18.440 PROBLEM SET 10

A. FROM TEXTBOOK CHAPTER NINE: Problems and Theoretical Exercises: 13, 15, 16, 17.

B. Read the Wikipedia articles on Black-Scholes, martingales, and the optional stopping theorem. Write a two paragraph summary of each. Then solve the following problems

1. Suppose Harriet has 7 dollars. Her plan is to make one dollar bets on fair coin tosses until her wealth reaches either 0 or 50, and then to go home. What is the expected amount of money that Harriet will have when she goes home? What is the probability that she will have 50 when she goes home?

2. Consider a contract that at time N will be worth either 100 or 0. Let $S(n)$ be its price at time n for $0 \le n \le N$. If $S(n)$ is a martingale, and $S(0) = 47$, then what is the probability that the contract will be worth 100 at time N?

3. Suppose $S(N)$ is with probability one either 100 or 0 and that $S(0)$ = 50. Suppose further there is at least a 60 percent probability that $S(n)$ will at some point dip to below 40 and then subsequently rise to above 60 before time N. Prove that $S(n)$ cannot be a martingale

C. Complete the derivation of the Black-Scholes formula for European call options, as outlined in the lecture slides, by explicitly computing

$$E[g(e^N)]e^{-rT},$$

where N is a normal random variable with mean $\mu = \log X_0 + (r - \sigma^2/2)T$ and $g(x) = \max\{0, x - K\}$.

D. Recall Pedro's story from lecture: there is a safe asset (whose value does not change) and a risky asset whose value goes up 15 percent with probability .53 and down 15 percent with probability .47 (independently of everything else) each month. But now suppose that Pedro has the option to divide his money between the risk-free and the risky assets.

1. If Pedro wishes to maximize the expected log of his wealth after one month, what fraction of his wealth should he invest in the risky asset?

2. If Pedro is allowed to rebalance his wealth between the two assets every month, what strategy should he use to maximize the expected log of the amount of money he will have after 100 years (1200 months)? (In particular, what fraction of his wealth should he invest in the risky asset each month? Should it be the same fraction every month?)

3. Suppose that Pedro uses this strategy for 100 years. Compute the mean and standard deviation of the log of Pedro's wealth at the end. Pedro's wealth itself after 100 years is a random variable. Estimate its median.

4. Read the Wikipedia article on the "Kelly criterion". Can you explain what the Kelly criterion has to do with Pedro's strategy?

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