \leq \nu_{\max}$, then $\dim_{\mathcal{H}} \Phi_\nu^+(\mathrm{CLE}_\kappa) \leq 2 - \gamma_\kappa(\nu)$ almost surely. If $\nu > \nu_{\max}$, then $\Phi_\nu^+(\mathrm{CLE}_\kappa) = \emptyset$ almost surely.

Proof. Observe that the unit disk can be written as a countable union Möbius transformations of $B(0,1/2)$. For example, for $q \in \mathbb{D} \cap \mathbb{Q}^2$, define φ_q to be the Riemann map for which $\varphi_q(0) = q$ and $\varphi_q'(0) > 0$. Then $\mathbb{D} = \bigcup_{q \in \mathbb{D}\cap\mathbb{Q}^2} \varphi_q(B(0,1/2))$. By Möbius invariance, therefore, it suffices to bound the Hausdorff dimension of $\Phi_\nu^\pm \cap B(0,1/2)$ for a CLE_κ in $\mathbb{D}$.

To upper bound the Hausdorff dimension, it suffices to find good covering sets. Let $r > 0$. Let $\mathcal{D}^r$ be the set of open balls in $\mathbb{C}$ which are centered at points of $r\mathbb{Z}^2 \cap B(0,1/2 + r/\sqrt{2})$ and have radius $(1 + 1/\sqrt{2})r$. For every point $z \in B(0,1/2)$, the closest point in $r\mathbb{Z}^2$ to z is the center of a ball $U \in \mathcal{D}^r$ for which $B(z,r) \subset U \subset B(z,(1+\sqrt{2})r)$.

For each ball $U \in \mathcal{D}^r$, let $z(U)$ be the center of U. We define

$$\begin{aligned} \mathcal{U}^{r,\nu+} &:= \left\{ U \in \mathcal{D}^r \,:\, \widetilde{\mathcal{N}}_{z(U)}(r) \geq \nu \right\} \\ \mathcal{U}^{r,\nu-} &:= \left\{ U \in \mathcal{D}^r \,:\, \widetilde{\mathcal{N}}_{z(U)}(r) \leq \nu \right\}. \end{aligned} \tag{2.4.1}$$

Recall that the conformal radius of $\mathbb{D}$ with respect to $z \in \mathbb{D}$ is $1 - |z|^2$. For $U \in \mathcal{D}^r$, we have $|z(U)| \leq 1/2 + r/\sqrt{2}$, so $\frac{1}{2} \leq \mathrm{CR}(z;\mathbb{D}) \leq 1$ provided $r \leq 1 - 1/\sqrt{2}$. Thus by Cramér's theorem (as in the proof of Lemma 2.2.18) and the continuity of $\gamma_\kappa(\nu)$, for $\nu > \nu_{\text{typical}}$ we have

$$\mathbb{P}[U \in \mathcal{U}^{r,\nu+}] \leq r^{\gamma_\kappa(\nu)+o(1)},$$

and for $\nu < \nu_{\text{typical}}$ we have

$$\mathbb{P}[U \in \mathcal{U}^{r,\nu-}] \leq r^{\gamma_\kappa(\nu)+o(1)},$$

where for fixed ν, the $o(1)$ terms tend to 0 as $r \to 0$, uniformly in U.

Next we define

$$\mathcal{C}^{m,\nu\pm} := \bigcup_{n \geq m} \mathcal{U}^{\exp(-n),\nu\pm}. \tag{2.4.2}$$

Suppose that $z \in \Phi_\nu^+(\Gamma) \cap B(0,1/2)$. Since $\liminf_{\varepsilon\to 0} \widetilde{\mathcal{N}_z}(\varepsilon) \geq \nu$, for any $\nu' < \nu$, for all large enough n, $\mathcal{N}_z((1+\sqrt{2})e^{-n}) \geq \nu' n$. There is a ball $U \in \mathcal{U}^{e^{-n}}$ for which $U \subset B(z,(1+\sqrt{2})e^{-n})$, and so $\mathcal{N}_{z(U)}((1+1/\sqrt{2})e^{-n}) \geq \mathcal{N}_z((1+\sqrt{2})e^{-n})$, so $U \in \mathcal{U}^{e^{-n},\nu'+}$. Hence, for any $m \in \mathbb{N}$ and $\nu' < \nu$, we conclude that $\mathcal{C}^{m,\nu'+}$ is a cover for $\Phi_\nu^+(\Gamma) \cap B(0,1/2)$.

Suppose that $z \in \Phi_\nu^-(\Gamma) \cap B(0,1/2)$. Since $\limsup_{\varepsilon\to 0} \widetilde{\mathcal{N}_z}(\varepsilon) \leq \nu$, for any $\nu' > \nu$, for all large enough n, $\mathcal{N}_z(e^{-n}) \leq \nu' n$. There is a ball $U \in \mathcal{U}^{e^{-n}}$ for which $B(z,e^{-n}) \subset U$, and so $\mathcal{N}_z(e^{-n}) \geq \mathcal{N}_{z(U)}((1+1/\sqrt{2})e^{-n})$, so $U \in \mathcal{U}^{e^{-n},\nu'-}$. Hence, for any $m \in \mathbb{N}$ and $\nu' > \nu$, we have that $\mathcal{U}^{m,\nu'-}$ is a cover for $\Phi_\nu^-(\Gamma)$.

We use these covers to bound the α-Hausdorff measure of $\Phi_\nu^\pm(\Gamma)$. If $m \in \mathbb{N}$ and $\nu' > \nu > \nu_{\text{typical}}$ (for $\Phi_\nu^+(\Gamma)$), or $\nu' < \nu < \nu_{\text{typical}}$ (for $\Phi_\nu^-(\Gamma)$),

$$\begin{aligned}
\mathbb{E}[\mathcal{H}_\alpha(\Phi_\nu^\pm(\Gamma))] &\leq \mathbb{E}\left[\sum_{U\in\mathcal{C}^{m,\nu'\pm}} (\operatorname{diam}(U))^\alpha\right] \\
&= \sum_{n\geq m} \sum_{U\in\mathcal{D}^{e^{-n}}} \left[(2+\sqrt{2})e^{-n}\right]^\alpha \mathbb{P}[U \in \mathcal{U}^{e^{-n},\nu'\pm}] \\
&\leq \sum_{n\geq m} e^{n(2-\alpha-\gamma_\kappa(\nu')+o(1))}.
\end{aligned} \tag{2.4.3}$$

If $\alpha > 2-\gamma_\kappa(\nu')$, the right-hand side tends to 0 as $m \to \infty$. Since m was arbitrary, we conclude that $\mathbb{E}[\mathcal{H}_\alpha(\Phi_\nu^\pm(\Gamma))] = 0$. Therefore, almost surely $\mathcal{H}_\alpha(\Phi_\nu^\pm(\Gamma)) = 0$. Any such α is an upper bound on $\dim_{\mathcal{H}} \Phi_\nu^\pm(\Gamma)$. The continuity of $\gamma_\kappa(\nu)$ then implies that almost surely $\dim_{\mathcal{H}} \Phi_\nu^+(\Gamma) \leq 2-\gamma_\kappa(\nu)$ (when $\nu > \nu_{\text{typical}}$) and $\dim_{\mathcal{H}} \Phi_\nu^-(\Gamma) \leq 2-\gamma_\kappa(\nu)$ (when $\nu < \nu_{\text{typical}}$). When $\nu = \nu_{\text{typical}}$, the dimension bound (which is 2) holds trivially. Finally, when $\nu > \nu_{\max}$, the bound in (2.4.3) shows that $\mathcal{H}_0(\Phi_\nu^+(\Gamma)) = 0$ almost surely. Therefore, $\Phi_\nu^+(\Gamma) = \emptyset$ almost surely. □

2.4.2 Lower bound

Next we lower bound $\dim_{\mathcal{H}}(\Phi_\nu(\Gamma))$. As we did for the upper bound, we assume without loss of generality that $D = \mathbb{D}$. We introduce a subset $P_\nu(\Gamma)$ of $\Phi_\nu(\Gamma)$ which has the property that the number and geometry of the loops which surround points in $P_\nu(\Gamma)$ are controlled at every length scale. This reduction is useful because the correlation structure of the loop counts for these special points is easier to estimate (Proposition 2.4.7) than that of arbitrary points in $\Phi_\nu(\Gamma)$. Then we prove that the Hausdorff dimension of this special class of points is at least $2-\gamma_\kappa(\nu)$ with positive probability. We complete the proof of the almost sure lower bound of $\dim_{\mathcal{H}} \Phi_\nu(\Gamma)$ using a zero-one argument.

Lemma 2.4.3. Let Γ be a CLE_κ in the unit disk $\mathbb{D}$, and fix $\nu \geq 0$. Then for functions $\delta(\varepsilon)$ converging to 0 sufficiently slowly as $\varepsilon \to 0$ and for sufficiently large $M > 1$, the event that

(i) there is a loop which is contained in the annulus $\overline{B(0,\varepsilon)} \setminus B(0,\varepsilon/M)$ and which surrounds $B(0,\varepsilon/M)$, and

(ii) the index J of the outermost such loop in this annulus $\overline{B(0,\varepsilon)} \setminus B(0,\varepsilon/M)$ satisfies $\nu \log \varepsilon^{-1} \le J \le (\nu + \delta(\varepsilon)) \log \varepsilon^{-1}$,

has probability at least $\varepsilon^{\gamma_\kappa(\nu)+o(1)}$ as $\varepsilon \to 0$.

Proof. We define $\delta(\varepsilon)$ to be 2 times the function denoted δ in Lemma 2.2.18. Let E_1 denote the event that between $\nu \log \varepsilon^{-1}$ and $(\nu + \frac{1}{2}\delta(\varepsilon)) \log \varepsilon^{-1}$ loops surround $B(z,\varepsilon)$, let E_2 denote the event that at most $\frac{1}{2}\delta(\varepsilon) \log \varepsilon^{-1}$ loops intersect the circle $\partial B(z,\varepsilon)$, and let E_3 denote the event that there is a loop winding around the closed annulus $\overline{B(0,\varepsilon)} \setminus B(0,\varepsilon/M)$.

Lemma 2.2.18 implies

$$\mathbb{P}[E_1] = \varepsilon^{\gamma_\kappa(\nu)+o(1)} \quad \text{as } \varepsilon \to 0\,. \tag{2.4.4}$$

Corollary 2.3.3 implies that for sufficiently small ε, we have

$$\mathbb{P}[E_2 \,|\, E_1] \ge \tfrac{3}{4}\,. \tag{2.4.5}$$

Lemma 2.3.6 applied to the log conformal radius increment sequence implies that for some large enough M,

$$\mathbb{P}\left[\mathrm{CR}\left(0; U_0^{J_{0,\varepsilon}^{\cap}}\right) \ge M^{-1/2}\varepsilon \,|\, E_1\right] \ge \tfrac{7}{8}\,. \tag{2.4.6}$$

Lemma 2.2.13 and Corollary 2.3.3 together imply that for large enough M

$$\mathbb{P}\left[\mathrm{CR}\left(0; U_0^{J_{0,\varepsilon}^{\subset}}\right) / \,\mathrm{CR}\left(0; U_0^{J_{0,\varepsilon}^{\cap}}\right) \ge M^{-1/2} \,|\, E_1\right] \ge \tfrac{7}{8}\,. \tag{2.4.7}$$

Combining (2.4.4), (2.4.5), (2.4.6), and (2.4.7), we arrive at

$$\mathbb{P}[E_1 \cap E_2 \cap E_3] = \varepsilon^{\gamma_\kappa(\nu)+o(1)} \quad \text{as} \quad \varepsilon \to 0\,.$$

Since $E_1 \cap E_2 \cap E_3$ implies the event described in the lemma, this concludes the proof. ☐

We define the set $P_\nu = P_\nu(\Gamma)$ as follows. For $z \in \mathbb{D}$ and $k \ge 0$ we inductively define

- Let $\tau_0 = 0$.
- Let $V_z^k = U_z^{\tau_k}$ be the connected component of $\mathbb{D} \setminus \mathcal{L}_z^{\tau_k}$ containing z. In particular, $V_z^0 = D = \mathbb{D}$.
- Let φ_z^k be the conformal map from V_z^k to $\mathbb{D}$ with $\varphi_z^k(z) = 0$ and $(\varphi_z^k)'(z) > 0$.
- Let $t_k = 2^{-(k+1)}$.
- Let τ_{k+1} be the smallest $j \in \mathbb{N}$ such that $\varphi_z^k(\mathcal{L}_z^j) \subset \overline{B(0,t_k)}$.

Let $\widetilde{\Gamma}_z^k$ be the image under φ_z^k of the loops of Γ which are surrounded by $\mathcal{L}_z^{\tau_k}$ and in the same component of $\mathbb{D} \setminus \mathcal{L}_z^{\tau_k}$ as z. Then $\widetilde{\Gamma}_z^k$ is a CLE_κ in $\mathbb{D}$.

Let $M > 1$ be a large enough constant for Lemma 2.4.3, and let E_z^k to be the event described in Lemma 2.4.3 for the CLE $\widetilde{\Gamma}_z^k$ and $\varepsilon = t_k$. We define

$$E_z^{k_1,k_2} := \bigcap_{k_1 \le k < k_2} E_z^k .$$

Throughout the rest of this section, we let

$$s_k = \prod_{0 \le i < k} t_i \quad \text{for} \quad k \ge 0 . \tag{2.4.8}$$

Lemma 2.4.4. There exist sequences $\{r_k\}_{k\in\mathbb{N}}$ and $\{R_k\}_{k\in\mathbb{N}}$ satisfying

$$\lim_{k\to\infty} \frac{\log r_k}{\log s_k} = \lim_{k\to\infty} \frac{\log R_k}{\log s_k} = 1 \tag{2.4.9}$$

such that for all $z \in \overline{B(0,1/2)}$ and $k \ge 0$, we have

$$B(z, r_k) \subset V_z^k \subset B(z, R_k) \tag{2.4.10}$$

on the event $E_z^{0,k}$.

Proof. For $0 < j \le k$, the chain rule implies that on the event $E_z^{0,k}$ we have

$$\mathrm{CR}(z; V_z^j) = \mathrm{CR}(0; \varphi_z^{j-1}(V_z^j))\, \mathrm{CR}(z; V_z^{j-1}) \le t_{j-1}\, \mathrm{CR}(z; V_z^{j-1}) , \tag{2.4.11}$$

where the inequality follows from Corollary 2.2.15. Iterating the inequality in (2.4.11), we see that

$$\begin{aligned} \mathrm{CR}(z; V_z^k) &\le t_{k-1}\, \mathrm{CR}(z; V_z^{k-1}) \le \cdots \le (t_{k-1} \cdots t_0)\, \mathrm{CR}(z; V_z^0) \\ &= s_k\, \mathrm{CR}(z; \mathbb{D}) . \end{aligned} \tag{2.4.12}$$

Since $|((\varphi_z^{k-1})^{-1})'(0)| = \mathrm{CR}(z; V_z^{k-1})$, it follows from the Koebe distortion theorem that $V_z^k \subseteq B(z, t_{k-1}/(1-t_{k-1})^2\, \mathrm{CR}(z; V_z^{k-1}))$. Since $\mathrm{CR}(z; V_z^{k-1}) \le s_{k-1}\, \mathrm{CR}(z; \mathbb{D})$, $\mathrm{CR}(z; \mathbb{D}) = 1 - |z|^2 \le 1$, and $t_{k-1} \le 1/2$, we see from (2.4.12) that $V_z^k \subseteq B(z, 4s_k)$, so we set $R_k = 4s_k$ to get the second inclusion in (2.4.10).

To find $\{r_k\}_{k\in\mathbb{N}}$ satisfying the first inclusion in (2.4.10), we observe that on $E_z^{0,k}$ we have

$$\begin{aligned} \mathrm{CR}(z; V_z^k) &\ge M^{-1} t_{k-1}\, \mathrm{CR}(z; V_z^{k-1}) \ge \cdots \ge M^{-k} (t_{k-1} \cdots t_0)\, \mathrm{CR}(z; V_z^0) \\ &= M^{-k} s_k\, \mathrm{CR}(z; \mathbb{D}) . \end{aligned}$$

Applying the upper bound of Corollary 2.2.15, we thus see that $\mathrm{inrad}(z; V_z^k) \ge \frac{1}{4} M^{-k} s_k\, \mathrm{CR}(z; \mathbb{D})$. Since $\mathrm{CR}(z; \mathbb{D}) \ge 3/4$ for $z \in \overline{B(0,1/2)}$, setting $r_k = \frac{3}{16} M^{-k} s_k$

gives (2.4.10).

A straightforward calculation confirms that these sequences $\{r_k\}_{k\in\mathbb{N}}$ and $\{R_k\}_{k\in\mathbb{N}}$ satisfy (2.4.9). □

We define $P_\nu(\Gamma) \subseteq \mathbb{D}$ by

$$P_\nu(\Gamma) := \bigcap_{n\geq 1} \{z \in \overline{B(0,1/2)} : E_z^{0,n} \text{ occurs}\}. \tag{2.4.13}$$

Next we show that elements of $P_\nu(\Gamma)$ are special points of $\Phi_\nu(\Gamma)$:

Lemma 2.4.5. For $\nu \geq 0$, always $P_\nu(\Gamma) \subseteq \Phi_\nu(\Gamma)$.

Proof. It follows from the definition of E_z^k that for $z \in P_\nu$, the number of loops surrounding V_z^k is $(\nu + o(1)) \log s_k^{-1}$ as $k \to \infty$. Lemma 2.4.4 then implies that the number of loops surrounding $B(0, s_k)$ is also $(\nu + o(1)) \log s_k^{-1}$ as $k \to \infty$.

If $0 < \varepsilon < 1$, we may choose $k = k(\varepsilon) \geq 0$ so that $s_{k+1} \leq \varepsilon \leq s_k$. Then

$$\frac{\mathcal{N}_z(s_k)}{\log s_k^{-1}} \cdot \frac{\log s_k^{-1}}{\log \varepsilon^{-1}} \leq \frac{\mathcal{N}_z(\varepsilon)}{\log \varepsilon^{-1}} \leq \frac{\mathcal{N}_z(s_{k+1})}{\log s_{k+1}^{-1}} \cdot \frac{\log s_{k+1}^{-1}}{\log \varepsilon^{-1}}.$$

Observe that $\log s_{k+1} / \log s_k \to 1$ as $k \to \infty$. From this we see that both the left hand side and right hand side converge to ν as $\varepsilon \to 0$, so the middle expression also converges to ν as $\varepsilon \to 0$, which implies $z \in \Phi_\nu(\Gamma)$. □

We use the following lemma, which establishes that the right-hand side of (2.4.13) is an intersection of closed sets.

Lemma 2.4.6. For each $n \in \mathbb{N}$, the set $P_{\nu,n} := \{z \in \overline{B(0,1/2)} : E_z^{0,n} \text{ occurs}\}$ is always closed.

Proof. Suppose that z is in the complement of $P_{\nu,n}$, and let k be the least value of j such that E_z^j fails to occur. Each of the two conditions in the definition of E_z^k (see Lemma 2.4.3) has the property that its failure implies that E_w^k also does not occur for all w in some neighborhood of z. (We need the continuity of φ_z^k in z, which may be proved by realizing φ_w^k as a composition of φ_z^k with a Möbius map that takes the disk to itself and the image of w to 0.) This shows that the complement of $P_{\nu,n}$ is open, which in turn implies that $P_{\nu,n}$ is closed. □

Proposition 2.4.7. Consider a CLE_κ in $\mathbb{D}$. There exists a function f (depending on κ and ν) such that (1) $f(s) = s^{\gamma_\kappa(\nu)+o(1)}$ as $s \to 0$, and (2) for all $z, w \in \overline{B(0,1/2)}$

$$\mathbb{P}[E_z^{0,n} \cap E_w^{0,n}] f(\max(s_n, |z-w|)) \leq \mathbb{P}[E_z^{0,n}] \mathbb{P}[E_w^{0,n}]. \tag{2.4.14}$$

Proof. Suppose $z, w \in \overline{B(0,1/2)}$. Let r_k and R_k be defined as in Lemma 2.4.4. If $|z-w| \leq R_n$, then we bound $\mathbb{P}[E_z^{0,n} \cap E_w^{0,n}] \leq \mathbb{P}[E_z^{0,n}]$ and, using Lemma 2.4.3 and the fact that $R_n = 4s_n$,

$$\mathbb{P}[E_z^{0,n}] \geq \prod_{k \leq n} t_k^{\gamma_\kappa(\nu)+o(1)} = s_n^{\gamma_\kappa(\nu)+o(1)} = \max(s_n, |z-w|)^{\gamma_\kappa(\nu)+o(1)},$$

which implies (2.4.14). Next suppose $|z-w| > R_n$. Let

$$u = \min\{k \in \mathbb{N} : R_k < |z-w|\}.$$

$$\mathbb{P}[E_z^{0,n} \cap E_z^{0,n}] = \mathbb{P}[E_z^{0,u} \cap E_w^{0,u}]\mathbb{P}[E_z^{u,n} \cap E_w^{u,n} | E_z^{0,u} \cap E_w^{0,u}]$$

By Lemma 2.4.4, $w \notin V_z^u$ and $z \notin V_w^u$, so we see that V_z^u and V_w^u are disjoint. By the renewal property of CLE, this implies that conditional on $E_z^{0,u} \cap E_w^{0,u}$, the events E_z^k and E_w^k for $k \geq u$ are independent. Thus

$$\begin{aligned} \mathbb{P}[E_z^{0,n} \cap E_z^{0,n}] &= \mathbb{P}[E_z^{0,u} \cap E_w^{0,u}]\mathbb{P}[E_z^{u,n}]\mathbb{P}[E_w^{u,n}] \\ &\leq \mathbb{P}[E_z^{0,u}]\mathbb{P}[E_z^{u,n}]\mathbb{P}[E_w^{u,n}] \\ \mathbb{P}[E_z^{0,n} \cap E_z^{0,n}]\mathbb{P}[E_w^{0,u}] &\leq \mathbb{P}[E_z^{0,n}]\mathbb{P}[E_w^{0,n}]. \end{aligned}$$

Since

$$\mathbb{P}[E_w^{0,u}] \geq s_u^{\gamma_\kappa(\nu)+o(1)} = \max(s_n, |z-w|)^{\gamma_\kappa(\nu)+o(1)},$$

(2.4.14) follows in this case as well. □

We take t_k as in Section 2.4.2 and s_k as in (2.4.8). We will prove Theorem 3.8.7 using Proposition 2.4.7 and the following general fact about Hausdorff dimension, the key ideas of which appeared in [19, 11, 23].

Proposition 2.4.8. Suppose $P_1 \supset P_2 \supset P_3 \supset \cdots$ is a random nested sequence of closed sets, and $\{s_n\}_{n \in \mathbb{N}}$ is a sequence of positive real numbers converging to 0. Suppose further that $0 < a < 2$, and $f(s) = s^{a+o(1)}$ as $s \to 0$. If for each $z, w \in \mathbb{D}$ and $n \geq 1$ we have $\mathbb{P}[z \in P_n] > 0$ and

$$\mathbb{P}[z, w \in P_n] f(\max(s_n, |z-w|)) \leq \mathbb{P}[z \in P_n]\mathbb{P}[w \in P_n], \tag{2.4.15}$$

then for any $\alpha < 2 - a$,

$$\mathbb{P}[\dim_{\mathcal{H}}(P) \geq \alpha] > 0 \quad \text{where} \quad P := \bigcap_{n \geq 1} P_n.$$

Proof. Let μ_n denote the (random) measure with density with respect to Lebesgue measure on $\mathbb{C}$ given by

$$\frac{d\mu_n(z)}{dz} = \frac{\mathbf{1}_{z \in P_n \cap \mathbb{D}}}{\mathbb{P}[z \in P_n]}.$$

Then $\mathbb{E}[\mu_n(\mathbb{D})] = \text{area}(\mathbb{D})$, and by (2.4.15),

$$\mathbb{E}[\mu_n(\mathbb{D})^2] = \iint_{\mathbb{D}\times\mathbb{D}} \frac{\mathbb{P}[z, w \in P_n]}{\mathbb{P}[z \in P_n]\mathbb{P}[w \in P_n]} \, dz \, dw \le C_1 < \infty$$

for some constant C_1 depending on the function f but not n.

Recall that for $\alpha \ge 0$, the *α-energy* of a measure μ on $\mathbb{C}$ is defined by

$$I_\alpha(\mu) := \iint_{\mathbb{C}\times\mathbb{C}} \frac{1}{|z-w|^\alpha} \, d\mu(z) \, d\mu(w) \, .$$

Recall also that if there exists a nonzero measure with finite α-energy supported on a set $P \subset \mathbb{C}$, usually called a Frostman measure, then the Hausdorff dimension of P is at least α [17, Theorem 4.13]. The expected α-energy of μ_n is

$$\mathbb{E}[I_\alpha(\mu_n)] = \iint_{\mathbb{D}\times\mathbb{D}} \frac{\mathbb{P}[z, w \in P_n]}{\mathbb{P}[z \in P_n]\mathbb{P}[w \in P_n]} \frac{1}{|z-w|^\alpha} \, dz \, dw \, ,$$

and when $\alpha < 2 - a$, the expected α-energy is bounded by a finite constant C_2 depending on f and α but not n.

Since the random variable $\mu_n(\mathbb{D})$ has constant mean and uniformly bounded variance, it is uniformly bounded away from 0 with uniformly positive probability as $n \to \infty$. Also, $\mathbb{P}[I_\alpha(\mu_n) \le d] \to 1$ as $d \to \infty$ uniformly in n. Therefore, we can choose b and d large enough that the probability of the event

$$G_n := \{b^{-1} \le \mu_n(\mathbb{D}) \le b \text{ and } I_\alpha(\mu_n) \le d\}$$

is bounded away from 0 uniformly in n. It follows that with positive probability infinitely many G_n's occur. The set of measures μ satisfying $b^{-1} \le \mu(\mathbb{D}) \le b$ and is weakly compact by Prohorov's compactness theorem. Therefore, on the event that G_n occurs for infinitely many n, there is a sequence of integers $k_1, k_2, \ldots$ for which μ_{k_ℓ} converges to a finite nonzero measure $\mu_\star$ on $\mathbb{D}$.

We claim that $\mu_\star$ is supported on P. To show this, we use the portmanteau theorem, which implies that if $\pi_\ell \to \pi$ weakly and U is open, then $\pi(U) \le \liminf_\ell \pi_\ell(U)$. Since P_n is closed for each $n \in \mathbb{N}$, we have

$$\mu_\star(\mathbb{C} \setminus P_n) \le \liminf_{\ell\to\infty} \mu_{k_\ell}(\mathbb{C} \setminus P_n) = 0 \, ,$$

where the last step follows because μ_{k_ℓ} is supported on $P_{k_\ell} \subset P_n$ for $k_\ell \ge n$. Therefore

$$\mu_\star(\mathbb{C} \setminus P) = \lim_{n\to\infty} \mu_\star(\mathbb{C} \setminus P_n) = 0 \, ,$$

so $\mu_\star$ is supported on P.

To see that $\mu_\star$ has finite α-energy, we again use the portmanteau theorem, which implies that

$$\int f \, d\mu \le \liminf_{\ell\to\infty} \int f \, d\mu_\ell$$

whenever f is a lower semicontinuous function bounded from below and $\mu_\ell \to \mu$ weakly. Taking $f(z,w) = |z-w|^{-\alpha}$, $\mu_\ell = \mu_{k_\ell}(dz)\mu_{k_\ell}(dw)$, and $\mu = \mu(dz)\mu(dw)$ concludes the proof. $\square$

Proof of Theorem 3.8.7. Recall that conformal invariance was proved in Proposition 2.4.1. We now show that $\dim_{\mathcal{H}} \Phi_\nu(\Gamma) = 2 - \gamma_\kappa(\nu)$ almost surely, when $0 \le \nu \le \nu_{\max}$. (The case $\nu = \nu_{\max}$ uses a separate argument.) We established the upper bound in Section 2.4.1, so we just need to prove the lower bound.

Suppose $\nu < \nu_{\max}$. For each connected component U in the complement of the gasket of Γ, let $z(U)$ be the lexicographically smallest rational point in U, and let φ_U be the Riemann map from $(U,z(U))$ to $(\mathbb{D},0)$ with positive derivative at $z(U)$. By Proposition 2.4.8, for any $\varepsilon > 0$, there exists $p(\varepsilon) > 0$ such that

$$\mathbb{P}[\dim_{\mathcal{H}}(P_\nu(\varphi_U(\Gamma|_U))) \ge 2 - \gamma_\kappa(\nu) - \varepsilon] \ge p(\varepsilon)\,.$$

By Lemma 2.4.5 $P_\nu(\varphi_U(\Gamma|_U)) \subset \Phi_\nu(\varphi_U(\Gamma|_U))$, and by conformal invariance,

$$\dim_{\mathcal{H}} \Phi_\nu(\varphi_U(\Gamma|_U)) = \dim_{\mathcal{H}} \Phi_\nu(\Gamma|_U),$$

which lower bounds $\dim_{\mathcal{H}} \Phi_\nu(\Gamma)$. Since there are infinitely many components U in the complement of the gasket, and the $\Gamma|_U$'s are independent, almost surely $\dim_{\mathcal{H}} \Phi_\nu(\Gamma) \ge 2 - \gamma_\kappa(\nu) - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that almost surely $\dim_{\mathcal{H}} \Phi_\nu(\Gamma) \ge 2 - \gamma_\kappa(\nu)$.

It remains to show that $\Phi_\nu(\Gamma)$ is dense in D almost surely, for $0 \le \nu < \nu_{\max}$. Let z be a rational point in D, and recall that U_z^k is the the complementary connected component of $D \setminus \mathcal{L}_z^k$ which contains z. Almost surely $\Phi_\nu(\Gamma|_{U_z^k})$ has positive Hausdorff dimension, and in particular is nonempty. Since there are countably many such pairs (z,k), almost surely $\Phi_\nu(\Gamma|_{U_z^k}) \ne \varnothing$ for each such z and k, and almost surely for each rational point z, $\operatorname{diameter}(U_z^k) \to 0$. $\square$

Theorem 2.4.9. For a CLE_κ Γ in a proper simply connected domain D, almost surely $\Phi_{\nu_{\max}}(\Gamma)$ is equinumerous with $\mathbb{R}$. Furthermore, almost surely $\Phi_{\nu_{\max}}(\Gamma)$ is dense in D.

Proof. As usual we assume without loss of generality that $D = \mathbb{D}$. We will describe a random injective map from the set $\{0,1\}^{\mathbb{N}}$ of binary sequences to $\mathbb{D}$ such that the image of the map is almost surely a subset of $\Phi_{\nu_{\max}}(\Gamma)$.

Let $M > 0$ be a large constant as described in Lemma 2.4.3. For a CLE Γ in $\mathbb{D}$, let $E_{0,\varepsilon}^{\mathbb{D}}(\nu)$ denote the event that there is a loop contained in $\overline{B(0,\varepsilon)} \setminus B(0,\varepsilon/M)$ surrounding $B(0,\varepsilon/M)$ and such that the index J of the outermost such loop is at least $\nu \log \varepsilon^{-1}$. If $(D,z) \ne (\mathbb{D},0)$, Γ is a CLE in D, and $\varepsilon > 0$, let $E_{z,\varepsilon}^{D}(\nu)$ be the event $E_{0,\varepsilon}^{\mathbb{D}}(\nu)$ occurs for the conformal image of Γ under a Riemann map from (D,z) to $(\mathbb{D},0)$. If $\{\varepsilon_j\}_{j\in\mathbb{N}}$ is a sequence of positive real numbers, let $E_z^{D,n}(\nu) = E_{z,\{\varepsilon_j\}_{j=1}^\infty}^{D,n}(\nu)$ denote the event that $E_{z,\varepsilon}^{D}(\nu)$ "occurs n times" for the first n values of ε in the sequence. More precisely, we define $E_z^{D,n}(\nu)$ inductively by $E_z^{D,1}(\nu) = E_{z,\varepsilon_1}^{D}(\nu)$ and

$$E_z^{D,n}(\nu) = E_{z,\varepsilon_1}^{D}(\nu) \cap E_{z,\{\varepsilon_j\}_{j=2}^{\infty}}^{U_z^j,n-1}(\nu), \tag{2.4.16}$$

for $n > 1$. For the remainder of the proof, we fix the sequence $\varepsilon_j := t_j = 2^{-j-1}$ and define the events $E_z^{D,n}(\nu)$ with respect to this sequence.

For a domain D with $z_0 \in D$, let φ be a conformal map from $(\mathbb{D}, 0)$ to (D, z_0), and let $F^{D,z_0,n}(\nu)$ denote the event that there is some point $z \in B(0, 1/2)$ for which $E_{\varphi(z)}^{D,n}(\nu)$ occurs. By Lemma 2.4.6 and Propositions 2.4.7 and 2.4.8, we see that there is some $p > 0$ (depending on $\kappa \in (8/3, 8)$ and $\nu < \nu_{\max}$), such that $\mathbb{P}[F^{D,z_0,n}(\nu)] \geq p$ for all n.

For each $k \in \mathbb{N}$, we choose $\nu_k \in (\nu_{\text{typical}}, \nu_{\max})$ so that $\gamma_\kappa(\nu_k) = 2 - 2^{-k-1}$. For each $k \in \mathbb{N}$ and $\ell \in \mathbb{N}$, we define $q_{k,\ell} = 2^{-2k-\ell}$.

Suppose $z \in \overline{B(0,1/2)}$ and $0 < r < 1/2$ and $0 < u < r/M$. For $n \in \mathbb{N}$ and $\nu < \nu_{\max}$, we say that the annulus $B(z,r) \setminus B(z,u)$ is (n,ν)-*good* if (i) there exists a loop contained in the annulus and surrounding z (say $\mathcal{L}_z^j$ is the outermost such loop), and (ii) the event $F^{U_z^j,z,n}(\nu)$ occurs.

For $q, r > 0$, define $u(q,r) = (q/C)^{1/\alpha} r^{1+2/\alpha}$, where C and α are chosen so that every annulus $B(z,r) \setminus B(z,u)$ contained in D contains a loop surrounding z with probability at least $1 - C(u/r)^\alpha$ (see Lemma 2.3.4). For $0 < r < 1/2$, let S_r be a set of $\frac{1}{100r^2}$ disjoint disks of radius r in $B(0,1/2)$. By our choice of $u(q,r)$, the event G that all the disks $B(z,r)$ in S_r contain a CLE loop surrounding $B(z,u)$ has probability at least $1 - q$. We choose $r_{k,\ell} > 0$ small enough so that for all $n \in \mathbb{N}$, with probability at least $1 - 2^{1-2k-\ell}$ there are two disks $B(z, r_{k,\ell})$ in $S_{r_{k,\ell}}$ such that $B(z, r_{k,\ell}) \setminus B(z, u(q_{k,\ell}, r_{k,\ell}))$ is an (n_k, ν_k)-good annulus. This is possible because on the event G, the disks in $S_{r_{k,\ell}}$ give us $\frac{1}{100r_{k,\ell}^2}$ independent trials to obtain a good annulus, and each has success probability at least p. Abbreviate $u_{k,\ell} = u(q_{k,\ell}, r_{k,\ell})$. Finally, we define a sequence $(n_k)_{k\in\mathbb{N}}$ growing sufficiently fast that

$$\lim_{k\to 0} \frac{\sum_{j=1}^{k} \log u_{j+1,1}^{-1}}{\sum_{j=1}^{k} \log s_{n_j}^{-1}} = 0, \tag{2.4.17}$$

Now suppose that Γ is a CLE in the unit disk. Define

$$A = A(\mathbb{D}, 0, r, u, q, n, \nu)$$

to be the event that there are at least two disks $B(z,r)$ and $B(w,r)$ in S_r such that $B(z,r) \setminus B(z,u)$ and $B(w,r) \setminus B(w,u)$ are both (n,ν)-good. If $(D,z) \neq (\mathbb{D}, 0)$ and Γ is a CLE in D, define $A = A(D, z, r, u, q, n, \nu)$ to be the event that $A(\mathbb{D}, 0, r, u, q, n, \nu)$ occurs for the conformal image of Γ under a Riemann map from (D,z) to $(\mathbb{D}, 0)$. Abbreviate $A(D, z, r_{k,\ell}, u_{k,\ell}, q_{k,\ell}, n_k, \nu_k)$ as $A_{k,\ell}(D,z)$

We define a random map $b \mapsto D_b$ from the set of terminating binary sequences to the set of subdomains of $\mathbb{D}$ as follows. If the event $A_{1,1}(\mathbb{D},0)$ occurs, we set $\ell(\mathbb{D}) = 1$ and define $D_0 = \varphi_z^{-1}(U_{\acute{z}}^{I(\acute{z})})$ and $D_1 = \varphi_w^{-1}(U_{\acute{w}}^{I(\acute{w})})$, where z and w are the centers of two (n_1, v_1)-good annuli, φ_z (resp. φ_w) is a Riemann map from $(\mathbb{D}, z)$ (resp. (D, w)) to $(\mathbb{D}, 0)$, $\acute{z} \in U_z^{J^{\subset}_{z,r_{1,1}}}$ and $\acute{w} \in U_w^{J^{\subset}_{w,r_{1,1}}}$ are points for which $E_{\acute{z}}^{U_z^{J^{\subset}_{z,r_{1,1}}}, n_1}(v_1)$ and $E_{\acute{w}}^{U_w^{J^{\subset}_{w,r_{1,1}}}, n_1}(v_1)$ occur, and $I(\acute{z})$ (resp. $I(\acute{w})$) is the index of the nth loop encountered in the definition of $E_{\acute{z}}^{D,n}(v_1)$ (resp. $E_{\acute{w}}^{D,n}(v_1)$) (in other words, the first such loop is denoted J in (2.4.16), the second such loop is the first one contained in the preimage of $B(0, \varepsilon_2)$ under a Riemann map from (U_z^J, z) to $(D, 0)$, and so on). If A does not occur, then we choose a disk $B(z, r_{1,1})$ in $S_{r_{1,1}}$ and consider whether the event $A_{1,2}(U_z^{J^{\subset}_{z,r_{1,1}}})$ occurs. If it does, then we set $\ell(\mathbb{D}) = 2$ and define D_0 and D_1 to be the conformal preimages of $U_z^{J^{\subset}_{z,r_{1,2}}}$ and $U_w^{J^{\subset}_{w,r_{1,2}}}$, respectively, where again z and w are centers of two (n_1, v_1)-good annuli. Continuing inductively in this way, we define $\ell(\mathbb{D}) \in \mathbb{N}$ and D_0 and D_1 (note that $\ell(\mathbb{D}) < \infty$ almost surely by the Borel-Cantelli lemma since $\sum_\ell 2^{1-2k-\ell} < \infty$). Repeating this procedure in D_0 and D_1 beginning with $k = 2$ and $\ell = 1$, we obtain $D_{i,j} \subset D_i$ for $i, j \in \{0,1\} \times \{0,1\}$. Again continuing inductively, we obtain a map $b \mapsto D_b$ with the property that $D_b \subset D_{b'}$ whenever b' is a prefix of b.

If $b \in \{0,1\}^{\mathbb{N}}$, we define $z_b = \bigcap_{b' \text{ is a prefix of } b} D_{b'}$. Since $\sum_k 2^k 2^{-2k-\ell} < \infty$, with probability 1 at most finitely many of the domains D_b have $\ell(D_b) > 0$. It follows from this observation and (2.4.17) that

$$\liminf_{t \to 0} \widetilde{\mathcal{N}}_{z_b}(t) \geq v_{\max}\,.$$

But by Proposition 2.4.2, almost surely every point z in $\mathbb{D}$ satisfies

$$\limsup_{t \to 0} \widetilde{\mathcal{N}}_z(t) \leq v_{\max}.$$

Therefore, $z_b \in \Phi_{v_{\max}}(\Gamma)$.

Since the set of binary sequences is equinumerous with $\mathbb{R}$, this concludes the proof that $\Phi_{v_{\max}}(\Gamma)$ is equinumerous with $\mathbb{R}$. The proof that $\Phi_{v_{\max}}(\Gamma)$ is dense now follows using the argument for density in Theorem 3.8.7. □

2.5 Weighted loops and Gaussian free field extremes

The main result of this section is Theorem 2.5.3, which generalizes Theorem 3.8.7 and highlights the connection between extreme loop counts and the extremes of the Gaussian free field [23]. Let Γ be a CLE_κ, and fix a probability measure μ on $\mathbb{R}$. Conditional on Γ, let $(\xi_{\mathcal{L}})_{\mathcal{L} \in \Gamma}$ be an i.i.d. collection of μ-distributed random variables indexed by Γ. For $z \in D$ and $\varepsilon > 0$, we let $\Gamma_z(\varepsilon)$ be the set of loops in Γ

which surround $B(z,\varepsilon)$ and define

$$\mathcal{S}_z(\varepsilon) = \sum_{\mathcal{L}\in\Gamma_z(\varepsilon)} \xi_{\mathcal{L}} \quad \text{and} \quad \widetilde{\mathcal{S}}_z(\varepsilon) = \frac{\mathcal{S}_z(\varepsilon)}{\log(1/\varepsilon)}.$$

For a CLE_κ Γ on a domain D and $\alpha \in \mathbb{R}$, we define $\Phi^\mu_\alpha(\Gamma) \subset D$ by

$$\Phi^\mu_\alpha(\Gamma) := \left\{z \in D : \lim_{\varepsilon\to 0} \widetilde{\mathcal{S}}_z(\varepsilon) = \alpha\right\}.$$

To study the Hausdorff dimension of $\Phi^\mu_\alpha(\Gamma)$, where Γ is a CLE_κ on D, we introduce for each $(\alpha,\nu) \in \mathbb{R}\times[0,\infty)$ the set

$$\Phi^\mu_{\alpha,\nu}(\Gamma) := \left\{z \in D : \lim_{\varepsilon\to 0} \widetilde{\mathcal{S}}_z(\varepsilon) = \alpha \quad \text{and} \quad \lim_{\varepsilon\to 0} \widetilde{\mathcal{N}}_z(\varepsilon) = \nu\right\}. \tag{2.5.1}$$

Let $\Lambda^\star_\mu$ be the Fenchel-Legendre transform of μ and let $\Lambda^\star_\kappa$ be the Fenchel-Legendre transform of the log conformal radius distribution (2.1.3). We define

$$\gamma_\kappa(\alpha,\nu) = \begin{cases} \nu\Lambda^\star_\mu\left(\frac{\alpha}{\nu}\right) + \nu\Lambda^\star_\kappa\left(\frac{1}{\nu}\right) & \nu > 0 \\ \lim_{\nu'\searrow 0}\gamma_\kappa(\alpha,\nu') & \nu = 0 \text{ and } \alpha \neq 0 \\ \lim_{\nu'\searrow 0}\gamma_\kappa(\nu') = 1 - \frac{2}{\kappa} - \frac{3\kappa}{32} & \nu = 0 \text{ and } \alpha = 0, \end{cases} \tag{2.5.2}$$

where the limits exist by the convexity of $\Lambda^\star_\kappa$ and $\Lambda^\star_\mu$ (Proposition 2.2.5((i))). Note that $\gamma_\kappa(\alpha,\nu)$ may be infinite for some (α,ν) pairs. Note also that the second and third limit expressions for $\alpha = 0$, $\nu = 0$ agree except when $\Lambda^\star_\mu(0) = \infty$, because $\lim_{\nu'\to 0}\nu'\Lambda^\star_\mu(0/\nu') = 0$ whenever $\Lambda^\star_\mu(0) < \infty$.

Theorem 2.5.1. Suppose $\nu \geq 0$, $\alpha \in \mathbb{R}$, $\Phi^\mu_{\alpha,\nu}(\mathrm{CLE}_\kappa)$ is given by (2.5.1), and $\gamma_\kappa(\alpha,\nu)$ is given by (2.5.2). If $\gamma_\kappa(\alpha,\nu) \leq 2$, then almost surely,

$$\dim_{\mathcal{H}} \Phi^\mu_{\alpha,\nu}(\mathrm{CLE}_\kappa) = 2 - \gamma_\kappa(\alpha,\nu). \tag{2.5.3}$$

If $\gamma_\kappa(\alpha,\nu) > 2$, then almost surely $\Phi^\mu_{\alpha,\nu}(\mathrm{CLE}_\kappa) = \emptyset$.

Proof. Suppose that $\Gamma \sim \mathrm{CLE}_\kappa$ in a proper simply connected domain $D \subset \mathbb{C}$. If $\alpha = \nu = 0$, then $\Phi^\mu_{\alpha,\nu}(\Gamma)$ contains the gasket of Γ, which implies $\dim_{\mathcal{H}} \Phi^\mu_{\alpha,\nu}(\Gamma) \geq 2 - \gamma_\kappa(0,0)$ [45]. Furthermore, $\Phi^\mu_{\alpha,\nu}(\Gamma) \subset \Phi_0(\Gamma)$, which implies by Theorem 3.8.7 that $\dim_{\mathcal{H}} \Phi^\mu_{\alpha,\nu}(\Gamma) \leq 2 - \gamma_\kappa(0,0)$. Therefore, (2.5.3) holds in the case $\alpha = \nu = 0$.

Suppose that $(\alpha,\nu) \neq (0,0)$, and assume $\gamma_\kappa(\alpha,\nu) \leq 2$. For the upper bound in (2.5.3), we follow the proof of Proposition 2.4.2. As before, we restrict our attention without loss of generality to the case that $D = \mathbb{D}$ and the set $\Phi^\mu_{\alpha,\nu}(\Gamma) \cap B(0,1/2)$.

For the remainder of the proof, we interpret the expression $0\Lambda^\star(\alpha/0)$ to mean $\lim_{\nu\to 0}\nu\Lambda^\star(\alpha/\nu)$ for $\Lambda^\star \in \{\Lambda^\star_\mu, \Lambda^\star_\kappa\}$ and $\alpha \in \mathbb{R}$. Fix $\varepsilon > 0$. We claim that for $\delta > 0$

sufficiently small,

$$\inf_{\nu' \in (\nu-\delta, \nu+\delta) \cap [0,\infty)} \nu' \Lambda_\kappa^* \left(\frac{1}{\nu'} \right) \geq \nu \Lambda_\kappa^* \left(\frac{1}{\nu} \right) - \frac{\varepsilon}{8}, \quad \text{and} \tag{2.5.4}$$

$$\inf_{\substack{\nu' \in (\nu-\delta, \nu+\delta) \cap [0,\infty), \\ \alpha' \in (\alpha-\delta, \alpha+\delta)}} \nu' \Lambda_\mu^* \left(\frac{\alpha'}{\nu'} \right) \geq 3 \wedge \left(\nu \Lambda_\mu^* \left(\frac{\alpha}{\nu} \right) - \frac{\varepsilon}{8} \right). \tag{2.5.5}$$

(We include the minimum with 3 on the right-hand side of (2.5.5) to handle the case that $\nu \Lambda_\mu^*(\alpha/\nu) = \infty$. The particular choice of 3 was arbitrary; any value strictly larger than 2 would suffice.)

The continuity of $\nu \Lambda_\kappa^*(1/\nu)$ on $[0, \infty)$ (Proposition 2.2.17) implies (2.5.4).

For (2.5.5), we consider three cases.

(i) If $\nu > 0$, then (2.5.5) follows from the lower semi-continuity of Λ_μ^* (See the definitions in the beginning of [12, Section 1.2] and [12, Lemma 2.2.5]).

(ii) If $\nu = 0$ (so that $\alpha \neq 0$) and $\lim_{x \to 0} x \Lambda_\mu^*(1/x) < \infty$, we write

$$\nu' \Lambda_\mu^* \left(\frac{\alpha'}{\nu'} \right) = \alpha' \cdot \left(\frac{\nu'}{\alpha'} \Lambda_\mu^* \left(\frac{\alpha'}{\nu'} \right) \right). \tag{2.5.6}$$

Assume that $\alpha' > 0$; the case that $\alpha' < 0$ is symmetric. If $\delta \in (0, \alpha)$, then $\alpha' \in (\alpha - \delta, \alpha + \delta)$ implies that α' is bounded away from 0. Therefore, (2.5.6) and the lower semi-continuity of Λ_μ^* imply that for all $\eta > 0$, there exists $\delta > 0$ such that

$$\frac{\nu'}{\alpha'} \Lambda_\mu^* \left(\frac{\alpha'}{\nu'} \right) \geq \lim_{x \to 0} x \Lambda_\mu^*(1/x) - \eta.$$

whenever $0 < \nu' < \delta$ and $\alpha' \in (\alpha - \delta, \alpha + \delta)$. Since $\alpha' > \alpha - \delta$, we can choose $\eta > 0$ and then $\delta > 0$ sufficiently small that (2.5.5) holds.

(iii) If $\nu = 0$ (so that $\alpha \neq 0$) and $\lim_{x \to 0} x \Lambda_\mu^*(1/x) = \infty$, then the lower semicontinuity of Λ_μ^* implies that there exists $\delta > 0$ such that (2.5.5) holds with 3 on the right-hand side.

We choose $\delta > 0$ so that (2.5.4) and (2.5.5) hold, and we replace the definition (2.4.1) of $\mathcal{U}^{r,\nu+}$ with

$$\mathcal{U}^{r,\nu,\alpha} := \left\{ U \in \mathcal{D}^r : \left| \widetilde{\mathcal{N}}_{z(U)}(r) - \nu \right| \leq \delta \text{ and } \left| \widetilde{S}_{z(U)}(r) - \alpha \right| \leq \delta \right\},$$

where $\mathcal{D}^r$ is defined as in Section 2.4.1 in the proof of Proposition 2.4.2. As in (2.4.2), $\mathcal{C}^{m,\nu,\alpha} = \bigcup_{n \geq m} \mathcal{U}^{\exp(-n),\nu,\alpha}$ is a cover of $\Phi_{\alpha,\nu}^\mu(\Gamma) \cap B(0, 1/2)$ for all $m \in \mathbb{N}$. Suppose that $\gamma_\kappa(\alpha, \nu) \leq 2$. Using Lemma 2.2.18 and Cramér's theorem, we see that for sufficiently large n,

$$\mathbb{P}[U \in \mathcal{U}^{\exp(-n),\nu,\alpha}] \le \mathbb{P}\left[\left|\widetilde{\mathcal{S}}_{z(U)}(e^{-n}) - \alpha\right| \le \delta \,\middle|\, \left|\widetilde{\mathcal{N}}_{z(U)}(e^{-n}) - \nu\right| \le \delta\right] \times$$
$$\mathbb{P}\left[\left|\widetilde{\mathcal{N}}_{z(U)}(e^{-n}) - \nu\right| \le \delta\right]$$
$$\le e^{-(\gamma_\kappa(\alpha,\nu)-\varepsilon/2)n}. \tag{2.5.7}$$

If $\gamma_\kappa(\alpha,\nu) > 2$, then the same analysis shows that $\mathbb{P}[U \in \mathcal{U}^{\exp(-n),\nu,\alpha}] \le e^{-cn}$ for some $c > 2$. The rest of the argument now follows the proof of Proposition 2.4.2.

For the lower bound we may assume $\gamma_\kappa(\alpha,\nu) \le 2$, which implies that $\nu\Lambda_\mu^\star(\alpha/\nu)$ is finite. We consider the events denoted by E_z^k in the discussion following Lemma 2.4.3, which we now denote by $E_z^k(1)$. We also define events on which we can control the sums associated with the loops in each annulus. More precisely, suppose that $(\delta_k)_{k\in\mathbb{N}}$ is a sequence of positive real numbers with $\delta_k \to 0$ as $k \to \infty$. We define

$$E_z^k(2) = \left\{\mathcal{S}_0(t_k;\widetilde{\Gamma}_z^k) \in \left((\alpha-\delta_k)\log t_k^{-1}, (\alpha+\delta_k)\log t_k^{-1}\right)\right\}.$$

(Recall the definition of $\widetilde{\Gamma}_z^k$ from Section 2.4.2 and that $\mathcal{S}_0(t_k;\widetilde{\Gamma}_z^k)$ represents the weighted loop count with respect to $\widetilde{\Gamma}_z^k$, where we define $\xi_{\mathcal{L}}$ for $\mathcal{L} \in \widetilde{\Gamma}_z^k$ to be equal to the weight of the conformal preimage of $\mathcal{L}$ in Γ.) We define the events $\acute{E}_z^k = E_z^k(1) \cap E_z^k(2)$ and $\acute{E}_z^{k_1,k_2} = \bigcap_{k=k_1}^{k_2} \acute{E}_z^k$ as before. Similar to (2.5.7), we have by Cramér's theorem

$$\mathbb{P}\left[E_z^k(2) \mid E_z^k(1)\right] = t_k^{\nu\Lambda_\mu^\star(\alpha/\nu)+o(1)},$$

provided $\delta_k \to 0$ slowly enough. We multiply both sides by $\mathbb{P}[E_z^k(1)] = t_k^{\nu\Lambda_\kappa^\star(1/\nu)+o(1)}$ and get

$$\mathbb{P}[\acute{E}_z^k] = t_k^{\gamma_\kappa(\alpha,\nu)+o(1)} \quad \text{as} \quad k \to \infty.$$

Thus Proposition 2.4.7 and its proof carry over with $\gamma_\kappa(\nu)$ replaced by $\gamma_\kappa(\alpha,\nu)$.

It remains to verify that $\acute{P}(\alpha,\nu;\Gamma) \subset \Phi_{\alpha,\nu}^\mu(\Gamma)$, where $\acute{P}(\alpha,\nu;\Gamma)$ is defined to be the set of points z for which $\acute{E}_z^{1,n}$ occurs for all n. We see that $\lim_{\varepsilon\to 0}\widetilde{\mathcal{N}}_z(\varepsilon) = \nu$ for the reasons explained in the proof of Lemma 2.4.5. Moreover, $\lim_{\varepsilon\to 0}\widetilde{\mathcal{S}}_z(\varepsilon) = \alpha$ for analogous reasons. By Proposition 2.4.8, this concludes the proof. □

In Theorem 2.5.3 we show that $\dim_{\mathcal{H}} \Phi_\alpha^\mu(\mathrm{CLE}_\kappa)$ is almost surely equal to the maximum of the expression given in Theorem 2.5.1 as ν is allowed to vary. In Theorem 2.5.2 we show that, with the exception of some degenerate cases, there is a unique value of ν at which this maximum is achieved.

Theorem 2.5.2. Let $\alpha \in \mathbb{R}$ and let μ be a probability measure on $\mathbb{R}$.

(i) If $\alpha = 0$, then $\nu \mapsto \gamma_\kappa(\alpha,\nu)$ has a unique nonnegative minimizer ν_0.

(ii) If $\alpha > 0$ and $\mu((0,\infty)) > 0$ or if $\alpha < 0$ and $\mu((-\infty,0)) > 0$, then $\nu \mapsto \gamma_\kappa(\alpha,\nu)$ has a unique minimizer ν_0. Furthermore, $\nu_0 > 0$.

(iii) If $\alpha > 0$ and $\mu((0,\infty)) = 0$ or $\alpha < 0$ and $\mu((-\infty,0)) = 0$, then for all $\nu \in [0,\infty)$ we have $\gamma_\kappa(\alpha,\nu) = \infty$. In this case we set $\nu_0 = 0$.

Proof. For part ((i)), note that when $\alpha = 0$, the expression we seek to minimize is $\nu\Lambda_\mu^\star(0) + \nu\Lambda_\kappa^\star(1/\nu)$. If $\Lambda_\mu^\star(0) < +\infty$, then this expression has a unique positive minimizer because its derivative with respect to ν differs from that of $\nu\Lambda_\kappa^\star(1/\nu)$ by the constant $\Lambda_\mu^\star(0)$ and therefore varies strictly monotonically from $-\infty$ to $+\infty$. If $\Lambda_\mu^\star(0) = +\infty$, then $\nu = 0$ is the unique minimizer.

For part ((iii)), observe by Cramér's theorem that $\Lambda_\mu^\star(x) = \infty$ when x and α have the same sign, so $\gamma_\kappa(\alpha,\nu) = \infty$.

For part ((ii)), we may assume without loss of generality that $\alpha > 0$ and $\mu((0,\infty)) > 0$. Define $a = \operatorname{ess\,inf} X$ and $b = \operatorname{ess\,sup} X$ for a μ-distributed random variable X, so that $-\infty \le a \le b \le +\infty$. Since $\mu((0,\infty)) > 0$, we have $b > 0$ by Proposition 2.2.5((v)).

We make some observations about the functions $f_\mu : (0,\infty) \to [0,\infty]$ and $f_\kappa : (0,\infty) \to [0,\infty]$ defined by

$$f_\mu(\nu) := \nu\Lambda_\mu^\star\left(\frac{\alpha}{\nu}\right) \quad \text{and} \quad f_\kappa(\nu) := \nu\Lambda_\kappa^\star\left(\frac{1}{\nu}\right).$$

First, they inherit convexity from $\Lambda_\mu^\star$ and $\Lambda_\kappa^\star$ by Lemma 2.2.8. Note that the sum $f(\nu) := f_\mu(\nu) + f_\kappa(\nu)$ is also convex.

By Proposition 2.2.5((viii)), $\Lambda_\mu^\star$ is continuously differentiable on (a,b). The chain rule gives

$$f_\mu'(\nu) = -\frac{\alpha}{\nu}(\Lambda_\mu^\star)'\left(\frac{\alpha}{\nu}\right) + \Lambda_\mu^\star\left(\frac{\alpha}{\nu}\right).$$

If $a > -\infty$, then Proposition 2.2.5((ix)) implies $(\Lambda_\mu^\star)'(x) \to -\infty$ as $x \searrow a$. Similarly, if $b < \infty$, Proposition 2.2.5((x)) implies $(\Lambda_\mu^\star)'(x) \to +\infty$ as $x \nearrow b$. In other words,

$$\lim_{\nu \nearrow \alpha/a} f_\mu'(\nu) = +\infty \quad \text{if} \quad a > 0, \quad \text{and} \tag{2.5.8}$$

$$\lim_{\nu \searrow \alpha/b} f_\mu'(\nu) = -\infty \quad \text{if} \quad b < \infty. \tag{2.5.9}$$

Recall from Proposition 2.2.10 that (note $f_\kappa = \gamma_\kappa$)

$$\{(\nu, f_\kappa(\nu) : 0 < \nu < \infty\} = \left\{\left(\frac{1}{\Lambda_\kappa'(\lambda)}, \lambda - \frac{\Lambda_\kappa(\lambda)}{\Lambda_\kappa'(\lambda)}\right) : -\infty < \lambda < 1 - \frac{2}{\kappa} - \frac{3\kappa}{32}\right\}.$$

Suppose $-\infty < \lambda_0 < 1 - 2/\kappa - 3\kappa/32$. If $\nu = 1/\Lambda_\kappa'(\lambda_0)$, then

$$f_\kappa'(\nu) = \frac{\frac{d}{d\lambda}\Big|_{\lambda=\lambda_0}[\lambda - \Lambda_\kappa(\lambda)/\Lambda_\kappa'(\lambda)]}{\frac{d}{d\lambda}\Big|_{\lambda=\lambda_0}[1/\Lambda_\kappa'(\lambda)]} = -\Lambda_\kappa(\lambda_0).$$

When we take $\lambda_0 \to -\infty$ (which corresponds to taking $\nu \to +\infty$) and $\lambda_0 \to 1 - 2/\kappa - 3\kappa/32$ (which corresponds to taking $\nu \to 0$), respectively, in the explicit formula (2.1.3) for Λ_κ, we obtain

$$\lim_{\nu \searrow 0} f'_\kappa(\nu) = -\infty, \quad \text{and} \tag{2.5.10}$$

$$\lim_{\nu \nearrow +\infty} f'_\kappa(\nu) = +\infty. \tag{2.5.11}$$

We conclude the proof of ((ii)) by treating five cases separately. For each of the cases (i)–(ii) and (iv)–(v), we argue that $f'(\nu)$ ranges from $-\infty$ to $+\infty$ for $\nu \in (\alpha/b, \alpha/\max(0,a))$ (if $a < 0$ so that $\max(0,a) = 0$ then we interpret $\alpha/0 = +\infty$). Upon showing this, continuous differentiability of f (Proposition 2.2.5((viii))) guarantees by the intermediate value theorem that the equation $f'(\nu) = 0$ has a solution. The convexity of f_μ and strict convexity of f_κ (Proposition 2.2.11) imply that the solution is unique. Case (iii) uses a separate (easy) argument.

(i) $a \le 0 < b < \infty$. Note that $f'_\mu(x) \to -\infty$ as $x \searrow \alpha/b$ and $f'_\kappa(x) \to +\infty$ as $x \to +\infty$. Since $f'_\kappa(x) \not\to \infty$ as $x \searrow \alpha/b$ and $f'_\mu(x) \not\to -\infty$ as $x \to +\infty$, we conclude that $f'((\alpha/b, +\infty)) = (-\infty, +\infty)$.

(ii) $a \le 0 < b = \infty$. We have $f'((0, +\infty)) = (-\infty, +\infty)$ since $f'_\kappa(x)$ goes to $-\infty$ as $x \searrow 0$ and to $+\infty$ as $x \to +\infty$.

(iii) $0 < a = b < \infty$. Since $a = b$, $\Lambda^\star_\mu(x) = +\infty$ for all $x \ne b$, so $\nu = \alpha/b$ is the unique minimizer of $\nu \mapsto \gamma_\kappa(\alpha, \nu)$.

(iv) $0 < a < b < \infty$. We have $f'((\alpha/b, \alpha/a)) = (-\infty, +\infty)$ since $f'_\mu(x)$ goes to $-\infty$ as $x \searrow \alpha/b$ and to $+\infty$ as $x \nearrow \alpha/a$.

(v) $0 < a < b = \infty$. We have $f'((0, \alpha/a)) = (-\infty, +\infty)$ since $f'_\kappa(x)$ goes to $-\infty$ as $x \searrow 0$ and $f'_\mu(x)$ goes to $+\infty$ as $x \nearrow \alpha/a$. □

Theorem 2.5.3. Let $\alpha \in \mathbb{R}$ and let μ be a probability measure on $\mathbb{R}$. Let $\nu_0 = \nu_0(\alpha)$ be the minimizer of $\nu \mapsto \gamma_\kappa(\alpha, \nu)$ from Theorem 2.5.2. If $\gamma_\kappa(\alpha, \nu_0(\alpha)) \le 2$, then almost surely

$$\dim_{\mathcal{H}} \Phi^\mu_\alpha(\mathrm{CLE}_\kappa) = 2 - \gamma_\kappa(\alpha, \nu_0(\alpha)). \tag{2.5.12}$$

If $\gamma_\kappa(\alpha, \nu_0(\alpha)) > 2$, then $\Phi^\mu_\alpha(\mathrm{CLE}_\kappa) = \varnothing$ almost surely.

Proof. The lower bound is immediate from Theorem 2.5.1, since

$$\Phi^\mu_\alpha(\Gamma) \subset \Phi^\mu_{\alpha,\nu_0(\alpha)}(\Gamma),$$

where Γ is a CLE_κ. For the upper bound, we follow the approach in the proof of Proposition 2.4.2. It suffices to consider the case where the domain is the unit disk $\mathbb{D}$, and without loss of generality we may consider the set $\Phi^\mu_\alpha(\Gamma) \cap B(0, 1/2)$. Observe that if $\alpha = 0$, then

$$\gamma_\kappa(0, \nu) = \nu \Lambda^\star_\mu(0) + \nu \Lambda^\star_\kappa(1/\nu). \tag{2.5.13}$$

If $\Lambda^\star_\mu(0) = \infty$, then the first term in (2.5.13) is infinite unless $\nu = 0$. It follows that $\nu_0(0) = 0$ in this case. If $\Lambda^\star_\mu(0) < \infty$, then the derivative of the first term with respect to ν is a nonnegative constant $\Lambda^\star_\mu(0)$, while the derivative of the second

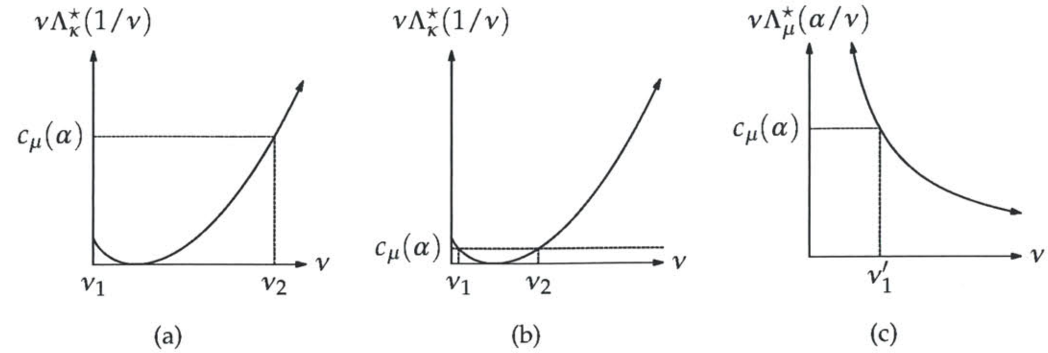


Figure 2-11: To obtain (2.5.14), we trim $[0,\infty)$ to a compact subinterval $[v_1', v_2']$ as follows. First, we remove an interval (v_2, ∞) on which $v\Lambda_\kappa^\star(1/v)$ is larger than $c_\mu(\alpha) = \gamma_\kappa(\alpha, v_0(\alpha))$ (panel (a)). Second, if $c_\mu(\alpha)$ is smaller than $1 - \frac{2}{\kappa} - \frac{3\kappa}{32}$, we remove an interval $[0, v_1)$ on which $v\Lambda_\kappa^\star(1/v)$ is larger than $c_\mu(\alpha)$ (panel (b)). If necessary, we remove intervals $[v_1, v_1')$ and/or $(v_2', v_2]$ on which $f_\mu(v)$ is larger than $c_\mu(\alpha)$ (panel (c) shows an example for which the former is necessary but not the latter).

term is a strictly increasing function going from $-\infty$ to ∞ as v goes from 0 to ∞. It follows that $\Lambda_\mu^\star(0) < \infty$ implies $v_0(0) > 0$. We first handle the case $\Lambda_\mu^\star(0) < \infty$.

Let $c_\mu(\alpha) = \gamma_\kappa(\alpha, v_0(\alpha))$. Since $v\Lambda_\kappa^\star(1/v)$ and $v\Lambda_\mu^\star(\alpha/v)$ are convex and lower semicontinuous, we may define v_1 and v_2 so that $v\Lambda_\kappa^\star(1/v) \le c_\mu(\alpha)$ if and only if $0 \le v_1 \le v \le v_2 < \infty$ (see Figure 2-11). Observe that $[v_1, v_2]$ is nonempty since it contains $v_0(\alpha)$. We also define $v_1' := \inf\{v \ge v_1 : v\Lambda_\mu^\star(\alpha/v) \le c_\mu(\alpha)\}$ and $v_2' := \sup\{v \le v_2 : v\Lambda_\mu^\star(\alpha/v) \le c_\mu(\alpha)\}$.

We claim that

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ so that } \forall (\alpha', v) \in [\alpha - \delta, \alpha + \delta] \times [v_1', v_2'] \text{ we have}$$
$$v\Lambda_\mu^\star(\alpha'/v) > v\Lambda_\mu^\star(\alpha/v) - \frac{\varepsilon}{4}. \qquad (2.5.14)$$

Using (2.5.13), observe that if $\alpha = 0$, then $c_\mu(\alpha)$ is less than $\gamma_\kappa(0,0)$, which implies that $v_1 > 0$. Therefore, (2.5.14) follows in the case $\alpha = 0$ from the lower semicontinuity of $\Lambda_\mu^\star$ at 0. For the case $\alpha > 0$, we observe that $v\Lambda_\mu^\star(\alpha/v)$ finite on $[v_1', v_2']$. By lower semicontinuity and convexity of $\Lambda_\mu^\star$, this implies that $v\Lambda_\mu^\star(\alpha/v)$ is continuous on $[v_1', v_2']$. Since $[v_1', v_2']$ is compact, we conclude that $v\Lambda_\mu^\star(\alpha/v)$ is uniformly continuous on $[v_1', v_2']$. Since $v\Lambda_\mu^\star(\alpha'/v)$ can be written as $\frac{\alpha'}{\alpha}\frac{v\alpha}{\alpha'}\Lambda_\mu^\star(\alpha/(v\alpha/\alpha'))$ (a straightforward limiting argument shows that this equality holds even when $v = 0$), the uniform continuity of $\Lambda_\mu^\star$ implies (2.5.14) except possibly at the endpoints v_1' and v_2'. However, since $\Lambda_\mu^\star$ is lower semi-continuous, (2.5.14) holds at v_1' and v_2' as well.

Recall the collection of balls $\mathcal{D}^r$ for $r > 0$ that we defined in the proof of Proposition 2.4.2. For $n \in \mathbb{N}$, let

$$\mathcal{Q}^n := \left\{ Q \in \mathcal{D}^{\exp(-n)} \,:\, \widetilde{\mathcal{S}}_{z(\varepsilon)}(e^{-n}) \in (\alpha - \delta, \alpha + \delta) \right\} .$$

Our goal is to show that for all $Q \in \mathcal{D}^{\exp(-n)}$ and n sufficiently large,

$$\mathbb{P}[Q \in \mathcal{Q}^n] \leq e^{-n(c_\mu(\alpha) - \varepsilon/2)} . \tag{2.5.15}$$

The rest of the proof is similar to that of Proposition 2.4.2. To prove (2.5.15), we abbreviate $\widetilde{\mathcal{N}}_{z(Q)}(e^{-n})$ as $\widetilde{\mathcal{N}}$ and write

$$\mathbb{P}[Q \in \mathcal{Q}^n] = \mathbb{E}\left[\mathbb{P}\left[\widetilde{\mathcal{S}}_{z(Q)}(e^{-n}) \in (\alpha - \delta, \alpha + \delta) \mid \widetilde{\mathcal{N}}\right]\right] .$$

We split the conditional probability according to the value of $\widetilde{\mathcal{N}}$:

$$\begin{aligned} \mathbb{P}[Q \in \mathcal{Q}^n] \leq\; & \mathbb{E}\left[\mathbf{1}_{\{\widetilde{\mathcal{N}} \notin [\nu_1, \nu_2]\}} \mathbb{P}\left[\widetilde{\mathcal{S}}_{z(Q)}(e^{-n}) \in (\alpha - \delta, \alpha + \delta) \mid \widetilde{\mathcal{N}}\right]\right] \\ & + \mathbb{E}\left[\mathbf{1}_{\{\widetilde{\mathcal{N}} \in [\nu_1, \nu_2] \setminus [\nu_1', \nu_2']\}} \mathbb{P}\left[\widetilde{\mathcal{S}}_{z(Q)}(e^{-n}) \in (\alpha - \delta, \alpha + \delta) \mid \widetilde{\mathcal{N}}\right]\right] \\ & + \mathbb{E}\left[\mathbf{1}_{\{\widetilde{\mathcal{N}} \in [\nu_1', \nu_2']\}} \mathbb{P}\left[\widetilde{\mathcal{S}}_{z(Q)}(e^{-n}) \in (\alpha - \delta, \alpha + \delta) \mid \widetilde{\mathcal{N}}\right]\right] . \end{aligned}$$

The first term on the right-hand side is bounded above by $e^{-n(c_\mu(\alpha) + o(1))}$, because of our choice of ν_1 and ν_2. Similarly, the second term is bounded above by $e^{-n(c_\mu(\alpha)+o(1))}$ by Cramér's theorem and our choice of ν_1' and ν_2'. Thus it remains to show that the third term is bounded above by $e^{-n(c_\mu(\alpha) - \varepsilon/2)}$ for all n sufficiently large. Multiplying and dividing by $e^{-n\widetilde{\mathcal{N}}\Lambda_\kappa^\star(1/\widetilde{\mathcal{N}})}$, applying Cramér's theorem, and using (2.5.14), we find that for large enough n, the third term is bounded above by

$$\begin{aligned} & \mathbb{E}\left[\mathbf{1}_{\{\widetilde{\mathcal{N}} \in [\nu_1', \nu_2]\}} e^{-n(\widetilde{\mathcal{N}}\Lambda_\mu^\star(\alpha/\widetilde{\mathcal{N}}) + \widetilde{\mathcal{N}}\Lambda_\kappa^\star(1/\widetilde{\mathcal{N}}) - \varepsilon/2)} e^{n\widetilde{\mathcal{N}}\Lambda_\kappa^\star(1/\widetilde{\mathcal{N}})}\right] \\ & \quad \leq e^{-n(c_\mu(\alpha) - \varepsilon/4)} \mathbb{E}\left[\mathbf{1}_{\widetilde{\mathcal{N}} \in [\nu_1', \nu_2']} e^{n\widetilde{\mathcal{N}}\Lambda_\kappa^\star(1/\widetilde{\mathcal{N}})}\right] . \end{aligned}$$

It remains to show that $\mathbb{E}\left[\mathbf{1}_{\widetilde{\mathcal{N}} \in [\nu_1', \nu_2']} e^{n\widetilde{\mathcal{N}}\Lambda_\kappa^\star(1/\widetilde{\mathcal{N}})}\right] = e^{o(n)}$. We claim that in fact $\mathbb{E}\left[\mathbf{1}_{\widetilde{\mathcal{N}} \in [\nu_1', \nu_2']} e^{n\widetilde{\mathcal{N}}\Lambda_\kappa^\star(1/\widetilde{\mathcal{N}})}\right]$ is bounded above independently of n. If $\nu_{\text{typical}} \leq \nu_2'$, we partition $[\nu_{\text{typical}}, \nu_2']$ into n intervals of equal length, with endpoints denoted by $x_0, \ldots, x_n$. Denote by $L_{\widetilde{\mathcal{N}}}^{(n)}$ the law of $\widetilde{\mathcal{N}}$. Applying (2.2.5) from Cramér's theorem and using an upper Riemann sum, we get

$$\int_{\nu_{\text{typical}}}^{\nu_2'} e^{nx\Lambda_\kappa^\star(1/x)}\, dL_{\widetilde{\mathcal{N}}}^{(n)}(x) \leq 2 \sum_{i=1}^{n} e^{n(x_i\Lambda_\kappa^\star(1/x_i) - x_{i-1}\Lambda_\kappa^\star(1/x_{i-1}))} \frac{\nu_2' - \nu_{\text{typical}}}{n} .$$

Letting C be the maximum of the derivative of f_κ on $[\nu_{\text{typical}}, \nu_2']$, we estimate difference in parentheses using the mean-value theorem. We get

$$\int_{v_{\text{typical}}}^{v_2'} e^{nx\Lambda_\kappa^\star(1/x)}\, dL_{\widetilde{\mathcal{N}}}^{(n)}(x) \le 2(v_2' - v_{\text{typical}}) \sum_{i=1}^{n} e^{Cn(x_i - x_{i-1})} n^{-1}$$
$$= 2(v_2' - v_{\text{typical}}) e^{C(v_2' - v_{\text{typical}})},$$

which does not grow with n. By an analogous computation, if $v_1' \le v_{\text{typical}}$ then the integral from v_1' to v_{typical} is also bounded in n. Writing

$$[v_1', v_2'] = [v_1', \min(v_{\text{typical}}, v_2')] \cup [\max(v_{\text{typical}}, v_1'), v_2']$$

(where we interpret an interval $[a,b]$ to be empty if $b < a$), we conclude that $\mathbb{E}\left[\mathbf{1}_{\widetilde{\mathcal{N}} \in [v_1', v_2']} e^{n\widetilde{\mathcal{N}}\Lambda_\kappa^\star(1/\widetilde{\mathcal{N}})}\right]$ is bounded in n, as desired.

Now consider the case $\Lambda_\mu^\star(0) = \infty$, which implies that $v_0(0) = 0$. As in the case $\Lambda_\mu^\star(0) < \infty$, it suffices to show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mathbb{P}\left[|\widetilde{S}_z(e^{-n})| \le \delta\right] \le e^{-n(\gamma_\kappa(0,0) - \varepsilon)}. \tag{2.5.16}$$

Choose $\eta > 0$ small enough that $v\Lambda_\kappa^\star(v) \ge 1 - 2/\kappa - 3\kappa/32 - \varepsilon/2$ whenever $v \in (0, \eta)$. Then choose $\delta > 0$ small enough that $\Lambda_\mu^\star(x) \ge 2/\eta$ for all $x \in (-\delta/\eta, \delta/\eta)$ (this is possible by lower semicontinuity of $\Lambda^\star$). Again abbreviating $\widetilde{\mathcal{N}}_z(e^{-n})$ as $\widetilde{\mathcal{N}}$, we write

$$\begin{aligned}
\mathbb{P}\left[|\widetilde{S}_z(e^{-n})| \le \delta\right] &= \mathbb{E}\left[\mathbb{P}\left[|\widetilde{S}_z(e^{-n})| \le \delta \mid \widetilde{\mathcal{N}}\right]\right] \\
&= \mathbb{E}\left[\mathbb{P}\left[|\widetilde{S}_z(e^{-n})| \le \delta \mid \widetilde{\mathcal{N}}\right] \mathbf{1}_{\widetilde{\mathcal{N}} \in [0,\eta)}\right] \\
&\quad + \mathbb{E}\left[\mathbb{P}\left[|\widetilde{S}_z(e^{-n})| \le \delta \mid \widetilde{\mathcal{N}}\right] \mathbf{1}_{\widetilde{\mathcal{N}} \in [\eta,\infty)}\right].
\end{aligned}$$

Bounding the conditional probability by 1 and using our choice of η, we see that the first term is bounded above by $e^{-n(\gamma_\kappa(0,0) - \varepsilon)}$. For the second term, we note by (2.2.5) in Cramér's theorem that the conditional probability is bounded above by

$$2\exp\left(-n\widetilde{\mathcal{N}} \inf_{|y| \le \delta/\widetilde{\mathcal{N}}} \Lambda^\star(y)\right).$$

On the event where $\widetilde{\mathcal{N}}$ is at least η, the factor $\widetilde{\mathcal{N}} \inf_{|y| \le \delta/\widetilde{\mathcal{N}}} \Lambda^\star(y)$ is at least 2, which implies that the second term is bounded by e^{-2n}. This establishes (2.5.16) and concludes the proof. ☐

Proof of Theorem 2.1.2. The logarithmic moment generating function of the signed Bernoulli distribution is $\Lambda_\mu(\eta) = \log\cosh(\sigma\eta)$. When $\kappa = 4$, formula (2.1.3) for Λ_κ

simplifies to

$$\Lambda_4(\lambda) = \begin{cases} -\log\cosh(\pi\sqrt{-2\lambda}) & \lambda < 0 \\ -\log\cos(\pi\sqrt{2\lambda}) & \lambda \geq 0\,. \end{cases}$$

Using the definition of the Fenchel-Legendre transform,

$$\nu_0(\alpha)\Lambda_\mu^\star\left(\frac{\alpha}{\nu_0(\alpha)}\right) + \nu_0(\alpha)\Lambda_\kappa^\star\left(\frac{1}{\nu_0(\alpha)}\right) = \inf_{\nu\geq 0}\sup_{\eta,\lambda}\left[\eta\alpha + \lambda - \nu(\Lambda_\kappa(\lambda) + \Lambda_\mu(\eta))\right]\,.$$

By the Minimax Theorem (see e.g., [54]), the right-hand side equals

$$\sup_{\eta,\lambda}\inf_{\nu\geq 0}\left[\eta\alpha + \lambda - \nu(\Lambda_\kappa(\lambda) + \Lambda_\mu(\eta))\right] = \sup_{\eta,\lambda\,:\,\Lambda_\kappa(\lambda)+\Lambda_\mu(\eta)\leq 0}\left[\eta\alpha + \lambda\right]\,.$$

Since $\Lambda_\kappa(\lambda)$ is continuous in λ and $\Lambda_\kappa(\lambda) \to \infty$ as $\lambda \to \infty$, if $\Lambda_\kappa(\lambda) + \Lambda_\mu(\eta) < 0$, then λ can be increased so that $\Lambda_\kappa(\lambda) + \Lambda_\mu(\eta) = 0$. Thus this last supremum can be replaced by the supremum over λ and η satisfying $\Lambda_\kappa(\lambda) + \Lambda_\mu(\eta) = 0$.

Observe that $\Lambda_\mu(\eta) \geq 0$ for all η, and $\Lambda_\kappa(\lambda) < 0$ only when $\lambda < 0$. It follows that if $\Lambda_\kappa(\lambda) + \Lambda_\mu(\eta) = 0$, then $\lambda < 0$ and we can use the formulas for Λ_κ and Λ_μ to conclude that

$$\Lambda_\kappa(\lambda) + \Lambda_\mu(\eta) = 0 \text{ implies } \sigma\eta = \pi\sqrt{-2\lambda}\,. \tag{2.5.17}$$

So we have

$$\begin{aligned}\nu_0(\alpha)\Lambda_\mu^\star\left(\frac{\alpha}{\nu_0(\alpha)}\right) + \nu_0(\alpha)\Lambda_\kappa^\star\left(\frac{1}{\nu_0(\alpha)}\right) &= \sup_{\eta,\lambda\,:\,\Lambda_\kappa(\lambda)+\Lambda_\mu(\eta)=0}(\eta\alpha + \lambda) \\ &= \sup_{\lambda<0}\left(\frac{\alpha\pi}{\sigma}\sqrt{-2\lambda} + \lambda\right) \\ &= \frac{\pi^2\alpha^2}{2\sigma^2}\,,\end{aligned}$$

since the supremum is achieved when $\lambda = -\alpha^2\pi^2/2\sigma^2$. □

Proof of Theorem 2.1.3. In light of Theorems 2.5.1 and 2.5.3, it suffices to show that the maximum of $2 - \gamma_\kappa(\alpha, \nu)$ is obtained when $\nu = \frac{\alpha}{\sigma}\coth(\frac{\pi^2\alpha}{\sigma})$. As in the proof of Theorem 2.1.2, we begin by writing

$$\gamma_\kappa(\alpha,\nu) = \nu\Lambda_\kappa^\star\left(\frac{\alpha}{\nu}\right) + \nu\Lambda_\mu^\star\left(\frac{1}{\nu}\right) = \sup_{\eta,\lambda}\left[\eta\alpha + \lambda - \nu(\Lambda_\kappa(\lambda) + \Lambda_\mu(\eta))\right]\,.$$

At the minimizing value of ν and the corresponding maximizing values of η and λ, the derivatives of the expression in brackets with respect to ν, λ, and η are all zero. Differentiating, we obtain the system

$$\Lambda_\kappa(\lambda) + \Lambda_\mu(\eta) = 0$$
$$\Lambda'_\kappa(\lambda) = \frac{1}{\alpha}\Lambda'_\mu(\eta) = \frac{1}{\nu}.$$

The first equation implies $\sigma\eta = \pi\sqrt{-2\lambda}$ as in (2.5.17). Substituting for λ in the equation $\Lambda'_\kappa(\lambda) = \frac{1}{\alpha}\Lambda'_\mu(\eta)$, we get $\alpha = \sigma^2\eta/\pi^2$. Finally, substituting into $\frac{1}{\alpha}\Lambda'_\mu(\eta) = 1/\nu$ gives $\nu = \frac{\alpha}{\sigma}\coth\left(\frac{\pi^2\alpha}{\sigma}\right)$, as desired. □

2.6 The weighted nesting field

We prove the existence and conformal invariance of the limit as $\varepsilon \to 0$ of the random function $z \mapsto \mathcal{N}_z(\varepsilon) - \mathbb{E}[\mathcal{N}_z(\varepsilon)]$ (with no additional normalization) in an appropriate space of distributions (Theorem 2.6.1). We refer to this object as the *nesting field* because, roughly, its value describes the fluctuations of the nesting of Γ around its mean. This result also holds when the loops are assigned i.i.d. weights. More precisely, we fix a probability measure μ on $\mathbb{R}$ with finite second moment, define $\Gamma_z(\varepsilon)$ to be the set of loops in Γ surrounding $B(z,\varepsilon)$, and define

$$\mathcal{S}_z(\varepsilon) = \sum_{\mathcal{L}\in\Gamma_z(\varepsilon)} \xi_{\mathcal{L}}, \tag{2.6.1}$$

where $\xi_{\mathcal{L}}$ are i.i.d. random variables with law μ. We show that $z \mapsto \mathcal{S}_z(\varepsilon) - \mathbb{E}[\mathcal{S}_z(\varepsilon)]$ converges as $\varepsilon \to 0$ to a distribution we call the *weighted nesting field*. When $\kappa = 4$ and μ is a signed Bernoulli distribution, the weighted nesting field is the GFF [44, 48]. Our result serves to generalize this construction to other values of $\kappa \in (8/3, 8)$ and weight measures μ. In Theorem 2.6.2, we answer a question asked in [65, Problem 8.2].

The weighted nesting field is a random distribution, or generalized function, on D. Informally, it is too rough to be defined pointwise on D, but it is still possible to integrate it against sufficiently smooth compactly supported test functions on D. More precisely, we prove convergence to the nesting field in a certain local Sobolev space $H^s_{\mathrm{loc}}(D) \subset C_c^\infty(D)'$ on D, where $C_c^\infty(D)$ is the space of compactly supported smooth functions on D, $C_c^\infty(D)'$ is the space of distributions on D, and the index $s \in \mathbb{R}$ is a parameter characterizing how smooth the test functions need to be. We review all the relevant definitions in Section 2.10.

Given $h \in C_c^\infty(D)'$ and $f \in C_c^\infty(D)$, we denote by $\langle h, f\rangle$ the evaluation of the linear functional h at f. Recall that the pullback $h \circ \varphi^{-1}$ of $h \in C_c^\infty(D)'$ under a conformal map φ^{-1} is defined by $\langle h\circ\varphi^{-1}, f\rangle := \langle h, |\varphi'|^2 f\circ\varphi\rangle$ for $f \in C_c^\infty(\varphi(D))$.

Theorem 2.6.1. Fix $\kappa \in (8/3, 8)$ and $\delta > 0$, and suppose μ is a probability measure on $\mathbb{R}$ with finite second moment. Let $D \subsetneq \mathbb{C}$ be a simply connected domain. Let Γ be a CLE_κ on D and $(\xi_{\mathcal{L}})_{\mathcal{L}\in\Gamma}$ be i.i.d. weights on the loops of Γ drawn from the distribution μ. Recall that for $\varepsilon > 0$ and $z \in D$, $\mathcal{S}_z(\varepsilon)$ denotes

$$\mathcal{S}_z(\varepsilon) = \sum_{\substack{\mathcal{L}\in\Gamma \\ \mathcal{L} \text{ surrounds } B(z,\varepsilon)}} \xi_{\mathcal{L}}.$$

Let

$$h_\varepsilon(z) = \mathcal{S}_z(\varepsilon) - \mathbb{E}[\mathcal{S}_z(\varepsilon)]. \tag{2.6.2}$$

There exists an $H^{-2-\delta}_{\text{loc}}(D)$-valued random variable $h = h(\Gamma, (\xi_{\mathcal{L}}))$ such that for all $f \in C_c^\infty(D)$, almost surely $\lim_{\varepsilon\to 0}\langle h_\varepsilon, f\rangle = \langle h, f\rangle$. Moreover, $h(\Gamma, (\xi_{\mathcal{L}}))$ is almost surely a deterministic conformally invariant function of the CLE Γ and the loop weights $(\xi_{\mathcal{L}})_{\mathcal{L}\in\Gamma}$: almost surely, for any conformal map φ from D to another simply connected domain, we have

$$h(\varphi(\Gamma), (\xi_{\varphi(\mathcal{L})})_{\mathcal{L}\in\Gamma}) = h(\Gamma, (\xi_{\mathcal{L}})_{\mathcal{L}\in\Gamma}) \circ \varphi^{-1}.$$

In Theorem 2.11.2, we prove a stronger form of convergence, namely almost sure convergence in the norm topology of $H^{-2-\delta}(D)$, when ε tends to 0 along any given geometric sequence.

We also consider the *step nesting* sequence, defined by

$$\mathfrak{h}_n(z) = \sum_{k=1}^n \xi_{\mathcal{L}_k(z)} - \mathbb{E}\left[\sum_{k=1}^n \xi_{\mathcal{L}_k(z)}\right], \quad n \in \mathbb{N},$$

where the random variables $(\xi_{\mathcal{L}})_{\mathcal{L}\in\Gamma}$ are i.i.d. with law μ. We may assume without loss of generality that μ has zero mean, so that $\mathfrak{h}_n(z) = \sum_{k=1}^n \xi_{\mathcal{L}_k(z)}$. We establish the following convergence result for the step nesting sequence, which parallels Theorem 2.6.1:

Theorem 2.6.2. Suppose that $D \subsetneq \mathbb{C}$ is a proper simply connected domain and $\delta > 0$. Assume that the weight distribution μ has a finite second moment and zero mean. There exists an $H^{-2-\delta}_{\text{loc}}(D)$-valued random variable $\mathfrak{h}$ such that $\lim_{n\to\infty}\mathfrak{h}_n = \mathfrak{h}$ almost surely in $H^{-2-\delta}_{\text{loc}}(D)$. Moreover, $\mathfrak{h}$ is almost surely determined by Γ and $(\xi_{\mathcal{L}})_{\mathcal{L}\in\Gamma}$.

Suppose that $\acute{D}$ is another simply connected domain and $\varphi\colon D \to \acute{D}$ is a conformal map. Let $\acute{h}$ be the random element of $H^{-2-\delta}_{\text{loc}}(\acute{D})$ associated with the CLE $\acute{\Gamma} = \varphi(\Gamma)$ on $\acute{D}$ and weights $(\xi_{\varphi^{-1}(\acute{\mathcal{L}})})_{\acute{\mathcal{L}}\in\acute{\Gamma}}$. Then $\acute{\mathfrak{h}} = \mathfrak{h}\circ\varphi^{-1}$ almost surely.

In Proposition 2.12.2, we show that the step nesting field and the weighted nesting field are equal, under the assumption that μ has zero mean.

When $\kappa = 4$, $\sigma = \sqrt{\pi/2}$, and $\mu = \mu_B$ where $\mu_B(\{\sigma\}) = \mu_B(\{-\sigma\}) = 1/2$ (as in Theorem 2.1.2) the distribution h of Theorem 2.6.1 is that of a GFF on D [44]. The existence of the distributional limit for other values of κ was posed in [65, Problem 8.2]. Note that in this context, $\frac{2}{\pi}\mathbb{E}[\mathcal{S}_z(\varepsilon)\mathcal{S}_w(\varepsilon)]$ is equal to the expected number of loops which surround both $B(z,\varepsilon)$ and $B(w,\varepsilon)$. Let $G_D(z,w)$ be the Green's function for the negative Dirichlet Laplacian on D. Since $\mathcal{S}_z(\varepsilon)$ converges to the GFF [44], it follows that $\frac{2}{\pi}\mathbb{E}[\mathcal{S}_z(\varepsilon)\mathcal{S}_w(\varepsilon)]$ converges to $\frac{2}{\pi}G_D(z,w)$ (see Section 2 in [11]). That is, the expected number of CLE_4 loops which surround both z and w is given by $\frac{2}{\pi}G_D(z,w)$.

One of the elements of the proof of Theorem 2.6.1 is an extension of this bound which holds for all $\kappa \in (8/3, 8)$. We include this as our final main theorem.

Theorem 2.6.3. Let Γ be a CLE_κ (with $8/3 < \kappa < 8$) on a simply connected proper domain D. For $z, w \in D$ distinct, let $\mathcal{N}_{z,w}$ be the number of loops of Γ which surround both z and w. For each integer $j \geq 1$, there exists a constant $C_{\kappa,j} \in (0, \infty)$ such that

$$\left|\mathbb{E}[\mathcal{N}_{z,w}^j] - (\nu_{\text{typical}}\, 2\pi\, G_D(z,w))^j\right| \leq C_{\kappa,j}(G_D(z,w)+1)^{j-1}. \tag{2.6.3}$$

2.7 Further CLE estimates

We record the following corollary of the proof of Lemma 2.3.5.

Lemma 2.7.1. Let $\{X_j\}_{j\in\mathbb{N}}$ be non-negative i.i.d. random variables whose law has a positive density with respect to Lebesgue measure on $(0,\infty)$ and for which there exists $\lambda_0 > 0$ such that $\mathbb{E}[e^{\lambda_0 X_1}] < \infty$. For $a \geq 0$, let $S_n^a = a + \sum_{j=1}^n X_j$, and for $a, M > 0$, let $\tau_M^a = \min\{n \geq 0 : S_n^a \geq M\}$. There exists a coupling between S^a and $\widehat{S}^b$ (identically distributed to S^b but not independent of it) and constants $C, c > 0$ so that for all $0 \leq a \leq b \leq M$, we have

$$\mathbb{P}\left[S^a_{\tau^a_M} = \widehat{S}^b_{\widehat{\tau}^b_M}\right] \geq 1 - Ce^{-cM}.$$

The following lemma provides a quantitative version of the statement that it is unlikely that there exists a CLE loop surrounding the inner boundary but not the outer boundary of a given small, thin annulus.

Lemma 2.7.2. Let Γ be a CLE_κ in $\mathbb{D}$. There exist constants $C > 0$, $\alpha > 0$, and $\varepsilon_0 > 0$ depending only on κ such that for $0 < \varepsilon < \varepsilon_0$ and $0 \leq \delta < 1/2$,

$$\mathbb{E}[\mathcal{N}_0(\varepsilon(1-\delta)) - \mathcal{N}_0(\varepsilon)] \leq C\delta + C\varepsilon^\alpha. \tag{2.7.1}$$

Proof. We couple the CLE_κ $\Gamma_\mathbb{D} = \Gamma$ in the disk with a whole-plane CLE_κ $\Gamma_\mathbb{C}$ as in Theorem 2.3.7. Index the loops of $\Gamma_\mathbb{C}$ surrounding 0 by $\mathbb{Z}$ in such a way that $\mathcal{L}_0^n(\Gamma_\mathbb{C})$ and $\mathcal{L}_0^n(\Gamma_\mathbb{D})$ are exponentially close for large n. For $n \in \mathbb{N}$ define $V_n^\mathbb{D} = -\log \operatorname{inrad} \mathcal{L}_0^n(\Gamma_\mathbb{D})$, and for $n \in \mathbb{Z}$ define $V_n^\mathbb{C} = -\log \operatorname{inrad} \mathcal{L}_0^n(\Gamma_\mathbb{C})$. Since whole-plane CLE_κ is scale invariant, the set $\{V_n^\mathbb{C} : n \in \mathbb{Z}\}$ is translation invariant. Using Corollary 2.2.15 to compare $(V_n^\mathbb{C})_{n\in\mathbb{Z}}$ to the sequence of log conformal radii of the loops of $\Gamma_\mathbb{C}$ surrounding the origin, the translation invariance implies

$$\mathbb{E}\left[\#\left\{n : a \leq V_n^\mathbb{C} < b\right\}\right] = \nu_{\text{typical}}(b-a).$$

Let α and the term *low distortion* be defined as in the statement of Theorem 2.3.7. With probability $1 - O(\varepsilon^\alpha)$ there is a low distortion map from $\Gamma_{\mathbb{D}}|_{B(0,\varepsilon)^+}$ to $\Gamma_{\mathbb{C}}|_{B(0,\varepsilon)^+}$, and on this event, we can and bound

$$\# \left\{ n \,:\, \log \frac{1}{\varepsilon} \le V_n^{\mathbb{D}} < \log \frac{1}{\varepsilon(1-\delta)} \right\} \le$$
$$\le \# \left\{ n \,:\, \log \frac{1}{\varepsilon} - O(\varepsilon^\alpha) \le V_n^{\mathbb{C}} < \log \frac{1}{\varepsilon(1-\delta)} + O(\varepsilon^\alpha) \right\} .$$

On the event that there is no such low distortion map, this can be detected by comparing the boundaries of $\Gamma_{\mathbb{D}}|_{B(0,\varepsilon)^+}$ and $\Gamma_{\mathbb{C}}|_{B(0,\varepsilon)^+}$, so that conditional on this unlikely event, $\Gamma_{\mathbb{D}}|_{B(0,\varepsilon)^+}$ is still an unbiased CLE_κ conformally mapped to the region surrounded by the boundary of $\Gamma_{\mathbb{D}}|_{B(0,\varepsilon)^+}$. In particular, the sequence of log-conformal radii of loops of $\Gamma_{\mathbb{D}}|_{B(0,\varepsilon)^+}$ surrounding 0 is a renewal process, which together with the Koebe distortion theorem and the bound $\delta \le 1/2$ imply

$$\mathbb{E}[\mathcal{N}_0(\varepsilon(1-\delta)) - \mathcal{N}_0(\varepsilon) \,|\, \text{no low distortion map}] \le \text{constant} .$$

Combining these bounds yields (2.7.1). □

Lemma 2.7.3. For each $\kappa \in (8/3, 8)$ and integer $j \in \mathbb{N}$, there are constants $C > 0$, $\alpha > 0$, and $\varepsilon_0 > 0$ (depending only on κ and j) such that whenever D is a simply connected proper domain, $z \in D$, φ is a conformal transformation of D, and $0 < \varepsilon < \varepsilon_0$, if Γ is a CLE_κ in D, then

$$\mathbb{E}\left[\left| \mathcal{N}_z(\varepsilon\, \mathrm{CR}(z; D); \Gamma) - \mathcal{N}_{\varphi(z)}(\varepsilon\, \mathrm{CR}(\varphi(z); \varphi(D)); \varphi(\Gamma)) \right|^j \right] < C \varepsilon^\alpha .$$

Proof. Observe that translating and scaling the domain D or its conformal image $\varphi(D)$ has no effect on the loop counts, so we assume without loss of generality that $z = 0$, $\varphi(z) = 0$, $\mathrm{CR}(z; D) = 1$, and $\mathrm{CR}(\varphi(z); \varphi(D)) = 1$. Observe also that it suffices to prove this lemma in the case that the domain D is the unit disk $\mathbb{D}$, since a general φ may be expressed as the composition $\varphi = \varphi_2 \circ \varphi_1^{-1}$ where φ_1 and φ_2 are conformal transformations of the unit disk with $\varphi_i(0) = 0$ and $\varphi_i'(0) = 1$, and the desired bound follows from the triangle inequality.

Let Γ be a CLE_κ on $\mathbb{D}$, and let $\acute{\Gamma} = \varphi(\Gamma)$. By the Koebe distortion theorem and the elementary inequality

$$1 - 3r \le \frac{1}{(1+r)^2} \le \frac{1}{(1-r)^2} \le 1 + 3r, \quad \text{for } r \text{ small enough}, \tag{2.7.2}$$

we have

$$B(0, \varepsilon - 3\varepsilon^2) \subset \varphi^{-1}(B(0, \varepsilon)) \subset B(0, \varepsilon + 3\varepsilon^2) ,$$

for small enough ε. Hence $\mathcal{N}_0(\varepsilon + 3\varepsilon^2; \Gamma) \le \mathcal{N}_0(\varepsilon; \acute{\Gamma}) \le \mathcal{N}_0(\varepsilon - 3\varepsilon^2; \Gamma)$, and so for

$$X := \mathcal{N}_0(\varepsilon - 3\varepsilon^2; \Gamma) - \mathcal{N}_0(\varepsilon + 3\varepsilon^2; \Gamma)$$

we have $|\mathcal{N}_0(\varepsilon;\acute{\Gamma}) - \mathcal{N}_0(\varepsilon;\Gamma)| \leq X$.

By Lemma 2.7.2 we have $\mathbb{E}[X] = O(\varepsilon^\alpha)$, which proves the case $j = 1$.

Notice that the conformal radius of every new loop after the first that intersects $B(0, \varepsilon + 3\varepsilon^2)$ has a uniformly positive probability of being less than $\frac{1}{4}(\varepsilon - 3\varepsilon^2)$, conditioned on the previous loop. By the Koebe quarter theorem, such a loop intersects $B(0, \varepsilon - 3\varepsilon^2)$. Thus for some $p < 1$ we have $\mathbb{P}[X \geq k+1] \leq p\mathbb{P}[X \geq k]$ for $k \geq 0$. Hence

$$\mathbb{E}[X^j] = \sum_{k=1}^{\infty} k^j \mathbb{P}[X = k] \leq \sum_{k=1}^{\infty} k^j p^k \mathbb{P}[X = 1] \leq \left(\sum_{k=1}^{\infty} k^j p^k \right) \mathbb{E}[X] = O(\varepsilon^\alpha),$$

which proves the cases $j > 1$. □

2.8 Co-nesting estimates

We use the following lemma in the proof of Theorem 2.6.3:

Lemma 2.8.1. Let $\lambda_0 > 0$, and suppose $\{X_j\}_{j \in \mathbb{N}}$ are nonnegative i.i.d. random variables for which $\mathbb{E}[X_1] > 0$ and $\mathbb{E}[e^{\lambda_0 X_1}] < \infty$. Let $\Lambda(\lambda) = \log \mathbb{E}[e^{\lambda X_1}]$ and let $S_n = \sum_{j=1}^n X_j$. For $x > 0$, define $\tau_x = \inf\{n \geq 0 : S_n \geq x\}$. For $\lambda < \lambda_0$, let

$$M_n^\lambda = \exp(\lambda S_n - \Lambda(\lambda) n).$$

Then for $\lambda < \lambda_0$ and $x \geq 0$, the random variables $\{M_{n \wedge \tau_x}^\lambda\}_{n \in \mathbb{N}}$ are uniformly integrable.

Proof. Fix $\beta > 1$ such that $\beta\lambda < \lambda_0$. By Hölder's inequality, any family of random variables which is uniformly bounded in L^p for some $p > 1$ is uniformly integrable. Therefore, it suffices to show that $\sup_{n \geq 0} \mathbb{E}[(M_{n \wedge \tau_x}^\lambda)^\beta] < \infty$. We have,

$$\begin{aligned}(M_{n \wedge \tau_x}^\lambda)^\beta &= \exp(\beta\lambda(S_{n \wedge \tau_x} - x)) \times \exp(\beta\lambda x - \beta\Lambda(\lambda)(n \wedge \tau_x)) \\ &\leq \exp(\beta\lambda(S_{\tau_x} - x)) \times \exp(\beta\lambda x).\end{aligned}$$

The result follows from Lemma 2.2.12. □

Proof of Theorem 2.6.3. Fix $z, w \in D$ distinct and $j \in \mathbb{N}$. Let $\varphi \colon D \to \mathbb{D}$ be the conformal map which sends z to 0 and w to $e^{-x} \in (0,1)$. Let G_D (resp. $G_{\mathbb{D}}$) be the Green's function for $-\Delta$ with Dirichlet boundary conditions on D (resp. $\mathbb{D}$). Explicitly,

$$G_{\mathbb{D}}(u,v) = \frac{1}{2\pi} \log \frac{|1 - \overline{u}v|}{|u - v|} \quad \text{for} \quad u, v \in \mathbb{D}.$$

In particular, $G_{\mathbb{D}}(0,u) = \frac{1}{2\pi} \log |u|^{-1}$ for $u \in \mathbb{D}$. By the conformal invariance of CLE_κ and the Green's function, i.e. $G_D(u,v) = G_{\mathbb{D}}(\varphi(u), \varphi(v))$, it suffices to show that there exists a constant $C_{j,\kappa} \in (0,\infty)$ which depends only on j and $\kappa \in (8/3, 8)$ such that

$$\left|\mathbb{E}[(\mathcal{N}_{0,e^{-x}})^j] - (\nu_{\text{typical}} x)^j\right| \le C_{j,\kappa}(x+1)^{j-1} \qquad \text{for all } x > 0. \tag{2.8.1}$$

Let $\{T_i\}_{i\in\mathbb{N}}$ be the sequence of log conformal radii increments associated with the loops of Γ which surround 0, let $S_k = \sum_{i=1}^k T_i$, and let $\tau_x = \min\{k \ge 1 : S_k \ge x\}$. Recall that $\Lambda_\kappa(\lambda)$ denotes the log moment generating function of the law of T_1. Let $M_n = \exp(\lambda S_n - \Lambda_\kappa(\lambda) n)$. By Lemma 2.8.1, $\{M_{n\wedge\tau_x}\}_{n\in\mathbb{N}}$ is a uniformly integrable martingale for $\lambda < 1 - \frac{2}{\kappa} - \frac{3\kappa}{32}$. By Lemma 2.2.12, we can write $S_{\tau_x} = x + X$ where $\mathbb{E}[e^{\lambda X}] < \infty$. By the optional stopping theorem for uniformly integrable martingales (see [85, § A14.3]), we have that

$$1 = \mathbb{E}[\exp(\lambda S_{\tau_x} - \Lambda_\kappa(\lambda)\tau_x)] = \mathbb{E}[\exp(\lambda x + \lambda X - \Lambda_\kappa(\lambda)\tau_x)]. \tag{2.8.2}$$

We argue by induction on j that

$$\mathbb{E}[(\Lambda'_\kappa(0)\tau_x)^j] = x^j + O((x+1)^{j-1}). \tag{2.8.3}$$

The base case $j = 0$ is trivial.

If we differentiate (2.8.2) with respect to λ and then evaluate at $\lambda = 0$, we obtain

$$0 = \mathbb{E}[(x + X - \Lambda'_\kappa(0)\tau_x)].$$

If we instead differentiate twice, we obtain

$$0 = \mathbb{E}[(x + X - \Lambda'_\kappa(0)\tau_x)^2 - \Lambda''_\kappa(0)\tau_x].$$

Similarly, if we differentiate j times with respect to λ and then evaluate at $\lambda = 0$, we obtain

$$0 = \mathbb{E}[(x + X - \Lambda'_\kappa(0)\tau_x)^j] + \sum_{\substack{i\ge 0, k\ge 1\\ i+2k\le j}} A_{\kappa,i,k}\mathbb{E}[(x + X - \Lambda'_\kappa(0)\tau_x)^i \tau_x^k], \tag{2.8.4}$$

where the $A_{\kappa,i,k}$'s are constant coefficients depending on the higher order derivatives of Λ_κ at 0. By our induction hypothesis, for $h < j$ we have $\mathbb{E}[\tau_x^h] = O((x+1)^h)$. Conditional on τ_x, X has exponentially small tails, so $\mathbb{E}[\tau_x^h X^\ell] = O((x+1)^h)$ as well. From this we obtain

$$0 = \mathbb{E}[(x - \Lambda'_\kappa(0)\tau_x)^j] + O((x+1)^{j-1}). \tag{2.8.5}$$

Using our induction hypothesis again for $h < j$, we obtain

$$0 = \sum_{h=0}^{j-1} \binom{j}{h} (-1)^h x^j + \mathbb{E}[(-\Lambda'_\kappa(0)\tau_x)^j] + O((x+1)^{j-1}), \tag{2.8.6}$$

from which (2.8.3) follows, completing the induction.

Recall that $J_{0,r}^{\cap}$ (resp. $J_{0,r}^{\subset}$) is the smallest index j such that $\mathcal{L}_0^j$ intersects (resp. is contained in) $B(0,r)$. It is straightforward that

$$\tau_{x-\log 4} \le J_{0,e^{-x}}^{\cap} \le \mathcal{N}_{0,e^{-x}} + 1 \le J_{0,e^{-x}}^{\subset}.$$

Since the τ's are stopping times for an i.i.d. sum, conditional on the value of $\tau_{x-\log 4}$, the difference $\tau_x - \tau_{x-\log 4}$ has exponentially decaying tails. Moreover, by Lemma 2.3.2, conditional on the value of τ_x, $J_{0,e^{-x}}^{\subset} - \tau_x$ has exponentially decaying tails. Thus $\mathbb{E}[\mathcal{N}_{0,e^{-x}}^j] = \mathbb{E}[\tau_x^j] + O((x+1)^{j-1})$. Finally, we recall that $1/\Lambda_\kappa'(0) = 1/\mathbb{E}[T_1] = \nu_{\text{typical}}$. □

By combining Theorem 2.6.3 and Corollary 2.3.3, we can estimate the moments of the number of loops which surround a ball in terms of powers of $G_D(z,w)$.

Corollary 2.8.2. There exists a constant $C_{j,\kappa} \in (0,\infty)$ depending only on $\kappa \in (8/3,8)$ and $j \in \mathbb{N}$ such that the following is true. For each $\varepsilon > 0$ and $z \in D$ for which $\operatorname{dist}(z,\partial D) \ge 2\varepsilon$ and $\theta \in \mathbb{R}$, we have

$$\left|\mathbb{E}[(\mathcal{N}_z(\varepsilon))^j] - (2\pi\nu_{\text{typical}} G_D(z,z+\varepsilon e^{i\theta}))^j\right| \le C_{j,\kappa}(G_D(z,z+\varepsilon e^{i\theta})+1)^{j-1}. \quad (2.8.7)$$

In particular, there exists constant a constant $C_\kappa \in (0,\infty)$ depending only on $\kappa \in (8/3,8)$ such that

$$\left|\mathbb{E}[\mathcal{N}_z(\varepsilon)] - \nu_{\text{typical}} \log \frac{\operatorname{CR}(z;D)}{\varepsilon}\right| \le C_\kappa. \quad (2.8.8)$$

Proof. Let $w = z + \varepsilon e^{i\theta}$. Corollary 2.3.3 implies that $|\mathcal{N}_{z,w} - \mathcal{N}_z(\varepsilon)|$ is stochastically dominated by a geometric random variable whose parameter p depends only on κ. Consequently, (2.8.7) is a consequence of Theorem 2.6.3. To see (2.8.8), we apply (2.8.7) for $j = 1$ and use that $G_D(u,v) = \frac{1}{2\pi}\log|u-v|^{-1} - \psi_u(v)$ where $\psi_u(v)$ is the harmonic extension of $v \mapsto \frac{1}{2\pi}\log|u-v|^{-1}$ from ∂D to D. In particular, $\psi_z(z) = \frac{1}{2\pi}\log \operatorname{CR}(z;D)$. □

2.9 Regularity of the ε-ball nesting field

A key estimate that we use in the proof of Theorem 2.6.1 is the following bound on how much the centered nesting field h_ε depends on ε. The proof of Theorem 2.9.1 and the remaining sections may be read in either order.

Theorem 2.9.1. Let D be a proper simply connected domain, and let $h_\varepsilon(z)$ be the centered weighted nesting around the ball $B(z,\varepsilon)$ of a CLE_κ on D, defined in (2.6.2). Suppose $0 < \varepsilon_1(z) \le \varepsilon$ and $0 < \varepsilon_2(z) \le \varepsilon$ on a compact subset $K \subset D$ of the domain. Then there is some $c > 0$ (depending on κ) and $C_0 > 0$ (depending on κ,

D, K, and the loop weight distribution) for which

$$\iint_{K\times K} \left|\mathbb{E}[(h_{\varepsilon_1(z)}(z)-h_{\varepsilon_2(z)}(z))\,(h_{\varepsilon_1(w)}(w)-h_{\varepsilon_2(w)}(w))]\right| dz\,dw \le C_0\varepsilon^c\,. \tag{2.9.1}$$

Proof. Let A, B, and C be the disjoint sets of loops for which $A \cup B$ is the set of loops surrounding $B(z,\varepsilon_1(z))$ or $B(z,\varepsilon_2(z))$ but not both, and $B \cup C$ is the set of loops surrounding $B(w,\varepsilon_1(w))$ or $B(w,\varepsilon_2(w))$ but not both. Letting $\xi_{\mathcal{L}}$ denote the weight of loop $\mathcal{L}$, then we have

$$\begin{aligned}
&\mathbb{E}[(h_{\varepsilon_1(z)}(z)-h_{\varepsilon_2(z)}(z))(h_{\varepsilon_1(w)}(w)-h_{\varepsilon_2(w)}(w))]\\
&\qquad= \operatorname{Cov}[h_{\varepsilon_1(z)}(z)-h_{\varepsilon_2(z)}(z), h_{\varepsilon_1(w)}(w) - h_{\varepsilon_2(w)}(w)]\\
&\qquad= \pm\operatorname{Cov}\left[\sum_{a\in A}\xi_a + \sum_{b\in B}\xi_b, \sum_{b\in B}\xi_b + \sum_{c\in C}\xi_c\right]\\
&\qquad= \pm\operatorname{Var}[\xi]\,\mathbb{E}[|B|] \,\pm \mathbb{E}[\xi]^2\operatorname{Cov}[|A|+|B|,|B|+|C|]\\
&\qquad= \pm\operatorname{Var}[\xi]\,\mathbb{E}[|B|] \,+ \mathbb{E}[\xi]^2\operatorname{Cov}(\mathcal{N}_z(\varepsilon_1)-\mathcal{N}_z(\varepsilon_2),\mathcal{N}_w(\varepsilon_1)-\mathcal{N}_w(\varepsilon_2))\,,
\end{aligned} \tag{2.9.2}$$

where the $\pm$ signs are the sign of $(\varepsilon_1(z)-\varepsilon_2(z))(\varepsilon_1(w)-\varepsilon_2(w))$.

Let $G_D^{\kappa,\varepsilon}(z,w)$ denote the expected number of loops surrounding z and w but surrounding neither $B(z,\varepsilon)$ nor $B(w,\varepsilon)$. Then $\mathbb{E}[|B|] \le G_D^{\kappa,\varepsilon}(z,w)$. In Lemma 2.9.3 we prove

$$\iint_{K\times K} G_D^{\kappa,\varepsilon}(z,w)\,dz\,dw \le C_1\varepsilon^c\,,$$

and in Lemma 2.9.7 we prove

$$\iint_{K\times K}\left|\operatorname{Cov}(\mathcal{N}_z(\varepsilon_1(z)) - \mathcal{N}_z(\varepsilon_2(z)),\mathcal{N}_w(\varepsilon_1(w)) - \mathcal{N}_w(\varepsilon_2(w)))\right| dz\,dw \le C_2\varepsilon^c\,,$$

where c depends only on κ and C_1 and C_2 depend only on κ, D, and K. Equation (2.9.1) follows from these bounds. ☐

In the remainder of this section we prove Lemmas 2.9.3 and 2.9.7.

Lemma 2.9.2. For any $\kappa \in (8/3,8)$ and $j \in \mathbb{N}$, there is a positive constant $c > 0$ such that, whenever $D \subsetneq \mathbb{C}$ is a simply connected proper domain, $z \in D$, and $0 < \varepsilon < r$, the j^{th} moment of the number of CLE_κ loops surrounding z which intersect $B(z,\varepsilon)$ but are not contained in $B(z,r)$ is $O((\varepsilon/r)^c)$.

Proof. If there is a loop $\mathcal{L} = \mathcal{L}_z^k$ surrounding z which is not contained in $B(z,r)$ and comes within distance ε of z, then $J_{z,\varepsilon}^{\cap} \le k$ and $J_{z,r}^{\subset} > k$, so $J_{z,\varepsilon}^{\cap} < J_{z,r}^{\subset}$. But from Corollary 2.3.3 $J_{z,r}^{\subset} - J_{z,r}^{\cap}$ is dominated by twice a geometric random variable, and by Lemma 2.2.12 in [48] together with the Koebe quarter theorem we have $J_{z,\varepsilon}^{\cap} - J_{z,r}^{\cap}$ is order $\log(r/\varepsilon)$ except with probability $O((\varepsilon/r)^{c_1})$, for some constant $c_1 > 0$ (depending on κ). Therefore, except with probability $O((\varepsilon/r)^{c_2})$ (with $c_2 = c_2(\kappa) > 0$), we have $J_{z,\varepsilon}^{\cap} \ge J_{z,r}^{\subset}$. In this case there is no loop $\mathcal{L}$ surrounding z, not contained in $B(z,r)$, and coming within distance ε of z. Finally, note that conditioned on the event that there is such a loop $\mathcal{L}$, the conditional expected number of such loops is by Corollary 2.3.3 dominated by twice a geometric random variable. □

Lemma 2.9.3. For some positive constant $c < 2$,

$$\iint_{K\times K} G_D^{\kappa,\varepsilon}(z,w)\,dz\,dw = O(\operatorname{area}(K)^{2-c/2}\varepsilon^c)\,. \tag{2.9.3}$$

Proof. Let $F_{z,w}^{\varepsilon}$ denote the number of loops surrounding both z and w but not $B(z,\varepsilon)$ or $B(w,\varepsilon)$. Then $G_D^{\kappa,\varepsilon}(z,w) = \mathbb{E}[F_{z,w}^{\varepsilon}]$.

Suppose $|z-w| \le \varepsilon$. Let $\mathcal{L}$ be the outermost loop (if any) surrounding both z and w but not $B(z,\varepsilon)$ or $B(w,\varepsilon)$. The number of additional such loops is $\mathcal{N}_{z,w}(\Gamma')$, where Γ' is a CLE_κ in $\operatorname{int}\mathcal{L}$, and by Theorem 2.6.3 we have $\mathbb{E}[\mathcal{N}_{z,w}(\Gamma')] \le C_1\log(\varepsilon/|z-w|) + C_2$ for some constants C_1 and C_2. Integrating the logarithm, we find that

$$\iint_{\substack{K\times K\\ |z-w|\le\varepsilon}} G_D^{\kappa,\varepsilon}(z,w)\,dz\,dw = O(\operatorname{area}(K)\varepsilon^2)\,. \tag{2.9.4}$$

Next suppose $|z-w| > \varepsilon$. Now $F_{z,w}^{\varepsilon}$ is dominated by the number of loops surrounding z which intersect $B(z,\varepsilon)$ but are not contained in $B(z,|z-w|)$, and Lemma 2.9.2 bounds the expected number of these loops by $O((\varepsilon/|z-w|)^c)$ for some $c > 0$. We decrease c if necessary to ensure $0 < c < 2$, and let $R = \operatorname{area}(K)^{1/2}$. Since $(\varepsilon/|z-w|)^c$ is decreasing in $|z-w|$, we can bound

$$\begin{aligned}\iint_{\substack{K\times K\\ |z-w|>\varepsilon}} G_D^{\kappa,\varepsilon}(z,w)\,dz\,dw &\le \iint_{\substack{R\mathbb{D}\times R\mathbb{D}\\ |z-w|>\varepsilon}} O((\varepsilon/|z-w|)^c)\,dz\,dw\\ &= O(\operatorname{area}(K)^{2-c/2}\varepsilon^c)\,.\end{aligned} \tag{2.9.5}$$

Combining (2.9.4) and (2.9.5), using again $c < 2$, we obtain (2.9.3). □

We let $S_{z,w}$ be the index of the outermost loop surrounding z which separates z from w in the sense that $w \notin U_z^{S_{z,w}}$. Note that $S_{z,w}$ is also the smallest index for which $z \notin U_w^{S_{z,w}}$:

$$S_{z,w} := \min\{k : w \notin U_z^k\} = \min\{k : z \notin U_w^k\}\,. \tag{2.9.6}$$

We let $\Sigma_{z,w}$ denote the σ-algebra

$$\Sigma_{z,w} := \sigma(\{\mathcal{L}_z^k : 1 \le k \le S_{z,w}\} \cup \{\mathcal{L}_w^k : 1 \le k \le S_{z,w}\})\,. \tag{2.9.7}$$

Lemma 2.9.4. There is a constant C (depending only on κ) such that if $z,w \in D$ are distinct, then

$$-C \le \mathbb{E}\left[\log\frac{\mathrm{CR}(z;U_z^{S_{z,w}})}{\min(|z-w|,\mathrm{CR}(z;D))}\right] \le C\,.$$

Proof. Let $r = \min(|z-w|, \mathrm{dist}(z, \partial D))$. By the Koebe distortion theorem,

$$\mathrm{CR}(z; U_z^{S_{z,w}}) \le 4r,$$

which gives the upper bound. By [48, Lemma 2.3.4], there is a loop contained in $B(z,r)$ but which surrounds $B(z, r/2^k)$ except with probability exponentially small k, which gives the lower bound. □

Lemma 2.9.5. There exists a constant $C > 0$ (depending only on κ) such that if $z, w \in D$ are distinct, and $0 < \varepsilon < \min(|z-w|, \mathrm{CR}(z; D))$, then on the event $\{\mathrm{CR}(z; U_z^{S_{z,w}}) \ge 8\varepsilon\}$,

$$\left| \mathbb{E}[J_{z,\varepsilon}^{\cap} - S_{z,w} \mid U_z^{S_{z,w}}] - \mathbb{E}[J_{z,\varepsilon}^{\cap} - S_{z,w}] - \nu_{\text{typical}} \log \frac{\mathrm{CR}(z; U_z^{S_{z,w}})}{\min(|z-w|, \mathrm{CR}(z;D))} \right| \le C. \quad (2.9.8)$$

Proof. Let $S = S_{z,w}$. By (2.8.8) of Corollary 2.8.2 we see that there exist $C_1 > 0$ such that on the event $\{\mathrm{CR}(z; U_z^S) \ge 8\varepsilon\}$ we have

$$\left| \mathbb{E}[J_{z,\varepsilon}^{\cap} - S_{z,w} \mid U_z^{S_{z,w}}] - \nu_{\text{typical}} \log \frac{\mathrm{CR}\left(z; U_z^{S_{z,w}}\right)}{\varepsilon} \right| \le C_1. \quad (2.9.9)$$

We can write

$$\mathbb{E}\left[J_{z,\varepsilon}^{\cap} - S\right] = \mathbb{E}\left[(J_{z,\varepsilon}^{\cap} - S)\mathbf{1}_{\{\mathrm{CR}(z;U_z^S) \ge 8\varepsilon\}}\right] + \mathbb{E}\left[(J_{z,\varepsilon}^{\cap} - S)\mathbf{1}_{\{\mathrm{CR}(z;U_z^S) < 8\varepsilon\}}\right]. \quad (2.9.10)$$

Applying (2.9.9), we can write the first term of (2.9.10) as,

$$\begin{aligned}
\mathbb{E}\left[(J_{z,\varepsilon}^{\cap} - S)\mathbf{1}_{\{\mathrm{CR}(z;U_z^S) \ge 8\varepsilon\}}\right] &= \mathbb{E}\left[\mathbb{E}[J_{z,\varepsilon}^{\cap} - S \mid U_z^S]\,\mathbf{1}_{\{\mathrm{CR}(z;U_z^S) \ge 8\varepsilon\}}\right] \\
&= \mathbb{E}\left[\left(\nu_{\text{typical}} \log \frac{\mathrm{CR}(z; U_z^S)}{\varepsilon} \pm C_1\right) \mathbf{1}_{\{\mathrm{CR}(z;U_z^S) \ge 8\varepsilon\}}\right] \\
&= \nu_{\text{typical}} \log \frac{\min(|z-w|, \mathrm{CR}(z;D))}{\varepsilon} \pm \text{const} \\
&\quad - \mathbb{E}\left[\left(\nu_{\text{typical}} \log \frac{\mathrm{CR}(z; U_z^S)}{\varepsilon}\right) \mathbf{1}_{\{\mathrm{CR}(z;U_z^S) < 8\varepsilon\}}\right].
\end{aligned}$$

Using [48, Lemma 2.3.4], there is a loop contained in $B(z,\varepsilon)$ which surrounds $B(z, \varepsilon/2^k)$ except with probability exponentially small in k, so the last term on the right is bounded by a constant (depending on κ).

If $J_{z,\varepsilon}^{\cap} \ge S$, then $J_{z,\varepsilon}^{\cap} - S$ counts the number of loops $(\mathcal{L}_z^k)_{k\in\mathbb{N}}$ after separating z from w before hitting $B(z,\varepsilon)$. If $J_{z,\varepsilon}^{\cap} \le S$, then $S - J_{z,\varepsilon}^{\cap}$ counts the number of loops $(\mathcal{L}_z^k)_{k\in\mathbb{N}}$ after intersecting $B(z,\varepsilon)$ before separating z from w. Consequently, by Corollary 2.3.3, we see that absolute value of the second term of (2.9.10) is bounded by some constant $C_2 > 0$. Putting these two terms of (2.9.10) together, we obtain

$$\left|\mathbb{E}\left[J^{\cap}_{z,\varepsilon} - S_{z,w}\right] - \nu_{\text{typical}} \log \frac{\min(|z-w|, \mathrm{CR}(z;D))}{\varepsilon}\right| \leq \text{const}. \tag{2.9.11}$$

Subtracting (2.9.11) from (2.9.9) and rearranging gives (2.9.8). □

Lemma 2.9.6. There exist constants $C_3, c > 0$ (depending only on κ) such that if $z, w \in D$ are distinct, and $0 < \varepsilon' \leq \varepsilon \leq r$ where $r = \min(|z-w|, \mathrm{CR}(z;D))$, then

$$\mathbb{E}\left[\left(\mathbb{E}[J^{\cap}_{z,\varepsilon} - J^{\cap}_{z,\varepsilon'} \,|\, U_z^{S_{z,w}}] - \mathbb{E}[J^{\cap}_{z,\varepsilon} - J^{\cap}_{z,\varepsilon'}]\right)^2\right] \leq C_3 \left(\frac{\varepsilon}{r}\right)^c. \tag{2.9.12}$$

Proof. We construct a coupling between three CLE$_\kappa$'s, Γ, $\widetilde{\Gamma}$, and $\acute{\Gamma}$, on the domain D. Let $S = S_{z,w}$, $\widetilde{S} = \widetilde{S}_{z,w}$, and $\acute{S} = \acute{S}_{z,w}$ denote the three corresponding stopping times. We take Γ and $\acute{\Gamma}$ to be independent. On $D \setminus \acute{U}_z^{\acute{S}}$, we take $\widetilde{\Gamma}$ to be identical to $\acute{\Gamma}$. In particular, $\widetilde{S} = \acute{S}$ and $\widetilde{U}_z^{\widetilde{S}} = \acute{U}_z^{\acute{S}}$. Within $\widetilde{U}_z^{\widetilde{S}}$, we couple $\widetilde{\Gamma}$ to Γ as follows. We sample so that the sequences

$$\left\{-\log \mathrm{CR}\left(z; U_z^{S+k}\right)\right\}_{k\in\mathbb{N}} \quad \text{and} \quad \left\{-\log \mathrm{CR}\left(z; \widetilde{U}_z^{\widetilde{S}+k}\right)\right\}_{k\in\mathbb{N}}$$

are coupled as in Lemma 2.7.1. Define

$$K = \min\left\{k \geq S : \mathrm{CR}\left(z; U_z^k\right) = \mathrm{CR}\left(z; \widetilde{U}_z^{\widetilde{k}}\right) \text{ for some } \widetilde{k} \geq \widetilde{S}\right\},$$

and let $\widetilde{K}$ be the value of $\widetilde{k}$ for which the conformal radius equality is realized. Let $\psi : U_z^K \to \widetilde{U}_z^{\widetilde{K}}$ be the unique conformal map with $\psi(z) = z$ and $\psi'(z) > 0$. We take $\widetilde{\Gamma}$ restricted to $\widetilde{U}_z^{\widetilde{K}}$ to be given by the image under ψ of the restriction of Γ to U_z^K.

Since $|\log \mathrm{CR}(z; U_z^S) - \log r|$ and $|\log \mathrm{CR}(z; \widetilde{U}_z^{\widetilde{S}}) - \log r|$ have exponential tails, and since the coupling time from Lemma 2.7.1 has exponential tails, each of $K - S$, $\widetilde{K} - \widetilde{S}$, and $|\log \mathrm{CR}(z; U_z^K) - \log r| = |\log \mathrm{CR}(z; \widetilde{U}_z^{\widetilde{K}}) - \log r|$ have exponential tails, with parameters depending only on κ.

Let

$$\Delta := \mathbb{E}[J^{\cap}_{z,\varepsilon} - J^{\cap}_{z,\varepsilon'} \,|\, U_z^S] - \mathbb{E}[\widetilde{J}^{\cap}_{z,\varepsilon} - \widetilde{J}^{\cap}_{z,\varepsilon'} \,|\, \widetilde{U}_z^{\widetilde{S}}].$$

In the above coupling U_z^S and $\widetilde{U}_z^{\widetilde{S}}$ are independent, so we have

$$\mathbb{E}[J^{\cap}_{z,\varepsilon} - J^{\cap}_{z,\varepsilon'} \,|\, U_z^S] - \mathbb{E}[J^{\cap}_{z,\varepsilon} - J^{\cap}_{z,\varepsilon'}] = \mathbb{E}[\Delta \,|\, U_z^S].$$

Therefore, the left-hand side of (2.9.12) is equal to $\mathbb{E}[(\mathbb{E}[\Delta | U_z^S])^2]$. Jensen's inequality applied to the inner expectation yields

$$\mathbb{E}[(\mathbb{E}[\Delta | U_z^S])^2] \leq \mathbb{E}[\mathbb{E}[\Delta^2 \,|\, U_z^S]] = \mathbb{E}[\Delta^2].$$

We can also write Δ as

$$\begin{aligned}\Delta &= \mathbb{E}[J^{\cap}_{z,\varepsilon} - J^{\cap}_{z,\varepsilon'} - \widetilde{J}^{\cap}_{z,\varepsilon} + \widetilde{J}^{\cap}_{z,\varepsilon'} \mid U^S_z, \widetilde{U}^{\widetilde{S}}_z] \\ &= \mathbb{E}\Big[J^{\cap}_{z,\varepsilon} - K - \widetilde{J}^{\cap}_{z,\varepsilon} + \widetilde{K} \mid U^S_z, \widetilde{U}^{\widetilde{S}}_z\Big] - \mathbb{E}\Big[J^{\cap}_{z,\varepsilon'} - K - \widetilde{J}^{\cap}_{z,\varepsilon'} + \widetilde{K} \mid U^S_z, \widetilde{U}^{\widetilde{S}}_z\Big] .\end{aligned}$$

and then use the inequality $(a+b)^2 \le 2(a^2+b^2)$ for $a, b \in \mathbb{R}$ to bound

$$\Delta^2 \le 2Y_\varepsilon + 2Y_{\varepsilon'},$$

where for $\hat{\varepsilon} \le \varepsilon$ we define

$$Y_{\hat{\varepsilon}} := \mathbb{E}\Big[J^{\cap}_{z,\hat{\varepsilon}} - K - \widetilde{J}^{\cap}_{z,\hat{\varepsilon}} + \widetilde{K} \mid U^S_z, \widetilde{U}^{\widetilde{S}}_z\Big]^2 .$$

We define the event

$$A = \{\mathrm{CR}(z; U^K_z) \ge \sqrt{r\varepsilon}\}.$$

Then

$$\begin{aligned}\mathbb{E}[Y_{\hat{\varepsilon}} \mathbf{1}_A] &= \mathbb{E}\Big[\mathbb{E}\Big[J^{\cap}_{z,\hat{\varepsilon}} - K - \widetilde{J}^{\cap}_{z,\hat{\varepsilon}} + \widetilde{K} \mid U^S_z, \widetilde{U}^{\widetilde{S}}_z\Big]^2 \mathbf{1}_A\Big] \\ &\le \mathbb{E}\Big[\mathbb{E}\Big[(J^{\cap}_{z,\hat{\varepsilon}} - K - \widetilde{J}^{\cap}_{z,\hat{\varepsilon}} + \widetilde{K})^2 \mathbf{1}_A \,\Big|\, U^S_z, \widetilde{U}^{\widetilde{S}}_z\Big]\Big] \\ &= \mathbb{E}\Big[(J^{\cap}_{z,\hat{\varepsilon}} - K - \widetilde{J}^{\cap}_{z,\hat{\varepsilon}} + \widetilde{K})^2 \mathbf{1}_A\Big] \\ &\le \text{const} \times (\varepsilon/r)^c\end{aligned}$$

where the last inequality follows from Lemma 2.7.3, for some $c > 0$ and for suitably large r/ε.

Next we apply Cauchy-Schwarz to find that

$$\mathbb{E}[Y_{\hat{\varepsilon}} \mathbf{1}_{A^c}] \le \sqrt{\mathbb{E}[Y^2_{\hat{\varepsilon}}]\mathbb{P}[A^c]}.$$

Lemma 2.7.1 and the construction of the coupling between Γ and $\widetilde{\Gamma}$ imply that $\mathbb{P}[A^c] \le \text{const} \times (\varepsilon/r)^c$ for some $c > 0$. It therefore suffices to show that $\mathbb{E}[Y^2_{\hat{\varepsilon}}] \le C$ for some constant C which does not depend on ε or ε'. By Jensen's inequality, it suffices to show that there exists C such that

$$\mathbb{E}[(J^{\cap}_{z,\hat{\varepsilon}} - K - \widetilde{J}^{\cap}_{z,\hat{\varepsilon}} + \widetilde{K})^4] \le C. \tag{2.9.13}$$

To prove (2.9.13), we consider the event $B = \{\mathrm{CR}(z; U^K_z) \ge \varepsilon\}$. By Lemma 2.7.3,

$$\mathbb{E}[(J^{\cap}_{z,\hat{\varepsilon}} - K - \widetilde{J}^{\cap}_{z,\hat{\varepsilon}} + \widetilde{K})^4 \mathbf{1}_B] \le \text{const}$$

where the constant depends only on κ.

Using $(a+b)^4 \le 8(a^4+b^4)$ for $a, b \in \mathbb{R}$, and the fact that $J^{\cap}_{z,\hat{\varepsilon}} - K$ and $\widetilde{J}^{\cap}_{z,\hat{\varepsilon}} - \widetilde{K}$

are equidistributed, we have

$$\mathbb{E}[(J^{\cap}_{z,\hat{\varepsilon}} - K - \widetilde{J}^{\cap}_{z,\hat{\varepsilon}} + \widetilde{K})^4 \mathbf{1}_{B^c}] \leq 16\,\mathbb{E}[(J^{\cap}_{z,\hat{\varepsilon}} - K)^4 \mathbf{1}_{B^c}]\,.$$

On the event B^c, we have $K \geq J^{\cap}_{z,\hat{\varepsilon}}$. Conditional on this, $K - J^{\cap}_{z,\hat{\varepsilon}}$ has exponentially decaying tails, so the above fourth moment is bounded by a constant (depending on κ), which completes the proof. □

Lemma 2.9.7. Suppose $0 < \varepsilon_1(z) \leq \varepsilon$ and $0 < \varepsilon_2(z) \leq \varepsilon$ on a compact subset $K \subset D$ of the domain D. Then there is some $c > 0$ (depending on κ) and $C_0 > 0$ (depending on κ, D, and K) for which

$$\iint_{K\times K} |\operatorname{Cov}(\mathcal{N}_z(\varepsilon_1(z)) - \mathcal{N}_z(\varepsilon_2(z)), \mathcal{N}_w(\varepsilon_1(w)) - \mathcal{N}_w(\varepsilon_2(w)))|\, dz\, dw \leq C_0 \varepsilon^c\,. \tag{2.9.14}$$

Proof. For a random variable X, we let $\mathring{X}$ denote

$$\mathring{X} = X - \mathbb{E}[X]\,. \tag{2.9.15}$$

We let Y_z denote

$$Y_z := J^{\cap}_{z,\varepsilon_1(z)} - J^{\cap}_{z,\varepsilon_2(z)}\,. \tag{2.9.16}$$

Recalling that $J^{\cap}_{z,r} = \mathcal{N}_z(r) + 1$, we see that

$$\mathbb{E}[\mathring{Y}_z \mathring{Y}_w] = \operatorname{Cov}(\mathcal{N}_z(\varepsilon_1(z)) - \mathcal{N}_z(\varepsilon_2(z)), \mathcal{N}_w(\varepsilon_1(w)) - \mathcal{N}_w(\varepsilon_2(w)))\,,$$

so we need to bound $\left|\mathbb{E}[\mathring{Y}_z \mathring{Y}_w]\right|$.

We treat two subsets of $K \times K$ separately: (1) the near regime $\{(z,w) : |z-w| \leq \varepsilon\}$, and (2) the far regime $\{(z,w) : \varepsilon < |z-w|\}$.

For the near regime, we first write

$$Y_z = Y^{(1)}_{z,w} + Y^{(2)}_{z,w}\,,$$

where $Y^{(1)}_{z,w}$ counts those loops surrounding $B(z, \min(\varepsilon_1(z), \varepsilon_2(z)))$ and intersecting $B(z, \max(\varepsilon_1(z), \varepsilon_2(z)))$ with index smaller than $S_{z,w}$, and $Y^{(2)}_{z,w}$ counts those loops with index at least $S_{z,w}$. Then $\Sigma_{z,w}$ determines $Y^{(1)}_{z,w}$ and $Y^{(1)}_{w,z}$, and conditional on $\Sigma_{z,w}$, $Y^{(2)}_{z,w}$ and $Y^{(2)}_{w,z}$ are independent (recall that $\Sigma_{z,w}$ was defined in (2.9.7)). Thus $Y^{(i)}_{z,w}$ and $Y^{(j)}_{w,z}$ are conditionally independent (given $\Sigma_{z,w}$) for $i, j \in \{1,2\}$.

Observe that

$$\left|\mathbb{E}[\mathring{Y}_z \mathring{Y}_w]\right| \leq \sum_{i,j\in\{1,2\}} \left|\mathbb{E}[\mathring{Y}^{(i)}_{z,w} \mathring{Y}^{(j)}_{w,z}]\right|\,. \tag{2.9.17}$$

For $i,j \in \{1,2\}$,

$$\begin{aligned}\left|\mathbb{E}\big[\mathring{Y}^{(i)}_{z,w}\mathring{Y}^{(j)}_{w,z}\big]\right| &= \left|\mathbb{E}\left[\mathbb{E}\big[\mathring{Y}^{(i)}_{z,w}\mathring{Y}^{(j)}_{w,z} \mid \Sigma_{z,w}\big]\right]\right| \\ &= \left|\mathbb{E}\left[\mathbb{E}\big[\mathring{Y}^{(i)}_{z,w} \mid \Sigma_{z,w}\big]\,\mathbb{E}\big[\mathring{Y}^{(j)}_{w,z} \mid \Sigma_{z,w}\big]\right]\right| \\ &\leq \mathbb{E}\left[\mathbb{E}\big[\mathring{Y}^{(i)}_{z,w} \mid \Sigma_{z,w}\big]^2\right]^{1/2}\mathbb{E}\left[\mathbb{E}\big[\mathring{Y}^{(j)}_{w,z} \mid \Sigma_{z,w}\big]^2\right]^{1/2}. \end{aligned} \tag{2.9.18}$$

For the index $i = 1$, we write

$$\mathbb{E}\left[\mathbb{E}\big[\mathring{Y}^{(1)}_{z,w} \mid \Sigma_{z,w}\big]^2\right] = \mathbb{E}\left[(\mathring{Y}^{(1)}_{z,w})^2\right] \leq \mathbb{E}\left[(Y^{(1)}_{z,w})^2\right] = \mathbb{E}\Big[\mathbb{E}\big[(Y^{(1)}_{z,w})^2\big] \,\big|\, U_z^{J^{\cap}_{z,\varepsilon}}\Big].$$

But

$$Y^{(1)}_{z,w} \leq 1 + \mathcal{N}_{z,w}\left(\Gamma|_{U_z^{J^{\cap}_{z,\varepsilon}}}\right).$$

By Theorem 2.6.3, $\mathbb{E}[(1+\mathcal{N}_{z,w}(\Gamma|_U))^2] \leq \text{const} + \text{const} \times G_U(z,w)^2$, where G_U denotes the Green's function for the Laplacian in the domain U. By the Koebe distortion theorem, the Green's function is in turn bounded by $G_U(z,w) \leq \text{const} + \text{const} \times \max(0, \log(\mathrm{CR}(z;U)/|z-w|))$. Therefore,

$$\mathbb{E}\left[(\mathring{Y}^{(1)}_{z,w})^2\right] \leq \mathbb{E}\left[O\left(1 + \log^2 \frac{|z-w|}{\mathrm{CR}\left(z; U_z^{J^{\cap}_{z,\varepsilon}}\right)}\right)\right].$$

By Lemma 2.2.12, $-\log \mathrm{CR}\left(z; U_z^{J^{\cap}_{z,\varepsilon}}\right) = -\log\varepsilon + X$ for some random variable X with exponentially decaying tails. It follows that

$$\mathbb{E}\left[\mathbb{E}\big[\mathring{Y}^{(1)}_{z,w} \mid \Sigma_{z,w}\big]^2\right] = \mathbb{E}\left[(\mathring{Y}^{(1)}_{z,w})^2\right] = O\left(1 + \log^2\frac{|z-w|}{\varepsilon}\right). \tag{2.9.19}$$

For the index $i = 2$, we express $Y^{(2)}_{z,w}$ in terms of $J_{z,\varepsilon_1(z)}$ and $J_{z,\varepsilon_2(z)}$ and use Lemma 2.9.5 twice (once with $\varepsilon_1(z)$ and once with $\varepsilon_2(z)$ playing the role of ε in the lemma statement) and subtract to write

$$\begin{aligned}\mathbb{E}\big[\mathring{Y}^{(2)}_{z,w} \mid \Sigma_{z,w}\big] = \mathbb{E}\big[\mathring{Y}^{(2)}_{z,w} \mid U_z^{S_{z,w}}\big] &\leq \text{const} \\ \mathbb{E}\left[\mathbb{E}\big[\mathring{Y}^{(2)}_{z,w} \mid \Sigma_{z,w}\big]^2\right] &\leq C. \end{aligned} \tag{2.9.20}$$

for some constant C depending only on κ.

Combining (2.9.17), (2.9.18), (2.9.19), and (2.9.20), we obtain

$$\left|\mathbb{E}[\mathring{Y}_z\mathring{Y}_w]\right| \leq \text{const} + \text{const} \times \log^2\frac{\varepsilon}{|z-w|},$$

which implies

$$\iint\limits_{\substack{K\times K\\|z-w|\le\varepsilon}} \left|\mathbb{E}[\mathring{Y}_z\mathring{Y}_w]\right| dz\,dw \le \text{const}\times\text{area}(K)\times\varepsilon^2\,. \tag{2.9.21}$$

For the far regime, we again condition on $\Sigma_{z,w}$, the loops up to and including the first ones separating z from w, and use Cauchy-Schwarz, as in (2.9.18), but without first expressing Y_z and Y_w as sums:

$$\left|\mathbb{E}\big[\mathring{Y}_z\mathring{Y}_w\big]\right| \le \mathbb{E}\left[\mathbb{E}[\mathring{Y}_z\,|\,\Sigma_{z,w}]^2\right]^{1/2}\mathbb{E}\left[\mathbb{E}[\mathring{Y}_w\,|\,\Sigma_{z,w}]^2\right]^{1/2}. \tag{2.9.22}$$

By Lemma 2.9.6, we have

$$\mathbb{E}\big[\mathbb{E}[\mathring{Y}_z\,|\,\Sigma_{z,w}]^2\big] \le C\left(\frac{\varepsilon}{\min(|z-w|,\mathrm{CR}(z;D))}\right)^c. \tag{2.9.23}$$

Integrating over $\{(z,w)\in K\times K : \varepsilon<|z-w|\}$ gives (2.9.14). ☐

2.10 Properties of Sobolev spaces

In this section we provide an overview of the distribution theory and Sobolev space theory required for the proof of Theorem 2.6.1. We refer the reader to [78] or [79] for a more detailed introduction.

Fix a positive integer d. Recall that the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is defined to be the set of smooth, complex-valued functions on $\mathbb{R}^d$ whose derivatives of all orders decay faster than any polynomial at infinity. If $\beta=(\beta_1,\beta_2,\dots,\beta_d)$ is a multi-index, then the partial differentiation operator ∂^β is defined by $\partial^\beta=\partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}\cdots\partial_{x_d}^{\beta_d}$. We equip $\mathcal{S}(\mathbb{R}^d)$ with the topology generated by the family of seminorms

$$\left\{\|\phi\|_{n,\beta}:=\sup_{x\in\mathbb{R}^d}|x|^n|\partial^\beta\phi(x)| \;:\; n\ge 0,\ \beta\text{ is a multi-index}\right\}.$$

The space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions is defined to be the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$. We write the evaluation of $f\in\mathcal{S}'(\mathbb{R}^d)$ on $\phi\in\mathcal{S}(\mathbb{R}^d)$ using the notation $\langle f,\phi\rangle$. For any Schwartz function $g\in\mathcal{S}(\mathbb{R}^d)$ there is an associated continuous linear functional $\phi\mapsto\int_{\mathbb{R}^d}g(x)\phi(x)\,dx$ in $\mathcal{S}'(\mathbb{R}^d)$, and $\mathcal{S}(\mathbb{R}^d)$ is a dense subset of $\mathcal{S}'(\mathbb{R}^d)$ with respect to the weak* topology.

For $\phi\in\mathcal{S}(\mathbb{R}^d)$, its Fourier transform $\widehat{\phi}$ is defined by

$$\widehat{\phi}(\xi)=\int_{\mathbb{R}^d}e^{-2\pi i x\cdot\xi}\phi(x)\,dx\quad\text{for }\xi\in\mathbb{R}^d\,.$$

Since $\phi \in \mathcal{S}(\mathbb{R}^d)$ implies $\widehat{\phi} \in \mathcal{S}(\mathbb{R}^d)$ [78, Section 1.13] and since

$$\langle \widehat{\phi}_1, \phi_2 \rangle = \iint \phi_1(x) e^{-2\pi i x \cdot y} \phi_2(y)\, dx\, dy = \langle \phi_1, \widehat{\phi}_2 \rangle$$

for all $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d)$, we may define the Fourier transform $\widehat{f}$ of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ by setting $\langle \widehat{f}, \phi \rangle := \langle f, \widehat{\phi} \rangle$ for each $\phi \in \mathcal{S}(\mathbb{R}^d)$.

For $x \in \mathbb{R}^d$, we define $\langle x \rangle := (1 + |x|^2)^{1/2}$. For $s \in \mathbb{R}$, define $H^s(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ to be the set of functionals f for which there exists $R_f^s \in L^2(\mathbb{R}^d)$ such that for all $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle \widehat{f}, \phi \rangle = \int_{\mathbb{R}^d} R_f^s(\xi) \phi(\xi) \langle \xi \rangle^{-s}\, d\xi. \tag{2.10.1}$$

Equipped with the inner product

$$\langle f, g \rangle_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} R_f^s(\xi) \overline{R_g^s(\xi)}\, d\xi, \tag{2.10.2}$$

$H^s(\mathbb{R}^d)$ is a Hilbert space. (The space $H^s(\mathbb{R}^d)$ is the same as the Sobolev space denoted $W^{s,2}(\mathbb{R}^d)$ in the literature.)

Recall that the support of a function $f : \mathbb{R}^d \to \mathbb{C}$ is defined to the closure of the set of points where f is nonzero. Define $\mathbb{T} = [-\pi, \pi]$ with endpoints identified, so that $\mathbb{T}^d$, the d-dimensional torus, is a compact manifold. If M is a manifold (such as $\mathbb{R}^d$ or $\mathbb{T}^d$), we denote by $C_c^\infty(M)$ the space of smooth, compactly supported functions on M. We define the topology of $C_c^\infty(M)$ so that $\psi_n \to \psi$ if and only if there exists a compact set $K \subset M$ on which each ψ_n is supported and $\partial^\alpha \psi_n \to \partial^\alpha \psi$ uniformly, for all multi-indices α [78]. We write $C_c^\infty(M)'$ for the space of continuous linear functionals on $C_c^\infty(M)$, and we call elements of $C_c^\infty(M)'$ *distributions* on M. For $f \in C_c^\infty(\mathbb{T}^d)'$ and $k \in \mathbb{Z}^d$, we define the Fourier coefficient $\widehat{f}(k)$ by evaluating f on the element $x \mapsto e^{-ik \cdot x}$ of $C_c^\infty(\mathbb{T}^d)$. For distributions f and g on $\mathbb{T}^d$, we define an inner product with Fourier coefficients $\widehat{f}(k)$ and $\widehat{g}(k)$:

$$\langle f, g \rangle_{H^s(\mathbb{T}^d)} := \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{2s} \widehat{f}(k) \overline{\widehat{g}(k)}. \tag{2.10.3}$$

If $f \in \mathcal{S}'(\mathbb{R}^d)$ is supported in $(-\pi, \pi)^d$, i.e. vanishes on functions which are supported in the complement of $(-\pi, \pi)^d$, then f can be thought of as a distribution on $\mathbb{T}^d$, and the norms corresponding to the inner products in (2.10.2) and (2.10.3) are equivalent [79] for such distributions f.

Note that $H^{-s}(\mathbb{R}^d)$ can be identified with the dual of $H^s(\mathbb{R}^d)$: we associate with $f \in H^{-s}(\mathbb{R}^d)$ the functional $g \mapsto \langle f, g \rangle$ defined for $g \in H^s(\mathbb{R}^d)$ by

$$\langle f, g \rangle := \int_{\mathbb{R}^d} R_f^{-s}(\xi) \overline{R_g^s(\xi)}\, d\xi.$$

This notation is justified by the fact that when f and g are in $L^2(\mathbb{R}^d)$, this is the same as the $L^2(\mathbb{R}^d)$ inner product of f and g. By Cauchy-Schwarz, $g \mapsto \langle f, g \rangle$ is a bounded linear functional on $H^s(\mathbb{R}^d)$. Observe that the operator topology on the dual $H^s(\mathbb{R}^d)$ coincides with the norm topology of $H^{-s}(\mathbb{R}^d)$ under this identification.

It will be convenient to work with local versions of the Sobolev spaces $H^s(\mathbb{R}^d)$. If $h \in \mathcal{S}'(\mathbb{R}^d)$ and $\psi \in C_c^\infty(\mathbb{R}^d)$, we define the product $\psi h \in \mathcal{S}'(\mathbb{R}^d)$ by $\langle \psi h, f\rangle = \langle h, \psi f\rangle$. Furthermore, if $h \in H^s(\mathbb{R}^d)$, then $\psi h \in H^s(\mathbb{R}^d)$ as well [2, Lemma 4.3.16]. For $h \in C_c^\infty(D)'$, we say that $h \in H^s_{\mathrm{loc}}(D)$ if $\psi h \in H^s(\mathbb{R}^d)$ for every $\psi \in C_c^\infty(D)$. We equip $H^s_{\mathrm{loc}}(D)$ with a topology generated by the seminorms $\|\psi \cdot\|_{H^s(\mathbb{R}^d)}$, which implies that $h_n \to h$ in $H^s_{\mathrm{loc}}(D)$ if and only if $\psi h_n \to \psi h$ in $H^s(\mathbb{R}^d)$ for all $\psi \in C_c^\infty(D)$.

The following proposition provides sufficient conditions for proving almost sure convergence in $H^{-d-\delta}_{\mathrm{loc}}(\mathbb{R}^d)$.

Proposition 2.10.1. Let $D \subset \mathbb{R}^d$ be an open set, let $\delta > 0$, and suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence of random measurable functions defined on D. Suppose further that for every compact set $K \subset D$, there exist a summable sequence $(a_n)_{n\in\mathbb{N}}$ of positive real numbers such that for all $n \in \mathbb{N}$, we have

$$\iint_{K\times K} \left|\mathbb{E}[(f_{n+1}(x) - f_n(x))(f_{n+1}(y) - f_n(y))]\right| dx\,dy \le a_n^3 . \tag{2.10.4}$$

Then there exists $f \in H^{-d-\delta}_{\mathrm{loc}}(\mathbb{R}^d)$ supported on the closure of D such that $f_n \to f$ in $H^{-d-\delta}_{\mathrm{loc}}(D)$ almost surely.

Before proving Proposition 2.10.1, we prove the following lemma. Recall that a sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets is called a *compact exhaustion* of D if $K_n \subset K_{n+1} \subset D$ for all $n \in \mathbb{N}$ and $D = \bigcup_{n\in\mathbb{N}} K_n$.

Lemma 2.10.2. Let $s > 0$, let $D \subset \mathbb{R}^d$ be an open set, suppose that $(K_j)_{j\in\mathbb{N}}$ is a compact exhaustion of D, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of elements of $H^{-s}(\mathbb{R}^d)$. Suppose further that $(\psi_j)_{j\in\mathbb{N}}$ satisfies $\psi_j \in C_c^\infty(D)$ and $\psi_j\big|_{K_j} = 1$ for all $j \in \mathbb{N}$. If for every j there exists $f^{\psi_j} \in H^{-s}(\mathbb{R}^d)$ such that $\psi_j f_n \to f^{\psi_j}$ as $n \to \infty$ in $H^{-s}(\mathbb{R}^d)$, then there exists $f \in H^{-s}_{\mathrm{loc}}(D)$ such that $f_n \to f$ in $H^{-s}_{\mathrm{loc}}(D)$.

Proof. We claim that for all $\psi \in C_c^\infty(D)$, the sequence ψf_n is Cauchy in $H^{-s}(\mathbb{R}^d)$. We choose j large enough that $\operatorname{supp}\psi \subset K_j$. For all $g \in H^s(\mathbb{R}^d)$,

$$|\langle \psi f_n, g\rangle - \langle \psi f_m, g\rangle| = |\langle \psi_j(f_n - f_m), \psi g\rangle| .$$

By hypothesis $\psi_j f_n$ converges in $H^{-s}(\mathbb{R}^d)$ as $n \to \infty$, so we may take the supremum over $\{g : \|g\|_{H^s(\mathbb{R}^d)} \le 1\}$ of both sides to conclude $\|\psi f_n - \psi f_m\|_{H^{-s}(\mathbb{R}^d)} \to 0$ as $\min(m,n) \to \infty$. Since $H^{-s}(\mathbb{R}^d)$ is complete, it follows that for every $\psi \in C_c^\infty(D)$, there exists $f^\psi \in H^{-s}(\mathbb{R}^d)$ such that $\psi f_n \to f^\psi$ in $H^{-s}(\mathbb{R}^d)$.

We define a linear functional f on $C_c^\infty(D)$ as follows. For $g \in C_c^\infty(D)$, set

$$\langle f, g\rangle := \langle f^\Psi, g\rangle , \tag{2.10.5}$$

where ψ is a smooth compactly supported function which is identically equal to 1 on the support of g. To see that this definition does not depend on the choice of ψ, suppose that $\psi_1 \in C_c^\infty(D)$ and $\psi_2 \in C_c^\infty(D)$ are both equal to 1 on the support of g. Then we have

$$\langle f^{\psi_1}, g\rangle - \langle f^{\psi_2}, g\rangle = \lim_{n\to\infty} \langle (\psi_1 - \psi_2) f_n, g\rangle = 0,$$

as desired. From the definition in (2.10.5), f inherits linearity from f^ψ and thus defines a linear functional on $C_c^\infty(D)$. Furthermore, $f \in H_{\mathrm{loc}}^{-s}(D)$ since $\psi f = f^\psi \in H^{-s}(R^d)$ for all $\psi \in C_c^\infty(D)$. Finally, $f_n \to f$ in $H_{\mathrm{loc}}^{-s}(D)$ since $\psi f_n \to \psi f = f^\psi$ in $H^{-s}(\mathbb{R}^d)$. □

Proof of Proposition 2.10.1. Fix $\psi \in C_c^\infty(D)$. Let D_ψ be a bounded open set containing the support ψ and whose closure is contained in D. Since D_ψ is bounded, we may scale and translate it so that it is contained in $(-\pi, \pi)^d$. We will calculate the Fourier coefficients of $\psi(f_{n+1} - f_n)$ in $(-\pi, \pi)^d$, identifying it with $\mathbb{T}^d$. By Fubini's theorem, we have for all $k \in \mathbb{Z}^d$

$$\begin{aligned}
&\mathbb{E}|\widehat{\psi f_{n+1} - \psi f_n}(k)|^2 \qquad (2.10.6)\\
&= \mathbb{E}\left[\left(\int_D f_{n+1}(x)\psi(x)e^{-ik\cdot x}dx - \int_D f_n(x)\psi(x)e^{-ik\cdot x}dx\right)^2\right]\\
&\le \|\psi\|_{L^\infty(\mathbb{R}^d)}^2 \iint_{D_\psi\times D_\psi} \left|\mathbb{E}[(f_{n+1}(x) - f_n(x))(f_{n+1}(y) - f_n(y))]\right| dx\,dy\\
&\le \|\psi\|_{L^\infty(\mathbb{R}^d)}^2\, a_n^3,
\end{aligned}$$

by (2.10.4). By Markov's inequality, (2.10.6) implies

$$\mathbb{P}\left[|\widehat{\psi f_{n+1} - \psi f_n}(k)| \ge a_n \langle k\rangle^{d/2+\delta/2}\right] \le \|\psi\|_{L^\infty(\mathbb{R}^d)}^2 \langle k\rangle^{-d-\delta} a_n.$$

The right-hand side is summable in k and n, so by the Borel-Cantelli lemma, the event on the left-hand side occurs for at most finitely many pairs (n, k), almost surely. Therefore, for sufficiently large n_0, this event does not occur for any $n \ge n_0$. For these values of n, we have

$$\begin{aligned}
\|\psi f_n - \psi f_{n+1}\|_{H^{-d-\delta}(\mathbb{T}^d)}^2 &= \sum_{k\in\mathbb{Z}^d} |\widehat{\psi f_n - \psi f_{n+1}}(k)|^2 \langle k\rangle^{-2(d+\delta)}\\
&\le \sum_{k\in\mathbb{Z}^d} a_n^2 \langle k\rangle^{d+\delta} \langle k\rangle^{-2d-2\delta}\\
&= O(a_n^2/\delta),
\end{aligned}$$

Applying the triangle inequality, we find that for $m, n \geq n_0$

$$\|\psi f_m - \psi f_n\|_{H^{-d-\delta}(\mathbb{T}^d)} = O\left(\delta^{-1/2} \sum_{j=m}^{n-1} a_j\right). \tag{2.10.7}$$

Recall that the $H^{-d-\delta}(\mathbb{T}^d)$ and $H^{-d-\delta}(\mathbb{R}^d)$ norms are equivalent for functions supported in $(-\pi, \pi)^d$ (see the discussion following (2.10.3)). The sequence $(a_n)_{n\in\mathbb{N}}$ is summable by hypothesis, so (2.10.7) shows that $(\psi f_n)_{n\in\mathbb{N}}$ is almost surely Cauchy in $H^{-d-\delta}(\mathbb{R}^d)$. Since $H^{-d-\delta}(\mathbb{R}^d)$ is complete, this implies that with probability 1 there exists $h^\psi \in H^{-d-\delta}(\mathbb{R}^d)$ to which ψf_n converges in the operator topology on $H^{-d-\delta}(\mathbb{R}^d)$.

Applying Lemma 2.10.2, we obtain a limiting random variable $f \in H^{-d-\delta}_{\mathrm{loc}}(\mathbb{R}^d)$ to which $(f_n)_{n\in\mathbb{N}}$ converges in $H^{-d-\delta}_{\mathrm{loc}}(\mathbb{R}^d)$. □

2.11 Convergence to limiting field

We have most of the ingredients in place to prove the convergence of the centered ε-nesting fields, but we need one more lemma.

Lemma 2.11.1. Fix $C > 0$, $\alpha > 0$, and $L \in \mathbb{R}$. Suppose that F, F_1, and F_2 are real-valued functions on $(0, \infty)$ such that

(i) F_1 is nondecreasing on $(0, \infty)$,

(ii) $|F_2(x + \delta) - F_2(x)| \leq C \max(\delta^\alpha, e^{-\alpha x})$ for all $x > 0$ and $\delta > 0$,

(iii) $F = F_1 + F_2$, and

(iv) For all $\delta > 0$, $F(n\delta) \to L$ as $n \to \infty$ through the positive integers.

Then $F(x) \to L$ as $x \to \infty$.

Proof. Let $\varepsilon > 0$, and choose $\delta > 0$ so that $C\delta^\alpha < \varepsilon$. Choose x_0 large enough that $Ce^{-\alpha x_0} < \varepsilon$ and $|F(n\delta) - L| < \varepsilon$ for all $n > x_0/\delta$. Fix $x > x_0$, and define $a = \delta\lfloor x/\delta \rfloor$. For $u \in \{F, F_1, F_2\}$, we write $\Delta u = u(a + \delta) - u(a)$. Observe that $|\Delta F_2| \leq \varepsilon$ by (ii). By (iii) and (iv), this implies

$$|\Delta F_1| = |\Delta F - \Delta F_2| \leq |\Delta F| + |\Delta F_2| < 3\varepsilon.$$

Since F_1 is monotone, we get $|F_1(x) - F_1(a)| < 3\varepsilon$. Furthermore, (ii) implies $|F_2(x) - F_2(a)| < \varepsilon$. It follows that

$$|F(x) - L| \leq |F_1(x) - F_1(a)| + |F_2(x) - F_2(a)| + |F(a) - L| < 5\varepsilon.$$

Since $x > x_0$ and $\varepsilon > 0$ were arbitrary, this concludes the proof. □

Theorem 2.11.2. Let $h_\varepsilon(z)$ be the centered weighted nesting of a CLE_κ around the ball $B(z, \varepsilon)$, defined in (2.6.2). Suppose $0 < a < 1$. Then $(h_{a^n})_{n\in\mathbb{N}}$ almost surely converges in $H^{-2-\delta}_{\mathrm{loc}}(D)$.

Proof. Immediate from Theorem 2.9.1 and Proposition 2.10.1. □

Proof of Theorem 2.6.1. We claim that for all $g \in C_c^\infty(D)$, we have $\langle h_\varepsilon, g\rangle \to \langle h, g\rangle$ almost surely. Suppose first that the loop weights are almost surely nonnegative and that $g \in C_c^\infty(D)$ is a nonnegative test function. Define $F(x) := \langle h_{e^{-x}}, g\rangle$, $F_1(x) := \langle S_z(e^{-x}), g\rangle$, and $F_2(x) := -\langle \mathbb{E}[S_z(e^{-x})], g\rangle$. We apply Lemma 2.11.1 with α as given in Lemma 2.7.2, which implies

$$\lim_{\varepsilon\to 0}\langle h_\varepsilon, g\rangle = \langle h, g\rangle \quad \text{for} \quad g \in C_c^\infty(D), g \geq 0\,. \tag{2.11.1}$$

For arbitrary $g \in C_c^\infty(D)$, we choose $\tilde{g} \in C_c^\infty(D)$ so that $\tilde{g}$ and $g + \tilde{g}$ are both nonnegative. Applying (2.11.1) to $\tilde{g}$ and $g + \tilde{g}$, we see that

$$\lim_{\varepsilon\to 0}\langle h_\varepsilon, g\rangle = \langle h, g\rangle \quad \text{for} \quad g \in C_c^\infty(D)\,. \tag{2.11.2}$$

Finally, consider loop weights which are not necessarily nonnegative. Define loop weights $\xi_{\mathcal{L}}^{\pm} = (\xi_{\mathcal{L}})^{\pm}$, where $x^+ = \max(0, x)$ and $x^- = \max(0, -x)$ denote the positive and negative parts of $x \in \mathbb{R}$. Define $h^{\pm}$ to be the weighted nesting fields associated with the weights $\xi_{\mathcal{L}}^{\pm}$ (associated with the same CLE). Then $\langle h_\varepsilon^{\pm}, g\rangle \to \langle h^{\pm}, g\rangle$ almost surely, and

$$\langle h_\varepsilon, g\rangle = \langle h_\varepsilon^+, g\rangle - \langle h_\varepsilon^-, g\rangle \to \langle h^+, g\rangle - \langle h^-, g\rangle = \langle h, g\rangle\,,$$

which concludes the proof that $\langle h_\varepsilon, g\rangle \to \langle h, g\rangle$ almost surely.

To see that the field h is measurable with respect to the σ-algebra Σ generated by the CLE_κ and the weights $(\xi_{\mathcal{L}})_{\mathcal{L}\in\Gamma}$, note that there exists a countable dense subset $\mathcal{F}$ of $C_c^\infty(D)$ [78, Exercise 1.13.6]. Observe that $h_{2^{-n}}$ is Σ-measurable and h is determined by the values $\{h_{2^{-n}}(g) : n \in \mathbb{N}, g \in \mathcal{F}\}$. Since h is an almost sure limit of $h_{2^{-n}}$, we conclude that h is also Σ-measurable.

To establish conformal invariance, let $z \in D$ and $\varepsilon > 0$ and define the sets of loops

$$\begin{aligned}
\Xi_1 &= \text{loops surrounding } B(\varphi(z), \varepsilon|\varphi'(z)|), \text{ and} \\
\Xi_2 &= \text{loops surrounding } \varphi(B(z,\varepsilon)) \\
\Xi_3 &= \Xi_1 \Delta\, \Xi_2\,,
\end{aligned}$$

where Δ denotes the symmetric difference of two sets. Since either $\Xi_1 \subset \Xi_2$ or $\Xi_2 \subset \Xi_1$,

$$h_\varepsilon(z) - \acute{h}_{\varepsilon|\varphi'(z)|}(\varphi(z)) = \pm \sum_{\xi\in\Xi_3} \xi_{\mathcal{L}}\,.$$

Multiplying by $g \in C_c^\infty(D)$, integrating over D, and taking $\varepsilon \to 0$, we see that by Lemma 2.7.3 and the finiteness of $\mathbb{E}[|\xi_{\mathcal{L}}|]$, the sum on the right-hand side goes to 0 in L^1 and hence in probability as $\varepsilon \to 0$. Furthermore, we claim that

$$\int_D \left[\acute{h}_{\varepsilon|\varphi'(z)|}(\varphi(z)) - \acute{h}_\varepsilon(\varphi(z))\right] g(z)\, dz \to 0$$

in probability as $\varepsilon \to 0$. To see this, we write the difference in square brackets as

$$\acute{h}_{\varepsilon|\varphi'(z)|}(\varphi(z)) - \acute{h}_{C\varepsilon}(\varphi(z)) + \acute{h}_{C\varepsilon}(\varphi(z)) - \acute{h}_{\varepsilon}(\varphi(z)),$$

where C is an upper bound for $|\varphi'(z)|$ as z ranges over the support of g. Note that $\int_D \left[\acute{h}_{C\varepsilon}(\varphi(z)) - \acute{h}_\varepsilon(\varphi(z))\right] g(z)\, dz \to 0$ in probability because for all $0 < \varepsilon' < \varepsilon$ and $\psi \in C_c^\infty(D)$, we have

$$\begin{aligned}
\mathbb{E}\|\psi h_\varepsilon - \psi h_{\varepsilon'}\|^2_{H^{-d-\delta}(\mathbb{T}^d)} &= \sum_{k\in\mathbb{Z}^d} \mathbb{E}|\widehat{\psi h_\varepsilon - \psi h_{\varepsilon'}}(k)|^2 \langle k\rangle^{-2(d+\delta)} \\
&\le \sum_{k\in\mathbb{Z}^d} \|\psi\|^2_{L^\infty(\mathbb{R}^d)} \iint_{D_\psi^2} \left|\mathbb{E}[(h_\varepsilon(x) - h_{\varepsilon'}(x))(h_\varepsilon(y) - h_{\varepsilon'}(y))]\right| dx\, dy \langle k\rangle^{-2(d+\delta)} \\
&\le \varepsilon^{\Omega(1)}/\delta;
\end{aligned}$$

see (2.10.6) for more details. The same calculation along with Theorem 2.9.1 show that

$$\int_D \left[\acute{h}_{C\varepsilon}(\varphi(z)) - \acute{h}_{\varepsilon|\varphi'(z)|}(\varphi(z))\right] g(z)\, dz \to 0,$$

in probability. It follows that $\langle h, g\rangle = \langle \acute{h} \circ \varphi, g\rangle$ for all $g \in C_c^\infty(D)$, as desired. □

2.12 Step nesting

In this section we prove Theorem 2.6.2. Suppose that D is a proper simply connected domain, and let Γ be a CLE_κ in D. Let μ be a probability measure with finite second moment and zero mean, and define

$$\mathfrak{h}_n(z) = \sum_{k=1}^{n} \xi_{\mathcal{L}_k(z)}, \quad n \in \mathbb{N}.$$

We call $(\mathfrak{h}_n)_{n\in\mathbb{N}}$ the *step nesting sequence* associated with Γ and $(\xi_\mathcal{L})_{\mathcal{L}\in\Gamma}$.

Lemma 2.12.1. For each $\kappa \in (8/3, 8)$ there are positive constants c_1, c_2, and c_3 (depending on κ) such that for any simply connected proper domain $D \subsetneq \mathbb{C}$ and points $z, w \in D$, for a CLE_κ in D,

$$\Pr\left[\mathcal{N}_{z,w} \ge c_1 \log \frac{\mathrm{CR}(z;D)}{|z-w|} + c_2 j + c_3\right] \le \exp[-j].$$

Proof. Let X_i be i.i.d. copies of the log conformal radius distribution, and let $T_\ell = \sum_{i=1}^{\ell} X_i$. Then

$$\Pr[T_\ell \le t] \le \mathbb{E}[e^{-X}]^\ell e^t$$
$$\Pr[T_\ell \le \log(\mathrm{CR}(z;D)/|z-w|)] \le \mathbb{E}[e^{-X}]^\ell \frac{\mathrm{CR}(z;D)}{|z-w|}.$$

If $T_\ell > \log(\mathrm{CR}(z;D)/|z-w|)$, then $J^{\cap}_{z,|z-w|} \le \ell$. But $\mathcal{N}_{z,w} < J^{\subset}_{z,|z-w|}$, and by Corollary 2.3.3, $J^{\subset}_{z,|z-w|} - J^{\cap}_{z,|z-w|}$ has exponential tails. □

Proof of Theorem 2.6.2. We check that (2.10.4) holds with $f_n = \mathfrak{h}_n$. Writing out each difference as a sum of loop weights and using the linearity of expectation, we calculate for $0 \le m \le n$ and $z, w \in D$,

$$\mathbb{E}[(\mathfrak{h}_m(z) - \mathfrak{h}_n(z))(\mathfrak{h}_m(w) - \mathfrak{h}_n(w))] = \sigma^2 \sum_{k=m+1}^{n} \mathbb{P}[\mathcal{N}_{z,w} \ge k].$$

Let $\delta(z)$ be the value for which $c_1 \log(\mathrm{CR}(z;D)/\delta(z)) + c_3 = k$, where c_1 and c_3 are as in Lemma 2.12.1. Let K be compact, and let $\delta = \max_{z\in K} \delta(z)$. Then

$$\delta \le \exp[-\Theta(k)] \times \sup_{z\in K} \mathrm{dist}(z, \partial D) \tag{2.12.1}$$

and

$$\iint\limits_{\substack{K\times K\\ |z-w|\ge\delta}} \Pr[\mathcal{N}_{z,w} \ge k]\, dz\, dw \le \exp(-k) \times \mathrm{area}(K)^2. \tag{2.12.2}$$

The integral of $\mathbb{P}[\mathcal{N}_{z,w} \ge k]$ over z, w which are closer than δ is controlled by virtue of the small volume of the domain of integration:

$$\iint\limits_{\substack{K\times K\\ |z-w|\le\delta}} \mathbb{P}[\mathcal{N}_{z,w} \ge k]\, dz\, dw \le \delta^2 \times \mathrm{area}(K). \tag{2.12.3}$$

Putting together (2.12.1), (2.12.2) and (2.12.3) establishes

$$\iint\limits_{K\times K} \mathbb{P}\left[\mathcal{N}_{z,w} \ge k\right]\ dz\, dw \le \exp[-\Theta(k)] \times C_{K,D} \tag{2.12.4}$$

as $k \to \infty$.

Having proved (2.12.4), we may appeal to Proposition 2.10.1 and conclude that $\mathfrak{h}_n$ converges almost surely to a limiting random variable $\mathfrak{h}$ taking values in $H^{-2-\delta}_{\mathrm{loc}}(D)$.

Since each $\mathfrak{h}_n$ is determined by Γ and $(\xi_\mathcal{L})_{\mathcal{L}\in\Gamma}$, the same is true of $\mathfrak{h}$. Similarly, for each $n \in \mathbb{N}$, $\mathfrak{h}_n$ inherits conformal invariance from the underlying CLE_κ. It follows that $\mathfrak{h}$ is conformally invariant as well. □

The following proposition shows that if the weight distribution μ has zero mean, then the step nesting field $\mathfrak{h}$ and the usual nesting field h are equal.

Proposition 2.12.2. Suppose that $D \subsetneq \mathbb{C}$ is a simply connected domain, and let μ be a probability measure with finite second moment and zero mean. Let Γ be a CLE_κ in D, and let $(\xi_\mathcal{L})_{\mathcal{L}\in\Gamma}$ be an i.i.d. sequence of μ-distributed random variables. The weighted nesting field $h = h(\Gamma, (\xi_{\mathcal{L}})_{\mathcal{L}\in\Gamma})$ from Theorem 2.6.1 and the step nesting field $\mathfrak{h} = \mathfrak{h}(\Gamma, (\xi_{\mathcal{L}})_{\mathcal{L}\in\Gamma})$ from Theorems 2.6.1 and 2.6.2 are almost surely equal.

Proof. Let $g \in C_c^\infty(D)$, $\varepsilon > 0$ and $n \in \mathbb{N}$. By Fubini's theorem, we have

$$\begin{aligned}&\mathbb{E}[(\langle h_\varepsilon, g\rangle - \langle \mathfrak{h}_n, g\rangle)^2] \\ &\quad= \iint_{D\times D} \mathbb{E}[(h_\varepsilon(z) - \mathfrak{h}_n(z))(h_\varepsilon(w) - \mathfrak{h}_n(w))]\, g(z)g(w)\, dz\, dw\,.\end{aligned} \tag{2.12.5}$$

Applying the same technique as in (2.9.2), we find that the expectation on the right-hand side of (2.12.5) is bounded by σ^2 times the expectation of the number $N_{z,w}(n,\varepsilon)$ of loops $\mathcal{L}$ satisfying both of the following conditions:

1. $\mathcal{L}$ surrounds $B_z(\varepsilon)$ or $\mathcal{L}$ is among the n outermost loops surrounding z, but not both.
2. $\mathcal{L}$ surrounds $B_w(\varepsilon)$ or $\mathcal{L}$ is among the n outermost loops surrounding w, but not both.

Using Fatou's lemma and (2.12.5), we find that

$$\begin{aligned}\mathbb{E}[(\langle h, g\rangle - \langle \mathfrak{h}, g\rangle)^2] &= \mathbb{E}\left[\lim_{\varepsilon\to 0}\lim_{n\to\infty}(\langle h_\varepsilon, g\rangle - \langle \mathfrak{h}_n, g\rangle)^2\right] \\ &\le \liminf_{\varepsilon\to 0}\liminf_{n\to\infty}\mathbb{E}[(\langle h_\varepsilon, g\rangle - \langle \mathfrak{h}_n, g\rangle)^2] \\ &\le \liminf_{\varepsilon\to 0}\liminf_{n\to\infty}\iint_{D\times D}\mathbb{E}[N_{z,w}(n,\varepsilon)]\, g(z)g(w)\, dz\, dw \\ &\le \limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\iint_{D\times D}\mathbb{E}[N_{z,w}(n,\varepsilon)]\, g(z)g(w)\, dz\, dw\,.\end{aligned}$$

If $z \neq w$, then $\mathcal{N}_{z,w} < \infty$ almost surely, so $\mathbb{E}[N_{z,w}(n,\varepsilon)]$ tends to 0 as $\varepsilon \to 0$ and $n \to \infty$. Furthermore, the observation $N_{z,w}(n,\varepsilon) \le \mathcal{N}_{z,w}$ implies by Theorem 2.6.3 that $\mathbb{E}[N_{z,w}(n,\varepsilon)]$ is bounded by $\nu_{\text{typical}} \log|z-w|^{-1} + \text{const}$ independently of n and ε. Since $(z,w) \mapsto \mathbb{E}[N_{z,w}(n,\varepsilon)]g(z)g(w)$ is dominated by the integrable function $(\nu_{\text{typical}} \log|z-w|^{-1} + \text{const})g(w)g(w)$, we may apply the reverse Fatou lemma to obtain

$$\begin{aligned}\mathbb{E}[(\langle h, g\rangle - \langle \mathfrak{h}, g\rangle)^2] &\le \iint_{D\times D}\limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\mathbb{E}[N_{z,w}(n,\varepsilon)]\, g(z)g(w)\, dz\, dw \\ &= 0\,,\end{aligned}$$

which implies

$$\langle h, g\rangle = \langle \mathfrak{h}, g\rangle \tag{2.12.6}$$

almost surely. The space $C_c^\infty(\mathbb{C})$ is separable [78, Exercise 1.13.6], which implies that $C_c^\infty(D)$ is also separable. To see this, consider a given countable dense subset of $C_c^\infty(\mathbb{C})$. Any sufficiently small neighborhood of a point in $C_c^\infty(D)$ is open in $C_c^\infty(\mathbb{C})$, and therefore intersects the countable dense set. Therefore, we may apply (2.12.6) to a countable dense subset of $C_c^\infty(D)$ to conclude that $h = \mathfrak{h}$ almost surely. □

Acknowledgements

We thank Amir Dembo for providing us with the computation which extracts Theorems 2.1.2 and 2.1.3 from Theorem 2.5.3. We also thank Nike Sun for helpful comments.

Chapter 3

CLE_4 and the Gaussian free field

This chapter presents joint works with Scott Sheffield and Hao Wu. Much of this content appears in [68].

In this chapter, we establish a coupling between a zero-boundary Gaussian free field and a CLE_4 growth process in which the boundaries of explored regions in the growth process correspond to level loops of the GFF. Sections 3.1 through 3.5 cover prerequisite tools involving conformal loop ensembles in doubly connected domains. Sections 3.6 through 3.8 describe the GFF/CLE_4 coupling, and the remainder of the chapter is devoted to proving a property of the CLE_4 growth process: the dynamics are determined by the CLE_4 loops.

3.1 Introduction

In [70], CLEs in simply connected domains are constructed from Brownian loop soup. The authors proved that CLE_κ for $\kappa \in (8/3, 4]$ are the only random collections of curves satisfying conformal invariance and the restriction property (see Section 3.2.3 for more details). They also showed that CLE is closely related to SLE: each loop in CLE_κ for $\kappa \in (8/3, 4]$ is an SLE_κ-type loop. In [25], nested CLE in the Riemann sphere is defined and shown to be invariant under Möbius transformations.

In this section, we study CLE in doubly connected domains. By the theory of conformal maps [53], every doubly connected domain is conformally equivalent to exactly one of the following standard domains:

(i) an annulus $\{z \in \mathbb{C} : r < |z| < 1\}$ for some $r \in (0,1)$,

(ii) the punctured disk $\mathbb{D} \setminus \{0\}$, or

(iii) the punctured plane $\mathbb{C} \setminus \{0\}$.

We construct CLE in each of these domains. We construct CLE in an annulus using the Brownian loop soup, and show that invariant under conformal maps from the annulus onto itself. We consider a limit as the inner radius of the annulus tends to 0 to construct CLE in the punctured disk, and we consider a limit of CLEs in punctured disks of radii tending to infinity to construct CLE in the punctured plane. We show that CLE in the punctured disk is also rotationally invariant, and we show that CLE in the punctured plane is invariant under scalings, rotations, and inversion. Having constructed CLE in standard doubly connected domains, we define CLE in an arbitrary doubly connected domain as the image of CLE under a conformal map from the standard domain to which it is conformally equivalent. The aforementioned invariances ensure that the resulting law does not depend on the choice of conformal map.

The main results about CLE in an annular domain, which we call annulus CLE, can be summarized as follows:

- The law of annulus CLE is conformally invariant and satisfies the restriction property.
- Annulus CLE and CLE in a simply connected domain are closely related: given a CLE in simply connected domain, consider the loop containing a given interior point. The collection of loops between this particular loop and the boundary of the domain has the same law as an annulus CLE.

CLE in the punctured disk satisfies the following properties.

- The law of CLE in the punctured disk is conformally invariant and satisfies the restriction property.
- CLE in the punctured disk may be regarded as a CLE in the unit disk conditioned on the event that no loop surrounds the origin (this event has probability zero; nevertheless this conditioning can be defined via a limiting procedure).
- If the loops that we are interested are far from the origin, then these loops are almost the same as the loops in CLE in simply connected domain (see Proposition 3.4.5 for a precise statement).

Finally, CLE in the punctured plane satisfies the following properties.

- The law of CLE in the punctured plane is conformally invariant and satisfies the restriction property.
- CLE in the punctured plane may be viewed as CLE in the Riemann sphere conditioned on the event that neither 0 nor ∞ is surrounded by a loop.

For CLE in the punctured plane, invariance under inversion $z \mapsto 1/z$ is true by construction. By contrast, reversibility for nested CLE in the whole plane is nontrivial—it is the main result in [25].

In [70], the authors described a procedure to discover the loops in a CLE by exploring in small steps from the boundary. See Figure 3-1 for an illustration of this exploration process in the case $\kappa = 4$. The construction of this exploration procedure makes extensive use of conformal invariance and the restriction property of CLE. We use the same procedure to explore the loops for CLE in doubly connected domains, and we prove a quantitative relation between the CLE explorations in simply and doubly connected domains.

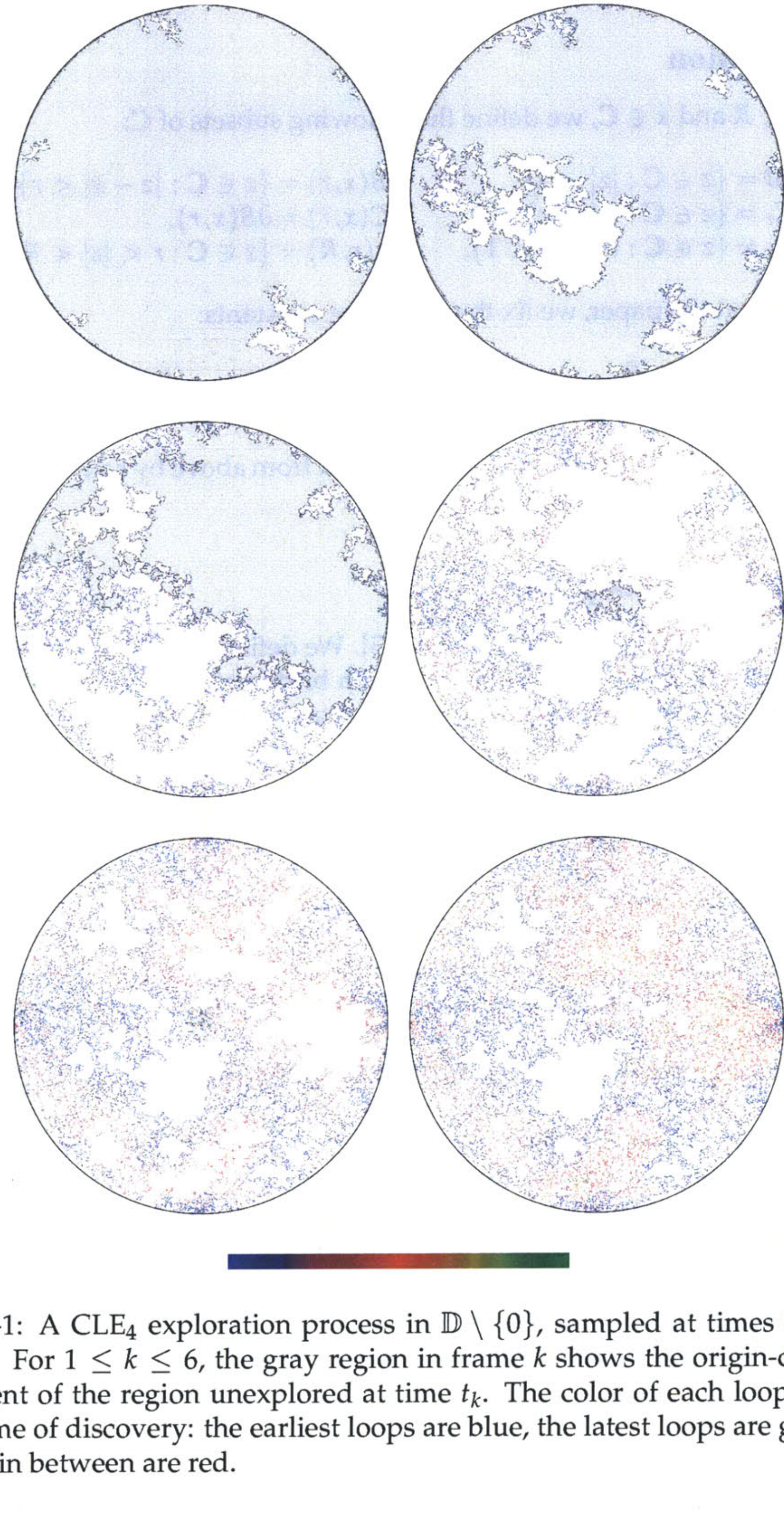


Figure 3-1: A CLE_4 exploration process in $\mathbb{D} \setminus \{0\}$, sampled at times $0 < t_1 < \cdots < t_6$. For $1 \leq k \leq 6$, the gray region in frame k shows the origin-containing component of the region unexplored at time t_k. The color of each loop indicates the its time of discovery: the earliest loops are blue, the latest loops are green, and the ones in between are red.

3.2 Preliminaries

3.2.1 Notation

For $0 < r < R$ and $x \in \mathbb{C}$, we define the following subsets of $\mathbb{C}$:

$$\begin{aligned} \mathbb{D} &= \{z \in \mathbb{C} : |z| < 1\}, & B(x,r) &= \{z \in \mathbb{C} : |z-x| < r\}, \\ C_r &= \{z \in \mathbb{C} : |z| = r\}, & C(x,r) &= \partial B(x,r), \\ \mathbb{A}_r &= \{z \in \mathbb{C} : r < |z| < 1\}, & \mathbb{A}(r,R) &= \{z \in \mathbb{C} : r < |z| < R\}. \end{aligned}$$

Throughout the paper, we fix the following constants:

$$\kappa \in (8/3, 4], \quad \beta = \frac{8}{\kappa} - 1, \quad \alpha = \frac{(8-\kappa)(3\kappa-8)}{32\kappa}, \quad c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}. \tag{3.2.1}$$

We write $f \lesssim g$ to mean that f/g is bounded from above by some constant, and we write $f \asymp g$ to mean that $f \lesssim g$ and $g \lesssim f$.

3.2.2 Brownian loop soup

We now briefly recall some results from [33]. We define for all $t \geq 0$ and $z \in \mathbb{C}$ the law $\mu_t(z,z)$ of the two-dimensional Brownian bridge of duration t that starts and ends at z. We define the infinite, σ-finite measure μ^{loop} on *unrooted* loops modulo time reparametrization by

$$\mu^{\text{loop}} = \int_{\mathbb{C}} \int_0^\infty \frac{\mu_t(z,z)}{2\pi t^2} \, dt \, dA(z),$$

where dA denotes area measure on $\mathbb{C}$ (see Chapter 5 of [30] for a rigorous construction of the integral of a measure-valued function). Then, μ^{loop} inherits a striking conformal invariance property. More precisely, define for all $D \subset \mathbb{C}$ the Brownian loop measure $\mu^{\text{loop}}(D)$ as the restriction of μ^{loop} to the set of loops contained in D. It is shown in [33] that

(i) for two domains $D' \subset D$, sampling from $\mu^{\text{loop}}(D)$ and restricting to the set of loops contained in D' is the same as sampling from $\mu^{\text{loop}}(D')$, and

(ii) for two connected domains D_1, D_2, the image of $\mu^{\text{loop}}(D_1)$ under a conformal map $\Phi : D_1 \to D_1$ has the same law as $\mu^{\text{loop}}(D_2)$.

The restriction property is an apparent consequence of the definition of $\mu^{\text{loop}}(D)$, and conformal invariance is inherited from the conformal invariance of planar Brownian motion.

We denote by $\Lambda(V_1, V_2; D)$ the measure under μ^{loop} of the set of loops contained in a domain D that intersect both $V_1 \subset \mathbb{C}$ and $V_2 \subset \mathbb{C}$.

Proposition 3.2.1. [28, Lemma 3.1, equation (22)] Suppose that $0 < r < 1$ and $R \geq 2$. Then

$$\Lambda(C_1, C_R; \mathbb{C} \setminus \mathbb{D}_r) = 2 \int_r^1 s^{-1} \rho(R/s) ds$$

for some function $\rho : (0, \infty) \to \mathbb{R}$ satisfying the following estimate: there exists a universal constant $C < \infty$ such that, for $u \geq 2$, we have

$$\left| \rho(u) - \frac{1}{2 \log u} \right| \leq \frac{C}{u \log u}.$$

For a fixed domain $D \subset \mathbb{C}$ and a constant $c > 0$, a **Brownian loop soup with intensity** c **in** D is a Poisson point process with intensity $c\mu^{\text{loop}}(D)$. The following property of the Brownian loop soup follows from properties of the Brownian loop measure: fix a domain D and a constant $c > 0$, and suppose D' is a subset of D and that $\mathcal{L}$ is a Brownian loop soup in D. Let $\mathcal{L}_1$ be the collection of loops in $\mathcal{L}$ that are contained in D' and let $\mathcal{L}_2 = \mathcal{L} \setminus \mathcal{L}_1$. Then $\mathcal{L}_1$ has the same law as Brownian loop soup in D', and $\mathcal{L}_1$ and $\mathcal{L}_2$ are independent. In fact, this assertion also holds for certain random domains D. In particular, we have the following proposition.

Proposition 3.2.2. Let $\mathcal{L}$ be a Brownian loop soup in D with intensity $c \in (0, 1]$, and let $(D_t)_{t \geq 0}$ be a (deterministic) decreasing family of subdomains of $\mathbb{D}$. For $t \geq 0$, define D_t^* to be the domain obtained by removing from $\mathbb{D}$ the closures of clusters of loops in $\mathcal{L}$ which are not contained in D_t. Let τ be a stopping time for the process $(D_t^*)_{t \geq 0}$. Then the conditional law of $\mathcal{L}$ restricted to D_τ^* given D_τ^* is that of a Brownian loop soup in D_τ^*.

Proof. The proof is a straightforward modification of the proof of Lemma 9.2 in [70], but for completeness we review it here. For $D \subset \mathbb{D}$ and $n \geq 1$ we define $F_n(D)$ to be the union of all squares in $2^{-n}\mathbb{Z}^2$ contained in D. Then for all such unions-of-squares $V \subset \mathbb{D}$, the event $\{F_n(D_\tau^*) = V\}$ depends only on the loops intersecting $\mathbb{D} \setminus V$ and is therefore independent of the loop configuration in V. This shows that for all $n \geq 1$, the loop configuration in $F_n(D_\tau^*)$ is a Brownian loop soup in $F_n(D_\tau^*)$. Since $\bigcup_n F_n(D_\tau) = D_\tau$, this shows that the configuration in D_τ^* is a Brownian loop soup. ☐

3.2.3 CLE in a simply connected domain

Definition and properties of CLE

Let us now briefly recall some features of the conformal loop ensembles for $\kappa \in (8/3, 4]$ – we refer to [70] for details and proofs of these statements. A simple CLE Γ in $\mathbb{D}$ is a random loop configuration $(\gamma_j, j \in J)$ in $\mathbb{D}$, measurable with respect to $\mathcal{F}$, whose law possesses the following two properties:

- *Conformal invariance.* For any Möbius transformation Φ of $\mathbb{D}$ onto itself, the laws of Γ and $\Phi(\Gamma)$ are the same. Thus, for any simply connected domain D, we may define CLE in D as the distribution of $\tilde{\Phi}(\Gamma)$, where $\tilde{\Phi} : \mathbb{D} \to D$ is a conformal map. Note that Möbius invariance implies that the resulting measure does not depend on the choice of conformal map $\tilde{\Phi}$ from $\mathbb{D}$ onto D.

- *Restriction.* For any simply connected domain $D \subset \mathbb{D}$, define the set $D^* = D^*(D, \Gamma)$ obtained by removing from D all the loops (and their interiors) of Γ are not contained in D. Then, conditionally on D^*, and for each connected component U of D^*, the law of those loops of Γ that do stay in U is that of a CLE in U.

In [70], the authors show that for each CLE, there exists a real number $\kappa \in (8/3, 4]$ so that all the loops in the CLE are almost surely SLE$_\kappa$-type loops. Furthermore for each such value of κ, there exists exactly one CLE distribution that has SLE$_\kappa$-type loops.

As explained in [70], a construction of these particular families of loops can be given in terms of outermost boundaries of clusters of the Brownian loops in a Brownian loop soup with intensity $c(\kappa) \in (0, 1]$, where

$$c = c(\kappa) = \frac{(6-\kappa)(3\kappa - 8)}{2\kappa}.$$

Throughout the paper, we will denote the law of simple CLE in simply connected domain D as $\mu_D^\sharp$.

Exploring CLE

Consider a simple CLE in the unit disk and a small disk $B(x, \varepsilon)$ of radius ε, where $x \in \partial\mathbb{D}$. Let γ^ε be the loop that intersects $B(x, \varepsilon)$ with largest harmonic measure as seen from the origin. Define the quantity

$$u(\varepsilon) := \mathbb{P}[\gamma^\varepsilon \text{ surrounds the origin}]. \tag{3.2.2}$$

As $\varepsilon \to 0$, we have $u(\varepsilon) = \varepsilon^{\beta + o(1)}$, where $\beta = \frac{8}{\kappa} - 1$ [70, Corollary 4.3]. We say that a loop is pinned at $x \in \partial D$ if the loop contains x and is contained in $D \cup \{x\}$. We refer to a loop which is pinned at some $x \in \partial D$ as a *bubble*.

Proposition 3.2.3. [70, Section 4] The law of γ^ε normalized by $1/u(\varepsilon)$ converges as $\varepsilon \to 0$ to a σ-finite measure on the set of loops pinned at x, which we denote $\nu^{bub}(\mathbb{D}; x)$ and call the **SLE bubble measure in $\mathbb{D}$ rooted at** x. Furthermore,

(i) the $\nu^{bub}(\mathbb{D}; x)$-measure of the set of loops surrounding the origin is 1, and

(ii) for r small enough, $\nu^{bub}(\mathbb{D}; x)(R(\gamma) \geq r) \asymp r^{-\beta}$ where $R(\gamma)$ is the smallest radius r such that γ is contained in $B(x, r)$.

Now we describe the discrete exploration process. Suppose we have a simple CLE loop configuration in the unit disk $\mathbb{D}$. We draw a small semi-disk of radius ε whose center is uniformly chosen on the unit circle. The loops that intersect this small semi-disk are said to be *discovered* on the first step. If we do not discover the loop containing the origin, we call the connected component of the remaining domain that surrounds the origin the to-be-explored domain. We define the conformal map f_1^ε from the to-be-explored domain onto the unit disk normalized at the origin (meaning that 0 maps to 0 and the derivative of the conformal map at 0 is positive). We also define γ_1^ε to be the loop we discovered with largest harmonic measure as seen from the origin. Because of the conformal invariance and restriction property of simple CLE, the image of the loops in the to-be-explored domain under the conformal map f_1^ε has the same law as simple CLE in the unit disc. Thus we can repeat the same procedure to the image of the loops under f_1^ε. We draw a small semi-disk of radius ε whose center is uniformly chosen on the unit circle. The loops that intersect the small semi-disk are the loops we discovered at the second step. If we do not discover the loop containing the origin, define the conformal map f_2^ε from the to-be-explored domain onto the unit disk normalized at the origin. The image of the loops in the to-be-explored domain under f_2^ε has the same law as simple CLE in the unit disk, and we may repeat. With probability 1, there is a finite step N at which we discover the loop containing the origin, and we define γ_N^ε to be the loop containing the origin discovered at this step and stop the exploration. We summarize the properties and notations for this discrete exploration below.

- Before time N, all the steps of discrete exploration are i.i.d.
- The random variable N has a geometric distribution:

$$\mathbb{P}(N > n) = \mathbb{P}(\gamma^\varepsilon \text{does not contain the origin})^n = (1 - u(\varepsilon))^n.$$

- We define the conformal map

$$\Phi^\varepsilon = f_{N-1}^\varepsilon \circ \cdots \circ f_2^\varepsilon \circ f_1^\varepsilon.$$

The discrete exploration converges as $\varepsilon \to 0$ to a Poisson point process of bubbles with intensity measure given by

$$\nu^{\text{bub}}(\mathbb{D}) = \int_{\partial\mathbb{D}} \nu^{\text{bub}}(\mathbb{D}; x)\, dx.$$

In fact, this Poisson point process of bubbles can be used to recover the CLE loops. More precisely, let $(\gamma_t, t \geq 0)$ be a Poisson point process on the Cartesian product of the set of bubbles and the nonnegative real line with intensity $\nu^{\text{bub}}(\mathbb{D})$ times Lebesgue measure. Define $\tau = \inf\{t \geq 0 : \gamma_t \text{ surrounds the origin}\}$. Then for each $t < \tau$, γ_t does not contain the origin, so we may define f_t to be the conformal map from the connected component of $\mathbb{D} \setminus \gamma_t$ containing the origin onto the unit disk and normalizing at the origin. For this Poisson point process, we have the following properties [70, Sections 4 and 7]:

- τ has exponential law: $\mathbb{P}(\tau > t) = e^{-t}$.
- For $r > 0$ small, let $t_1(r), t_2(r), ..., t_j(r)$ be the times t before τ at which the bubble γ_t has diameter greater than r. Define $\Psi^r = f_{t_j(r)} \circ \cdots \circ f_{t_1(r)}$. Then Ψ^r almost surely converges as r goes to zero to some conformal map Ψ with

respect to the Carathéodory topology seen from the origin. We write $\Psi = \circ_{t<\tau} f_t$.

- More generally, for each $t \leq \tau$, we can define $\Psi_t = \circ_{s<t} f_s$. Then $(L_t := \Psi_t^{-1}(\gamma_t), 0 \leq t \leq \tau)$ is a collection of loops in the unit disk, and L_τ surrounds the origin.

The relationship between this Poisson point process of bubbles and the discrete exploration process we described above is given via the following proposition.

Proposition 3.2.4. Φ^ϵ converges in distribution to Ψ with respect to the Carathéodory topology seen from the origin. Furthermore, L_τ has the same law as the loop in simple CLE containing the origin.

Denote $D_t = \Psi_t^{-1}(\mathbb{D})$ for $t \leq \tau$. We call the sequence of domains $(D_t, t \leq \tau)$ **the uniform** CLE_κ **exploration process** in $\mathbb{D}$, targeted at the origin.

3.2.4 Conformal radius

A proper simply connected domain D is an open subset of $\mathbb{C}$ such that D and its complement in $\mathbb{C}$ are both nonempty and connected . From Riemann Mapping Theorem, we know that for any proper simply connected domain D and an interior point $z \in D$, there exists a unique conformal map Φ from D onto the unit disk $\mathbb{D}$ such that $\Phi(z) = 0$ and $\Phi'(z) > 0$. We define the **conformal radius** of D seen from z by

$$\text{CR}(D; z) = 1/\Phi'(z).$$

We abbreviate $\text{CR}(D) := \text{CR}(D; 0)$.

Consider a closed subset K of $\mathbb{D}$ such that $\mathbb{D} \setminus K$ is simply connected and $0 \in \mathbb{D} \setminus K$. There exists a unique conformal map Φ_K from $\mathbb{D} \setminus K$ onto $\mathbb{D}$ which is normalized at the origin: $\Phi_K(0) = 0, \Phi_K'(0) > 0$. By the Schwarz lemma, we have $\Phi_K'(0) \geq 1$, from which it follows that

$$\text{CR}(\mathbb{D} \setminus K) = 1/\Phi_K'(0) \leq 1.$$

The Schwarz lemma and the Koebe one quarter theorem imply that

$$d \leq \text{CR}(\mathbb{D} \setminus K) \leq 4d, \tag{3.2.3}$$

where $d = \text{dist}(0, K)$ is the distance from the origin to K.

Define the **capacity of** K **in** $\mathbb{D}$ **seen from the origin** as

$$\text{cap}(K) = -\log \text{CR}(\mathbb{D} \setminus K) \geq 0.$$

We adopt the convention that $\text{CR}(\mathbb{D} \setminus K) = 0$ and $\text{cap}(K) = \infty$ if $0 \in K$. When K is small, for example when the radius $R(K)$ of K is less than $1/2$, we have that

$$\text{cap}(K) \lesssim R(K)^2.$$

An *annular domain* A is a connected open subset of $\mathbb{C}$ such that its complement in the Riemann sphere has two connected components and both of them contain more than one point. If A is an annular domain, then there exists a unique constant $r \in (0,1)$ such that A can be conformally mapped onto the standard annulus $\mathbb{A}_r = \{z : r < |z| < 1\}$. We define the **conformal modulus** of A, denoted by $\mathrm{mod}(A)$, to be r^{-1}.

The following standard lemma describes the relationship between the conformal radius of simply connected domain and the conformal modulus of an annular domain.

Lemma 3.2.5. Suppose K is a closed subset of $\overline{\mathbb{D}}$ such that $\mathbb{D} \setminus K$ is simply connected and $0 \in \mathbb{D} \setminus K$. Clearly $\mathbb{A}_r \setminus K$ is an annular domain for r small enough. We have

$$\lim_{r \to 0} \frac{\mathrm{mod}(\mathbb{A}_r \setminus K)}{\mathrm{mod}(\mathbb{A}_r)} = \mathrm{CR}(\mathbb{D} \setminus K).$$

3.2.5 Overshoot estimates for compound Poisson processes

Recall that the total variation distance between two probability measures ρ_1 and ρ_2 on a common measurable space is defined by

$$\|\rho_1 - \rho_2\|_{\mathrm{TV}} := \sup\{|\rho_1(A) - \rho_2(A)| : A \text{ measurable}\}.$$

For any coupling (X_1, X_2) of ρ_1 and ρ_2 (i.e. for any coupling (X_1, X_2) where the marginal law of X_i is ρ_i, $i = 1, 2,$), we have

$$\|\rho_1 - \rho_2\|_{\mathrm{TV}} \leq \mathbb{P}[X_1 \neq X_2],$$

and there exists coupling (X_1, X_2) such that the equality holds (see [36, Proposition 4.7]).

Suppose $(\sigma(t), t \geq 0)$ is a compound Poisson process starting from 0 with jump measure Π that is supported on $(0, \infty)$ and satisfies

$$\int (1 \wedge x)\Pi(dx) < \infty,$$

where $a \wedge b$ denotes the minimum of two real numbers a and b. The process can be written as

$$\sigma(t) = \sum_{0 \leq s \leq t} \Delta_s \quad \text{for all } t \geq 0,$$

where $(\Delta_s, s \geq 0)$ is a Poisson point process with intensity measure Π. The Campbell formula states that the Laplace transform of σ is given by

$$\mathbb{E}[\exp(-\lambda\sigma(t))] = \exp\left(-t \int (1 - e^{-\lambda x})\Pi(dx)\right) \quad \text{for all } \lambda > 0, t > 0.$$

Define the local time process $(L_x, x > 0)$ and the first passage process $(D_x, x > 0)$

of $(\sigma(t), t \geq 0)$ by

$$L_x := \inf\{t \geq 0 : \sigma(t) > x\}, \quad D_x = \sigma(L_x) \quad \text{for } t > 0.$$

We have the following estimates of the overshoot $D_x - x$:

Proposition 3.2.6. Suppose that Π is absolutely continuous with respect to Lebesgue measure and that there exists a constant $\lambda_0 > 0$ such that

$$\int (e^{\lambda_0 x} - 1)\Pi(dx) < \infty.$$

Then there exists $C \in (0, \infty)$ such that

$$\mathbb{P}[D_x - x \geq y] \leq Ce^{-\lambda_0 y}, \text{ for all } x \geq 0, y > 0. \tag{3.2.4}$$

If we define ρ_x to be the law of $D_x - x$, then there exist $C \in (0, \infty)$ and $\delta > 0$ such that ρ_x converges exponentially fast in total variation to some measure ρ_∞:

$$||\rho_x - \rho_\infty||_{\text{TV}} \leq Ce^{-\delta x} \text{ for all } x \geq 0. \tag{3.2.5}$$

Proof. For finite measures Π, the result is established in [46, Lemma 2.12, Lemma 3.5]. This result implies the result for the case $\Pi((0, \infty)) = +\infty$. To see this, define for all $\varepsilon > 0$ a compound Poisson point process σ^ε by aggregating each sequence of consecutive jumps of size less than ε into a single jump. Then with probability tending to 1 as $\varepsilon \to 0$, the overshoot of σ^ε equals the overshoot of σ. $\square$

3.3 Annulus CLE

3.3.1 Definition and properties of annulus CLE

We say that a family of measures

$$\{\mu_D^\sharp : D \subset \mathbb{C} \text{ is an annular or simply connected domain}\}$$

is an **annulus CLE** if (i) $\mu(D)$ is a measure on the set of configurations of simple loops whose exteriors are contained in $\mathbb{C} \setminus D$, (ii) the family of measures satisfies conformal invariance, and (iii) the family of measures satisfies the restriction property. In this section, we will for each $\kappa \in (8/3, 4]$ construct an annulus CLE with SLE_κ-type loops.

Our construction proceeds by first defining random loop configurations in the standard annuli $\mathbb{A}_r$ for $r \in (0,1)$. We will do so by constructing simple CLE in $\mathbb{D}$ from the Brownian loop soup as in [70]. For $\kappa \in (8/3, 4]$, let $\mathcal{L}(\mathbb{A}_r)$ be a Brownian loop soup with intensity $c(\kappa)$ in $\mathbb{A}_r$. Define the event $E(\mathcal{L}(\mathbb{A}_r))$ that there is no cluster of $\mathcal{L}(\mathbb{A}_r)$ that disconnects the inner boundary from the outer boundary.

On the event $E(\mathcal{L}(\mathbb{A}_r))$, let $\Gamma(\mathbb{A}_r)$ be the collection of outermost boundaries of clusters of $\mathcal{L}(\mathbb{A}_r)$, so that $\Gamma(\mathbb{A}_r)$ is a collection of disjoint simple loops in $\mathbb{A}_r$.

The event $E(\mathcal{L}(\mathbb{A}_r))$ contains the event that the loop surrounding the origin for CLE in the disk has inradius less than r, and this event has positive probability—indeed, the conformal radius of the loop surrounding the origin has a distribution whose density with respect to Lebesgue measure is positive on $(0,1)$ [62]. Therefore, $E(\mathcal{L}(\mathbb{A}_r))$ has positive probability and we may define annulus CLE_κ in $\mathbb{A}_r$ as the law of $\Gamma(\mathbb{A}_r)$ conditioned on the event $E(\mathcal{L}(\mathbb{A}_r))$.

For any annular domain A with conformal radius r, let φ be a conformal map from $\mathbb{A}_r$ onto A and define $\mu_A^\sharp$ to be the law of the loop configuration $\varphi(\Gamma)$ where Γ is an annulus CLE_κ in $\mathbb{A}_r$.

We denote by $p(A)$ the probability of the event $E(\mathcal{L}(A))$ and abbreviate $p(\mathbb{A}_r)$ as $p(r)$. The following lemma summarizes the asymptotic behavior of $p(r)$ as r goes to zero. Recall the definition of α in terms of κ in (3.2.1).

Proposition 3.3.1. [52, Lemma 7, Corollary 8] The function p is nondecreasing, and there exists a universal constant $C < \infty$ such that, for $0 < r, r' < 1$,

$$\frac{1}{C}p(r)p(r'/C) \leq p(rr') \leq p(r)p(r'). \tag{3.3.1}$$

Furthermore,

1. There exists a constant $C \geq 1$ such that, for r small enough, $r^\alpha \leq p(r) \leq Cr^\alpha$.
2. For any constant $\lambda > 0$, the limit of $p(\lambda r)/p(r)$ exists as r goes to zero and $\lim_{r\to 0} p(\lambda r)/p(r) = \lambda^\alpha$.

Proposition 3.3.2. For $r\ in(0,1)$, the law of CLE_κ in $\mathbb{A}(r, \frac{1}{r})$ is invariant under rotations $z \mapsto e^{i\theta}z$ and under the map $z \mapsto 1/z$.

Proof. The result follows directly from the conformal invariance of the Brownian loop soup. □

Annulus CLE_κ also satisfies the restriction property:

Proposition 3.3.3. Suppose Γ is an annulus CLE_κ in $\mathbb{A}_r$ and that D is an open subset of $\mathbb{A}_r$. Let D^* be the set obtained by removing from D all the loops (and their interiors) in Γ that are not contained in D.

(i) If D is simply connected, then conditionally on D^*, for each connected component U of D^*, the conditional law of the loops in Γ that stay in U is that of a CLE_κ in U.

(ii) If D is an annular region, then conditionally on D^*, for each connected component U of D^*, the conditional law of the loops of Γ that are contained in U is that of a CLE in U.

Furthermore, the loop configurations in different components U are independent.

Remark 3.3.4. Note that in case (i), U is necessarily simply connected. In case (ii), U may be simply connected or annular.

Proof. Let $\mathcal{L}$ be a Brownian loop soup in $\mathbb{A}_r$. We define $\tilde{\mathcal{L}}$ to be the Brownian loop configuration obtained by sampling $\mathcal{L}$, determining D^*, and then sampling independent Brownian loop soups in each connected component of D^*. By Proposition 3.8.9, $\tilde{\mathcal{L}}$ has the same distribution as $\mathcal{L}$.

Similarly, define Γ to be a CLE_κ in $\mathbb{A}_r$ and define $\tilde{\Gamma}$ to be the loop configuration obtained by sampling Γ and then independently resampling the loop ensembles in each connected component of D^*. The proposition statement is equivalent to the assertion that Γ and $\tilde{\Gamma}$ have the same law.

Let $B \in \mathcal{F}$ (recall that we defined the σ-algebra $\mathcal{F}$ on the space of loop configurations in Section 3.2.3). Let E be the set of simple loop configurations in $\mathbb{A}_r$ with no loops surrounding C_r, and write $\tilde{\mu}^\sharp_{\mathbb{A}_r}$ for the law of $\tilde{\Gamma}$. Then by the definition of annulus CLE, we have

$$\begin{aligned} \mu^\sharp_{\mathbb{A}_r}[B] &= \frac{\mathbb{P}[\Gamma(\mathcal{L}) \in B \cap E]}{\mathbb{P}[\Gamma(\mathcal{L}) \in E]} \\ &= \frac{\mathbb{P}[\Gamma(\tilde{\mathcal{L}}) \in B \cap E]}{\mathbb{P}[\Gamma(\tilde{\mathcal{L}}) \in E]}, \end{aligned}$$

where in the second line we have used the fact that $\Gamma(\mathcal{L})$ and $\Gamma(\tilde{\mathcal{L}})$ have the same law.

Define the event E_1 that there are no loops in $\Gamma(\tilde{\mathcal{L}})$ which intersect $\mathbb{A}_r \setminus D$ and surround C_r, and let E_2 be the event that there are no loops in $\Gamma(\tilde{\mathcal{L}})$ which are contained in D and surround C_r. Since E_1 is measurable with respect to the σ-algebra generated by the set $\mathcal{L}^*$ of loop clusters intersecting $\mathbb{A}_r \setminus D$ and E_2 is measurable with respect to the loops in D^*, we conclude that the following two procedures give a loop configuration with the same law:

(i) Sample $\mathcal{L}^*$ and the loop configuration inside D^* from their joint law, and condition on the event $E_1 \cap E_2$.

(ii) Sample $\mathcal{L}^*$ from its marginal distribution conditioned on E_1, and then sample the loop configuration inside D^* from its conditional law given $\mathcal{L}^*$ and E_2.

Since $\mathcal{L}^*$ determines D^*, this gives

$$\frac{\mathbb{P}[\Gamma(\tilde{\mathcal{L}}) \in B \cap E]}{\mathbb{P}[\Gamma(\tilde{\mathcal{L}}) \in E]} = \tilde{\mu}^\sharp_{\mathbb{A}_r}[B],$$

which concludes the proof. □

Propositions 3.3.5 and 3.3.6 describe two ways to find annulus CLE in CLE in a simply connected domain.

Proposition 3.3.5. Suppose Γ is a CLE_κ in $\mathbb{D}$ and $D \subset \mathbb{D}$ is an annulus. Let D^* be the set obtained by removing from D all the loops in Γ the closure of whose interiors that are not contained in D. Then conditionally on D^*, for each connected component U of D^*, the conditional law of the loops in Γ that stay in U is that of a CLE in U. Furthermore, the CLEs in different components U are independent.

Proof. Realize Γ as the set of outermost clusters of a Brownian loop soup $\mathcal{L}$ in $\mathbb{D}$. We define the following procedure for finding D^*: denote by D_1^* the complement of the union of the loop clusters not contained in D. By Proposition 3.8.9, the law of the Brownian loops contained in D_1^* is that of a Brownian loop soup in D_1^*.

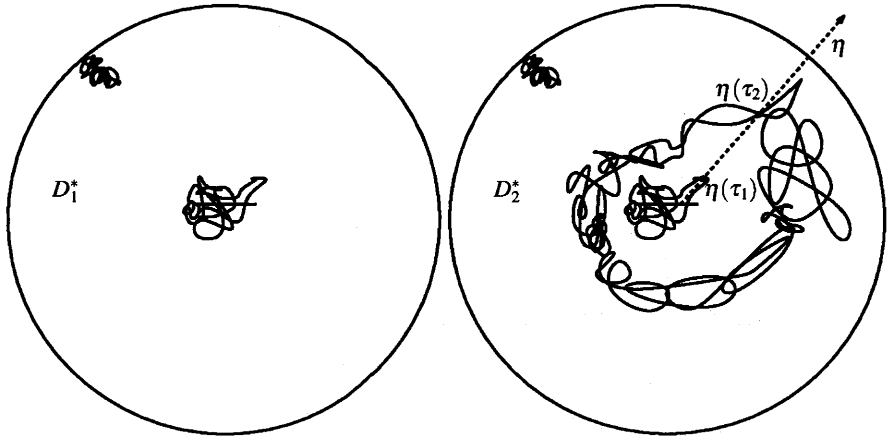


Figure 3-2: The first panel shows the domain D_1^*, where $D = \mathbb{D} \setminus \{\text{a union of two slits}\}$. The second panel shows the exploration process we use to discover the innermost cluster surrounding the inner boundary of D: we choose a ray whose endpoint is on the interior slit and define τ_1 to be the first time the ray hits the boundary of D_1^*, and from $\eta(\tau_1)$ we explore outward along the ray η until we exit D_1^* or discover the first loop cluster $\mathcal{C}$ winding around the annulus, which happens in the diagram above at time τ_2. We define D_2^* to be the intersection of D_1^* and the unbounded component of the complement of $\mathcal{C}$. We continue in this way to define a random sequence of domains $D_1^*, D_2^*, \ldots, D_R^*$.

Observe that D^* and D_1^* are not necessarily equal, because, with positive probability, D_1^* has an annular component U containing one or more loop clusters surrounding the inner boundary of U (and so the interior of the corresponding CLE loop would not be in D–see Figure 3-2). We will discover any such clusters as follows. Choose an arbitrary ray emanating from an arbitrary point in the bounded component of $\mathbb{C} \setminus D$ and let $(\eta_t : t \geq 0)$ be a parameterization of the ray. For $t \geq 0$, denote by $\mathcal{L}_t$ the set of loop clusters of $\mathcal{L}$ not contained in $D \setminus \eta[0, t]$. Define

$$\tau_1 = \inf\{t \geq 0 : \eta(t) \in D_1^*\}.$$

Define the (possibly empty) set of random times $\tau_2, \ldots, \tau_R$. to be the sequence of times at which the growing cluster $\mathcal{L}_t$ acquires a cluster K_j surrounding the inner boundary of D. Here R is a random variable taking values in $\{1, 2, \ldots, \infty\}$. Define D_j^* to be the intersection of D_{j-1}^* and the unbounded component of $\mathbb{C} \setminus K_{j-1}$ (see Figure 3-2).

Proposition 3.8.9 implies that the law of the configuration of loops in D_j^* is distributed like a Brownian loop soup in D_j^*. This shows that $R < \infty$ almost surely, because at each step there is a positive probability (indeed, a probability tending to 1 in j) that there are no clusters surrounding the inner boundary of D_j^*.

Since R is the least index j for which the Brownian loop soup in D_j^* has no loops surrounding the inner boundary, our exploration may be viewed as a rejection sampling procedure which conditions on the event that no loop clusters surround the inner boundary. Therefore, the loop configuration in D_R^* has the law of a Brownian loop soup in D_R^* conditioned to have no loop clusters surrounding the inner boundary. Since $D_R^* = D^*$, this concludes the proof. □

Proposition 3.3.6. Let Γ be a CLE_κ in $\mathbb{D}$, and let $\gamma(0)$ be the loop in Γ that surrounds the origin. Let D^* be the subset of $\mathbb{D}$ obtained by removing from $\mathbb{D}$ the loop $\gamma(0)$ and its interior. Then conditioned on D^*, the loop configuration $\Gamma \setminus \{\gamma(0)\}$ is distributed as an annulus CLE_κ in D^*.

Proof. The idea of this proof is to apply Proposition 3.3.5 to $\mathbb{A}_r$ and let $r \to 0$. Let $\widetilde{\Gamma}$ be a loop configuration obtained by sampling $\gamma(0)$ and then sampling an annulus CLE_κ in D^*; our goal is to show that the law of $\widetilde{\Gamma}$ is that of a CLE_κ.

For each $r > 0$, define $\widetilde{\Gamma}_r$ to be the loop configuration obtained as follows: sample the loops in Γ whose interiors are not contained in $\mathbb{A}_r \subset \mathbb{D}$, define $D_r^* = \mathbb{D} \setminus \{\gamma(0)$ and its interior$\}$, and resample a CLE in D_r^* in such a way that if $D_r^* = D^*$, then $\widetilde{\Gamma}_r = \widetilde{\Gamma}$. Since $0 \notin \gamma(0)$ almost surely, we have $\lim_{r \to 0} \widetilde{\Gamma}_r = \widetilde{\Gamma}$ almost surely. Therefore, dominated convergence gives us

$$\mathbb{P}[\widetilde{\Gamma} \in B] = \lim_{r \to 0} \mathbb{P}[\Gamma_r \in A] = \lim_{r \to 0} \mu_D^\sharp[B] = \mu_D^\sharp[B]$$

for all measurable sets $B \in \mathcal{F}$ of loop configurations. □

3.3.2 Uniform annulus CLE exploration

Let Γ_r be an annulus CLE_κ in $\mathbb{A}_r$. Fix $x \in \partial\mathbb{D}$ and consider the loops in Γ_r that intersect $B(x, \varepsilon)$. Suppose γ_r^ε is the largest of these loops, according to harmonic measure seen from the origin. Then we have the following counterpart of Proposition 3.2.3 (recall the definition $u(\varepsilon) := \mathbb{P}[\gamma^\varepsilon$ surrounds the origin$]$ from (3.2.2)):

Proposition 3.3.7. The law of γ_r^ε normalized by $1/u(\varepsilon)$ converges as $\varepsilon \to 0$ to a measure on loops in $\mathbb{A}_r$ pinned at x, which we denote by $\nu^{\text{bub}}(\mathbb{A}_r; x)$ and call **SLE bubble measure in $\mathbb{A}_r$ rooted at x**. Furthermore, the Radon-Nikodym derivative of $\nu^{\text{bub}}(\mathbb{A}_r; x)$ with respect to $\nu^{\text{bub}}(\mathbb{D}; x)$ is given by

$$\frac{d\nu^{\text{bub}}(\mathbb{A}_r; x)}{d\nu^{\text{bub}}(\mathbb{D}; x)}(\gamma) = 1_{\{\overline{\text{int}(\gamma)} \subset \mathbb{A}_r\}} \frac{p(\mathbb{A}_r \setminus \gamma)}{p(\mathbb{A}_r)} \exp(c\Lambda(r\mathbb{D}, \gamma; \mathbb{D})).$$

For a proof, we refer the reader to [68, Proposition 3.6].

3.4 CLE in the punctured disk

3.4.1 Existence and properties of CLE in $\mathbb{D} \setminus \{0\}$

Lemma 3.4.1. There exists a universal constant $C < \infty$ such that for all $0 < r' < r < \delta^2 < 1$ and $D \subset \mathbb{A}_\delta$ an annular domain, there exists a coupling between a CLE_κ Γ_r in $\mathbb{A}_r$ and a CLE_κ $\Gamma_{r'}$ in $\mathbb{A}_{r'}$ such that

$$\mathbb{P}[D_r^* \neq D_{r'}^*] \leq C\frac{\log(1/\delta)}{\log(1/r)},$$

where D_r^* (resp. $D_{r'}^*$) is the set obtained by removing from D all loops (and their interiors) of Γ_r (resp. $\Gamma_{r'}$) that are not contained in D, and that on the event $\{D_r^* = D_{r'}^*\}$, the collection of loops of Γ_r restricted to D_r^* is the same as the collection of loops of $\Gamma_{r'}$ restricted to $D_{r'}^*$.

Proof. Suppose $\mathcal{L}$ is a Brownian loop-soup in $\mathbb{A}_{r'}$. Denote by $\mathcal{L}_1$ the collection of loops of $\mathcal{L}$ that are contained in $\mathbb{A}_r$, and define $\mathcal{L}_2 = \mathcal{L} \setminus \mathcal{L}_1$. Note that $\mathcal{L}_1$ and $\mathcal{L}_2$ are independent. On the event $E(\mathcal{L})$, define Γ (resp. Γ_1) as the collection of outer boundaries of outmost clusters of $\mathcal{L}$ (resp. $\mathcal{L}_1$). Note that, conditioned on $E(\mathcal{L})$, Γ (resp. Γ_1) has the same law as CLE_κ in $\mathbb{A}_{r'}$ (resp. $\mathbb{A}_r$). Let D^* (resp. D_1^*) be the set obtained by removing from D all loops (and their interiors) of Γ (resp. Γ_1) that are not contained in D. Then by construction, we have $D^* \subset D_1^* \subset \mathbb{A}_\delta$.

Note that on the event $E(\mathcal{L})$, if no loop in $\mathcal{L}_2$ intersects D_1^*, then we have $D^* = D_1^*$. Define $S(\mathcal{L}_2, \mathbb{A}_\delta)$ to be the event that some loop in $\mathcal{L}_2$ intersects $\mathbb{A}_\delta$. Let E_1 be the event that no loop of Γ_1 disconnects C_r from C_1; and let E_2 be the event that $\mathcal{L}$ that no cluster contained in $A(r,r')$ disconnects $C_{r'}$ from C_r. We have

$$\begin{aligned}
&\mathbb{P}[\{D^* \neq D_1^*\} \cap E(\mathcal{L})]/p(r') \\
&\quad \leq \mathbb{P}[S(\mathcal{L}_2, \mathbb{A}_\delta) \cap E(\mathcal{L})]/p(r') \\
&\quad \leq \mathbb{P}[S(\mathcal{L}_2, \mathbb{A}_\delta) \cap E_1 \cap E_2]/p(r')
\end{aligned}$$

Since they are measurable with respect to disjoint sets of loops, E_1, E_2, and $S(\mathcal{L}_2, \mathbb{A}_\delta)$ are independent events, and the probabilities of E_1 and E_2 are $p(r)$ and $p(r'/r)$, respectively. Thus we have

$$\begin{aligned}
&\mathbb{P}[S(\mathcal{L}_2, \mathbb{A}_\delta), E_1, E_2]/p(r') \\
&\quad \leq \mathbb{P}[S(\mathcal{L}_2, \mathbb{A}_\delta)]p(r)p(r'/r)/p(r') \\
&\quad \lesssim \mathbb{P}[S(\mathcal{L}_2, \mathbb{A}_\delta)],
\end{aligned}$$

where the implied constant can be expressed in terms of the constant in Proposition 3.3.1 and is universal. So we only need to show that

$$\mathbb{P}[S(\mathcal{L}_2, \mathbb{A}_\delta)] \lesssim \frac{\log \delta^{-1}}{\log r^{-1}}.$$

Note that $S(\mathcal{L}_2, \mathbb{A}_\delta)$ is the same as the event that there exists loop in $\mathcal{L}$ intersecting both C_r and C_δ. The latter event has the probability

$$1 - \exp(-c\Lambda(C_r, C_\delta; \mathbb{A}_{r'})).$$

From Proposition 3.2.1, we have that

$$\begin{aligned}
\mathbb{P}[S(\mathcal{L}_2, \mathbb{A}_\delta)] &= 1 - \exp(-c\Lambda(C_r, C_\delta; \mathbb{A}_{r'})) \\
&\lesssim \Lambda(C_r, C_\delta; \mathbb{A}_{r'}) \\
&= \Lambda(C_r, C_\delta; \mathbb{C} \setminus r'\mathbb{D}) - \Lambda(C_r, C_1; \mathbb{C} \setminus r'\mathbb{D}) \\
&= 2\int_{r'}^{r} \frac{1}{s}\left(\rho\left(\frac{\delta}{s}\right) - \rho\left(\frac{1}{s}\right)\right) ds \\
&\lesssim \int_{r'}^{r} \frac{\log(1/\delta)}{s\left(\log\frac{1}{s}\right)^2} ds \\
&\lesssim \frac{\log(1/\delta)}{\log(1/r)}.
\end{aligned}$$

□

Theorem 3.4.2. For $\kappa \in (8/3, 4]$, there exists a unique measure on collections of disjoint simple loops contained (along with their interiors) in $\mathbb{D} \setminus \{0\}$, which we call **CLE$_\kappa$ in $\mathbb{D} \setminus \{0\}$**, to which CLE$_\kappa$ in $\mathbb{A}_r$ converges as $r \to 0$ in the following sense. There exists a universal constant $C < \infty$ such that for any $\delta > 0$ and any annular region $D \subset A_\delta$, a CLE$_\kappa$ $\Gamma^\dagger$ in $\mathbb{D} \setminus \{0\}$ and a CLE$_\kappa$ Γ_r in $\mathbb{A}_r$ can be coupled so that

$$\mathbb{P}[D^{\dagger,*} \neq D_r^*] \leq C\frac{\log(1/\delta)}{\log(1/r)},$$

where $D^{\dagger,*}$ (resp. D_r^*) is the set obtained by removing from D all the loops of $\Gamma^\dagger$ (resp. Γ_r), along with their interiors, that are not contained in D, and such that on the event $\{D^{\dagger,*} = D_r^*\}$ the collection of loops of $\Gamma^\dagger$ restricted to $D^{\dagger,*}$ is the same as the collection of loops of Γ_r restricted to D_r^*.

Proof. For $k \in \mathbb{N}$, define $r_k = 1/e^{e^k}$. For $k \geq 1$, suppose Γ_k is an annulus CLE$_\kappa$ in $\mathbb{A}_{r_k}$ and D_k^* is the set obtained by removing from D all loops of Γ_k that are not contained in D. From Lemma 3.4.1, Γ_k and Γ_{k+1} can be coupled so that the probability of $\{D_k^* \neq D_{k+1}^*\}$ is at most

$$C\log\left(\frac{1}{\delta}\right)e^{-k},$$

and on the event $D_k^* = D_{k+1}^*$, the collection of loops of Γ_k restricted to D_k^* is the same as the collection of loops of Γ_{k+1} restricted to D_{k+1}^*. Suppose that, for each $k \geq 1$, Γ_k and Γ_{k+1} are coupled in this way. Then with probability 1, for all but finitely many couplings, we have that $D_k^* = D_{k+1}^*$, so suppose that this is true for all $k \geq l$, and define, for $k \geq l$,

$$D^{\dagger,*} = D_k^*,$$

and define $\Gamma^\dagger$ restricted to $D^{\dagger,*}$ to be the collection of loops of Γ_k restricted to D_k^*.

Then, for any $k_0 \geq 1$, the probability of $D^{\dagger,*} \neq D^*_{k_0}$ is at most

$$\sum_{k \geq k_0} C \log\left(\frac{1}{\delta}\right) e^{-k} \lesssim \log\left(\frac{1}{\delta}\right) e^{-k_0}.$$

For any $r > 0$, suppose $r_{k_0} \leq r \leq r_{k_0-1}$, annulus CLE_κ Γ_r in $\mathbb{A}_r$ can be coupled with Γ_{k_0} so that the probability of $\{D^*_r \neq D^*_{k_0}\}$ is at most

$$C\frac{\log(1/\delta)}{\log(1/r)},$$

where D^*_r is the set obtained by removing from D all loops of Γ_r that are not contained in D. And on the event $\{D^*_r = D^*_{k_0}\}$, the collection of loops of Γ_r restricted to D^*_r is the same as the collection of loops of Γ_{k_0} restricted to $D^*_{k_0}$. Therefore, the probability that $D^{\dagger,*} \neq D^*_r$ is at most

$$C\log(1/\delta)e^{-k_0} + C\frac{\log(1/\delta)}{\log(1/r)} \lesssim \frac{\log(1/\delta)}{\log(1/r)}.$$

This completes the proof. □

Clearly, CLE_κ in $\mathbb{D} \setminus \{0\}$ is invariant under rotations $z \mapsto e^{i\theta}z$. We define CLE_κ in any domain conformally equivalent to a punctured disk as the conformal image of CLE_κ in $\mathbb{D} \setminus \{0\}$. The rotational invariance ensures that the resulting law does not depend on the choice of conformal map.

Propositions 3.4.3 and 3.4.4 describe the restriction property of CLE_κ in $\mathbb{D} \setminus \{0\}$.

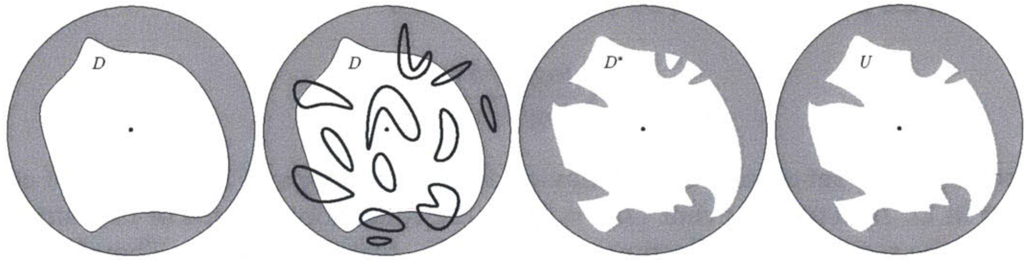
Proposition 3.4.3. Let $D \subset \mathbb{D}$ be domain which is either simply connected or annular, the closure of which does not contain the origin, and let $\Gamma^\dagger$ be a CLE_κ in $D \setminus \{0\}$. Define $D^{\dagger,*}$ to be the set obtained by removing from D all loops of $\Gamma^\dagger$ that are not contained in D. Then, conditionally on $D^{\dagger,*}$, for each connected component U of $D^{\dagger,*}$, the conditional law of the loops in $\Gamma^\dagger$ that stay in U is that of a CLE_κ in U.

Proof. The conclusion is direct consequence of the construction of conditioned CLE_κ in Theorem 3.4.2 and the restriction property of annulus CLE_κ in Proposition 3.3.3. □

Proposition 3.4.4. For any simply connected domain $D \subset \mathbb{D}$ such that $0 \in D$, let $D^{\dagger,*}$ be the set obtained by removing from D all loops of a CLE_κ $\Gamma^\dagger$ in $\mathbb{D} \setminus \{0\}$ that are not contained in D. If we denote by U the connected component of $D^{\dagger,*}$ that contains the origin, the conditional law given $D^{\dagger,*}$ of the loops in $\Gamma^\dagger$ that stay in U is the same as a CLE_κ in $U \setminus \{0\}$.

Proof. For $r > 0$ sufficiently small, define $D_r := D \cap \mathbb{A}_r$, and define D^*_r to be the set obtained by removing from D_r all loops of $\Gamma^\dagger$ that are not contained in D_r. Note that with probability tending to 1 in r, $\Gamma^\dagger$ has no loop intersecting both $\mathbb{D} \setminus D$ and

$r\mathbb{D}$. Suppose there is no such loop and let U_r be the connected component of D_r^* that is contained in U (see Figure 3-3). From Proposition 3.4.3, the collection of loops of $\Gamma^\dagger$ restricted to U_r has the same conditional law given U_r as a CLE_κ in U_r. To complete the proof, it suffices to note that $\text{mod}(U_r) \to \infty$ almost surely as $r \to 0$. □



(a) The first panel shows a domain D containing the origin. The second panel depicts a sample of CLE_κ in $\mathbb{D} \setminus \{0\}$. The third panel shows the corresponding set D^*, and the last panel shows the connected component U.

(b) The first panel shows the set $D_r = D \cap \mathbb{A}_r$. The second panel depicts the corresponding set D_r^*. The last panel shows the connected component U_r.

Figure 3-3: Restriction property of CLE_κ in $\mathbb{D} \setminus \{0\}$.

The following proposition describes the relationship between CLE_κ in $\mathbb{H} \setminus \{i\}$ and CLE_κ in $\mathbb{H}$: loops very far from the singular point i are look similar.

Proposition 3.4.5. Define $D = \mathbb{D} \cap \mathbb{H}$, and let $y > 0$. There exists a universal constant $C < \infty$ such that a CLE_κ $\Gamma_y^\dagger$ in $\mathbb{H} \setminus \{yi\}$ can be coupled with a CLE_κ Γ in $\mathbb{H}$ so that

$$\mathbb{P}[D^{\dagger,*} \neq D^*] \leq C/\log y,$$

where D^* (resp. $D_y^{\dagger,*}$) is the set obtained by removing from D all loops of Γ (resp. $\Gamma_y^\dagger$), along with their interiors, that are not contained in D. And on the event $\{D_y^{\dagger,*} = D^*\}$, the collection of loops of $\Gamma_y^\dagger$ restricted to $D_y^{\dagger,*}$ is the same as the collection of loops of Γ restricted to D^*.

Proof. Suppose Γ is a CLE_κin $\mathbb{H}$ and $\gamma(yi)$ is the loop in Γ that contains the point yi. We write $\overline{\gamma(yi)}$ to denote the union of the loop and its interior. We fix a constant $\eta > 1/\beta$, and set $R = y/(\log y)^\eta$.

From Proposition 3.3.6, we know that, given $\overline{\gamma(yi)}$, the collection of loops in Γ restricted to $\mathbb{H} \setminus \overline{\gamma(yi)}$, denoted as Γ_1, has the same law as annulus CLE. Given $\overline{\gamma(yi)}$ and on the event that $\overline{\gamma(yi)} \cap C_R = \emptyset$, we have $D^* = D_1^*$ where D^* (resp. D_1^*) is the set obtained by removing from D all loops of Γ (resp. Γ_1) that are not contained in D.

With an idea similar to the one used in the proof of Lemma 3.4.1, annulus CLE$_\kappa$ Γ_1 can be coupled with annulus CLE$_\kappa$ Γ_2 in $\mathbb{H} \setminus B(yi, 1)$ so that

$$\mathbb{P}[D_1^* \neq D_2^*] \leq C\Lambda(\overline{\gamma(yi)}, C_1; \mathbb{H} \setminus B(yi, 1)),$$

where D_2^* is the set obtained by removing from D all loops of Γ_2 that are not contained in D. On the event $\{\overline{\gamma(yi)} \cap C_R = \emptyset\}$, this quantity is less than

$$C\Lambda(C_R, C_1; \mathbb{H}).$$

And, by [28, Lemma 4.5], we have that

$$\Lambda(C_R, C_1; \mathbb{H}) \lesssim 1/\log R.$$

From Theorem 3.4.2, an annulus CLE Γ_2 can be coupled with CLE$_\kappa$ $\Gamma_y^\dagger$ in $\mathbb{H} \setminus \{yi\}$ so that

$$\mathbb{P}[D_2^* \neq D_y^{\dagger,*}] \leq C/\log y.$$

So we see that a CLE$_\kappa$ Γ in $\mathbb{H}$ and a CLE$_\kappa$ $\Gamma_y^\dagger$ in $\mathbb{H} \setminus \{yi\}$ can be coupled so that

$$\mathbb{P}[D^* \neq D_y^{\dagger,*}] \leq C\left(\left(\frac{R}{y}\right)^{-\beta+o(1)} + \frac{1}{\log y} + \frac{1}{\log R}\right) \lesssim 1/\log y. \qquad \square$$

3.4.2 Uniform exploration of CLE in $\mathbb{D} \setminus \{0\}$

We will explore CLE$_\kappa$ in $\mathbb{D} \setminus \{0\}$ in a manner similar to the uniform exploration of CLE$_\kappa$ in $\mathbb{D}$. Suppose $\Gamma^\dagger$ is a CLE$_\kappa$ in $\mathbb{D} \setminus \{0\}$. We choose $x \in \partial D$ and explore the loops of $\Gamma^\dagger$ that intersect $B(x, \varepsilon)$. Let $\gamma^{\dagger,\varepsilon}$ be the largest of these, in the sense of harmonic measure as seen from the origin. We have the following counterpart of Proposition 3.2.3 (recall the definition of $u(\varepsilon)$ in (3.2.2):

Proposition 3.4.6. The law of $\gamma^{\dagger,\varepsilon}$ normalized by $1/u(\varepsilon)$ converges to a measure, which we denote by $\nu^{\text{bub}}(\mathbb{D} \setminus \{0\}; x)$ and call the **SLE bubble measure in $\mathbb{D} \setminus \{0\}$ rooted at** x. The Radon-Nikodym derivative of $\nu^{\text{bub}}(\mathbb{D} \setminus \{0\}; x)$ with respect to $\nu^{\text{bub}}(\mathbb{D}; x)$ is given by

$$\frac{d\nu^{\text{bub}}(\mathbb{D} \setminus \{0\}; x)}{d\nu^{\text{bub}}(\mathbb{D}; x)}(\gamma) = 1_{E(\gamma)} \, \text{CR}(\mathbb{D} \setminus \gamma)^{-\alpha},$$

where $E(\gamma)$ is the event that γ does not surround the origin.

Proof. A combination of Proposition 3.3.7 and Lemma 3.2.5 implies the conclusion. □

Suppose γ is a loop in $\mathbb{D} \setminus \{0\} \cup \partial\mathbb{D}$ rooted at $x \in \partial D$, recall the definition of $R(\gamma)$ as the infimum of all values of $r > 0$ such that γ is contained in the disc centered at x with radius r. Recall the constants defined in (3.2.1); we have the following quantitative result for $\nu^{\text{bub}}(\mathbb{D} \setminus \{0\}; x)$:

Lemma 3.4.7. Whenever $\eta > \beta$, we have

$$\int R(\gamma)^{\eta} \nu^{\text{bub}}(\mathbb{D} \setminus \{0\}; x)(d\gamma) < \infty.$$

Furthermore, since $\beta \in [1, 2)$, we have

$$\int R(\gamma)^{2} \nu^{\text{bub}}(\mathbb{D} \setminus \{0\}; x)(d\gamma) < \infty.$$

Proof. The conclusion is direct consequence of Proposition 3.4.6 and Proposition 3.2.3. □

Now, we can describe the uniform exploration process of CLE_κ in $\mathbb{D} \setminus \{0\}$. Most of the proofs are similar to the proofs in [70] for CLE_κ in simply connected domains. For completeness we rewrite the proofs in the present setting.

Suppose $(\gamma_t^{\dagger}, t \geq 0)$ is a Poisson point process with intensity

$$\nu^{\text{bub}}(\mathbb{D} \setminus \{0\}) = \int_{\partial\mathbb{D}} \nu^{\text{bub}}(\mathbb{D} \setminus \{0\}; x)\, dx.$$

For $t \geq 0$, let $f_t^{\dagger}$ be the conformal map from $\mathbb{D} \setminus \gamma_t^{\dagger}$ onto $\mathbb{D}$ and is normalized at the origin. For any fixed $T > 0$ and $r > 0$, let $t_1(r) < ... < t_j(r)$ be the times t before T at which $R(\gamma_t^{\dagger})$ is greater than r. Define

$$\Psi_T^{\dagger, r} = f_{t_j(r)}^{\dagger} \circ \cdots \circ f_{t_1(r)}^{\dagger}.$$

Then $\Psi_T^{\dagger, r}$ converges as $r \to 0$:

Lemma 3.4.8. $\Psi_T^{\dagger, r}$ converges almost surely in the Carathéodory topology seen from the origin to some conformal map, denoted by $\Psi_T^{\dagger}$, as r goes to zero.

Proof. Lemma 3.4.7 guarantees that

$$\begin{aligned} & \mathbb{E}\left[\sum_{t<T} \text{cap}(\gamma_t^{\dagger}) 1_{R(\gamma_t^{\dagger}) \leq 1/2}\right] \\ = \; & T\nu^{\text{bub}}(\mathbb{D} \setminus \{0\}) \left(\text{cap}(\gamma^{\dagger}) 1_{R(\gamma^{\dagger}) \leq 1/2}\right) \\ \lesssim \; & T\nu^{\text{bub}}(\mathbb{D} \setminus \{0\}) \left(R(\gamma^{\dagger})^2 1_{R(\gamma^{\dagger}) \leq 1/2}\right) < \infty. \end{aligned}$$

Since there are only finitely many times t before T at which $R(\gamma_t^\dagger) > 1/2$, we have that, almost surely,

$$\sum_{t<T} \mathrm{cap}(\gamma_t^\dagger) < \infty,$$

and this implies the convergence in Carathéodory topology (see [70, Stability of Loewner chains]). □

Define $D_t^\dagger = (\Psi_t^\dagger)^{-1}(\mathbb{D})$ for $t \geq 0$. Then $(D_t^\dagger)_{t\geq 0}$ is a decreasing family of simply connected domains containing the origin, which we call the **uniform CLE$_\kappa$ exploration process in $\mathbb{D} \setminus \{0\}$**. We define $L_t^\dagger = (\Psi_t^\dagger)^{-1}(\gamma_t^\dagger)$ for $t \geq 0$. It is clear that

$$D_t^\dagger = \bigcap_{s\leq t} D_s^\dagger, \quad D_{t+}^\dagger := \bigcup_{s>t} D_s^\dagger = D_t^\dagger \setminus L_t^\dagger.$$

Suppose $(\gamma_t, t \geq 0)$ is a Poisson point process with intensity $\nu^{\mathrm{bub}}(\mathbb{D})$ and that $(D_t, t \leq \tau)$ is the uniform CLE$_\kappa$ exploration process in $\mathbb{D}$ defined from the process $(\gamma_t, t \geq 0)$ in the manner described in Section 3.2.3. Define, for $\eta > 0$,

$$\theta(\eta) := \int (e^{\eta\, \mathrm{cap}(\gamma)} - 1) 1_{E(\gamma)} \nu^{\mathrm{bub}}(\mathbb{D})(d\gamma), \tag{3.4.1}$$

where $E(\gamma)$ is the event that γ does not surround the origin. Then the relationship between the process $(D_t^\dagger, t \geq 0)$ and the process $(D_t, t \leq \tau)$ is described in Proposition 3.4.11 below. First, however, we show that $\theta(\alpha)$ is finite and positive.

Lemma 3.4.9. The quantity $\theta(\eta)$, which is defined in Equation (3.4.1), is finite whenever $\eta \leq 1 - \kappa/8$. In particular, $\theta(\alpha)$ is finite.

Proof. The integral may blow up when $R(\gamma)$ is small or when γ is close to the origin. We will control the two parts separately.

To handle the integral over the set of loops γ for which $R(\gamma)$ is small, we calculate

$$\begin{aligned} &\int (e^{\eta\, \mathrm{cap}(\gamma)} - 1) 1_{R(\gamma)\leq 1/2} \nu^{\mathrm{bub}}(\mathbb{D})(d\gamma) \\ &\quad \lesssim \int \mathrm{cap}(\gamma) 1_{R(\gamma)\leq 1/2} \nu^{\mathrm{bub}}(\mathbb{D})(d\gamma) \\ &\quad \lesssim \int R(\gamma)^2 1_{R(\gamma)\leq 1/2} \nu^{\mathrm{bub}}(\mathbb{D})(d\gamma) < \infty. \end{aligned}$$

For the set of loops passing near the origin.

$$\int (e^{\eta\, \mathrm{cap}(\gamma)} - 1) 1_{\{R(\gamma)>1/2\}} 1_{E(\gamma)} \nu^{\mathrm{bub}}(\mathbb{D})(d\gamma) \leq \int \mathrm{CR}(\mathbb{D}\setminus\gamma)^{-\eta} 1_{\{R(\gamma)>1/2\}} 1_{E(\gamma)} \nu^{\mathrm{bub}}(\mathbb{D})(d\gamma).$$

Conditioned on $\{R(\gamma) > 1/2\} \cap E(\gamma)$, we can parameterize γ clockwise by the capacity seen from the origin starting from the root and ending at the root: $(\gamma(t), 0 \leq t \leq T)$. Suppose S is the first time that γ exits the ball $B(x, 1/2)$ where $x \in \partial\mathbb{D}$ is the root of γ. Then we know that, given $\gamma[0, S]$, the future part of the curve $\gamma[S, T]$

has the same law as a chordal SLE in $\mathbb{D} \setminus \gamma[0,S]$ from $\gamma(S)$ to the root of γ. Thus we only need to show that the integral is finite when we replace the curve by chordal SLE curve.

Precisely, suppose $\gamma = (\gamma_t, t \geq 0)$ is a chordal SLE in the upper-half plane $\mathbb{H}$ from 0 to ∞ (parameterized by the half-plane capacity), we only need to show that

$$\mathbb{E}[\mathrm{CR}(\mathbb{H} \setminus \gamma; i)^{\kappa/8-1}] < \infty \tag{3.4.2}$$

where $\mathrm{CR}(\mathbb{H} \setminus \gamma; i)$ is the conformal radius of $\mathbb{H} \setminus \gamma$ seen from i. Suppose g_t is the conformal map from $\mathbb{H} \setminus \gamma[0,t]$ onto $\mathbb{H}$ normalized at infinity, so that $(g_t(z) - z)z \to 2t$ as $z \to \infty$. And let W_t be the image of the tip $\gamma(t)$ under g_t. Define

$$Z_t = g_t(i) - W_t, \quad \Theta_t = \arg Z_t, \quad S_t = \sin \Theta_t.$$

And define

$$M_t = \mathrm{CR}(\mathbb{H} \setminus \gamma[0,t]; i)^{\kappa/8-1} \times S_t^{8/\kappa-1}.$$

Then M_t is a local martingale (see [80, Proposition 6.1]). Denote $\mathbb{P}^*$ as the law of chordal SLE weighted by the martingale M. We also know that $\mathbb{E}^*[S_t^{1-8/\kappa}]$ is bounded from above by some universal constant C (see [80, Equation (6.9)]). Thus

$$\mathbb{E}[\mathrm{CR}(\mathbb{H} \setminus \gamma; i)^{\kappa/8-1}] \leq C,$$

which completes the proof. □

Corollary 3.4.10. The quantity

$$\int (e^{\eta \, \mathrm{cap}(\gamma^\dagger)} - 1) \nu^{\mathrm{bub}}(\mathbb{D} \setminus \{0\})(d\gamma^\dagger) \tag{3.4.3}$$

is finite as long as $\eta \leq 2/\kappa - \kappa/32$.

Proof. The combination of Proposition 3.4.6 and the proof of Lemma 3.4.9 gives the result that the quantity in Equation (3.4.3) is finite as long as $\eta + \alpha \leq 1 - \kappa/8$. □

Proposition 3.4.11. For any $t > 0$, the law of $(\gamma_s^\dagger, s < t)$ is the same as the law of $(\gamma_s, s < t)$ conditioned on $(\tau \geq t)$ and weighted by M_t where

$$M_t = \exp\left(\alpha \sum_{s<t} \mathrm{cap}(\gamma_s) - \theta(\alpha) t \right). \tag{3.4.4}$$

In particular, for any $t > 0$, the law of $D_t^\dagger$ is the same as the law of D_t conditioned on $(\tau \geq t)$ and weighted by

$$\mathrm{CR}(D_t)^{-\alpha} e^{-\theta(\alpha) t}.$$

Proof. We first note that the process $(\gamma_s, s < t)$ conditioned on $(\tau \geq t)$ has the same law as a Poisson point process with intensity $1_{E(\gamma)} \nu^{\mathrm{bub}}(\mathbb{D})$ restricted to the time interval $[0,t)$. Suppose $(\widehat{\gamma}_s, s \geq 0)$ is a Poisson point process with intensity $1_{E(\gamma)}\nu^{\text{bub}}(\mathbb{D})$, and define

$$\widehat{M}_t = \exp\left(\alpha \sum_{s<t} \text{cap}(\widehat{\gamma}_s) - \theta(\alpha)t\right).$$

So we only need to show that, for any function f on the set of bubbles such that both of the following integrals are finite,

$$\mathbb{E}\left[\exp\left(-\sum_{s<t} f(\widehat{\gamma}_s)\right)\widehat{M}_t\right] = \exp\left(-t\int(1-e^{-f(\gamma)})e^{\alpha\,\text{cap}(\gamma)}1_{E(\gamma)}\nu^{\text{bub}}(\mathbb{D})(d\gamma)\right).$$

This can be obtained by direct calculation:

$$\begin{aligned}
&-\log\mathbb{E}\left[\exp(-\sum_{s<t} f(\widehat{\gamma}_s))\widehat{M}_t\right] \\
&= -\log\mathbb{E}\left[\exp\left(-\sum_{s<t}(f(\widehat{\gamma}_s)-\alpha\,\text{cap}(\widehat{\gamma}_s))\right)\right] + \theta(\alpha)t \\
&= t\int(1-e^{-f(\gamma)+\alpha\,\text{cap}(\gamma)})1_{E(\gamma)}\nu^{\text{bub}}(\mathbb{D})(d\gamma) + \theta(\alpha)t \\
&= t\int(1-e^{-f(\gamma)})e^{\alpha\,\text{cap}(\gamma)}1_{E(\gamma)}\nu^{\text{bub}}(\mathbb{D})(d\gamma).
\end{aligned}$$ □

The following proposition describes the transience of the uniform exploration process of CLE_κ in $\mathbb{D}\setminus\{0\}$.

Proposition 3.4.12. Suppose $(D_t^\dagger, t \geq 0)$ is a uniform CLE_κ exploration process in $\mathbb{D}\setminus\{0\}$. Denote by $R(D_t^\dagger)$ the least value of R such that $D_t^\dagger$ is contained in $R\mathbb{D}$. Then, almost surely,

$$R(D_t^\dagger) \to 0, \quad \text{as } t \to \infty.$$

Proof. Suppose $(D_t, t \leq \tau)$ is a uniform CLE exploration process in the unit disk. From Proposition 3.4.11, we have that

$$\mathbb{P}[\partial D_{T+}^\dagger \cap \partial\mathbb{D} = \emptyset] \geq \mathbb{P}[\partial D_T^\dagger \cap \partial\mathbb{D} = \emptyset] \geq e^{-\theta(\alpha)T}\mathbb{P}[\partial D_T \cap \partial\mathbb{D} = \emptyset \mid \tau \geq T] > 0.$$

Define $D_1^\dagger = D_{T+}^\dagger$; then there exist $r \in (0,1)$ and $p > 0$ such that

$$\mathbb{P}[D_1^\dagger \subset r\mathbb{D}] \geq p.$$

For $k \geq 1$, define

$$T_k = kT, \quad D_k^\dagger = D_{T_k+}^\dagger,$$

and let φ_k be the conformal map from $D_k^\dagger$ onto $\mathbb{D}$ normalized at the origin: $\varphi_k(0) =$

$0, \varphi_k'(0) > 0$. For $k \geq 1$, the events $\{\varphi_k(D^\dagger_{k+1}) \subset r\mathbb{D}\}$ are i.i.d. Thus

$$\sum_k \mathbb{P}[\varphi_k(D^\dagger_{k+1}) \subset r\mathbb{D}] = \infty.$$

Thus, almost surely, there exists a sequence $K_j \to \infty$ such that

$$\varphi_{K_j}(D^\dagger_{K_j+1}) \subset r\mathbb{D}, \quad \text{for all } j.$$

Since $D^\dagger_{K_{j+1}} \subset D^\dagger_{K_j+1}$, we have that

$$\varphi_{K_j}(D^\dagger_{K_{j+1}}) \subset r\mathbb{D}, \quad \text{for all } j.$$

Then we can prove by induction that $D^\dagger_{K_j} \subset r^{j-1}\mathbb{D}$: Suppose it is true for some $j \geq 1$. Since φ_{K_j} is the conformal map from $D^\dagger_{K_j} \subset r^{j-1}\mathbb{D}$ onto $\mathbb{D}$, let ψ_1 be the conformal map from $D^\dagger_{K_j}$ onto $r^{j-1}\mathbb{D}$ normalized at the origin and let ψ_2 be the conformal map from $r^{j-1}\mathbb{D}$ onto $\mathbb{D}$ normalized at the origin (in fact, $\psi_2(z) = z/r^{j-1}$). Then $\varphi_{K_j} = \psi_2 \circ \psi_1$. Thus

$$D^\dagger_{K_{j+1}} \subset (\varphi_{K_j})^{-1}(r\mathbb{D}) = \psi_1^{-1} \circ \psi_2^{-1}(r\mathbb{D}) = \psi_1^{-1}(r^j\mathbb{D}) \subset r^j\mathbb{D}.$$

The last relation follows from the inequality $|\psi_1(z)| \geq |z|$ for all z.

We have proved that, almost surely, there exists a sequence $K_j \to \infty$ such that $D^\dagger_{K_j} \subset r^j\mathbb{D}$. Since the sequence of domains $(D^\dagger_t, t \geq 0)$ is decreasing, this implies the conclusion. □

We conclude this section by explaining how the loops $(L^\dagger_t, t \geq 0)$ obtained from the point process of bubbles $(\gamma^\dagger_t, t \geq 0)$ (the Poisson point process with intensity $\nu^{\text{bub}}(\mathbb{D} \setminus \{0\})$) correspond to the loops in a CLE. We first remove from $\mathbb{D}$ all loops $L^\dagger_t$ (with their interiors) for $t \geq 0$, and then, in each connected component, sample independent CLEs. We will argue that the collection of these loops from CLE together with the sequence $(L^\dagger_t, t \geq 0)$ has the same law as the collection of loops in a CLE in $\mathbb{D} \setminus \{0\}$. The idea is very similar to the one used in [70, Section 7] to show that the loops obtained from the bubbles have the same law as the loops in CLE in $\mathbb{D}$.

Suppose $\Gamma^\dagger$ is a CLE in $\mathbb{D} \setminus \{0\}$. Fix a point $z \in \mathbb{D} \setminus \{0\}$. Let $L^\dagger(z)$ be the loop in $\Gamma^\dagger$ that contains z. We will describe a discrete exploration of $\Gamma^\dagger$ for discovering $L^\dagger(z)$. Fix $\varepsilon > 0$ small and $\delta > \varepsilon$ small. Sample $x_1 \in \partial\mathbb{D}$ uniformly from the circle. The loops of $\Gamma^\dagger$ that intersect $B(x_1, \varepsilon)$ are the loops we discovered. Call the connected component of the remaining domain that contains the origin the to-be-explored domain and let $f_1^{\dagger,\varepsilon}$ be the conformal map from the to-be-explored domain onto the unit disc normalized at the origin. Let $\gamma_1^{\dagger,\varepsilon}$ be the discovered loop with largest harmonic measure seen from the origin. The image of the loops in the to-be-explored domain under $f_1^{\dagger,\varepsilon}$ has the same law as CLE. Thus we can repeat the same procedure, define $f_2^{\dagger,\varepsilon}, \gamma_2^{\dagger,\varepsilon}$ etc. For $k \geq 1$, define

$$\Phi_k^{\dagger,\varepsilon} = f_k^{\dagger,\varepsilon} \circ \cdots \circ f_1^{\dagger,\varepsilon}.$$

We also need to keep track of the point z: let $z_k = \Phi_k^{\dagger,\varepsilon}(z)$, and let K be the largest k such that $z \in (\Phi_k^{\dagger,\varepsilon})^{-1}(\mathbb{D})$. Define another auxiliary stopping time $K' \leq K$ as the first step k at which either $|z_k| \geq 1 - \delta$ or $k = K$. If $K' < K$, this means that the point z is conformally far from the origin and is likely to be cut off in the discrete exploration. We first address the case that z is discovered at the step $K + 1$. Note that $\Phi_K^{\dagger,\varepsilon}$ will converge in distribution to some conformal map $\Psi_S^\dagger$ (for reasons analogous to those given in the proof of Proposition 3.2.4) obtained from the Poisson point process $(\gamma_t^\dagger, t \geq 0)$. This implies that $L^\dagger(z)$ has the same law as $(\Psi_S^\dagger)^{-1}(\gamma_S^\dagger)$, as we expected.

Next we deal with the case that z is cut off from the origin: we stop the discrete exploration at step $K' - 1$. At the step K', instead of discovering the loops intersecting the ball of radius ε, we discover the loops intersecting the circle with radius $\sqrt{\delta}$. After this step, we continue the discrete exploration (of size ε) by targeting the image of the point z, following the discrete CLE exploration procedure. We discover the point z at some step. Letting ε and δ tend to zero appropriately, we can also prove the conclusion for $L^\dagger(z)$ in this case.

More generally, for fixed $z_1, ..., z_k \in \mathbb{D} \setminus \{0\}$, we also need to demonstrate the conclusion for the joint law of $(L^\dagger(z_1), ..., L^\dagger(z_k))$. The argument is almost the same as above and is omitted.

3.5 CLE in the punctured plane

3.5.1 Existence and properties of CLE in the punctured plane

In this section we discuss CLE in the final standard doubly connected planar domain: the punctured plane. The following lemma is analogous to Lemma 3.4.1. For a proof, we refer the reader to [68, Lemma 5.1].

Lemma 3.5.1. There exists universal constant $C < \infty$ such that the following is true. For any $\delta \in (0,1), 0 < r' < r < \delta^2$, and annular region $D \subset \mathbb{A}(\delta, \frac{1}{\delta})$, there exists a coupling between an annulus CLE_κ Γ_r in $\mathbb{A}(r, \frac{1}{r})$ and an annulus CLE_κ $\Gamma_{r'}$ in $\mathbb{A}(r', \frac{1}{r'})$ such

$$\mathbb{P}[D_r^* = D_{r'}^*] \leq C\frac{\log(1/\delta)}{\log(1/r)},$$

where D_r^* (resp. $D_{r'}^*$) is the set obtained by removing from D all loops (and their interiors) of Γ_r (resp. $\Gamma_{r'}$) that are not contained in D. And on the event $D_r^* = D_{r'}^*$, the collection of loops of Γ_r restricted to D_r^* is the same as the collection of loops of $\Gamma_{r'}$ restricted to $D_{r'}^*$.

The following theorem, analogous to Theorem 3.4.2, establishes the existence of CLE in the punctured plane.

Theorem 3.5.2. For $\kappa \in (8/3, 4]$, there exists a unique measure on collections of disjoint simple loops in $\mathbb{C} \setminus \{0\}$, which we call **CLE$_\kappa$ in** $\mathbb{C} \setminus \{0\}$, to which CLE$_\kappa$ in $\mathbb{A}(r, \frac{1}{r})$ converges in the following sense. There exists universal constant $C < \infty$ such that for any $\delta > 0$ and for any annulus $D \subset A(\delta, \frac{1}{\delta})$, one can couple a CLE$_\kappa$ $\Gamma^\dagger$ in $\mathbb{C} \setminus \{0\}$ and a CLE$_\kappa$ Γ_r in $\mathbb{A}(r, \frac{1}{r})$ so that

$$\mathbb{P}[D^{\dagger,*} \neq D_r^*] \leq C \frac{\log(1/\delta)}{\log(1/r)},$$

where $D^{\dagger,*}$ (resp. D_r^*) is the set obtained by removing from D all loops of $\Gamma^\dagger$ (resp. Γ_r) that are not contained in D, and so that on the event $\{D^{\dagger,*} = D_r^*\}$, the collection of loops of $\Gamma^\dagger$ restricted to $D^{\dagger,*}$ is the same as the collection of loops of Γ_r restricted to D_r^*.

It is clear that CLE$_\kappa$ in $\mathbb{C} \setminus \{0\}$ can also be viewed as the limit of CLE$_\kappa$ in $R\mathbb{D} \setminus \{0\}$ as $R \to \infty$ or the limit of CLE$_\kappa$ in $\mathbb{C} \setminus r\mathbb{D}$ as $r \to 0$. Theorem 3.5.2 may be proved by making small modifications to the proof of Theorem 3.4.2.

The following proposition is a direct consequence of the construction given in Theorem 3.5.2.

Proposition 3.5.3. CLE$_\kappa$ in $\mathbb{C} \setminus \{0\}$ is invariant under the conformal maps
1. $z \mapsto \lambda z$, for all $\lambda \in \mathbb{C}$, and
2. $z \mapsto 1/z$.

3.5.2 Uniform exploration of CLE in the punctured plane

For $R > 1$, suppose $(D_t^{\dagger,R}, t \geq 0)$ is a uniform CLE$_\kappa$ exploration process in $R\mathbb{D} \setminus \{0\}$ with $D_0^{\dagger,R} = R\mathbb{D}$. For $s \in \mathbb{R}$, define

$$T_s^R = \inf\{t : \mathrm{CR}(D_{t+}^{\dagger,R}) \leq e^{-s}\}, \quad \mathcal{D}_s^\dagger(R) = D_{T_s^R+}^{\dagger,R}. \tag{3.5.1}$$

We abbreviate $T^R := T_0^R$ and $\mathcal{D}^\dagger(R) := \mathcal{D}_0^\dagger(R)$.

Lemma 3.5.4. For all $s \in \mathbb{R}$, there exists a decreasing function $\delta(R)$ such that $\delta(R) \to 0$ as $R \to \infty$ and that

$$\mathbb{P}[\mathcal{D}_s^\dagger(\tilde{R}) \subset R\mathbb{D}] \geq 1 - \delta(R) \quad \text{as long as} \quad \tilde{R} \geq R^2.$$

Proof. Without loss of generality, we may assume $s = 0$. From the scale invariance and the transience of the uniform exploration process (Proposition 3.4.12), there exists a decreasing function $\delta_1(R)$ such that $\delta_1(R) \to 0$ as $R \to \infty$ and that

$$\mathbb{P}\left[\mathcal{D}^\dagger(R) \subset \frac{1}{4} R\mathbb{D}\right] \geq 1 - \delta_1(R). \tag{3.5.2}$$

Let $R \gg 1$, and choose $\tilde{R} \geq R^2$. Suppose $(\tilde{D}_t^\dagger, t \geq 0)$ is a uniform CLE_κ exploration in $\tilde{R}\mathbb{D} \setminus \{0\}$. Define

$$\begin{aligned}\tilde{T} &= \inf\{t : \mathrm{CR}(\tilde{D}_{t+}^\dagger) \leq R\},\\ \tilde{D}_R^\dagger &= \tilde{D}_{\tilde{T}+}^\dagger, \quad \text{and}\\ \mathcal{R} &= \mathrm{CR}(\tilde{D}_R^\dagger).\end{aligned}$$

And let φ be the conformal map from $\mathcal{R}\mathbb{D}$ onto $\tilde{D}_R^\dagger$ normalized at the origin so that $\varphi(0) = 0$ and $\varphi'(0) = 1$.

Let $(D_t^\dagger, t \geq 0)$ be an independent uniform CLE_κ exploration in $\mathcal{R}\mathbb{D} \setminus \{0\}$, and define $\mathcal{D}^\dagger(\mathcal{R})$ as in (3.5.1) with respect to $(D_t^\dagger, t \geq 0)$. Then we have that

$$\varphi(\mathcal{D}^\dagger(\mathcal{R})) \stackrel{d}{=} \mathcal{D}^\dagger(\tilde{R}).$$

To show the conclusion, we only need to control the domain $\varphi(\mathcal{D}^\dagger(\mathcal{R}))$.

The process $(-\log \mathrm{CR}(\tilde{D}_t^\dagger), t \geq 0)$ is a compound Poisson process starting from $-\log \tilde{R}$. Furthermore, its jump distribution is absolutely continuous with respect to Lebesgue, because by Proposition 3.4.6 its jump distribution is absolutely continuous with respect to the log conformal radius jump distribution for the uniform CLE exploration in the disk. Therefore, by Proposition 3.2.6 and Corollary 3.4.10, there exists a positive constant $\eta > 0$ such that

$$\mathbb{P}[\mathcal{R} \geq \rho R] \geq 1 - \rho^\eta, \quad \text{for all} \quad \rho \in [0, 1/2],\ \tilde{R} \geq R^2. \tag{3.5.3}$$

We will fix ρ later. Set

$$E_1 = [\mathcal{R} \geq \rho R].$$

From (3.5.2), we have that on E_1,

$$\mathbb{P}\left[\mathcal{D}^\dagger(\mathcal{R}) \subset \frac{1}{4}\mathcal{R}\mathbb{D}\right] \geq 1 - \delta_1(\rho R) \quad \text{for all } \mathcal{R}. \tag{3.5.4}$$

Set

$$E_2 = \left\{\mathcal{D}^\dagger(\mathcal{R}) \subset \frac{1}{4}\mathcal{R}\mathbb{D}\right\}.$$

By Corollary 2.2.16,

$$|\varphi(z) - z| \leq 4|z|^2/\mathcal{R}, \quad \text{for all } |z| \leq \mathcal{R}/4. \tag{3.5.5}$$

Thus, on E_2 we have

$$\varphi(\mathcal{D}^\dagger(\mathcal{R})) \subset R\mathbb{D}.$$

Combining (3.5.3) and (3.5.4), we have that

$$\mathbb{P}[\mathcal{D}^\dagger(\tilde{R}) \subset R\mathbb{D}] \geq \mathbb{P}[E_1 \cap E_2] \geq 1 - \rho^\eta - \delta_1(\rho R).$$

The conclusion is proved by setting $\rho = \log R / R$, and $\delta(R) = \rho^\eta + \delta_1(\rho R)$. □

Lemma 3.5.5. For all $s \in \mathbb{R}$, there exists a decreasing function $\delta(R)$ such that $\delta(R) \to 0$ as $R \to \infty$ and that the following is true. There exists a coupling between uniform exploration processes in $R_1\mathbb{D}$ and $R_2\mathbb{D}$ with corresponding domains $\mathcal{D}_s^\dagger(R_1)$ and $\mathcal{D}_s^\dagger(R_2)$ such that the conformal map ψ from $\mathcal{D}_s^\dagger(R_1)$ onto $\mathcal{D}_s^\dagger(R_2)$, normalized at the origin so that $\psi(0) = 0$ and $\psi'(0) > 0$, satisfies

$$\mathbb{P}\left[\|\psi - \mathrm{id}\|_\infty \le 1/R\right] \ge 1 - \delta(R), \tag{3.5.6}$$

where $\|f - g\|_\infty$ is the supremum norm for two conformal maps defined on the closure of $D = \mathcal{D}_s^\dagger(R_1)$:

$$\|f - g\|_\infty = \sup\{|f(z) - g(z)| : z \in \overline{D}\}.$$

Proof. Without loss of generality we assume $s = 0$. Suppose $R_1, R_2 \ge R^8$, where $R \ge 128$. Set $\tilde{R} = R^4$. For $i \in \{1, 2\}$, let $(D_t^{\dagger,i}, t \ge 0)$ be a uniform CLE_κ exploration process in $R_i\mathbb{D} \setminus \{0\}$. Define

$$\begin{aligned} T^i &= \inf\{t : \mathrm{CR}(D_{t+}^{\dagger,i}) \le \tilde{R}\}, \\ D^{\dagger,i} &= D_{T^i+}^{\dagger,i}, \text{ and} \\ \mathcal{R}_i &= \mathrm{CR}(D^{\dagger,i}). \end{aligned}$$

Let φ_i be the conformal map from $\mathcal{R}_i\mathbb{D}$ onto $D^{\dagger,i}$ normalized at the origin so that $\varphi_i(0) = 0$ and $\varphi_i'(0) = 1$.

By Proposition 3.2.6, there exists $\eta > 0$ such that there is a coupling between $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfying

$$\mathbb{P}[\mathcal{R}_1 = \mathcal{R}_2] \ge 1 - \tilde{R}^{-\eta}. \tag{3.5.7}$$

We couple $(D_{t+}^{\dagger,1}, 0 \le t \le T^1)$ and $(D_{t+}^{\dagger,2}, 0 \le t \le T^2)$ so that (3.5.7) is satisfied, and we define $E_1 = \{\mathcal{R}_1 = \mathcal{R}_2\}$. On E_1, denote $\mathcal{R} = \mathcal{R}_1 = \mathcal{R}_2$. Set $\rho = 64/R$ and

$$E_2 = \{\mathcal{R} \ge \rho\tilde{R}\}.$$

We also have that

$$\mathbb{P}[E_2] \ge 1 - \rho^\eta. \tag{3.5.8}$$

On $E_1 \cap E_2$, let $(D_t^\dagger, t \ge 0)$ be an independent uniform CLE_κ exploration in $\mathcal{R}\mathbb{D} \setminus \{0\}$, and let $\mathcal{D}^\dagger(\mathcal{R})$ be the domain defined in (3.5.1) with respect to the exploration $(D_t^\dagger, t \ge 0)$. We couple $(D_t^{\dagger,1}, t \ge T^1)$ and $(D_t^{\dagger,2}, t \ge T^2)$ so that each is the conformal image of $(D_t^\dagger, t \ge 0)$ on the event $E_1 \cap E_2$, and we couple them as independent explorations on the complement of $E_1 \cap E_2$. Set

$$E_3 = [\mathcal{D}^\dagger(\mathcal{R}) \subset R\mathbb{D}].$$

From Lemma 3.5.4, there exists a decreasing function $\delta_1(R)$ such that $\delta_1(R)$ goes to zero as $R \to \infty$ and, on $E_1 \cap E_2$, we have

$$\mathbb{P}[E_3] \geq 1 - \delta_1(R), \tag{3.5.9}$$

since $\rho\tilde{R} \geq R^2$. For $i \in \{1,2\}$,

$$\varphi_i(\mathcal{D}^\dagger(\mathcal{R})) \stackrel{d}{=} \mathcal{D}^\dagger(R_i).$$

Set $\psi = \varphi_2 \circ \varphi_1^{-1}$. Then ψ is the conformal map from $D^{\dagger,1}$ onto $D^{\dagger,2}$ normalized at the origin and

$$\psi(\varphi_1(\mathcal{D}^\dagger(\mathcal{R}))) = \varphi_2(\mathcal{D}^\dagger(\mathcal{R})).$$

Thus we only need to show that ψ satisfies (3.5.6).

First, on $E_1 \cap E_2 \cap E_3$, since

$$|\varphi_1(z) - z| \leq 4|z|^2/(\rho\tilde{R}), \quad \text{for all} \quad |z| \leq R,$$

we know that

$$\varphi_1(\mathcal{D}^\dagger(\mathcal{R})) \subset 2R\mathbb{D}.$$

Second, on $E_1 \cap E_2 \cap E_3$, by the Koebe 1/4 Theorem, we have that

$$D^{\dagger,1} \supset \frac{1}{4}\rho\tilde{R}\mathbb{D} = 16R^3\mathbb{D}.$$

By (3.5.5), we have

$$|\psi(z) - z| \leq \frac{4|z|^2}{16R^3} \quad \text{for all} \quad |z| \leq 2R.$$

Combining these, we have that

$$|\psi(z) - z| \leq 1/R, \quad \text{for all} \quad z \in \varphi_1(\mathcal{D}^\dagger(\mathcal{R})).$$

Finally, we combine (3.5.7), (3.5.8), and (3.5.9) to obtain

$$\mathbb{P}\left[|\psi(z) - z| \leq 1/R \;\; \forall z \in \varphi_1(\mathcal{D}^\dagger(\mathcal{R}))\right] \geq \mathbb{P}[E_1 \cap E_2 \cap E_3] \geq 1 - \tilde{R}^{-\eta} - \rho^\eta - \delta_1(R),$$

which completes the proof. □

Theorem 3.5.6. There exists a coupling of a CLE$_\kappa$ Γ in $\mathbb{C} \setminus \{0\}$ and a random family of domains $(D_t)_{t\in\mathbb{R}}$, which we call the **uniform exploration of** CLE **in the punctured plane**, such that

(i) for all $t \in \mathbb{R}$, the restriction $\Gamma|_{D_t}$ is a CLE$_\kappa$ in D_t,
(ii) $(D_{t+s})_{s\geq 0}$ is a uniform exploration of $\Gamma|_{D_t}$ in D_s, and
(iii) $\mathrm{CR}(D_t) \leq 1$ if and only if $t \geq 0$.

Remark 3.5.7. Conditions (i) and (ii) in Theorem 3.5.6 determine the law of $(D_t)_{t\in\mathbb{R}}$ up to translations of the form $(D_t)_{t\in\mathbb{R}} \mapsto (D_{t+t_0})_{t\in\mathbb{R}}$ for $t_0 \in \mathbb{R}$, and condition (iii) is included to remove this ambiguity.

Proof. Suppose $\delta(R)$ is the decreasing function in Lemma 3.5.5. For $k \geq 1$, define R_k inductively so that $\delta(R_k) \leq 2^{-k}$, $R_k \geq 2^k$, and $R_{k+1} \geq R_k^8$.

By the proof of Lemma 3.5.5, for all $k \geq 1$ there exists a coupling between a uniform exploration in $R_{k+1}\mathbb{D}$ and a uniform exploration in $R_{k+2}\mathbb{D}$ so that the probability of the event E_k is at least $1 - 2^{-k}$, where E_k is the intersection of the events

- $\mathcal{D}^\dagger(R_{k+1}) \subset 2R_k\mathbb{D}$, and

- there exists a conformal map ψ defined on $16R_k^3\mathbb{D}$, normalized at the origin, that maps $\mathcal{D}^\dagger(R_{k+1})$ onto $\mathcal{D}^\dagger(R_{k+2})$ and satisfies

$$|\psi_k(z) - z| \leq R_k^{-1} \quad \text{for all} \quad |z| \leq 2R_k.$$

Since $\sum_{k=1}^{\infty} 2^{-k} < \infty$, the Borel-Cantelli lemma implies that there almost surely exists a positive integer l such that E_k holds for all $k \geq l$. For $m \geq l$, define

$$F_m = \psi_m \circ \cdots \circ \psi_l.$$

Clearly, for $m \geq l$ and $n \geq 1$ we have

$$|F_m(z) - z| \leq 2/R_l, \quad \text{for all } |z| \leq 2R_l\mathbb{D}, \text{ and}$$

$$|F_{m+n}(z) - F_m(z)| \leq 2/R_m, \quad \text{for all } |z| \leq 2R_l\mathbb{D}.$$

Thus F_m converges uniformly to a map F_∞ on $2R_l\mathbb{D}$. Set

$$D_0 := \mathcal{D}^\dagger(\infty) := F_\infty(\mathcal{D}^\dagger(R_{l+1})).$$

Our coupling is consistent in that D_0 does not depend on the choice of l. For $t \geq 0$, define $(D_t)_{t\geq 0}$ to be the image of the exploration in D_0 under F_∞.

To define D_t for negative values of t, we choose a monotone sequence s_n tending to $-\infty$ and repeat the above construction (using the same coupling) with $\mathcal{D}^\dagger_{s_n}$ in place of $\mathcal{D}^\dagger$. In this way we obtain for each $n \geq 1$ a domain $\mathcal{D}^\dagger_{s_n}(\infty)$ and an exploration process $(\tilde{D}_u^{s_n})_{u\geq 0}$ therein. The aforementioned consistency of our coupling ensures that D_0 appears as one of the domains in each of these exploration processes. In other words, for all $n \geq 1$, we have $D_0 = \tilde{D}_{u_n}^{s_n}$ for some sequence of real numbers u_n. By Lemma 3.5.4, $u_n \to \infty$ as $n \to \infty$. We set $D_{-t} = \tilde{D}_{u_n - t}^{s_n}$, where n is chosen to be large enough that $u_n > t$. This completes the construction of the domains $(D_t)_{t\in\mathbb{R}}$, and conditions (i)–(iii) follow immediately from the construction. □

3.6 A coupling of the GFF and the CLE exploration process

3.6.1 The topographic map of the Gaussian free field

Let $D \subsetneq \mathbb{C}$ be a domain, and let $h : D \to \mathbb{R}$ be a continuous function. One natural way to encode h is to specify the collection of all level sets $\{z : h(z) = t\}$ for $t \in \mathbb{R}$, in the same way that cartographers use a topographic map to represent terrain. We will construct such a collection of all level sets of the zero-boundary Gaussian free field (GFF) h, a conformally invariant random generalized function which is a fundamental object in the study of two-dimensional statistical physics models.

Since h is a generalized function but not a function, some work is required to make the notion of a level set precise. One approach, carried out in [59], is to project the GFF onto the space of functions piecewise affine on a triangulation of small mesh size $\delta > 0$. The image h^δ of h under this projection is a continuous function, so its level sets are well-defined. Although our construction works directly in the continuum setting and does not make reference to the discrete GFF (as in [60]), it is nevertheless instructive to begin by considering the level sets of the discrete GFF.

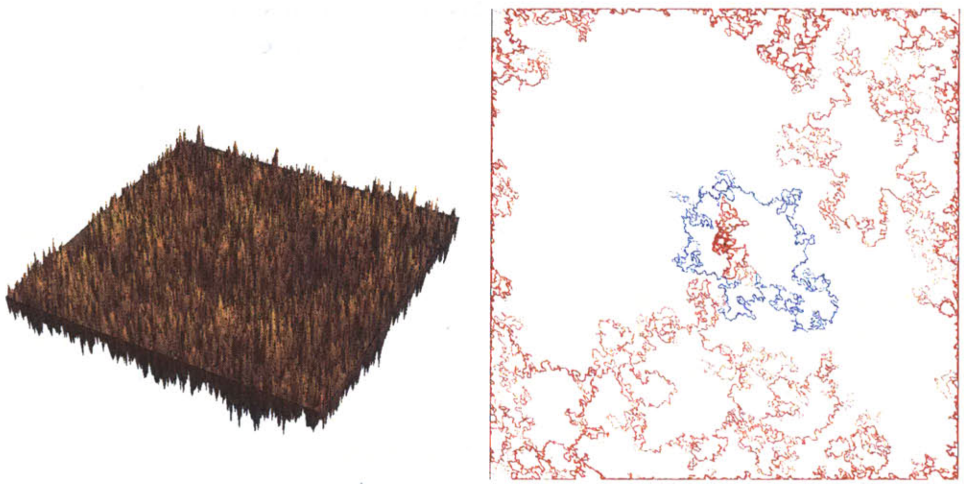
For simplicity, we let $D = [-1,1]^2$, $n \in \mathbb{N}$, and $\delta = 2/n$. We triangulate D by tiling D with $\delta \times \delta$ squares and dividing each square into two right triangles with lines of slope -1 (so each tile looks like ◳). For each face F in this triangulation, h^δ is almost surely non-constant along all three edges of F, which means that the level sets of $h^\delta\big|_F$ are line segments. It follows that the level loops of h^δ are polygonal curves which we may equip with a natural orientation, specified by the rule that we trace out the loop so that the values of h immediately to the right of the curve are larger than the values of h immediately to the left of the curve. In other words, the values of h infinitesimally outside a counterclockwise oriented loop are larger than the values of h infinitesimally inside the loop, and vice versa for a clockwise loop. In Figure 3-4, we show a surface plot of a GFF sample[1], along with the collection of level loops surrounding the origin, with the colors red and blue used to indicate counterclockwise and clockwise orientation, respectively.

We are particularly interested in the outermost blue loop surrounding the origin, as well as the collection Γ^δ of all the outermost blue loops. We will show that the continuum analogue of Γ^δ, which we denote Γ, is a CLE_4 (for more details regarding the conformal loop ensemble, see Section 3.6.2). We will also give an interpretation of the red loops in Figure 3-4, relating them to a uniform exploration of Γ using a Poisson point process of SLE bubbles, introduced in [84].

3.6.2 The uniform CLE exploration and the GFF

The Schramm-Loewner evolution (SLE) processes were introduced by Oded Schramm [58] as candidates for the scaling limits of discrete statistical physics models in

[1]To simulate the GFF height gap, we add $\lambda = \sqrt{\pi/8}$ to the values of h on the boundary of the square. This has the effect of specifying the orientation of the boundary of the domain.



(a) (b)

Figure 3-4: Panel (a) shows a surface plot of a discrete GFF h^δ on a 250×250 grid. Panel (b) shows the level lines of h^δ surrounding the origin. Red loops are oriented counterclockwise (with the values of h larger outside than inside), and blue loops are oriented clockwise (with values of h larger inside). Observe that the loops surrounding a given point appear in an alternating sequence of red and blue bands, where each band consists of a nested progression of loops of the same color.

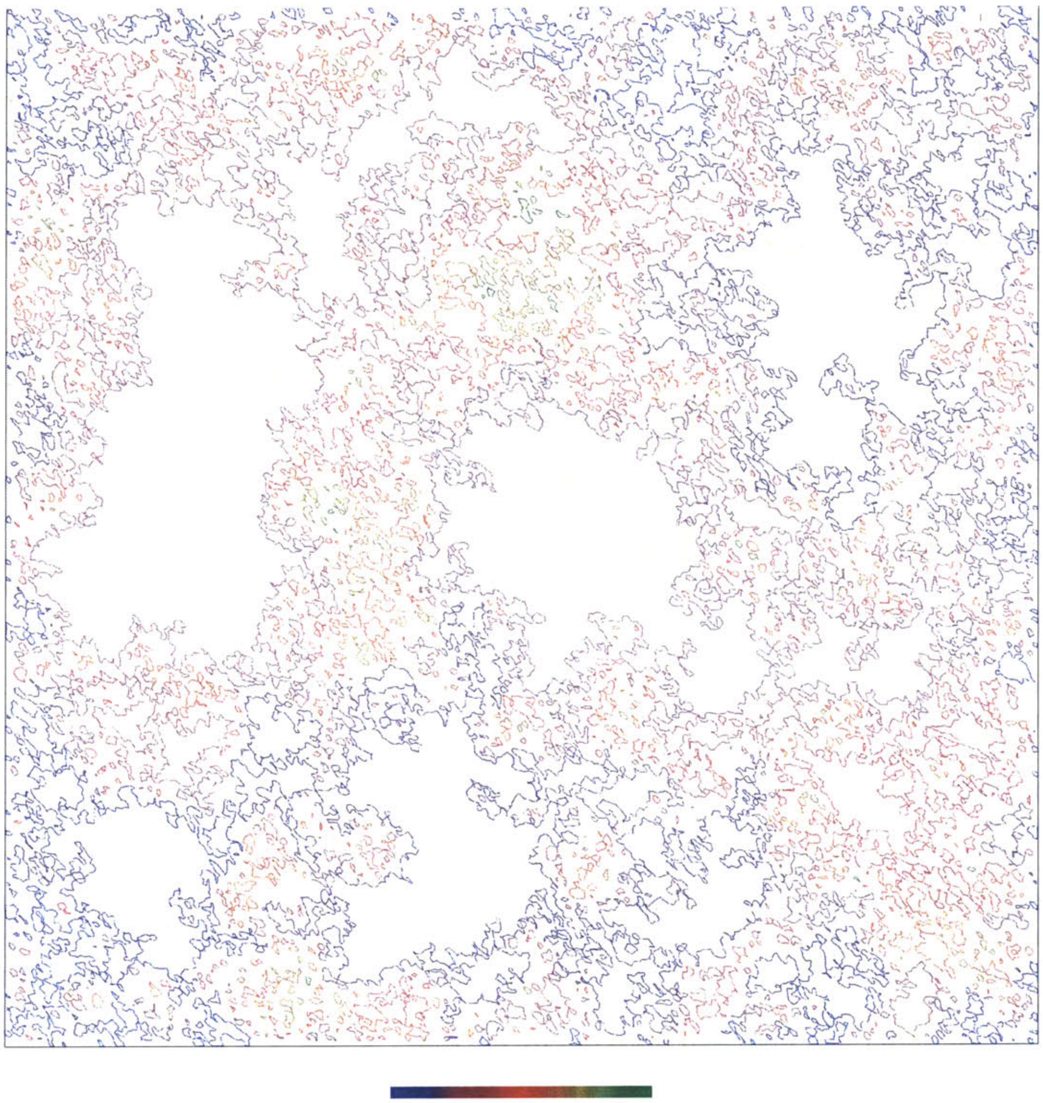


Figure 3-5: A picture of all the clockwise loops which are surrounded by no other clockwise loop, colored from blue to red to green in decreasing order of height. The continuum analogue of this height-indexed collection of loops is a uniform CLE_4 exploration process in $[-1,1]^2$.

two dimensions. A chordal SLE is a random non-self-traversing curve in a simply connected domain, joining two prescribed boundary points of the domain. SLE curves, indexed by $\kappa \geq 0$, are the only random planar curves that satisfy conformal invariance and the domain Markov property. SLE processes have been proved to be the scaling limits of many discrete models. For example, SLE_4 is the scaling limit of a level line of the discrete GFF with certain boundary conditions [59].

The conformal loop ensembles (CLE) were introduced as candidates for the scaling limit of the collection of all of the interfaces in the discrete model, in contrast to the single-interface model SLE. A CLE is a countable random collection of simple loops that are disjoint and non-nested. CLEs, indexed by $\kappa \in (8/3, 4]$, are defined and studied in [66, 70]. The CLEs are the only collections of simple loops in a simply connected domain that satisfy both conformal invariance and the domain Markov property: if Γ is a CLE in the unit disk, then Γ satisfies

- conformal invariance: Let φ be any conformal map from $\mathbb{D}$ onto itself, $\varphi(\Gamma)$ has the same law as Γ. This makes it possible to define CLE in any simply connected domain D via conformal images.

- the domain Markov property: For any simply connected domain $D \subset \mathbb{D}$, let D^* be the set obtained by removing from D all loops of Γ that are not totally contained in D. Then, given D^*, for any simply connected component U of D^*, the conditional law of the loops of Γ contained in U is the same as the law of CLE in U.

Each loop in CLE_κ is an SLE_κ-type loop. This implies, for example, that the Hausdorff dimension of each loop is $1 + \frac{\kappa}{8}$ almost surely [4].

The original construction of CLE_κ [65] involves choosing a root $R \in \partial D$ and considering an *exploration tree*, which is a collection $\{\eta^{R \to z} : z \in D\}$ of $\mathrm{SLE}_\kappa(\kappa - 6)$ processes (a variant of SLE_κ), each starting from R and targeted at a point $z \in D$. The collection of processes is coupled in such a way that for every $z, w \in D$, the processes $\eta^{R \to z}$ and $\eta^{R \to w}$ agree (up to time change) until the first time τ that z and w lie in different connected components of $D \setminus \eta^{R \to z}[0, \tau]$. The loops are constructed using branches of the exploration tree. We refer the reader to [65] for more details. Observe that, because of the choice of root, the exploration tree is not invariant under any map that does not fix the root.

In [84], the authors constructed a more symmetric exploration process for $\kappa \in (8/3, 4]$. Loosely speaking, they define $\eta^{R \to z}$ as follows: (1) choose a root R according to harmonic measure on ∂D, and (2) after each time η closes a loop, resample a fresh starting point for the next loop according the harmonic measure on the boundary of the unexplored component containing z. Such a process carries two natural time parameterizations: (1) log conformal radius time, for which $t = -\log \mathrm{CR}(D \setminus \eta^{R \to z}[0, t], z) + \log \mathrm{CR}(D, z)$, and (2) the local time of the log conformal radius time, which is obtained from log conformal radius time by excising each interval during which a loop is traced.

For $\kappa = 4$, this exploration process is target invariant in the sense that it is possible to couple a collection of such processes $\eta^{R \to z}$ targeted at every point $z \in$

D. Furthermore, in this coupling, the local time associated with each loop in Γ is independent of the target point z used to define it [84]. We encode this process by associating with each loop $\mathcal{L} \in \Gamma$ the local time $t_{\mathcal{L}}$ when that loop was added. We refer to this process as the **uniform exploration of** CLE_4.

3.6.3 Statement of coupling theorems

Our first result establishes the existence of a coupling of the type described in Section 3.6.1, in which the CLE_4 loops may be viewed as level loops of the Gaussian free field. In this coupling, the height of each level loop $\mathcal{L}$ corresponds to its time $t(\mathcal{L})$ in the uniform exploration. We write $\operatorname{int}\mathcal{L}$ to denote the bounded region enclosed by $\mathcal{L}$.

Theorem 3.6.1. There exists a coupling between a zero-boundary GFF h in the unit disk and a uniform CLE_4 exploration process $((\mathcal{L}, t(\mathcal{L})), \mathcal{L} \in \Gamma)$ such that the following is true. Given the exploration process $((\mathcal{L}, t(\mathcal{L})), \mathcal{L} \in \Gamma)$, the conditional law of $h|_{\operatorname{int}\mathcal{L}}$ is that of a GFF with boundary value $2\lambda(1 - t(\mathcal{L}))$. Furthermore, the family $(h|_{\operatorname{int}\mathcal{L}} : \mathcal{L} \in \Gamma)$ is conditionally independent given the exploration process.

We will also show that level loops and heights are deterministic functions of the free field in this coupling. This result is what one would expect from the discrete motivation, since level loops of a random continuous function are determined by the function.

Theorem 3.6.2. In the coupling of Theorem 3.6.1, the CLE loops and the exploration process are deterministic functions of the field.

We also prove a whole plane version of Theorems 3.6.1 and 3.6.2. Let Γ be a nested CLE_4 in $\mathbb{C}$ [25, 49]. We define a subset $\Gamma_0 \subset \Gamma$, which we call the **origin-nested whole plane CLE**, as follows. Let $\mathcal{L}_0$ be the outermost loop $\mathcal{L} \in \Gamma$ surrounding the origin for which $\mathrm{CR}(\mathcal{L}, 0) \leq 1$, and let $(\mathcal{L}_k : k \in \mathbb{Z})$ be the doubly-infinite sequence of all loops surrounding the origin arranged in nested order (so that $\mathcal{L}_j$ surrounds $\mathcal{L}_k$ for all $j < k$). Denote by Γ_0 the union of $(\mathcal{L}_k : k \in \mathbb{Z})$ and the set of all loops $\mathcal{K} \in \Gamma$ such that no loop with the same orientation as $\mathcal{K}$ surrounds $\mathcal{K}$ and is surrounded by the innermost loop in $(\mathcal{L}_k : k \in \mathbb{Z})$ surrounding $\mathcal{K}$. We say that $(t_{\mathcal{L}} : \mathcal{L} \in \Gamma_0)$ is an exploration of Γ if for all $k \in \mathbb{Z}$, either $((-1)^k(t_{\mathcal{L}} - t_{\mathcal{L}_k}) : \mathcal{L} \in \operatorname{int}\mathcal{L}_k \setminus \operatorname{int}\mathcal{L}_{k+1})$ or $((-1)^{k+1}(t_{\mathcal{L}} - t_{\mathcal{L}_k}) : \mathcal{L} \in \operatorname{int}\mathcal{L}_k \setminus \operatorname{int}\mathcal{L}_{k+1})$ is a CLE_4 exploration process in $\operatorname{int}\mathcal{L}_k$. We denote by $\sigma(\mathcal{L}) \in \{-1, 1\}$ the orientation of a level loop $\mathcal{L}$ of a GFF, where $\sigma(\mathcal{L}) = -1$ if $\mathcal{L}$ is clockwise and $\sigma(\mathcal{L}) = +1$ if $\mathcal{L}$ is counterclockwise.

Theorem 3.6.3. There exists an origin-nested whole plane uniform CLE_4 exploration process $(t(\mathcal{L}) : \mathcal{L} \in \Gamma_0)$ and a coupling of this process with a whole plane Gaussian free field h with the property that for all $k \in \mathbb{Z}$, the field $h|_{\operatorname{int}\mathcal{L}_k}$ and the exploration $(\sigma(\mathcal{L}_k)(t_{\mathcal{L}} - t_{\mathcal{L}_k}) : \mathcal{L} \subset \operatorname{int}\mathcal{L}_k \setminus \operatorname{int}\mathcal{L}_{k+1})$ are coupled as in Theorem 3.6.1. Furthermore, in this coupling the exploration is a deterministic function of the field.

3.6.4 Comparison with a symmetric GFF/CLE_4 coupling

In this section, we relate our coupling between GFF and CLE_4 to a coupling between CLE_4 and the Gaussian free field introduced in [42]. We begin by reviewing a standard result about Brownian motion.

Consider a one-dimensional standard Brownian motion $(B(t), t \geq 0)$, and define the reflected Brownian motion $Y(t) = |B(t)|$ for $t \geq 0$. Then Y can be decomposed into countably many Brownian excursions (a Brownian excursion $(e(t), 0 \leq t \leq \tau)$ is a Brownian path with $e(0) = 0, e(\tau) = 0$ and $e(t) > 0$ for $0 < t < \tau$). We define the local time process $(L(t), t \geq 0)$ of the Brownian motion, which is a nondecreasing function which is constant on the interior of each excursion. If we parameterize these Brownian excursions by the local time process of the Brownian motion, then we obtain a Poisson point process of Brownian excursions $(e_u, u \geq 0)$.

We can also reverse this procedure. There are two ways to construct a Brownian motion from a Poisson point process of Brownian excursions $(e_u, u \geq 0)$.

(i) Sample i.i.d. coin tosses σ_u for each excursion e_u, multiply the excursion by the sign σ_u, and concatenate these signed excursions. The process we get is a Brownian motion.

(ii) Concatenate all the excursions to obtain a reflected Brownian motion $(Y(t), t \geq 0)$. Define the local time process $(L(t), t \geq 0)$ of Y. Then the process $(Y(t) - L(t), t \geq 0)$ has the same law as a Brownian motion.

The coupling in [42] is defined as follows. We let Γ be a CLE_4 in $\mathbb{D}$. For each loop $\mathcal{L} \in \Gamma$, sample an independent random variable $\sigma(\mathcal{L})$ to be $+2\lambda$ or -2λ with equal probability. We think of $\sigma(\mathcal{L})$ as the orientation of $\mathcal{L}$, that is, $\sigma(\mathcal{L}) = +2\lambda$ (resp. $\sigma(\mathcal{L}) = -2\lambda$) corresponds to $\mathcal{L}$ being oriented clockwise (resp. counterclockwise). The law on the obtained sample $((\mathcal{L}, \sigma(\mathcal{L})), \mathcal{L} \in \Gamma)$ is called **CLE_4 with symmetric orientations**. The following theorems are analogous to Theorems 3.6.1 and 3.6.2 for the GFF/CLE_4 exploration coupling.

Theorem 3.6.4. There exists a coupling between zero-boundary GFF h in the unit disk and CLE_4 with symmetric orientations $((\mathcal{L}, \sigma(\mathcal{L})), \mathcal{L} \in \Gamma)$ in the unit disk such that the following is true. We denote the restriction of h inside the loop $\mathcal{L}$ by $h|_{\mathcal{L}}$. Given the loop configuration with orientations $((\mathcal{L}, \sigma(\mathcal{L})), \mathcal{L} \in \Gamma)$, for each loop $\mathcal{L}$, the conditional law of $h|_{\mathcal{L}}$ is that of a GFF with boundary value $2\lambda\sigma(\mathcal{L})$. Furthermore, the family $(h|_{\mathcal{L}} : \mathcal{L} \in \Gamma)$ is conditionally independent.

Theorem 3.6.5. In the coupling given by Theorem 3.6.4, the loop configuration with orientations $((\mathcal{L}, \sigma(\mathcal{L})), \mathcal{L} \in \Gamma)$ is a deterministic function of the field h.

Because of the additional randomness of the signs $\sigma(\mathcal{L})$, the coupling between the GFF and CLE_4 given in [42] is analogous to construction (i) of Brownian motion from the Itō excursions. The uniform exploration coupling is analogous to construction (ii).

Acknowledgements. We thank Wendelin Werner and Jason Miller for helpful discussions.

Remark 3.6.6. In this paper, we focus on $\kappa = 4$. The necessary and sufficient condition for η to have positive probability to hit the interior of the interval $(x^{j,R}, x^{j+1,R})$ (resp. $(x^{j+1,L}, x^{j,L})$) is

$$\sum_{i=0}^{j} \rho^{i,R} \in (-2,0) \quad (\text{resp. } \sum_{i=0}^{j} \rho^{i,L} \in (-2,0)),$$

with the convention $\rho^{0,L} = \rho^{0,R} = 0$, $x^{0,L} = 0^-$, $x^{l+1,L} = -\infty$, $x^{0,R} = 0^+$, $x^{r+1,R} = \infty$. See [13, Lemma 15].

3.6.5 The Gaussian free field

The zero-boundary Gaussian free field h on a domain $D \subsetneq \mathbb{C}$ is a random distribution, or generalized function, on D. Loosely speaking, this means that h is too rough for h to be defined pointwise, but it is possible to integrate h against sufficiently regular test functions. More precisely, h is a random element of the space of continuous linear functionals of the space $C_c^\infty(D)$ of smooth functions compactly supported in D. We use the notation (h, ρ) for the evaluation of h at $\rho \in C_c^\infty(D)$. We refer the reader to the survey articles [64] and [37] for more details about the construction of the GFF.

The law of the zero-boundary GFF on D is characterized by its covariance kernel, which is the function $G_D : D \times D \to \mathbb{R}$ for which

$$\text{Cov}[(h, \rho_1), (h, \rho_2)] = \int_D \int_D \rho_1(x) \rho_2(y) G_D(x,y)\, dx\, dy.$$

In fact, the GFF covariance kernel G_D is the Green's function of the Dirichlet Laplacian on D, which may be written as

$$G_D(x,y) = -\frac{1}{2\pi} \log|x-y| + \widetilde{G}_D(x,y),$$

where for each $x \in \mathbb{D}$, $y \mapsto \widetilde{G}_D(x,y)$ is the harmonic extension of the restriction of the function $y \mapsto -\frac{1}{2\pi}\log|x-y|$ to ∂D. Alternatively, $y \mapsto G_D(x,y)$ can be expressed as the density of the occupation measure of Brownian motion started at $x \in D$, which assigns to each measurable set $B \subset D$ the expected amount of time spent in B by a Brownian motion started at x and stopped upon exiting D. By the conformal invariance of planar Brownian motion, G_D is conformally invariant, and therefore the zero-boundary GFF is also conformally invariant.

3.7 Coupling between the GFF and radial SLE_4

3.7.1 Level lines of the GFF

SLE_4 curves can be viewed as level lines of the GFF, but since the GFF is not a function, some care is required to make this notion precise. The results in this subsection collected from [59, 60, 42, 40].

Theorem 3.7.1. Fix weights $(\underline{\rho}^L; \underline{\rho}^R)$ and corresponding force points $(\underline{x}^L; \underline{x}^R)$. Denote by $(K_t)_{t\geq 0}$ an $\mathrm{SLE}_4(\underline{\rho}^L; \underline{\rho}^R)$ process. There exists a coupling (K, ξ) where ξ is a zero boundary GFF on $\mathbb{H}$ such that the following is true. Let τ be any stopping time which is almost surely less than the continuation threshold of K. Let h_t be the harmonic function in $\mathbb{H}$ with boundary values:

$$-\lambda\left(1+\sum_{i=0}^{j}\rho^{i,L}\right) \quad \text{if} \quad x \in [f_t(x^{j+1,L}), f_t(x^{j,L})),$$
$$+\lambda\left(1+\sum_{i=0}^{j}\rho^{i,R}\right) \quad \text{if} \quad x \in [f_t(x^{j,R}), f_t(x^{j+1,R})),$$

where $\rho^{0,L} = \rho^{0,R} = 0$, $x^{0,L} = 0^-$, $x^{l+1,L} = -\infty$, $x^{0,R} = 0^+$, $x^{r+1,R} = \infty$ (see Figure 3-6). Then the conditional law of $\xi + h_0$ restricted to $\mathbb{H} \setminus K_\tau$ given K_τ is equal to the law of $\xi \circ f_\tau + h_\tau$.

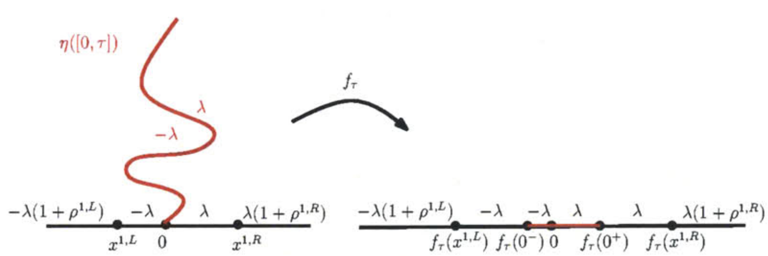


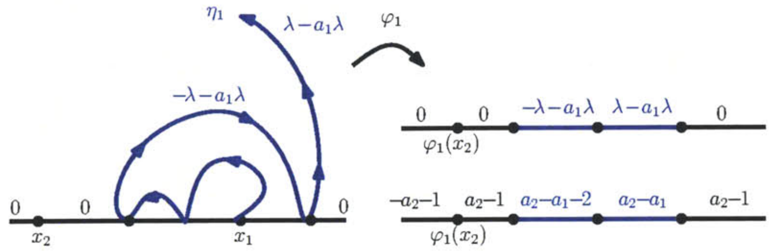
Figure 3-6: The function h_τ in Theorem 3.7.1 is the harmonic extension of the boundary value in the right panel.

Denote by η the curve generating the Loewner chain in the coupling of Theorem 3.7.1. The height of the field on the left and right sides of η are not the same, but in fact differ by 2λ. This *height gap* is a reflection of the fact that the GFF is a distribution rather than a function. Nevertheless, since the heights of the field on the two sides of η are constant and average to zero, we may interpret η as a height-zero level curve of ξ. Another feature of level curves we expect η to satisfy is the property of being determined by the field:

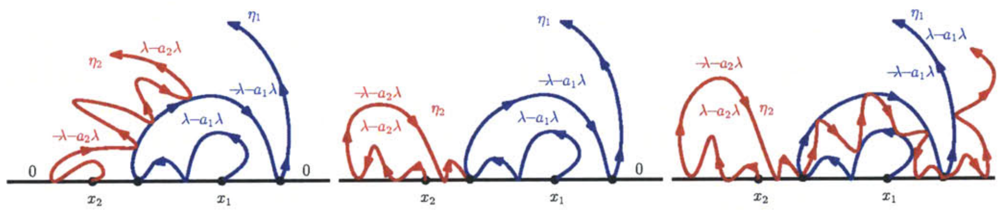
Theorem 3.7.2. Suppose η is an $\mathrm{SLE}_4(\underline{\rho}^L; \underline{\rho}^R)$ and ξ is a zero-boundary GFF on $\mathbb{H}$. In the coupling (η, ξ) of Theorem 3.7.1, η is almost surely determined by ξ.

By Theorems 3.7.1 and 3.7.2, we may say that in the coupling (η, ξ), the curve η is the zero level line of $\xi + h_0$. In particular, if we let ξ be a zero-boundary GFF on $\mathbb{H}$ and $a \in (-1, +1)$, then the level line of ξ with height $a\lambda$ is a chordal $\mathrm{SLE}_4(-a - 1; a - 1)$ curve with force points at $(0^-; 0^+)$. Note that when $a \in (-1, 1)$, both $-a - 1$ and $a - 1$ are above the continuation threshold, so the curve can be continued all the way to ∞.

The following results describe the interaction behavior of several level lines (see Figure 3-7). For simplicity, we only state the results in zero-boundary GFF. They can be generalized to the GFF with piecewise constant boundary values.



(a) Let ξ be a zero-boundary GFF. Fix $x_1 > x_2$. η_i is the level line of ξ with height $a_i\lambda$ starting from x_i targeted at ∞, $a_i \in (-1, 1)$, for $i = 1, 2$.



(b) If $a_2 > a_1$, then η_2 stays to the left of η_1.

(c) If $a_2 = a_1$, then η_2 merges with η_1 upon intersecting.

(d) If $a_2 < a_1$, then η_2 crosses η_1 upon intersecting and never crosses back.

Figure 3-7: Let ξ be a zero-boundary GFF. Fix $x_1 > x_2$. η_i is the level line of ξ with height $a_i\lambda$ starting from x_i targeted at ∞, $a_i \in (-1, 1)$, for $i \in \{1, 2\}$.

For proofs of the following two propositions, see [81].

Proposition 3.7.3. Suppose that ξ is zero-boundary GFF on $\mathbb{H}$, fix $x_1 > x_2$, and let $a_1, a_2 \in (-1, 1)$. Let η_i be the level line of ξ with height $a_i\lambda$ starting from x_i targeted at ∞, for $i \in \{1, 2\}$.

1. If $a_2 > a_1$, then η_2 almost surely stays to the left of η_1.
2. If $a_2 = a_1$, then η_2 may intersect η_1, and upon intersecting, the two curves merge and never separate.
3. If $a_2 < a_1$, then η_2 may intersect η_1, and upon intersecting, crosses and never crosses back.

In particular, if $x_1 = x_2$, that is the two level lines starting from the same point, then η_2 almost surely stays to the left of η_1 when $a_2 > a_1$ and η_2 almost surely stays to the right of η_1 when $a_2 < a_1$.

The following result describes the conditional law of one level line given the other level lines (see Figure 3-8).

Proposition 3.7.4. Let ξ be a zero-boundary GFF on $\mathbb{H}$. Fix $-1 \le a < b < c \le 1$, and let η_a, η_b, η_c be the level lines of ξ starting from 0 targeted at ∞ with height $a\lambda, b\lambda, c\lambda$ respectively. Then the conditional law of η_b given η_a and η_c is chordal $\mathrm{SLE}_4(c-b-2; b-a-2)$.

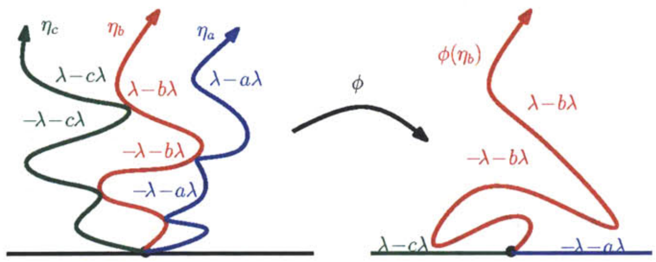


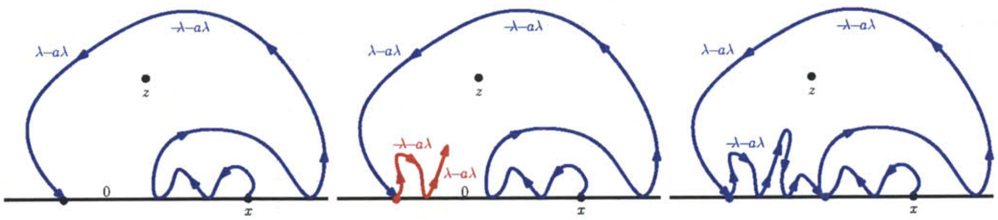
Figure 3-8: Fix $-1 < a < b < c < 1$, the conditional law of η_b given η_a, η_c is $\mathrm{SLE}_4(c-b-2; b-a-2)$.

From Theorems 3.7.1 and 3.7.2, the GFF level line starting from 0 and targeted at ∞ is well-defined and is a deterministic function of the field. By conformal invariance, we can define the level line of the GFF in any simply connected domain starting from a boundary point and targeted at another boundary point. In this section, we describe a level line of the GFF starting from a boundary point and targeted at an interior point.

We let h be a zero-boundary GFF in the upper half plane, and we fix a starting point $x \in \mathbb{R}$, a target point $z \in \mathbb{H}$, and a constant $a \in (-1,1)$. The level line of h starting from x targeted at ∞ with height $a\lambda$ is a chordal $\mathrm{SLE}_4(-a-1; a-1)$. In particular, this chordal curve is target independent, which means that we can construct the level line $\eta = \eta_a^{x\to z}$ targeted at z in the following way: Run the curve starting from x targeted at ∞ until the first time t_1 that $\eta([0,t_1])$ disconnects z from ∞, and denote by H_1 the connected component of $\mathbb{H} \setminus \eta([0,t_1])$ containing z. We choose any point x_1 on the boundary of H_1, and continue the curve by targeting at x_1 until the first time t_2 that $\eta([t_1,t_2])$ disconnects z from x_1 in H_1. Denote by H_2 the connected component of $H_1 \setminus \eta([t_1,t_2])$ containing z. We choose any point x_2 on the boundary of H_2 and continue the curve by targeting at x_2 until the first time t_3 that $\eta([t_2,t_3])$ disconnects z from x_2, and so on. This procedure can be continued until we reach the continuation threshold: at some time t_K, the boundary value on H_K is constant and is either $\lambda - a\lambda$ or $-\lambda - a\lambda$; see Figure 3-9 and recall Remark 3.6.6.

Once we reach the continuation threshold, we can no longer continue the level line towards z. We define $\eta_a^{x\to z}$ to be $\eta[0,t_N]$ and reparameterize the curve by the capacity seen from z, yielding the curve $(\eta_a^{x\to z}(t), 0 \le t \le \tau_a^{x\to z})$. Note that, although $(t_k, H_k, k \in \mathbb{N})$ depends on the choice of the target points $(x_k, k \in \mathbb{N})$, the process $\eta_a^{x\to z}$ does not. We call the curve $(\eta_a^{x\to z}(t), 0 \le t \le \tau_a^{x\to z})$ the **level line of h with height $a\lambda$ starting from x targeted at z.**

Denote by $H = H_a(x,z)$ the connected component of $\mathbb{H} \setminus \eta_a^{x\to z}$ containing z. Note that, when we complete the level line η with height $a\lambda$ targeted at z, there are two possibilities for the boundary value of h on the interior side of ∂H: either $\lambda - a\lambda$ or $-\lambda - a\lambda$. In the former case we say that H is a **plateau**, and in the latter case we say H is a **valley**. We may also say that z is an a plateau or in a valley, if x and a are clear from the context. The level line $\eta_a^{x\to z}$ has the following basic properties.



(a) Run the level line with height $a\lambda$ starting from x targeted at ∞ until the first time t_1 that η disconnects z from ∞. We denote by H_1 the connected component of $\mathbb{H} \setminus \eta$ containing z.

(b) Choose any point x_1 on the boundary of H_1. We continue the level line with height $a\lambda$ from $\eta(t_1)$ inside H_1 by targeting at x_1 until the first time t_2 that η disconnects z from x_1.

(c) At some time t_n, we meet the continuation threshold: the boundary values of h on the inside of H_n are constant, equal to either $\lambda - a\lambda$ or $-\lambda - a\lambda$.

Figure 3-9: The construction of the level line with height $a\lambda$ starting from $x \in \mathbb{R}$ targeted at $z \in \mathbb{H}$. If we reparameterize the curve by capacity seen from z, then this level line does not depend on the choice of target points $(x_k, k \in \mathbb{N})$ in the process of the construction.

Lemma 3.7.5. Suppose h is a zero-boundary GFF in $\mathbb{H}$ and fix $x \in \mathbb{R}$, $z \in \mathbb{H}$, and $a \in (-1,1)$. The level line of h with height $a\lambda$ starting from x and targeted at z satisfies the following properties:

- It is a continuous curve up to and including the continuation threshold.
- It is a deterministic function of the field.
- The probability that $H_z(x,z)$ is a plateau is $(1+a)/2$.

Proof. The first two properties are immediate from the construction. The third property follows from the optional stopping theorem and the fact that $(h_t(z))_{t\ge 0}$ is a martingale, where h_t is defined as in the statement of Theorem 3.7.1. See [67] for details. □

By target independence, we also have the following.

Lemma 3.7.6. Suppose h is a zero-boundary GFF in $\mathbb{D}$ and fix $a \in (-1,1)$. The level line of h with height $a\lambda$ starting from 1 and targeted at the origin has the same law as radial $\text{SLE}_4(-a-1; a-1)$ from 1 to the origin with two force points $(1^+;1^-)$, up to the first **closing time** τ_1:

$$\tau_1 = \inf\{t > 0 : W_t = V_t^+ = V_t^-\} \tag{3.7.1}$$

where W is the driving function, and the processes V^+ and V^- track the evolution of the force points under the Loewner flow

Next we describe the relationship between the GFF and radial SLE after the first closing time. Suppose h is a zero-boundary GFF in $\mathbb{D}$ and fix $a = -1 + 2\varepsilon$ for $\varepsilon > 0$ small. The level line of h starting from 1 and targeted at the origin with height $a\lambda$ has the same law as radial $\text{SLE}_4(-a-1; a-1)$, and we denote it by $\eta([0,\tau_1])$. Let D_1 be the connected component of $\mathbb{D} \setminus \eta([0,\tau_1])$ containing the origin. Note that the probability of the event that the origin is on a plateau is ε. If this is true, we stop the curve. If not, given $\eta([0,\tau_1])$, the law of h restricted to D_1 is the same as GFF in D_1 with boundary value $-2\varepsilon\lambda$. We continue the curve in D_1 by the following the level line of height $-2\varepsilon\lambda + a\lambda$ of the field in D_1 starting from $\eta(\tau_1)$ targeted at the origin until the continuation threshold is hit, and denote this part of the curve by $\eta([\tau_1,\tau_2])$. Let D_2 be the connected component of $D_1 \setminus \eta([\tau_1,\tau_2])$ containing the origin. If the origin is not on a plateau, then we continue the curve by the level line of height $-4\varepsilon\lambda + a\lambda$, and so on. At some finite step N the origin is on a plateau, and we stop. See Figure 3-10. We call the path $(\eta(t), 0 \le t \le \tau_N)$ the **ε-exploration process of h starting from 1 and targeted at the origin** stopped at the **discovery time** τ_N. It follows from the construction that N has a geometric distribution:

$$\mathbb{P}[N > n] = (1-\varepsilon)^n, \quad \text{for all} \quad n \ge 0.$$

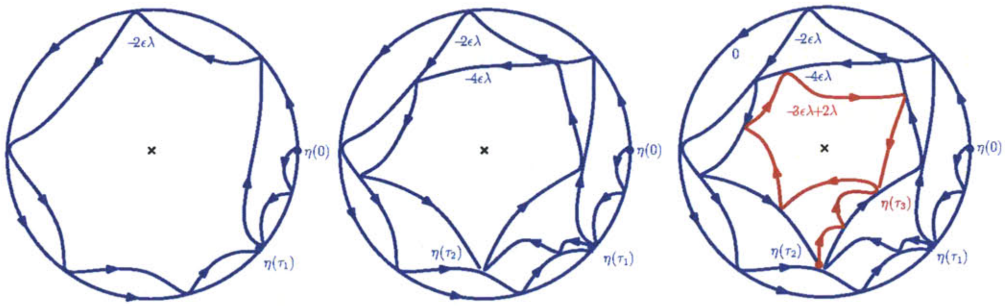
Recall that D_N is the connected component of $\mathbb{D} \setminus \eta([0,\tau_N])$ containing the origin. In Section 3.8, we show that D_N converges in distribution as $\varepsilon \to 0$ to the CLE_4 loop containing the origin.

From the domain Markov property, we know the existence of the entire radial $\text{SLE}_4(-a-1; a-1)$:

Corollary 3.7.7. Fix $a \in (-1,1)$. Radial $\text{SLE}_4(-a-1; a-1)$ starting from 1 with force points $(1^+;1^-)$ is generated by a continuous curve $(\eta(t), t \ge 0)$ which is transient in the sense that

$$\lim_{t\to\infty} |\eta(t)| = 0.$$

Proof. We only need to show the transience of the path. Suppose $a \in (-1,0]$ and $a = -1 + 2\varepsilon$. Let η be a radial $\text{SLE}_4(-a-1; a-1)$ and $(\tau_k, k \ge 1)$ be its successive closing times. Let N be the random index for which τ_N is η's discovery time. For $t > 0$, denote by H_t the connected component of $\mathbb{D} \setminus \eta([0,t])$ containing the origin, and define g_t to be the conformal map from H_t onto $\mathbb{D}$ normalized at the origin so that $g_t(0) = 0$ and $g_t'(0) = e^t$.



(a) Given η up to time τ_1, if the origin is in a valley, the mean height of the field at the origin is $-2\varepsilon\lambda$.

(b) Given η up to time τ_2, if the origin is in a valley, the mean height of the field at the origin is $-4\varepsilon\lambda$.

(c) Given η up to time τ_N, the mean height of the field at the origin is $-2N\varepsilon\lambda + 2\lambda$.

Figure 3-10: Fix $a = -1 + 2\varepsilon$, where $\varepsilon > 0$ is small. We start the radial $\mathrm{SLE}_4(-a-1; a-1)$ by the level line of the field with height $a\lambda$, denoted as $\eta([0, \tau_1])$. If the origin is in a valley. We continue the curve by following the level line of height $-2\varepsilon\lambda + a\lambda$. If the origin is still in the valley, we continue the process until at some finite step N, the origin is on a plateau.

We claim that for $T > 0$ sufficiently large, we have

$$\mathbb{P}[\partial H_T \cap \partial \mathbb{D} = \emptyset] > 0. \tag{3.7.2}$$

Define $n = \lfloor \frac{1+a}{2\varepsilon} \rfloor + 1$. Clearly, for T large enough, the probability of the event $\{T > \tau_n \text{ and } N > n\}$ is positive. On this event, the conditional law of boundary of H_{τ_n} given $\eta([0, \tau_n])$ is the law of the level line of zero-boundary GFF in $\mathbb{D}$ with height $-2\lambda n\varepsilon + a\lambda$. Since

$$-2\lambda n\varepsilon + a\lambda < -\lambda,$$

the probability that this level lines hits $\partial\mathbb{D}$ is zero by Remark 3.6.6. This implies (3.7.2).

From (3.7.2), there exist $r \in (0,1)$ and $p \in (0,1)$ such that

$$\mathbb{P}[H_T \subset r\mathbb{D}] \geq p.$$

Denote

$$D_k = H_{kT}, \quad \varphi_k = g_{kT}, \quad E_k = [\varphi_k(D_{k+1}) \subset r\mathbb{D}], \quad \text{for} \quad k \geq 1.$$

From the conformal invariance and domain Markov property of η we know that the events $(E_k, k \geq 1)$ are i.i.d. Thus $\sum_k \mathbb{P}[E_k] = \infty$, which implies that there almost surely exists a sequence $n_j \to \infty$ such that for all j, the event E_{n_j} holds. We will show by induction that for all $j \geq 1$,

$$D_{n_j} \subset r^{j-1}\mathbb{D}, \tag{3.7.3}$$

which implies the transience of the path.

Suppose (3.7.3) is true for $j \geq 1$. Note that

$$D_{n_{j+1}} \subset D_{n_j+1} \subset \varphi_{n_j}^{-1}(r\mathbb{D}).$$

We only need to show

$$\varphi_{n_j}^{-1}(r\mathbb{D}) \subset r^j\mathbb{D}. \tag{3.7.4}$$

We know that φ_{n_j} is the conformal map from $D_{n_j} \subset r^{j-1}\mathbb{D}$ onto $\mathbb{D}$ normalized at the origin. Let ψ_1 be the conformal map from D_{n_j} onto $r^{j-1}\mathbb{D}$ normalized at the origin and let $\psi_2(z) = z/r^{j-1}$. Then $\varphi_{n_j} = \psi_2 \circ \psi_1$, and (3.7.4) follows by noting that $|\psi_1(z)| \geq |z|$ for all z. □

3.8 Exploration of the GFF

3.8.1 The boundary-branching GFF exploration tree

In this section we describe a way of exploring the GFF which is closely related to the uniform CLE_4 exploration from Section 3.2.3. To this end, we first construct an object we call the boundary-branching GFF exploration tree.

Suppose h is a zero-boundary GFF in $\mathbb{D}$. Fix $a = -1 + 2\varepsilon$ with $\varepsilon > 0$. The level line of h starting from 1 targeted at -1 with height $a\lambda$ has the same law as chordal $\text{SLE}_4(-a-1; a-1)$ with force points $(1^-; 1^+)$. Given two target points $y_1, y_2 \in \partial\mathbb{D}$; the relationship between the level line targeted at y_1 and the level line targeted at y_2 is the following: the two curves coincide up to the first time that y_1, y_2 are disconnected. After this time, the two paths evolve towards their target points respectively. We denote by $\Upsilon^{B,a}$ the union of all level lines starting from some $x \in \partial\mathbb{D}$ and targeted at some $y \in \partial\mathbb{D}$. We call $\Upsilon^{B,a}$ the **boundary-branching exploration tree of GFF** with height $a\lambda$.

Lemma 3.8.1. Let h be a zero-boundary GFF in the unit disk. Fix $a \in (-1, 1)$. The boundary-branching exploration tree $\Upsilon^{B,a}$ with height $a\lambda$ of h has the following properties:

- $\Upsilon^{B,a}$ is almost surely a deterministic function of h.
- The law of $\Upsilon^{B,a}$ is conformal invariant, that is, for any conformal map φ from $\mathbb{D}$ onto itself, $\varphi(\Upsilon^{B,a})$ has the same law as $\Upsilon^{B,a}$.
- $\Upsilon^{B,a}$ separates $\mathbb{D}$ into countably many connected components. Given $\Upsilon^{B,a}$, the conditional law of h restricted in these components are independent GFF's; some of these GFF's have boundary value $\lambda(1-a)$ and the others have boundary value $\lambda(-1-a)$. We call the components with boundary value $\lambda(1-a)$ *plateaux* and the other components *valleys*.
- The probability of the event that the origin is in a plateau is $(1+a)/2$.

In fact, the connected component of $\mathbb{D} \setminus \Upsilon^{B,a}$ containing the origin has the same law as the connected component of $\mathbb{D} \setminus \eta([0, \tau_1])$ containing the origin, where η is radial $\mathrm{SLE}_4(-a-1; a-1)$ and τ_1 is its first closing time. Recall that $a = -1 + 2\varepsilon$ with $\varepsilon > 0$ small. Given $\Upsilon^{B,a}$, let γ^ε be the plateau with largest harmonic measure seen from the origin.

Lemma 3.8.2. The law of γ^ε normalized by $1/\varepsilon$ converges vaguely to M, the SLE_4-bubble measure in $\mathbb{D}$ uniformly rooted over the circle, with respect to the Hausdorff metric.

Proof. By conformal invariance, it suffices to show that the law of γ^ε conditioned on the event that γ^ε contains the origin converges to

$$M(\cdot \mid \gamma \text{ contains the origin}).$$

Let $a = -1 + 2\varepsilon$, and suppose $\Upsilon^{B,a}$ is the boundary-branching exploration tree of the zero-boundary GFF in $\mathbb{D}$. Define D^ε to be the connected component of $\mathbb{D} \setminus \Upsilon^{B,a}$ containing the origin. Denote by η the corresponding radial $\mathrm{SLE}_4(-a-1; a-1)$ started at point chosen uniformly at random on the boundary of the disk.

We condition on the event that D^ε is a plateau, and let σ and τ be the random times for which $\eta[\sigma, \tau]$ is the excursion of η which surrounds the origin. For all $\delta > 0$, we can sample a curve with the same law as $\eta[\sigma, \tau]$ by first sampling $\eta[0, \sigma + \delta]$ and then sampling an SLE_4 curve in $\mathbb{D} \setminus \eta[0, \sigma + \delta]$ from $\eta(\sigma + \delta)$ to $\eta(\sigma)$ conditioned to surround the origin. Since the conformal map from $\mathbb{D} \setminus \eta[0, \sigma + \delta]$ to $\mathbb{D}$ converges on compact subsets of $\mathbb{D}$ to the identity as $\varepsilon, \delta \to 0$, we see that the law of D^ε conditioned on $\eta(\sigma)$ converges to the SLE_4 bubble measure rooted at $\eta(\sigma)$. By symmetry, however, $\eta(\sigma)$ is uniform on $\partial\mathbb{D}$. Therefore, D^ε conditioned to be a plateau converges to $M(\cdot \mid \gamma \text{ contains the origin})$, as desired. □

Now we can describe the discrete GFF exploration process. Suppose h is a zero-boundary GFF in $\mathbb{D}$. Fix $a = -1 + 2\varepsilon$, where $\varepsilon > 0$ is small. Let Υ^B_1 be the boundary-branching exploration tree for h with height $a\lambda$. And define $\tilde{D}_1$ to be the connected component of $\mathbb{D} \setminus \Upsilon^B_1$ containing the origin. Let f_1 be the conformal map from $\tilde{D}_1$ onto $\mathbb{D}$ normalized at the origin, and denote by γ_1 the plateau with largest harmonic measure seen from the origin. If $\tilde{D}_1$ is a plateau (in other words $\gamma_1 = \tilde{D}_1$), we stop. Otherwise, let h_1 be the image of h restricted to $\tilde{D}_1$ under f_1. Then h_1 has the same law as a GFF with boundary value $-2\varepsilon\lambda$. Let Υ^B_2 be the boundary-branching exploration tree for h_1 with height $-2\varepsilon\lambda + a\lambda$. Denote by $\tilde{D}_2$ the connected component of $\mathbb{D} \setminus \Upsilon^B_2$ containing the origin. Let f_2 be the conformal map from $\tilde{D}_2$ onto $\mathbb{D}$ normalized at the origin, and γ_2 be the plateau with largest harmonic measure seen from the origin. If $\tilde{D}_2$ is a plateau, we stop. If not, we continue, etc. At some finite step N, $\tilde{D}_N = \gamma_N$ is a plateau, and we stop. Note that, when the origin is in the plateau, the mean height of the field in $\tilde{D}_N$ is $-2N\varepsilon\lambda + 2\lambda$.

We summarize notation and properties of the discrete GFF exploration GFF below:

- The steps of discrete exploration are i.i.d. In particular, $((f_n, \gamma_n), n < N)$ have the same law.
- The first step N when the origin is in some plateau has a geometric distribution:
$$\mathbb{P}(N > n) = \mathbb{P}(\text{The origin is in some valley})^n = (1-\varepsilon)^n.$$
- We define the conformal map
$$\Phi^\varepsilon = f^\varepsilon_{N-1} \circ \cdots \circ f^\varepsilon_2 \circ f^\varepsilon_1.$$

Recall the terminology in Section 3.7; in fact, $D_N := (\Phi^\varepsilon)^{-1}(\gamma_N)$ has the same law as the connected component of $\mathbb{D} \setminus \eta([0,T])$ where η is a radial $\text{SLE}_4(-a-1; a-1)$ and T is the discovery time time.

By arguments analogous to those for Theorem 3.8.2 and Proposition 3.2.4, we have the following conclusions. Suppose $(\gamma_t, t \geq 0)$ is a Poisson point process with intensity M. Define $\tau = \inf\{t : \gamma_t \text{ contains the origin}\}$. For each $t < \tau$, let f_t be the conformal map from the connected component of $\mathbb{D} \setminus \gamma_t$ containing the origin onto D normalized at the origin. And, for each $t \leq \tau$, set $\Psi_t = \circ_{s<t} f_s$, $L_\tau = (\Psi_\tau)^{-1}(\gamma_\tau)$.

Lemma 3.8.3. The relationship between the discrete exploration of GFF and the Poisson point process of bubbles is the following:

- Φ^ε converges in distribution to Ψ_τ in Carathéodory topology seen from the origin as $\varepsilon \to 0$.
- $D_N = (\Phi^\varepsilon)^{-1}(\gamma_N)$ converges in distribution to L_τ, the loop in CLE_4 containing the origin, in Carathéodory topology seen from the origin as $\varepsilon \to 0$.
- Given $(\Upsilon^B_n, 1 \leq n \leq N)$, the mean height of the field on $D_N = (\Phi^\varepsilon)^{-1}(\gamma_N)$, which is $2\lambda(1 - N\varepsilon)$, converges in distribution to $2\lambda(1-\tau)$.

Corollary 3.8.4. Set $a = -1 + 2\varepsilon$ with $\varepsilon > 0$. Suppose η is a radial $\text{SLE}_4(-a-1; a-1)$ and T is its discovery time. Define L^ε to be the connected component of $\mathbb{D} \setminus \eta([0,T])$ containing the origin. Then L^ε converges in distribution to the loop in CLE_4 containing the origin in the Carathéodory topology seen from the origin.

3.8.2 The GFF exploration and proofs of main theorems

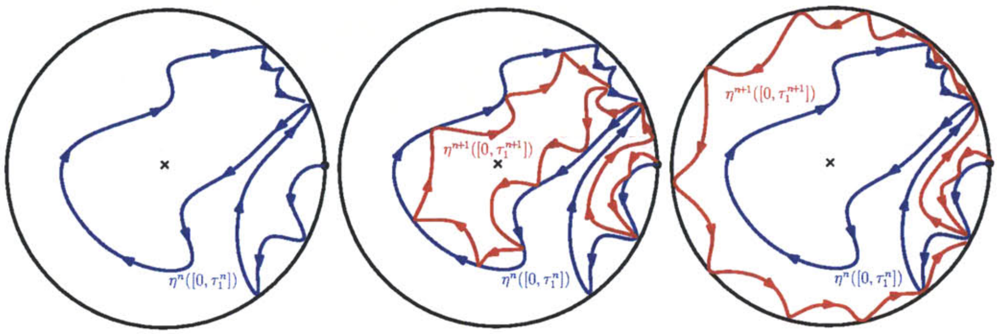
We first explain the relationship between the GFF ε-exploration process and $\varepsilon/2$-exploration process (starting from 1 and targeted at the origin), introduced in Section 3.7.

Suppose h is a zero-boundary GFF in $\mathbb{D}$. Fix $n \in \mathbb{N}$ and $\varepsilon = 2^{-n}$. Let η^n be the ε-exploration process of h starting from 1 targeted at the origin, and τ^n_1 be its first closing time. Denote by D^n_1 the connected component of $\mathbb{D} \setminus \eta^n([0, \tau^n_1])$ containing the origin. Let η^{n+1} be the $\varepsilon/2$-exploration process of h starting from 1 targeted at the origin, and let τ^{n+1}_1 and τ^{n+1}_2 be its first and second closing times. Define

D_i^{n+1} to be the connected component of $\mathbb{D} \setminus \eta^{n+1}([0, \tau_i^{n+1}])$ containing the origin, for $i \in \{1, 2\}$.

On the event that D_1^n is a valley, Proposition 3.7.3 implies that almost surely,

$$D_2^{n+1} = D_1^n \subset D_1^{n+1}. \tag{3.8.1}$$



(a) D_1^n is a plateau. (b) Case 1. D_1^{n+1} is a plateau. (c) Case 2. D_1^{n+1} is a valley.

Figure 3-11: On the event that D_1^n is a plateau, there are two cases for D_1^{n+1}, D_2^{n+1}.

On the event that D_1^n is a plateau, there are two cases for D_1^{n+1}, D_2^{n+1}. (i) If D_1^{n+1} is a plateau, then almost surely, (see Figure 3-11)

$$D_1^{n+1} \subset D_1^n. \tag{3.8.2}$$

(ii) If D_1^{n+1} is a valley and D_2^{n+1} is a plateau, then almost surely, we have (see Figure 3-11)

$$D_2^{n+1} = D_1^n \subset D_1^{n+1}. \tag{3.8.3}$$

Now suppose h is a zero-boundary GFF in $\mathbb{D}$. Fix $\varepsilon = 2^{-n}$ for some $n \in \mathbb{N}$. Let η^n (resp., η^{n+1}) be the ε-exploration process (resp. $\varepsilon/2$-exploration process) of h starting from 1 targeted at the origin, and $(\tau_k^n, k \geq 1)$ (resp., $(\tau_k^{n+1}, k \geq 1)$) be its successive closing times, and N^n (resp., N^{n+1}) is the integer for which

$$T^n = \tau_{N^n}^n \quad (\text{resp},. \quad T^{n+1} = \tau_{N^{n+1}}^{n+1})$$

is its discovery time. Also, denote by L^n (resp. L^{n+1}) the connected component of $\mathbb{D} \setminus \eta^n([0, T^n])$ (resp. $\mathbb{D} \setminus \eta^{n+1}([0, T^{n+1}])$) containing the origin.

From (3.8.1), (3.8.2), and (3.8.3), we have that, almost surely,

$$L^{n+1} \subseteq L^n, \tag{3.8.4}$$

and

$$N^{n+1} \in \{2N^n, 2N^n - 1\}. \tag{3.8.5}$$

To complete the proof of Theorems 3.6.1 and 3.6.2, we introduce the full branching

GFF exploration tree. Fix $\varepsilon > 0$ small and $a = -1 + 2\varepsilon$. Suppose h is a zero-boundary GFF in $\mathbb{D}$. Recall the construction of ε-exploration process of the field starting from 1 targeted at the origin, introduced in Section 3.7. This construction can be easily generalized to the exploration process starting from any point on the boundary targeted at any interior point. Note that, if $x \in \partial\mathbb{D}$ and $y_1, y_2 \in \mathbb{D}$, the relation between the exploration process η_1 starting from x targeted at y_1 and the exploration process η_2 starting from x targeted at y_2 is the following: the two paths coincide up to the first time that y_1 and y_2 are disconnected. After this time, the two paths evolve toward their respective target points (stopped at their discovery times respectively). Now we define **full branching GFF ε-exploration tree of the field** to be the union of all ε-exploration trees starting from all $x \in \partial\mathbb{D}$ and targeted at all $z \in \mathbb{D}$.

Lemma 3.8.5. Let h be a zero-boundary GFF in $\mathbb{D}$. Fix $\varepsilon > 0$. The full-branching ε-exploration tree $\Upsilon^{F,\varepsilon}$ of h has the following properties:

- $\Upsilon^{F,\varepsilon}$ is almost surely a deterministic function of h.
- The law of $\Upsilon^{F,\varepsilon}$ is conformal invariant, that is for any conformal map φ from $\mathbb{D}$ onto itself, $\varphi(\Upsilon^{F,\varepsilon})$ has the same law as $\Upsilon^{F,\varepsilon}$.
- Define $L^\varepsilon(z)$ to be the connected component of $\mathbb{D} \setminus \Upsilon^{F,\varepsilon}$ containing $z \in \mathbb{D}$. Then $L^\varepsilon(0)$ is, almost surely, the same as the connected component of $\mathbb{D} \setminus \eta([0,T])$ where η is the ε-exploration process of h starting from 1 targeted at the origin and T is its discovery time.
- Given $\Upsilon^{F,\varepsilon}$, the conditional law of h restricted to $L^\varepsilon(0)$ is the same as a GFF with mean height $h^c(0) = 2\lambda(1 - \varepsilon N^c)$ for some integer N^c determined by $\Upsilon^{F,\varepsilon}$.

Proof of Theorems 3.6.1 and 3.6.2. Suppose h is a zero-boundary GFF in $\mathbb{D}$. For any $n \in \mathbb{N}$, set $\varepsilon = 2^{-n}$. Run the full-branching exploration tree $\Upsilon^{F,n}$ of h. Note that $\Upsilon^{F,n}$ is a deterministic function of h. For any $z \in \mathbb{D}$, define $L^n(z)$ to be the connected component of $\mathbb{D} \setminus \Upsilon^{F,n}$ containing z. Denote by $N^n(z)$ the integer such that, given $\Upsilon^{F,n}$, the mean height of h restricted to $L^n(z)$ is

$$h^n(z) := \lambda(1 - 2^{-n}N^n(z)).$$

From (3.8.4) and (3.8.5), for any fixed $z \in \mathbb{D}$, we have almost surely,

$$L^{n+1}(z) \subset L^n(z), \quad |h^{n+1}(z) - h^n(z)| \le 2^{-n}\lambda. \tag{3.8.6}$$

Note that $L^n(z)$ and $h^n(z)$ are deterministic functions of h. Fix some countable dense subset $\mathcal{D}$ of $\mathbb{D}$.

On one hand, almost surely, (3.8.6) holds for all $z \in \mathcal{D}$. On this event, define

$$\Upsilon^{F,\infty} = \overline{\bigcup_n \Upsilon^{F,n}}, \text{ and } \quad h^\infty(z) = \lim_n h^n(z).$$

Define $L^\infty(z)$ to be the connected component of $\mathbb{D} \setminus \Upsilon^{F,\infty}$ containing $z \in \mathcal{D}$. Note that $((L^\infty(z), h^\infty(z)), z \in \mathcal{D})$ is almost surely determined by h. From Lemma 3.8.5, we know that, given $\Upsilon^{F,n}$, for all $z \in \mathbb{D}$, the conditional law of h restricted to $L^n(z)$ is a GFF with mean height $h^n(z)$. In fact, h^n (resp. h^∞) can be defined almost everywhere by setting $h^n(w) = h^n(z)$ (resp. $h^\infty(w) = h^\infty(z)$) when $w \in L^n(z)$ (resp. $w \in L^\infty(z)$). Fix a smooth function ρ, compactly supported in $\mathbb{D}$. We know that (h^n, ρ) plus an independent Gaussian with mean zero and variance $E^n(\rho)$ has the same law as a Gaussian with mean zero and variance $E(\rho)$, where

$$E(\rho) = \iint_{\mathbb{D}\times\mathbb{D}} \rho(x)\rho(y)G_{\mathbb{D}}(x,y)\,dx\,dy, \text{ and}$$

$$E^n(\rho) = \iint_{\mathbb{D}\times\mathbb{D}} \rho(x)\rho(y)G^n(x,y)\,dx\,dy, \quad G^n(x,y) = \sum_{L^n} G_{L^n}(x,y)$$

Here G_D is the Green's function of the Dirichlet Laplacian on the domain D, and the sum $\sum_{L^n}$ is taken over all connected components of $\mathbb{D} \setminus \Upsilon^{F,n}$. From (3.8.6), (h^n, ρ) converges to (h^∞, ρ) almost surely. Furthermore, $E^n(\rho)$ is decreasing in n thus has a limit, which we denote by $E^\infty(\rho)$. Then (h^∞, ρ) plus an independent Gaussian with mean zero and variance $E^\infty(\rho)$ has the same law as a Gaussian with mean zero and variance $E(\rho)$. This implies that h^∞ plus independent zero-boundary GFF's in each connected component of $\mathbb{D} \setminus \Upsilon^{F,\infty}$ has the same law as a zero-boundary GFF in $\mathbb{D}$. And given $\Upsilon^{F,\infty}$, the conditional law of h restricted to $L^\infty(z)$ is the same as a GFF with mean $h^\infty(z)$ for any $z \in \mathbb{D}$.

On the other hand, from Lemma 3.8.3, we know that $L^n(z)$ converges in distribution to the loop in CLE_4 containing z in the Carathéodory topology seen from z. Thus $L^\infty(z)$ has the same law as the loop $L(z)$ in CLE_4 containing z. Moreover, $h^\infty(z)$ has the same law as $2\lambda(1 - t(L(z)))$ where $t(L(z))$ is the time parameter of the loop $L(z)$ in CLE_4 with time parameter.

Combining these two observations, we conclude the proof of Theorem 3.6.1. Since $((L^\infty(z), h^\infty(z)), z \in \mathcal{D})$ is almost surely determined by h, we have also proved Theorem 3.6.2. □

Remark 3.8.6. If we consider the setting of Lemma 3.8.3, the proof of Theorems 3.6.1 and 3.6.2 actually gives the almost sure convergence of $D_N = (\Phi^\varepsilon)^{-1}(\gamma_N)$ along the subsequence $\varepsilon_n = 2^{-n}$.

We conclude by proving Theorem 3.6.3.

Proof of Theorem 3.6.3. Let Γ be a nested CLE_4 in $\mathbb{C}$, and denote by $(\mathcal{L}_k : k \in \mathbb{Z})$ the sequence of loops surrounding the origin as defined in Section 3.6.3. Choose $\sigma(\mathcal{L}_0)$ uniformly at random from the set $\{+1, -1\}$, and let $\sigma(\mathcal{L}_k) = (-1)^k \sigma(\mathcal{L}_0)$ for all $k \in \mathbb{Z}$. For each $k \in \mathbb{Z}$, we sample an annulus CLE_4 exploration process

$$(t^k_{\mathcal{L}} : \mathcal{L} \text{ is between } \mathcal{L}_k \text{ and } \mathcal{L}_{k+1})$$

in $\operatorname{int} \mathcal{L}_k \setminus \overline{\operatorname{int} \mathcal{L}_{k+1}}$. We define $(t_{\mathcal{L}_k} : k \in \mathbb{Z})$ inductively by $t_{\mathcal{L}_0} = 0$ and $t_{\mathcal{L}_{k+1}} = t_{\mathcal{L}_k} + (-1)^{\sigma(\mathcal{L}_k)} t^k_{\mathcal{L}_{k+1}}$. For every CLE_4 loop $\mathcal{L}$ between $\mathcal{L}_k$ and $\mathcal{L}_{k+1}$, we define $t_{\mathcal{L}} =$

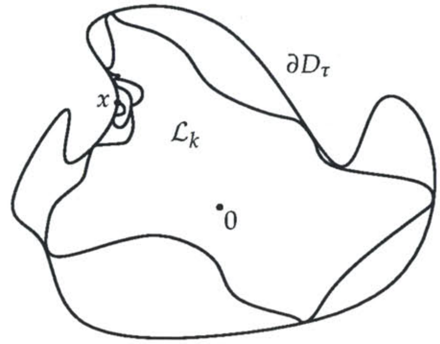


Figure 3-12: To show that the level lines are determined by the free field in the whole-plane exploration coupling, we sample an outside-in exploration $(D_t)_{t\in\mathbb{R}}$ (from ∞ to 0) and an inside-out exploration (from 0 to ∞) with the same GFF h, conditionally independent of one another. Each level loop $\mathcal{L}_k$ surrounding the origin for the inside-out process is equal to D_t for some $t \in \mathbb{R}$. To see this, we run the outside-in process until it hits $\mathcal{L}_k$ at some point x and some time τ, and we then run a level line of the GFF inside D_τ starting at x and with height equal to the height of $\mathcal{L}_k$. By pathwise uniqueness of level lines, this level line equals $\mathcal{L}_k$.

$t_{\mathcal{L}_k} + (-1)^{\sigma(\mathcal{L})} t^k_{\mathcal{L}}$. It is clear from the construction that the resulting process satisfies the definition of an origin-nested CLE_4 exploration.

For each loop $\mathcal{L}$ in the origin-nested CLE_4, we define an independent Gaussian free field $h_{\mathcal{L}}$ with boundary conditions corresponding to the height $t_{\mathcal{L}}$ and the orientation of $\mathcal{L}$. Then $h = \sum_{\mathcal{L}} h_{\mathcal{L}}$, considered as a tempered distribution modulo global additive constant, has the law of a GFF on $\mathbb{C}$ because the restriction of h to the interior of $\mathcal{L}_k$ has the law of a zero-boundary GFF (modulo additive constant), and zero-boundary GFFs on a sequence of domains tending to $\mathbb{C}$ converge in law to the whole plane GFF [43, Proposition 2.10].

To show that the level loops are determined by h in this coupling, we couple h with an exploration process from ∞ to 0 and an exploration process from 0 to ∞ in such a way that the exploration processes are conditionally independent given h, following [14]. This is possible since h conformally invariant and in particular is invariant under the inversion $z \mapsto 1/z$. We claim that the two exploration processes are in fact equal, which implies that each is determined by h.

To see this, let $\mathcal{L}_k$ be one of the loops surrounding the origin for the inside-out process, and let $t_{\mathcal{L}_k}$ be the height corresponding to $\mathcal{L}_k$. Encode the outside-in process as a nested progression $(D_t)_{t\in\mathbb{R}}$ of domains as defined in Theorem 3.5.6 in [68]. Let $\tau = \sup\{t \in \mathbb{R} : \mathcal{L}_k \subset D_t\}$ be the hitting time of $\mathcal{L}_k$ for the process $(\partial D_t)_{t\in\mathbb{R}}$; see Figure 3-12. Since level lines cannot cross one another except at the boundary of the domain, ∂D_τ and $\mathcal{L}_k$ touch rather than crossing. Since ∂D_τ and $\mathcal{L}_k$ touch, they have the same orientation and heights which differ by less than λ. Let $x \in \partial D_\tau \cap \mathcal{L}_k$, and consider the level line in D_τ starting from x of height $t_{\mathcal{L}_k}$. By pathwise uniqueness of GFF level lines, this level line traces out $\mathcal{L}_k$. It follows that every loop surrounding the origin for the inside-out process is equal to D_t for some $t \in \mathbb{R}$. Since $k \in \mathbb{Z}$ was arbitrary, the result follows from Theorem 3.6.2. □

In the remainder of the thesis, we will prove the following theorem.

Theorem 3.8.7. In a conformally invariant exploration process of non-nested CLE_4 in a simply connected domain $D \subsetneq \mathbb{C}$, the exploration is a deterministic function of the CLE_4 loops.

We use Theorem 3.8.7 to construct a conformally invariant metric on CLE_4.

Theorem 3.8.8. There exists a conformally invariant metric on the collection of CLE_4 loops in $\mathbb{D}$.

Proposition 3.8.9. Let $\mathcal{L}$ be a Brownian loop soup in D with intensity $c \in (0,1]$, and let $(D_t)_{t\geq 0}$ be a decreasing family of subdomains of $\mathbb{D}$. For $t \geq 0$, define D_t^* to be the domain obtained by removing from $\mathbb{D}$ the closures of clusters of loops in $\mathcal{L}$ which are not contained in D_t. Let τ be a stopping time for the process $(D_t^*)_{t\geq}$. Then the conditional law of $\mathcal{L}$ restricted to D_τ^* given D_τ^* is that of a Brownian loop soup in D_τ^*.

Proof. The proof is a straightforward modification of the proof of Lemma 9.2 in [70], but for completeness we review it here. For $D \subset \mathbb{D}$ and $n \geq 1$ we define $F_n(D)$ to be the union of all squares in $2^{-n}\mathbb{Z}^2$ contained in D. Then for all such unions-of-squares $V \subset \mathbb{D}$, the event $\{F_n(D_\tau^*) = V\}$ depends only on the loops intersecting $\mathbb{D} \setminus V$ and is therefore independent of the loop configuration in V. This shows that for all $n \geq 1$, the loop configuration in $F_n(D_\tau^*)$ is a Brownian loop soup in $F_n(D_\tau^*)$. Since $\bigcup_n F_n(D_\tau) = D_\tau$, this shows that the configuration in D_τ^* is a Brownian loop soup. □

3.9 The loops determine the exploration process

In this section we show that the conformally invariant CLE_4 exploration process is a deterministic function of the CLE_4 loops. Our strategy is inspired by the one used by Dubédat in [13, 14] to prove that the Gaussian free field determines the SLE trace in the usual GFF/SLE coupling. The idea is to explore from $\partial\mathbb{D}$ to the loop γ surrounding the origin and from γ to $\partial\mathbb{D}$, with these explorations sampled conditionally independently given the CLE_4 loops. We show that the amounts of exploration time in each direction in this coupling are almost surely equal. This implies that the discovery time of γ is determined by the CLE_4 loops.

Denote by $\mathcal{K}$ the set of compact subsets of $\mathbb{C}$. For $K, K' \in \mathcal{K}$, we define the Hausdorff metric

$$d_{\text{Hausdorff}}(K, K') = \max\left\{ \sup_{x\in K} \inf_{y\in K'} |x-y|, \sup_{y\in K'} \inf_{x\in K} |x-y| \right\},$$

We regard a CLE_4 Γ as a compact set by considering its gasket, which is the set of points surrounded by no loop of Γ. Denote by $\mathcal{T}$ the space of processes $(D_t)_{t\geq 0}$ of nested domains, and we define

$$d_{\mathcal{T}}(D, \widetilde{D}) = \sup_{t\geq 0} d_{\text{Hausdorff}}(D_t, \widetilde{D}_t).$$

and equip $\mathcal{T}$ with its Borel σ-algebra. The metric $d_{\mathcal{T}}$ inherits completeness and separability from the Hausdorff metric [50], so $\mathcal{T}$ is Polish. Theorem 9.2.1 in [77] establishes the existence of regular conditional probability distributions for Polish space valued random elements, and we will use this theorem implicitly when we sample random elements from their conditional laws given certain σ-algebras.

3.9.1 Annulus CLE exploration

Proposition 3.9.1. Let $U \subset \mathbb{D}$ be an open set containing the origin, and let $(D_t)_{t\geq 0}$ be a uniform CLE_4 exploration in $\mathbb{D}$ with associated CLE_4 Γ. Define

$$T_U = \inf\{t \geq 0 : U \not\subseteq D_t\}.$$

Then, given the loops of Γ intersecting $\mathbb{D} \setminus U$, the loops of Γ contained in U and the exploration $(D_t)_{0\leq t\leq T_U}$ are conditionally independent.

Proof. Denote by Γ_U the loops of Γ contained in U. We couple $(D_t)_{0\leq t\leq T_U}$ and Γ as follows: sample $(D_t)_{0\leq t\leq T_U}$ from the law of a CLE_4 exploration stopped upon hitting U. Then sample Γ by sampling independent CLEs in the components unexplored by $(D_t)_{0\leq t\leq T_U}$. By the domain Markov property of CLE, Γ_U is distributed as a CLE_4 in the origin-containing complementary component of $\Gamma \setminus \Gamma_U$. In particular, Γ_U is conditionally independent of $(D_t)_{0\leq t\leq T_U}$ given $\Gamma \setminus \Gamma_U$. □

Proposition 3.9.2. Let $(D_t)_{t\in\mathbb{R}}$ be a uniform CLE_4 exploration in $\mathbb{D}$ with associated CLE Γ. Suppose that K is a random simply connected subset of $\mathbb{D}$ coupled with a CLE_4 in such a way that K contains the origin and almost surely no loop of Γ intersects both K and its complement. Then the conditional law of the exploration up to the hitting time of K is conditionally independent given Γ of the loops of Γ inside K.

Proof. For $n \in \mathbb{N}$, define the random domain U_n to be the simply connected open set of minimal area which contains K whose closure is a union of closed squares in the grid $2^{-n}\mathbb{Z}^2$. Denote by A_n the exploration up to T_{U_n}, by B_n the loops of Γ inside U_n, and by C_n the loops of Γ not inside U_n. Applying Proposition 3.9.1 to each domain U such that $\mathbb{P}[U_n = U] > 0$, we find that for all bounded continuous functions f and g,

$$\mathbb{E}[f(A_n)g(B_n)|C_n] = \mathbb{E}[f(A_n)|C_n]\,\mathbb{E}[g(B_n)|C_n]. \tag{3.9.1}$$

As $n \to \infty$, we have $A_n \to A$ almost surely, where A is the exploration until the hitting time of K. Similarly, we have $B_n \to B$ and $C_n \to C$, where B denotes

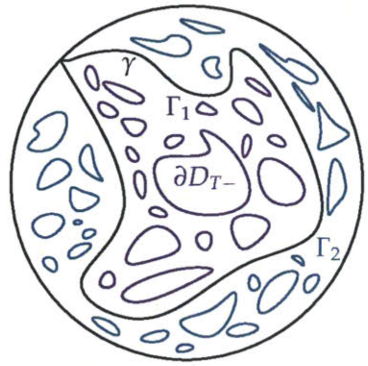


Figure 3-13: Define a loop ensemble by first sampling a pinned loop γ surrounding the origin, and then inside of γ sample an exploration until just before the loop surrounding the origin is discovered at time T. This procedure is a reversal of the CLE_4 exploration process in $\mathbb{D}$, wherein we explore loops not surrounding the origin for an exponential amount of time and then sample a pinned loop surrounding the origin. Proposition 3.9.3 says that loops discovered by the exploration along with ∂D_{T-} (shown in purple) along with an independent CLE_4 outside γ (shown in cyan) form a CLE_4 in $\mathbb{D}$ (Proposition 3.9.3).

the loops of Γ inside K and B denotes the loops of Γ not inside K. Hunt's lemma [10, Theorem 5.45] states that if $(X_n)_{n\in\mathbb{N}}$ is an L^1-dominated sequence of random variables almost surely converging to X and $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a monotone sequence of σ-algebras with $\mathcal{F} = \sigma\left(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n\right)$, then $\mathbb{E}[X_n \mid \mathcal{F}_n] \to \mathbb{E}[X \mid \mathcal{F}]$ almost surely (and in L^1). Applying this lemma to take $n \to \infty$ on both sides of (3.9.1), we obtain the desired result. □

Proposition 3.9.3. Let γ be a loop sampled from the bubble measure $\nu^{\text{bub}}(\mathbb{D})$ restricted to loops surrounding the origin. Let $(D_t)_{t\geq 0}$ be an independent CLE_4 exploration inside γ with $\widetilde{\Gamma}$ the associated CLE_4, and let T be the time when the loop containing the origin is discovered. Define Γ_1 to be the loops of $\widetilde{\Gamma}$ outside of D_{T-}, as well as ∂D_{T-}. Sample an independent CLE_4 Γ_2 in the simply connected region between γ and $\partial\mathbb{D}$. Then the union Γ of Γ_1 and Γ_2 is a CLE_4 in $\mathbb{D}$.

Proof. Consider two consecutive CLE_4 loops $\mathcal{L}_2$ and $\mathcal{L}_4$ surrounding the origin in an origin-nested CLE_4 coupled with a whole-plane Gaussian free field, as defined in Theorem 3.6.3 (see Figure 3-14). Define $\mathcal{L}_1$ and $\mathcal{L}_3$ to be the exploration frontiers just before the discovery times of $\mathcal{L}_1$ and $\mathcal{L}_3$, respectively. We will refer to touching pairs of oppositely-oriented loops, such as $(\mathcal{L}_1, \mathcal{L}_2)$, *figure eights*. The proposition is equivalent to the claim that $\mathcal{L}_3$ along with the loops between $\mathcal{L}_1$ and $\mathcal{L}_3$ are distributed as a CLE_4.

By inversion invariance of the GFF, loops $\mathcal{L}_3$ and $\mathcal{L}_1$ are consecutive CLE_4 loops surrounding ∞ in the exploration of h from 0 to ∞. Therefore, the loops between $\mathcal{L}_3$ and $\mathcal{L}_1$ are distributed as an annulus CLE. Furthermore, $\mathcal{L}_3$ has the law of the

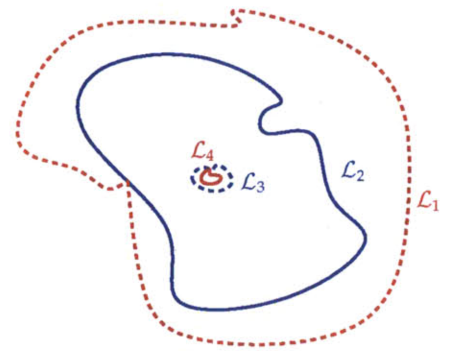


Figure 3-14: Consider two consecutive figure eights in a whole-plane Gaussian free field. The CLE_4 loops are illustrated with solid lines, and the exploration frontiers just before the discovery time of each CLE_4 loop are shown dashed. The loops are colored according to their orientation as GFF level lines.

origin-surrounding loop for a CLE_4 in $\mathcal{L}_1$ by the inversion invariance of CLE_4, which is proved in [26]. □

3.9.2 The two-way annulus exploration

Let $\mathcal{L}$ be a CLE_4 loop surrounding the origin in an origin-nested whole plane CLE_4, sample a CLE_4 Γ_0 inside $\mathcal{L}$, and let $\mathcal{K}$ be the CLE_4 loop surrounding the origin. Denote by Ω the annular region between $\mathcal{L}$ and $\mathcal{K}$, and denote by Γ the loop ensemble in Ω obtained by removing $\mathcal{L}$ from Γ_0. By Proposition 3.3.6, Γ is an annulus CLE in Ω. We sample an outside-in annulus exploration $(D_t)_{0\leq t\leq D}$ of Γ from its conditional law given Γ, and conditionally independent of $(D_s)_{0\leq s\leq S}$ we sample an outside-in exploration $(E_t)_{0\leq t\leq T}$ from its law given Γ. Denote by μ the resulting law of $(\Omega, (D_s)_{0\leq s\leq S}, (E_t)_{0\leq t\leq T})$. Denote by μ_1 the measure on quintuples

$$(\Omega, (D_s)_{0\leq s\leq S}, (E_t)_{0\leq t\leq T}, \sigma, \tau)$$

obtained by (1) sampling $(\Omega, (D_s)_{0\leq s\leq S}, (E_t)_{0\leq t\leq T})$ from $S\,d\mu/\mu(S)$, (2) choosing σ from the uniform distribution on $[0, S]$ (and conditionally independent given S of everything else), and (3) defining

$$\tau := \sup\{t \geq 0 \,:\, E_t \not\subseteq \Omega \setminus D_\sigma\}.$$

We define μ_2 similarly except that τ is selected uniformly from $[0, T]$ and

$$\sigma := \sup\{s \geq 0 \,:\, D_s \not\subseteq \Omega \setminus E_\tau\}.$$

Lemma 3.9.4. The marginal laws of

$$(\Omega, (D_s)_{0\leq s\leq \sigma}, (E_t)_{0\leq t\leq \tau})$$

under μ_1 and μ_2 are equal.

Proof. We begin by describing a way to sample $(\Omega, (D_s)_{0\le s\le\sigma}, (E_t)_{0\le t\le\tau})$ from its μ_1 law. Sample the region Ω and exploration $(D_s)_{0\le s\le S}$ from its law under μ_1. Choose σ uniformly in $[0, S]$, and note that conditioned on ∂D_σ and ∂D_S, the CLE_4 loops between ∂D_σ and ∂D_S are distributed as an annulus CLE Γ in that region. By Proposition 3.9.3 and 3.9.2, we may sample an annulus CLE exploration from ∂D_S to ∂D_σ from its conditional law given Γ. If we denote this exploration by $(E_t)_{0\le t\le\tau}$, then by Proposition 3.9.2 the law of $(\Omega, (D_s)_{0\le s\le\sigma}, (E_t)_{0\le t\le\tau})$ is the same as its μ_1-law.

Define $(\mathcal{L}_1, \mathcal{L}_2)$, $(\mathcal{L}_3, \mathcal{L}_4)$, and $(\mathcal{L}_5, \mathcal{L}_6)$ to be three consecutive figure eights in a whole-plane GFF, numbered outside-in. Define a measure ν on triples $(\Omega, (D_s)_{0\le s\le\sigma}, (E_t)_{0\le t\le\tau})$ by letting Ω be the annular region between $\mathcal{L}_2$ and $\mathcal{L}_5$, letting $(D_s)_{0\le s\le\sigma}$ be the GFF exploration from $\mathcal{L}_2$ to $\mathcal{L}_3$, and letting $(E_t)_{t\le\tau}$ be the GFF exploration from $\mathcal{L}_5$ to $\mathcal{L}_4$. In the following paragraph we argue that the μ_1-law of $(\Omega, (D_s)_{0\le s\le\sigma}, (E_t)_{0\le t\le\tau})$ is equal to ν.

The μ-law of S is a unit mean exponential random variable, so the μ_1-law of S is $te^{-t}\,dt$. Thus the μ_1-law of σ, which is uniformly distributed from 0 to S, is also a unit mean exponential and is therefore the same as the μ-law of S. Thus the μ_1- and ν-laws of the exploration $(D_s)_{0\le s\le\sigma}$ agree. Furthermore, the ν and μ_1 conditional laws of the inner boundary of Ω given D_σ are both given by $Q(\partial D_\sigma)$, where $Q = \mathcal{L} \mapsto \mathcal{K}$ denotes the transition kernel mapping a loop $\mathcal{L}$ surrounding the origin to the law of the loop $\mathcal{K}$ surrounding the origin for a CLE_4 in $\mathcal{K}$. This is true by definition for ν, and for μ_1 it may be seen by noting that the μ_1-law of $S - \sigma$ given σ is a unit mean exponential independent of σ. Finally, the conditional laws of the explorations $(E_t)_{t\le\tau}$ given $(\partial D_s)_{0\le s\le\sigma}$ and ∂D_S are both given by annulus explorations from ∂D_S to ∂D_σ.

By inversion invariance of the whole-plane Gaussian free field, the law of $(\Omega, (D_s)_{0\le s\le\sigma}, (E_t)_{0\le t\le\tau})$ under μ_2 is also equal to ν. This concludes the proof. □

Lemma 3.9.5. $\mu_1(S) = \mu_1(T)$

Proof. We will abbreviate $D_{\le s} := (D_u)_{u\le s}$ and similarly for E. We claim that

$$\mu_1[S \mid \Omega, \Gamma, D_{\le\sigma}, E_{\le\tau}] = \mu_1[T \mid \Omega, \Gamma, D_{\le\sigma}, E_{\le\tau}]. \tag{3.9.2}$$

The lemma follows from (3.9.2) and Lemma 3.9.4.

For $s \in \mathbb{R}$, we define the function

$$g(\Omega, \Gamma, D_{\le s}) = \mu[S \mid \Omega, \Gamma, D_{\le s}].$$

Then the left-hand side of (3.9.2) equals $\mu_1[S \mid \Omega, \Gamma, D_{\le\sigma}]$, since E is conditionally independent of S. This in turn equals $\mu[S \mid \Omega, \Gamma, D_{\le\sigma}]$, because the μ- and μ_1-laws of the evolution of D after time σ are the same. Thus the left-hand side of (3.9.2) equals $g(\Omega, \Gamma, D_{\le\sigma})$.

The right hand side can be written as

$$\mu_1[T \mid \Omega, \Gamma, D_{\le\sigma}, E_{\le\tau}] = \mu[T \mid \Omega, \Gamma, D_{\le\sigma}, E_{\le\tau}] = \mu[T \mid \Omega, \Gamma, D, \sigma, E_{\le\tau}].$$

because the forward exploration is conditionally independent of S given Ω and Γ. We define for $t \geq 0$

$$M_t := \mu[T \mid \Omega, \Gamma, D, \sigma, E_{\leq t \wedge T}] = g(\iota(\Omega, \Gamma, E_{\leq t})), \tag{3.9.3}$$

where ι denotes the inversion $z \mapsto 1/z$. Since (3.9.3) holds for all t, it holds for all stopping times taking countably many values. Note that M is a nonnegative martingale, which by Doob's upcrossing lemma implies that M_t has a most countably many discontinuities. Since the conditional law of τ given Ω, Γ, and E is absolutely continuous with respect to Lebesgue measure (by Lemma 3.9.4), τ is almost surely a continuity point of M_t. Therefore, we may define $\tau_n = 2^{-n}\lfloor \tau 2^n \rfloor$, apply (3.9.3) with $t = \tau_n$, and take $n \to \infty$ to conclude that the right-hand side of (3.9.2) is the same as $g(\iota(\Omega, \Gamma, E_{\leq \tau}))$. The result then follows from Lemma 3.9.4. $\square$

Theorem 3.9.6. We have $S = T$ almost surely.

Proof. From Lemma 3.9.5, $\mu(S^2) = \mu_1(S) = \mu_1(T) = \mu(ST)$. By symmetry, $\mu(S^2) = \mu(T^2)$. Therefore, by the Cauchy-Schwarz inequality, $S = \lambda T$ almost surely for some $\lambda \in \mathbb{R}$. Since $\mu(S^2) = \mu(T^2)$, we have $S = T$ almost surely. $\square$

3.10 The CLE_4 metric

In this section we use result that the CLE_4 loops determine their exploration to define a metric on CLE_4 loops. In this metric, the balls of radius t correspond to the set of loops discovered by the exploration up to time t. We begin by describing a simple way to recover a metric from its metric balls.

3.10.1 An alternate metric space axiomatization

If (X, d) is a metric space, then for every $x \in X$ and $r \geq 0$, the closed d-ball of radius r centered at x is defined by

$$B_r(x) = \{y \in X : d(x,y) \leq r\}.$$

The collection $\{B_r(x) : x \in X, r \geq 0\}$ satisfies the following properties:
(i) (identity of indiscernables) $B_0(x) = \{x\}$,
(ii) (nesting) $B_r(x) \subset B_s(x)$ whenever $0 \leq r \leq s$,
(iii) (symmetry) for all $x, y \in X$, $\sup\{r \geq 0 : y \notin B_r(x)\} = \sup\{r \geq 0 : x \notin B_r(y)\} < \infty$, and
(iv) (triangle inequality) for all $x, y \in X$ and $r, s \geq 0$ such that $r + s < \sup\{t \geq 0 : y \notin B_t(x)\}$, we have $B_r(x) \cap B_s(y) = \emptyset$.

It is straightforward to verify that if $\{B_r(x) : x \in X, r \geq 0\}$ is a collection of sets satisfying these properties, then there exists a unique metric d on X such that for all $x \in X$ and $r \geq 0$, the closed d-ball of radius r centered at x is $B_r(x)$.

The following proposition establishes a sufficiency condition for a collection of subsets of X to satisfy the metric ball properties enumerated above.

Proposition 3.10.1. Let X be a nonempty set, and suppose that $\{B_r(x) : x \in X, r \geq 0\}$ is a collection of subsets of X satisfying properties (i) and (ii). Also, suppose that for all $x, y \in X$ and for all $0 < r < \sup\{t \geq 0 : x \notin B_t(y)\}$, we have

$$r + \sup\{s \geq 0 : B_r(x) \cap B_s(y) = \emptyset\} = \sup\{t \geq 0 : y \notin B_t(x)\} \in \mathbb{R}. \tag{3.10.1}$$

Then there exists a unique metric on x with respect to which the closed ball of radius r centered at x is $B_r(x)$.

Proof. We define $d(x,y) = \sup\{t \geq 0 : y \notin B_t(x)\}$. Note that (3.10.1) implies

$$d(x,y) \leq r + \sup\{s : B_r(x) \cap B_s(y) = \emptyset\} \leq r + d(y,x)$$

for $r > 0$ arbitrary. This establishes $d(x,y) \leq d(y,x)$. Reversing the roles of x and y shows that $d(y,x) \leq d(x,y)$, so property (iii) is satisfied.

To demonstrate property (iv), we note that if $r + s < d(x,y)$, then (3.10.1) implies that

$$s < \sup\{s \geq 0 : B_r(x) \cap B_s(y) = \emptyset\}.$$

By property (ii), this implies that $B_r(x) \cap B_s(y) = \emptyset$. □

3.10.2 The CLE_4 metric space

By Theorem 3.9.6, the discovery time of the loop containing the origin in a CLE_4 in the disk is a deterministic function of the CLE_4 loops. By conformal invariance, the discovery time of every loop, and hence the entire exploration, is a deterministic function of the CLE_4 loops.

The boundary of $\mathbb{D}$ plays a privileged role as the loop from which we explore in the CLE exploration process. However, it is also possible to explore from any CLE loop. When used to describe a loop, the term *shape* will be defined to be its equivalence class modulo scalings of the form $z \mapsto \lambda z$, where $\lambda > 0$. We say that a loop γ is stationary of the law of its shape is equal to the law of the shape of a loop in a nested CLE_4 in $\mathbb{C}$ (note that by scale-invariance, all such loop shapes have the same law). The construction in the following definition is illustrated in Figure 3-15.

Definition 3.10.2. Let Γ be a CLE_4 in $\mathbb{D}$. For $t \geq 0$ and $\mathcal{L} \in \Gamma$, we define $B_t(\mathcal{L}) \subset \Gamma$ as follows. Let γ be a stationary loop, independent of Γ, and conformally map Γ to the interior of γ. Choose a point z_0 in the interior of the image of $\mathcal{L}$, apply the inversion $z \mapsto 1/(z - z_0)$. Finally, conformally map the resulting configuration to the unit disk $\mathbb{D}$. Since this loop ensemble is a CLE_4 in $\mathbb{D}$, we may define an exploration process from $\partial\mathbb{D}$ which is determined by the CLE_4 loops. We define $B_t(\mathcal{L})$ to be the preimage under this composition of conformal maps of the set of loops discovered up to and including time t in this exploration process.

By conformal invariance, the resulting set $B_t(\mathcal{L})$ does not depend on γ, the choice of z_0, or the two choices of conformal map between $\mathbb{D}$ and a stationary loop (the first and third maps illustrated in Figure 3-15).

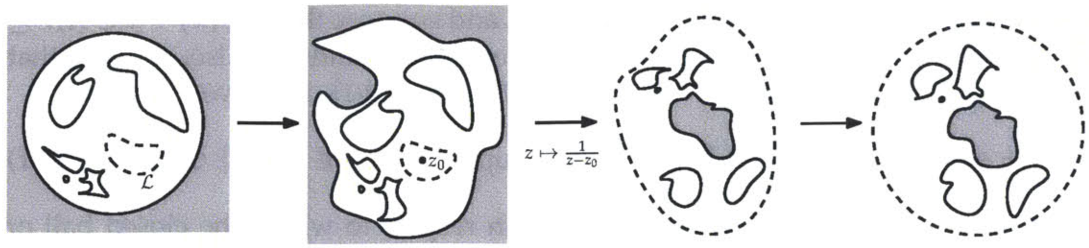


Figure 3-15: To define the ball of radius t centered at $\mathcal{L}$ in terms of the CLE exploration, we apply a series of conformal maps to obtain a CLE configuration with (the image of) $\mathcal{L}$ as the outer boundary. The first map sends the configuration to the interior of an independent, stationary loop. The second map is an inversion that positions the image of the $\mathcal{L}$ as the outermost loop in the configuration. Finally, we conformally map to $\mathbb{D}$.

Proposition 3.10.3. There exists a metric δ on Γ whose closed ball of radius r centered at $\mathcal{L}$ is equal to $B_r(\mathcal{L})$, for all $r \geq 0$, $\mathcal{L} \in \Gamma$.

Proof. It suffices to show that $\{B_r(\mathcal{L}) : r \geq 0, \mathcal{L} \in \Gamma\}$ satisfies the hypotheses of Proposition 3.10.1. Conditions (i) and (ii) are immediate from the construction.

Let $(D_{s\geq S})$ be the exploration from ∂D to the loop $\mathcal{L}$ surrounding the origin, and let $(E_t)_{t\geq 0}$ be the exploration from $\mathcal{L}$ to ∂D. Define

$$F(s) = \inf\{t \geq 0 : \partial E_t \nsubseteq D_s\}.$$

By Theorem 3.9.6, $F(0) = S$ and $F(T) = S$. Furthermore, by Lemma 3.9.4, the conditional laws of σ and $F(\sigma)$ given Γ are both the uniform distribution on $[0, S]$. It follows that $F(s) = S - s$, which implies (3.10.1). □

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