

MIT Open Access Articles

Symmetry relations in viscoplastic drag laws

The MIT Faculty has made this article openly available. **Please share** how this access benefits you. Your story matters.

Citation: Kamrin, K., and J. D. Goddard. "Symmetry Relations in Viscoplastic Drag Laws." Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 470, no. 2172 (October 8, 2014): 20140434–20140434.

As Published: <http://dx.doi.org/10.1098/rspa.2014.0434>

Publisher: Royal Society

Persistent URL: <http://hdl.handle.net/1721.1/97530>

Version: Original manuscript: author's manuscript prior to formal peer review

Terms of use: Creative Commons Attribution-Noncommercial-Share Alike



Symmetry relations in viscoplastic drag laws

K. Kamrin* and J. Goddard**

*Department of Mechanical Engineering
Massachusetts Institute of Technology

**Department of Mechanical and Aerospace Engineering
University of California, San Diego

Abstract

The following note shows that the symmetry of various resistance formulae, often based on Lorentz reciprocity for linearly viscous fluids, applies to a wide class of non-linear viscoplastic fluids. This follows from Edelen's non-linear generalization of the Onsager relation for the special case of *strongly dissipative* rheology, where constitutive equations are derivable from his dissipation potential. For flow domains with strong dissipation in the interior and on a portion of the boundary this implies strong dissipation on the remaining portion of the boundary, with strongly dissipative traction-velocity response given by a dissipation potential. This leads to a non-linear generalization of Stokes resistance formulae for a wide class of viscoplastic fluid problems. We consider the application to non-linear Darcy flow and to the effective slip for viscoplastic flow over textured surfaces.

Introduction

Symmetry occupies an important position in the classical linear theories of elasticity, viscosity and viscoelasticity, where it is synonymous with self-adjointness and Lorentz reciprocity.

In the case of (hyper)elastic materials, the symmetry of the linear-elastic modulus is a consequence of the existence of a strain-energy function, whereas in the case of linearly viscous fluids, the symmetry of the viscosity tensor represents Rayleigh-Onsager symmetry. As shown by Day (Day, 1971), the same symmetry applies to the linear-viscoelastic memory function for materials that exhibit time-reversibility on certain closed strain paths¹.

¹It is not too difficult to show that for general linear-response theory, this implies a Hermitian matrix

For the special case of Newtonian fluids there have been numerous applications of Lorentz reciprocity to various problems of Stokes (inertialess) flow (Happel and Brenner, 1965; Hinch, 1972; Brunet and Ajdari, 2004) to obtain various symmetry restrictions on the associated drag laws. In a similar spirit, this same principle has been applied to the symmetries of apparent slip in the far-field above arbitrarily textured surfaces with arbitrary local Navier slip distribution (Kamrin and Stone, 2011). This problem area has received attention in recent day due to applications in microfluidics; see references in (Kamrin et al., 2010) for a large body of related work.

While the above symmetry is bound up with variational principles based on the associated quadratic forms, there exist more general nonlinear variants. Thus, in the case of non-linear hyperelasticity, there exist well-known elastostatic variational principles, with elastic stress given by the gradient of a strain-energy function, or by an associated pseudo-linear form involving a symmetric (tangent) modulus based on the Hessian of a complementary energy.

Less well known are the analogous forms for strictly dissipative nonlinear systems given by the general theory of Edelen (Edelen, 1972, 1973; Goddard, 2014) as a generalization of Onsager symmetry. In particular, Edelen proves that, modulo a gyroscopic or “powerless” force, the dissipative force \mathbf{f} is given as the gradient $\partial_{\mathbf{v}}\psi(\mathbf{v})$ of a dissipation potential $\psi(\mathbf{v})$ depending on a generalized velocity \mathbf{v} or, again, by a pseudo-linear form based on the Hessian of a complimentary potential. Whenever the gyroscopic force is identically zero, we call the system *strongly dissipative* or, by analogy to the elastic case, *hyperdissipative*. For later reference, we note the dual form $\mathbf{v} = \partial_{\mathbf{f}}\varphi(\mathbf{f})$ where φ is the (Legendre-Fenchel) convex conjugate of ψ .

The main goal of the current note is to identify and exploit connections between strongly dissipative *local* properties of a system (i.e. constitutive relations and/or surface interactions) and strongly dissipative global properties, which often take the form of homogenized macroscale relations. In particular, we will demonstrate how this result restricts (i) Darcy-like laws for porous flow, and (ii) effective slip relations over textured surfaces, when the fluid rheology and surface interactions are non-linear while maintaining a strongly dissipative form. Throughout, we emphasize the ubiquity of strong dissipation among commonly used viscoplastic fluid models and, hence, the generality of the results to be presented.

As discussed elsewhere (Goddard, 2014), Edelen’s work has interesting implications for rate-independent rigid plasticity, where plastic potentials are often based on rather special

$\hat{\mathbf{L}}(\omega)$ in the Fourier description $\hat{\mathbf{f}}(\omega) = \hat{\mathbf{L}}(\omega)\hat{\mathbf{v}}(\omega)$ in the frequency domain of the linear relation connecting generalized force $\mathbf{f}(t)$ and velocity $\mathbf{v}(t)$ in the time domain.

physical arguments. Moreover, Edelen's work provides a rigorous mathematical extension to the phenomenological viscoplastic potentials introduced by others (Rice, 1970), which leads to interesting analogies between viscoplasticity and non-linear elasticity. In both domains, variational principles govern quasi-static stress equilibrium, and the associated symmetry carries over to various symmetries in global force laws. Some existing applications of highly special viscoplastic potentials involve extremum principles for non-Newtonian flow (Johnson, 1961) and bounds for nonlinear homogenization (Dormieux et al., 2006).

The purpose of the present work is to introduce variational forms into strongly dissipative flow problems and to explore the consequences for some special cases involving viscoplastic flow within porous media and over textured surfaces. In each case, a set of symmetry conditions and differential constraints arise as necessary conditions on the homogenized resistance formulae.

Strongly dissipative rheology and mobile boundaries

In what follows $\boldsymbol{\sigma}$ represents Cauchy stress, \mathbf{v} material velocity, and $\mathbf{D} = \text{sym}(\nabla \mathbf{v})$ associated strain-rate tensor, all fields depending on spatial position and time, \mathbf{x} and t . We employ the prime ' to denote the deviator, with

$$\mathbf{S} \equiv \boldsymbol{\sigma}' \equiv \boldsymbol{\sigma} - \frac{1}{3}(\text{tr } \boldsymbol{\sigma})\mathbf{1}, \quad \text{and} \quad \mathbf{E} = \mathbf{D}' = \mathbf{D} - \frac{1}{3}(\text{tr } \mathbf{D})\mathbf{1}, \quad (1)$$

where $\mathbf{1}$ denotes the three-dimensional identity tensor.

We begin by defining a strongly dissipative incompressible viscoplastic fluid as one which obeys

$$\begin{aligned} \mathbf{S} = \mathbf{S}(\mathbf{E}) = \partial_{\mathbf{E}}\psi(\mathbf{E})', \quad \text{with } \mathfrak{D} = \mathbf{E} : \mathbf{S}(\mathbf{E}) \geq 0, \\ \text{and } \mathfrak{D} > 0 \text{ for } |\mathbf{E}| \neq 0 \ \& \ \mathfrak{D} = 0 \text{ for } |\mathbf{E}| = 0. \end{aligned} \quad (2)$$

\mathfrak{D} denotes dissipation (per volume), and the inequality represents a convexity condition on ψ . Here as below, the colon $:$ denotes tensorial contraction, with e.g. $\mathbf{A} : \mathbf{B} = \text{tr } \mathbf{A}\mathbf{B}^T$ representing the (Euclidian) scalar product of real tensors. Also, we write $\partial_X = (\partial / \partial X)^T$ and the tensorial derivative carries the usual definition, i.e. $[\partial_{\mathbf{E}}\psi(\mathbf{E})]_{ij} = \partial_{E_{ij}}\psi(\mathbf{E})$.

The function ψ represents a *dissipation potential* according to the definition of Edelen. Owing to its convexity, the roles of the dependent and independent variables can be swapped, since there exists a dual potential or Legendre convex conjugate $\varphi(\mathbf{S}) = \mathbf{S} : \mathbf{E}(\mathbf{S}) - \psi(\mathbf{E}(\mathbf{S}))$,

Fluid Model	Deviatoric stress \mathbf{S}	Dissipation potential ψ
Newtonian	$2\eta\mathbf{E}$	$\eta \mathbf{E} ^2$
Power-law	$2K \mathbf{E} ^{n-1}\mathbf{E}$	$2K \mathbf{E} ^{(n+1)}/(n+1)$
Bingham plastic	$\mu\mathbf{E}/ \mathbf{E} + 2\eta\mathbf{E}$	$\mu \mathbf{E} + \eta \mathbf{E} ^2$
Herschel-Bulkley	$\mu\mathbf{E}/ \mathbf{E} + 2K \mathbf{E} ^{n-1}\mathbf{E}$	$\mu \mathbf{E} + 2K \mathbf{E} ^{1+n}/(1+n)$

Table 1: Common incompressible fluid models exhibiting a strongly dissipative form

such that the inverse of (2) is given by

$$\mathbf{E} = \mathbf{E}(\mathbf{S}) = \partial_{\mathbf{S}}\varphi(\mathbf{S})', \quad \text{with } \mathbf{S}: \mathbf{E}(\mathbf{S}) \geq 0 \quad (3)$$

We note that this duality is singular in the case of certain non-smooth rheologies, such as rate-independent plasticity, where $\psi(\mathbf{E})$ is a homogeneous function of degree one (Goddard, 2014).

It turns out that many common non-Newtonian models have the strongly dissipative form. As examples, Table 1 lists some standard models of viscoplastic fluids, with $|\mathbf{E}| = (\mathbf{E}:\mathbf{E})^{1/2}$. The Bingham and Herschel-Bulkley models represent a class of “yield-stress fluids”, which have indeterminate stress at the rest state $\mathbf{E} = \mathbf{0}$, unless the unit director $\mathbf{E}/|\mathbf{E}|$ is specified. The long-standing problem of determining the spatial location of yield surfaces is the subject of ongoing research cited in a recent review article (Denn and Bonn, 2011).

The models displayed in the table are special cases of a potential $\psi(\mathbf{E}) = \Psi(I_2, I_3)$, where $I_2 = |\mathbf{E}|^2/2$ and $I_3 = \det \mathbf{E}$ are the non-zero isotropic invariants of \mathbf{E} . This potential yields the incompressible isotropic (Reiner-Rivlin) model:

$$\mathbf{S} = (\partial_{I_2}\Psi)\mathbf{E} + (\partial_{I_3}\Psi)(\mathbf{E}^2 - \frac{1}{3}|\mathbf{E}|^2\mathbf{1}) = \boldsymbol{\eta}(\mathbf{E}) : \mathbf{E}, \quad (4)$$

where $\boldsymbol{\eta}(\mathbf{E}) = [\eta_{ijkl}(\mathbf{E})]$ is a non-negative fourth-rank viscosity tensor representing a *pseudo-linear* form (Goddard, 2014) of a type that appears frequently in the following.

We recall that the dissipation potentials in Table 1 follow from the special forms considered by Hunter (Hunter, 1976), who may have been unaware of the general theory of Edelen, and we note that certain variational principles have been formulated for the special case of the “generalized Newtonian fluid” (Johnson, 1961) $\mathbf{S} = \eta(|\mathbf{E}|)\mathbf{E}$ involving a scalar viscosity depending on a single scalar invariant.

All the above isotropic fluid models are special cases of a more general *anisotropic* fluid, with $\boldsymbol{\sigma} = \partial_{\mathbf{D}}\psi(\mathbf{D}, \boldsymbol{\mathcal{S}}) = \boldsymbol{\eta}(\mathbf{D}, \boldsymbol{\mathcal{S}}):\mathbf{D}$, where Edelen’s dissipation potential ψ depends on the joint isotropic invariants of \mathbf{D} and a set of “structure tensors” $\boldsymbol{\mathcal{S}}$. Moreover, such tensors may depend on the history of flow, with evolution described by a set of objective Lagrangian ODEs depending on \mathbf{D} . For example, in the case of a single “fabric” tensor \mathbf{A} , the evolution equation takes the form

$$\overset{\circ}{\mathbf{A}} = \boldsymbol{\alpha}(\mathbf{A}, \mathbf{D}). \quad (5)$$

where superposed “o” denotes the Jaumann or corotational derivative. The joint isotropic invariants of \mathbf{A}, \mathbf{D} are well known, and are represented by a finite set of traces of the form $\text{tr}(\mathbf{A}^n \mathbf{D}^m)$. Models of this type also allow for change of density and include Reiner’s “dilatant” isotropic fluid (Reiner, 1945) as well various anisotropic variants with application to dilatant granular media. That said, we focus attention in the present work on incompressible fluids.

It is worth recalling several previous works on non-linear flow in porous media, including less general models of non-Newtonian flow in Bear (Bear, 1988, p. 128) and Dormieux et al. (Dormieux et al., 2006) as well as turbulent flow of Newtonian fluids, where one can also define a dissipation potential. See e.g. Joseph et al. (1982) or (Bear, 1988, p. 177) and Dormieux et al. (2006).

Boundary conditions

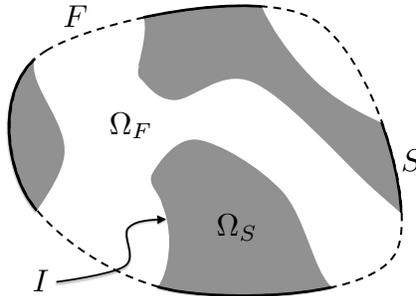


Figure 1: 2D schematic of flow in a porous medium $\Omega = \Omega_S \cup \Omega_F$ or in a (simply connected) fluid region Ω_F above a textured surface I (See Fig. 2).

We turn now from bulk constitutive relations to a consideration of boundary conditions. Thus, we imagine a region of space $\Omega = \Omega_F \cup \Omega_S$ with $I \cap F = \emptyset$, consisting of disjoint

surfaces I and F , where I represents the interface between the fluid and solid region and F is the bounding surface lying in the fluid. Then, the boundary of the fluid domain is $\partial\Omega_F = I \cup F$.

For example, in the porous solid illustrated in Fig. 1, F represents that part of $\partial\Omega$ lying in the fluid, with I representing the interface between Ω_S and Ω_F on which there is partial slip with no permeation. It may be possible to extend our analysis so as to permit permeation through I , but we shall specialise shortly to the case of an impermeable surface I .

Considering the traction of fluid on solid $\mathbf{t}_I = -(\boldsymbol{\sigma}\mathbf{n})$ for \mathbf{n} the interfacial outward normal (pointing into solid), we set down the following definitions:

A *globally dissipative surface* I is one for which

$$\int_I \mathbf{t}_I \cdot \mathbf{v} \, dS \geq 0, \quad (6)$$

a *locally dissipative surface* I is one for which

$$\mathbf{t}_I(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \geq 0, \quad \forall \mathbf{x} \in I, \quad (7)$$

and a *strongly dissipative surface* I is locally dissipative surface described by a dissipative surface potential $\psi_I(\mathbf{v}_I; \mathbf{x})$, such that

$$\mathbf{t}_I(\mathbf{x}) = \partial_{\mathbf{v}} \psi_I(\mathbf{v}; \mathbf{x}), \quad \text{with } \mathbf{t}_I(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \geq 0 \quad \text{for all } \mathbf{x} \in I \quad (8)$$

The relation (6) can be expected to apply on the surface I of a porous medium with internal viscoplastic flow in pores whose walls are locally dissipative, whereas (7) and (8) may fail to apply because of the local power input to the fluid (“pV” work) associated with outflow from I .

Owing to the impermeability of the solid interface, \mathbf{t}_I can be replaced by the vector of shear traction $\boldsymbol{\tau}_I = -(\mathbf{1} - \mathbf{n} \otimes \mathbf{n})\boldsymbol{\sigma}\mathbf{n}$ which provides the power associated with surface slip. Letting φ_I represent the dual potential to ψ_I , we note for example that $\varphi_I(\boldsymbol{\tau}_I) = \ell|\boldsymbol{\tau}|^2/2\eta$ gives the *Navier* slip relation for the slip velocity $\mathbf{v} = \partial_{\boldsymbol{\tau}_I} \varphi_I = \ell\boldsymbol{\tau}_I/\eta$ in a Newtonian fluid with viscosity η and slip-length ℓ , for which the no-slip condition is given by $\ell \equiv 0$. As an extension of Navier slip, the strongly dissipative slip relation proposed here describes a wide range of wall-slip phenomena, including nonlinear, anisotropic and spatially inhomogeneous slip, with inhomogeneity represented by dependence on $\mathbf{x} \in I$.

For example, when the anisotropy is determined by a symmetric surface tensor $\mathbf{\Lambda} : \mathcal{T}_I \rightarrow \mathcal{T}_I$, transforming vectors in the tangent space \mathcal{T}_I of I we may take $\varphi_I(\boldsymbol{\tau})$ to be a function of the joint isotropic invariants of $\boldsymbol{\tau}, \mathbf{\Lambda}$:

$$|\boldsymbol{\tau}|, \text{tr } \mathbf{\Lambda}, \text{tr } \mathbf{\Lambda}^2, \boldsymbol{\tau} \cdot \mathbf{\Lambda} \boldsymbol{\tau}, \dots$$

an instructive special case being

$$\varphi_I(\boldsymbol{\tau}; \mathbf{x}) = f(\boldsymbol{\tau} \cdot \mathbf{\Lambda} \boldsymbol{\tau}), \quad \text{with } \mathbf{v}_I = 2f'(\boldsymbol{\tau}_I \cdot \mathbf{\Lambda} \boldsymbol{\tau}_I) \mathbf{\Lambda} \boldsymbol{\tau}_I, \quad (9)$$

where the prime denotes a derivative, and $\mathbf{\Lambda} = \mathbf{\Lambda}(\mathbf{x})$ is allowed depend on position \mathbf{x} on I . When $f(s)$ is given by a power law in s one obtains an anisotropic power-law for \mathbf{v}_I in terms of $\boldsymbol{\tau}_I$. Were we to consider a permeable surface I , then $\mathbf{\Lambda}$ would have to be replaced by a more general linear transformation $\mathbb{R}^3 \rightarrow \mathbb{R}^3$.

From locally to globally strong dissipation

The local considerations above have immediate implications for the existence of systems that are strongly dissipative in a global sense.

Suppose that a strongly dissipative fluid occupies a region Ω_F discussed above. A flow field $\mathbf{v}(\mathbf{x})$ in Ω_F is said to be admissible, if $\nabla \cdot \mathbf{v} = \mathbf{0}$ and its associated stress field satisfies $\nabla \cdot \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0}$ for $\forall \mathbf{x} \in \Omega_F$. We next suppose that the fluid exhibits a strongly dissipative slip on I and shall then prove that *all admissible solutions for the flow in Ω_F must satisfy strongly dissipative boundary conditions on F* .

As a general convention, we denote vector-valued functionals by arguments enclosed in brackets $[\]$, e.g. $\mathbf{f} = \mathbf{f}[\mathbf{w}]$ denotes a map $\mathfrak{F} \rightarrow \mathbb{R}^n$ from the vector field $\mathbf{w}(\mathbf{x})$ to the real vector \mathbf{f} , where $\mathbf{w}(\mathbf{x})$ is an element of function space \mathfrak{F} , e.g. a Banach space. We denote maps $\mathfrak{F} \rightarrow \mathfrak{F}$ of one vector field to another by curly brackets $\{ \ } = \$, i.e. $\mathbf{f} = \mathbf{f}\{\mathbf{w}\} = \mathbf{f}[\mathbf{w}](\mathbf{x})$ denotes a map from the vector field $\mathbf{w}(\mathbf{x})$ to the vector field $\mathbf{f}(\mathbf{x})$. Further, for a given vector-valued functional $\mathbf{f}[\mathbf{w}]$, we make use of the *Fréchet (or variational) derivative* of $\mathbf{f}[\mathbf{w}]$, denoted $\delta_{\mathbf{w}} \mathbf{f}\{\mathbf{w}\}$, which is the mapping $\mathfrak{F} \rightarrow \mathfrak{F}$ that satisfies

$$\langle \delta_{\mathbf{w}} \mathbf{f}\{\mathbf{w}\}, \mathbf{h} \rangle \equiv \int_{\mathcal{R}} \mathbf{h}(\mathbf{x}) \cdot \delta_{\mathbf{w}} \mathbf{f}\{\mathbf{w}\}(\mathbf{x}) \, d\mathcal{R}(\mathbf{x}) = \lim_{\epsilon \rightarrow 0} \frac{\mathbf{f}[\mathbf{w} + \epsilon \mathbf{h}] - \mathbf{f}[\mathbf{w}]}{\epsilon} \quad (10)$$

for all test functions $\mathbf{h} \in \mathfrak{F}$, where \mathcal{R} is the domain of the vector field $\mathbf{w}(\mathbf{x})$. The bilinear

pairing above defines a functional of \mathbf{h} referred to as the *Fréchet differential* denoted $\delta_{\mathbf{w}}\mathbf{f}[\mathbf{h}] = \langle \delta_{\mathbf{w}}\mathbf{f}\{\mathbf{w}\}, \mathbf{h} \rangle$.

Our main results pertain to the nature of the mapping from boundary velocity fields $\mathbf{v}_F(\mathbf{x}) : F \rightarrow \mathbb{R}^3$ to the corresponding boundary traction field $\mathbf{t}_F(\mathbf{x}) : F \rightarrow \mathbb{R}^3$, i.e. the mapping $\mathbf{t}_F = \mathbf{t}_F\{\mathbf{v}_F\}$. To be specific, $\mathbf{t}_F = \boldsymbol{\sigma}\mathbf{n}$ is the surface traction field on F that arises from the admissible bulk flow that has \mathbf{v}_F as boundary condition. Due to incompressibility, the set of boundary velocities is further restricted to be solenoidal, i.e. $\int_F \mathbf{v}_F \cdot \mathbf{n} \, dS = 0$, where subscript F denotes a function whose domain is restricted to the surface F .

We shall show that

$$\int_F \mathbf{t}_F\{\mathbf{v}_F\} \cdot \mathbf{v}_F \, dS \geq 0, \quad (11)$$

and secondly, that there exists a functional $\Psi[\mathbf{v}_F]$ such that

$$\delta_{\mathbf{v}_F} \Psi\{\mathbf{v}_F\} = \mathbf{t}_F\{\mathbf{v}_F\}. \quad (12)$$

which establishes a direct analogy to (2). Note that $\mathbf{t}_F\{\mathbf{v}_F\}$ is defined up to the addition of an inessential uniform surface pressure.

Proof:

Assuming sufficiently smooth potentials and fields, we may express the quasi-static stress balance in terms of *virtual power* as

$$\int_F \mathbf{t}_F\{\mathbf{v}_F\} \cdot \mathbf{u}_F \, dS = \int_I \mathbf{t}_I(\mathbf{v}) \cdot \mathbf{u}_I \, dS + \int_{\Omega_F} \mathbf{S}(\mathbf{E}) : \nabla \mathbf{u} \, dV, \quad (13)$$

for any admissible velocity field $\mathbf{v}(\mathbf{x})$, with $\mathbf{E} = \text{sym } \nabla \mathbf{v}$, where $\mathbf{u}(\mathbf{x})$ is any solenoidal “test field”.

Now, we assume here and in the following that the choice of boundary field $\mathbf{v}_F(\mathbf{x})$, $\mathbf{x} \in F$, uniquely determines the bulk flow $\mathbf{v}(\mathbf{x})$ and strain-rate $\mathbf{E}(\mathbf{x})$. In the case of multiplicity, a possibility we do not entertain here, one could presumably restrict the solution to a particular solution branch.

In the case of an impermeable surface, \mathbf{t}_I can be replaced by $\boldsymbol{\tau}_I$ above, and, invoking locally strong dissipation in $\Omega_F \cup I$, we have the local relations

$$\mathbf{S} = \partial_{\mathbf{E}}\psi(\mathbf{E}; \mathbf{x}), \quad \boldsymbol{\tau}_I = \partial_{\mathbf{v}}\psi_I(\mathbf{v}; \mathbf{x}), \quad (14)$$

where $v_n = \mathbf{v} \cdot \mathbf{n} = 0$ on I . We have allowed the potential ψ to depend on position \mathbf{x} to reflect a possible dependence on inhomogeneous and/or evolutionary structure tensors of the type discussed above. Note that time variations in velocity do not imply instantaneous variations in these tensors, which are presumably governed by smooth ODEs of the form (5).

The preceding relations yield a linear functional of the test velocity \mathbf{u}_F on F :

$$L_{\mathbf{v}_F}[\mathbf{u}_F] := \int_F \mathbf{t}_F\{\mathbf{v}_F\} \cdot \mathbf{u}_F \, dS = \int_{\Omega_F} \partial_{\mathbf{E}}\psi(\mathbf{E}; \mathbf{x}) : \nabla \mathbf{u} \, dV + \int_I \partial_{\mathbf{v}}\psi_I(\mathbf{v}; \mathbf{x}) \cdot \mathbf{u}_I \, dS \quad (15)$$

It is evident that in the case where $\mathbf{u} \equiv \mathbf{v}$ in Ω_F that the dissipation of $\mathbf{v}(\mathbf{x})$ is given by

$$\mathfrak{D}[\mathbf{v}] = L_{\mathbf{v}_F}[\mathbf{v}_F] = \int_F \mathbf{t}_F\{\mathbf{v}_F\} \cdot \mathbf{v}_F \, dS = \int_{\Omega_F} \partial_{\mathbf{E}}\psi(\mathbf{E}; \mathbf{x}) : \mathbf{E} \, dV + \int_I \partial_{\mathbf{v}}\psi_I(\mathbf{v}_I; \mathbf{x}) \cdot \mathbf{v}_I \, dS \geq 0 \quad (16)$$

as all integrands in the last expression are necessarily positive by local strong dissipation.

By employing (10), it can also be easily proved that:

$$L_{\mathbf{v}_F}[\mathbf{u}_F] = \langle \delta_{\mathbf{v}_F} \Psi, \mathbf{u}_F \rangle, \quad \text{where } \Psi[\mathbf{v}_F] = \int_{\Omega_F} \psi(\mathbf{E}; \mathbf{x}) \, dV + \int_I \psi_I(\mathbf{v}; \mathbf{x}) \, dS \quad (17)$$

the latter relation following from the fact that $\mathbf{v}(\mathbf{x})$ and $\mathbf{E}(\mathbf{x})$ are determined by \mathbf{v}_F . The above, together with (15) implies the desired result

$$\delta_{\mathbf{v}_F} \Psi\{\mathbf{v}_F\} = \mathbf{t}_F\{\mathbf{v}_F\}. \quad (18)$$

In the Appendix, we indicate that the above results follow from a tentative extension of Edelen's formula for finite dimensional vector spaces to infinite dimensional (e.g. Banach) function spaces.

Remarks:

- i. In the dual description with interfacial boundary condition $\mathbf{v}_I = \partial_{\boldsymbol{\tau}_I} \phi_I(\boldsymbol{\tau}_I)$ on I , we can swap variables and show that dual functional $\Phi[\mathbf{t}]$ exists, mapping traction fields \mathbf{t} on F to a scalar, such that

$$\delta_{\mathbf{t}_F} \Phi\{\mathbf{t}_F\} = \mathbf{v}_F\{\mathbf{t}_F\}. \quad (19)$$

- ii. The relation (12), or its dual (19), confirms the perhaps intuitively obvious fact that the dissipation and dissipation potential are given directly by the boundary field \mathbf{v}_F

on F . To compute the corresponding boundary traction \mathbf{t}_F in full, one would need to compute the actual fluid velocity field $\mathbf{v}(\mathbf{x}) = \mathbf{v}\{\mathbf{v}_F\}$ in Ω_F . However, the fact that the boundary field \mathbf{t}_F must arise as a Fréchet derivative places a restriction on \mathbf{t}_F that can be exploited without specification of the full flow field. Based on simplified boundary conditions, the following sections will make use of this derivative in a form appropriate to finite dimensional vector spaces.

- iii. In the case of unbounded regions, we should replace integrals like those in (16)-(17) by appropriate averages over Ω_F, I and F , which we do not bother to define explicitly.
- iv. There are direct analogies between the theorem proven above and the global theorems of hyperelasticity, owing to the fact that both are based on constitutive potentials. We note however, that the non-trivial slip boundary conditions in this treatment would correspond to an unconventional *Robin* traction-displacement condition in the elastic analogy. As in hyperelasticity, the admissible solution minimizes the functional for appropriate boundary conditions. Unlike hyperelasticity, where the functional represents total potential energy, the functional Ψ provides in fact a *lower bound* for total dissipation, with equality in the case of a linear rheology/slip condition, or proportionality in the case of homogeneous potentials. The lower-bound property is easily proved based on the guaranteed convexity of the underlying local dissipation potentials.
- v. The proof above relies on C^1 smoothness of the underlying potentials, such that a unique derivative and therefore a unique stress can be assumed in Eq 2. However, we note that certain dissipation potentials φ may become “kinked”, with loss of convexity, such as those representing the above-mentioned yield-stress fluids. Although the stress may fail to be uniquely defined at zero strain rate, we shall assume that the solution to the variational problem, including the spatial location of the associated yield surfaces, can be rendered unique by various techniques, such as those employed to determine “singular minimizers” for non-convex hyperelasticity (Ball, 2001). In this respect, we note that existing treatments (Denn and Bonn, 2011) of yield-stress fluid appear to overlook the associated extremum principles and the applicability of variational methods.
- vi. For the sake of definiteness, we have adopted in (13) a relatively strong form of virtual power. However, certain fluids may exhibit material instability, arising from a loss of convexity of the dissipation potential and leading to the formation of singular surfaces. At such surfaces the velocity field may become non-differentiable and even discontinuous,

e.g. across an infinitely thin “shear band”. Indeed, the assumed slip on the surface I has this same character. To cover such singular behavior, we may express (13) in the weaker form involving jumps $[[\mathbf{u}]]_J$ in \mathbf{u} on a discrete set of singular surfaces J across which the traction \mathbf{t}_F is continuous:

$$\int_F \mathbf{t}_F \{ \mathbf{v}_F \} \cdot \mathbf{u} dS = \sum_J \int_J \mathbf{t}_J \cdot [[\mathbf{u}]]_J dS + \int_{\Omega_F} \mathbf{S}(\mathbf{E}) : \nabla \mathbf{u} dV, \text{ where } \mathbf{t}_J = \boldsymbol{\sigma} \cdot \mathbf{n}|_J \quad (20)$$

In this case, we admit discontinuous test velocity fields, and (13) is a special case in which $\mathbf{u} = \mathbf{0}$ on I , with jump $[[\mathbf{u}]] = \mathbf{u}_I$, and is otherwise continuous.

We now illustrate the utility of the above results by the application to two physically interesting problems involving flow around a geometrically complex solid surface.

Generalized Darcy flow in porous media

For the application to viscoplastic flow in a porous medium, we apply the above to two types of boundary condition on the fluid surface F of a given porous body, namely prescribed velocity and prescribed stress.

Traction BC

We first consider the boundary condition

$$\mathbf{t}_F = -(\mathbf{g} \cdot \mathbf{x})\mathbf{n} + \mathbf{T}\mathbf{n}, \quad \mathbf{x} \in F, \quad (21)$$

for a constant vector \mathbf{g} and constant tensor \mathbf{T} . Above, \mathbf{g} represents a pressure gradient applied to the fluid boundary of the domain and \mathbf{T} is a symmetric boundary stress tensor.

As in the preceding subsection, we assume now that the boundary condition (21) determines the flow in Ω_F . The functionals of \mathbf{t}_F in (19) now reduce to functions of \mathbf{g} and \mathbf{T} ,

with dissipation and potentials obeying

$$\begin{aligned}
\mathcal{D}[\mathbf{t}_F] &= D(\mathbf{g}, \mathbf{T}) = V_F(\mathbf{g} \cdot \bar{\mathbf{v}} + \mathbf{T} : \bar{\mathbf{D}}), \quad \text{with} \\
\bar{\mathbf{v}}(\mathbf{g}, \mathbf{T}) &= \frac{1}{V_F} \int_{\Omega_F} \mathbf{v} \, dV = \partial_{\mathbf{g}} \Phi(\mathbf{g}, \mathbf{T}), \\
\bar{\mathbf{D}}(\mathbf{g}, \mathbf{T}) &= \frac{1}{V_F} \text{sym} \left\{ \int_{\Omega_F} \nabla \mathbf{v} \, dV + \int_I \mathbf{n}_I \otimes \mathbf{v}_I \, dS \right\} = \partial_{\mathbf{T}} \Phi(\mathbf{g}, \mathbf{T}), \\
\text{and } \Phi(\mathbf{g}, \mathbf{T}) &= \Phi[\mathbf{t}_F(\mathbf{x})] \Big|_{\mathbf{t}_F(\mathbf{x}) = -(\mathbf{g}\mathbf{x})\mathbf{n} + \mathbf{T}\mathbf{n}}
\end{aligned} \tag{22}$$

where \mathbf{n}_I is the inner normal to the surface I . The second and third lines of (22) follow from the vanishing of $v_n = \mathbf{v} \cdot \mathbf{n}$ on I , and the relations

$$\begin{aligned}
\int_F \mathbf{x} v_n \, dS &= \int_{\partial\Omega_F} \mathbf{x} v_n \, dS = \int_{\Omega_F} \nabla \cdot (\mathbf{v} \otimes \mathbf{x}) \, dV = \int_{\Omega_F} \mathbf{v} \, dV, \\
\text{and } \int_F \mathbf{n} \otimes \mathbf{v} \, dS &= \int_{\partial\Omega_F} \mathbf{n} \otimes \mathbf{v} \, dS - \int_I \mathbf{n} \otimes \mathbf{v} \, dS = \int_{\Omega_F} \nabla \mathbf{v} \, dV + \int_I \mathbf{n}_I \otimes \mathbf{v}_I \, dS
\end{aligned} \tag{23}$$

These results establish the volume average velocity, $\bar{\mathbf{v}}$, and deformation rate, $\bar{\mathbf{D}}$, as respective conjugates of \mathbf{g} and \mathbf{T} . Note that we could have also obtained the result for average deformation gradient by considering a singular test field with $\mathbf{u} = \mathbf{0}$ and $[[\mathbf{u}]] = \mathbf{u}_I$ on I , as subsumed by (20). The fact that the drag laws for $\bar{\mathbf{v}}$ and $\bar{\mathbf{D}}$ both must emerge from a single potential function Φ is a significant restriction on the functional forms of each.

Velocity BC

We now consider flow induced by an applied velocity boundary condition

$$\mathbf{v}(\mathbf{x}) = \mathbf{q} + \mathbf{L}\mathbf{x}, \quad \mathbf{x} \in F, \tag{24}$$

where \mathbf{q} is a constant velocity vector and \mathbf{L} is a constant velocity-gradient tensor. The incompressibility constraint on \mathbf{v} then requires that

$$\int_F v_n \, dS = \mathbf{a} \cdot \mathbf{q} + \mathbf{A} : \mathbf{L} = 0, \quad \text{where } \mathbf{a} = \int_F \mathbf{n} \, dS \text{ and } \mathbf{A} = \int_F \mathbf{n} \otimes \mathbf{x} \, dS \tag{25}$$

The vector \mathbf{a} and the second rank tensor \mathbf{A} represent an ostensibly new class of structure tensors for a representative volume element (RVE), given generally by the n th rank moment

tensors:

$$\mathcal{A}^{(m)} = \int_F \mathbf{n}(\otimes \mathbf{x})^{m-1} dS, \quad m = 1, 2, \dots \quad (26)$$

The lowest moment \mathbf{a} equals the vectorial excess of out-flow over in-flow area, and \mathbf{A} can be regarded as a second-rank “fabric” tensor, by loose analogy to a tensor employed to describe the anisotropy of granular media. It is clear that the overall dissipation potential must eventually be given as a function $\Psi(\mathbf{q}, \mathbf{L}, \mathbf{a}, \mathbf{A})$, subject to the restriction (25).

We now specialise to isotropic media, defining an isotropic medium of *degree* M as one for which the structure tensors of order $m = 1, 2, \dots, M$ are invariant under the transformation $\mathbf{n}, \mathbf{x} \rightarrow \mathbf{Q}\mathbf{n}, \mathbf{Q}\mathbf{x}$, where \mathbf{Q} is an arbitrary constant orthogonal tensor. This requires that $\mathcal{A}^{(m)}$ be a scalar multiple of the isotropic tensor of order m , for $m = 1, 2, \dots, M$ and, hence, that $\mathbf{a} = \mathbf{0}$ and $\mathbf{A} \propto \mathbf{I}$ for the case $M = 2$ considered here.

Hence, for an isotropic medium of degree 2, (25) implies that \mathbf{q} and \mathbf{L} , with $\text{tr } \mathbf{L} = 0$, can be chosen independently. A common example from homogenization theory is a periodic porous structure, for an RVE consisting of a single periodic cell, discussed briefly below.

Assuming that the boundary condition (24) determines a unique solution to the flow in Ω_F , the functionals of \mathbf{v}_F in (16) reduce to ordinary functions of \mathbf{q} and \mathbf{L} with dissipation given by

$$\begin{aligned} \mathcal{D}[\mathbf{v}_F] &= D(\mathbf{q}, \mathbf{L}) = \mathbf{f} \cdot \mathbf{q} + \boldsymbol{\Sigma} : \mathbf{L}, \quad \text{with} \\ \mathbf{f}(\mathbf{q}, \mathbf{L}) &= \int_F \mathbf{t}_F\{\mathbf{v}_F\} dS = \partial_{\mathbf{q}} \Psi(\mathbf{q}, \mathbf{L}), \\ \boldsymbol{\Sigma}(\mathbf{q}, \mathbf{L}) &= \int_F \mathbf{t}_F\{\mathbf{v}_F\} \otimes \mathbf{x} dS = \partial_{\mathbf{L}} \Psi(\mathbf{q}, \mathbf{L}), \\ \text{and } \Psi(\mathbf{q}, \mathbf{L}) &= \Psi[\mathbf{v}_F(\mathbf{x})] \Big|_{\mathbf{v}_F(\mathbf{x})=\mathbf{q}+\mathbf{L}\mathbf{x}}. \end{aligned} \quad (27)$$

Here \mathbf{f} is the force and $\boldsymbol{\Sigma}$ the moment of traction acting on F , which can be regarded as the contribution of F to the volume average stress

$$\frac{1}{V_F} \int_{\Omega_F} \boldsymbol{\sigma} dV = \frac{1}{V_F} \int_{\Omega_F} \text{div}(\mathbf{x} \otimes \boldsymbol{\sigma}) dV = \frac{1}{V_F} \int_{\partial\Omega_F} \mathbf{t} \otimes \mathbf{x} dS \quad (28)$$

The relations (27)-(28) provide yet another extension of Darcy’s law which, in contrast to the case of stress boundary conditions, bears a rather opaque relation to the linear version. Still, it bears emphasizing that the emergence of $\boldsymbol{\Sigma}(\mathbf{q}, \mathbf{L})$ and $\mathbf{f}(\mathbf{q}, \mathbf{L})$ from a single potential $\psi(\mathbf{q}, \mathbf{L})$ is a notable restriction on the functional forms of both quantities.

In closing here we note that higher gradient theories would involve higher-order structure tensors of the type (26).

Permeability relations

The prior subsections apply to global flow relations, and we can apply a key result displayed in (22), namely,

$$\bar{\mathbf{v}} = \partial_{\mathbf{g}}\Phi, \quad (29)$$

to describe the average or *homogenized* permeability of a non-Newtonian fluid within a regular porous solid, with surface slip between fluid and solid that may be both non-linear and non-uniform.

As one example of a structure with a single well-defined permeability, consider an idealized periodic RVE, composed of a repeated tiling of box-shaped porous elements. Owing to periodicity, the permeability of a large sample of such a material can be defined by treating the flow through one such element induced by an applied pressure on the element of the form (21), with $\mathbf{T} = \mathbf{0}$. The pressure gradient \mathbf{g} provides an arbitrary pressure difference between parallel faces of the cell, and (29) shows that mean fluid velocity $\bar{\mathbf{v}}$ and the applied pressure gradient, \mathbf{g} are connected through the derivative of a potential $\Phi = \Phi(\mathbf{g})$.

Following a previous mathematical analysis (Goddard, 2014), the preceding relationship (29) can also be expressed in a *pseudo-linear* form in terms of a permeability tensor $\mathbf{K} = \mathbf{K}(\mathbf{g})$ defined by

$$\bar{\mathbf{v}} = \mathbf{K}(\mathbf{g}) \mathbf{g}. \quad (30)$$

where \mathbf{K} must take the form of the Hessian $\partial_{\mathbf{g}\mathbf{g}}^2\chi$ of a complimentary function $\chi(\mathbf{g})$ derived from $\Phi(\mathbf{g})$ (Goddard, 2014). This in turn implies the following symmetry and differential compatibility conditions on the permeability tensor:

$$\mathbf{K}(\mathbf{g}) = \mathbf{K}(\mathbf{g})^T \quad \text{and} \quad \frac{\partial K_{ij}}{\partial g_k} = \frac{\partial K_{jk}}{\partial g_i} \quad (31)$$

for all $i, j, k \in \{1, 2, 3\}$.

Thus, by means of (30) we extend the standard linear (Onsager) symmetry to the flow of rheologically non-linear fluids through porous solids with non-uniform and nonlinear interfacial slip. In contrast to the linear case, \mathbf{K} depends on \mathbf{q} , but as with linear case \mathbf{K} is non-negative definite, reflecting non-negativity of dissipation. The constraints imposed by (31) represent a severe restriction on the form of $\mathbf{K}(\mathbf{q})$, as a substantial extension of the

linear theory.

Nonlinear mobility in effective-slip problems

As a generalization of the corresponding Stokes-flow problem (Kamrin and Stone, 2011), we consider the far-field condition on the top surface F of a viscoplastic shear flow bounded below by a textured surface solid surface I on which strongly dissipative slip may occur. The interface I represents an impermeable solid with possibly non-uniform partial slip distribution.

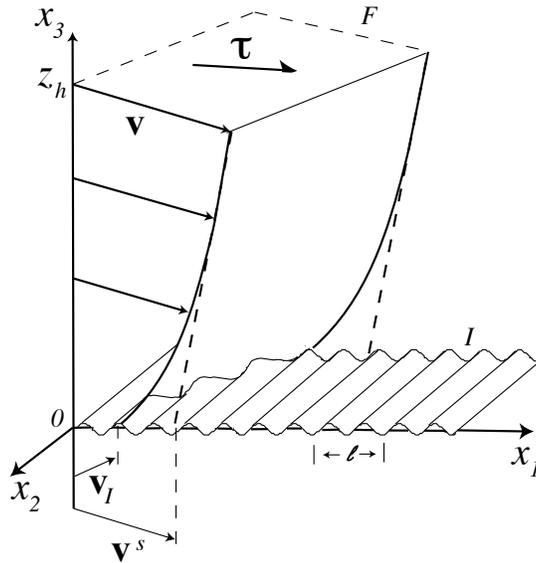


Figure 2: Effective slip above a textured surface.

Adopting Cartesian coordinates x_i with unit basis \mathbf{e}_i for $i = 1, 2, 3$, we consider the case of a surface I of infinite extent having mean elevation $x_3 = 0$. As illustrated in Fig. 2, the layer is bounded above by a flat plane F situated at an arbitrary position $x_3 = z_H$, and driven by a uniform parallel shearing traction $\boldsymbol{\tau}$, with $\boldsymbol{\tau} \cdot \mathbf{e}_3 = 0$. We assume the surface texture is a repeating pattern, and hence the induced flow field is periodic in horizontal planes. That said, we place no constraints on the period length so this could still represent an arbitrarily large surface pattern.

In the absence of external body forces or overall pressure gradient, and given $x_3 \gg \ell$, where ℓ is a characteristic length scale related to the surface texture, we assume that an

admissible velocity field will exhibit far field behavior consisting of uniform simple shear plus an apparent slip relative to the surface I :

$$\mathbf{v}(x_1, x_2, x_3 \gg \ell) = \dot{\boldsymbol{\gamma}} x_3 + \mathbf{v}^s, \quad \text{with } \dot{\boldsymbol{\gamma}} \cdot \mathbf{e}_3 = 0, \quad \mathbf{v}^s \cdot \mathbf{e}_3 = 0, \quad \mathbf{E} = \text{sym}(\dot{\boldsymbol{\gamma}} \otimes \mathbf{e}_3), \quad (32)$$

where the shear rate vector, $\dot{\boldsymbol{\gamma}}$, and the effective slip velocity, \mathbf{v}^s , are assumed to become asymptotically independent of position x_2, x_3 on F .

Given the fluid rheology, the surface pattern, and the slip relation on I , the problem is to find the relation between \mathbf{v}^s and $\boldsymbol{\tau}$ on F . This relation may be taken to represent an effective slip boundary condition for the large-scale description of the flow on length scales $\gg \ell$. In the case of a Newtonian fluid with linear slip on I , it has been shown (Kamrin and Stone, 2011) that there exists a positive symmetric 2×2 *surface (Onsager) mobility tensor* $\mathbf{M} = [M_{ij}]$, such that:

$$\mathbf{v}^s = \mathbf{M} \boldsymbol{\tau}, \quad \text{or } v_i^s = M_{ij} \tau_j, \quad i, j = 1, 2 \quad (33)$$

on a Cartesian basis. Here, we extend the analysis to nonlinear viscoplastic fluids with strongly dissipative nonlinear interfacial slip.

As as a helpful albeit not essential device, we define our domain Ω as a cell $[-\ell_1, \ell_1] \times [-\ell_2, \ell_2] \times [z_L, z_H]$, where z_L lies within the rigid solid beneath I and ℓ_1 and ℓ_2 are the respective period lengths in the horizontal plane. Owing to the flow periodicity, the vertical walls need not be considered part of $\partial\Omega$, and we have $F = [-\ell_1, \ell_1] \times [-\ell_2, \ell_2] \times z_H$ and $S = [-\ell_1, \ell_1] \times [-\ell_2, \ell_2] \times z_L$. Then, the formulae in (27) for constant velocity \mathbf{q} on F are applicable, and one readily finds, upon choosing \mathbf{q} to have a vanishing x_3 component, that

$$D = \boldsymbol{\tau} \cdot \mathbf{q} = \boldsymbol{\tau} \cdot \dot{\boldsymbol{\gamma}} x_3 + \boldsymbol{\tau} \cdot \mathbf{v}^s, \quad \boldsymbol{\tau} = \partial_{\mathbf{q}} \Psi(\mathbf{q}), \quad \mathbf{q} = \partial_{\boldsymbol{\tau}} \Phi(\boldsymbol{\tau}). \quad (34)$$

Here, the surface force \mathbf{f} in (27) is $\boldsymbol{\tau} A_F$, and one absorbs $A_F = 4\ell_2\ell_2$ into the potential. The last relation above follows as the Legendre convex conjugate of the prior relation, and we obtain the same potential Φ from the Legendre conjugate as the one described in (27). Because the fluid rheology is strongly dissipative, with $\mathbf{D}(\mathbf{S}) = \partial_{\mathbf{S}} \phi$ for some given dissipation potential ϕ , the flow at the top boundary satisfies

$$\dot{\boldsymbol{\gamma}}(\boldsymbol{\tau}) = \frac{\partial \phi^H}{\partial \boldsymbol{\tau}}(\boldsymbol{\tau}) \quad \text{for} \quad \phi^H(\boldsymbol{\tau}) \equiv 2 \phi(\boldsymbol{\tau} \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \boldsymbol{\tau}). \quad (35)$$

Hence, by the definition of effective slip, we may write

$$\mathbf{q} = z_H \frac{\partial \phi^H}{\partial \boldsymbol{\tau}}(\boldsymbol{\tau}) + \mathbf{v}^s. \quad (36)$$

Combining (34) and (36), and defining $\phi^s(\boldsymbol{\tau}) = \Phi(\boldsymbol{\tau}) - z^H \phi^H(\boldsymbol{\tau})$, we arrive at

$$\mathbf{v}^s = \frac{\partial \phi^s}{\partial \boldsymbol{\tau}}. \quad (37)$$

We thus find that the slip velocity must arise from the shear traction as the gradient of a *slip potential*. As shown in a previous analysis (Goddard, 2014) this implies the existence of a function χ^M such that the pseudo-linear form

$$\mathbf{v}^s = \mathbf{M}(\boldsymbol{\tau}) \boldsymbol{\tau} \quad (38)$$

arises from a mobility obeying $\mathbf{M}(\boldsymbol{\tau}) = \partial^2 \chi^M(\boldsymbol{\tau}) / \partial \boldsymbol{\tau}^2$. Accordingly the mobility must obey the following symmetry and differential compatibility conditions:

$$\mathbf{M}(\boldsymbol{\tau}) = \mathbf{M}(\boldsymbol{\tau})^T \quad \text{and} \quad \frac{\partial M_{ij}}{\partial \tau_k} = \frac{\partial M_{jk}}{\partial \tau_i} \quad (39)$$

for all $i, j, k \in \{1, 2\}$. In the linear case, the constant \mathbf{M} tensor can be shown to be symmetric via Lorentz reciprocity (Kamrin and Stone, 2011), whereas (39) establishes the symmetry of the mobility tensor for the more general case of any strongly dissipative rheology and interfacial slip relation.

Conclusions

The foregoing analysis addresses a general class of problems involving viscoplastic flow adjacent to solid boundaries with nonlinear and possibly nonuniform slip conditions. When the bulk rheology and interfacial slip take on strongly dissipative forms, the relationship between the velocity and traction on fluid boundaries are interrelated by global variational derivatives. Exploiting this fact, we have established symmetries of viscoplastic drag laws for two illustrative applications.

In the first example, involving flow in porous media, the global conjugacy between traction and velocity on the external fluid boundary is employed to determine permeability relations for strongly dissipative fluids exhibiting slip along the pore walls. The permeability is shown

to be characterized by a symmetric positive-definite tensor that depends on the flow, with partial derivatives satisfying a set of compatibility conditions.

In the second example, we have explored the implications of strong dissipation for viscoplastic flow over textured surfaces. As an extension of the linear case, we find once again a symmetric positive-definite mobility tensor which connects the far-field shear traction to the apparent slip velocity and satisfies once more a set of differential compatibility conditions.

While the foregoing analysis has focused on impenetrable solid boundaries and incompressible fluids, many of the results may remain qualitatively correct whenever these conditions are relaxed. In particular, note that on replacing \mathbf{S} by $\boldsymbol{\sigma}$ and \mathbf{E} by \mathbf{D} one can obtain a theory for viscoplastic flow with variable density, such as might occur in a dilatant granular media or particle suspension. Such variants on the current model may engender additional complications that seem worthy of further analysis.

Although we have not explored various consequences of the associated extremum principles for strongly dissipative fluids, these may prove a convenient tool for other applications. Important examples are the the derivation of continuum models for dispersions of rigid or viscoplastic particles in viscoplastic fluids, or the determination of the spatial configuration of yield surfaces in yield-stress fluids.

Appendix: Functional for the dissipation potential

Edelen's formula (Goddard, 2014) for the dissipation potential $\psi(\mathbf{v})$ in a finite vector space \mathbb{R}^n is given in terms of the dissipation $D(\mathbf{v})$ by

$$\psi(\mathbf{v}) = \int_0^1 D(\lambda\mathbf{v})\lambda^{-1}d\lambda, \quad (40)$$

with conjugate force given by $\mathbf{f} = \partial_{\mathbf{v}}\psi$ in the dual space.

Without attempting a rigorous proof here, we offer as a conjecture a generalization to a function space (e.g. Banach space) \mathfrak{F} of vector-valued velocities $\mathbf{v}(\mathbf{x})$ and (dual space) of vector-valued forces $\mathbf{f}\{\mathbf{v}\}$, $\mathbf{x} \in \mathcal{R}$. It appears that one might achieve this generalization rigorously by extending Edelen's differential geometric treatment to Banach spaces (Lang, 1999)². Thus, with dissipation defined by the pairing

$$\mathcal{D}[\mathbf{v}] = \langle \mathbf{v}, \mathbf{f} \rangle := \int_{\mathcal{R}} \mathbf{v}(\mathbf{x}) \cdot \mathbf{f}\{\mathbf{v}\} d\mathcal{R} := \int_{\mathcal{R}} \mathbf{v}(\mathbf{x}) \cdot \mathbf{f}[\mathbf{v}](\mathbf{x}) d\mathcal{R}, \quad (41)$$

²We are indebted to Professor Reuven Segev for this reference.

the corresponding function-space forms for potential and associated force are:

$$\Psi[\mathbf{v}] = \int_0^1 \mathcal{D}[\lambda\mathbf{v}]\lambda^{-1}d\lambda, \text{ with } \delta_{\mathbf{v}}\Psi[\mathbf{q}] = \langle \mathbf{f}, \mathbf{q} \rangle = \int_{\mathcal{R}} \mathbf{f}\{\mathbf{v}\} \cdot \mathbf{q} \, d\mathcal{R}, \text{ i.e. } \mathbf{f}\{\mathbf{v}\} = \delta_{\mathbf{v}}\Psi\{\mathbf{v}\}. \quad (42)$$

It is not too difficult to verify the final relation in (42) by making use of the properties of Fréchet derivatives, whose existence is guaranteed by *Moreau's Theorem* for convex functionals (?).

For example, with solenoidal velocity field \mathbf{v} and associated deviatoric straining \mathbf{E} , we have

$$\mathcal{D}[\mathbf{v}] = \int_{\Omega} D(\mathbf{E}) \, dV, \text{ where } D(\mathbf{E}) = \mathbf{E} : \mathbf{S}(\mathbf{E}) \quad (43)$$

Substitution of the resulting expression for $\mathcal{D}[\lambda\mathbf{v}]$ of into the integrand in (42) gives

$$\Psi[\mathbf{v}] = \int_{\Omega} \psi(\mathbf{E}) \, dV, \text{ since } \psi(\mathbf{E}) = \int_0^1 D(\lambda\mathbf{E})\lambda^{-1}d\lambda, \quad (44)$$

By taking the domain $\mathcal{R} = \Omega_F \cup I$, one obtains by similar reasoning the final relation in (16). It seems plausible that one might be able further to establish a pseudo-linear form for the force appearing in (42):

$$\mathbf{f}\{\mathbf{v}\} := \mathbf{f}[\mathbf{v}](\mathbf{x}) = \int_{\mathcal{R}} \mathcal{L}[\mathbf{v}](\mathbf{x}, \mathbf{y})\mathbf{v}(\mathbf{y}) \, d\mathcal{R}(\mathbf{y}), \quad (45)$$

involving a positive-definite matrix $\mathcal{L}[\mathbf{v}](\mathbf{x}, \mathbf{y})$. However, this is a matter requiring a more thorough mathematical treatment.

Literature Cited

Ball, J. M. (2001). Singularities and computation of minimizers for variational problems. In DeVore, R., Iserles, A., and Süli, E., editors, *Foundations of Computational Mathematics*, volume 284 of *London Mathematical Society Lecture Note Series*, pages 1–20. Cambridge University Press, Cambridge.

Bear, J. (1988). *Dynamics of fluids in porous media*. Dover, New York.

Brunet, E. and Ajdari, A. (2004). Generalized Onsager relations for electrokinetic effects in anisotropic and heterogeneous geometries. *Phys. Rev. E*, 69:016306.

- Day, W. A. (1971). Time-reversal and the symmetry of the relaxation function of a linear viscoelastic material. *Arch. Ratl. Mech. Anal.*, 40(3):155–59.
- Denn, M. and Bonn, D. (2011). Issues in the flow of yield-stress liquids. *Rheologica Acta*, 50(4):307–315.
- Dormieux, L., Kondo, D., and Ulm, F.-J. (2006). *Microporomechanics*. Wiley.
- Edelen, D. G. B. (1972). A nonlinear Onsager theory of irreversibility. *Int. J. Eng. Sci.*, 10(6):481–90.
- Edelen, D. G. B. (1973). On the existence of symmetry relations and dissipation potentials. *Arch. Ratl. Mech. Anal.*, 51:218–27.
- Goddard, J. D. (2014). Edelen’s dissipation potentials and the visco-plasticity of particulate media. *Acta Mech.*, accepted and to appear.
- Happel, J. and Brenner, H. (1965). *Low Reynolds number hydrodynamics*. Prentice-Hall, Englewood Cliffs, N.J.
- Hinch, E. J. (1972). Note on the symmetries of certain material tensors for a particle in Stokes flow. *J. Fluid Mech.*, 54:423–425.
- Hunter, S. C. (1976). *Mechanics of Continuous Media*. Halsted Press, Sydney.
- Johnson, M. W. (1961). On variational principles for non-newtonian fluids. *Trans. Soc. Rheol.*, 5:9–21.
- Joseph, D. D., Nield, D. A., and Papanicolaou, G. (1982). Nonlinear equation governing flow in a saturated porous medium. *Water Resources Research*, 18(4):1049–52.
- Kamrin, K., Bazant, M. Z., and Stone, H. A. (2010). Effective slip boundary conditions for arbitrary periodic surfaces: the surface mobility tensor. *Journal of Fluid Mechanics*, 658:409–437.
- Kamrin, K. and Stone, H. A. (2011). The symmetry of mobility laws for viscous flow along arbitrarily patterned surfaces. *Phys. Fluids*, 23:031701.
- Lang, S. (1999). *Fundamentals of differential geometry*, volume 191 of *Graduate texts in mathematics*. Springer, New York.

Reiner, M. (1945). A mathematical theory of dilatancy. *Am. J. Math.*, 67(3):350–62.

Rice, J. R. (1970). On the structure of stress- strain relations for time-dependent plastic deformation in metals. *Journal of Applied Mechanics*, 37:728–737.