Some Problems in Graph Ramsey Theory

by

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Abstract

A graph G is r-Ramsey minimal with respect to a graph H if every r-coloring of the edges of G yields a monochromatic copy of H, but the same is not true for any proper subgraph of G. The study of the properties of graphs that are Ramsey minimal with respect to some H and similar problems is known as graph Ramsey theory; we study several problems in this area.

Burr, Erdős, and Lovász introduced $s(H)$, the minimum over all G that are 2-Ramsey minimal for H of $\delta(G)$, the minimum degree of G. We find the values of $s(H)$ for several classes of graphs H, most notably for all 3-connected bipartite graphs which proves many cases of a conjecture due to Szabó, Zumstein, and Zürcher.

One natural question when studying graph Ramsey theory is what happens when, rather than considering all 2-colorings of a graph G, we restrict to a subset of the possible 2-colorings. Erdős and Hajnal conjectured that, for any fixed color pattern C, there is some $\varepsilon > 0$ so that every 2-coloring of the edges of a K_n, the complete graph on n vertices, which doesn't contain a copy of C contains a monochromatic clique on n^ε vertices. Hajnal generalized this conjecture to more than 2 colors and asked in particular about the case when the number of colors is 3 and C is a rainbow triangle (a K_3 where each edge is a different color); we prove Hajnal's conjecture for rainbow triangles.

One may also wonder what would happen if we wish to cover all of the vertices with monochromatic copies of graphs. Let $\mathcal{F} = \{F_1, F_2, \ldots\}$ be a sequence of graphs such that F_n is a graph on n vertices with maximum degree at most Δ. If each F_n is bipartite, then the vertices of any 2-edge-colored complete graph can be partitioned into at most $2^{C\Delta}$ vertex disjoint monochromatic copies of graphs from $\mathcal{F}$, where C is an absolute constant. This result is best possible, up to the constant C.

Thesis Supervisor: Jacob Fox
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Contents

Chapter 1

Introduction

A graph G is *H-Ramsey* with r colors, denoted by $G \to_r H$, if any r-coloring of the edges of G contains a monochromatic copy of H. If $r = 2$ we simply write $G \to H$. The fact that for every graph H there is a graph G such that G is H-Ramsey was first proved by Ramsey [79] in 1930 and rediscovered independently by Erdős and Szekeres a few years later [36]. Ramsey theory is currently one of the most active areas of combinatorics with connections to number theory, geometry, analysis, logic, and computer science.

A fundamental problem in graph Ramsey theory is to understand the graphs G satisfying G is K_k-Ramsey, where K_k denotes the complete graph on k vertices. The Ramsey number $r(H)$ is the minimum number of vertices of a graph G which is H-Ramsey. The most famous question in this area is that of estimating the Ramsey number $r(K_k)$. Classical results of Erdős [31] and Erdős and Szekeres [36] show that $2^{k/2} \leq r(K_k) \leq 2^{2k}$. While there have been several improvements on these bounds (see, for example, [21]), despite much attention, the constant factors in the above exponents remain the same. Given these difficulties, the field has naturally stretched in different directions. In this thesis we prove new results in three of these directions. It should be noted that the text of this thesis, including the abstract and introduction, may closely follow or be directly taken from various papers of the author including [41, 42, 44, 51, 52].

1.1 Chapter 2: Minimum Degrees of Minimal Ramsey Graphs

The first such direction is to understand, for a fixed graph H, properties of the collection of graphs that are Ramsey for H. Clearly, for a fixed graph H, the collection of graphs that are Ramsey for it is upwards closed. That is, if G is Ramsey for H and G is a subgraph of G', then G' is also Ramsey for H. Therefore, to understand the collection of graphs that are Ramsey for H, it is sufficient to understand the collection $\mathcal{M}(H)$ of graphs that are minimal subject to being Ramsey for H; these graphs are called *Ramsey minimal* for H.

For a graph G, let $\delta(G)$ denote the minimum degree of the vertices of G. Our interest in Chapter 2 lies in $s(H)$, which is the minimum of $\delta(G)$ over all graphs $G \in \mathcal{M}(H)$. This parameter was first introduced and studied by Burr, Erdős, and Lovász in 1976 [12]. A simple upper bound is $s(H) \leq r(H) - 1$. Indeed, one may take any Ramsey-minimal graph on $r(H)$ vertices, and the minimum degree of this graph is at most $r(H) - 1$. Since $r(K_t)$ is exponential in t, the result of Burr, Erdős, and Lovász [12] that $s(K_t) = (t-1)^2$ may be surprising.

Fox and Lin [43] observed the simple lower bound $s(H) \geq 2\delta(H) - 1$ which holds for every graph H. We say a graph H is *Ramsey simple* if this lower bound is tight, that is if $s(H) = 2\delta(H) - 1$. In recent years, the study of $s(H)$ has received increased attention. Fox and Lin [43] present an alternative proof that $s(K_t) = (t-1)^2$ and also show that $K_{s,t}$ is Ramsey simple, where $K_{s,t}$ is a complete bipartite graph with parts of size s and t. Szabó, Zumstein, and Zürcher [88] conjectured that every bipartite graph without isolated vertices is Ramsey simple.

They prove this conjecture for a variety of bipartite graphs including trees, even cycles, and bipartite graphs where every vertex in one of the parts has degree $\delta(H)$. They also prove the conjecture for connected bipartite graphs with parts A of size a and B of size b with $b \geq a$ in which A contains a minimum degree vertex. It is worth noting that they also address the case of isolated vertices.

In Chapter 2 we prove this conjecture for all 3-connected bipartite graphs, as

well as for several other graphs. We also find the first examples of graphs that are Ramsey simple but are not bipartite and the first examples of connected graphs H, H' satisfying $H \subseteq H'$ with $s(H) > s(H')$, a rather surprising property.

1.2 Chapter 3: The Erdős-Hajnal Conjecture for Rainbow Triangles

In Chapter 3 we pursue a different variant when studying which graphs satisfy $G \to H$, namely by considering what happens when, rather than considering all colorings of G, we restrict which colorings are allowed. Erdős and Hajnal [35] famously conjecture that, for any 2-coloring C of a complete graph, there is a $\varepsilon = \varepsilon(C) > 0$ such that, for every n, every 2-coloring of the edges of a complete graph on n vertices satisfying that the 2-coloring contains no copy of C does contain a monochromatic K_{n^ε}. In other words, if we only considered those colorings which don't contain C, Ramsey numbers would grow polynomially rather than exponentially.

There are now several partial results on the Erdős-Hajnal conjecture. Erdős and Hajnal [35] proved that, for each fixed coloring C, there is $\epsilon = \epsilon(C) > 0$ such that every coloring of a K_n which does not contain a copy of C has a monochromatic clique with $e^{\epsilon\sqrt{\log n}}$ vertices. Fox and Sudakov [45], strengthening an earlier result of Erdős and Hajnal, proved that for each fixed coloring C there is $\epsilon = \epsilon(C) > 0$ such that every coloring of a K_n which does not contain a copy of C has either a balanced complete bipartite graph in the first color or a clique of order n^ϵ in the second color.

Erdős and Hajnal also proposed studying a multicolor generalization of their conjecture. It states that for every fixed k-coloring of the edges C of a complete graph, there is an $\epsilon = \epsilon(C) > 0$ such that every k-coloring of the edges of the complete graph on n vertices without a copy of C contains a clique of order n^ϵ which only uses $k - 1$ colors. They proved a weaker estimate, replacing n^ϵ by $e^{\epsilon\sqrt{\log n}}$. Note that the case of two colors is what is typically referred to as the Erdős-Hajnal conjecture.

Hajnal [59] conjectured that the following special case of the multicolor generaliza-

tion of the Erdős-Hajnal conjecture holds. There is $\epsilon > 0$ such that every 3-coloring of the edges of the complete graph on n vertices without a rainbow triangle (that is, a triangle with all its edges different colors), contains a set of order n^ϵ which uses at most two colors. We prove Hajnal's conjecture, and further determine a tight bound on the order of the largest guaranteed 2-colored set in any such coloring; this size is $\Theta(n^{1/3} \log^2 n)$. Indeed, we actually find asymptotically tight bounds for every $r \geq s \geq 1$ for the largest size of a clique that uses at most s colors in an r-coloring of a K_n that contains no rainbow triangles.

1.3 Chapter 4: Packing Vertex-Disjoint Monochromatic Copies of Sparse Graphs

In Chapter 4 we pursue another variant of Ramsey theory where, rather than taking a coloring of a graph G and finding a single monochromatic copy of some graph H, we instead wish to partition all of the vertices of G into monochromatic copies of graphs.

An area that has attracted much interest is the study of Ramsey numbers for bounded degree graphs. In 1975, Burr and Erdős [11] raised the problem that every graph G with n vertices and maximum degree Δ has a linear Ramsey number, so $r(G) \leq C(\Delta)n$, for some constant $C(\Delta)$ depending only on Δ. This was proved by Chvátal, Rödl, Szemerédi and Trotter [20] in one of the earliest applications of Szemerédi's celebrated Regularity Lemma [89]. Because the proof uses the Regularity Lemma, the bound on $C(\Delta)$ is quite weak; it is of tower type in Δ.

Recently, Conlon [22] and, independently, Fox and Sudakov [45] have shown how to prove the bound of $C_B(\Delta) \leq 2^{O(\Delta)}$ in the bipartite case. For the non-bipartite graph case, the current best bound is due to Conlon, Fox, and Sudakov [25] $C(\Delta) \leq 2^{O(\Delta \log \Delta)}$.

Graham, Rödl, and Ruciński [49] proved that there are bipartite graphs with n vertices and maximum degree Δ for which the Ramsey number is at least $2^{\Omega(\Delta)}n$,

meaning that the upper bound of Conlon, Fox, and Sudakov is best possible, up to the constant in the exponent.

It is a natural question (initiated by András Gyárfás) to ask how many monochromatic members from a bounded-degree graph family are needed to partition the vertex set of a 2-edge-colored complete graph. In Chapter 4 we study this problem and related questions. Given $\mathcal{F} = \{F_1, F_2, \ldots\}$ a sequence of graphs, we say it is a proper graph sequence if F_n is a graph on n vertices. We say it has some graph property if every graph of $\mathcal{F}$ has that property (e.g. $\mathcal{F}$ is bipartite if F_n is bipartite for every n).

We prove, for bipartite proper graph sequences $\mathcal{F}$ with maximum degree at most Δ, that the vertices of any 2-edge-colored complete graph may be partitioned into $2^{O(\Delta)}$ monochromatic copies of graphs from $\mathcal{F}$, and that this is best possible up to the constant in the exponent. We further prove that for proper graph sequences $\mathcal{F}$ with maximum degree at most Δ, that the vertices of any 2-edge-colored complete graph may be partitioned into $2^{O(\Delta \log \Delta)}$ monochromatic copies of graphs from $\mathcal{F}$. Finally, we generalize this to arrangeable graph sequences by showing that, for proper graph sequences $\mathcal{F}$ with arrangeability a and chromatic number k satisfying that the maximum degree of F_n is at most $\sqrt{n}/\log n$, the vertices of any 2-edge-colored complete graph may be partitioned into $2^{O(a^4 k^2)}$ monochromatic copies of graphs from $\mathcal{F}$.

Chapter 2

Minimum Degrees of Minimal Ramsey Graphs

2.1 Introduction

For graphs G and H we write $G \to H$ and say G is *Ramsey* for H if in any 2-coloring of the edges of G there exists a monochromatic copy of H. Ramsey's theorem [79] states that for any H there is a G that is Ramsey for H.

Clearly, for a fixed graph H, the collection of graphs that are Ramsey for it is upwards closed. That is, if G is Ramsey for H and G is a subgraph of G', then G' is also Ramsey for H. Therefore, to understand the collection of graphs that are Ramsey for H, it is sufficient to understand the collection $\mathcal{M}(H)$ of graphs that are minimal subject to being Ramsey for H; these graphs are called *Ramsey minimal* for H.

Many fundamental problems in graph theory concern the study of $\mathcal{M}(H)$. The most famous of these, one of the driving motivations for the field of Ramsey theory, is to compute or estimate the Ramsey number $r(H)$ for various H, where $r(H)$ is the smallest number of vertices of any graph in $\mathcal{M}(H)$. Of particular interest is $r(K_t)$, the Ramsey number of the complete graph on t vertices. Classical results of Erdős and Szekeres [36] and Erdős [31] imply that $2^{t/2} \leq r(K_t) \leq 2^{2t}$ for $t \geq 2$. Despite much interest (see, e.g., [21]), there have been no improvements in the constant factors in

the above exponents.

This impasse has naturally led to the study of other problems related to $\mathcal{M}(H)$. For example, the size Ramsey number $\hat{r}(H)$ is the minimum number of edges of any graph in $\mathcal{M}(H)$. This parameter was introduced by Erdős et al. [32]. While this parameter is still far from understood, there are now several beautiful results about size Ramsey numbers of sparse graphs (see, e.g., [5], [64], [80]). Another related problem asks which graphs H are Ramsey-infinite, that is for which graphs H is the family $\mathcal{M}(H)$ infinite (see, e.g., the book [50]).

For a graph G, let $\delta(G)$ denote the minimum degree of the vertices of G. Our interest in this chapter lies in $s(H)$, which is the minimum of $\delta(G)$ over all graphs $G \in \mathcal{M}(H)$. This parameter was first introduced and studied by Burr, Erdős, and Lovász in 1976 [12]. A simple upper bound is $s(H) \leq r(H) - 1$. Indeed, one may take any Ramsey-minimal graph on $r(H)$ vertices, and the minimum degree of this graph is at most $r(H) - 1$. Since $r(K_t)$ is exponential in t, the result of Burr, Erdős, and Lovász [12] that $s(K_t) = (t - 1)^2$ may be surprising.

Fox and Lin [43] observed the simple lower bound $s(H) \geq 2\delta(H) - 1$ which holds for every graph H. To see this, assume for contradiction there is some minimal Ramsey graph G for H with a vertex v of degree at most $2\delta(H) - 2$. By minimality, there must be some red-blue coloring of the edges of $G - v$ without a monochromatic copy of H; we may extend this to an edge-coloring of G by coloring at most $\delta(H) - 1$ of the edges incident to v blue and at most $\delta(H) - 1$ of the edges incident to v red. As G is Ramsey for H, it follows that this coloring must have a monochromatic copy of H containing v. However, v has degree less than $\delta(H)$ in any monochromatic subgraph, contradicting that v is in a monochromatic copy of H, and completing the proof. We say a graph H is *Ramsey simple* if this lower bound is tight, that is if $s(H) = 2\delta(H) - 1$.

In recent years, the study of $s(H)$ has received increased attention. Fox and Lin [43] present an alternative proof that $s(K_t) = (t - 1)^2$ and also show that $K_{s,t}$ is Ramsey simple. Szabó, Zumstein, and Zürcher [88] conjectured that every bipartite graph without isolated vertices is Ramsey simple.

Conjecture 2.1.1. *[88] If H is bipartite with no isolated vertices, then H is Ramsey simple.*

They prove this conjecture for a variety of bipartite graphs including trees, even cycles, and bipartite graphs where every vertex in one of the parts has degree $\delta(H)$. They also prove the conjecture for connected bipartite graphs with parts A of size a and B of size b with $b \geq a$ in which A contains a minimum degree vertex. It is worth noting that they also address the case of isolated vertices: they show that for any graph H on n vertices (not necessarily bipartite), if we denote the graph obtained from H by adding t isolated vertices by $H + tK_1$, then $s(H + tK_1) = s(H)$ if $t \leq r(H) - n$, and $s(H + tK_1) = 0$ if $t > r(H) - n$.

In this chapter we prove Conjecture 2.1.1 for all 3-connected bipartite graphs.

Theorem 2.1.2. *If H is bipartite and 3-connected, then $s(H) = 2\delta(H) - 1$.*

In the process of proving the above result, we prove the following theorem which gives the first examples of Ramsey simple graphs that are not bipartite.

Theorem 2.1.3. *If H is 3-connected and has some vertex v of degree $\delta(H)$ so that the neighbors of v are contained in an independent set of size $2\delta(H) - 1$, then H is Ramsey simple.*

In a fairly wide range of values of p, the binomial random graph $G(n, p)$ satisfies the conditions of the above theorem, and hence we have the following corollary.

Corollary 2.1.4. *If $n^{-1} \log n \ll p \ll n^{-2/3}$, then $G(n, p)$ is almost surely Ramsey simple.*

The lower bound on p originates from the need to be 3-connected (for $p \ll n^{-1} \log n$ there will almost surely be isolated vertices). The upper bound on p guarantees that the expected number of triangles is $o(n)$ and hence the neighborhood of a typical vertex is an independent set. Further, one expects that there is a minimum degree vertex v whose neighborhood is an independent set, and can be extended to an independent set twice larger. The details of this proof are given in Section 2.9. Note also

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that in this range $G(n, p)$ is almost surely not bipartite, so these graphs do not fall under the assumptions of Theorem 2.1.2, and are examples of Ramsey simple graphs that are not bipartite.

There has also been much interest in the value of $s(K_t \cdot K_2)$, where $H \cdot K_2$ denotes the collection of graphs obtained by adding a new vertex v to H, picking a vertex u of H, and connecting v to u. It was shown [88] that $s(K_t \cdot K_2) \geq t - 1$, and they conjecture that $s(K_t \cdot K_2) = s(K_t) = (t - 1)^2$, for t sufficiently large. The motivation for this conjecture is that, for t sufficiently large, it intuitively may be the case that any graph which is Ramsey for K_t is also Ramsey for $K_t \cdot K_2$. This conjecture was disproved in [42], where it is shown that $s(K_t \cdot K_2) = t - 1$. In this chapter, we generalize the lower bound of [88] that $s(K_t \cdot K_2) \geq t - 1$ to graphs other than K_t, and we find upper bounds on $s(H \cdot K_2)$ for many graphs H. Most notably, we find that $s(K_{t,t} \cdot K_2) = 1$, where $K_{t,t}$ is the complete bipartite graph with parts of size t. This is strong support for Conjecture 2.1.1, as $K_{t,t} \cdot K_2$ was thought to be the best candidate for a counterexample.

Theorem 2.1.2 and Theorem 2.1.3 use a powerful tool originating from [12] and generalized in [13] which requires the graphs to be 3-connected. In Section 2.3, we present the tools necessary to prove Theorems 2.1.2 and 2.1.3 and then present their proofs. We defer the proof that Corollary 2.1.4 follows from Theorem 2.1.3 and some basic facts about $G(n, p)$ to Section 2.9. In Section 2.4, we prove lower bounds for $s(H \cdot K_2)$ for many graphs H. In Section 2.5, we show that $s(K_{t,t} \cdot K_2) = 1$, and in Section 2.6 we give an upper bound for $s(H \cdot K_2)$ for many graphs H. Finally, in Section 2.7, we give the first examples of connected graphs $H \subseteq H'$ with $s(H) > s(H')$.

2.2   Preliminaries

In this section, we introduce notation and tools that we use to prove bounds on $s(H)$. Unless stated otherwise, all our colourings are red-blue colourings of the edges of a graph. We call two edge-disjoint graphs R, B on the same vertex set V a *colour*

pattern on V. For a graph H, a colour pattern is called *H-free* if neither R nor B contains H as a subgraph. Let G be a graph that contains $R \cup B$ as a subgraph, where R, B is a colour pattern. We say a colouring c of G *extends* (or *has*) *colour pattern* $R \cup B$ if R and B are both monochromatic with different colours. For a graph H, we call a colouring c H-free if there is no monochromatic copy of H in c. Given a graph G which contains some G_0 as an induced subgraph, we say the pair (G, G_0) is H-robust if any graph which is obtained from G by adding some vertices S to G and adding edges within $S \cup V(G_0)$ satisfies that any copy of H is either contained entirely within $S \cup V(G_0)$ or is contained entirely within G.

Burr, Erdős, and Lovász [12] introduced a powerful tool in determining $s(K_t)$. It states that, given any K_t-free colour pattern R, B, there is some graph $G \supseteq R \cup B$ which is not Ramsey for K_t, but any K_t-free coloring of G extends the colour pattern $R \cup B$. For a graph H and a colour pattern R, B, we call a graph $\mathbf{B} = \mathbf{B}(H, R, B)$ a *BEL gadget for H (with colour pattern $R \cup B$)* if $\mathbf{B}$ contains $R \cup B$ as an induced subgraph, $\mathbf{B} \nrightarrow H$ and any H-free colouring of $\mathbf{B}$ has colour pattern $R \cup B$. Burr, Nešetřil and Rödl [13] extended the proofs in [12] in the following way.

Lemma 2.2.1. *[13] For any 3-connected graph H and any H-free colour pattern $R \cup B$, there exists a graph $\mathbf{B} = \mathbf{B}(H, R, B)$ that is a BEL gadget for H with colour pattern R, B so that $(\mathbf{B}, R \cup B)$ are H-robust. Furthermore, if H and $R \cup B$ are bipartite, then so is $\mathbf{B}$.*

Let us say that *BEL gadgets exist for H* if for any H-free colour pattern R, B there is a BEL gadget for H with colour pattern $R \cup B$.

2.3   A large class of Ramsey-simple graphs

In this section, we prove Theorem 2.1.2 and Theorem 2.1.3. In the following, we give a sufficient condition for a graph H to be Ramsey-simple.

Lemma 2.3.1. *Let H be a graph and suppose H has no isolated vertices and BEL gadgets exist for H. If there is an H-free graph G with an independent set S of size*

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$2\delta(H) - 1$ *so that any graph obtained from G by adding a vertex v and connecting v to $\delta(H)$ vertices of S contains a copy of H, then H is Ramsey simple.*

Proof. Take G_0 to be a red copy R of G with distinguished set S and a blue copy B of G with the same distinguished set S (note this creates no conflicts since S is an independent set). Note that R, B is a colour pattern that is H-free, since both R and B consist of a copy of G along with isolated vertices. Since there exist BEL gadgets for H, we may create a graph $\mathbf{B}$ so that $\mathbf{B} \nrightarrow H$, but any H-free coloring of $\mathbf{B}$ extends the colour pattern $R \cup B$. Now, add a vertex v to $\mathbf{B}$ and add edges from v to all of S. Call the resulting graph G'. The degree of v in G' is $2\delta(H) - 1$, and the graph obtained by removing v is not Ramsey for H. In any two-coloring of the edges containing v, at least $\delta(H)$ of those edges must have the same colour, say, without loss of generality, red. Then, by assumption on G, v along with these $\delta(H)$ neighbours in S and the red copy of G contain a monochromatic copy of H. Hence, G' is Ramsey for H. Any Ramsey-minimal subgraph of G' must contain v, and so $s(H) \leq 2\delta(H) - 1$, i.e., H is Ramsey simple. $\qquad\square$

In the next lemma, we show how to construct G under certain assumptions on H.

Lemma 2.3.2. *Let H be a graph and suppose H has no isolated vertices and BEL gadgets exist for H. If there is a vertex u of degree $\delta(H)$ in H whose neighbourhood is contained in an independent set of size $2\delta(H) - 1$, then H is Ramsey simple.*

Proof. Let n be the number of vertices of H. Consider a graph G on $n-1$ vertices that is complete except for an independent set S on $2\delta(H) - 1$ vertices; that is, the graph consists of an independent set on $2\delta(H) - 1$ vertices and a clique on $n - 1 - (2\delta(H) - 1)$ vertices, and there is a complete bipartite graph between them. Notice that $H - u$ is a subgraph of G for any H satisfying the assumptions of the theorem. Adding a vertex v to G and connecting v to any $\delta(H)$ vertices of S creates a copy of H where v acts as a copy of u. The graph G is H-free since it has only has $n - 1$ vertices. Therefore, we can apply Lemma 2.3.1 to conclude that H is Ramsey simple. $\qquad\square$

Note that Theorem 2.1.3 is a corollary of Lemma 2.3.2 and Lemma 2.2.1. We now

apply Lemma 2.2.1 to show that every bipartite 3-connected graph is Ramsey simple (Theorem 2.1.2). We again prove a slightly stronger result.

Theorem 2.3.3. *Suppose H is a connected bipartite graph with at least two vertices. If BEL gadgets exist for H, then H is Ramsey simple.*

Proof. Let A, B be a bipartition of H with $|A| \leq |B|$, and take $a = |A|, b = |B|$, so $a \leq b$. Let $n = a + b = |V(H)|$. Let $\delta = \delta(H)$. If B contains only vertices of degree δ, if A contains a vertex of degree δ, or if $a = b$, then it was proved in [88] that H is Ramsey simple, as desired. So we may assume that B contains some vertex of degree larger than δ, that A contains no vertex of degree δ, and that $b > a$; this also means that B must contain some vertex u of degree δ. Under these assumptions we will show that there is a graph G satisfying the assumptions of Lemma 2.3.1, thus completing the proof. That is, G is H-free and has an independent set S of size $2\delta - 1$ so that adding a vertex v to G and connecting v to any δ vertices of S creates a copy of H.

If $b \geq 2\delta$, then we may take G to be a complete bipartite graph with both parts of size $b - 1$. We will take S to be any $2\delta - 1$ vertices from one of the parts. G will not contain a copy of H, as neither of its parts has size at least b. Adding a vertex v and connecting it to any δ vertices of S creates a copy of H with v serving as a copy of u.

Therefore, we may assume that $a < b < 2\delta$. In particular, this means that any two vertices in B have a common neighbour and any two vertices in A have a common neighbour. Now, we will instead consider the graph G obtained as follows. Take an independent set S on $2\delta - 1$ vertices. For any set S' of δ vertices of S, add a copy of $H - u$ where S' is $N(u)$. Formally, G will have vertex set $S \cup \left(\binom{S}{\delta} \times [n - \delta - 1] \right)$. Enumerate the vertices of H as $u_1, \ldots, u_n$ so that $u = u_n$ and $N(u) = \{u_{n-\delta}, \ldots, u_{n-1}\}$. For each set $S' \subseteq S$ of size δ, fix some ordering $v_{S',n-\delta}, v_{S',n-\delta+1}, \ldots, v_{S',n-1}$ of the vertices of S'. The edges of G that are not incident to S are those pairs of vertices of the form $\{(S', k_1), (S', k_2)\}$ where S' is a set of δ vertices from S and $\{u_{k_1}, u_{k_2}\}$ is an edge of H. The edges that are incident to S are those pairs of the form $\{v_{S',k_1}, (S', k_2)\}$

where $\{u_{k_1}, u_{k_2}\}$ is an edge of H. Note that G is bipartite.

The subgraph of G consisting of those edges incident to S is bipartite with one part being S. In the other part, all of the vertices have degree at most δ (since the vertex (S', k) can be adjacent only to vertices in S'). Therefore, if this subgraph contains a copy of H, one of the parts of H must contain only vertices of degree δ, contradicting the assumption.

Hence, any copy of H in G must contain some edge $\{(S', k_1), (S', k_2)\}$ not incident to S. Note that there are only $n - 1$ vertices in S' or of the form (S', k). Therefore, a copy of H must contain some other vertex.

However, a copy of H cannot contain any vertex of S other than those in S'. Indeed, such a vertex would share no common neighbours with both (S', k_1) and (S', k_2). However, as (S', k_1) and (S', k_2) are adjacent, they must be in different parts of the bipartition, contradicting the assumption that any vertex of the copy of H must have a common neighbour with each vertex in the same part.

Therefore, a copy of H must contain a vertex of the form (S'', ℓ_1) with S'' distinct from S'. However, since S'' and S' are distinct sets of size δ, they can intersect in at most $\delta - 1$ vertices. Since the only vertices from S in a copy of H must be contained in S', and the only neighbours of (S'', ℓ_1) in S are in S'', we must have that all neighbours in S of (S'', ℓ_1) used by this copy of H must be contained in $S' \cap S''$. Since this is at most $\delta - 1$ vertices, the vertex (S'', ℓ_1) must have degree at least δ and so must have another neighbour in the copy of H. In G, the only neighbours of (S'', ℓ_1) not contained in S are of the form (S'', ℓ_2), and so H must contain some vertex (S'', ℓ_2) as a neighbour of (S'', ℓ_1). In particular, in this copy of H, the vertices of the form (S', k) contain vertices from both parts of H, as do vertices of the form (S'', k). In order to have that any two vertices from the same part share a common neighbour, we must have that $S' \cap S''$ contains vertices from both parts of H. However, G is bipartite and $S' \cap S''$ is contained in one of the bipartitions, contradicting that a copy of the connected graph H can have vertices from both parts in $S' \cap S''$. Therefore, G has no copy of H. By construction, adding a vertex to G and connecting it to any δ vertices of S creates a copy of H. Therefore, G has the desired properties, and so,

by Lemma 2.3.1, H is Ramsey simple. $\qquad\square$

2.4 Stronger lower bounds for graphs with hanging edges

For a collection $\mathcal{H}$ of graphs, we say that a graph G is $\mathcal{H}$-*Ramsey*, and denote this by $G \to \mathcal{H}$, if in any two colouring of the edges of G there exists a monochromatic copy of some H in $\mathcal{H}$. If G is minimal with this property, we call G $\mathcal{H}$-*Ramsey minimal*. We denote by $\mathcal{M} = \mathcal{M}(\mathcal{H})$ the class of $\mathcal{H}$-Ramsey minimal graphs. Note this class does not need to relate by inclusion to any of the classes $\mathcal{M}(H)$ for $H \in \mathcal{H}$ due to the minimality assumption. We also set $s(\mathcal{H}) := \min_{G \in \mathcal{M}(\mathcal{H})} \delta(G)$, as before.

Given a graph $H = (V, E)$, we denote by $H \cdot K_2$ the collection of graphs obtained by picking some vertex v of H, adding some new vertex w, and adding an edge between v and w. This is a slight abuse of notation as we have already defined $K_k \cdot K_2$ to be a single graph, but a graph G is Ramsey for the previous definition of $K_k \cdot K_2$ if and only if it is Ramsey for the new definition, so the notation is consistent. Note that if H is vertex transitive, then $H \cdot K_2$ contains, up to isomorphism, only one graph.

The following proof closely follows the ideas of the proof that $s(K_t \cdot K_2) \geq t - 1$ in [88].

For a graph H, let

$$\mathcal{F}(H) := \{C \subseteq H[N(x)] \ : \ x \in V(H), C \text{ is a connected component of } H[N(x)]\}$$

denote the collection of all connected graphs that appear in the neighbourhood of any vertex x of H. We will prove the following theorem:

Theorem 2.4.1. *Let $H = (V, E)$ be a graph on n vertices, and assume H has the following properties:*

1. *H is connected.*

2. *H has minimum degree at least two.*

3. There is a colouring of $K_{\alpha(H)}$ that is C-free for every $C \in \mathcal{F}(H)$.

Then $s(H \cdot K_2) \geq \delta(H)$.

Remark 2.4.2. Note that Condition (3) trivially fails, e.g., when H is bipartite, or more generally, if there exists a vertex x of H so that $H[N(x)]$ contains an isolated vertex. Note also, that when H is the complete graph, then Condition (3) is trivially satisfied. In Section 2.10, we give a large class of non-trivial examples of sparse (vertex-transitive) graphs H that fulfill all conditions of the theorem, meaning that the single graph $H \cdot K_2$ satisfies $s(H \cdot K_2) \geq \delta(H)$.

Proof. Let G' be an $H \cdot K_2$-Ramsey minimal graph. We want to show that $\delta(G') \geq \delta(H)$. Assume the opposite and remove some vertex from G' of degree $\delta(G') < \delta(H)$. This leaves some graph $G = (V, E)$. By minimality of G', there is a two colouring χ of $E(G)$ such that there is no monochromatic copy of F for any $F \in H \cdot K_2$. Call the two colours red and blue.

We say a vertex of G is critical under some colouring ψ if it is contained in both a red and a blue copy of H. We will show below that we can convert χ to a colouring ψ of G with no monochromatic $F \in H \cdot K_2$ and without critical vertices. Let us first show how the existence of such a colouring implies Theorem 2.4.1.

Claim 2.4.3. *If there is a colouring ψ of G with no monochromatic $F \in H \cdot K_2$ and with no critical vertices, then there is a colouring of G' with no monochromatic $H \cdot K_2$.*

Proof. If v is the vertex we removed from G' of degree less than $\delta(H)$, we define the colouring ψ' of G' as follows: ψ' agrees with ψ on G, and an edge vw is coloured blue if w is contained under ψ in a red copy of H, and otherwise the edge is coloured red.

Assume G' has a monochromatic copy F' of some $F \in H \cdot K_2$. By choice of ψ, this F' must use v. Since $d(v) < \delta(H)$, v must be the hanging vertex. Suppose F' were red. Then $F' - v$ is red copy of H, so the pending edge vw must be coloured blue in ψ', a contradiction. On the other hand, if F' were blue, then vw would be blue and by definition of ψ' w is also contained in a red copy of H under ψ. Then w would be critical in G under ψ, a contradiction. $\qquad\square$

It remains to show the existence of a colouring ψ of $E(G)$ without a monochromatic $F \in H \cdot K_2$ and no critical vertices, completing the proof. Let us first study the structure of χ, the given colouring of G.

Lemma 2.4.4. *Let χ be any colouring of G without a monochromatic copy of any $F \in H \cdot K_2$.*

(i) *For any red (blue) copy H_1 of H in G, all edges between $V(H_1)$ and $V(G) \backslash V(H_1)$ are blue (red).*

(ii) *Given H_1, H_2 two monochromatic copies of H in G of the same colour, either $V(H_1) = V(H_2)$ or $V(H_1) \cap V(H_2) = \emptyset$.*

(iii) *Let H_1 be a red and H_2 be a blue copy of H in G. Then there are no edges in G between $V(H_1) \setminus V(H_2)$ and $V(H_2) \setminus V(H_1)$.*

(iv) *For any critical vertex v, if it is contained in a red copy H_1 and a blue copy H_2 of H, v is not adjacent in G to any vertex w not in $V(H_1) \cup V(H_2)$.*

Proof. By assumption, there is no monochromatic $F \in H \cdot K_2$, so (i), (iii) and (iv) are immediate. (ii) follows by connectivity of H. $\qquad\qquad\square$

We would like to keep track of the positions of monochromatic copies of H. Therefore, for a red-blue-colouring ψ of G, we set

$$\mathcal{V}_{\text{red}}(\psi) := \{V' \subseteq V(G) : \text{There is a red copy of } H \text{ in } G \text{ such that } V(H) = V'\}, \qquad \text{and}$$

$$\mathcal{V}_{\text{blue}}(\psi) := \{V' \subseteq V(G) : \text{There is a blue copy of } H \text{ in } G \text{ such that } V(H) = V'\}.$$

In other words, $\mathcal{V}_{\text{red}}(\psi)$ $(\mathcal{V}_{\text{blue}}(\psi))$ is the collection of V' so that V' is the vertex set of a red (blue) copy of H.

We are now ready to describe the recolouring algorithm. The main motivation behind the definition of ψ below is that we have strong control over the kinds of structures that contain edges incident to critical vertices, and so have much leeway when recolouring said edges. Indeed, we will use this structure to remove all monochromatic copies of H that contain critical vertices.

I. Let $\mathcal{V}_{\mathrm{red}}(\chi) = \{V_0, V_1, \ldots\}$ be the hosts of red copies of H under χ. Define χ' to agree with χ on any edge that is not internal to any V_i, and to be red on any edge internal to some V_i.

II. Let $\mathcal{V}_{\mathrm{blue}}(\chi') = \{W_0, W_1, \ldots\}$ be the hosts of blue copies of H under χ'. Define χ'' to be blue on any edge that is internal to some W_i but not internal to any V_i, and to agree with χ' on all other edges.

Note that by Lemma 2.4.4 *(ii)*, the elements of $\mathcal{V}_{\mathrm{red}}(\chi)$ are pairwise disjoint, and by 2.4.4 *(i)* all edges between any two V_i are blue. Now, χ' only colours edges red inside the V_i. Further, χ'' only changes edges not inside the V_i to blue. We therefore have

$$\mathcal{V}_{\mathrm{red}}(\chi) = \mathcal{V}_{\mathrm{red}}(\chi'') =: \mathcal{V}_{\mathrm{red}}.$$

Further note, since G has no monochromatic $H \cdot K_2$ under χ and χ' only coloured edges red inside the V_i, G has no monochromatic $H \cdot K_2$ under χ'. Therefore, again by Lemma 2.4.4 *(i)* and *(ii)*, the elements $W_j \in \mathcal{V}_{\mathrm{blue}}(\chi')$ are pairwise disjoint, and all edges between any two W_j are red. Now, since χ'' recolours only edges inside $W_j \in \mathcal{V}_{\mathrm{blue}}(\chi')$ to blue, the new hosts of blue H are the same as before. That is,

$$\mathcal{V}_{\mathrm{blue}}(\chi') = \mathcal{V}_{\mathrm{blue}}(\chi'') =: \mathcal{V}_{\mathrm{blue}}.$$

For each V_i containing a critical vertex under χ'', choose some critical $v_i \in V_i$. For each W_j containing a critical vertex, if possible, choose some critical $w_j \in W_j$ such that for every i, $w_j \neq v_i$. If this is not possible, choose any critical vertex $w_j \in W_j$. Take A to be the set of v_i and B the set of w_j. We will now describe the final recolouring step. Since the V_i are pairwise disjoint, as are the W_j, the sets of the form $V_i \cap W_j$ are pairwise disjoint. So we may colour their internal edges independently of each other.

IV. Our final colouring ψ will be obtained by recolouring some edges of χ''. First, recolour the edges internal to any $(V_i \cap W_j) \setminus (A \cup B)$ so that it contains no

monochromatic copy of any connected component of $H[N(v)]$ for any $v \in V(H)$ (we will show that this is possible). ψ will also recolour edges incident to some v_i, w_j. Given any v_i, it is a critical vertex under χ'', so take j to be the unique index such that $v_i \in W_j$. For any vertex $v' \in V_i$ with $v_i v' \in E$, colour the edge $v_i v'$ blue if $v' \neq w_j$. If $v' = w_j$, colour the edge arbitrarily.

Given any w_j, it is a critical vertex under χ'', so take i to be the unique index such that $w_j \in V_i$. For any vertex $w' \in W_j$ with $w_j, w' \in E$, colour the edge $w_j w'$ red if $w' \neq v_i$, and colour it arbitrarily if $w' = v_i$. (We will later check that this is well-defined, i.e. that we haven't coloured any edges twice, except those coloured arbitrarily.)

ψ will agree with χ'' on all other edges.

We now begin proving that ψ is well-defined and has the desired properties. We first note some properties of χ''.

Observation 2.4.5. χ'' above satisfies

(i) χ'' has no monochromatic $H \cdot K_2$;

(ii) if an edge e is internal to V_i and not internal to any W_j, then e has colour red;

(iii) if an edge e is internal to W_i and not internal to any V_j, then e has colour blue.

Proof. We noted above already that G contains no monochromatic $H \cdot K_2$ under χ'. Now, χ'' only recolours edges inside $\mathcal{V}_{\text{blue}}$ to blue, so (i) follows. (ii) and (iii) are immediate from the colouring procedure. □

Note that each property listed by the lemma above is symmetric with respect to the colours.

Lemma 2.4.6. ψ *is well-defined.*

Proof. Let us first note that for each i and j, $|V_i \cap W_j| \leq \alpha(H)$. To see this, note any edge internal to $V_i \cap W_j$ is coloured red by χ'', so since there is a blue copy of H on W_j by definition, we must have that in this copy $V_i \cap W_j$ forms an independent set,

so $|V_i \cap W_j| \leq \alpha(H)$. Therefore, by assumption, $V_i \cap W_j$ and so $V_i \cap W_j \setminus (A \cup B)$ may be coloured so there is no monochromatic copy of any connected component of $H[N(v)]$ for any $v \in V$.

We now check that we do not ask for ψ to colour an edge both red and blue; note if an edge does not contain a v_i or w_j then it is only coloured once by ψ. Otherwise, assume it has the form $\{v_i, v\}$ with $v \in V_i \setminus B$ and so was coloured blue. But then for it to be coloured red we must have $v_i = w_j$ for some j and $v \in W_j \setminus A$. But if $v_i = w_j$ we must have that w_j could not have been chosen such that $w_j \neq v_i$, so we must have $|V_i \cap W_j| = 1$, which is impossible since $v, v_i \in V_i \cap W_j$. $\qquad\square$

We collect some immediate facts about the colouring ψ.

Observation 2.4.7.

The red degree of v_i in V_i under ψ is at most one.

The blue degree of w_j in W_j under ψ is at most one.

Any edge in V_i that is not internal to any W_j and is blue under ψ is incident to v_i.

Any edge in W_j that is not internal to any V_i and is red under ψ is incident to w_j.

Lemma 2.4.8. *If v was a critical vertex under χ'', then v is not contained in any monochromatic copy of H under ψ.*

Proof. Let H_1 be a monochromatic red copy of H in G under ψ (the case for blue is symmetric) and assume it contains a vertex critical under χ''. If $V(H_1) = V_i$ for some i, then the red-degree of v_i in V_i is at most 1, giving a contradiction since $\delta(H) > 1$.

Therefore, H_1 needs to use some edge $e = vw$ that has been recoloured by ψ to red. But ψ only recoloured edges that were incident to some critical vertices, that is $v \in V_i \cap W_j$ and $w \in V_i \cup W_j$ for some $V_i \in \mathcal{V}_{\text{red}}$, $W_j \in \mathcal{V}_{\text{blue}}$. Since $V(H_1) \neq V_i$, we may assume that e is an edge leaving V_i, i.e. $w \in W_j \setminus V_i$. Hence e is not contained in $V_i \cap W_j$, so by the definition of ψ, e is incident to w_j, the critical vertex we chose for W_j. That is, $v = w_j$. Note that there are no red edges between $W_j \setminus V_i$ and V_i excluding those incident to w_j. We consider now the neighbourhood of w_j in $V(H_1) \cap W_j$. Note that property (3) of Theorem 2.4.1 implies in particular that

there are no isolated vertices in $H_1[N(w_j)]$. Therefore some connected component C of $H_1[N(w_j)]$ containing at least one edge is contained in W_j. But the only red edges of W_j are either incident to w_j or contained in some $V_{i'} \cap W_j$ (where i and i' may or may not coincide). Therefore by connectivity, we must have that C is contained in some $(V_{i'} \cap W_j) \setminus \{w_j\}$.

By construction $(V_{i'} \cap W_j) \setminus (A \cup B) = (V_{i'} \cap W_j) \setminus \{v_{i'}, w_j\}$ contains no red copy of C. But $v_{i'}$ is blue to all vertices of $V_{i'}$ except possibly w_j, so by connectivity $(V_{i'} \cap W_j) \setminus \{w_j\}$ contains no red copy of C, but this is a contradiction as the neighbourhood of w_j must have a connected component in a set of this form. $\qquad\square$

Lemma 2.4.9. ψ *contains no critical vertices.*

Proof. Let a vertex v be given, and assume v is critical under ψ. Then v must be contained in ψ in some red copy H_1 of H and in some blue copy H_2 of H. By Lemma 2.4.8, neither H_1 nor H_2 may contain any vertices that were critical under χ''. However, any recoloured edge of ψ is incident to a critical vertex of χ''; therefore, the colourings of H_1 and H_2 agree with those of χ'', but then v is critical under χ'', a contradiction. $\qquad\square$

Lemma 2.4.10. ψ *contains no monochromatic copy of $H \cdot K_2$.*

Proof. We already know that under χ'', G does not contain a monochromatic copy of any $F \in H \cdot K_2$. Suppose F_1 is a monochromatic copy of some $F \in H \cdot K_2$ under ψ. Then F_1 must use some edge $e = vw$, where v is critical under χ'', since ψ only recoloured such edges. By Lemma 2.4.8, e must be the pending edge and v the pending vertex. Thus, w is contained in a monochromatic copy H_1 of H under ψ. Again by Lemma 2.4.8, w cannot be a critical vertex under χ''. Also, since ψ only recoloured edges which contained a critical vertex under χ'', and none of these vertices are contained in a monochromatic copy of H under ψ, none of the edges internal to H_1 were recoloured by ψ. Therefore, H_1 was already a monochromatic copy of H in χ''. But since ψ recoloured e, it needs to be internal to some V_i or some W_j, none of which are equal to $V(H_1)$. But that means w is critical under χ'', a contradiction. $\qquad\square$

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Now, Claim 2.4.3, Lemma 2.4.9, and Lemma 2.4.10 prove Theorem 2.4.1. $\qquad\square$

2.5 Upper bounds for complete bipartite graphs with hanging edges

In this section we will show that, for every $t \geq 2$, we have $s(K_{t,t} \cdot K_2) = 1$. Since $K_{t,t} \cdot K_2$ is not 3-connected, we cannot simply apply Lemma 2.2.1 to create BEL gadgets for it. We will instead use BEL gadgets for $K_{t,t}$ to construct a weaker version of BEL gadgets for $K_{t,t} \cdot K_2$. However, first we must show that BEL gadgets do exists for $K_{2,2}$, as $K_{2,2}$ is not 3-connected. Since the proof is almost identical for $K_{2,t}$, we prove the more general version here. Throughout this and the next section, we call a graph $S = S(H, e, f)$ a *negative (positive) signal sender* if $S \nrightarrow H$ and in any H-free colouring of S, the edges e and f receive a different (the same) colour. The two edges e and f of S are called *signal edges*.

Lemma 2.5.1. *For $t \geq 2$, let R, B be a $K_{2,t}$-free colour pattern. Then there exists a graph $\mathbf{B} = \mathbf{B}(K_{2,t}, R, B)$ that is a BEL gadget for $K_{2,t}$ so that $(\mathbf{B}, R \cup B)$ is $K_{2,t}$-robust. Furthermore, if $R \cup B$ is bipartite, then so is $\mathbf{B}$.*

Proof. Let $s = 6(t-1) + 1$. We first show that the graph $K_{3,s}$ with one edge removed is a negative signal sender for $K_{2,t}$ in which the two signal edges are adjacent. It is known [43] that $K_{3,s}$ is Ramsey minimal for $K_{2,t}$. Take a copy of $K_{3,s-1}$ and name the three vertices a, b, c from the part of size three. Add a vertex v to the graph and connect it to both a and b. Call this graph S^-. We claim that S^- is a negative signal sender for $K_{2,t}$ with signal edges va and vb. To see this, assume there is some colouring of S^- in which va and vb have the same colour, say red, and there is no monochromatic copy of $K_{2,t}$. Then we may add an edge from v to c and colour it blue. This graph is a copy of $K_{3,s}$ and so must have a monochromatic copy of $K_{2,t}$. This copy must use the added edge and therefore v, but v has blue-degree 1, a contradiction.

Furthermore, there is a $K_{2,t}$-free colouring of S^- (in which va and vb necessarily have different colours), since $K_{3,s}$ is Ramsey-minimal.

Once the existence of a negative signal sender S^- has been established in which the signal edges are adjacent, it follows along the lines in [13] that BEL-gadgets for $K_{2,2}$ exist, with the modification that, rather than using 3-connectivity, we use that the graph S^- above is bipartite and therefore has girth at least 4. The general approach is to glue several copies of S^- along their signal edges to obtain signal senders (both positive and negative) in which the signal edges are arbitrarily far apart. Due to the similarity, we omit this argument. $\qquad\square$

Next, we show the existence of a "weak" BEL gadget for $K_{t,t} \cdot K_2$ conditioned on $s(K_{t,t} \cdot K_2) > 1$. We need this weak version to construct "strong" signal senders in Lemma 2.5.3.

Lemma 2.5.2. *For any $t \geq 2$, let R, B be a colour pattern. There is a graph $\widetilde{\mathbf{B}}$ with an induced copy of $R \cup B$ so that $(\widetilde{\mathbf{B}}, R \cup B)$ is $K_{t,t}$-robust and so that the following hold.*

(1) *Any $K_{t,t}$-free colouring in which R is red and B is blue extends to a $K_{t,t}$-free colouring of $\widetilde{\mathbf{B}}$.*

(2) *Any $K_{t,t} \cdot K_2$-free colouring of $\widetilde{\mathbf{B}}$ has colour pattern $R \cup B$.*

Furthermore, if $R \cup B$ is bipartite, then so is $\widetilde{\mathbf{B}}$.

Proof. Let $\mathbf{B} = \mathbf{B}(K_{t,t}, R, B)$ be a BEL gadget for $K_{t,t}$. $\mathbf{B}$ exists by Lemma 2.2.1 for $t \geq 3$, and by Lemma 2.5.1 for $t = 2$. If $R \cup B$ is bipartite, we may assume that $\mathbf{B}$ is bipartite. Note that $\mathbf{B}$ satisfies Property (1) and satisfies Property (2) with $K_{t,t} \cdot K_2$ replaced by $K_{t,t}$.

We now modify $\mathbf{B}$ to create the desired weak signal sender. To do this, if $\mathbf{B}$ is bipartite with parts A, B, then for every set S of t vertices contained either entirely within A or entirely within B, we add a new set of $t + 1$ vertices V_S and add a complete bipartite graph between V_S and S. If $\mathbf{B}$ is not bipartite, we take every set S of t vertices and add a set V_S as above. Note that the degree of each of the vertices
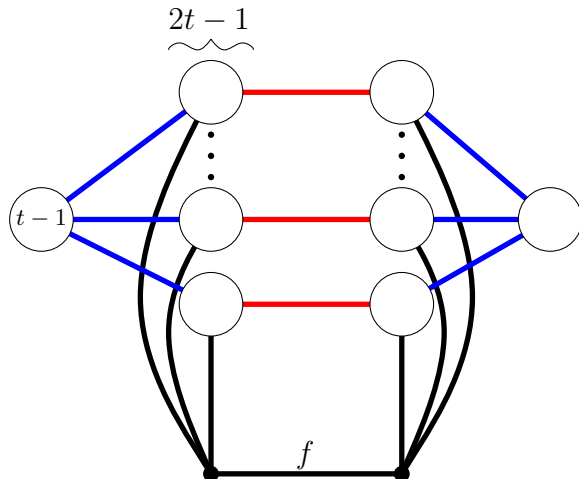
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Figure 2-1: The graph G_0 in the proof of Lemma 2.5.3 with *special* signal edge f. The white circles are all sets of $t-1$ vertices, and thick lines indicate that vertices between those sets are pairwise connected. The blue edges are all edges in B and the red edges are all edges in R.

of V_S is t, so any colouring of $\mathbf{B}$ without a monochromatic copy of $K_{t,t}$ extends to a colouring of the modified graph without a monochromatic copy of $K_{t,t}$ by giving every vertex added this way degree $t-1$ in red and degree 1 in blue. However, if there is a monochromatic copy of $K_{t,t}$ in $\mathbf{B}$, without loss of generality in colour red, then, picking one of the parts of t vertices, call it S, from the monochromatic copy, either one of the edges from S to V_S is red and we have a red $K_{t,t} \cdot K_2$, or all of the edges are blue and the complete bipartite graph between S and V_S contains a blue $K_{t,t} \cdot K_2$. Note also that we maintain robustness when adding the various V_S. $\qquad\square$

We now construct a version of a signal sender which we call "strong" (negative or positive) signal sender. The reason for this name is that we have control of the colours of the edges incident to one of the signal edges.

Lemma 2.5.3. *For $t \geq 2$ there is a bipartite graph $S^- = S^-(K_{t,t}, e, f)$ with two independent edges e, f so that any $K_{t,t} \cdot K_2$-free colouring of S^- satisfies that e and f have different colours, and there is a $K_{t,t}$-free colouring of S^- so that every edge incident to f has a different colour from f.*

Proof. We first describe a colour pattern R, B and then apply the previous lemma; we

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will then add some vertices and edges to $R \cup B$ to obtain a graph G_0; an illustration of G_0 with colour pattern $R \cup B$ can be found in Figure 2-1.

Let $A_0, A_1, \ldots, A_{2t-1}, B_0, B_1, \ldots, B_{2t-1}$ be disjoint sets of $t-1$ elements each. For each $1 \leq i \leq 2t-1$, add a complete bipartite graph between B_0 and A_i, between A_0 and B_i, and between A_i and B_i. Furthermore, add a new edge f, say between new vertices v_a and v_b, and add all edges between v_b and $\bigcup_{1 \leq i \leq 2t-1} A_i$, and add all edges between v_a and $\bigcup_{1 \leq i \leq 2t-1} B_i$. This is the graph G_0. All edges between $A_0 \cup B_0$ and $\bigcup_{1 \leq i \leq 2t-1} A_i \cup B_i$ form the subgraph B. All edges between $\bigcup_{1 \leq i \leq 2t-1} A_i$ and $\bigcup_{1 \leq i \leq 2t-1} B_i$ form the subgraph R.

Now, apply Lemma 2.5.2 to obtain a graph $\widetilde{\mathbf{B}}$ that contains $R \cup B$ as an induced subgraph such that Property (1) and (2) of the lemma hold. The graph S^- is obtained by adding to $\widetilde{\mathbf{B}}$ the vertices v_a, v_b and the edges incident to them, as described above. Let e be an arbitrary edge in R.

By construction, there exists a $K_{t,t}$-free colouring of $\widetilde{\mathbf{B}}$ that has the colour pattern $R \cup B$. Without loss of generality, we may assume that R is red and B is blue in this colouring. We extend this to a $K_{t,t}$-free colouring of S^- in the following way. Colour f blue and colour all other edges incident to v_a and v_b red. By construction, f has a different colour than all edges adjacent to it. It is easy to see that there is no monochromatic $K_{t,t}$ in this colouring of G_0, and, therefore, by robustness there is no monochromatic $K_{t,t}$.

It remains to prove that any $K_{t,t} \cdot K_2$-free colouring of S^- satisfies that e and f have different colours. Let c be a $K_{t,t} \cdot K_2$-free colouring of S^-. By construction, the colouring c has colour pattern R, B, say without loss of generality that R is red and B is blue. Assume for a contradiction that f is red. If there exists $1 \leq i \leq 2t-1$ such that all the edges $v_a x$ and $v_b y$ for $x \in B_i$, $y \in A_i$ are red, then $A_i \cup B_i \cup \{v_a, v_b\}$ forms a red $K_{t,t}$. If f is adjacent to one more red edge, this forms a red $K_{t,t} \cdot K_2$, a contradiction. If v_b has blue degree at least t, then it and its neighbors along with B_0 and any other blue edge out of B_0 form a monochromatic $K_{t,t} \cdot K_2$. The symmetric statement holds for v_a. However, in any colouring in which v_b and v_a both have blue degree less than t, there must be some i so that the edges from $\{v_a, v_b\}$ are

monochromatic in red to $A_i \cup B_i$; then $A_i \cup B_i$ along with v_a and v_b and one more red edge incident to v_a form a red copy of $K_{t,t} \cdot K_2$, a contradiction. □

As in the case of Lemma 2.5.1 and Lemma 2.5.2, the above lemma will imply the existence a stronger version of BEL gadgets for $K_{t,t} \cdot K_2$, and we omit the proof.

Lemma 2.5.4. *For any $t \geq 2$, let R, B be a colour pattern. There is a graph $\widetilde{\mathbf{B}}$ with an induced copy of $R \cup B$ so that $(\widetilde{\mathbf{B}}, R \cup B)$ is $K_{t,t}$-robust and so that the following hold.*

(1) *Any $K_{t,t} \cdot K_2$-free colouring in which R is red and B is blue and no $K_{t,t}$ in R is incident or contains any edges of B and no $K_{t,t}$ in B is incident or contains any edges of R extends to a $K_{t,t} \cdot K_2$-free colouring of $\widetilde{\mathbf{B}}$ in which no vertex of $R \cup B$ is contained in a monochromatic copy of a $K_{t,t}$, except those vertices contained in a $K_{t,t}$ within R or within B.*

(2) *Any $K_{t,t} \cdot K_2$-free colouring of $\widetilde{\mathbf{B}}$ has colour pattern R, B.*

Furthermore, if $R \cup B$ is bipartite, then so is $\widetilde{\mathbf{B}}$.

We are now ready to prove the main theorem of this section.

Theorem 2.5.5. *For $t \geq 2$, $s(K_{t,t} \cdot K_2) = 1$.*

Proof. We first describe a graph G_0 together with a colour pattern $R \cup B \subseteq G_0$ and then apply Lemma 2.5.4 to force this colour pattern. An illustration of the graph G_0 with colour pattern $R \cup B$ can be found in Figure 2-2.

Let U be a set of size $2t-1$, and let $\{U_T : T \in \binom{U}{t}\}$ be a collection of disjoint sets of size $2t$, indexed by the t-subsets of U. Form a graph F on the union of those sets in the following way. The subgraph $F[U]$ forms a clique K_{2t-1}, and each subgraph $F[U_T]$ for $T \in \binom{U}{t}$ forms a clique K_{2t}. Furthermore, for each $T \in \binom{U}{t}$, we choose a subset S_T of size $t-1$ in U_T and add all edges between this set and T; we also add, for each T, a new vertex and connect it to one vertex of S_T.

Take two copies, F and F', of the above graph. Add a vertex v and add all edges between v and $U \cup U'$. This graph is G_0. The colour pattern $R \cup B$ we define as
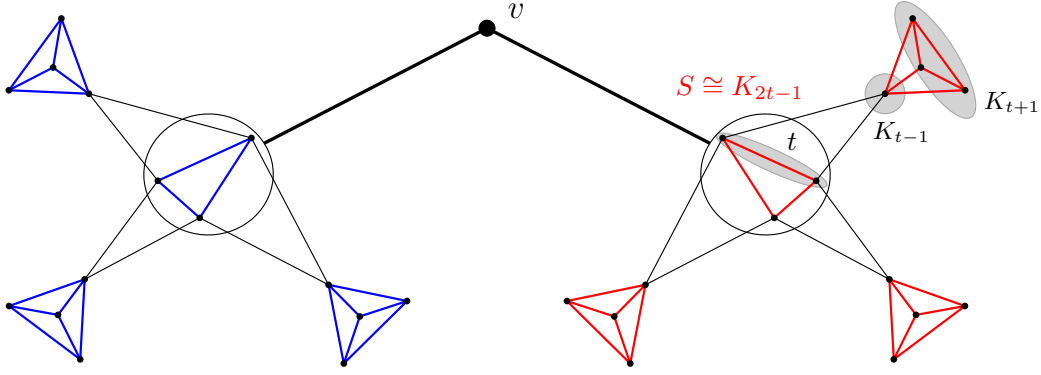
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Figure 2-2: The graph G_0 in the proof of Theorem 2.5.5 for $t = 2$ with colour pattern $R \cup B$ where B consists of all blue edges and R consists of all red edges.

follows:

$$R = \binom{U}{2} \cup \bigcup_{T \in \binom{U}{t}} \binom{U_T}{2} \subseteq F,$$

$$B = \binom{U'}{2} \cup \bigcup_{T \in \binom{U'}{t}} \binom{U'_T}{2} \subseteq F'.$$

We claim there is a $K_{t,t} \cdot K_2$-free colouring of the edges of G_0 that extends the colour pattern $R \cup B$ so that any copy of $K_{t,t}$ in R is not incident to any edge of B and any copy of $K_{t,t}$ in B is not incident to any edge in R. To see this, colour R red and B blue. Further, colour all remaining edges of F blue and all remaining edges of F' red. Finally, colour all of the edges from v to U red and from v to U' blue. The only monochromatic copies of $K_{t,t}$ are contained within one of the U_T or U'_T or are contained in v along with U or v along with U'. The edges touching those monochromatic copies are either contained themselves in a monochromatic copy of $K_{t,t}$ or are not contained in $R \cup B$.

We now show that any $K_{t,t} \cdot K_2$-free colouring of the edges of G_0 that extends the colour pattern $R \cup B$ must satisfy that v is contained in both a red and a blue copy of $K_{t,t}$. In any such colouring, all of the edges of R must have the same colour, without loss of generality red. Then, since the colouring has no monochromatic $K_{t,t} \cdot K_2$, all of the edges leaving any U_T must be blue. Therefore, any t vertices of U form one of the

parts in a blue $K_{t,t-1}$. If v had blue degree at least t to U, then v along with t of its blue neighbours would be contained in a blue $K_{t,t} \cdot K_2$, contradicting our assumption. Therefore, v must have red degree at least t to U. In this case, v is contained in a red $K_{t,t}$ with the vertices of U. By symmetry, v is contained in a blue $K_{t,t}$ with the vertices of U'.

Now, applying Lemma 2.5.4 to G_0 with R, B and adding a vertex w to $\mathbf{B}$ and connecting w only to v gives the desired result. $\qquad\square$

2.6 Upper bounds for graphs with hanging edges

In this section, we generalize the methods of the previous section. Throughout this section, let H be a graph that is *sufficiently connected*, which we define to be a graph that is either 3-connected or isomorphic to the complete bipartite graph $K_{2,t}$ with $t \geq 2$. Further, let $H' \in H \cdot K_2$. We call a vertex w a distinguished vertex of H if attaching a pendant edge to H at w yields a copy of H'. Clearly, if H is vertex-transitive, any vertex in it is distinguished.

Let $\delta_2(G)$ be the second-smallest degree in G. Note that $\delta_2(H') \leq \delta(H) + 1$. In this section we show that $s(H') \leq \delta_2(H')$, and if H is bipartite then $s(H') = 1$. Since H' is not 3-connected, we cannot directly apply Lemma 2.2.1 to construct BEL gadgets for it. By applying Lemma 2.2.1 to H, we will get a weaker version of BEL gadgets for H'. First, however, we need an even simpler lemma.

Lemma 2.6.1. *If H is a 2-connected graph and $H' \in H \cdot K_2$, then either $s(H') = 1$ or there is a graph F with a vertex u satisfying that*

(1) *there is an H'-free colouring of F in which u is not a distinguished vertex of any monochromatic copy of H, and*

(2) *in any H'-free colouring of F, u is incident to edges of both colours.*

Furthermore, if H is bipartite, so is F.

Proof. Let t be the number of vertices of H. Let $\widetilde{F}$ be a minimal Ramsey graph for H' (if H is bipartite, we may take $\widetilde{F}$ to be bipartite as well) and obtain a graph F'

by removing an edge $e = \{w_0, w_1\}$ from $\widetilde{F}$, and adding a pendant edge to both w_0 and w_1. By minimality, the graph $\widetilde{F} - e$ is not Ramsey for H'. Therefore, if F' is Ramsey for H', then one of the pendant edges is necessary for being Ramsey, and thus $s(H') = 1$. Otherwise, F' is not Ramsey for H'. If F' satisfies Property (1) and (2) with $u = w_0$ then we are done.

We now split the argument into two cases, based on which of the two properties F' fails to possess. In both cases we conclude that one of the following holds:

1. $s(H) = 1$,

2. There is a graph F (different from F') with the properties desired by the lemma,

3. There is a graph F'' with a special property (P).

After this, we will show that property (P) implies the existence of F, as desired.

The Property (P) is the following. F'' contains $t-1$ vertices $u = v_0, v_1, v_2, \ldots, v_{t-2}$ that are at pairwise distance at least t so that

(a) there is an H'-free colouring χ of F'' in which all edges containing $u = v_0$ are red and, for every $0 \leq i \leq t - 2$, v_i is not a distinguished vertex of any red copy of H; and

(b) in any colouring of F'' in which u is incident to edges of only one colour, all of $v_1, \ldots, v_{t-2}$ are contained as distinguished vertices in a monochromatic copy of H.

Assume first that in any H'-free colouring of F', the vertex w_0 is a distinguished vertex in a monochromatic copy of H. It is possible that in every H'-free colouring of F', the vertex w_0 is a distinguished vertex in a monochromatic copy of H in both colours. In this case, $s(H) = 1$ (just add a pendant edge to w_0). Otherwise, there is an H-free colouring in which w_0 is a distinguished vertex in a monochromatic copy of H in only one colour. In the latter case, the graph F'' may be obtained by taking $t - 2$ copies of F taking $v_1, \ldots, v_{t-2}$ to be the copies of w_0, and taking u to be an isolated vertex.

For the second case, assume that there is a colouring of F' without a monochromatic copy of H' in which w_0 is not a distinguished vertex in any monochromatic copy of H and that there is a colouring of F' without a monochromatic copy of H' in which w_0 is incident to edges of only one colour. Note that if we add the edge e to F' then F' is Ramsey for H' and so, in particular, if we take the colouring without a monochromatic copy of H' in which w_0 is incident to edges of only one colour, say C_0, and colour e in the other colour, say C_1, we must create a copy of H'; this copy must contain e and be of colour C_1. But the degree of w_0 in colour C_1 in this colouring is 1, and so w_0 cannot be contained in a monochromatic copy of H in colour C_1. Therefore, the edge e must be a pendant edge in a monochromatic copy of H'. Note further that in this colouring w_0 is not a distinguished vertex in a monochromatic copy of H, since in F' we added a pendant edge to w_0 and this would create a copy of H' in the colouring of F', so in the colouring of F' we must have that w_1 is a distinguished vertex in a monochromatic copy of H. Note that, since we added a pendant edge to w_1, it cannot be a distinguished vertex in monochromatic copies in both colours, for otherwise there would be a copy of H' (in the colour of the pendant edge).

Now, obtain F'' as follows. Take $|V(H)|\,(t-1)$ copies of F', call them $F'_0, F'_1, \ldots,$ $F'_{|V(H)|(t-1)}$. For each copy F'_i, associate the copy of w_1 in F'_i with the copy of w_0 in F'_{i+1}. Take u to be the copy of w_0 from F'_0 and take v_i to be the copy of w_0 from $F'_{i*|V(H)|}$.

The distance from u to any v_i and between any two v_i, v_j is at least $|V(H)|$ by construction. There is a colouring without a monochromatic copy of H' in which u is incident to edges of only one colour and each v_i is not contained as the distinguished vertex in a monochromatic copy of H in that same colour; to see this, colour each F'_i independently without a monochromatic copy of H' so that its copy of w_0 is monochromatic in colour C_0. The connectivity condition on H guarantees that any monochromatic copy of H must be internal to some copy of F'_i, and we do not create any monochromatic copies of H' since we have said that in any colouring of F' in which there is no monochromatic copy of H' and in which w_0 is incident to edges of only one colour, w_1 must not be contained in a monochromatic copy of H' in the other

colour. We also see by induction that each v_i must be contained as a distinguished vertex in a monochromatic copy of H; indeed, we see this for each copy of w_0 in each of the F_i', as by induction the vertex w_1 in F_{i-1}' is contained as a distinguished vertex in a monochromatic copy of H, and so w_0 in F_i' must be incident to only edges of the other colour in F_i', and so the copy of w_1 in F_i' is also contained as a distinguished vertex in a monochromatic copy of H.

We now show how to obtain a graph F with the desired properties from F'' that has Property (P). To F'', add two isolated vertices v and v_1' and put a copy of H on the vertex set $u, v, v_1, \ldots, v_{t-2}$ in which v_1 is a distinguished vertex and so that uv forms an edge. This is possible since H is 2-connected. Finally, add an edge between v_1' and v_1. This is the graph F.

To see that there is an H'-free colouring of F in which u is not contained in a monochromatic copy of H as a distinguished vertex, let χ be the colouring of F'' from (a). Now colour all edges of the copy of H on vertex set $u, v, v_1, \ldots, v_{t-2}$ red, except for the edge uv which we colour blue. Note that we have not created any new monochromatic copies of H since H is 2-connected and the distance in F'' between any v_i and v_j $(0 \leq i < j \leq t-2)$ is large. Since none of $u, v_1, \ldots, v_{t-2}$ is contained in a red copy of H as a distinguished vertex, we have also not created a monochromatic copy of H'. Thus, Property (1) follows.

Finally, let χ' be a colouring of F in which u is incident to edges of only one colour, say red. By Property (b), each v_i, $1 \leq i \leq t-2$, is a distinguished vertex in a monochromatic copy of H in F''. Suppose, one of those copies, say H_i that "hangs" at vertex v_i is red. Then, either it forms a red copy with another edge containing v_i (and there is nothing to prove), or all other edges not in H_i that are incident to v_i must be blue. But then, on the shortest path between u and v_i that misses v (which exists by 2-connectivity of H) there is some v_j which is incident to edges of both colours, and since it is the distinguished vertex of a monochromatic copy of H in F'', there is a monochromatic H'. So suppose that all the monochromatic copies of H in F'', of which the vertices $v_1, \ldots, v_{t-2}$ are distinguished, are blue. Again, either we find a blue copy of H', or all edges between the v_i, including the edge $v_1'v_1$ are coloured red.

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By assumption, all edges containing u are red, including the edge uv, yielding a red copy of H' on the vertex set $\{u, v, v_1', v_1, \dots, v_{t-2}\}$. $\qquad\square$

We now prove a (weak) generalization of Lemma 2.5.2 by using BEL gadgets for H.

Lemma 2.6.2. *Let H be a graph that is sufficiently connected and let $H' \in H \cdot K_2$. Either $s(H') = 1$ or the following holds. Given an H-free colour pattern R, B there is a graph G with an induced copy of $R \cup B$ so that $(G, R \cup B)$ is H-robust and:*

(1) *There exists an H'-free colouring of G that extends the colour pattern R, B in which none of the vertices of $R \cup B$ are distinguished vertices of a monochromatic copy of H.*

(2) *Any H'-free colouring of G has the colour pattern R, B.*

Furthermore, if $R \cup B$ and H are bipartite, then so is G.

Proof. Let $\mathbf{B} = \mathbf{B}(H, R, B)$ be a BEL gadget for H which exists by Lemma 2.2.1 when H is 3-connected, and by Lemma 2.5.1 when $H = K_{2,t}$ for some $t \geq 2$. Furthermore, if H is bipartite and if $R \cup B$ is bipartite, we may assume that $\mathbf{B}$ is bipartite. Note that $\mathbf{B}$ satisfies Property (1) and Property (2) if we replace in the properties H' by H.

For every vertex u of $\mathbf{B}$, add a copy of the graph F given by the previous lemma (which exists unless $s(H') = 1$) and identify the distinguished vertex of F with u; this is the graph G. Consider the colouring in which R is red and B is blue. Since $\mathbf{B}$ is a BEL gadget for H, this colouring extends to an H-free colouring of $\mathbf{B}$. This colouring extends to an H'-free colouring of G in which no vertex of $\mathbf{B}$ is a distinguished vertex in a monochromatic copy of H. This follows from Property (1) in Lemma 2.6.1, since any copy of F meets $\mathbf{B}$ only in its distinguished vertex u, and since H is connected. Therefore, property (1) holds.

To see that property (2) holds, let χ be an H'-free colouring of G. Then by property (2) in Lemma 2.6.1, every vertex u of $\mathbf{B}$ is incident to edges of both colours

that do not lie inside **B**. Therefore, the colouring must be H-free on **B**. And thus, by the property of a BEL gadget, χ must extend the colour pattern $R \cup B$. $\qquad\square$

Using the above lemma, we will construct signal senders in which we have some control over the structure of edges incident to the signal edges, similar to the graphs in Lemma 2.5.3.

Lemma 2.6.3. *Let H be a graph that is sufficiently connected, let $H' \in H \cdot K_2$ and assume that $s(H') \neq 1$. Take t to be the number of vertices of H Then there is a graph $S^- = S^-(H', e, f)$ with two independent edges e, f so that any H'-free colouring of S^- satisfies that e and f have different colours, and there is an H'-free colouring of S^- so that every edge incident to f has a different colour from f, and so that any vertex in e or f is not the distinguished vertex in a monochromatic copy of H. Furthermore, if H is triangle-free, then e and f are not contained in any triangles.*

Proof. Let χ be the chromatic number of H and let $\overline{\chi}$ be the set of all χ-colorings of H. Define σ to be the size of the smallest color class over all χ-colorings of B; i.e., $\sigma = \min\limits_{\chi \in \overline{\chi}} \min\limits_{i \in [\chi]} |\chi^{-1}(i)|$.

If $\chi > 2$, let $\mathcal{H}$ be any $(\chi - 1)$-uniform linear hypergraph having girth greater than t and chromatic number at least 3. If $\chi = 2$, we take $V(\mathcal{H}) = \{v_1\}$ and $E(\mathcal{H}) = \{\{v_1\}\}$. In either case, set $N = |V(\mathcal{H})|$ and $\{v_1, v_2, \ldots, v_N\} = V(\mathcal{H})$.

From $\mathcal{H}$ we build a graph G_0 on $V(\mathcal{H})$ by joining two vertices if and only if they are both contained in some hyperedge of $\mathcal{H}$. Throughout this construction we will keep the underlying structure of $\mathcal{H}$ in mind. At this point, each hyperedge in $\mathcal{H}$ corresponds to a clique on $(\chi - 1)$ vertices.

We now build a graph G_1 by blowing up each vertex $v_i \in V(G_0)$ into an independent set V_i of size $2t(t-1)$ vertices; that is $V(G_1) = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_N$ with $|V_i| = 2t(t-1)$ and $xy \in E(G_1)$ if and only if there exists a pair of integers $i, j \in [N]$ such that $x \in V_i$, $y \in V_j$, and $v_i v_j \in E(G_0)$. Each hyperedge of $\mathcal{H}$ now corresponds to a complete $(\chi - 1)$-partite graph and the intersection of two hyperedges is either the empty set or some independent set V_i.

To $V(G_1)$ add an independent set θ of $\sigma - 1$ vertices and join every vertex in θ to every vertex in $V(G_1)$. Add, for every vertex v from $V(G_1)$, a new vertex w and make vw an edge. Add two more vertices and an edge e between them. Let G_2 be the graph obtained by this procedure. Take $B = G_2$.

If H has an edge r not contained in any K_3, take x_1 and x_2 to be the endpoints of r; note such an edge exists if H is bipartite. Otherwise, take r to be any edge of H and x_1, x_2 to be its endpoints. Define $\tilde{H}$ to be the (labeled) graph on $t - 1$ vertices obtained from H' by removing the vertices x_1 and x_2. We obtain G_3 from G_2 by inserting the edges of $2t$ vertex-disjoint (labeled) copies of $\tilde{H}$ into each V_i. Take R to be the edges we just added, that is the edges of G_3 that are not in G_2.

Let $\mathbf{B}$ be the graph obtained by applying Lemma 2.6.2 to the colour pattern R, B (we will show later that R, B has the necessary properties). The vertex set of S^- is obtained by adding two vertices x_1' and x_2' to $V(\mathbf{B})$. We add the edge $f := x_1' x_2'$ and, for each i and for each copy of $\tilde{H}$ that we added to V_i, we connect x_1' to the neighbors of x_1 in that copy, and x_2' to the neighbors of x_2 in that copy.

We will now check that the colour pattern R, B does not contain a monochromatic H.

Note that R is the disjoint union of connected components that contain at most $t - 1$ vertices, so R cannot contain a copy of H. Note also that B, excluding vertices of degree 1, is a χ-chromatic graph where one of the parts has size $\sigma - 1$, so B cannot contain a monochromatic copy of H by definition of σ.

Indeed, we argue that we may extend R, B to colour the edges incident to x_1', x_2' without creating a monochromatic copy of H. To do this, colour all of R red, colour all of B blue, colour f blue, and colour all of the edges incident to f red. Clearly, we do not create a blue copy of H. Note that, in the case H is 3-connected, any red copy of H using x_1' and/or x_2' must remain connected after removing x_1' and x_2', and so vertices of the copy, excluding x_1' and x_2', must be contained entirely within the vertex set of one of the copies of $\tilde{H}$ inside one of the V_i. If we fix any copy of $\tilde{H}$ and consider the graph induced by this copy along with x_1' and x_2', this graph is isomorphic to H' in which r is coloured blue rather than red. This graph has no red

copy of H, as the vertex of degree 1 cannot be used in a copy of H, and then the number of red edges remaining after this vertex is omitted is fewer than the number of edges in H.

Finally, we wish to show that if the edges of B are coloured blue, the edges of R are coloured red, and f is coloured red, then there must be a monochromatic copy of H. Recall that the induced graph on V_i consists of $2t(t-1)$ vertex-disjoint red copies of $\tilde{H}$. For each such copy of $\tilde{H}$ in V_i, if the bipartite graph between $\tilde{H}$ and $\{x'_1, x'_2\}$ is colored entirely red, then this forms a red copy of H'. If this is not the case, for each V_i and all the copies of $\tilde{H}$ in V_i, there is at least one blue edge in the bipartite graph between $\tilde{H}$ and $\{x'_1, x'_2\}$. Hence the bipartite graph on $V_i \cup \{x'_1, x'_2\}$ contains at least $2(t-1)$ blue edges, which implies that either the bipartite graph $V_i \cup \{x'_1\}$ has at least has $t-1$ blue edges or the bipartite graph $V_i \cup \{x'_1\}$ has at least $t-1$ blue edges. In the case $\chi = 2$, then the graph B along with the $t-1$ blue edges contain a blue $K_{\sigma, t-1}$, which must contain a blue copy of H, and there are other blue edges incident to either part, so taking one of those forms a blue H', as desired.

The remaining case is when $\chi > 2$. In this case, we define a 2-colouring of $\mathcal{H}$; we colour vertex v_i with colour 1 if x'_1 has at least $t-1$ blue edges to V_i, and otherwise x'_2 has at least $t-1$ blue edges to V_i and we colour vertex v_i with colour 2. Because $\mathcal{H}$ has chromatic number at least 3, there must exist a monochromatic edge under this colouring. Therefore, there is some $j \in [2]$ and some edge s of the hypergraph so that x'_j has blue-degree at least $t-1$ to each of the V_i with $i \in j$. However, this forms a complete blue multipartite graph with one part of size σ and the remaining parts of size $t-1$, which must contain a blue H'. $\qquad\square$

As discussed around Lemma 2.5.2, the above lemma immediately gives the following version of BEL gadgets for H'.

Lemma 2.6.4. *If H is sufficiently connected and $H' \in H \cdot K_2$ with $s(H') \neq 1$, let R, B be a colour pattern. There is a graph $\widetilde{\mathbf{B}}$ with an induced copy of $R \cup B$ so that $(\widetilde{\mathbf{B}}, R \cup B)$ is H-robust and so that the following hold.*

(1) *Any H'-free colouring in which R is red and B is blue and no H in R is incident*

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or contains any edges of B and no H in B is incident or contains any edges of R extends to a H′-free colouring of $\widetilde{\mathbf{B}}$ in which no vertex of $R \cup B$ is contained as a distinguished vertex in a monochromatic copy of a H, except those vertices contained as a distinguished vertex in a copy of H within R or within B.

(2) *Any H′-free colouring of $\widetilde{\mathbf{B}}$ has colour pattern R, B.*

The previous lemma will allow us to show the upper bound $s(H') \leq \delta_2(H')$.

Theorem 2.6.5. *If H is sufficiently connected and $H' \in H \cdot K_2$, we have $s(H') \leq \delta_2(H')$.*

Proof. If $s(H) = 1$, we are done, and otherwise we may apply Lemma 2.6.4. Take t to be the number of vertices of H. Take B to be $t - 1$ vertex-disjoint cliques, each on t vertices. Apply Lemma 2.6.4 to this colour pattern (R is empty) to get a graph **B**. Pick vertices $v_1, \ldots, v_{t-1}$, one from each of the cliques from B. Pick a vertex $v_t \neq v_1$ from the first clique. Add edges to form a clique on $v_1, \ldots, v_{t-1}$, and add one more edge between v_1 and v_t. Observe that this is not Ramsey for H', as we may colour all of B blue and all of the edges added to B red; in blue there is no copy of H' by construction, and in red we have a connected component on t vertices, but one of those vertices has degree 1 so there is no red H in this part of the graph. We will now add one more vertex v of degree $\delta_2(H')$ and show that with this vertex the graph is Ramsey for H'. We first consider the case where there is a vertex of degree $\delta_2(H')$ that is not incident to the vertex of degree 1 in H'. Add a vertex v and connect it to $v_1, \ldots, v_{\delta_2(H')}$. Otherwise, if the vertex of degree $\delta_2(H')$ is incident to the vertex of degree 1, then connect v to $v_1, \ldots, v_{\delta_2(H')-1}$ and to v_t. In either case, if B is coloured blue, then if v has any outgoing blue edges it is contained in a blue H' as the pendant vertex, and, otherwise, all of its outgoing edges are red and it is contained in a red H'. $\square$

2.7 The complete graph with an added vertex

We define $H_{t,d}$ to be the graph on $t+1$ vertices that contains a K_t and in which the remaining vertex (not in the K_t) has degree d, with its neighbors being any d vertices of the K_t.

Note $H_{d,d}$ is isomorphic to K_{d+1}, for which $s(K_{d+1})$ is known to be d^2 [12]. For $d = 1$, it was recently shown that $s(H_{t,1}) = t - 1$ [42]. For $d = 0$, it was found $s(H_{t,0}) = s(K_t) = (t-1)^2$ [88]. A natural question that arises is how $s(H_{t,d})$ behaves when d is between 1 and t. We now state the main result of this section.

Theorem 2.7.1. *For all $1 < d < t$ we have*

$$s(H_{t,d}) = d^2.$$

The proof of this theorem is presented in two parts. In the first part, we prove that $s(H_{t,d}) \geq d^2$ for all values of d. The second part expands on the ideas in [12] and [42] and deals with the upper bound on $s(H_{t,d})$ for $d \geq 2$: we construct a graph G with a vertex v of degree d^2 that is Ramsey for $H_{t,d}$ such that $G - v \nrightarrow H_{t,d}$. It follows from this that $s(H_{t,d}) \leq d^2$, and so we obtain $s(H_{t,d}) = d^2$ for all $1 < d < t$. We now begin with the first part of our proof, which closely follows the ideas of [12].

Lemma 2.7.2. *Let H be a graph such that, for all $v \in V(H)$, the neighborhood of v contains a copy of K_d. Then $s(H) \geq d^2$.*

Proof. Suppose there exists $F \in \mathcal{M}(H)$ and some $v \in V(F)$ with $\deg(v) < d^2$. Since F is minimal, we can 2-color the edges of $F - v$ so that there is no monochromatic copy of H. Consider any such 2-coloring of $F - v$. In this coloring, let S denote the neighborhood of v and let $T_1, \ldots, T_k$ be a maximal set of vertex-disjoint red copies of K_d in S. Since $\deg(v) < d^2$, we must have $|S| < d^2$, and so $k \leq d - 1$. Now we color all the edges connecting v to $T_1, \ldots, T_k$ blue, and all other edges incident to v red. We claim that no monochromatic copy of H arises in such a coloring. Note that such a copy would need to use v. We will now show that there is no red d-clique in the

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red neighborhood of v and that there is no blue d-clique in the blue neighborhood of v, thus showing that v cannot be contained in any monochromatic copy of H.

Any red d-clique in S must intersect one of $T_1, \ldots, T_k$ and therefore would have a blue edge from v. On the other hand, suppose there exists a blue d-clique in the blue neighborhood of v, which is precisely $T_1 \cup \cdots \cup T_k$. Since $k \leq d - 1$, by the pigeonhole principle, at least two vertices of this blue d-clique must be contained in the same T_i. These two vertices, however, are connected by a red edge, a contradiction. It follows that such an $F \in \mathcal{M}(H)$ cannot exist, and hence $s(H) \geq d^2$. $\qquad \square$

Since the neighborhood of each vertex in $H_{t,d}$ contains a copy of K_d, we have the following corollary.

Corollary 2.7.3. *For all values of d we have $s(H_{t,d}) \geq d^2$.*

This completes the first part of our proof, establishing a lower bound on the value of $s(H_{t,d})$.

For the upper bound, we wish to construct an H-minimal graph with vertex of degree exactly d^2 for $d \geq 2$. To that end, we wish to show that $H_{t,d}$ has BEL gadgets. Theorem 2.2.1 implies this in the case $d \geq 3$, but not when $d = 2$; the majority of the work in this section is proving that $H_{t,2}$ has BEL gadgets.

Theorem 2.7.4. *For all $2 \leq d \leq t$, the graph $H_{t,d}$ has BEL gadgets.*

We postpone the proof of this theorem to the end of the section; let us first see why it implies the desired upper bound on $s(H_{t,d})$.

Lemma 2.7.5. *For all $2 \leq d \leq t$ there exists a graph F' with vertex v of degree d^2 so that $F' \rightarrow H_{t,d}$ but $F' - v \not\rightarrow H_{t,d}$.*

Proof. If $d = t$ then $s(H_{t,d}) = d^2$ by [12], which immediately implies the lemma; we will henceforth assume $d < t$.

The graph $H_{t,d}$ has BEL gadgets by Theorem 2.7.4. This means that, for any graph G and 2-coloring ψ of G without a monochromatic copy of H, there exists a graph $F \not\rightarrow H_{t,d}$ with an induced copy of G such that every 2-coloring of F without

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a monochromatic copy of $H_{t,d}$ agrees with ψ on the copy of G, up to permutation of colors. We describe our graph G together with its coloring ψ for our BEL gadget as follows:

1. G contains d disjoint red copies $T_1, \ldots, T_d$ of K_t,

2. For each distinct pair i and j, there is a complete blue bipartite graph between T_i and T_j, and

3. For each way there is to choose a d-tuple $T = (t_1, \ldots, t_d) \in T_1 \times \cdots \times T_d$ by taking one vertex from each T_i, we add a set of $t-d$ vertices $S_T = \{v_1^T, \ldots, v_{t-d}^T\}$; we add blue edges between all pairs of vertices in S_T so that S_T becomes a blue clique, and add more blue edges so that there is a complete blue bipartite graph between S_T and T. For distinct d-tuples T and T', S_T and $S_{T'}$ are disjoint.

An example of this G with coloring ψ is shown in Figure 2-3. We first claim that this coloring ψ contains no monochromatic copy of $H_{t,d}$. The connected components in red are all copies of K_t, so there is no red copy of $H_{t,d}$. We also claim there is no blue copy of $H_{t,d}$. If we omit the vertices that are contained in the various S_T, the blue graph is d-partite and so contains no K_t, as $d < t$. Therefore, any blue copy of $H_{t,d}$ must use some vertex w in some S_T as part of a blue K_t. Note that the blue degree of w is $t-1$, and therefore this blue K_t must consist precisely of w and its neighborhood. However, any vertex that is not w or contained in the blue neighborhood of w has degree at most $d-1$ to the neighborhood of w by construction, and so cannot be the vertex of degree d in $H_{t,d}$. Therefore, there is no blue copy of $H_{t,d}$.

Consider a graph $F \not\to H_{t,d}$ with an induced copy of G such that any 2-coloring of F without a monochromatic copy of $H_{t,d}$ restricts to the coloring ψ on the induced copy of G, up to permutation of the colors; this exists by Theorem 2.7.4. We now modify F to F' by adding a vertex v, and adding d edges from v to each T_i in the induced copy of G. The vertex v clearly has degree d^2. We claim that this modified graph F' is Ramsey for $H_{t,d}$. Consider any 2-coloring of F'. In this 2-coloring, if there is a monochromatic copy of $H_{t,d}$ in the subgraph $F = F' - v$, then we are done.

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Otherwise suppose the 2-coloring does not yield a monochromatic copy of $H_{t,d}$ in F. Then the induced graph G must have coloring ψ, up to permutation of colors. Let us assume without loss of generality that each T_i forms a red clique and the remaining edges are blue.

If v had red degree d to some T_i, then v together with T_i would be a red copy of $H_{t,d}$. Thus, at least one edge from v to each copy of T_i must be colored blue. Choose one vertex t_i from each T_i so that v has a blue edge to t_i and take $T = (t_1, \ldots, t_d)$. Then these vertices t_i together with S_T forms a blue K_t, and adding v creates a blue $H_{t,d}$. $\qquad\square$

This immediately gives the desired upper bound on $s(H_{t,d})$.

Corollary 2.7.6. *For every $2 \le d \le t$, we have $s(H_{t,d}) \le d^2$.*

Proof. By the previous lemma, there is a graph F' with a vertex v of degree d^2 which is Ramsey for $H_{t,d}$ so that $F' - v$ is not Ramsey for $H_{t,d}$. Take F'' to be a subgraph of F' which is minimal subject to the constraint that F'' is Ramsey for F. F'' must contain v, and so $s(H_{t,d}) \le \delta(F'') \le d^2$, as desired. $\qquad\square$
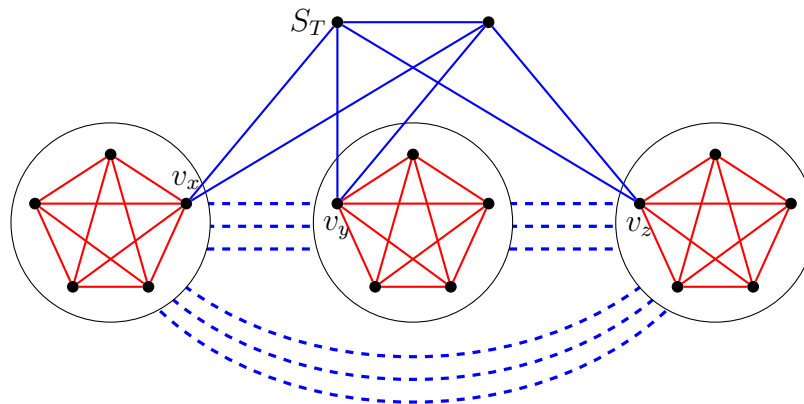

Figure 2-3: Example of G with the coloring ψ for $t = 5$ and $d = 3$. Here, only one set S_T is shown, corresponding to the triple $T = (v_x, v_y, v_z)$. The dashed blue edges represent complete blue bipartite graphs. When we add the external vertex v, we will connect it to three vertices from each copy of K_5, making its degree $d^2 = 9$.

We now prove that $H_{t,2}$ has BEL gadgets. Note that, for $t = 2$, the graph $H_{2,2}$ is isomorphic to K_3, for which it is known that BEL gadgets exist [12]. Henceforth, we will assume that $t \geq 3$. The ideas behind the proof of BEL gadgets for $H_{t,2}$ stems from a strategy in [42]. We now introduce the main tool that we will need.

Definition 2.7.7. *Write $F \xrightarrow{\epsilon} H$ to mean that, for every $S \subseteq V(F)$ such that $|S| \geq \epsilon|V(F)|$, the subgraph of F induced by S is Ramsey for H (i.e. $F[S] \to H$).*

The following lemma, which is a strengthening of a theorem in [76], is proven in [42].

Lemma 2.7.8. *For any graph H and every $\epsilon > 0$ and $t > 2$, if $\omega(H) < t$ then there exists a graph F that is K_t-free such that $F \xrightarrow{\epsilon} H$.*

We are now ready to construct a graph G_0 so that, for every coloring of G_0 without a monochromatic copy of $H_{t,2}$, a particular copy of some (arbitrary) graph R_0 is forced to be monochromatic. Furthermore, there is a coloring of G_0 where R_0 is red, all of the edges leaving R_0 are blue, there is no red $H_{t,2}$, and there is no blue K_t. The proof of this lemma closely follows the arguments in [42].

Lemma 2.7.9. *Let R_0 be a graph that has no copy of $H_{t,2}$. Then there exists a graph G_0 with an induced copy of R_0 and the following properties:*

1. *There is a 2-coloring of G_0 without a red copy of $H_{t,2}$ and without a blue copy of K_t in which the edges of R_0 are red, and all of the edges incident to, but not contained in, R_0 are blue, and*

2. *Every 2-coloring of G_0 without a monochromatic copy of $H_{t,2}$ results in R_0 being monochromatic.*

Proof. Take $\epsilon = 2^{-n-t^2}$, where n is the number of vertices in R_0. Let $F_1, F_2, \ldots, F_{t-2}$ be copies of the graph as defined in Lemma 2.7.8 when applied to $H = H_{t-1,1}$. We claim that the graph $G_0 := F_1 \boxtimes F_2 \boxtimes \cdots \boxtimes F_{t-2} \boxtimes R_0$ satisfies both desired conditions (see Figure 2-4).

To see the first property, color all the edges internal to any of $F_1, F_2, \ldots, F_{t-2}, R_0$ red and the remaining edges blue. There can be no monochromatic red copy of $H_{t,2}$, since each F_i is K_t-free and R_0 is $H_{t,2}$-free. Furthermore, there is no blue K_t, since the graph induced by the blue edges is $(t-1)$-chromatic.

To see the second property, we consider some 2-coloring ψ of G_0 so that G_0 does not have a monochromatic copy of $H_{t,2}$. We show that this forces R_0 to be monochromatic. For a subset S of the vertices and some vertex $v \notin S$, define the *color pattern c_v with respect to S* to be the function with domain S that maps a vertex $w \in S$ to the color of the edge (v, w). This method was utilized in [42].

For a vertex $v \in F_1$, consider its color pattern c_v with respect to $V(R_0)$. There are 2^n possible color patterns, so at least a 2^{-n} fraction of the vertices in F_1 have the same color pattern with respect to $V(R_0)$. Call the set of these vertices S_1. Then $|S_1| \geq 2^{-n} \cdot |V(F_1)| \geq \epsilon \cdot |V(F_1)|$, so there must exist a monochromatic copy H_1 isomorphic to $H_{t-1,1}$ in S_1. Without loss of generality, suppose H_1 is monochromatic in red. We claim that all the edges going from S_1 to R_0 (and in particular from H_1 to R_0) are blue. Indeed, since all vertices $v \in S_1$ have the same color pattern with respect to R_0, then for a fixed vertex $i \in R_0$ the edges (i, v) have the same color for all $v \in S_1$. If that color is red, then i along with all the vertices of H_1 would form a monochromatic red copy of $H_{t,2}$, which contradicts our definition of ψ. We now proceed inductively. Suppose we have identified red copies of $H_{t-1,1}$ labeled $H_1, \ldots, H_{k-1}$ in $F_1, \ldots, F_{k-1}$ with vertex sets $V_1, \ldots, V_{k-1}$ respectively, and that all edges between these copies as well as to R_0 are blue. In F_k, at least a $2^{-n-t(k-1)} > \epsilon$ fraction of the vertices S_k have the same color pattern with respect to $V(R_0) \cup V(H_1) \cup \cdots \cup V(H_{k-1})$. Since $|S_k| > \epsilon \cdot |V(F_k)|$, we have $F[S_k] \to H_{t-1,1}$. Find a monochromatic copy of $H_{t-1,1}$ and call it H_k. Suppose H_k is blue. Then, as in the case before, all the edges between H_k and R_0, as well as to $H_1, \ldots, H_{k-1}$, would have to be red, otherwise there would be a monochromatic blue copy of $H_{t,2}$. But if all these edges are red, then any vertex of H_k along with H_1 forms a monochromatic copy of $H_{t,2}$, a contradiction. Thus, H_k must be red, and consequently all edges between H_k and $H_1, \ldots, H_{k-1}, R_0$ must be blue, completing the inductive step. After applying

this argument $t-2$ times, we have a collection $(H_1, \ldots, H_{t-2})$ of red copies of $H_{t-1,1}$ with complete bipartite blue graphs between any two of them. Now, suppose some edge in R_0 was blue. Then this edge, along with one vertex in each of $H_1, \ldots, H_{t-2}$ and one other arbitrary vertex in H_1 forms a monochromatic blue copy of $H_{t,2}$. Thus, all the edges in R_0 must be colored red, as required. $\square$
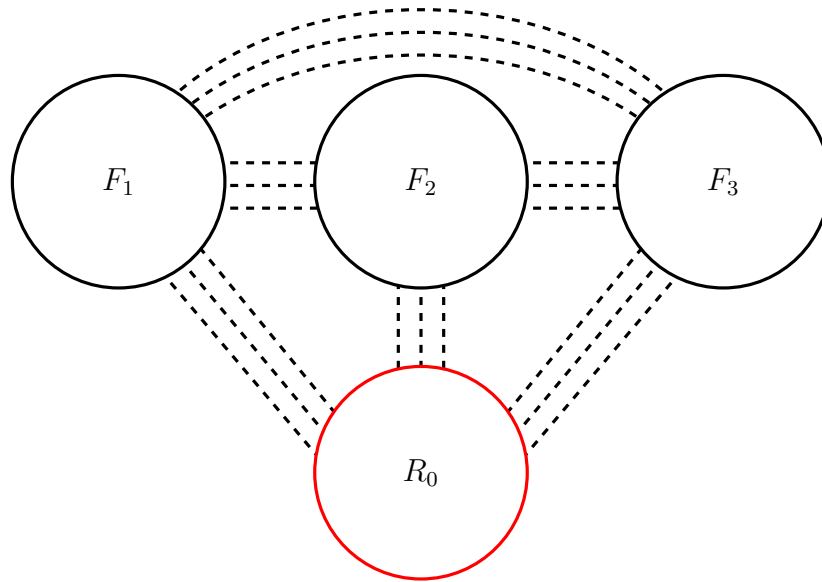

Figure 2-4: Construction of the gadget graph G_0 for $t = 5$ and $d = 2$. The dashed lines represent complete bipartite graphs.

We now introduce a lemma which is a stronger version of an idea first introduced in [12] known as a positive signal sender.

Lemma 2.7.10. *There is a graph G with two independent edges e and f so that, in any 2-coloring of G without a monochromatic copy of $H_{t,2}$, both edges e and f must have the same color. Furthermore, there is a 2-coloring of G with no red $H_{t,2}$ and no blue K_t in which both edges e and f are red, and in which all of the edges incident to either of e or f are blue. Furthermore, there are no edges incident to both e and f.*

Proof. This follows by taking R_0 in the previous lemma to be two disjoint edges, e and f. $\square$

We now take the above lemma and use it to prove a slight strengthening of itself.

Lemma 2.7.11. *There is a graph G with two independent edges e and f so that in any 2-coloring of G without a monochromatic copy of $H_{t,2}$ both edges e and f must have the same color. Furthermore, there is a 2-coloring of G with no red $H_{t,2}$ and no blue K_t in which both edges e and f are red, and in which all of the edges incident to either of e or f are blue. Furthermore, any path between a vertex of e and a vertex of f has length at least 3.*

Proof. Lemma 2.7.10 gave us a graph that satisfied all of these constraints except for the last one. Take two copies G', G'' of this graph from Lemma 2.7.10, with distinguished pairs of edges (e', f') and (e'', f''), respectively. Identify f' with e'' and take $e = e'$ and $f = f''$, and call the resulting (combined) graph G. By construction, any path between a vertex of e and a vertex of f has length at least 3. Also by construction, in any 2-coloring of G without a monochromatic copy of $H_{t,2}$, we must have that $e = e'$ and f' have the same color, and $f' = e''$ and $f'' = f$ have the same color, and so e and f have the same color. Finally, if we color e, f', and f all red, then we may extend this to colorings of G' and G'' so that neither G' nor G'' contains a red $H_{t,2}$ or a blue K_t so that all edges incident to either of e or f are blue. This coloring contains no red $H_{t,2}$, as every connected component in red is contained entirely within at least one of G' and G'', and neither one of these graphs has a red copy of $H_{t,2}$. There is no blue copy of K_t, as every blue triangle is contained either entirely within G' or entirely within G'', and neither one contains a blue copy of K_t. $\square$

The next lemma uses these so-called *strong positive signal senders* to construct a weaker version of BEL gadgets for $H_{t,2}$. It is weaker because it does not guarantee that we can agree with a given coloring ψ of a graph up to permutation of colors; it only guarantees that in a monochromatic $H_{t,2}$-free coloring of the graph, the edges that are red in ψ all end up with one color α_1 and the edges that are blue in ψ all end up with one color α_2. The two colors α_1 and α_2 may be the same. After proving this lemma, we will then show that the existence of this weaker version of BEL gadgets implies the full strength of the BEL theorem, completing the proof.

Lemma 2.7.12. *Given edge-disjoint graphs G_0 and G_1 on the same vertex set that are both $H_{t,2}$-free, there is a graph G with an induced copy of $G_0 \cup G_1$ so that there is a 2-coloring of G without a monochromatic copy of $H_{t,2}$ in which G_0 is red and G_1 is blue. Furthermore, in any 2-coloring of G without a monochromatic $H_{t,2}$, all the edges in G_0 have the same color and all the edges in G_1 have the same color.*

Proof. Take F to be a copy of the graph given by Lemma 2.7.11.

Form a graph G as follows. Start with $G_0 \cup G_1$ on the same vertex set. Add two edges e_0 and e_1 independent from both G_0 and G_1. For any edge f_0 in G_0, we add a copy of F with e_0 and f_0 as the distinguished edges. For any edge f_1 in G_1, we add a copy of F with e_1 and f_1 as the distinguished edges. By construction, in any 2-coloring of G without a monochromatic $H_{t,2}$, all of the edges in G_0 have the same color and all of the edges in G_1 have the same color.

Consider coloring all edges of G_0 as well as e_0 red and all edges of G_1 as well as e_1 blue. By construction of F, we may extend this coloring to a coloring of G in which every copy of F attached to two edges in G_0 contains no blue K_t and no red $H_{t,2}$ and in which all of the edges of F that are incident to the two edges are blue. Symmetrically, in this coloring every copy of F attached to two edges in G_1 contains no red K_t and no blue $H_{t,2}$ and satisfies that all of the edges of F that are incident to the two edges are red.

We claim there is no blue $H_{t,2}$. By symmetry it will follow that there is also no red $H_{t,2}$. First, observe that if we pick any two edges (e, f) to which a copy of F is attached, the vertices of any triangle in G are either contained entirely in F or entirely in the graph G' obtained by removing the vertices of F except e and f; this follows immediately from the construction. Note further that any triangle that is not contained entirely in G' must use some vertex w that belongs to F but not to G'; since there is no vertex in F that has as a neighbor both a vertex of e and a vertex of f, such a triangle may not use both a vertex of e and a vertex of f; in particular, this means that all of the edges used by the triangle are contained in F (note that there are no edges between e and f that are not contained in F, by the way we constructed G_0 and G_1). Therefore, any copy of K_t must be contained entirely in the

edges of F or in entirely in G'. Since there is no blue K_t in the copies of F attached to edges from G_0, any blue copy of $H_{t,2}$ must have its copy of K_t contained entirely in G_1 or entirely in some copy of F attached to an edge of G_1. If we take a blue K_t contained in some copy of F attached an edge e and some edge f in G_1, then, since all of the edges incident to both e and f are red, if we take the connected component corresponding to the blue subgraph of G containing this copy of K_t, we see that it is contained entirely in this copy of F. But by assumption this copy of F has no $H_{t,2}$, and so this blue K_t is not contained in any copy of $H_{t,2}$. Therefore, any blue copy of $H_{t,2}$ must have its K_t contained in G_1. By assumption, G_1 contains no copy of $H_{t,2}$, so this copy must have some vertex outside of G_1 that has blue degree at least 2 to this copy of K_t. Such a vertex cannot be contained in the copies of F attached to an edge of G_1, as these are completely red to G_1. Therefore, this vertex must be contained in some copy of F attached to an edge e and an edge f of G_0. But neither e nor f may be edges of the blue clique, since they are both red, and so this vertex must have a blue neighbor in e and a blue neighbor in f, but this contradicts our assumptions on F, concluding the proof. $\qquad\square$

If a graph H satisfies the conclusions of the above lemma, we say it has *weak BEL gadgets*. We now prove that this is enough to get *strong BEL gadgets* for $H_{t,2}$, thus completing the proof of the upper bound.

Lemma 2.7.13. *If H is connected and has weak BEL gadgets, then H has BEL gadgets.*

Proof. Consider a graph G with a given 2-coloring ψ. Let G be composed of the graphs G'_0 and G'_1, where G'_0 is the graph induced by the blue edges of G and G'_1 is the graph induced by the red edges of G. Take t to be the number of vertices in H.

Define a graph G_0 by taking G'_0, adding to it some set S of t vertices, and adding edges to S so it forms a copy of H with one edge removed. Define G_1 by taking G'_1, adding to it S, and adding to S the edge that was removed from H. We will show that this resulting graph can be made a strong BEL gadget for H. Note that neither G_0 nor G_1 contains a copy of H; the connected components are either connected

components of G_0 or G_1, or are in S. Note further that in any 2-coloring of $G_0 \cup G_1$ in which all of the edges in G_0 have the same color and all of the edges of G_1 have the same color, if G_0 and G_1 have the same color then there is a monochromatic copy of H, namely on vertex set S. Now, taking a weak BEL gadget for G_0 and G_1 yields the desired strong BEL gadget for G_0' and G_1'. $\qquad\square$

2.8   Concluding Remarks

We have shown that all 3-connected bipartite graphs are Ramsey simple. However, Conjecture 2.1.1 remains open.

Conjecture 2.8.1. *[88] If H is bipartite with no isolated vertices, then H is Ramsey simple.*

We have also demonstrated the first class of graphs that is not bipartite but is Ramsey simple, namely those graphs that have BEL, that don't have isolated vertices, and that contain a vertex of minimum degree δ whose neighbourhood is contained in an independent set of size $2\delta - 1$. One may hope to use similar techniques to those found in the proof of Theorem 2.1.2 to get a corresponding result for triangle-free graphs. This leads us to the following conjecture.

Conjecture 2.8.2. *Every connected triangle-free graph without isolated vertices is Ramsey simple.*

Based on the techniques used here, it may be significantly easier to prove the conjecture just for 3-connected graphs.

Conjecture 2.8.3. *Every 3-connected triangle-free graph is Ramsey simple.*

2.9   Sparse random graphs are Ramsey simple

We prove Corollary 2.1.4 that the random graph $G(n,p)$ with $n^{-1}\log n \ll p \ll n^{-2/3}$ is Ramsey simple with high probability. This follows from showing that $G(n,p)$ satisfies the conditions of Theorem 2.1.3 with high probability.

Proof of Corollary 2.1.4. Let $n^{-1} \log n \ll p \ll n^{-2/3}$ and assume that the vertex set of $G(n, p)$ is $[n]$.

We will use the following basic facts about $G(n, p)$, for p in the aforementioned range.

(1) For every d, $P[\delta(G(n, p)) = d] = o(1)$.

(2) $P[\delta(G(n, p)) \leq pn] = 1 - o(1)$.

(3) With high probability, there exists a unique vertex in $G(n, p)$ of minimum degree.

(4) With high probability, $G(n, p)$ is 3-connected.

(5) For any d, we have $P[\delta(G(n-1, p)) \geq d - 1] \geq P[\delta(G(n, p)) \geq d] - o(1)$.

Facts (1), (2), and (3) follow from Theorem 3.9(i) and its proof in [8], and (4) follows from Theorem 7.7 in [8]. We now prove fact (5).

Consider the vertex $n \in [n]$. Since with high probability there is a unique vertex of minimum degree by fact (3), the probability that n is this vertex is $o(1)$. Then, in the distribution obtained by taking $G(n, p)$ and removing the vertex n, every vertex has degree at least its degree in the $G(n, p)$ minus 1. Since with high probability the minimum degree vertex of $G(n, p)$ was in $[n - 1]$, we have that the minimum degree of this $G(n - 1, p)$ is at least $\delta(G(n, p)) - 1$. This completes the proof of (5).

We wish to show that, with high probability, $G = G(n, p)$ contains a minimum degree vertex, call its degree δ, whose neighborhood is contained in an independent set of size $2\delta - 1$. Combining this with fact (4) and applying Theorem 2.1.3 immediately implies Corollary 2.1.4.

For any $0 < \varepsilon < 1$, pick d minimal so that $P[\delta(G(n, p)) \leq d] \geq 1 - \varepsilon$. We will let ϵ very slowly tend to 0. By (2) we must have that $d \leq pn$. By (1) we must have that

$$1 - \varepsilon \leq P[\delta(G(n, p)) \leq d] \leq 1 - \varepsilon + o(1).$$

For every set $S \subseteq [n]$ with $|S| \leq d$ and every vertex $v \in [n] \setminus S$, take $T_{S,v}$ to be some set containing S of size $2|S| - 1$ so that $T_{S,v}$ does not contain v. For any such S and v, let $A_{S,v}$ be the event that v is the unique vertex of minimum degree in $G(n, p)$ and that its neighbourhood is S. Let $B_{S,v}$ be the event that $T_{S,v}$ is not an independent

set. Note that the probability that there is some vertex v of minimum degree whose neighbourhood is contained in an independent set of size $2\deg(v) - 1$ is at least the probability that there exists a set S of size at most pn and a vertex v not in S so that $A_{S,v}$ holds and $B_{S,v}$ does not hold. We will show that this is true with high probability.

Note that the events $A_{S,v}$ are disjoint. With probability at least $1 - \epsilon - o(1)$, which is $1 - o(1)$ as ϵ tends to 0, the minimum degree is at most d and there is a unique vertex of minimum degree. Equivalently, exactly one of the events $A_{S,v}$ occurs. We next show that the conditional probability $P[B_{S,v}|A_{S,v}]$ is $o(1)$, which completes the proof.

We consider the distribution $G(n,p)|A_{S,v}$. For convenience, we will assume that $v = n$. Then, in this distribution, the neighbours of v are S, and the remaining vertices form the distribution $G(n-1,p)$ conditioned on the event that all vertices in S have degree at least δ and all vertices in $[n-1] \setminus S$ have degree at least $\delta + 1$; call this event C_S. Hence, $P[B_{S,v}|A_{S,v}] = P[B_{S,v}|C_S]$, where the first probability is taken with respect to $G(n,p)$ and the second probability is taken with respect to $G(n-1,p)$.

By definition, we have $P[B_{S,v}|C_S] = P[B_{S,v} \wedge C_S]/P[C_S] \leq P[B_{S,v}]/P[C_S]$. The expected number of edges in $T_{S,v}$ in $G(n-1,p)$ is

$$\binom{2|S| - 1}{2}p \leq \binom{2d}{2}p \leq 4d^2p \leq 4p^3n^2 = o(1),$$

where the last inequality uses $d \leq pn$, and the last equality uses $p = o(n^{-2/3})$. Since the probability that there is an edge in $T_{S,v}$ is at most the expected number of edges in $T_{S,v}$, we have $P[B_{S,v}] = o(1)$.

We next give a lower bound for $P[C_S]$. Note that C_S holds if $\delta(G(n-1,p)) \geq \delta + 1$. Hence,

$$
\begin{aligned}
P[C_S] \;\geq\; & P\left[\delta(G(n-1,p)) \geq \delta + 1\right] \geq P\left[\delta(G(n,p)) \geq \delta + 2\right] - o(1) \\
\geq\; & P\left[\delta(G(n,p)) \geq d + 2\right] - o(1) \geq P\left[\delta(G(n,p)) \geq d + 1\right] - o(1) \geq \epsilon - o(1)
\end{aligned}
$$

where the second inequality is by fact (5), the third inequality uses $\delta \leq d$, the fourth inequality follows from fact (2), and the last inequality holds by the choice of d. Putting this together, we have

$$P_{G(n,p)}[B_{S,v}|A_{S,v}] = P_{G(n-1,p)}[B_{S,v}|C_S] \leq$$

$$P_{G(n-1,p)}[B_{S,v}]/P_{G(n-1,p)}[C_S] = o(1)/(\epsilon - o(1)) = o(1),$$

where the last inequality is by taking ϵ to tend to 0 slower than $P_{G(n,p)}[B_{S,v}]$ tends to 0. This completes the proof. $\qquad\square$

2.10 Random Caley graphs with a pendant edge

We give a non-trivial example of a vertex-transitive graph H such that $H \cdot K_2$ is not Ramsey simple.

Let G be a group, and S be a subset of elements of G, called the set of *generators*. The (undirected) Caley graph $X(G,S)$ has vertex set G and edge set $\{\{g,gs\} : g \in G, s \in S\}$. Clearly, $X(G,S)$ is vertex-transitive and the degree of a vertex is $|S \cup S^{-1}|$. For $0 \leq p \leq 1$, we denote by $H \sim X(G,p)$ a random Caley graph $H = X(G,S)$ where S is chosen by including every element of G with probability p.

Theorem 2.10.1. *Let G be a group on n elements, let $p \to 0$ such that $p \gg \sqrt{\frac{\ln n}{n}}$, and let $H \sim X(G,p)$. Then a.a.s. $s(H \cdot K_2) \geq \delta(H) > 1$.*

Proof. We prove that H satisfies the conditions of Theorem 2.4.1 a.a.s. Let S denote the set of generators of H chosen by including every element of G with probability p.

It is clear that $\delta(H) \geq 2$ a.a.s. since a.a.s. $\delta(H) \geq |S| \geq pn/2 \gg 1$.

For condition (1) and (3) of Theorem 2.4.1, let C be a large constant to be determined later. We pick the set S in C rounds. Let $q \in [0,1]$ be such that $p = 1 - (1-q)^C$. Note that $q = (1 + o(1))p/C$ if $p \to 0$. For $1 \leq i \leq C$, pick a set R_i of elements each with probability q of being in the set, choices for different i being independent. Let $S = R_1 \cup \ldots \cup R_C$ and note that this is an equivalent way of picking the set S of elements each with probability p of being in the set.

For an element $x \in G$ and a subset $A \subseteq G$, we use the shorthand notation $x.A := \{xa : a \in A\}$.

Lemma 2.10.2. *Let $x \in G$ be a vertex of H, and let $R_1, \ldots, R_C$ be as above. Then with probability tending to one, the set $x.R_1$ is connected.*

Before we prove the lemma, let us show how it implies that H is connected a.a.s., i.e. condition (1) of Theorem 2.4.1 holds.

Let $x \in G$ be an arbitrary element of G. We claim that a.a.s. $x.R_1$ is a dominating set of H, i.e. for every $y \in G$, there is an edge between y and $x.R_1$. By definition, there is an edge between y and $x.R_1$ if there exists a generator $s \in S$ and an element $r_1 \in R_1$ such that $ys = xr_1$. For fixed x and y, there are $|R_1|$ possible values $s \in G$ such that $ys = xr_1$. With probability tending to one, $|x.R_1| \geq nq/2$. Therefore, choosing $R_2, \ldots, R_C$, the probability that there exists some $y \in G$ such that there is no edge between $x.R_1$ and y is at most

$$n(1-q)^{(C-1)nq/2} + \mathbf{Pr}(|x.R_1| < nq/2) \leq e^{-\frac{C-1}{2}nq^2 + \ln n} + o(1) = o(1),$$

since $q \gg \sqrt{\frac{\ln n}{n}}$. Together with Lemma 2.10.2 it follows that a.a.s. the graph H is connected.

To prove that $x.R_1$ is connected, we need to show that a.a.s. for every nontrivial partition $A \dot\cup B$ of R_1, there is an edge between $x.A$ and $x.B$. That is, there exists $a \in A$, $b \in B$ and $y \in S$ such that $b = ay$. We show first that with high probability the set $Y_{A,B} := \{a^{-1}b : a \in A, b \in B\}$ of distinct such values $y \in G$ is large enough for any partition of R_1, and then use the random choices of $R_2, \ldots, R_C$ to show that at least one solution survives with very high probability. In fact, we show something slightly stronger.

Lemma 2.10.3. *There exists a universal constant $c > 0$ such that the following holds. Let $q \gg \sqrt{\frac{\ln n}{n}}$. Further, let $1 \leq n_A \leq nq$, $nq/3 \leq n_B \leq 2nq$, and let $A \subseteq G$ of size $|A| = n_A$. Let B be a set of size n_B chosen uniformly at random from $G \setminus A$. Then with probability at least $1 - 2^{-100n_B}$ we have that $|Y_{A,B}| \geq \min\{c \cdot n, c \cdot n_A n_B\}$.*

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Proof. Let $c > 0$ be a constant to be chosen later. For disjoint sets $A, B \subseteq G$ let $U_{A,B}$ be the event that $|Y_{A,B}| < \min\{cn, c|A||B|\}$. Let us choose B one element at a time and analyse how the value of $Y_{A,B}$ changes. To be precise, for $1 \le t \le n_B$ choose an element $b_t \in G \setminus (A \cup B_{t-1})$ uniformly at random, where $B_0 := \emptyset$ and $B_t := B_{t-1} \cup \{b_t\}$. Let $X_t := |Y_{A,B_t} \setminus Y_{A,B_{t-1}}|$ be the random variable that counts the "new" values $a^{-1}b$ with $a \in A$, $b \in B$ after step t. If $|Y_{A,B_{t-1}}| \ge cn$, we are done. So we may condition on the event that $|Y_{A,B}| < cn$ (and hence $|Y_{A,B_t}| < cn$ for any $t \le n_B$) which we abbreviate with Z. Then for a fixed element $a \in A$

$$\mathbf{Pr}\big(a^{-1}b_t \in Y_{A,B_{t-1}} \big| \, Z\big) < \frac{cn}{|G - A - B_{t-1}|} < 2c$$

for n large enough since $|A| + |B_t| \le 3nq = o(n)$. Hence, the expected size of X_t conditioning on the event Z is at least $(1 - 2c)|A|$ and therefore, by Markov's Inequality,

$$\mathbf{Pr}\left(X_t < \frac{|A|}{2} \Big| \, Z\right) = \mathbf{Pr}\left(|A| - X_t > \frac{|A|}{2} \Big| \, Z\right)$$
$$\le \frac{2 \cdot \mathbb{E}(|A| - X_t | \, Z)}{|A|}$$
$$\le 4c.$$

That is, the random variable X_t takes value (at least) $|A|/2$ with probability (at least) $1 - 4c$, independent of the history $X_1, \dots, X_{t-1}$. Let $Y_1, \dots, Y_{n_B}$ be independent Bernoulli experiments that are one with probability $4c$ and zero otherwise. Then the sum $\sum_{1 \le t \le n_B} Y_t$ is a lower bound on the number of steps that X_t fails to have value

at least $|A|/2$. Therefore,

$$\mathbf{Pr}(U_{A,B}) \leq \mathbf{Pr}\left(|Y_{A,B}| < c|A||B| \mid |Y_{A,B}| < cn\right)$$

$$= \mathbf{Pr}\left(X_t < |A|/2 \text{ at least } (1-2c)n_B \text{ times}\right)$$

$$\leq \mathbf{Pr}\left(\sum_{1 \leq t \leq n_B} Y_t \geq (1-2c)n_B\right)$$

$$\leq 2^{n_B} \cdot (4c)^{(1-2c)n_B}$$

$$< 2^{-100n_B}$$

for c small enough. $\qquad\square$

Proof of Lemma 2.10.2. Let $c > 0$ be the constant from Lemma 2.10.3 and let Z denote the event that $N := |R_1| \in [\frac{qn}{2}, 2qn]$ and that for every non-trivial partition $A \dot\cup B$ of R_1 the set $Y_{A,B} = \{a^{-1}b : a \in A, b \in B\}$ has size at least $\min\{cn, c|A||B|\}$. We claim that the probability of Z is $1 - o(1)$. Certainly, $N \in [\frac{qn}{2}, 2qn]$ a.a.s. Let now $A \dot\cup B = R$ be a partition of R, let $n_A := |A|$, $n_B := |B|$ and assume without loss of generality that $n_A \leq n_B$. Note that by Lemma 2.10.3, the probability that $|Y_{A,B}| < \min\{cn, c|A||B|\}$ is at most $2^{-100n_B} \leq 2^{-20nq}$. It follows that the probability that Z fails to hold is at most

$$\sum_{N=nq/2}^{2nq} \sum_{A \subseteq R, |A| \leq N/2} 2^{-20nq} + o(1) \leq 2nq2^{-19nq} + o(1) \to 0$$

since $nq \to \infty$. We now condition on Z. Fix now a non-trivial partition $R_1 = A \dot\cup B$, say $|A| = n_A$, $|B| = n_B$, and assume without loss of generality that $n_A \leq N/2$. By assumption, there are at least $\min\{cn, c|A||B|\}$ (nontrivial) solutions for $ay = b$ with $a \in A$ and $b \in B$. Let $X_{A,B}$ be the random variable that counts the number of elements in $Y_{A,B}$ that are chosen to be in $R_2 \cup \ldots \cup R_C$. Since for a particular element $y \in Y_{A,B}$ the choices for $R_2, \ldots, R_C$ are independent, and since choices for distinct elements in $Y_{A,B}$ are independent, it follows that

$$\mathbf{Pr}_{R_2,\ldots,R_C}(X_{A,B} = 0) = (1-q)^{(C-1)|Y_{A,B}|} \leq \exp\left[-(C-1)q\min\{cn, c|A||B|\}\right].$$

It follows that

$\mathbf{Pr}(R_1 \text{ is not connected})$

$= \mathbf{Pr}(R_1 \text{ is not connected} \mid Z) + o(1)$

$= \mathbf{Pr}(\exists \text{ a partition } A \cup B = R_1 \text{ such that } X_{A,B} = 0 \mid Z) + o(1)$

$\leq \sum_{N=nq/2}^{2nq} \sum_{n_A=1}^{N/2} \sum_{A \in \binom{R_1}{n_A}} \mathbf{Pr}_{R_2,\ldots,R_C}(X_{A,B} = 0 \mid Z) + o(1)$

$\leq \sum_{N=nq/2}^{2nq} \left(\sum_{\substack{1 \leq n_A \leq N/2 \\ n_A(N-n_A) \leq n}} \binom{N}{n_A} e^{-(C-1)qcn_A(N-n_A)} + \sum_{\substack{1 \leq n_A \leq N/2 \\ n_A(N-n_A) > n}} \binom{N}{n_A} e^{-(C-1)qcn} \right) + o(1)$

$\leq \sum_{N=nq/2}^{2nq} \left(\sum_{\substack{1 \leq n_A \leq N/2 \\ n_A(N-n_A) \leq n}} e^{n_A(\ln N - \Theta(qN))} + 2^N e^{-(C-1)qcn} \right) + o(1)$

$\leq 2(nq)^2 e^{-\Theta(q^2 n)} + 2nq 2^N e^{-(C-1)qcn} + o(1),$

since $\ln N \leq \ln n \ll q^2 n/2 \leq qN$. Therefore, the probability that R_1 is not connected is at most

$$\exp\left[O(\ln(qn)) - \Theta(q^2 n)\right] + \exp\left[\ln(2nq) + 2nq - (C-1)cnq\right] + o(1) = o(1),$$

again since $\ln(nq) \ll q^2 n$ and if we choose $C = C(c) \geq 3/c$. $\qquad\square$

We now prove condition (3) of Theorem 2.4.1. We need to analyse the family

$$\mathcal{F}(H) = \{F \subseteq H[N(x)] \;:\; x \in V(H), F \text{ is a connected component of } H[N(x)]\}$$

and show that there is a 2-colouring of $K_{\alpha(H)}$ that does not contain a monochromatic copy of F, for any $F \in \mathcal{F}(H)$. Since H is vertex-transitive, the graph $H[N(x)]$ is the same graph for every $x \in G$. Let us denote this graph by H_N. We show first that H_N is connected a.a.s. which implies that $\mathcal{F}(H)$ consists of a single graph.

Lemma 2.10.4. *Let $G, p, q, H, R_1, \ldots, R_C$ be as above. Then, H_N is connected a.a.s.*

Proof. Fix a vertex $x \in G$. Note that the neighbourhood $N(x) = x.(S \cup S^{-1})$ is connected since $x.R_1$ is connected a.a.s. by Lemma 2.10.2, and since $x.R_1$ is a dominating set in H. $\qquad \square$

Lemma 2.10.5. *Let G, p, H be as in Theorem 2.10.1. Then, H_N has at least $m = p^3 n^2 (1 + o(1))$ edges a.a.s.*

Proof. Fix $x \in G$ and let S be the set we pick by including each element of G with probability p, all choices being independent. We will count only the edges in $x.S \subseteq N(x)$. Note that an edge in $x.S$ corresponds to a triple $(a, b, c) \in S^3$ such that $a = bc$. We shall see that a.a.s the number of solutions to $a = bc$ with $a, b, c \in S$ is large. There are a total of n^2 choices for b and c, giving a total of n^2 solutions for $a = bc$ in the group G of order n. For simplicity, let us only consider *nontrivial triples*, i.e. solutions for $a = bc$ where a, b, c are distinct. There are still $n^2 - O(n)$ such solutions. For a nontrivial triple (a, b, c) let $X_{a,b,c}$ be the indicator random variable which evaluates to one if and only if all of a, b, c are chosen to be in S. Let $X = \sum X_{a,b,c}$ be the random variable counting the number of these solutions with a, b, c chosen to be generators and a, b, c distinct. The expected number of X is $p^3 n^2 - O(np^2)$ by linearity of expectation, as $X_{a,b,c} = 1$ with probability p^3 for each nontrivial triple. To see that X is concentrated about its mean, we use the second moment method and bound the variance of X. For two (distinct) nontrivial triples write $(a, b, c) \sim (a', b', c')$ if $X_{a,b,c}$ and $X_{a',b',c'}$ are not independent. Let

$$\Delta = \sum_{(a,b,c) \sim (a',b',c')} \mathbf{Pr}(X_{a,b,c} = 1 \ \wedge \ X_{a',b',c'} = 1),$$

where the sum runs over all ordered pairs of non-trivial triples. We note that $\mathrm{Var}(X) \leq \mathbb{E}(X) + \Delta$, see e.g. Chapter 4.3 in [3]. A pair $(X_{a,b,c}, X_{a',b',c'})$ is not independent if and only if the sets $\{a, b, c\}$ and $\{a', b', c'\}$ have nonempty intersection.

If two nontrivial triples $\{a, b, c\}$ and $\{a', b', c'\}$ intersect in one element then $\mathbf{Pr}(X_{a,b,c} = 1 \ \wedge \ X_{a',b',c'} = 1) = p^5$. The number of such (pairs of) triples is $O(n^3)$. If two nontrivial triples $\{a, b, c\}$ and $\{a', b', c'\}$ intersect in two or three elements then

$\mathbf{Pr}(X_{a,b,c} = 1 \ \wedge \ X_{a',b',c'} = 1) \leq p^3$. The number of such (pairs of) triples is $O(n^2)$.

It follows that $\mathrm{Var}(X) = O(p^3 n^2) + O(p^5 n^3) = o(\mathbb{E}(X)^2)$ since $p \gg n^{-2/3}$. Therefore, by Corollary 4.3.3 in [3], $X = \mathbb{E}(X)(1 + o(1)) = p^3 n^2 (1 + o(1))$ a.a.s. $\qquad \square$

We are ready to prove condition (3) and thus finish the proof of Theorem 2.10.1.

Let H_N be as before. By Lemma 2.10.4, H_N is a.a.s. connected, and by Lemma 2.10.5, H_N has a.a.s. at least $p^3 n^2 (1 + o(1))$ edges, and a.a.s. at most $2np$ vertices. The Ramsey number of a graph G with M edges and N vertices is at least $2^{(M-1)/N}$ (take a random colouring of the complete graph on $2^{(M-1)/N}$ vertices and see that the expected number of monochromatic copies of G is $o(1)$).

Now, $\alpha(H) \leq n$ since H has n vertices. Therefore, $\alpha(H) \ll R(H_N)$ a.a.s., since $p \gg \sqrt{\frac{\ln n}{n}}$, and it follows that Property (3) holds a.a.s. $\qquad \square$

Chapter 3

The Erdős-Hajnal Conjecture for Rainbow Triangles

3.1 Introduction

A classical result of Erdős and Szekeres [36], which is a quantitative version of Ramsey's theorem [79], implies that every graph on n vertices contains a clique or an independent set of order at least $\frac{1}{2}\log n$. In the other direction, Erdős [31] showed that a random graph on n vertices almost surely contains no clique or independent set of order $2\log n$.

An *induced subgraph* of a graph is a subset of its vertices together with all edges with both endpoints in this subset. There are several results and conjectures indicating that graphs which do not contain a fixed induced subgraph are highly structured. In particular, the most famous conjecture of this sort by Erdős and Hajnal [35] says that for each fixed graph H there is $\epsilon = \epsilon(H) > 0$ such that every graph G on n vertices which does not contain a fixed induced subgraph H has a clique or independent set of order n^{ϵ}. This is in stark contrast to general graphs, where the order of the largest guaranteed clique or independent set is only logarithmic in the number of vertices.

There are now several partial results on the Erdős-Hajnal conjecture. Erdős and Hajnal [35] proved that for each fixed graph H there is $\epsilon = \epsilon(H) > 0$ such that every

graph G on n vertices which does not contain an induced copy of H has a clique or independent set of order $e^{\epsilon\sqrt{\log n}}$. Fox and Sudakov [45], strengthening an earlier result of Erdős and Hajnal, proved that for each fixed graph H there is $\epsilon = \epsilon(H) > 0$ such that every graph G on n vertices which does not contain an induced copy of H has a balanced complete bipartite graph or an independent set of order n^ϵ. All graphs on at most four vertices are known to satisfy the Erdős-Hajnal conjecture, and Chudnovsky and Safra [18] proved it for the 5-vertex graph known as the bull. Alon, Pach, and Solymosi [2] proved that if H_1 and H_2 satisfy the Erdős-Hajnal conjecture, then for every v of H_1, the graph formed from H by substituting v by a copy of H_2 satisfies the Erdős-Hajnal conjecture. The recent survey [17] discusses many further related results on the Erdős-Hajnal conjecture.

A natural restatement of the Erdős-Hajnal conjecture is that for every fixed red-blue edge-coloring χ of a complete graph, there is an $\epsilon = \epsilon(\chi) > 0$ such that every red-blue edge-coloring of the complete graph on n vertices without a copy of χ contains a monochromatic clique of order n^ϵ. Indeed, for the graphs H and G, we can color the edges red and the nonadjacent pairs blue.

Erdős and Hajnal also proposed studying a multicolor generalization of their conjecture. It states that for every fixed k-coloring of the edges of χ of a complete graph, there is an $\epsilon = \epsilon(\chi) > 0$ such that every k-coloring of the edges of the complete graph on n vertices without a copy of χ contains a clique of order n^ϵ which only uses $k-1$ colors. They proved a weaker estimate, replacing n^ϵ by $e^{\epsilon\sqrt{\log n}}$. Note that the case of two colors is what is typically referred to as the Erdős-Hajnal conjecture.

Hajnal [59] conjectured that the following special case of the multicolor generalization of the Erdős-Hajnal conjecture holds. There is $\epsilon > 0$ such that every 3-coloring of the edges of the complete graph on n vertices without a rainbow triangle (that is, a triangle with all its edges different colors), contains a set of order n^ϵ which uses at most two colors. We prove Hajnal's conjecture, and further determine a tight bound on the order of the largest guaranteed 2-colored set in any such coloring. A *Gallai coloring* is a coloring of the edges of a complete graph without rainbow triangles, and a *Gallai r-coloring* is a Gallai coloring that uses r colors.

Theorem 3.1.1. *Every Gallai 3-coloring on n vertices contains a set of order* $\Omega(n^{1/3} \log^2 n)$ *which uses at most two colors, and this bound is tight up to a constant factor.*

To give an upper bound, we use lexicographic products. We will let $[m] = \{1, \ldots, m\}$ denote the set consisting of the first m positive integers.

Definition 3.1.2. *Given edge-colorings F_1 of K_{m_1} and F_2 of K_{m_2} using colors from R, the* lexicographic product coloring *$F_1 \otimes F_2$ of $E(K_{m_1 m_2})$ is defined on any edge $e = \{(u_1, v_1), (u_2, v_2)\}$ (where we take the vertex set of $K_{m_1 m_2}$ to be $[m_1] \times [m_2]$) to be $F_1(u_1, u_2)$ if $u_1 \neq u_2$, and otherwise $v_1 \neq v_2$ and it is defined to be $F_2(v_1, v_2)$.*

That is, there are m_1 disjoint copies of F_2 and they are connected by edge colors defined by F_1.

The upper bound in Theorem 3.1.1 is obtained by taking the lexicographic product of three 2-edge-colorings of the complete graph on $n^{1/3}$ vertices, where each pair of colors is used in one of the colorings, and the largest monochromatic clique in each of the colorings is of order $O(\log n)$. A simple lemma in the next section shows that, in a lexicographic product coloring $F = F_1 \otimes F_2$, the largest set of vertices using only colors red and blue (for example) in F has size equal to the product of the size of the largest set of vertices using only colors red and blue in F_1 with the size of the largest set of vertices using only colors red and blue in F_2. For any set S of two of the three colors, the largest such set has order $O(n^{1/3})O(\log n)O(\log n) = O(n^{1/3} \log^2 n)$.

In the other direction, we will utilize the following important structural result of Gallai [48] on edge-colorings of complete graphs without rainbow triangles.

Lemma 3.1.3. *An edge-coloring F of a complete graph on a vertex set V with $|V| \geq 2$ is a Gallai coloring if and only if V may be partitioned into nonempty sets $V_1, \ldots, V_t$ with $t > 1$ so that each V_i has no rainbow triangles under F, at most two colors are used on the edges not internal to any V_i, and the edges between any fixed pair (V_i, V_j) use only one color. Furthermore, any such substitution of Gallai colorings for vertices of a 2-edge-coloring of a complete graph K_t yields a Gallai coloring.*

Gallai colorings naturally arise in several areas including in information theory [70], in the study of partially ordered sets, as in Gallai's original paper [48], and in

the study of perfect graphs [14]. There are now a variety of papers which consider Ramsey-type problems in Gallai colorings (see, e.g., [19, 47, 56, 58]). However, these works mainly focus on finding various monochromatic subgraphs in such colorings.

Because it may be of independent interest to the reader, we first present a particularly simple approach that will prove Hajnal's conjecture, but will not give tight bounds.

A graph is a *cograph* if it has at most one vertex, or if it or its complement is not connected, and all of its induced subgraphs have this property. In other words, the family of cographs consists of all those graphs that can be obtained from an isolated vertex by successively taking the disjoint union of two previously constructed cographs, G_1 and G_2, or by the join of them that we get by adding all edges between G_1 and G_2. It was shown by Seinsche [87] that cographs are precisely those graphs which do not contain the path with three edges as an induced subgraph. It is easy to check by induction that every cograph is a perfect graph, that is, the chromatic number of every induced subgraph is equal to its clique number.

Proposition 3.1.4. *In any Gallai 3-coloring of a complete graph, there is an edge-partition of the complete graph into three 2-colored subgraphs, each of which is a cograph.*

Proof: This follows from Gallai's structure theorem by induction on the number of vertices. The result is trivial for edge-colorings of complete graphs with fewer than two vertices, which serves as the base case. Using Lemma 3.1.3, we get a nontrivial vertex partition of the Gallai 3-coloring of the complete graph into parts $V_1, \ldots, V_t$ such that only two colors appear between the parts. By the induction hypothesis, we can partition the edge-set of the complete graph on V_i into three cographs, each which is two-colored. For the two colors that go between the parts, we take the graph which is the join of the cographs in each V_i, that is, add all edges between the parts, and for each of the other two pairs of colors, we just take the disjoint union of the cographs of those two colors from each part. Since the join or disjoint union of cographs are cographs, this completes the proof by induction.   □

The following corollary verifies Hajnal's conjecture and, apart from the two logarithmic factors, gives the lower bound in Theorem 3.1.1.

Corollary 3.1.5. *Every Gallai 3-coloring of $E(K_n)$ contains a 2-colored clique with at least $n^{1/3}$ vertices.*

Proof: Indeed, applying Proposition 3.1.4, if the first cograph (which is 2-colored) contains a clique of order $n^{1/3}$ then we are done; otherwise, it contains no clique of order $n^{1/3}$ and, since cographs are perfect graphs, has chromatic number at most $n^{1/3}$, in which case it contains an independent set of order $n^{2/3}$. In this latter case, this independent set of order $n^{2/3}$ in the first cograph contains in the second cograph a clique of order $n^{1/3}$ or an independent set (which is a clique in the third cograph) of order $n^{1/3}$. We thus get a clique of order $n^{1/3}$ in one of the three cographs, which is a 2-colored set. □

Improving the lower bound further to Theorem 3.1.1 appears to be considerably harder, and uses a different proof technique, relying on a weighted version of Ramsey's theorem and a carefully chosen induction argument. The weighted version of Ramsey's theorem shows that if each vertex of a complete graph on t vertices is given a positive red weight and a positive blue weight whose product is one, then in any red-blue edge-coloring of K_t, there is a red clique S and a blue clique U such that the product of the red weight of S (the sum of the red weights of the vertices in S) and the blue weight of U (the sum of the blue weights of the vertices in U) is $\Omega\left(\log^2 t\right)$. Note that this extends the quantitative version of Ramsey's theorem as the case in which all the red and blue weights are one implies that there is a monochromatic clique of order $\Omega(\log t)$.

We further consider a natural generalization of this problem to more colors, and give a tight bound in the next theorem. In order to state the result more succinctly,

we introduce some notation: for positive integers r and s with $s \leq r$, let

$$c_{r,s} = \begin{cases} 1 & \text{if } 1 = s < r \text{ or if } s = r - 1 \text{ and } r \text{ is even;} \\ s(r-s) & \text{if } 1 < s < r - 1; \\ 1 + \frac{3}{r} & \text{if } s = r - 1 \text{ and } r \text{ is odd;} \\ 0 & \text{if } s = r. \end{cases}$$

Theorem 3.1.6. *Let r and s be fixed positive integers with $s \leq r$. Every r-coloring of the edges of the complete graph on n vertices without a rainbow triangle contains a set of order $\Omega(n^{\binom{s}{2}/\binom{r}{2}} \log^{c_{r,s}} n)$ which uses at most s colors, and this bound is tight apart from the constant factor.*

We next give a brief discussion of the proof of Theorem 3.1.6. The case $s = r$ is trivial as the complete graph uses at most r colors. The case $s = 1$ is easy. Indeed, in this case, by the Erdős-Szekeres bound on Ramsey numbers for r colors, there is a monochromatic set of order $\Omega(\log n)$, where the implied positive constant factor depends on r. In the other direction, we give a construction which we conjecture is tight.

The Ramsey number $r(t)$ is the minimum n such that every 2-coloring of the edges of the complete graph on n vertices contains a monochromatic clique of order t. The bounds mentioned in the beginning of the introduction give $2^{t/2} \leq r(t) \leq 2^{2t}$ for $t \geq 2$. For r even, consider a lexicographic product of $r/2$ colorings, each a 2-edge coloring of the complete graph on $r(t) - 1$ vertices with no monochromatic K_t. This gives a Gallai r-coloring of the edges of the complete graph on $(r(t) - 1)^{r/2}$ vertices with no monochromatic clique of order t. A similar construction for r odd gives a Gallai r-coloring of the edges of the complete graph on $(t-1)(r(t) - 1)^{(r-1)/2}$ vertices with no monochromatic clique of order t. The following conjecture which states that these bounds are best possible seems quite plausible. It was verified by Chung and Graham [19] in the case $t = 3$.

Conjecture 3.1.7. *Let $N(r,t) = (r(t) - 1)^{r/2}$ for r even and*

$N(r, t) = (t - 1) (r(t) - 1)^{(r-1)/2}$ *for* r *odd. For* $n > N(r, t)$, *every* r-*coloring of the edges of the complete graph on* n *vertices has a rainbow triangle or a monochromatic* K_t.

Having verified the easy cases $s = 1$ and $s = r$ of Theorem 3.1.6, for the rest of the chapter, we assume $1 < s < r$. A natural upper bound on the size of the largest set using at most s colors comes from the following construction. We will let $[r]$ be the set of colors. Consider the complete graph on $[r]$, where each edge P gets a positive integer weight n_P such that the product of all n_P is n. For each edge P of this complete graph, we consider a 2-coloring c_P of the edges of the complete graph on n_P vertices using the colors in P and whose largest monochromatic clique has order $O(\log n_P)$, which exists by Erdős lower bound [31] on Ramsey numbers. We then consider the Gallai r-coloring c of the complete graph on n vertices which is the lexicographic product of the $\binom{r}{2}$ colorings of the form c_P. For each set S of colors, the largest set of vertices in this edge-coloring of K_n using only colors in S has order

$$\prod_{P \in S} n_P \prod_{|P \cap S| = 1} O(\log n_P).$$

The order of the largest set using at most s colors in coloring c is thus the maximum of the above expression over all subsets S of colors of size s. Therefore, we want to choose the various n_P to minimize this maximum. For $s < r - 1$, we give a second moment argument which shows that the best choice is essentially that the n_P are all equal, i.e., $n_P = n^{1/\binom{r}{2}}$ for all P. In this case, the above expression, for each choice of S, matches the claimed upper bound in Theorem 3.1.6. The case $s = r - 1$ turns out to be more delicate. For r even, the optimal choice turns out to be $n_P = n^{2/r}$ for P an edge of a perfect matching of the complete graph with vertex set $[r]$, and otherwise $n_P = 1$. For r odd, we have three different edge weights. The graph on $[r]$ whose edges consist of those pairs with weight not equal to 1 consist of a disjoint union of a triangle and a matching with $(r - 3)/2$ edges. The edges of the triangle each have weight $n^{1/r} (\log n)^{(r-3)/2r}$ and the edges of the matching each have weight $n^{2/r} (\log n)^{-3/r}$. It is straightforward to check that these choices of weights give the

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claimed upper bound in Theorem 3.1.6.

Similar to the case $r = 3$ and $s = 2$ mentioned above, using Gallai's structure theorem, we observe that, in any r-coloring of the edges of the complete graph on n vertices without a rainbow triangle, the complete graph can be edge-partitioned into $\binom{r}{2}$ subgraphs, each of which is a 2-colored perfect graph. A simple argument then shows that there is a vertex subset of at least $n^{\binom{s}{2}/\binom{r}{2}}$ vertices which uses at most s colors, which verifies the lower bound in Theorem 3.1.6 apart from the logarithmic factors. Improving the lower bound further to Theorem 3.1.6 is more involved, using a weighted version of Ramsey's theorem and a carefully chosen induction argument to prove this.

The rest of the chapter is organized as follows. In the next section, we prove some basic properties of lexicographic product colorings. In Section 3.3, we give simple proofs of lower and upper bounds in the direction of Theorem 3.1.1 which match apart from two logarithmic factors. In order to close the gap and obtain Theorem 3.1.1, in Section 3.4 we prove a weighted extension of Ramsey's theorem. We complete the proof of Theorem 3.1.1 in Section 3.5 by establishing a tight lower bound on the size of the largest 2-colored set of vertices in any Gallai 3-coloring of the complete graph on n vertices. The remaining sections are devoted to the proof of Theorem 3.1.6. In Section 3.6, we prove the upper bound for Theorem 3.1.6. In Section 3.7, we give a simple proof of a lower bound which matches Theorem 3.1.6 apart from the logarithmic factors. In Section 3.8.1, using the second moment method, we establish an auxiliary lemma which gives a tight bound on the minimum possible number of nonzero weights in a graph with non-negative edge weights such that no set of s vertices contains sufficiently more than the average weight of a subset of s vertices. We give the lower bound for Theorem 3.1.6 in Section 3.8.2, which completes the proof of this theorem. The proofs of some of the auxiliary lemmas which involve lengthy calculations are deferred to the end. All logarithms in this chapter are base 2, unless otherwise indicated. All colorings are edge-colorings of complete graphs, unless otherwise indicated. For the sake of clarity of presentation, we systematically omit floor and ceiling signs whenever they are not crucial. We also do not make any

serious attempt to optimize absolute constants in our statements and proofs.

3.2   Lexicographic product colorings

In this section, we will prove some simple results about lexicographic product colorings (Definition 3.1.2). These will be useful in constructing examples of r-colorings that do not contain large vertex sets that use at most s colors.

For such a lexicographic product coloring $F_1 \otimes F_2$ with F_1 on m_1 vertices and F_2 on m_2 vertices, we will view the vertex set interchangeably as $[m_1 \times m_2]$ and $[m_1] \times [m_2]$. For the sake of brevity, we often refer to a lexicographic product coloring as simply a product coloring.

Definition 3.2.1. *For F an edge-coloring of K_n and $S \subseteq R$ a set of colors, we write that a set Z of vertices is S-subchromatic in F if every edge internal to Z takes colors (under F) only from S.*

When F and S are clear from context, we shall simply say that Z is subchromatic. We will write $g_{S,F}$ to be the size of the largest subchromatic set of vertices.

If F is an edge-coloring constructed via a product of two other colorings F_1, F_2, then the next lemma allows us to determine $g_{S,F}$ in terms of g_{S,F_1} and g_{S,F_2}.

Lemma 3.2.2. *For any r-colorings F_1, F_2 of $E(K_{n_1}), E(K_{n_2})$, respectively, and any set $S \subseteq R$ of colors, $g_{S,F} = g_{S,F_1} \cdot g_{S,F_2}$, where $F = F_1 \otimes F_2$.*

Proof: Let Z a set of subchromatic vertices in F (so $Z \subseteq V(K_{n_1 \times n_2})$) be given. We will first show $|Z| \le g_{S,F_1} \cdot g_{S,F_2}$.

Take $U \subseteq [n_1]$ to be the set of $u \in [n_1]$ such that there is some $v \in [n_2]$ with $(u, v) \in Z$; that is, U is the subset of $[n_1]$ that is used in Z. For any $u \in [n_1]$, take $V_u \subseteq [n_2]$ to be the set of $v \in [n_2]$ such that $(u, v) \in Z$, that is, V_u is the subset of $[n_2]$ that is paired with u in Z. By construction, we have $Z = \bigcup_{u \in U} \{u\} \times V_u$.

Therefore, the set U must be subchromatic in F_1, as given distinct $u_1, u_2 \in U$

there are v_1, v_2 so that $(u_1, v_1), (u_2, v_2) \in Z$, and hence:

$$F_1(u_1, u_2) = F((u_1, v_1), (u_2, v_2)) \in S.$$

Thus, $|U| \leq g_{S,F_1}$.

Furthermore, given $u \in U$ we must have that V_u is subchromatic in F_2, as given distinct $v_1, v_2 \in V_u$ we have that

$$F_2(v_1, v_2) = F((u, v_1), (u, v_2)) \in S.$$

Therefore, $|V_u| \leq g_{S,F_2}$.

Hence,

$$|Z| = \left| \bigcup_{u \in U} \{u\} \times V_u \right| = \sum_{u \in U} |V_u| \leq \sum_{u \in U} g_{S,F_2} = |U| \, g_{S,F_2} \leq g_{S,F_1} \cdot g_{S,F_2}.$$

Since Z was arbitrary, we get $g_{S,F} \leq g_{S,F_1} \cdot g_{S,F_2}$.

We now prove that $g_{S,F} \geq g_{S,F_1} g_{S,F_2}$, thus giving the desired result: take $U \subseteq [n_1]$ a subchromatic set under F_1 and $V \subseteq [n_2]$ a subchromatic set under F_2. We claim that $U \times V$ is subchromatic under F. Consider any distinct pairs $(u_1, v_1), (u_2, v_2) \in U \times V$. If $u_1 \neq u_2$ then

$$F((u_1, v_1), (u_2, v_2)) = F_1(u_1, u_2) \in S,$$

and if $u_1 = u_2$ then

$$F((u_1, v_1), (u_2, v_2)) = F_2(v_1, v_2) \in S.$$

If we choose U to have size g_{S,F_1} and V to have size g_{S,F_2}, we get $g_{S,F_1} \cdot g_{S,F_2} = |U \times V| \leq g_{S,F}$. $\qquad \square$

The next lemma states that the property of being a Gallai coloring is preserved under taking product colorings.

Lemma 3.2.3. *If F_1, F_2 are Gallai r-colorings of $E(K_{n_1}), E(K_{n_2})$, respectively, then if $F = F_1 \otimes F_2$ then F is a Gallai coloring.*

Proof: Let any three vertices $u = (u_1, u_2), v = (v_1, v_2), w = (w_1, w_2) \in [n_1] \times [n_2]$ be given. We will show that they do not form a rainbow triangle under F. If $u_1 = v_1 = w_1$ then $F(u,v) = F_2(u_2, v_2), F(u,w) = F_2(u_2, w_2), F(v,w) = F_2(v_2, w_2)$; therefore, u, v, w do not form a rainbow triangle by the assumption that F_2 is a Gallai coloring. If u_1, v_1, w_1 are pairwise distinct then $F(u,v) = F_1(u_1, v_1), F(u,w) = F_1(u_1, w_1), F(v,w) = F_1(v_1, w_1)$ and so u, v, w do not form a rainbow triangle by the assumption that F_1 is a Gallai coloring. Otherwise, exactly one pair of u_1, v_1, w_1 are equal. Assume without loss of generality that $u_1 = v_1$, $u_1 \neq w_1$, and $v_1 \neq w_1$. We have:

$$F(u,w) = F_1(u_1, w_1) = F_1(v_1, w_1) = F(v,w),$$

so again u, v, w do not form a rainbow triangle. $\square$

The following corollary states that we may take a product of any number of 2-colorings and the result will be a Gallai coloring; since all 2-colorings are Gallai colorings, it follows by induction from the previous lemma.

Corollary 3.2.4. *If $F_1, \ldots, F_k$ are 2-edge-colorings, then $F_1 \otimes \cdots \otimes F_k$ is a Gallai coloring.*

3.3 Simple bounds for three colors

In this section we will demonstrate simple upper and lower bounds in the case $r = 3$ and $s = 2$. We first apply the techniques of the previous section to demonstrate a Gallai 3-coloring with no large 2-colored vertex set; we will use R for the set of colors.

Theorem 3.3.1. *There is a Gallai 3-coloring on m vertices so that, for every two colors $S \in \binom{R}{2}$, every vertex set Z using colors from S satisfies $|Z| \leq (4/9 + o(1))m^{1/3} \log^2 m$.*

Proof: Take $t = \lceil m^{1/3} \rceil$; then t^3 is at least m. For every pair of colors $P \in \binom{R}{2}$, take F_P to be a 2-coloring of $E(K_t)$ using colors from P so that the largest monochromatic clique has size at most $2 \log t$. Such a coloring exists by the lower bound on Ramsey

numbers proved by Erdős and Szekeres in [36]. We define F a coloring on t^3 vertices by taking $F = F_{\{R_1,R_2\}} \otimes F_{\{R_2,R_3\}} \otimes F_{\{R_1,R_3\}}$ where R_1, R_2, R_3 are such that $R = \{R_1, R_2, R_3\}$. This is a Gallai coloring by Corollary 3.2.4. Fixing any set S of two colors, two of the above three colorings have S-subchromatic sets of size at most $2 \log t$, and the remaining one has size t, so the size of the largest S-subchromatic set in F is at most $t(2 \log t)^2$ by Lemma 3.2.2. Since S is arbitrary, the size of the largest S-subchromatic set for any $S \in \binom{R}{2}$ is at most $t(2 \log t)^2$.

Restricting F to any m vertices will be a 3-Gallai coloring with no subchromatic set of size larger than $t(2 \log t)^2$. Note that since $t = \lceil m^{1/3} \rceil$, we have $t = (1 + o(1))m^{1/3}$, so

$$t(2 \log t)^2 = (1 + o(1))m^{1/3}(2 \log(m^{1/3}))^2 = (4/9 + o(1))m^{1/3} \log^2 m.$$

$\square$

We now proceed to prove that any Gallai 3-coloring on m vertices contains a subchromatic set on two colors of size at least $m^{1/3}$. Indeed, the next theorem is a strengthening of this statement, as it states that the geometric average over $S \in \binom{R}{2}$ of $g_{S,F}$ must be at least $m^{1/3}$.

We have three colors and we refer to them as red, blue, and yellow.

Theorem 3.3.2. *For any Gallai 3-coloring F on m vertices, we have $\prod_{S \in \binom{R}{2}} g_{S,F} \geq m$.*

Proof: We proceed by induction on m to prove the theorem.

Define g to be the size of the largest subchromatic set using only the colors blue and yellow, o to be the size of the largest subchromatic set using only the colors red and yellow, and p to be the size of the largest subchromatic set using only the colors red and blue. (A note on nomenclature: g stands for "green," as blue and yellow form green when mixed. Similarly, o stands for "orange" and p for "purple.") We wish to show that $gop \geq m$.

If $m = 1$, then $g = o = p = 1$ and $gop = m$.

Otherwise, $m > 1$ and by the structure theorem for Gallai colorings there is a non-trivial partition of the vertex set into parts $V_1, \ldots, V_t$ and a pair of colors $Q \in \binom{R}{2}$

satisfying that for any distinct $i, j \in [t]$ there is a $q \in Q$ so that every edge between V_i and V_j has color q. Take m_i to be the size of V_i. Take g_i to be the size of the largest set using only the colors blue and yellow from V_i, o_i to be the size of the largest set using only the colors red and yellow from V_i, and p_i to be the size of the largest set using only the colors red and blue from V_i. Without loss of generality we assume that Q contains colors blue and yellow.

We have $g = \sum_i g_i$. Indeed, we may combine all the largest sets using colors blue and yellow from each V_i to obtain a set of size $\sum_i g_i$ that only uses blue and yellow.

Furthermore, $o \geq \max_i o_i$ and $p \geq \max_i p_i$. This gives:

$$gop = \sum_i g_i op \geq \sum_i g_i o_i p_i \geq \sum_i m_i = m,$$

where the last inequality follows by the induction hypothesis applied to F restricted to V_i. □

Note that we use $o \geq \max_i o_i$ and $p \geq \max_i p_i$. It is on these inequalities that we will, in the next sections, gain multiple factors of $\log m$; if, for example, we find some set $U \subseteq [t]$ satisfying that, for each distinct $i, j \in U$, the edges between V_i, V_j are all yellow, then $o \geq \sum_{i \in U} o_i$. If it were the case that the o_i, p_i were all pairwise equal, then we would get by the Erdős-Szekeres bound for Ramsey numbers that $op = \Omega(\log^2 t \max_i o_i p_i)$; this motivates the approach in the next two sections, where we handle the general case in which it may not be true that the o_i, p_i are all pairwise equal.

3.4 A weighted Ramsey theorem

In this section we will prove a version of Ramsey's theorem that will apply to graphs in which the weight of a vertex may depend on the color of the clique that contains the vertex. The next lemma is a convenient statement of a quantitative bound on the classical Ramsey Theorem.

Lemma 3.4.1. *In every 2-coloring of the edges of K_t, for some k and ℓ there is a red clique of order k and a blue clique of order ℓ with $k\ell \geq \frac{1}{4}\log^2 t$.*

Proof: Take k to be the order of the largest red clique and ℓ to be the order of the largest blue clique. We must have

$$t < R(k+1, \ell+1) \leq \binom{k+\ell}{k}.$$

In Section 3.9 we indicate how to prove that $\binom{k+\ell}{k} \leq 2^{2\sqrt{k\ell}}$; combining this with the above inequality gives the desired result. $\qquad\square$

For the rest of this chapter, let $M := 2^{16}$. The following lemma, which we call the weighted Ramsey theorem, states that if vertex i contributes weight α_i to any red clique in which it is contained and weight β_i to any blue clique in which it is contained, then we may give a lower bound for the product of the sizes of the largest (weighted) red and blue cliques.

Lemma 3.4.2. *Given a 2-coloring of the edges of a complete graph on t vertices with $t \geq M$ and vertex weights (α_i, β_i), take $\gamma_i = \alpha_i\beta_i$ and $\gamma = \min_i \gamma_i$. There is a red clique S and a blue clique U with*

$$\left(\sum_{s \in S} \alpha_s\right)\left(\sum_{u \in U} \beta_u\right) \geq \frac{\gamma}{32}\log^2 t.$$

Proof: We will dyadically partition the vertices based on their pair of weights (α_i, β_i), and then apply the classical Erdős-Szekeres bound on Ramsey numbers in the form of the previous lemma. That is, we will find a large set of vertices A so that any two vertices in A have similar values for α_i and β_i. By applying Lemma 3.4.1 to this set we will obtain the desired result.

Take $\alpha = \max_i \alpha_i$ and $\beta = \max_i \beta_i$.

If $\alpha\beta \geq \frac{\gamma}{32}\log^2 t$ we may take $S = \{i\}$ with $\alpha_i = \alpha$ and $U = \{j\}$ with $\beta_j = \beta$. Otherwise, $\alpha\beta/\gamma < \frac{1}{32}\log^2 t$. Observe that for each i we have $\alpha_i \leq \alpha$, $\beta_i \leq \beta$, and $\alpha_i\beta_i \geq \gamma$.

This gives $\gamma/\beta \le \alpha_i \le \alpha$ and $\gamma/\alpha \le \beta_i \le \beta$. Note that we may partition $[\gamma/\beta, \alpha]$ into $m_1 \le \log(\alpha\beta/\gamma)+1$ intervals $I_1, \ldots, I_{m_1}$ such that, within any interval I_i, we have $\sup(I_i)/\inf(I_i) \le 2$. Similarly, we may partition $[\gamma/\alpha, \beta]$ into $m_2 \le \log(\alpha\beta/\gamma) + 1$ intervals $I'_1, \ldots, I'_{m_2}$ with $\sup(I'_i)/\inf(I'_i) \le 2$. By the pigeonhole principle, there must be some pair (j, j') such that, taking $A := \{i : \alpha_i \in I_j, \beta_i \in I'_{j'}\}$, we have $|A| \ge t/(m_1 m_2)$.

Applying the previous lemma to A, we get that there is a red clique S of size k and a blue clique U of size ℓ with $k\ell \ge \frac{1}{4} \log^2(t/(m_1 m_2))$.

Note that, since $t \ge M$, we get $m_1 m_2 \le (\log(\frac{1}{32} \log^2 t)+1)^2 = \log^2(\frac{1}{16} \log^2 t) \le t^{1/4}$. Therefore, we get

$$\frac{1}{4} \log^2(t/(m_1 m_2)) \ge \frac{1}{4} \log^2(t^{3/4}) \ge \frac{1}{8} \log^2 t.$$

Take $\alpha_A = \min_{i \in A} \alpha_i$ and $\beta_A = \min_{i \in A} \beta_i$. For any $i \in A$, $\alpha_i \in I_j$ and hence $\alpha_A \ge \alpha_i/2$. Similarly, for any $i \in A$ we have $\beta_A \ge \beta_i/2$. Therefore, fixing any $i \in A$, we get $\alpha_A \beta_A \ge \frac{\alpha_i}{2} \frac{\beta_i}{2} \ge \gamma/4$. Therefore,

$$\left(\sum_{s \in S} \alpha_s \right) \left(\sum_{u \in U} \beta_u \right) \ge \left(\sum_{s \in S} \alpha_A \right) \left(\sum_{u \in U} \beta_A \right) = k\alpha_A \ell \beta_A \ge k\ell\gamma/4 \ge \frac{\gamma}{32} \log^2 t.$$

$\square$

Since, in the statement of the weighted Ramsey theorem, we take $\gamma = \min_i \alpha_i \beta_i$, it provides good bounds when $\alpha_i \beta_i$ does not vary much between the vertices. Therefore, when we wish to use it in the upcoming sections, we will first dyadically partition the vertices based on $\alpha_i \beta_i$ and then apply the lemma to each partition.

Note that we chose $\gamma = \min_i \alpha_i \beta_i$. We may hope to be able to use other functions of α_i, β_i in this expression. However, it is not as robust as one may hope. In particular, we want to observe that the function $\alpha_i + \beta_i$ will not yield an analogous theorem, as if we have many vertices of weight $(0, 1)$ and color all of the edges red, then the largest red clique has size 0 and the largest blue clique has size 1, but for each i we

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have $\alpha_i + \beta_i = 1$. Fortunately, using $\alpha_i\beta_i$ will suffice for our purposes.

3.5 Tight lower bound for three colors

In this section we will show that any Gallai 3-coloring on m vertices has a 2-colored set of size $\Omega(m^{1/3}\log^2 m)$. This matches the upper bound up to a constant factor.

We will refer to the three edge colors as red, blue, and yellow.

For the rest of this section, fix an integer $m \in \mathbb{N}$. We remark that in this section there is an inductive argument for which it is important to note that m remains fixed throughout.

Let

$$f(n) := \begin{cases} c\log^2(Cn) & \text{if } 0 < n \le m^{4/9} \\ c^2\log^2(m^{4/9})\log^2(Cnm^{-4/9}) & \text{if } m^{4/9} < n \le m^{8/9} \\ c^3\log^4(m^{4/9})\log^2(Cnm^{-8/9}) & \text{if } m^{8/9} < n \le m, \end{cases},$$

where $D = 2^{2048}$, $C = 2^{D^8}$, and $c = \log^{-2}(C^2) = D^{-16}/4$. We will have a further discussion about f and its properties shortly. For now, simply note that $f(m) = \Omega(\log^6 m)$.

We will prove the following theorem.

Theorem 3.5.1. *For any $n \in [m]$, a Gallai coloring F on n vertices satisfies either $\max_S g_{S,F} \ge m^{7/18}/8$ or $\prod_S g_{S,F} \ge nf(n)$.*

Before we prove Theorem 3.5.1, we show how it implies the existence of a large subchromatic set.

Theorem 3.5.2. *Every Gallai 3-coloring of $E(K_m)$ has a two colored set of size $\Omega(m^{1/3}\log^2 m)$.*

Proof: By Theorem 3.5.1, we have that either $\max_S g_{S,F} \ge m^{7/18}/8 = \Omega(m^{1/3}\log^2 m)$, or

$$\prod_S g_{S,F} \ge mf(m) = c^3 m\log^4(m^{4/9})\log^2(Cm^{1/9})$$

$$\geq c^3 m 2^{-6} (\log^4 m) 2^{-9} (\log^2 m) = 2^{-15} c^3 m \log^6 m.$$

As we have a lower bound on the product of three numbers, one of these numbers must be at least the cubed root. Hence, $\max_S g_{S,F} \geq 2^{-5} c m^{1/3} \log^2 m = \Omega(m^{1/3} \log^2 m)$, as desired. □

We will now proceed with a further discussion about f. We call $(0, m^{4/9}], (m^{4/9}, m^{8/9}],$ $(m^{8/9}, m]$ the "intervals of f." Note that on each interval, $f(n) = \gamma \log^2(\delta n)$ for some constants γ, δ (where m is viewed as a constant). Intuitively, C is large so that we avoid the range of values in which log is poorly behaved, and c is small both so that we may assume n is large and to make the transitions between intervals easier. f was chosen so that it satisfies certain properties, the more interesting of which we explicitly enumerate below. All of these properties are formalizations of the statement "f does not grow too quickly."

Lemma 3.5.3. *If $m \geq C$, then the following statements hold about f for any integer n with $1 < n \leq m$.*

1. *For any $\alpha \in \left[\frac{1}{n}, 1\right]$, $f(\alpha n) \geq \alpha f(n)$.*

2. *For any $\alpha_1, \alpha_2, \alpha_3 \in \left[\frac{1}{n}, 1\right]$ such that $\sum_i \alpha_i = 1$ we have, taking $n_i = \alpha_i n$,*

$$n f(n) - \sum_i n_i f(n_i) \leq \frac{8}{\log C} n f(n).$$

3. *For $i \geq 0$ and $m^{7/18} \geq 2^j \geq 1$ we have $f(2^i) \log^2(D 2^j) \geq 512 f(2^{i + \frac{8}{7} j})$.*

4. *For $1 \leq \tau \leq n \leq D^3 \tau$, we have $f(\tau) \geq f(n)/2$.*

5. *For any $\alpha \in \left[\frac{1}{n}, \frac{1}{32}\right]$, $f(\alpha n) \geq 16 \alpha f(n)$.*

These properties are collectively referred to as "the facts about f" and are proved in Section 3.10.

We now proceed with a proof of Theorem 3.5.1.

Proof of Theorem 3.5.1: We proceed by induction on n. Define g to be the size of the largest set in F using only the colors blue and yellow, o to be the size of the

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largest set in F using only the colors red and yellow, and p to be the size of the largest set in F using only the colors red and blue. We wish to show that either $gop \geq nf(n)$ or $\max(g, o, p) \geq m^{7/18}/8$.

Our base cases are those n for which $f(n) \leq 1$, as for these cases by Theorem 3.3.2 $gop \geq n \geq nf(n)$. Since $c = \log^{-2}(C^2)$, any $n < C$ is a base case.

If we are not in a base case, we have $n \geq C$ (and $f(n) \geq 1$).

Since F is a Gallai coloring, there is a non-trivial partition $V(K_n) = V_1 \cup \ldots \cup V_t$ with $|V_1| \geq \ldots \geq |V_t| \geq 1$ such that there is some 2-coloring χ of $[t]$ such that for every distinct $i, j \in [t]$ and $u \in V_i$, $v \in V_j$, the color under F of $\{u, v\}$ is $\chi(i, j)$.

Suppose, without loss of generality, that χ only uses the colors blue and yellow. The proof will split into three cases.

Cases 1 and 2, Preliminary discussion: These will be the cases in which V_1 has a substantial portion of the vertices. Let $U_1 = V_1$, U_2 denote the union of V_j over $j \neq 1$ such that $\chi(1, j)$ is yellow, and U_3 denote the union of V_j over $j \neq 1$ such that $\chi(1, j)$ is blue. We have that U_1, U_2, U_3 is a non-trivial partition of V. Let $n_i = |U_i|$. Let $\alpha_i = |U_i|/n = n_i/n$ for $i = 1, 2, 3$, so $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

For $i = 1, 2, 3$, let F_i be the coloring F restricted to U_i. Let g_i be the size of the largest subchromatic set in F_i using only the colors blue and yellow, o_i be the size of the largest subchromatic set in F_i using only the colors red and yellow, and p_i be the size of the largest subchromatic set in F_i using only the colors red and blue. Suppose without loss of generality $n_2 \geq n_3$, so $\alpha_2 \geq (1 - \alpha_1)/2$ and $\max(\alpha_1, \alpha_2) \geq 1/3$. By the induction hypothesis, for $i = 1, 2, 3$, we have that either one of g_i, o_i, p_i is at least $m^{7/18}/8$, in which case we may use $g \geq \max_i g_i$, $o \geq \max_i o_i$, $p \geq \max_i p_i$ to complete the induction, or

$$g_i o_i p_i \geq n_i f(n_i).$$

Assume we are in this latter case. Since the U_i are connected only by yellow and blue edges, we may take the largest subchromatic set using only yellow and blue from each U_i, giving $g \geq g_1 + g_2 + g_3$ (in fact, equality holds). Since U_1 and U_2 are connected with yellow edges, we may take the largest subchromatic set using only red and yellow

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from both U_1 and U_2, or we may simply take the largest such subchromatic set from U_3, so we get $o \geq \max(o_1 + o_2, o_3)$. Similarly, $p \geq \max(p_1 + p_3, p_2)$.

Note

$$gop \geq g_1op + g_2op + g_3op \geq g_1(o_1 + o_2)(p_1 + p_3) + g_2(o_1 + o_2)p_2 + g_3o_3(p_1 + p_3)$$

$$\geq g_1(o_1 + o_2)p_1 + g_2(o_1 + o_2)p_2 + g_3o_3p_3 = g_1o_2p_1 + g_2o_1p_2 + \sum_{i=1}^{3} g_io_ip_i.$$

We thus have

$$gop - \sum_{i=1}^{3} g_io_ip_i \geq g_1o_2p_1 + g_2o_1p_2 \geq 2\sqrt{(g_1o_2p_2)(g_2o_1p_1)} = 2\sqrt{(g_1o_1p_1)(g_2o_2p_2)}$$

$$\geq 2\sqrt{(n_1f(n_1))(n_2f(n_2))} \geq 2\sqrt{\alpha_1\alpha_2}n\sqrt{f(n_1)f(n_2)},$$

where the second inequality is an instance of the arithmetic-geometric mean inequality.

Case 1: $\alpha_1, \alpha_2 \geq (\log C)^{-1/4}$. In this case, we have

$$gop - \sum_{i=1}^{3} g_io_ip_i \geq 2\sqrt{\alpha_1\alpha_2}n\sqrt{f(n_1)f(n_2)} \geq 2\alpha_1\alpha_2nf(n) \geq 2nf(n)/\sqrt{\log C}$$

$$\geq \frac{8}{\log C}nf(n) \geq nf(n) - \sum_{i=1}^{3} n_if(n_i),$$

where the second inequality is by the first fact about f, the third inequality is by substituting lower bounds on α_1 and α_2, and the last inequality is by the second fact about f. Hence,

$$gop \geq \sum_{i=1}^{3} g_io_ip_i + nf(n) - \sum_{i=1}^{3} n_if(n_i) \geq nf(n),$$

where the last inequality is by the induction on hypothesis applied to U_i for $i = 1, 2, 3$. This completes this case.

Case 2: $\alpha_1 \geq (\log C)^{-1/4} \geq \alpha_2$. Before we proceed with this case, we prove a simple

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claim.

Claim 3.5.4. $nf(n) + n_1 f(n_1) - 2n_1 f(n) > 0$.

Proof: Note $nf(n) - n_1 f(n) = (1 - \alpha_1)nf(n)$. Therefore,

$$n_1 f(n_1) - n_1 f(n) \geq \alpha_1 n_1 f(n) - n_1 f(n) = \alpha_1^2 nf(n) - \alpha_1 nf(n) = -\alpha_1 (1 - \alpha_1)nf(n),$$

where the first inequality follows from the first fact about f. From this we get

$$nf(n) + n_1 f(n_1) - 2n_1 f(n) \geq (1 - \alpha_1)nf(n) - \alpha_1(1 - \alpha_1)nf(n) = (1 - \alpha_1)^2 nf(n) > 0.$$

$\square$

In this case we have $\alpha_1 \geq 1 - (\alpha_2 + \alpha_3) \geq 1 - 2\alpha_2 \geq 1 - 2(\log C)^{-1/4} \geq 1/2$ and hence

$$gop - \sum_{i=1}^{3} g_i o_i p_i \geq 2\sqrt{\alpha_1 \alpha_2} n \sqrt{f(n_1)f(n_2)} \geq 8\alpha_1 \alpha_2 nf(n) \geq 4\alpha_2 nf(n)$$

$$\geq 2(\alpha_2 + \alpha_3)nf(n) = 2(n - n_1)f(n)$$

$$\geq 2(n - n_1)f(n) - (nf(n) + n_1 f(n_1) - 2n_1 f(n))$$

$$= nf(n) - n_1 f(n_1) \geq nf(n) - \sum_{i=1}^{3} n_i f(n_i),$$

where the second inequality is by both the first fact about f applied to $f(n_1)$ and the fifth fact about f applied to $f(n_2)$, the third inequality is by $\alpha_1 \geq 1/2$, and the second-to-last one is by the claim.

Hence,

$$gop \geq \sum_{i=1}^{3} g_i o_i p_i + nf(n) - \sum_{i=1}^{3} n_i f(n_i) \geq nf(n),$$

where the last inequality is by the induction on hypothesis applied to U_i for $i = 1, 2, 3$. This completes this case.

Case 3: $\alpha_1 < (\log C)^{-1/4}$. This is the sparse case, when each part is at most a $(\log C)^{-1/4} = D^{-2}$ fraction of the total.

Take $n_i = |V_i|$. Take F_i to be the coloring F restricted to V_i. Take g_i to be the size of the largest subchromatic set in F_i using only the colors blue and yellow, o_i to be the size of the largest subchromatic set in F_i using only the colors red and yellow, and p_i to be the size of the largest subchromatic set in F_i using only the colors red and blue.

We reorder the V_i so that if $i \leq j$ then $o_i p_i \leq o_j p_j$.

Take $\tau = \lfloor \log(2D^{-2}n) \rfloor$, so $\max_i n_i \leq (\log C)^{-1/4}n \leq D^{-2}n \leq 2^\tau \leq 2D^{-2}n$. Define, for $i \leq \tau$, $I_i := [2^i, 2^{i+1}]$. Take $\Phi(i) = \{j : n_j \in I_i\}$. The $\Phi(i)$ are dyadically partitioning the indices; we will eventually use these partitions to construct sets to which we will apply the weighted Ramsey theorem.

Note that $g = \sum_j g_j$, so we have $gop = \sum_j g_j op$.

We now present the idea behind the argument for the rest of this case. Fix i so that $\Phi(i)$ has at least $2D2^{\frac{7}{8}(\tau-i)}$ elements and $i \geq \log(nm^{-7/18})$ (we will show that most vertices v are contained in V_j as j varies over the $\Phi(i)$ that have this property). We will define a weighted graph whose vertices are the indices and whose coloring is χ. Given an index j its weight will be (o_j, p_j). If we find a yellow clique in χ then the sum of the o_j in the clique gives a lower bound on o, and, similarly, if we find a blue clique in χ then the sum of the p_j in the clique gives a lower bound on p. We will apply the weighted Ramsey theorem to half of the indices in $\Phi(i)$ (to the indices that are larger than the median of $\Phi(i)$, to be precise); from this, we will be able to conclude that if j is an index smaller than the median, then $op/(o_j p_j) \geq D'f(n)/f(n_j)$ for some large constant D' and so $g_j op \geq D'g_j o_j p_j f(n)/f(n_j) \geq D'n_j f(n)$. We now proceed with the argument.

When we count, we wish to omit parts $\Phi(i)$ that don't satisfy desired properties; take

$$B' := \{i \leq \tau : |\Phi(i)| \leq 2D2^{\frac{7}{8}(\tau-i)}\},$$
$$B'' := \{i \leq \log(nm^{-7/18})\}.$$

Take $B = B' \cup B''$. We will show that a large fraction of the vertices are not contained in V_j for $j \in \Phi(i)$ where i ranges over B.

$$\sum_{i \in B'} \sum_{j \in \Phi(i)} n_j \leq \sum_{i \leq \tau} 2^{i+1}(2D2^{\frac{7}{8}(\tau-i)}) = 4D2^{\frac{7}{8}\tau} \sum_{i \leq \tau} 2^{\frac{i}{8}} \leq 4D2^{\frac{7}{8}\tau} \frac{1}{2^{1/8}-1} \cdot 2^{(\tau+1)/8}$$

$$\leq 8D \frac{1}{2^{1/8}-1} 2^{\tau} \leq 128D2^{\tau} \leq \frac{256}{D} n \leq n/4,$$

where the fourth inequality follows from $2^{1/8} \geq (1 + 1/16)$.

Note, if $\sum_i g_i \geq m^{7/18}/8$, then we may complete the induction; assume this is not the case. In particular, we get $t \leq m^{7/18}/8$ (since $g_i \geq 1$). Therefore,

$$\sum_{i \in B''} \sum_{j \in \Phi(i)} n_j \leq \sum_{i \in B''} \sum_{j \in \Phi(i)} 2nm^{-7/18} \leq 2tnm^{-7/18} \leq n/4.$$

Hence,

$$\sum_{i \in B} \sum_{j \in \Phi(i)} n_j \leq \sum_{i \in B'} \sum_{j \in \Phi(i)} n_j + \sum_{i \in B''} \sum_{j \in \Phi(i)} n_j \leq n/4 + n/4 \leq n/2.$$

As a corollary we get $\sum_{i \notin B} \sum_{j \in \Phi(i)} n_j \geq n/2$.

For any fixed $i \leq \tau$ such that $i \notin B$, take β_i to be the median of $\Phi(i)$ (if $\Phi(i)$ has an even number of elements, take β_i to be the larger of the two medians). Consider $\{(o_j, p_j) : j \in \Phi(i), j \geq \beta_i\}$. By $i \notin B$, this has at least $D2^{\frac{7}{8}(\tau-i)} \geq M$ elements (recall from the weighted Ramsey theorem that $M = 2^{16}$), so we get, by applying the weighted Ramsey theorem to this set, that $op \geq o_{\beta_i} p_{\beta_i} \log^2 \left(D2^{\frac{7}{8}(\tau-i)} \right) /32$. Finally, observe that either one of the o_j, p_j, g_j is at least $m^{7/18}/8$ in which case we may conclude the induction, or by the induction hypothesis we may assume $o_j p_j g_j \geq n_j f(n_j)$. Therefore,

$$\sum_{j \in \Phi(i)} g_j o p \geq \sum_{j \in \Phi(i)} g_j o_{\beta_i} p_{\beta_i} \log^2 \left(D 2^{\frac{7}{8}(\tau - i)} \right) / 32 \geq \sum_{j \in \Phi(i): j \leq \beta_i} g_j o_{\beta_i} p_{\beta_i} \log^2 \left(D 2^{\frac{7}{8}(\tau - i)} \right) / 32$$

$$\geq \sum_{j \in \Phi(i): j \leq \beta_i} g_j o_j p_j \log^2 \left(D 2^{\frac{7}{8}(\tau - i)} \right) / 32 \geq \sum_{j \in \Phi(i): j \leq \beta_i} n_j f(n_j) \log^2 \left(D 2^{\frac{7}{8}(\tau - i)} \right) / 32$$

$$\geq \sum_{j \in \Phi(i): j \leq \beta_i} n_j f \left(2^{\log n_j} \right) \log^2 \left(D 2^{\frac{7}{8}(\tau - \log n_j)} \right) / 32 \geq \sum_{j \in \Phi(i): j \leq \beta_i} 16 n_j f(2^\tau)$$

$$\geq \sum_{j \in \Phi(i): j \leq \beta_i} 8 n_j f(n),$$

where the third inequality is by $o_j p_j \leq o_{j'} p_{j'}$ for $j \leq j'$, the fourth inequality is by the induction hypothesis applied to V_j, the sixth inequality is by the third fact about f, and the seventh inequality is by the fourth fact about f and noting $2^\tau \geq D^{-3} n$.

We now consider for any set $J \subseteq \Phi(i)$:

$$\sum_{j \in J} n_j \geq 2^i |J|.$$

$$\sum_{j \in \Phi(i)} n_j \leq 2^{i+1} |\Phi(i)|.$$

This gives:
$$\frac{\sum_{j \in J} n_j}{\sum_{j \in \Phi(i)} n_j} \geq \frac{|J|}{2 |\Phi(i)|}.$$

Noting that $|\{j \in \Phi(i) : j \leq \beta_i\}| \geq |\Phi(i)| / 2$:

$$\sum_{j \in \Phi(i): j \leq \beta_i} 8 n_j f(n) \geq \frac{1}{4} \sum_{j \in \Phi(i)} 8 n_j f(n) = 2 f(n) \sum_{j \in \Phi(i)} n_j.$$

Therefore,

$$gop \geq \sum_j g_j op \geq \sum_{i \leq \tau} \sum_{j \in \Phi(i)} g_j op \geq \sum_{i \leq \tau : i \notin B} \sum_{j \in \Phi(i)} g_j op \geq \sum_{i \leq \tau : i \notin B} 2f(n) \sum_{j \in \Phi(i)} n_j$$

$$= 2f(n) \sum_{i \leq \tau : i \notin B} \sum_{j \in \Phi(i)} n_j \geq 2f(n)\frac{n}{2} = nf(n).$$

We have thus concluded the induction. □

We informally refer to B'' in the above proof as large if a large fraction of the vertices are contained in a V_j for $j \in \Phi(i)$ where i ranges over B''. The case in which B'' was large easily implied the desired result. In extending this result in Section 3.8 to more colors, the primary difficulty is the following: when s is not 2, it is not obvious that there is a large s-colored set as a result of B'' being large.

3.6   Upper bound for many colors

In this section we will give asymptotically tight upper bounds for how large of a subchromatic set must exist in an edge coloring on m vertices. We will first show how to construct such colorings from weighted graphs with vertex set R, and then we will choose such graphs to finish the construction. The next theorem states that if we have a weighted graph on r vertices with edge weights w_P, then we can find a coloring F so that $g_{s,F}$ is, up to logarithmic factors, $\prod_{P \subseteq S} w_P$.

Lemma 3.6.1. *Given a weighted graph $(R, \mathcal{P})$ on r vertices with integer edge weights $\{w_P\}_{P \in \mathcal{P}}$, taking $m := \prod_{P \in \mathcal{P}} w_P$, there is a Gallai r-coloring on m vertices so that for any $S \subseteq R$, the size of the largest subchromatic set with colors in S is at most $\prod_{P \in \mathcal{P} : P \subseteq S} w_P \cdot \prod_{P \in \mathcal{P} : |P \cap S| = 1} 2 \log w_P$.*

Proof: We may define a Gallai r-coloring on m vertices as follows: take $P_1, \ldots, P_k$ an arbitrary enumeration of $\mathcal{P}$. For each edge P, take F_P to be a 2-coloring of $E(K_{w_P})$ using colors from P so that the largest monochromatic clique has order at most $2 \log w_P$ (such a coloring exists by the Erdős-Szekeres bound for Ramsey numbers

[36]). We define a coloring F on m vertices by

$$F = F_{P_1} \otimes F_{P_2} \otimes \cdots \otimes F_{P_k}.$$

F is a Gallai coloring by Corollary 3.2.4. Given any $S \subseteq R$, note that $g_{S,F_P} = w_P$ if $P \subseteq S$, as F_P uses only colors from P. If $|P \cap S| = 1$, then the largest subchromatic set in F_P using colors from $P \cap S$ is at most $2 \log w_P$ by choice of F_P, so $g_{S,F_P} \leq 2 \log w_P$. If $|P \cap S| = 0$, then $g_{S,F_P} = 1$ as any two distinct vertices are connected by an edge the color of which is not in S. Therefore,

$$g_{S,F} = \prod_i g_{S,F_{P_i}} \leq \prod_{P \in \mathcal{P}: P \subseteq S} w_P \cdot \prod_{P \in \mathcal{P}: |P \cap S| = 1} 2 \log w_P.$$

$\square$

The condition in the above lemma that the edge weights are integers is slightly cumbersome; we will now eliminate it.

Lemma 3.6.2. *Let $(R, \mathcal{P})$ be a weighted graph on r vertices ($r \geq 3$) and weights w_P satisfying $w_P = \omega(1)$ for every $P \in \mathcal{P}$. Letting $m := \prod_{P \in \mathcal{P}} w_P$, if m is an integer and each w_P satisfies $w_P \geq \omega(1)$, then there is a Gallai r-coloring on m vertices such that, for any $S \subseteq R$, the size of the largest subchromatic set is at most $(1 + o(1)) \prod_{P \in \mathcal{P}: P \subseteq S} w_P \cdot \prod_{P \in \mathcal{P}: |P \cap S| = 1} 2 \log w_P$.*

Proof: Take $w'_P = \lceil w_P \rceil$. Since $w_P \geq \omega(1)$, we get $w'_P \leq (1 + o(1)) w_P$. We may apply the previous lemma to the w'_P to get an r-Gallai coloring on $\prod_P w'_P \geq m$ vertices so that for any $S \subseteq R$ the size of the largest subchromatic set is at most

$$\prod_{P \in \mathcal{P}: P \subseteq S} w'_P \cdot \prod_{P \in \mathcal{P}: |P \cap S| = 1} 2 \log w'_P \leq (1 + o(1)) \prod_{P \in \mathcal{P}: P \subseteq S} w_P \cdot \prod_{P \in \mathcal{P}: |P \cap S| = 1} 2 \log w_P.$$

Restrict this coloring to any m vertices; it is still a Gallai r-coloring and for any $S \subseteq R$ the size of the largest subchromatic set is at most $(1 + o(1)) \prod_{P \in \mathcal{P}: P \subseteq S} w_P \cdot \prod_{P \in \mathcal{P}: |P \cap S| = 1} 2 \log w_P.$ $\square$

Now, if we wish to obtain colorings without large subchromatic sets, we need only construct appropriate weighted graphs. Intuitively, we would like to minimize the number of edges in such a graph (while still being able to maintain that all the $S \subseteq R$ have approximately the same value of $\prod_{P \subseteq S} w_P$), as every edge creates extra log factors. This observation motivates the following bounds.

Theorem 3.6.3. *There is a Gallai r-coloring on m vertices such that for any $S \in \binom{R}{s}$ the size of the largest subchromatic set is at most $(1 + o(1))m^{\binom{s}{2}/\binom{r}{2}} \log^{c_{r,s}} m$, where*

$$
c_{r,s} = \begin{cases} s(r-s) & \text{if } s < r-1, \\ 1 & \text{if } s = r-1 \text{ and } r \text{ is even}, \\ (r+3)/r & \text{if } s = r-1 \text{ and } r \text{ is odd}. \end{cases}
$$

Proof: If $s < r - 1$, we may apply the previous lemma to a clique on r vertices with edge weights $m^{1/\binom{r}{2}}$. Any $S \subseteq R$ of size s has $\binom{s}{2}$ internal edges and $s(r-s)$ edges intersecting it in one vertex. By the previous lemma, we may find a Gallai r-coloring where the size of the largest subchromatic set is asymptotically at most:

$$
m^{\binom{s}{2}/\binom{r}{2}} \left(2 \log \left(m^{1/\binom{r}{2}} \right) \right)^{s(r-s)} \leq m^{\binom{s}{2}/\binom{r}{2}} (\log m)^{s(r-s)}.
$$

If $s = r - 1$ and r is even, we may consider a perfect matching on r vertices where each edge has weight $m^{2/r}$; any subset of size $r - 1$ contains $r/2 - 1$ edges and there is one edge with which it shares exactly one vertex. By the previous lemma, we may find a Gallai r-coloring where the size of the largest subchromatic set is asymptotically at most:

$$
m^{(r/2-1)/(r/2)} 2 \log(m^{1/(r/2)}) \leq m^{(r/2-1)/(r/2)} \log m = m^{\binom{s}{2}/\binom{r}{2}} \log m.
$$

If $s = r - 1$ and r is odd, we may consider a graph formed by taking the disjoint union of a triangle on 3 vertices and a matching with $(r - 3)/2$ edges. The edges of the triangle will each have weight $w_1 := m^{1/r}(\log m)^{(r-3)/2r}$ and the edges of the matching will each have weight $w_2 := m^{2/r}(\log m)^{-3/r}$. Note that the product of the weights is $w_1^3 w_2^{(r-3)/2} = m$. Let $S \subseteq R$ of size $s = r - 1$ be given.

If the vertex not contained in S is part of the triangle, then S contains $(r-3)/2$ edges of weight w_2 and 1 edge of weight w_1. Furthermore, there are two edges each of weight w_1 that S intersects in one vertex. In the graph obtained from the previous lemma the size of the largest subchromatic set taking colors from S is asymptotically at most:

$$
\begin{aligned}
w_1 w_2^{(r-3)/2}(2\log w_1)^2 &= m^{(r-2)/r}(\log m)^{-(r-3)/r}(2\log(m^{1/r}(\log m)^{(r-3)/2r}))^2 \\
&\leq m^{(r-2)/r}(\log m)^{-(r-3)/r}(\log m)^2 \\
&= m^{\binom{s}{2}/\binom{r}{2}}(\log m)^{(r+3)/r}.
\end{aligned}
$$

If the vertex not contained in S is part of the matching then S contains $(r-5)/2$ edges of weight w_2 and 3 edges of weight w_1. Furthermore, there is one edge of weight w_2 that intersects S in one vertex. In the graph obtained from the previous lemma the size of the largest subchromatic set taking colors from S is asymptotically at most:

$$
\begin{aligned}
w_1^3 w_2^{(r-5)/2}(2\log w_2) &= m^{(r-2)/r}(\log m)^{3/r}(2\log(m^{2/r}(\log m)^{-3/r})) \\
&\leq m^{(r-2)/r}(\log m)^{3/r}(\log m) \\
&= m^{\binom{s}{2}/\binom{r}{2}}(\log m)^{(r+3)/r}.
\end{aligned}
$$

$\square$

3.7 Weak lower bound for many colors

We now provide a simple lower bound for the largest size of a subchromatic set in any r-coloring of $E(K_m)$ that shows our upper bounds are tight up to polylogarithmic factors; we show that any Gallai r-coloring on m vertices contains a subchromatic set of size at least $m^{\binom{s}{2}/\binom{r}{2}}$. The following is a common generalization of Hölder's inequality that we will find useful.

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Lemma 3.7.1. *If $\mathcal{S}$ is a finite set of indices and, for each $S \in \mathcal{S}$, g_S is a function mapping $[t]$ to the non-negative reals, then*

$$\prod_{S \in \mathcal{S}} \sum_i g_S(i) \geq \left(\sum_i \prod_{S \in \mathcal{S}} g_S(i)^{1/|\mathcal{S}|} \right)^{|\mathcal{S}|}$$

Using the above lemma, we will prove a lower bound on the product of the $g_{S,F}$ for F a Gallai r-coloring. This will easily imply the desired lower bound.

Theorem 3.7.2. *For any Gallai r-coloring F on m vertices,*

$$\prod_{S \in \binom{R}{s}} g_{S,F} \geq m^{\binom{r-2}{s-2}}.$$

Proof: Take $g_S = g_{S,F}$. We proceed by induction on m. If $m = 1$, then each g_S is 1 as is their product, while $m^{\binom{r-2}{s-2}}$ is also 1. If $m > 1$, we may find some pair of colors Q and some non-trivial partition of the vertices $V_1, \ldots, V_t$ such that for each pair of distinct i, j in $[t]$, there is a $q \in Q$ so that all of the edges between V_i and V_j have color q.

Define, for $i \in [t]$, F_i to be the restriction of F to V_i. Take $g_{S,i} := g_{S,F_i}$. By induction, for each i we have $\prod_S g_{S,i} \geq m_i^{\binom{r-2}{s-2}}$, where $m_i = |V_i|$.

Note that if $Q \subseteq S$ then $g_S \geq \sum_i g_{S,i}$, since we may combine the largest subchromatic sets from each F_i. For every S we have $g_S \geq \max_i g_{S,i}$, so

$$\prod_S g_S \geq \left(\prod_{S:Q \subseteq S} \sum_i g_{S,i} \right) \prod_{S:Q \not\subseteq S} g_S \geq \left(\sum_i \prod_{S:Q \subseteq S} g_{S,i}^{1/\binom{r-2}{s-2}} \right)^{\binom{r-2}{s-2}} \prod_{S:Q \not\subseteq S} g_S$$

$$= \left(\sum_i \prod_{S:Q \subseteq S} g_{S,i}^{1/\binom{r-2}{s-2}} \prod_{S:Q \not\subseteq S} g_S^{1/\binom{r-2}{s-2}} \right)^{\binom{r-2}{s-2}} \geq \left(\sum_i \prod_S g_{S,i}^{1/\binom{r-2}{s-2}} \right)^{\binom{r-2}{s-2}}$$

$$\geq \left(\sum_i m_i \right)^{\binom{r-2}{s-2}} = m^{\binom{r-2}{s-2}},$$

where the first inequality follows by $g_S \geq \sum_i g_{S,i}$ if $Q \subseteq S$, the second inequality

follows by the preceding lemma and noting $|S| = \binom{r-2}{s-2}$, the third inequality follows by $g_S \geq g_{S,i}$, and the fourth inequality follows by the induction hypothesis. $\qquad\square$

Note that, in proving this bound, if $|S \cap Q| = 1$, we simply use $g_S \geq g_{S,F_i}$. As in the $r = 3, s = 2$ case, if we can find a set of indices $V_{i_1}, \ldots, V_{i_k}$ so that between any two of them the edges use the color contained in $S \cap Q$, we may obtain a stronger lower bound on g_S.

We now conclude the argument.

Theorem 3.7.3. *In any Gallai r-coloring F on m vertices, there is some $S \in \binom{R}{s}$ with $g_{S,F} \geq m^{\binom{s}{2}/\binom{r}{2}}$*

Proof: By the previous theorem, $\prod_{S \in \binom{R}{s}} g_{S,F} \geq m^{\binom{r-2}{s-2}}$. As this is a product over $\binom{r}{s}$ numbers, there must be some S with

$$g_{S,F} \geq m^{\binom{r-2}{s-2}/\binom{r}{s}} = m^{\binom{s}{2}/\binom{r}{2}}.$$

$\qquad\square$

3.8 Lower bound for many colors

In this section we show that our upper bounds on sizes of subchromatic sets in Gallai colorings are tight up to constant factors (where we view r and s as constant).

3.8.1 Discrepancy lemma in edge-weighted graphs

The lemma in this subsection has the following form: either a given weighted graph has many edges of non-zero weight or it has some set S of size s whose weight is significantly larger than average. In the next subsection we will show how to reduce the problem of lower bounding the size of the largest subchromatic set in a Gallai r-coloring to a problem regarding the number of non-zero edges in a graph that doesn't contain vertex subsets S whose weight is significantly larger than average, so this lemma will be useful.

Lemma 3.8.1. *Given weights w_P for $P \in \binom{R}{2}$ with $w_P \geq 0$, take $w = \sum_P w_P$. Take $a_0 = \binom{r}{2}$ if $s < r - 1$, $a_0 = r/2$ if $s = r - 1$ and r is even, and $a_0 = (r+3)/2$ if $s = r - 1$ and r is odd. Either there are at least a_0 pairs P with $w_P > 0$ or there is some $S \subseteq R$ of size s satisfying*

$$\sum_{P \subseteq S} w_P \geq \left(1 + \left(4r \binom{r}{2}^2 \right)^{-1} \right) \frac{\binom{s}{2}}{\binom{r}{2}} w.$$

The proof of the above lemma uses elementary techniques along with the second moment method and is deferred to Section 3.11.

3.8.2 Proof of lower bound for many colors

Let

$$d = \frac{\binom{r-2}{s-1}}{\binom{r-2}{s-2}} = \frac{r-s}{s-1},$$

$$C = 32r \binom{r}{2}^3 d,$$

$$\delta = \left(4 \binom{r-2}{s-2} C \right)^{-1},$$

$$\delta_0 = C^{-1} \binom{r-2}{s-1} \binom{r-2}{s-2}^{-1} \left(\binom{r}{2} + 1 \right)^{-1} = C^{-1} d \left(\binom{r}{2} + 1 \right)^{-1},$$

$$\delta_1 = 2^{-\binom{r}{2}-2} \left(\delta_0^{-1} + 1 \right)^{-\binom{r}{2}-1} \binom{r-2}{s-1}^{-1},$$

$$c = (\delta/4)^2 \delta_1^{1/d}.$$

d is an appropriately chosen scaling factor; why it is appropriate will become evident later. C should be thought of as a large constant, and $\delta, \delta_0, \delta_1$, and c should be thought of as small constants. We provide some bounds on the above; although we will not explicitly reference these, they are useful for verifying various inequalities:

$$r^{-1} \leq d \leq r,$$
$$C \leq 4r^8,$$
$$\delta \geq 2^{-8r},$$
$$\delta_0 \geq r^{-11}/4,$$
$$\delta_1 \geq 2^{-11r^3},$$
$$c \geq 2^{-12r^4}.$$

When we constructed the upper bound via product colorings, there was a weighted graph (namely the one used to construct the coloring) so that for any $S \subseteq R$ we could approximate the size of the largest clique using colors from S by the product of the weights of edges contained in S. It is tempting to believe that the structure of any Gallai coloring F can be approximated this way. Though this is not true in general, the next theorem states that if it is not true then $\prod_S g_{S,F}$ must be large. Take for the rest of this chapter

$$m_0 := 2^{2^{2^{2^{2^{8r^2}}}}}. \tag{3.1}$$

Theorem 3.8.2. *If $m \geq m_0$, then for any Gallai coloring F on $n \leq m$ vertices, there are $f \geq 1$, $\epsilon \geq 0$, $\mathcal{P} \subseteq \binom{R}{2}$, and, for $P \in \mathcal{P}$, weights $w_P \in [1, \infty)$ satisfying:*

1. *For every $S \in \binom{R}{s}$, $g_{S,F} \geq \prod_{P \in \binom{S}{2} \cap \mathcal{P}} w_P$.*

2. *$\prod_{P \in \mathcal{P}} w_P \geq m^{-\epsilon} n$.*

3. *$\prod_{S \in \binom{R}{s}} g_{S,F} \geq (nf)^{\binom{r-2}{s-2}}$.*

4. *$f \geq (\log m)^{C\epsilon}$.*

5. *Taking a to be the size of $\mathcal{P}$, $f \geq \left(c \log^2 m\right)^{ad}$.*

From the above theorem we will quickly be able to conclude Theorem 3.1.1. Note that, if f is large enough, then by condition (3) we conclude that $\prod_{S \in \binom{R}{s}} g_{S,F}$ is large and so some $g_{S,F}$ is large. Otherwise, by condition (4) we have an upper bound on the

size of ϵ, so by conditions (1) and (2) the structure of the coloring is well-approximated by the w_P. This latter case will allow us to apply our work on weighted graphs from the previous subsection to get a lower bound on a, and then we will apply condition (5) to conclude that some $g_{S,F}$ is large.

Proof: We will write g_S for $g_{S,F}$. We will take $w_P = 1$ for any P in $\binom{R}{2}$ but not in $\mathcal{P}$; this way, for any $T \subseteq \binom{R}{2}$, we have $\prod_{P \in T \cap \mathcal{P}} w_P = \prod_{P \in T} w_P$.

We proceed by induction on n.

Base Case: If $n = 1$, then we may take $f = 1, \epsilon = 0$, and $\mathcal{P} = \emptyset$. Letting $a = |P| = 0$,

1. For every $S \in \binom{R}{s}$, $g_S = 1 = \prod_{P \in \binom{S}{2}} w_P$.

2. $\prod_{P \in \binom{R}{2}} w_P = 1 = m^{-\epsilon} n$.

3. $\prod_{S \in \binom{R}{s}} g_S = 1 = (nf)^{\binom{r-2}{s-2}}$.

4. $f = 1 = (\log m)^{C\epsilon}$.

5. $f = 1 = \left(c \log^2 m \right)^{ad}$.

Preliminary discussion: If $n > 1$, there is some pair of colors $Q = \{Q_1, Q_2\}$ and there is a non-trivial partition $V(K_n) = V_1 \cup \ldots \cup V_t$ with $|V_1| \geq \ldots \geq |V_t|$ such that there is some 2-coloring $\chi : \binom{t}{2} \to Q$ such that for every distinct $i, j \in [t]$ and $u \in V_i$, $v \in V_j$, the color under F of $\{u, v\}$ is $\chi(i, j)$ (which is in Q).

Given $\epsilon > 0$, define $f_\epsilon(\ell) := (\log m)^{C \frac{\log(\alpha m^\epsilon)}{\log m}}$, where $\alpha = \ell/n$. Note that we may rewrite $f_\epsilon(\ell) = (\log m)^{C\epsilon + C \frac{\log \alpha}{\log m}}$; we will move between the two expressions freely. Note also that $f_\epsilon(\ell)$ is an increasing function of ℓ. We will need some lemmas about f_ϵ, all of which are formalizations of the statement "f_ϵ does not grow too quickly".

Lemma 3.8.3. *The following statements hold about f_ϵ for every choice of $\epsilon \geq 0$, $m \geq m_0$, and $1 < n \leq m$.*

1. *For any $\alpha \in [\frac{1}{n}, 1]$,*
$$f_\epsilon(\alpha n) \geq \alpha^{1/\left(2\binom{r-2}{s-2}\right)} f_\epsilon(n).$$

In particular, $f_\epsilon(\alpha n) \geq \alpha f_\epsilon(n)$.

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2. *For any $\alpha_1, \alpha_2, \alpha_3 \in [\frac{1}{n}, 1]$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$, taking $n_i = \alpha_i n$,*

$$n f_\epsilon(n) \leq \sum_i n_i f_\epsilon(n_i) + 3(\log^{-3/4} m) n f_\epsilon(n).$$

3. *For $i \geq 0$ and $m^\delta \geq 2^j \geq 1$ we have $f_\epsilon(2^i) \log^{2/\binom{r-2}{s-2}}((\log^{1/4} m)2^j) \geq 256\binom{r}{2} f_\epsilon(2^{i+2j})$.*

4. *For any $\alpha \geq \log^{-1} m$, $f_\epsilon(\alpha n) \geq f_\epsilon(n)/2$.*

We will refer to the above collectively as the facts about f_ϵ; we prove them in Section 3.12.

The proof will split into four cases.

Cases 1 and 2, Preliminary discussion: For these cases, a simple numerical claim will be useful.

Claim 3.8.4. *For positive reals a, b with $a \leq 1$, we have*

$$(1 + a)^b \geq 1 + ab/2.$$

Proof: Since $0 \leq a \leq 1$ we have $1 + a \geq e^{a/2}$. Then

$$(1 + a)^b \geq e^{ab/2} \geq 1 + ab/2.$$

$\square$

Cases 1 and 2 will be those cases in which V_1 is large.

Take $U_1 = V_1$. Take U_2 to be the union of the V_j such that the edges between V_1 and V_j are of color Q_1. Take U_3 to be the union of the V_j such that the edges between V_1 and V_j are of color Q_2. We may assume without loss of generality that $|U_2| \geq |U_3|$. Then $\{U_1, U_2, U_3\}$ is a partition of V.

Define, for $i = 1, 2, 3$, F_i to be the restriction of F to U_i. Take $g_{S,i} := g_{S,F_i}$. Define $n_i = |U_i|$ and $\alpha_i = n_i/n$. By the induction hypothesis, for each F_i there are appropriate choices of $f_i, \epsilon_i, \mathcal{P}_i$, and $w_{P,i}$. Take $a_i = |\mathcal{P}_i|$.

The general approach for these cases as well as for Case 3 will be to choose some index i and simply use the same graph to approximate our coloring. That is, we will take $\mathcal{P} = \mathcal{P}_i$ and $w_P = w_{P,i}$, and then we will show that ϵ and f may be chosen appropriately.

We now proceed: if, for some index i, we take $\mathcal{P} = \mathcal{P}_i$ and $w_P = w_{P,i}$, since $g_S \geq g_{S,i}$, we will have property (1): for every $S \in \binom{R}{s}$, $g_S \geq g_{S,i} \geq \prod_{P \subseteq S} w_{P,i} = \prod_{P \subseteq S} w_P$. Furthermore,

$$\prod_{P \in \binom{R}{2}} w_P = \prod_{P \in \binom{R}{2}} w_{P,i} \geq m^{-\epsilon_i} n_i = m^{-\epsilon_i} \alpha_i n = m^{-\epsilon_i - \log(1/\alpha_i)/\log m} n.$$

If we take $\epsilon = \epsilon_i + \frac{\log(1/\alpha_i)}{\log m}$, then the above shows that property (2) will hold.

Define

$$x_i := \max\left((\log m)^{C\left(\epsilon_i + \frac{\log(1/\alpha_i)}{\log m}\right)}, (c \log^2 m)^{a_i d} \right).$$

If i is the index minimizing x_i, then we will take:

$$\epsilon = \epsilon_i + \frac{\log(1/\alpha_i)}{\log m},$$

$$\mathcal{P} = \mathcal{P}_i,$$

$$w_P = w_{P,i},$$

and $f = x_i$; we will show that this satisfies properties (4) and (5), so choosing i to minimize x_i minimizes our f. Take $a = |\mathcal{P}|$. We have already observed that properties (1) and (2) will hold.

Take $\epsilon' = \max\left(\epsilon, \frac{\log((c \log^2 m)^{ad})}{C \log \log m} \right)$. Note that $f = x_i = (\log m)^{C\epsilon'} = f_{\epsilon'}(n)$. In this case properties (4) and (5) hold by the choice of f:

$$f \geq (\log m)^{C\epsilon'} \geq (\log m)^{C\epsilon}.$$

$$f \geq (\log m)^{C\epsilon'} \geq (\log m)^{C \frac{\log((c\log^2 m)^{ad})}{C \log\log m}} = 2^{\log((c\log^2 m)^{ad})} = (c \log^2 m)^{ad}.$$

We have only to show that, with this choice of f, property (3) holds. We claim

that each f_i satisfies $f_i \geq f_{\epsilon'}(n_i)$.

If for some i we have $\epsilon_i < \epsilon' + \frac{\log \alpha_i}{\log m}$, then we must have $(c \log^2 m)^{a_i d} \geq f = (\log m)^{C\epsilon'}$, for otherwise we would have $x_i < f$, contradicting our choice of ϵ. Therefore, for such an index i,

$$f_i \geq (c \log^2 m)^{a_i d} \geq (\log m)^{C\epsilon'} = f_{\epsilon'}(n) \geq f_{\epsilon'}(n_i).$$

Otherwise, $\epsilon_i \geq \epsilon' + \frac{\log \alpha_i}{\log m}$ so

$$f_i \geq (\log m)^{C\epsilon_i} \geq (\log m)^{C\left(\epsilon' + \frac{\log \alpha_i}{\log m}\right)} = f_{\epsilon'}(n_i).$$

We have, for each S satisfying $Q \subseteq S$, that $g_S \geq g_{S,1} + g_{S,2} + g_{S,3}$. For each S satisfying $Q_1 \in S$, we have

$$g_S \geq \max(g_{S,1} + g_{S,2}, g_{S,3}) \geq g_{S,1} + g_{S,2}.$$

Similarly, if $Q_2 \in S$ then $g_S \geq g_{S,1} + g_{S,3}$. Finally, for all S we have $g_S \geq \max_i g_{S,i}$.

We have by the generalization of Hölder's inequality (Lemma 3.7.1):

$$\prod_S g_S \geq \prod_{S:Q \subseteq S} \sum_i g_{S,i} \prod_{S:Q \not\subseteq S} g_S \geq \left(\sum_i \left(\prod_{S:Q \subseteq S} g_{S,i} \prod_{S:Q \not\subseteq S} g_S \right)^{1/\binom{r-2}{s-2}} \right)^{\binom{r-2}{s-2}}.$$

Therefore, we need only check that

$$\sum_i \left(\prod_{S:Q \subseteq S} g_{S,i} \prod_{S:Q \not\subseteq S} g_S \right)^{1/\binom{r-2}{s-2}} \geq nf.$$

Fix $T \in \binom{R}{s}$ so that $T \cap Q = \{Q_1\}$. We get $g_T \geq g_{T,1} + g_{T,2}$.

Case 1: $\alpha_1, \alpha_2 \geq \log^{-1/2} m$. The argument from the weak lower bound case applied

here only gives:

$$\sum_i \left(\prod_{S:Q\subseteq S} g_{S,i} \prod_{S:Q\not\subseteq S} g_S \right)^{1/\binom{r-2}{s-2}} \geq \sum_i n_i f_{\epsilon'}(n_i).$$

The main idea behind solving this case is to observe that it is sufficient to gain a constant factor on either the largest or second largest term of the above sum.

Consider:

$$\sum_i \left(\prod_{S:Q\subseteq S} g_{S,i} \prod_{S:Q\not\subseteq S} g_S \right)^{1/\binom{r-2}{s-2}}$$

$$\geq \left((g_{T,1} + g_{T,2}) \prod_{S\neq T} g_{S,1} \right)^{1/\binom{r-2}{s-2}} + \left((g_{T,1} + g_{T,2}) \prod_{S\neq T} g_{S,2} \right)^{1/\binom{r-2}{s-2}} + \left(\prod_S g_{S,3} \right)^{1/\binom{r-2}{s-2}}.$$

We will handle the case $g_{T,1} \leq g_{T,2}$; the case where $g_{T,1} \geq g_{T,2}$ has a symmetric argument. Then, since $g_{T,1} + g_{T,2} \geq 2g_{T,1}$, the previous is at least:

$$2^{1/\binom{r-2}{s-2}} \prod_S g_{S,1}^{1/\binom{r-2}{s-2}} + \prod_S g_{S,2}^{1/\binom{r-2}{s-2}} + \prod_S g_{S,3}^{1/\binom{r-2}{s-2}}$$

$$= \sum_i \prod_S g_{S,i}^{1/\binom{r-2}{s-2}} + \left(2^{1/\binom{r-2}{s-2}} - 1 \right) \prod_S g_{S,1}^{1/\binom{r-2}{s-2}}$$

$$\geq \sum_i n_i f_i + ((1+1)^{1/\binom{r-2}{s-2}} - 1)n_1 f_1$$

$$\geq \sum_i n_i f_{\epsilon'}(n_i) + \left(2\binom{r-2}{s-2} \right)^{-1} n_1 f_{\epsilon'}(n_1)$$

$$\geq \sum_i n_i f_{\epsilon'}(n_i) + \left(2\binom{r-2}{s-2} \right)^{-1} (\log^{-1/2} m) n f_{\epsilon'}((\log^{-1/2} m)n)$$

$$\geq \sum_i n_i f_{\epsilon'}(n_i) + \left(4\binom{r-2}{s-2} \right)^{-1} (\log^{-1/2} m) n f_{\epsilon'}(n)$$

$$\geq \sum_i n_i f_{\epsilon'}(n_i) + 3(\log^{-3/4} m) n f_{\epsilon'}(n) \geq n f_{\epsilon'}(n) = nf,$$

where the first follows from the induction hypothesis, the second follows from Claim 3.8.4, the third follows from the lower bound on α_1, the fourth follows from the fourth

fact about $f_{\epsilon'}$, the fifth follows from $m \geq m_0$, and the sixth follows from the second fact about $f_{\epsilon'}$.

Case 2: $\log^{-1/2} m \geq \alpha_2$. In this case we have $\alpha_1 = 1 - (\alpha_2 + \alpha_3) \geq 1 - 2\alpha_2 \geq 1 - 2\log^{-1/2} m \geq 3/4$.

Again, the argument from the weak lower bound only gives:

$$\sum_i \left(\prod_{S:Q\subseteq S} g_{S,i} \prod_{S:Q\nsubseteq S} g_S \right)^{1/\binom{r-2}{s-2}} \geq \sum_i n_i f_{\epsilon'}(n_i).$$

The main idea behind this case is to observe that it is sufficient to gain either a factor of $(1 + 8\alpha_2)$ on the first term (which is much larger than the others) or a factor of $4\alpha_2^{-1/\left(2\binom{r-2}{s-2}\right)}$ on the second term. We will do the former if $g_{T,2}/g_{T,1}$ is large enough, and otherwise we may accomplish the latter.

If $g_{T,2} \geq 16\binom{r-2}{s-2}\alpha_2 g_{T,1}$, we have

$$
\begin{aligned}
&\sum_i \left(\prod_{S:Q\subseteq S} g_{S,i} \prod_{S:Q\nsubseteq S} g_S \right)^{1/\binom{r-2}{s-2}} \\
&\geq \left(\prod_{S:Q\subseteq S} g_{S,1} \prod_{S:Q\nsubseteq S} g_S \right)^{1/\binom{r-2}{s-2}} \\
&\geq (g_{T,1} + g_{T,2})^{1/\binom{r-2}{s-2}} \prod_{S\neq T} g_{S,1}^{1/\binom{r-2}{s-2}} \geq \left(1 + 16\binom{r-2}{s-2}\alpha_2 \right)^{1/\binom{r-2}{s-2}} \prod_S g_{S,1}^{1/\binom{r-2}{s-2}} \\
&\geq \left(1 + \frac{16\binom{r-2}{s-2}\alpha_2}{2\binom{r-2}{s-2}} \right) n_1 f_1 = n_1 f_1 + 8\alpha_2 n_1 f_1,
\end{aligned}
$$

where the last inequality is by Claim 3.8.4.

We know $f_i \geq f_{\epsilon'}(n_i)$, so the above is at least:

$$
\begin{aligned}
n_1 f_{\epsilon'}(n_1) + 8\alpha_2 n_1 f_{\epsilon'}(n_1) &\geq \alpha_1^2 n f_{\epsilon'}(n) + 8\alpha_2 \alpha_1^2 n f_{\epsilon'}(n) \\
&\geq (1 - 2\alpha_2)^2 n f_{\epsilon'}(n) + 4\alpha_2 n f_{\epsilon'}(n) \\
&> n f_{\epsilon'}(n) = nf,
\end{aligned}
$$

where the first inequality follows from the first fact about $f_{\epsilon'}$ and the second inequality from substituting lower bounds on α_1.

Otherwise, we have $g_{T,2} \leq 16\binom{r-2}{s-2}\alpha_2 g_{T,1}$, so

$$\sum_i \left(\prod_{S:Q\subseteq S} g_{S,i} \prod_{S:Q\not\subseteq S} g_S \right)^{1/\binom{r-2}{s-2}} \geq \prod_S g_{S,1}^{1/\binom{r-2}{s-2}} + \left((g_{T,1} + g_{T,2}) \prod_{S\neq T} g_{S,2} \right)^{1/\binom{r-2}{s-2}}.$$

Then the latter term is at least:

$$\left(g_{T,1} \prod_{S\neq T} g_{S,2} \right)^{1/\binom{r-2}{s-2}} \geq \left(\frac{1}{16\binom{r-2}{s-2}\alpha_2} \prod_S g_{S,2} \right)^{1/\binom{r-2}{s-2}}$$

$$\geq \left(\frac{1}{16\binom{r-2}{s-2}\alpha_2} \right)^{1/\binom{r-2}{s-2}} n_2 f_2 \geq 4\alpha_2^{-1/\left(2\binom{r-2}{s-2}\right)} n_2 f_2$$

$$\geq 4\alpha_2^{-1/\left(2\binom{r-2}{s-2}\right)} n_2 f_{\epsilon'}(n_2) \geq 4\alpha_2^{-1/\left(2\binom{r-2}{s-2}\right)} n_2 \left(\alpha_2^{1/\left(2\binom{r-2}{s-2}\right)} f_{\epsilon'}(n) \right)$$

$$= 4n_2 f_{\epsilon'}(n),$$

where the third inequality follows from the upper bound on α_2 and from $m \geq m_0$ and the fifth inequality from the first fact about $f_{\epsilon'}$.

Therefore,

$$\sum_i \left(\prod_{S:Q\subseteq S} g_{S,i} \prod_{S:Q\not\subseteq S} g_S \right)^{1/\binom{r-2}{s-2}} \geq \prod_S g_{S,1}^{1/\binom{r-2}{s-2}} + 4n_2 f_{\epsilon'}(n) \geq n_1 f_{\epsilon'}(n_1) + 4n_2 f_{\epsilon'}(n)$$

$$\geq \alpha_1^2 n f_{\epsilon'}(n) + 4\alpha_2 n f_{\epsilon'}(n)$$

$$\geq (1 - 2\alpha_2)^2 n f_{\epsilon'}(n) + 4\alpha_2 n f_{\epsilon'}(n)$$

$$\geq n f_{\epsilon'}(n) = nf,$$

where the third inequality follows from the first fact about $f_{\epsilon'}$.

Cases 3 and 4, Preliminary discussion: These will be the cases in which none of the V_i are large. For these cases, we will take $n_i = |V_i|$ and $\alpha_i = n_i/n$. We will take

F_i to be F restricted to V_i and take $g_{S,i} = g_{S,F_i}$. By induction, for each F_i there are appropriate choices of $f_i, \epsilon_i, \mathcal{P}_i$, and $w_{P,i}$. Take $a_i = |\mathcal{P}_i|$.

Since in these cases we have many indices, we will be able to apply the weighted Ramsey theorem to appropriately selected subsets of them. The rest of the preliminary discussion for Cases 3 and 4 is based on doing so.

For each non-negative integer i let $I_i := [2^i, 2^{i+1}]$. Take $\Phi(i) = \{j : n_j \in I_i\}$. The $\Phi(i)$ form a dyadic partition of the indices which will eventually determine how the indices are clustered when we apply the weighted Ramsey theorem. Take $B'' := \{i \leq \log(nm^{-\delta})\}$.

Take $\tau = \lfloor \log(2(\log^{-1/2} m)n) \rfloor$, so $\max_i n_i' \leq (\log^{-1/2} m)n \leq 2^\tau \leq 2(\log^{-1/2} m)n$ and $\Phi(i)$ is empty for $i > \tau$.

For any pair T, T' satisfying $Q \cap T = \{Q_1\}$ and $Q \cap T' = \{Q_2\}$, take $g_i = \prod_{S \notin \{T,T'\}} g_{S,i}, o_i = g_{T,i}, p_i = g_{T',i}$ and $g = \prod_{S \notin \{T,T'\}} g_S, o = g_T, p = g_{T'}$. Take $G_{T,T'}$ to be the set of indices j with $op \geq o_j p_j \log^2((\log^{1/4} m)2^{(\tau-i)/2})/32$, where i is such that $j \in \Phi(i)$.

$G_{T,T'}$ is the collection of indices j for which $g_T g_{T'}$ is substantially larger than $g_{T,j} g_{T',j}$, where the meaning of "substantially larger" depends on the size of n_j. The following lemma states that almost all vertices are contained in some V_j as j varies over the indices of $G_{T,T'}$.

Claim 3.8.5. $\sum_{j \in G_{T,T'}} n_j \geq (1 - \delta_1)n$.

Proof: Take $\{V_i'\}_{i \leq t}$ to be a reordering of $\{V_i\}_{i \leq t}$ so that if $i \leq j$ then $g_{T,V_i'} g_{T',V_i'} \leq g_{T,V_j'} g_{T',V_j'}$. That is, the V_i' are in increasing order based on the value of $g_{T,V_i'} g_{T',V_i'}$. Let $g_j', o_j', p_j', \Phi(i)', n_j', G_{T,T'}'$ be defined as before (so, for example, $o_i' = g_{T,V_i'}$).

When we count, we wish to omit certain intervals that do not satisfy desired properties. Let

$$B := \{i \leq \tau : |\Phi(i)'| \leq 4\delta_1^{-1}(\log^{1/4} m)2^{(\tau-i)/2}\}.$$

We will show that a large fraction of the vertices are not contained in $\bigcup_{i \in B} \bigcup_{j \in \Phi(i)'} V_j'$;

that is, most vertices are not contained in V'_j where j is an index in $\phi(i)'$ for some $i \in B$.

$$\sum_{i \in B} \sum_{j \in \Phi(i)'} n'_j \leq \sum_{i \leq \tau} 2^{i+1}(4\delta_1^{-1}(\log^{1/4} m)2^{(\tau-i)/2}) = 8\delta_1^{-1}(\log^{1/4} m)2^{\tau/2} \sum_{i \leq \tau} 2^{i/2}$$

$$\leq 8\delta_1^{-1}(\log^{1/4} m)2^{\tau/2}4 \cdot 2^{\tau/2} = 32\delta_1^{-1}(\log^{1/4} m)2^{\tau} \leq 64\delta_1^{-1}(\log^{-1/4} m)n$$

$$\leq \delta_1 n/2.$$

Thus,

$$\sum_{i \in B} \sum_{j \in \Phi(i)'} n'_j \leq \delta_1 n/2. \tag{3.2}$$

For any fixed $i \leq \tau$ such that $i \notin B$, enumerate $\Phi(i)'$ as $\phi_{i,1}, \phi_{i,2}, \ldots, \phi_{i,|\Phi(i)'|}$ with $o'_{\phi_{i,j_1}} p'_{\phi_{i,j_1}} \leq o'_{\phi_{i,j_2}} p'_{\phi_{i,j_2}}$ if $j_1 \leq j_2$. That is, this enumeration is so that the $V'_{\phi_{i,j}}$ are listed in increasing order with respect to their $o'_j p'_j$ values. Take β_i to be $\phi_{i,(1-\delta_1/4)|\Phi(i)'|}$. Consider $\{(o'_j, p'_j) : j \in \Phi(i)', j \geq \beta_i\}$. By $i \notin B$, this has at least $(\log^{1/4} m)2^{(\tau-i)/2} \geq M$ elements. We get by applying the weighted Ramsey theorem to this set (with the coloring given by χ) that:

$$op \geq o'_{\beta_i} p'_{\beta_i} \frac{\log^2\left((\log^{1/4} m)2^{(\tau-i)/2}\right)}{32}.$$

For any $j \in \Phi(i)'$, we have that if $j \leq \beta_i$ then $o'_j p'_j \leq o'_{\beta_i} p'_{\beta_i}$, so the above is at least $o'_j p'_j \frac{\log^2\left((\log^{1/4} m)2^{(\tau-i)/2}\right)}{32}$, so $j \in G'_{T,T'}$.

If $i \notin B$ we obtain

$$\sum_{j \in \Phi(i)': j \notin G'_{T,T'}} n'_j \leq \sum_{j \in \Phi(i)': j > \beta_i} n'_j \leq \frac{\delta_1}{4}|\Phi(i)'|2^{i+1}$$

$$= |\Phi(i)'|2^i \delta_1/2 \leq \frac{\delta_1}{2} \sum_{j \in \Phi(i)'} n_j$$

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Therefore,

$$\sum_{j \in \Phi(i)' \cap G'_{T,T'}} n'_j \geq (1 - \delta_1/2) \sum_{j \in \Phi(i)'} n'_j.$$

Thus,

$$\sum_{j \in G'_{T,T'}} n'_j \geq \sum_{i \notin B} \sum_{j \in \Phi(i)' \cap G'_{T,T'}} n'_j \geq (1 - \delta_1/2) \sum_{i \notin B} \sum_{j \in \Phi(i)'} n'_j$$

$$\geq (1 - \delta_1/2)^2 \, n \geq (1 - \delta_1) n,$$

where the third inequality follows from (3.2).

As $\sum_{j \in G_{T,T'}} n'_j = \sum_{j \in G_{T,T'}} n_j$, this completes the proof of the claim.

$\square$

Case 3: $\alpha_1 \leq \log^{-1/2} m$ and $\sum_{i \in B''} \sum_{j \in \Phi(i)} n_j \leq n/2$. Fix any pair $T, T' \in \binom{R}{s}$ with $T \cap Q = Q_1$ and $T' \cap Q = Q_2$. The idea behind this case will be to choose some subset G_a of the indices so that, as j varies over G_a, the value of a_j does not change, to intersect this set with $G_{T,T'}$, and to use this to show that $\prod_S g_S$ is large. In this case, as in Cases 1 and 2, we will simply choose some j appropriately and take $\mathcal{P} = \mathcal{P}_j$ and $w_P = w_{P,j}$.

Note that $s_i \in \left[\binom{r}{2}\right]$, so by the pigeonhole principle there must be some value s such that

$$\sum_{i \notin B''} \sum_{j \in \Phi(i): s_i = s} n_j \geq \binom{r}{2}^{-1} \sum_{i \notin B''} \sum_{j \in \Phi(i)} n_j \geq \binom{r}{2}^{-1} n/2,$$

where the last inequality follows by the assumptions for Case 3. Take G_a to be the set of indices j with $a_j = a$ and, taking i to be the index with $j \in \Phi(i)$, i is not in B''; by the above, $\sum_{j \in G_a} n_j \geq \binom{r}{2}^{-1} n/2$. Then take $\epsilon = \min_{i \in G_a} \epsilon_i + \frac{\log(1/\alpha_i)}{\log m}$. Note that this is the same as taking $\epsilon = \epsilon_i + \frac{\log(1/\alpha_i)}{\log m}$ where i is an index in G_a minimizing

$$x_i := \max\left((\log m)^{C\left(\epsilon_i + \frac{\log(1/\alpha_i)}{\log m}\right)}, (c \log^2 m)^{a_i d} \right),$$

as all the a_i are equal to a.

Take, with i as above, $\mathcal{P} = \mathcal{P}_i$ and $w_P = w_{P,i}$. Then, as in Cases 1 and 2,

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properties (1) and (2) hold. Furthermore, as in Cases 1 and 2, taking

$$\epsilon' = \max\left(\epsilon, \frac{\log\left((c\log^2 m)^{ad}\right)}{C\log\log m}\right),$$

we have for $i \in G_a$ that $f_i \geq f_{\epsilon'}(n_i)$, and taking $f = f_{\epsilon'}(n)$, properties (4) and (5) hold. We need only check that property (3) holds.

Then take $G = G_a \cap G_{T,T'}$.

We have

$$\sum_{j\in G} n_j \geq n - \sum_{j\notin G_a} n_j - \sum_{j\notin G_{T,T'}} n_j \geq \left(\binom{r}{2}^{-1}/2 - \delta_1\right)n \geq \left(\binom{r}{2}^{-1}/4\right)n. \quad (3.3)$$

Take $g_i = \prod_{S\notin\{T,T'\}} g_{S,i}, o_i = g_{T,i}, p_i = g_{T',i}$ and $g = \prod_{S\notin\{T,T'\}} g_S, o = g_T, p = g_{T'}$.

We have:

$$\sum_j (g_j op)^{1/\binom{r-2}{s-2}} \geq \sum_i \sum_{j\in G\cap\Phi(i)} (g_j op)^{1/\binom{r-2}{s-2}}$$

$$\geq \sum_i \sum_{j\in G\cap\Phi(i)} \left(g_j o_j p_j \frac{\log^2((\log^{1/4} m)2^{(\tau-i)/2})}{32}\right)^{1/\binom{r-2}{s-2}}$$

$$\geq \sum_i \sum_{j\in G\cap\Phi(i)} n_j f_j \log^{2/\binom{r-2}{s-2}}((\log^{1/4} m)2^{(\tau-i)/2})32^{-1/\binom{r-2}{s-2}}$$

$$\geq \sum_i \sum_{j\in G\cap\Phi(i)} n_j f_{\epsilon'}(n_j) \log^{2/\binom{r-2}{s-2}}((\log^{1/4} m)2^{(\tau-i)/2})32^{-1/\binom{r-2}{s-2}}$$

$$\geq \sum_i \sum_{j\in G\cap\Phi(i)} n_j f_{\epsilon'}(2^i) \log^{2/\binom{r-2}{s-2}}((\log^{1/4} m)2^{(\tau-i)/2})32^{-1}$$

$$\geq \sum_i \sum_{j\in G\cap\Phi(i)} 8\binom{r}{2} n_j f_{\epsilon'}(2^\tau)$$

$$\geq \sum_i \sum_{j\in G\cap\Phi(i)} 8\binom{r}{2} n_j f_{\epsilon'}(n \log^{-1/2} m)$$

$$\geq \sum_i \sum_{j\in G\cap\Phi(i)} 4\binom{r}{2} n_j f_{\epsilon'}(n) \geq n f_{\epsilon'}(n) = nf,$$

where the second inequality follows from $G \subseteq G_{T,T'}$, the sixth inequality follows by

the third fact about $f_{\epsilon'}$, the eighth inequality follows by the fourth fact about $f_{\epsilon'}$, and the ninth inequality follows from (3.3).

Note that here we used only one pair T, T'; we can afford to do this because we gain a large amount due to B'' being small. In the next case, we will use all of the relevant pairs.

Case 4: $\alpha_1 \leq \log^{-1/2} m$ and $\sum_{i \in B''} \sum_{j \in \Phi(i)} n_j \geq n/2$. In this case there are many vertices contained in very small parts; this is the case where we will not simply take $\mathcal{P}$ to be some $\mathcal{P}_i$.

The idea behind this case is to choose a set G_a of many indices j with similar values of $\mathcal{P}_j, w_{P,j}$, and ϵ_j. We will be able to take $w_Q = \sum_{j \in G_a} w_{Q,j}$, which is a significant improvement over simply taking for some j each $w_P = w_{P,j}$. We will intersect G_a with some collection of $G_{T,T'}$ where each T with $T \cap Q = \{Q_1\}$ and each T' with $T' \cap Q = \{Q_2\}$ will occur exactly once (so we pair up the sets T, T' with $T \cap Q = \{Q_1\}$ and $T' \cap Q = \{Q_2\}$; if an index is in $G_{T,T'}$, then we have gained a large factor on that index). This allows us to lower bound $\prod_S g_s$.

We may partition $[0, 1]$ into at most $\delta_0^{-1} + 1$ intervals J_i of length at most δ_0. Furthermore, we may partition $[1, m]$ into at most $\delta_0^{-1} + 1$ intervals H_i with $\sup(H_i)/\inf(H_i) \leq m^{\delta_0}$.

We partition the indices in $\bigcup_{i \in B''} \Phi(i)$ by saying two indices j, j' are in the same part if and only if $\epsilon_j, \epsilon_{j'}$ are in the same interval J_i, $\mathcal{P}_j = \mathcal{P}_{j'}$, and for each $P \in \mathcal{P}_j = \mathcal{P}_{j'}$, $w_{P,j}$ and $w_{P,j'}$ are in the same interval H_i.

Then the total number of possible partitions is at most $2^{\binom{r}{2}}(\delta_0^{-1} + 1)^{\binom{r}{2}+1}$. Therefore, there is some part $G_a \subseteq \bigcup_{i \in B''} \Phi(i)$ where

$$\sum_{j \in G_a} n_j \geq 2^{-\binom{r}{2}}(\delta_0^{-1}+1)^{-\binom{r}{2}-1} \sum_{i \in B''} \sum_{j \in \Phi(i)} n_j \geq 2^{-\binom{r}{2}-1}(\delta_0^{-1}+1)^{-\binom{r}{2}-1} n = 2\binom{r-2}{s-1}\delta_1 n.$$

Then take $\epsilon_0 = \max_{i \in G_a} \epsilon_i$. Take $w_Q = \sum_{i \in G_a} w_{Q,i}$ and for $P \neq Q$ take $w_P = \min_{i \in G_a} w_{P,i}$. Take $\mathcal{P} = \mathcal{P}_i \cup \{Q\}$ for any $i \in G_a$. Take $a = |\mathcal{P}|$ and note $a \leq a_i + 1$ for any $i \in G_a$. We check that property (1) holds. For each S with $Q \subseteq S$,

$$g_S \geq \sum_i g_{S,i} \geq \sum_i \prod_{P \subseteq S} w_{P,i} \geq \sum_{i \in G_a} w_{Q,i} \prod_{P \subseteq S, P \neq Q} \min_{i \in G_a} w_{P,i} = w_Q \prod_{P \subseteq S, P \neq Q} w_P = \prod_{P \subseteq S} w_P.$$

For each S with $Q \nsubseteq S$, fixing any $i \in G_a$,

$$g_S \geq g_{S,i} \geq \prod_{P \subseteq S} w_{P,i} \geq \prod_{P \subseteq S} \min_{i \in G} w_{P,i} = \prod_{P \subseteq S} w_P.$$

Now, we choose $\epsilon = \binom{r}{2}\delta_0 + \epsilon_0$ and check that property (2) holds:

$$\prod_P w_P = \sum_{i \in G_a} w_{Q,i} \prod_{P \neq Q} w_P \geq \sum_{i \in G_a} w_{Q,i} \prod_{P \neq Q} (m^{-\delta_0} w_{P,i})$$
$$= m^{-(\binom{r}{2}-1)\delta_0} \sum_{i \in G_a} \prod_P w_{P,i} \geq m^{-(\binom{r}{2}-1)\delta_0 - \epsilon_0} \sum_{i \in G_a} n_i$$
$$\geq m^{-(\binom{r}{2}-1)\delta_0 - \epsilon_0} 2 \binom{r-2}{s-1} \delta_1 n \geq m^{-\binom{r}{2}\delta_0 - \epsilon_0} n,$$

where the first inequality is valid since for $j, j' \in G_s$ we have $w_{P,j}/w_{P,j'} \leq m^{\delta_0}$, the second inequality follows by choice of $\epsilon_0 = \max_{j \in G_a} \epsilon_j$, and the last inequality uses $m^{\delta_0} \geq 2\binom{r-2}{s-1}\delta_1$ which follows from $m \geq m_0$ (m_0 is defined in Equation 3.1) and the choices of δ_0 and δ_1.

Fix a bijection π between $\{S \in \binom{R}{s} : S \cap Q = \{Q_1\}\}$ and $\{S \in \binom{R}{s} : S \cap Q = \{Q_2\}\}$ (one such bijection takes any S in the first set and removes Q_1 and adds Q_2.) Take G to be the intersection of G_a and all sets of the form $G_{S,\pi(S)}$ where $S \in \binom{R}{s}$ satisfies $S \cap Q = \{Q_1\}$. There are $\binom{r-2}{s-1}$ pairs $S, \pi(S)$, so by Claim 3.2 we have

$$\sum_{j \in G} n_j \geq \sum_{j \in G_a} n_j - \sum_{S: S \cap Q = \{Q_1\}} \sum_{j \notin G_{S,\pi(S)}} n_j \geq \left(2\binom{r-2}{s-1}\delta_1 - \binom{r-2}{s-1}\delta_1 \right) n$$

$$\geq \binom{r-2}{s-1}\delta_1 n \geq \delta_1 n.$$

Note that, if $j \in G$, then for any S with $S \cap Q = \{Q_1\}$, since $G \subseteq G_{S,\pi(S)}$, we

have $g_S g_{\pi(S)} \geq g_{S,j} g_{\pi(S),j} \log^2((\log^{1/4} m) 2^{(\tau-i)/2}) \geq g_{S,j} g_{\pi(S),j} \log^2(2^{(\tau-i)/2})$ where i is such that $j \in \Phi(i)$. However, if $j \in G$ then $i \in B''$, so

$$2^{(\tau-i)/2} = \left(\frac{2^\tau}{2^i}\right)^{1/2}$$

$$\geq \left(\frac{n \log^{-1/2} m}{nm^{-\delta}}\right)^{1/2} \geq m^{\delta/4}.$$

Therefore, $\log(2^{(\tau-i)/2}) \geq \delta(\log m)/4$.

This gives:

$$\sum_j \left(\prod_{S:Q\subseteq S} g_{S,j} \prod_{S:Q\nsubseteq S} g_S\right)^{1/\binom{r-2}{s-2}}$$

$$\geq \sum_{j\in G} \left(\prod_{S:Q\subseteq S} g_{S,j} \prod_{S:Q\nsubseteq S} g_S\right)^{1/\binom{r-2}{s-2}}$$

$$\geq \sum_{j\in G} \left(\prod_{S:|Q\cap S|\neq 1} g_{S,j} \prod_{S:|Q\cap S|=1} g_S\right)^{1/\binom{r-2}{s-2}}$$

$$= \sum_{j\in G} \left(\prod_{S:|Q\cap S|\neq 1} g_{S,j} \prod_{S:Q\cap S=\{Q_1\}} g_S g_{\pi(S)}\right)^{1/\binom{r-2}{s-2}}$$

$$\geq \sum_{j\in G} \left(\prod_{S:|Q\cap S|\neq 1} g_{S,j} \prod_{S:Q\cap S=\{Q_1\}} \frac{\delta^2}{16}(\log^2 m) g_{S,j} g_{\pi(S),j}\right)^{1/\binom{r-2}{s-2}}$$

$$= \sum_{j\in G} \left((\delta(\log m)/16)^{2\binom{r-2}{s-1}} \prod_S g_{S,j}\right)^{1/\binom{r-2}{s-2}}$$

$$\geq \sum_{j\in G} n_j f_j \left(\delta(\log m)/4\right)^{2\binom{r-2}{s-1}/\binom{r-2}{s-2}}$$

$$= \sum_{j\in G} n_j f_j \left(\delta(\log m)/4\right)^{2d}.$$

Take $f' = (\log m)^{C\epsilon}$, $f'' = (c\log^2 m)^{ad}$ and $f = \max(f', f'')$. Note that $f \geq f'$ guarantees that property (4) holds and $f \geq f''$ guarantees that property (5) holds, so we need only check that property (3) holds. There will be two cases, that in which

$f = f'$ and that in which $f = f''$.

If $f = f'$, for each $j \in G_a$ we have $\epsilon_j \geq \epsilon_0 - \delta_0$, so $f_j \geq (\log m)^{C(\epsilon_0 - \delta_0)}$. Then we get:

$$
\begin{aligned}
\sum_{j \in G} n_j f_j \left(\delta(\log m)/4\right)^{2d} &\geq \sum_{j \in G} n_j (\log m)^{C(\epsilon_0 - \delta_0)} \left(\delta(\log m)/4\right)^{2d} \\
&\geq \sum_{j \in G} n_j (\log m)^{C(\epsilon_0 + \binom{r}{2}\delta_0)} \left(\delta(\log^{1/2} m)/4\right)^{2d} \\
&\geq \delta_1 n (\log m)^{C(\epsilon_0 + \binom{r}{2}\delta_0)} \left(\delta(\log^{1/2} m)/4\right)^{2d} \\
&\geq n(\log m)^{C\left(\epsilon_0 + \binom{r}{2}\delta_0\right)} = n(\log m)^{C\epsilon} = nf' = nf.
\end{aligned}
$$

Otherwise, $f = f''$. For each $j \in G_a$ we have $f_j \geq (c \log^2 m)^{(a-1)d}$, as $a \leq a_j + 1$. This gives:

$$
\begin{aligned}
\sum_{j \in G} n_j f_j \left(\delta(\log m)/4\right)^{2d} &\geq \sum_{j \in G} n_j (c \log^2 m)^{(a-1)d} \left(\delta(\log m)/4\right)^{2d} \\
&= (\delta/4)^{2d} c^{-d} (c \log^2 m)^{ad} \sum_{j \in G} n_j \geq (\delta/4)^{2d} c^{-d} (c \log^2 m)^{ad} \delta_1 n \\
&\geq n(c \log^2 m)^{ad} = nf''.
\end{aligned}
$$

$\square$

Take a_0 to be $\binom{r}{2}$ if $s < r - 1$, $r/2$ if $s = r - 1$ and r is even, and $(r+3)/2$ if $s = r - 1$ and r is odd. Take $f_0 = (c \log^2 m)^{a_0 d}$. The following theorem states that either some g_s is large or their product is large.

Theorem 3.8.6. *If $m \geq m_0$, either $\prod_S g_s \geq (mf_0)^{\binom{r-2}{s-2}}$ or there is some $S \subseteq R$ of size s with $g_s \geq (mf_0)^{\binom{s}{2}/\binom{r}{2}}$.*

Proof: Choose $f, \epsilon, \mathcal{P}, w_P$ as given by the previous theorem, then we need only show $f \geq f_0$. If $\epsilon \geq \left(16r\binom{r}{2}^2\right)^{-1}$ then we have $C\epsilon \geq 2\binom{r}{2}d \geq 2a_0 d$ so $f \geq (\log m)^{C\epsilon} \geq (\log m)^{2a_0 d} \geq f_0$.

Otherwise, $\epsilon < \left(16r\binom{r}{2}^2\right)^{-1}$. Define a weighted graph on vertex set R where an edge $e \in \binom{R}{2}$ has weight $\log w_e$. Note that this graph has non-negative edge weights and if an edge is not in $\mathcal{P}$, then it has weight 0.

By Lemma 3.8.1, either this graph has at least a_0 edges or there is some set S on s vertices with

$$\sum_{P \subseteq S} \log w_P \geq \left(1 + \left(4r\binom{r}{2}^2\right)^{-1}\right) \frac{\binom{s}{2}}{\binom{r}{2}} \sum_{P} \log w_P.$$

If the graph has at least a_0 edges, then $|\mathcal{P}| \geq a_0$ so $f \geq (c \log^2 m)^{ap_0 d}$, as desired. Otherwise, there is some S so that

$$\sum_{P \subseteq S} \log w_P \geq \left(1 + \left(4r\binom{r}{2}^2\right)^{-1}\right) \frac{\binom{s}{2}}{\binom{r}{2}} \sum_{P} \log w_P.$$

Then we have

$$\prod_{P \subseteq S} w_P \geq \prod_{P} w_P^{\left(1+\left(4r\binom{r}{2}^2\right)^{-1}\right)\frac{\binom{s}{2}}{\binom{r}{2}}} \geq m^{(1-\epsilon)\left(1+\left(4r\binom{r}{2}^2\right)^{-1}\right)\frac{\binom{s}{2}}{\binom{r}{2}}}$$

$$\geq m^{\left(1-\left(16r\binom{r}{2}^2\right)^{-1}\right)\left(1+\left(4r\binom{r}{2}^2\right)^{-1}\right)\frac{\binom{s}{2}}{\binom{r}{2}}} \geq m^{\left(1+\left(8r\binom{r}{2}^2\right)^{-1}\right)\frac{\binom{s}{2}}{\binom{r}{2}}} \geq (mf_0)^{\binom{s}{2}/\binom{r}{2}},$$

where the second to last inequality follows from $(1+b)(1-b/4) \geq 1 + b/2$ for any $b \in [0,1]$. $\square$

The previous theorem easily implies a general lower bound for the largest value of g_s.

Theorem 3.8.7. *If $m \geq m_0$, there is some $S \subseteq R$ of size s with $g_s \geq (mf_0)^{\binom{s}{2}/\binom{r}{2}}$.*

Proof: By the previous theorem, either such an S exists or $\prod_{S \subseteq R} g_s \geq (mf_0)^{\binom{r-2}{s-2}}$. In this latter case, since this is the product of $\binom{r}{s}$ numbers, there must be some S with $g_s \geq (mf_0)^{\binom{r-2}{s-2}/\binom{r}{s}} = (mf_0)^{\binom{s}{2}/\binom{r}{2}}$. $\square$

Before we proceed, note that:

$$d = \binom{r-2}{s-1} / \binom{r-2}{s-2} = \frac{r-s}{s-1}.$$

We now simply rewrite the statement of the previous theorem in more familiar notation.

Theorem 3.8.8. *Every Gallai coloring of a complete graph on m vertices has a vertex subset using at most s colors of order $\Omega\left(m^{\binom{s}{2}/\binom{r}{2}} \log^{c_{r,s}} m\right)$, where*

$$c_{r,s} = \begin{cases} s(r-s) & \text{if } s < r-1, \\ 1 & \text{if } s = r-1 \text{ and } r \text{ is even}, \\ (r+3)/r & \text{if } s = r-1 \text{ and } r \text{ is odd}. \end{cases}$$

Proof: If $s < r-1$ and $m \geq m_0$, by Theorem 3.8.7 the coloring has a subchromatic set of order at least $m^{\binom{s}{2}/\binom{r}{2}} \left(c\log^2 m\right)^{\binom{s}{2}d}$. As $2\binom{s}{2}d = s(s-1)\frac{r-s}{s-1} = s(r-s)$, this gives the desired bound in this case.

If $s = r-1$, r is even, and $m \geq m_0$, by Theorem 3.8.7, the coloring has a subchromatic set of order at least $m^{\binom{s}{2}/\binom{r}{2}} \left(c\log^2 m\right)^{(r/2)d}$. As

$$2(r/2)\binom{s}{2}\binom{r}{2}^{-1}d = r\frac{s(s-1)}{r(r-1)}\frac{r-s}{s-1} = r\frac{(r-1)(r-2)}{r(r-1)}\frac{1}{r-2} = 1,$$

this gives the desired bound in this case.

If $s = r-1$, r is odd, and $m \geq m_0$, by Theorem 3.8.7, the coloring has a subchromatic set of order at least $m^{\binom{s}{2}/\binom{r}{2}} \left(c\log^2 m\right)^{((r+3)/2)d}$. As

$$2((r+3)/2)\binom{s}{2}\binom{r}{2}^{-1}d = (r+3)\frac{s(s-1)}{r(r-1)}\frac{r-s}{s-1} = (r+3)\frac{(r-1)(r-2)}{r(r-1)}\frac{1}{r-2} = (r+3)/r,$$

this gives the desired bound in this case. $\qquad \square$

3.9   Proof of Lemma 3.4.1

Lemma 3.9.1. *For all $k, \ell \geq 1$ we have $\binom{k+\ell}{\ell} \leq 2^{2\sqrt{k\ell}}$*

Proof: We first observe that $\binom{k+\ell}{\ell} \le \frac{(k+\ell)^{k+\ell}}{k^k \ell^\ell}$, or equivalently that $\frac{k^k}{k!} \frac{\ell^\ell}{\ell!} \le \frac{(k+\ell)^{k+\ell}}{(k+\ell)!}$. To see this, note that the function $f(n) = n \log n - \log(n!)$ is super-additive (over the positive integers), that is it satisfies $f(n_1) + f(n_2) \le f(n_1 + n_2)$. This follows from using Stirling's formula, which is equivalent to $f(n) = n \log e - \frac{1}{2} \log n - \frac{1}{2} \log 2\pi + o(1)$, and known estimates on the $o(1)$ term as well as the sub-additive nature of the logarithm function. Taking $n_1 = k, n_2 = \ell$ gives

$$(k \log k - \log(k!)) + (\ell \log \ell - \log(\ell!)) \le (k + \ell) \log(k + \ell) - \log((k + \ell)!).$$

Exponentiating both sides gives $\frac{k^k}{k!} \frac{\ell^\ell}{\ell!} \le \frac{(k+\ell)^{k+\ell}}{(k+\ell)!}$, as desired.

Therefore, it now suffices to show that $\frac{(k+\ell)^{k+\ell}}{k^k \ell^\ell} \le 2^{2\sqrt{k\ell}}$. This is equivalent to showing that, for all $k, \ell \ge 1$,

$$\left(\frac{k+\ell}{k}\right)^{k/\sqrt{k\ell}} \left(\frac{k+\ell}{\ell}\right)^{\ell/\sqrt{k\ell}} \le 4.$$

Taking $x = k/\ell$, note that $x > 0$ and that we may rewrite the left hand side of the above as

$$g(x) := \left(1 + x^{-1}\right)^{\sqrt{x}} (1 + x)^{\sqrt{x^{-1}}}.$$

Note that $g(x) = g(1/x)$, so it now suffices to show that, for all $x \ge 1$, we have $g(x) \le 4$. Note that $g'(1) = 0$. We claim that, for $x \in [1, 1.5]$ we have $g''(x) < 0$ and for $x \in [1.5, 5]$ we have $g'(x) < 0$. These claims may be verified numerically, as the inequalities are strict and the relevant functions are uniformly continuous over their respective compact sets. This gives that, in the interval $[1, 5]$, $g(x)$ is maximized at $g(1) = 4$. Then, for $x > 5$, since $1 + x^{-1} \le e^{x^{-1}}$ we have $(1 + x^{-1})^{\sqrt{x}} \le e^{\sqrt{x^{-1}}} < 1.6$ and, in this range, $1 + x \le 2.5^{\sqrt{x}}$ so we have $(1 + x)^{\sqrt{x^{-1}}} \le 2.5$, giving that $g(x) \le 2.5 \cdot 1.6 = 4$ for $x > 5$. $\qquad \square$

3.10 Proof of Lemma 3.5.3

For convenience, we restate both the definition of f and the statement of the lemma here:

$$f(n) := \begin{cases} c\log^2(Cn) & \text{if } 0 < n \leq m^{4/9} \\ c^2\log^2(m^{4/9})\log^2(Cnm^{-4/9}) & \text{if } m^{4/9} < n \leq m^{8/9} \\ c^3\log^4(m^{4/9})\log^2(Cnm^{-8/9}) & \text{if } m^{8/9} < n \leq m, \end{cases}$$

where $D = 2^{2048}$, $C = 2^{D^8}$, and $c = \log^{-2}(C^2) = D^{-16}/4$.

Lemma 3.10.1. *If $m \geq C$, then the following statements hold about f for any integer n with $1 < n \leq m$.*

1. *For any $\alpha \in \left[\frac{1}{n}, 1\right]$, we have $f(\alpha n) \geq \alpha f(n)$.*

2. *For any $\alpha_1, \alpha_2, \alpha_3 \in \left[\frac{1}{n}, 1\right]$ such that $\sum_i \alpha_i = 1$ we have, taking $n_i = \alpha_i n$,*

$$nf(n) - \sum_i n_i f(n_i) \leq \frac{8}{\log C} nf(n).$$

3. *For $i \geq 0$ and $m^{7/18} \geq 2^j \geq 1$, we have $f(2^i)\log^2(D2^j) \geq 512f(2^{i+\frac{8}{7}j})$.*

4. *For $1 \leq \tau \leq n \leq D^3\tau$, we have $f(\tau) \geq f(n)/2$.*

5. *For any $\alpha \in \left[\frac{1}{n}, \frac{1}{32}\right]$, we have $f(\alpha n) \geq 16\alpha f(n)$.*

Proof: Observe that $f(n)$ has two points of discontinuity: $p_0 = m^{4/9}$ and $p_1 = m^{8/9}$. Recall that the three intervals of f are $(0, p_0], (p_0, p_1], (p_1, m]$; name these I_0, I_1, I_2, respectively.

If t is either p_0 or p_1, then we have $f_+(t) := \lim_{n \to t^+} f(n) \leq \lim_{n \to t^-} f(n) =: f_-(t)$.

Observe further that, if n is in some interval I of f, then for any $t \in I$ we have $f(t) = \gamma\log^2(\delta t)$ for constants γ, δ with $\delta t \geq C$.

Proof of Fact 1: We first argue that it is sufficient to show Fact 1 in the case that both n and αn are in the same interval of f. Intuitively, the points of discontinuity

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only help us. If n is in I_1, αn is in I_0, and we have shown that Fact 1 holds when n and αn are in the same interval, then

$$f(\alpha n) \geq \frac{\alpha n}{p_0} f_+(p_0) \geq \frac{\alpha n}{p_0} f_-(p_0) \geq \frac{\alpha n}{p_0} \frac{p_0}{n} f(n) = \alpha f(n).$$

The case where n is in I_2 and αn is in I_1 and the case where n is in I_2 and αn is in I_0 hold by essentially the same argument.

We next show Fact 1 in the case that both n and αn are in the same interval I of f. We have, choosing γ and δ to be such that $f(t) = \gamma \log^2(\delta t)$ on I, that

$$f(\alpha n) = \gamma \log^2(\alpha \delta n),$$
$$\alpha f(n) = \gamma \alpha \log^2(\delta n) = \gamma(\sqrt{\alpha} \log(\delta n))^2.$$

Thus, it is sufficient to show that $\log(\alpha \delta n) - \sqrt{\alpha} \log(\delta n) \geq 0$. Note that equality holds if $\alpha = 1$. We consider the first derivative with respect to α; we will show that it is negative for $\alpha \geq \frac{4}{\ln^2(\delta n)}$. The first derivative is:

$$\frac{1}{\alpha \ln(2)} - \frac{1}{2\sqrt{\alpha}} \log(\delta n) = \frac{2 - \sqrt{\alpha} \ln(\delta n)}{\alpha \ln(4)}.$$

Note that the above is negative if $2 - \sqrt{\alpha} \ln(\delta n) \leq 0$, which is equivalent to $\alpha \geq \frac{4}{\ln^2(\delta n)}$.

Therefore, for $\alpha \in \left[\frac{4}{\ln^2(\delta n)}, 1\right]$, assuming $\alpha n \in I$, we have $f(\alpha n) \geq \alpha f(n)$. If $\alpha < \frac{4}{\ln^2(\delta n)}$ with $\alpha n \in I$ then,

$$\alpha f(n) < \frac{4}{\ln^2(\delta n)} \gamma \log^2(\delta n) = \frac{4}{\ln^2(2)} \gamma \leq \log^2 C\gamma \leq \gamma \log^2(\delta \alpha n) = f(\alpha n),$$

where the first inequality follows by the assumed upper bound on α and the last one by αn in I (and so $\delta \alpha n \geq C$).

Proof of Fact 2: Let γ, δ be such that for t in the interval I_j containing n we have $f(t) = \gamma \log^2(\delta t)$. We define a new function f_2 whose domain is $[n \log^{-1} C, n]$. For any t in the domain of f_2 that is in I_j, we define $f_2(t) = f(t)$, and for any t in the domain of

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f_2 that is not in I_j, we define $f_2(t) = \gamma \log^2 C$. If there is some point t in $[n \log^{-1} C, n]$ that is not in I_j, then t must be in I_{j-1}, as $t \geq n \log^{-1} C \geq p_{j-1} \log^{-1} C > p_{j-2}$. Then note that we have chosen, for t not in I_j, $f_2(t) = f_+(p_{j-1})$. Therefore, f_2 is continuous. Also, $tf_2(t)$ is convex (this is easy to see by looking at the first derivative). The main idea behind the proof will be to replace f by f_2 and then apply convexity to get the bounds.

We claim $f(t) \geq f_2(t)$ for all t in the domain of f_2. If t is in I_j, then $f_2(t) = f(t)$. Otherwise, t is in I_{j-1}. For any $t \in [n \log^{-1} C, n]$, note that $\delta t \geq \delta n \log^{-1} C \geq C \log^{-1} C$. Therefore,

$$f(t) = \frac{\gamma}{c \log^2(m^{4/9})} \log^2(\delta t m^{4/9}) \geq \frac{\gamma}{c \log^2(m^{4/9})} \log^2\left(m^{4/9} C \log^{-1} C\right)$$

$$\geq \frac{\gamma}{c} \geq \gamma \log^2 C = f_2(t),$$

where the first equality follows by $t \in I_{j-1}$ and the first inequality by $\delta t \geq \log^{-1} C$.

Take $S = \{i : \alpha_i \geq \log^{-1} C\}$. For $i \in S$ we have that $\alpha_i n$ is in the domain of f_2. Take κ such that $\sum_{i \in S} \alpha_i = \kappa$. Since $\sum_i \alpha_i = 1$, $\kappa = 1 - \sum_{i \notin S} \alpha_i \geq 1 - 3 \log^{-1} C$. Hence,

$$\sum_i n_i f(n_i) \geq \sum_{i \in S} n_i f(n_i) \geq \sum_{i \in S} n_i f_2(n_i) \geq \sum_{i \in S} \frac{\kappa}{|S|} n f_2\left(\frac{\kappa}{|S|} n\right)$$

$$= \kappa n f_2\left(\frac{\kappa}{|S|} n\right) \geq \kappa n f_2\left(\frac{1 - 3 \log^{-1} C}{3} n\right) \geq \kappa n f_2(n/4) \geq \kappa \gamma n \log^2(\delta n/4),$$

where the third inequality follows Jensen's inequality applied to the convex function $tf_2(t)$ and the fourth inequality holds since f_2 is an increasing function.

This gives

$$\kappa n f(n) - \sum_{i \in S} n_i f(n_i) \leq \kappa \gamma n \log^2(\delta n) - \kappa \gamma n \log^2(\delta n/4) = \kappa \gamma n (\log^2(\delta n) - \log^2(\delta n/4)).$$

We now consider

$$\log^2(\delta n) - \log^2(\delta n/4) = (\log(\delta n) + \log(\delta n/4))(\log(\delta n) - \log(\delta n/4))$$

$$\leq 2\log(\delta n)\log 4 = 4\log(\delta n).$$

Noting that $\log(\delta n) \geq \log C$, we get

$$\kappa \gamma n (4\log(\delta n)) \leq \frac{4}{\log C}\gamma n \log^2(\delta n) = \frac{4}{\log C}\gamma n f(n).$$

Thus,

$$nf(n) - \sum_i n_i f(n_i) \leq (1-\kappa)nf(n) + \kappa nf(n) - \sum_{i\in S} n_i f(n_i)$$
$$\leq \frac{1}{2\log C}nf(n) + \frac{4}{\log C}nf(n) \leq \frac{8}{\log C}nf(n).$$

Proof of Fact 3: Take γ, δ such that for t in the interval of f containing 2^i we have $f(t) = \gamma \log^2(\delta t)$. Take $j' = \frac{8}{7}j$. If 2^i and $2^{i+j'}$ are in the same intervals of f, then we get

$$f(2^i)\log^2(D2^j) = \gamma \log^2(\delta 2^i)\log^2(D2^j) = \gamma \log^2(2^{i+\log \delta})\log^2(2^{j+\log D})$$
$$= \gamma \log^2(2^{(j+\log D)(i+\log \delta)}) \geq 512\gamma \log^2(\delta 2^{i+j'}) = 512f(2^{i+\frac{8}{7}j}),$$

where the last inequality follows from $i + \log \delta \geq \log C \geq \log D$ and $j + \log D \geq \log D$. Therefore,

$$(j+\log D)(i+\log \delta) \geq \frac{\log D}{2}(j+i+\log \delta) \geq \frac{\log D}{4}(2j+i+\log \delta) \geq 512(j'+i+\log \delta).$$

If 2^i and $2^{i+j'}$ are in different intervals of f, then they are in adjacent intervals

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since $2^{j'} \le m^{4/9}$. Therefore,

$$f(2^{i+\frac{8}{7}j}) = f(2^{i+j'}) = c\gamma \log^2(m^{4/9}) \log^2(\delta m^{-4/9} 2^{i+j'}) \le c\gamma \log^2(\delta 2^i) \log^2(2^{j'})$$

$$\le 2c\left(\gamma \log^2(\delta 2^i)\right) \log^2(2^j) \le \frac{1}{512} f(2^i) \log^2(2^j) \le \frac{1}{512} f(2^i) \log^2(D2^j),$$

where the first inequality follows by the fact that if $a_0 \ge a_1 \ge b_1 \ge b_0 \ge 2$ and if $a_0 b_0 = a_1 b_1$ then $(\log a_0)(\log b_0) \le (\log a_1)(\log b_1)$. To see this last fact about logarithms, one may take the logarithm of both sides and apply the concavity of the logarithm function.

Proof of Fact 4: Choose γ, δ such that for t in the same interval I_j as τ we have $f(t) = \gamma \log^2(\delta t)$. If n and τ are in different intervals of f, then n must be in I_{j+1} as $n/\tau \le D^3 < m^{4/9}$. Furthermore, for n and τ to be in different intervals, we must have $D^3 \delta \tau \ge C m^{4/9}$ and so $\delta \tau \ge m^{4/9}$. This gives:

$$f(\tau) = \gamma \log^2(\delta \tau) \ge \gamma \log^2(m^{4/9}) \ge c \log^2(m^{4/9})\gamma \log^2(CD^3)$$

$$\ge c \log^2(m^{4/9})\gamma \log^2(\delta m^{-4/9} n) = f(n),$$

where the second inequality follows by $CD^3 \le C^2$ and $c = \log^{-2}(C^2)$.

Otherwise, τ and n are in the same interval. Then

$$f(n) = \gamma \log^2(\delta n) \le \gamma \log^2(\delta D^3 \tau) \le \gamma \log^2((\delta \tau)^{4/3}) \le 2\gamma \log^2(\delta \tau) = 2f(\tau),$$

where the second inequality follows by $\delta \tau \ge C$ and $C^{1/3} \ge D$.

Proof of Fact 5: Let $\alpha \le \frac{1}{32}$ be given. Consider:

$$16\alpha f(n) = \frac{1}{2}(32\alpha f(n)) \le \frac{1}{2} f(32\alpha n) \le f(\alpha n),$$

where the first inequality is by the first fact about f and the second inequality is by the fourth fact about f.

$\square$

3.11   Proof of Lemma 3.8.1

We argue that if a weighted graph on r vertices deviates in structure from the complete graph with edges of equal weight and if $s < r - 1$, then there is some set of vertices S of size s so that the sum of the weights of the edges contained in S is substantially larger than average.

Lemma 3.11.1. *Given weights w_P for $P \in \binom{R}{2}$ with $w_P \geq 0$, take $w = \sum_P w_P$. Then if $s < r - 1$, if some w_P differs from $\binom{r}{2}^{-1} w$ by at least F, then there is some $S \subseteq R$ of size s satisfying*

$$\sum_{P \subseteq S} w_P \geq \frac{\binom{s}{2}}{\binom{r}{2}} w + \frac{\binom{s}{2} F}{r \binom{r}{2}^2}.$$

Proof: We will directly handle the case $s = r - 2$, from which the other cases will follow. We are interested in finding an $S \subseteq R$ of size $r - 2$ with a large value for the total weight of edges in S. For each S we give this value a name: $Z_S = \sum_{P \subseteq S} w_P$. Note that Z_S is closely related the following: for $Q \in \binom{R}{2}$, define Y_Q to be the weight of edges incident to at least one vertex of Q: $Y_Q = \sum_{P \in \binom{R}{2} : P \cap Q \neq \emptyset} w_P$. Then, if we take S to be $R \setminus Q$, we have $Y_Q + Z_S$ is the total weight of all the edges. Thus, to show that there is a large Z_S, it is sufficient to show that there is a small Y_Q. Towards this end, choose $Q \in \binom{R}{2}$ uniformly at random; we will now compute the variance of Y_Q.

Take, for $P \in \binom{R}{2}$, X_P to be w_P if $P \cap Q \neq \emptyset$ and 0 otherwise. Then, taking $w = \sum_P w_P$, we get

$$\mathbb{E}[X_P] = \mathbf{Pr}[P \cap Q \neq \emptyset] w_P = \frac{2r - 3}{\binom{r}{2}} w_P.$$

By linearity of expectation, we have

$$\mathbb{E}[Y_Q] = \sum_{P \in \binom{R}{2}} \mathbb{E}[X_P] = \frac{2r - 3}{\binom{r}{2}} \sum_{P \in \binom{R}{2}} w_P = \frac{2r - 3}{\binom{r}{2}} w,$$

and

$$\mathbb{E}\left[Y_Q^2\right] = \sum_{P \in \binom{R}{2}} \sum_{P' \in \binom{R}{2}} \mathbb{E}\left[X_P X_{P'}\right]$$
$$= \sum_{P \in \binom{R}{2}} \mathbb{E}\left[X_P^2\right] + \sum_{v \in R} \sum_{(P,P') \in \binom{R}{2}^2 : v \in P, v \in P', P \neq P'} \mathbb{E}\left[X_P X_{P'}\right]$$
$$+ \sum_{(P,P') \in \binom{R}{2}^2 : P \cap P' = \emptyset} \mathbb{E}\left[X_P X_{P'}\right],$$

where the last equality follows by partitioning the pairs P, P' into those which are equal, those which are distinct but intersect in some vertex v, and those which are disjoint.

We now look at these terms individually.

$$\mathbb{E}\left[X_P^2\right] = \mathbf{Pr}[P \cap Q \neq \emptyset]w_P^2 = \frac{2r-3}{\binom{r}{2}}w_P^2.$$

For $P = \{v, u\}, P' = \{v, u'\}$ distinct and intersecting, the event $P \cap Q \neq \emptyset$ and $P' \cap Q \neq \emptyset$ can occur if either $v \in Q$ or $Q = \{u, u'\}$; the first of these has probability $\frac{r-1}{\binom{r}{2}}$ and the second has probability $\frac{1}{\binom{r}{2}}$, and they are disjoint events. So, if P and P' intersect in a vertex we get:

$$\mathbb{E}\left[X_P X_{P'}\right] = \frac{r}{\binom{r}{2}}w_P w_{P'}.$$

If P, P' are disjoint, then for $X_P X_{P'}$ to be non-zero we must have that Q has an element from P and an element from P', which occurs with probability $\frac{4}{\binom{r}{2}}$, so in this case:

$$\mathbb{E}\left[X_P X_{P'}\right] = \frac{4}{\binom{r}{2}}w_P w_{P'}.$$

Therefore, taking for $v \in R$ the (weighted) degree $d(v)$ to be $\sum_{P \in \binom{R}{2} : v \in P} w_P$, $\mathbb{E}\left[Y_Q^2\right]$ is equal:

$$\sum_{P\in\binom{R}{2}}\frac{2r-3}{\binom{r}{2}}w_P^2 + \sum_{v\in R}\sum_{(P,P')\in\binom{R}{2}^2:v\in P,v\in P',P\neq P'}\frac{r}{\binom{r}{2}}w_Pw_{P'} + \sum_{(P,P')\in\binom{R}{2}:P\cap P'=\emptyset}\frac{4}{\binom{r}{2}}w_Pw_{P'}$$

$$=\frac{2r-3}{\binom{r}{2}}\sum_{P\in\binom{R}{2}}w_P^2 + \frac{r}{\binom{r}{2}}\sum_{v\in R}\left(d(v)^2 - \sum_{P\in\binom{R}{2}:v\in P}w_P^2\right)$$

$$+\frac{4}{\binom{r}{2}}\left(\left(\sum_{P\in\binom{R}{2}}w_P\right)^2 - \sum_{v\in R}d(v)^2 + \sum_{P\in\binom{R}{2}}w_P^2\right)$$

$$=\frac{2r-3}{\binom{r}{2}}\sum_{P\in\binom{R}{2}}w_P^2 + \frac{r}{\binom{r}{2}}\sum_{v\in R}d(v)^2 - 2\frac{r}{\binom{r}{2}}\sum_{P\in\binom{R}{2}}w_P^2 + \frac{4}{\binom{r}{2}}w^2 - \frac{4}{\binom{r}{2}}\sum_{v\in R}d(v)^2 + \frac{4}{\binom{r}{2}}\sum_{P\in\binom{R}{2}}w_P^2$$

$$=\frac{4}{\binom{r}{2}}w^2 + \frac{1}{\binom{r}{2}}\sum_{P\in\binom{R}{2}}w_P^2 + \frac{r-4}{\binom{r}{2}}\sum_{v\in R}d(v)^2$$

Note that $\sum_{v\in R}d(v)^2$ is minimized subject to the constraint $\sum_{v\in R}d(v) = 2w$ when the $d(v)$ are pairwise equal by the Cauchy-Schwarz inequality, so $\sum_{v\in R}d(v)^2 \geq \sum_{v\in R}\left(\frac{2w}{r}\right)^2$, so the above is at least

$$\frac{4}{\binom{r}{2}}w^2 + \frac{1}{\binom{r}{2}}\sum_{P\in\binom{R}{2}}w_P^2 + \frac{r-4}{\binom{r}{2}}\sum_{v\in R}\left(\frac{2w}{r}\right)^2$$

$$=\frac{4}{\binom{r}{2}}w^2 + \frac{1}{\binom{r}{2}}\sum_{P\in\binom{R}{2}}w_P^2 + r\frac{r-4}{\binom{r}{2}}\left(\frac{2w}{r}\right)^2$$

$$=\frac{8r-16}{r\binom{r}{2}}w^2 + \frac{1}{\binom{r}{2}}\sum_{P\in\binom{R}{2}}w_P^2.$$

The variance of Y_Q satisfies:

$$\mathrm{Var}\,(Y_Q) = \mathbb{E}\left[Y_Q^2\right] - \mathbb{E}\left[Y_Q\right]^2 \geq \frac{8r-16}{r\binom{r}{2}}w^2 + \frac{1}{\binom{r}{2}}\sum_{P\in\binom{R}{2}}w_P^2 - \left(\frac{2r-3}{\binom{r}{2}}w\right)^2$$

$$= \frac{1}{\binom{r}{2}} \sum_{P \in \binom{R}{2}} w_P^2 - \frac{1}{\binom{r}{2}^2} w^2.$$

Note we may rewrite the variance as:

$$\mathrm{Var}\,(Y_Q) \geq \frac{1}{\binom{r}{2}} \sum_{P \in \binom{R}{2}} w_P^2 - \frac{1}{\binom{r}{2}^2} w^2 = \mathrm{Var}\,(w_Q).$$

If some w_P is far from $w/\binom{r}{2} = \mathbb{E}\,[w_Q]$, then the variance will be large. Assume that for some P' there is a non-zero real F so that $w_{P'} = w/\binom{r}{2} + F$. Note $\mathrm{Var}\,(Y_Q) \geq \mathrm{Var}\,(w_Q) = \mathbb{E}\,\left[(w_Q - w/\binom{r}{2})\right]$ and that $(w_Q - w/\binom{r}{2})^2$ is a non-negative random variable. If $Q = P'$ (which occurs with probability $\binom{r}{2}^{-1}$), then this random variable has value F^2, so its expectation is at least $\binom{r}{2}^{-1} F^2$. That is, the variance of w_Q is at least $\binom{r}{2}^{-1} F^2$.

Thus, there must be some Q so that

$$\left| Y_Q - \frac{2r-3}{\binom{r}{2}} w \right| \geq \binom{r}{2}^{-1/2} F \geq F/r.$$

If $Y_Q - \frac{(2r-3)w}{\binom{r}{2}} \geq F/r$, since there are $\binom{r}{2}$ different Y_Q and the average is $\frac{(2r-3)w}{\binom{r}{2}}$, there must be some Q' so that

$$Y_{Q'} - \frac{2r-3}{\binom{r}{2}} w \leq \frac{-F}{r\left(\binom{r}{2} - 1\right)} \leq -\left(r\binom{r}{2}\right)^{-1} F.$$

The other case is that

$$Y_Q - \frac{2r-3}{\binom{r}{2}} w \leq -F/r \leq -\left(r\binom{r}{2}\right)^{-1} F.$$

Therefore, there is some Q with $Y_Q \leq \frac{(2r-3)w - F/r}{\binom{r}{2}}$.

Define $S = R \setminus Q$. We get $Z_S + Y_Q = w$ so $Z_S = w - Y_Q$. By the above, there is some S with

$$Z_S \geq w - \frac{(2r-3)w - F/r}{\binom{r}{2}} = \frac{\binom{r-2}{2} w + F/r}{\binom{r}{2}}.$$

Taking S as above, choosing a random $S' \in \binom{S}{s}$, we get that $\mathbb{E}\left[Z_{S'}\right] \geq \frac{\binom{s}{2}}{\binom{r-2}{2}} \frac{\binom{r-2}{2}w + F/r}{\binom{r}{2}}$.
Therefore, there must be some $S' \in \binom{S}{s}$ with

$$Z_{S'} \geq \frac{\binom{s}{2}}{\binom{r}{2}} w + \frac{\binom{s}{2} F}{r \binom{r}{2}\binom{r-2}{2}} \geq \frac{\binom{s}{2}}{\binom{r}{2}} w + \frac{\binom{s}{2} F}{r \binom{r}{2}^2}.$$

$\square$

The case where $s < r - 1$ in Lemma 3.8.1 is an immediate corollary.

Lemma 3.11.2. *Given weights w_P for $P \in \binom{R}{2}$ with $w_P \geq 0$, take $w = \sum_P w_P$. If $s < r - 1$, then either there are at least $\binom{r}{2}$ pairs P (i.e. all of them) with $w_P > 0$ or there is some $S \subseteq R$ of size s satisfying*

$$\sum_{P \subseteq S} w_P \geq \left(1 + \left(4r\binom{r}{2}^2\right)^{-1}\right) \frac{\binom{s}{2}}{\binom{r}{2}} w.$$

Proof: Assume there is some P' with $w_{P'} = 0$. We may apply the previous lemma with $F = w/\binom{r}{2}$, since $w_{P'}$ differs from $w/\binom{r}{2}$ by F. This gives that there is some set $S \subseteq R$ of size s satisfying:

$$\sum_{P \subseteq S} w_P \geq \left(\frac{\binom{s}{2}}{\binom{r}{2}} + \frac{\binom{s}{2}}{r \binom{r}{2}^3}\right) w \geq \left(1 + \left(4r\binom{r}{2}^2\right)^{-1}\right) \frac{\binom{s}{2}}{\binom{r}{2}} w.$$

$\square$

The following lemma states that if in a weighted graph there is a vertex whose degree deviates from the average, then there is a set $S \subseteq R$ of size $r - 1$ so that the sum of the weights of the edges contained in S is substantially larger than average.

Lemma 3.11.3. *Given weights w_P for $P \in \binom{R}{2}$ with $w_P \geq 0$, take $w = \sum_P w_P$. For $v \in R$, define $d(v) := \sum_{P:v \in P} w_P$. If there is some $v \in R$ for which $d(v)$ differs from $2w/r$ by at least F, then there is some $S \subseteq R$ of size $r - 1$ with*

$$\sum_{P \subseteq S} w_P \geq \frac{\binom{r-1}{2}}{\binom{r}{2}} w + F/r.$$

Proof: Choose a vertex v for which $|d(v) - 2w/r| \geq F$. If $d(v) \leq 2w/r - F$, then we may take $S = V \setminus \{v\}$. This gives:

$$\sum_{P \subseteq S} w_P = w - d(v) \geq w - (2w/r - F) = \frac{\binom{r-1}{2}}{\binom{r}{2}} w + F \geq \frac{\binom{r-1}{2}}{\binom{r}{2}} w + F/r.$$

Otherwise, we have $d(v) \geq 2w/r + F$. Since $\sum_u d(u) = 2w$,

$$\sum_{u \neq v} d(u) \leq 2w - (2w/r + F) = ((r-1)/r)2w - F.$$

Since the average is $2w/r$, there is some u with

$$d(u) \leq \left(\frac{r-1}{r}2w - F\right)/(r-1) = 2w/r - F/(r-1) \leq 2w/r - F/r.$$

We may take $S = V \setminus \{u\}$. We get:

$$\sum_{P \subseteq S} w_P = w - d(u) \geq w - (2w/r - F/r) = \frac{\binom{r-1}{2}}{\binom{r}{2}} w + F/r.$$

$\square$

A strengthening of the case $s = r - 1$ and r is even in Lemma 3.8.1 is a corollary.

Lemma 3.11.4. *Given weights w_P for $P \in \binom{R}{2}$ with $w_P \geq 0$, take $w = \sum_P w_P$. Either there are at least $r/2$ pairs P for which $w_P \geq w/r^2$ or there is some $S \subseteq R$ of size $r - 1$ with*

$$\sum_{P \subseteq S} w_P \geq \left(1 + \left(4r\binom{r}{2}^2\right)^{-1}\right) \frac{\binom{r-1}{2}}{\binom{r}{2}} w.$$

Proof: If there are fewer than $r/2$ pairs P for which $w_P \geq w/r^2$, then we must have that there is some vertex v not adjacent to any such pair. For this v, $d(v) \leq (r-1)w/r^2 \leq w/r$. The previous lemma gives that there is some set S of size $r - 1$ with:

$$\sum_{P \subseteq S} w_P \geq \left(\frac{\binom{r-1}{2}}{\binom{r}{2}} + \frac{1}{r^2}\right) w \geq \left(1 + \left(4r\binom{r}{2}^2\right)^{-1}\right) \frac{\binom{r-1}{2}}{\binom{r}{2}} w.$$

$\square$

Finally, we prove a strengthening of the case $s = r - 1$ and r is odd in Lemma 3.8.1:

Lemma 3.11.5. *Given weights w_P for $P \in \binom{R}{2}$ with $w_P \geq 0$, take $w = \sum_P w_P$. If r is odd either there are at least $(r+3)/2$ pairs P for which $w_P > 0$ or there is some $S \subseteq R$ of size $r - 1$ with*

$$\sum_{P \subseteq S} w_P \geq \left(1 + \left(4r \binom{r}{2}^2 \right)^{-1} \right) \frac{\binom{r-1}{2}}{\binom{r}{2}} w.$$

Proof: Assume there is no $S \subseteq R$ of size $r - 1$ with $\sum_{P \subseteq S} w_P \geq \left(1 + \left(4r \binom{r}{2}^2 \right)^{-1} \right) \frac{\binom{r-1}{2}}{\binom{r}{2}} w$.

In this case there is no $S \subseteq R$ of size $r - 1$ with $\sum_{P \subseteq S} w_P \geq \frac{\binom{r-1}{2}}{\binom{r}{2}} w + w/(4r^3)$, as this latter term is larger than $\left(1 + \left(4r \binom{r}{2}^2 \right)^{-1} \right) \frac{\binom{r-1}{2}}{\binom{r}{2}} w$.

We define an unweighted graph $G = (V, E)$ by taking $V = R$ and a possible edge $e \in \binom{R}{2}$ is in E if and only if $w_e \geq w/(4r^3)$. By the previous lemma, G has at least $r/2$ edges e satisfying $w_e \geq w/r^2$, and, indeed, the proof of the previous lemma shows that every vertex must have degree at least 1. Since r is odd, G must have at least $(r+1)/2$ edges e with $w_e \geq w/r^2$, and so it must have some vertex v incident to two such edges.

Fix two neighbors v_1, v_2 of v so that $w_{\{v,v_1\}}$ and $w_{\{v,v_2\}}$ both have weight at least w/r^2. We claim that both v_1 and v_2 have degree at least 2 in G. Assume at least one of them, without loss of generality v_1, has degree one. We must have $d(v) \leq 2w/r + w/(2r^2)$, for otherwise we have a contradiction by Lemma 3.11.3. However, this gives that, since $w_{\{v,v_2\}} \geq w/r^2$, we must have $w_{\{v,v_1\}} \leq d(v) - w_{\{v,v_2\}} \leq 2w/r - w/(2r^2)$. Then all other P incident to v_1 have weight at most $w/(4r^3)$, so

$$w_{\{v,v_1\}} \leq w_{\{v,v_1\}} + (r-2)w/(4r^3) \leq 2w/r - w/(2r^2) + rw/(4r^3) = 2w/r - w/(4r^2).$$

Then by Lemma 3.11.3 we have reached a contradiction.

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Therefore, we must have that there are at least 3 vertices of degree 2 in G and that every vertex has degree at least 1. Then the sum of the degrees is at least $6 + (r - 3) = r + 3$ and so the number of edges of G must be at least $(r + 3)/2$, as desired. $\qquad\square$

3.12   Proof of Lemma 3.8.3

For convenience, we restate both the lemma and definition of f_ϵ here:

$$f_\epsilon(\ell) = (\log m)^{C\left(\frac{\log(\alpha m^\epsilon)}{\log m}\right)}, \text{ where } \alpha = \ell/n. \text{ Recall also } m_0 \text{ from Equation 3.1.}$$

Lemma 3.12.1. *The following statements hold about f_ϵ for every choice of $\epsilon \geq 0$, $n > 1$, and $m \geq m_0$.*

1. *For any $\alpha \in [\frac{1}{n}, 1]$, we have*

$$f_\epsilon(\alpha n) \geq \alpha^{1/\left(2\binom{r-2}{s-2}\right)} f_\epsilon(n).$$

In particular, $f_\epsilon(\alpha n) \geq \alpha f_\epsilon(n)$.

2. *For any $\alpha_1, \alpha_2, \alpha_3 \in [\frac{1}{n}, 1]$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$, taking $n_i = \alpha_i n$ we have*

$$n f_\epsilon(n) \leq \sum_i n_i f_\epsilon(n_i) + 3(\log^{-3/4} m) n f_\epsilon(n).$$

3. *For $i \geq 0$ and $m^\delta \geq 2^j \geq 1$, we have $f_\epsilon(2^i) \log^{2/\binom{r-2}{s-2}}((\log^{1/4} m)2^j) \geq 256\binom{r}{2} f_\epsilon(2^{i+2j})$.*

4. *For any $\alpha \geq \log^{-1} m$, we have $f_\epsilon(\alpha n) \geq f_\epsilon(n)/2$.*

Proof of 1: Note

$$f_\epsilon(\alpha n) = (\log m)^{C\left(\epsilon + \frac{\log \alpha}{\log m}\right)} = (\log m)^{C\frac{\log \alpha}{\log m}}(\log m)^{C\epsilon} = \alpha^{C\frac{\log \log m}{\log m}} f_\epsilon(n).$$

Since $0 < \alpha \leq 1$, it is sufficient to show that $C(\log \log m)/\log m \leq \left(2\binom{r-2}{s-2}\right)^{-1}$. This holds because C, $\log \log m$ and $2\binom{r-2}{s-2}$ are at most $\log^{1/3} m$, since $m \geq m_0$.

Proof of 2: Note that the function $\ell f_\epsilon(\ell)$ is a convex function of ℓ. Indeed,

$$f_\epsilon(\ell) = (\log m)^{C\epsilon}(\log m)^{(\log \ell)/\log m} = (\log m)^{C\epsilon}\ell^{(\log \log m)/\log m}.$$

That is, $f_\epsilon(\ell)$ is a polynomial of degree greater than 0 in ℓ, so $\ell f_\epsilon(\ell)$ is a polynomial of degree greater than 1 in ℓ and so is convex.

Take $S = \{i : \alpha_i \geq \log^{-3/4} m\}$. Take κ such that $\sum_{i \in S} \alpha_i = \kappa$. Note $\kappa \geq 1 - 2 \log^{-3/4} m$.

$$\sum_i n_i f_\epsilon(n_i) \geq \sum_{i \in S} n_i f_\epsilon(n_i) \geq \sum_{i \in S} \frac{\kappa}{|S|} n f_\epsilon\left(\frac{\kappa}{|S|} n\right)$$

$$= \kappa n f_\epsilon\left(\frac{\kappa}{|S|} n\right) \geq \kappa n f_\epsilon\left(\frac{1 - 2\log^{-1} m}{3} n\right) \geq \kappa n f_\epsilon(n/4),$$

where the second inequality follows by Jensen's inequality applied to the convex function $\ell f_\epsilon(\ell)$.

This gives

$$\kappa n f_\epsilon(n) - \sum_{i \in S} n_i f_\epsilon(n_i) \leq \kappa n f_\epsilon(n) - \kappa n f_\epsilon(n/4) = \kappa n (f_\epsilon(n) - f_\epsilon(n/4)).$$

We now consider

$$f_\epsilon(n) - f_\epsilon(n/4) = (\log m)^{C\epsilon} - (\log m)^{C\left(\epsilon + \frac{\log(1/4)}{\log m}\right)} = (\log m)^{C\epsilon}\left(1 - (\log m)^{-2C/\log m}\right).$$

The second factor satisfies:

$$1 - (\log m)^{-2C/\log m} = 1 - 2^{-2C(\log \log m)/\log m}$$

$$\leq 1 - (1 - 2C(\log \log m)/\log m) = 2C(\log \log m)/\log m \leq \log^{-3/4} m,$$

where the first inequality follows by $2^x \geq 1 + x$ for $x \leq 0$.

Thus,

$$nf_\epsilon(n) - \sum_i n_i f_\epsilon(n_i) \le (1-\kappa)nf_\epsilon(n) + \kappa n f_\epsilon(n) - \sum_{i\in S} n_i f(n_i)$$

$$\le 2(\log^{-3/4} m)nf_\epsilon(n) + (\log^{-3/4} m)nf_\epsilon(n) \le 3(\log^{-3/4} m)nf_\epsilon(n).$$

Proof of 3: We prove a slightly stronger statement. Take $j' = j + \frac14(\log\log m)$ so that $2^{j'} = (\log m)^{1/4}2^j$. We will show that

$$f_\epsilon(2^i)\log^{2/\binom{r-2}{s-2}}(2^{j'}) \ge f_\epsilon(2^{i+2j'}).$$

This is indeed stronger than the original statement as $f_\epsilon(\ell)$ is an increasing function of ℓ.

Consider

$$f_\epsilon(2^i) = (\log m)^{C\left(\epsilon + \frac{\log(2^i/n)}{\log m}\right)} = (\log m)^{C\epsilon}(\log m)^{C\frac{i}{\log m}}(\log m)^{-C\frac{\log n}{\log m}}.$$

Similarly,

$$f_\epsilon(2^{i+2j'}) = (\log m)^{C\epsilon}(\log m)^{C\frac{i+2j'}{\log m}}(\log m)^{-C\frac{\log n}{\log m}}.$$

Therefore, it is sufficient to show that

$$(\log m)^{C\frac{i}{\log m}}\log^{2/\binom{r-2}{s-2}}(2^{j'}) \ge 256\binom{r}{2}(\log m)^{C\frac{i+2j'}{\log m}},$$

or equivalently that

$$\log^{2/\binom{r-2}{s-2}}(2^{j'}) \ge 256\binom{r}{2}(\log m)^{C\frac{2j'}{\log m}}.$$

Taking logarithms of both sides, we see that it is sufficient to have

$$2(\log j')/\binom{r-2}{s-2} \ge 2Cj'(\log\log m)/(\log m) + \log\left(256\binom{r}{2}\right),$$

or equivalently

$$2(\log j')/\binom{r-2}{s-2} - 2Cj'(\log\log m)/(\log m) - \log\left(256\binom{r}{2}\right) \geq 0.$$

We consider the first derivative of this with respect to j': it is $2(\ln(2)j')^{-1}/\binom{r-2}{s-2} - 2C(\log\log m)/(\log m)$. Note that this derivative is monotone decreasing for $j' \in [1,\infty)$, so the minimum of $2(\log j')/\binom{r-2}{s-2} - 2Cj'(\log\log m)/(\log m) - \log\left(256\binom{r}{2}\right)$ must be achieved at either the largest or smallest possible value of j'. We have assumed $m^\delta \log^{1/4} m \geq 2^{j'} \geq \log^{1/4} m$, so $2\delta \log m \geq \delta(\log m) + (\log\log m)/4 \geq j' \geq (\log\log m)/4$. We consider the two extrema.

If $j' = (\log\log m)/4$, we have

$$2\log((\log\log m)/4)/\binom{r-2}{s-2} - \frac{C}{2}\log^2(\log m)/(\log m) - \log\left(256\binom{r}{2}\right)$$
$$\geq 2\log((\log\log m)/4)/\binom{r-2}{s-2} - 1 - \log\left(256\binom{r}{2}\right) \geq 0,$$

where the last inequality follows from $m \geq m_0$.

If $j' = 2\delta \log m$:

$$2\log(2\delta \log m)/\binom{r-2}{s-2} - 4\delta C(\log\log m) - \log\left(256\binom{r}{2}\right)$$
$$= 2\log(2\delta \log m)/\binom{r-2}{s-2} - (\log\log m)/\binom{r-2}{s-2} - \log\left(256\binom{r}{2}\right) \geq 0,$$

where the last inequality follows from $m \geq m_0$.

Proof of 4: Since f_ϵ is increasing, it is sufficient to show this for $\alpha = \log^{-1} m$. Then

$$f_\epsilon(\log^{-1} m) = (\log m)^{C(\epsilon - (\log\log m)/(\log m))}$$
$$= (\log m)^{C\epsilon} 2^{-(\log\log m)^2/\log m}$$
$$\geq (\log m)^{C\epsilon} 2^{-1} = f_\epsilon(n)/2.$$

Chapter 4

Packing Vertex-Disjoint Monochromatic Copies of Sparse Graphs

4.1 Introduction

Let K_n be a complete graph on n vertices whose edges are colored with r colors ($r \geq 1$). How many monochromatic cycles (single vertices and edges are considered to be cycles) are needed to partition the vertex set of K_n? This question received much attention in the last few years. Let $p(r)$ denote the minimum number of monochromatic cycles needed to partition the vertex set of any r-colored K_n. It is not obvious that $p(r)$ is a well-defined function. That is, it is not obvious that there always is a partition whose cardinality is independent of n. However, in [33] Erdős, Gyárfás and Pyber proved that there exists a constant C such that $p(r) \leq Cr^2 \log r$ (throughout this chapter log denotes the natural logarithm). Furthermore, in [33] (see also [53]) the authors conjectured that $p(r) = r$.

The special case $r = 2$ of this conjecture was asked earlier by Lehel and, for $n \geq n_0$, was first proved by Łuczak, Rödl, and Szemerédi [73]. Allen improved on the value of n_0 [1] and recently Bessy and Thomassé [6] proved the original conjecture for all values

of n with $r = 2$. For general r the current best bound is due to Gyárfás, Ruszinkó, Sárközy, and Szemerédi [54] who proved that, for $n \geq n_0(r)$, we have $p(r) \leq 100r \log r$. For $r = 3$, in [55] it was proved that all but $o(n)$ of the vertices may be covered by 3 monochromatic cycles. Surprisingly, Pokrovskiy [77] found a counterexample to the conjecture for all $r \geq 3$. However, in the counterexample, all but one vertex can be covered by r vertex-disjoint monochromatic cycles. Thus, a slightly weaker version of the conjecture still can be true, say that, apart from a constant number of vertices, the vertex set can be covered by r vertex-disjoint monochromatic cycles.

Let us also note that the above problem was generalized in various directions; for hypergraphs (see [57] and [84]), for complete bipartite graphs (see [33] and [61]), for graphs which are not necessarily complete (see [4] and [83]), and for vertex partitions by monochromatic connected k-regular subgraphs (see [85] and [86]).

Another area that attracted much interest is the study of Ramsey numbers for bounded degree graphs. For a graph G, the *Ramsey number* $R(G)$ is the smallest positive integer N such that, if the edges of a complete graph K_N are partitioned into two color classes, then one color class has a subgraph isomorphic to G. The existence of such a positive integer is guaranteed by Ramsey's classical result [79]. Determining $R(G)$ even for very special graphs is notoriously hard (see e.g. [50] or [78]).

In 1975, Burr and Erdős [11] raised the problem that every graph G with n vertices and maximum degree Δ has a linear Ramsey number, so $R(G) \leq C(\Delta)n$, for some constant $C(\Delta)$ depending only on Δ. This was proved by Chvátal, Rödl, Szemerédi and Trotter [20] in one of the earliest applications of Szemerédi's celebrated Regularity Lemma [89]. Because the proof uses the Regularity Lemma, the bound on $C(\Delta)$ is quite weak; it is of tower type in Δ. This was improved by Eaton [30] who proved, using a variant of the Regularity Lemma, that the function $C(\Delta)$ can be taken to be of the form $2^{2^{O(\Delta)}}$.

Soon after, Graham, Rödl, and Ruciński [49] improved this further to $C(\Delta) \leq 2^{O(\Delta \log^2 \Delta)}$ and for bipartite graphs $C_B(\Delta) \leq 2^{O(\Delta \log \Delta)}$. They also proved that there are bipartite graphs with n vertices and maximum degree Δ for which the Ramsey number is at least $2^{\Omega(\Delta)}n$. Recently, Conlon [22] and, independently, Fox and Su-

dakov [45] have shown how to remove the $\log \Delta$ factor in the exponent, achieving an essentially best possible bound of $C_B(\Delta) = 2^{\Theta(\Delta)}$ in the bipartite case. For the non-bipartite graph case, the current best bound is due to Conlon, Fox, and Sudakov [25] $C(\Delta) \le 2^{O(\Delta \log \Delta)}$. Similar results have been proven for hypergraphs: [26, 27, 75] use the hypergraph regularity lemma and [24] improves the bounds by avoiding the regularity lemma.

Similar results also hold for a-arrangeable graphs. An a-arrangeable graph is one in which the vertices may be ordered as $v_1, \ldots, v_n$ such that, for any index i, if we consider those neighbors of v_i in the set $\{v_{i+1}, \ldots, v_n\}$, they have at most a neighbors in the set $\{v_1, \ldots, v_i\}$. Chen and Schelp [16] proved that, for every a, there is some constant $C(a)$ so that the Ramsey number of any a-arrangeable graph on n vertices is at most $C(a)n$. The best bound that is known for $C(a)$, again due to Graham, Rödl, and Ruciński [49], is $C(a) \le 2^{Ca \log^2 a}$.

It is a natural question (initiated by András Gyárfás) to combine the studies of packing monochromatic cycles and of computing Ramsey numbers of sparse graphs and ask how many monochromatic members from a bounded-degree graph family are needed to partition the vertex set of a 2-edge-colored K_N. In this chapter we study this problem and related questions. Given $\mathcal{F} = \{F_1, F_2, \ldots\}$ a sequence of graphs, we say it is a proper graph sequence if F_n is a graph on n vertices. We say it has some graph property if every graph of $\mathcal{F}$ has that property (e.g. $\mathcal{F}$ is bipartite if F_n is bipartite for every n).

We derive the following lower bound from the result that, for n sufficiently large, there are bipartite graphs on n vertices of maximum degree at most Δ with Ramsey number $2^{\Omega(\Delta)}n$ [49]. We prove this bound in Section 4.6.

Theorem 4.1.1. *There exists an absolute constant c such that, for every Δ, there is a bipartite proper graph sequence $\mathcal{F}$ with maximum degree at most Δ and, for every n sufficiently large, there is a 2-edge-coloring of K_n so that covering the vertices of K_n using monochromatic copies of graphs from $\mathcal{F}$ requires at least $2^{c\Delta}$ such copies.*

This matches the upper bound we find.

Theorem 4.1.2. *There exists an absolute constant C such that, for every Δ and every bipartite proper graph sequence $\mathcal{F}$ with maximum degree at most Δ, every 2-edge-colored complete graph can be partitioned into at most $2^{C\Delta}$ vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$.*

We can actually prove that if $\mathcal{F}$ has chromatic number at most k then the bound above may be taken to be $2^{Ck\Delta}$. If $k \leq \log\Delta$, then this is the best bound we know how to get. However, for larger values of k, we can get a better bound that depends only on Δ.

Theorem 4.1.3. *There exists an absolute constant C such that, for every Δ and every proper graph sequence $\mathcal{F}$ with maximum degree at most Δ, every 2-edge-colored complete graph can be partitioned into at most $2^{C\Delta\log\Delta}$ vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$.*

If we are interested in graphs with bounded arrangeability rather than bounded degree, we get a slightly weaker bound, assuming the graphs satisfy an additional degree-bound, which is required for one of the tools we use.

Theorem 4.1.4. *There exists an absolute constant C such that, for every a there and every a-arrangeable proper graph sequence $\mathcal{F} = \{F_1, F_2, \ldots\}$ satisfying that the maximum degree of F_n is at most $\sqrt{n}/\log n$, every 2-edge-colored complete graph on at least N vertices can be partition into at most 2^{Ca^6} vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$.*

Theorems 4.1.2 and 4.1.3 give, perhaps surprisingly, that we have the same phenomenon for these classes of graphs as for cycles; we can partition into monochromatic graphs from $\mathcal{F}$ such that the average size of the parts (which are each monochromatic copies of graphs from $\mathcal{F}$) is roughly as large as the size of the single largest monochromatic graph we know how to find (i.e. the one given by our best bounds for Ramsey numbers). It is particularly interesting to note that the conditions of Theorem 4.1.4 are satisfied with high probability by $F_n = G(n, d/n)$ for d a constant, as Fox and Sudakov [46] showed that, with high probability, the arrangeability of $G(n, d/n)$ is $\Theta(d^2)$.

It would be desirable to close the gap between the upper bound given by Theorem 4.1.3 and the lower bound in Theorem 4.1.1, though doing so may require improved bounds for the Ramsey numbers of bounded degree graphs.

Finally, let us mention one interesting special case of Theorem 4.1.3. The kth *power* of a cycle C is the graph obtained from C by joining every pair of vertices with distance at most k in C. Density questions for powers of cycles have generated a lot of interest; in particular the famous Pósa-Seymour conjecture (see e.g. [15, 37, 38, 39, 40, 65, 68, 69, 71]). Theorem 4.1.3 implies the following result on the partition number by monochromatic powers of cycles.

Corollary 4.1.5. *There exists an absolute constant C so that, for every k, every 2-colored complete graph can be partitioned into at most $2^{Ck \log k}$ vertex-disjoint monochromatic kth powers of cycles.*

4.2   Notation

$V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. (A, B, E) denotes a bipartite graph $G = (V, E)$, where $V = A \cup B$ and $E \subset A \times B$. For a graph G and a subset U of its vertices, $G|_U$ is the restriction to U of G. $N(v)$ is the set of neighbors of $v \in V$. Hence, $|N(v)| = deg(v) = deg_G(v)$, the degree of v. $\delta(G)$ stands for the minimum and $\Delta(G)$ for the maximum degree in G. When A, B are subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of G with one endpoint in A and the other in B. We write $deg(v, U) = e(\{v\}, U)$ for the number of edges from v to U. For non-empty A and B,

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

is the *density* of the graph between A and B.

Definition 4.2.1. *The bipartite graph $G = (A, B, E)$ is ε-regular if*

$$X \subset A, \ Y \subset B, \ |X| > \varepsilon|A|, \ |Y| > \varepsilon|B| \ imply \ |d(X, Y) - d(A, B)| < \varepsilon.$$

We will often say simply that "the pair (A, B) is ε-regular" with the graph G implicit.

Definition 4.2.2. (A, B) *is* (ε, d, δ)-*super-regular if it is* ε-*regular, satisfies* $d(A, B) \geq d$, *and*

$$deg(a) > \delta|B| \ \forall \, a \in A, \qquad deg(b) > \delta|A| \ \forall \, b \in B.$$

Definition 4.2.3. *Given a* k-*partite graph* $G = (V, E)$ *with* k-*partition* $V = V_1 \cup \ldots \cup V_k$, *the* k-*cylinder* $V_1 \times \ldots \times V_k$ *is* ε-*regular (*(ε, d, δ)-*super-regular) if all the* $\binom{k}{2}$ *pairs of subsets* (V_i, V_j), $1 \leq i < j \leq k$, *are* ε-*regular (*(ε, d, δ)-*super-regular). If we wish to say a cylinder is* $(\varepsilon, \delta, \delta)$-*super-regular, we simply say it is* (ε, δ)-*super-regular, and in this case it is not necessary to check the density condition. Given* $\alpha \geq 0$, *the* k-*cylinder* $V_1 \times \ldots \times V_k$ *is* α-*balanced if, for every* $i < j$, $||V_i| - |V_j|| \leq \alpha \min(|V_i|, |V_j|)$.

We say a graph G on n vertices is a-nicely-arrangeable if it is a-arrangeable and satisfies that $\Delta(G) \leq \sqrt{n}/\log n$. We say it is χ-chromatically equitable if there is a proper χ-coloring of the vertices of G (one where no two adjacent vertices have the same color) in which the size of any two color classes differs by at most 1. We say it is R-linearly-Ramsey if $R(G) \leq Rn$.

4.3 Regularity and blow-up lemmas

Some of our main tools for finding monochromatic copies of graphs are regularity and blowup lemmas. Regularity lemmas allow us to find monochromatic regular cylinders, and blowup lemmas allow us to cover the vertices of such a cylinder with a sparse graph.

Instead of the Regularity Lemma of Szemerédi [89], we will use the following lemmas which Conlon and Fox [23] argued as consequences of the Duke, Lefmann, and Rödl weak Regularity Lemma [29].

Lemma 4.3.1 ([29] and Lemma 5.3 in [23])**.** *For each* $0 < \varepsilon < 1/2$, *given any graph* $G = (V, E)$ *on* $n \geq k$ *vertices we may find disjoint sets of vertices* $V_1, \ldots, V_k$ *so that*

the induced k-partite cylinder is 0-balanced and ε-regular; the size of each part is at least $\frac{1}{2k}\varepsilon^{k^2\varepsilon^{-5}}n$.

We will use the following corollary of this lemma.

Lemma 4.3.2 (Lemma 5.4 in [23]). *For each $0 < \varepsilon < 1/2$, given any 2-colored complete graph on $n \geq 2^{2k}$ vertices we may find vertex-disjoints sets $V_1, \ldots, V_k$ so that the induced multipartite graph is, in one of the colors (say in red), an $(\varepsilon, 1/2, 0)$-super-regular 0-balanced cylinder (i.e. one with no minimum degree constraint and parts of equal size), where the size of each part is at least $\frac{1}{2(2^{2k})}\varepsilon^{2^{4k}\varepsilon^{-5}}n$.*

Indeed, to get this one applies Lemma 4.3.1 for the red subgraph with 2^{2k} in place of k to get an ε-regular 2^{2k}-cylinder. Then we may consider the complete graph whose vertices i correspond to the parts of the cylinder V_i and we color the edge (i, j) by the majority color in the pair (V_i, V_j). We then apply $R(K_k) \leq 2^{2k}$ and use the fact that, if (V_i, V_j) is regular in one color, then it is also regular in the other color.

Our main tool for dealing with bounded degree graphs is a quantitative version of the Blow-up Lemma (see [66, 67, 82]).

Lemma 4.3.3 (Quantitative Blow-up Lemma). *For every constant α there exists a constant $C = C(\alpha)$ such that, given a graph R of order $r \geq 2$ and positive parameters δ, d, and Δ, for any $0 < \varepsilon < \left(\frac{\delta}{r\Delta}d^\Delta\right)^C$ the following holds. Let us replace the vertices of R with pairwise disjoint α-balanced sets $V_1, V_2, \ldots, V_r$ (blowing up). We construct two graphs on the same vertex set $V = \bigcup V_i$. The graph R' is obtained by replacing all edges of R with copies of the complete bipartite graph, and a sparser graph G is constructed by replacing the edges of R with some (ε, d, δ)-super-regular pairs. If a graph H with $\Delta(H) \leq \Delta$ is embeddable into R', then it is embeddable into G.*

To deal with a-nicely-arrangeable graphs, we use a version of the blowup lemma due to Böttcher, Kohayakawa, Taraz, and Würfl [9]. It is worth noting that, in their paper, the authors of [9] allow more parameters and compute more explicit bounds than what we state below.

Theorem 4.3.4 (Blow-up Lemma for Arrangeable Graphs [9]). *For every constant α there exists a constant $C = C(\alpha)$ such that, given a graph R of order $r \geq 2$ and positive parameters δ and a, if we take Δ_R to be a bound on the maximum degree of R, then, for any $0 < \varepsilon < \delta^{-Ca^2\Delta_R} \cdot 2^{-Ca^4\Delta_R^2}$, taking $n_0 := 2^{2^{Ca^4}rC}\delta^{-Ca^2}\epsilon^{-C}$, the following holds. Let us replace the vertices of R with pairwise-disjoint α-balanced sets $V_1, V_2, \ldots, V_r$ (blowing up) each of size at least n_0. We construct two graphs on the same vertex set $V = \bigcup V_i$. The graph R' is obtained by replacing all edges of R with copies of the complete bipartite graph, and a sparser graph G is constructed by replacing the edges of R with some (ε, δ)-super-regular pairs. If an a-nicely-arrangeable graph H is embeddable into R', then it is embeddable into G.*

4.4   Proofs of Theorems 4.1.2, 4.1.3, and 4.1.4

We basically follow the *greedy-absorbing* proof technique that originated in [33] and is used in many papers in this area (e.g. [54], [61], [86]).

Given an R-linearly-Ramsey proper graph sequence $\mathcal{F}$, we will show how to decompose any 2-edge-coloring of a K_n into useful structures, in pursuit of the goal of eventually decomposing it into monochromatic copies of graphs from $\mathcal{F}$.

The structures we will use are monochromatic copies from $\mathcal{F}$, monochromatic super-regular cylinders, and something resembling a union of almost-complete multipartite graphs. Such a decomposition is useful as we will use a blow-up lemma to cover the super-regular cylinders and it is easy to greedily embed graphs into almost-complete multipartite graphs. For technical reasons, we don't actually use complete multipartite graphs, but a structure that will serve a similar purpose that we call a branching degree cylinder.

Definition 4.4.1. *Given positive integers $k_1, \ldots, k_\ell$, the $(k_1, \ldots, k_\ell)$-branching tree is a rooted tree defined recursively as follows: the ()-branching tree is a single vertex, the root, and the $(k_1, \ldots, k_{\ell+1})$-branching tree is the graph obtained from the $(k_1, \ldots, k_\ell)$-branching tree by adding $k_{\ell+1}$ neighbors to each leaf (and the root vertex remains the same). If $k := k_1 = k_2 = \cdots = k_\ell$, we simply say it is a k-branching tree with ℓ levels.*

In other words, the branching tree is a full and complete rooted tree, and the parameters $(k_1, \ldots, k_\ell)$ determine the branching factor at each level.

Definition 4.4.2. *Given a 2-edge-coloring of a K_n and positive integers $k_1, \ldots, k_\ell$, a $(k_1, \ldots, k_\ell)$-branching $(1 - \gamma)$ degree cylinder with scaling ε assigns to every vertex v of the $(k_1, \ldots, k_\ell)$-branching tree a collection of vertices from the K_n; if v and w are not ancestors of each other in the branching tree, then V_v and V_w are disjoint, and otherwise, if w is an ancestor of v, then $V_v \subseteq V_w$. Furthermore, for every pair w and v with w the parent of v, there is some color and some set $S_v \subseteq V_w \setminus V_v$ so that every vertex in V_v has degree at least $(1 - \gamma)|S_v|$ to S_v in that color; we call S_v the parental neighborhood of V_v. Finally, $|V_v| \leq \varepsilon|S_v|$. If $k := k_1 = k_2 = \cdots = k_\ell$, then we say it is a k-branching cylinder with ℓ levels.*

We are now ready to state the way in which we will decompose the vertices of a 2-edge-colored K_n.

Lemma 4.4.3. *There exists an absolute constant C such that, given any 2-edge-coloring of a complete graph, any $k, \ell > 0$, any $0 < \varepsilon < 1/2$ and any $0 < \delta < 1/2 - \varepsilon$, and any R-linearly-Ramsey proper graph sequence $\mathcal{F}$, we may find*

- $\sum_{i=0}^{\ell-1} k^i R 2^{Ck} \varepsilon^{-C}$ *vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$*

- *A k-branching $(1 - \delta - k\varepsilon)$ degree cylinder with scaling $k\varepsilon$ and with ℓ levels in which, for v a vertex of the underlying branching tree, V_v is the corresponding set of vertices with parental neighborhood S_v, and*

- *Inside of each V_w with w not a leaf of the branching tree, a $(2k\varepsilon, 1/2 - 2k\varepsilon, \delta - k\varepsilon)$ super-regular cylinder C_w with k parts which are $k\varepsilon$-balanced and which satisfies that, for any v a child of w, $S_v \subseteq C_w$.*

The above structures satisfy two additional properties. The first is that every vertex of the complete graph is contained in one of the monochromatic copies of a graph from $\mathcal{F}$, one of the super-regular cylinders, or in V_v for v a leaf in the branching tree. The second is that, for every V_w with w not a leaf of the branching tree, the associated

super-regular cylinder C_w does not intersect V_v for any v a descendant of w and C_w does not intersect any of the monochromatic copies of graphs from $\mathcal{F}$.

We first prove the above decomposition lemma, and then we show how to apply it to prove Theorems 4.1.2, 4.1.3, and 4.1.4.

4.4.1   Proof of Lemma 4.4.3

To prove the above theorem, we will proceed by induction on ℓ. One essential part of the inductive step is the following lemma.

Lemma 4.4.4. *There exists an absolute constant C such that, given any R-linearly-Ramsey proper graph sequence $\mathcal{F}$, any $\varepsilon > 0$, any positive integer k, and any 2-edge-coloring of a K_n, we may partition the vertices into at most $R2^{Ck}\varepsilon^{-C}$ monochromatic copies of graphs from $\mathcal{F}$ and sets $V_1, \ldots, V_k$ so that the multipartite graph induced by one of the colors is a 0-balanced, $(\varepsilon, 1/2 - \varepsilon, 0)$-super-regular cylinder.*

To prove Lemma 4.4.4, we first find a monochromatic $\varepsilon/2$-regular cylinder so that the density between any two pairs is at least $1/2$ by Lemma 4.3.2; then, we will cover most of the remaining vertices with monochromatic copies of graphs from $\mathcal{F}$. We simply add the uncovered vertices to the cylinder; they do not significantly harm the regularity. To this end, we prove the following observation about R-linearly-Ramsey proper graph sequences.

Lemma 4.4.5. *For any R-Ramsey proper graph sequence, given a 2-edge-colored K_n and an $\alpha > 0$, we may cover all but an α fraction of the vertices of the K_n using at most $2R \log(e/\alpha)$ vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$.*

Proof.   Given any 2-coloring of a K_n for $n \geq 2R$ we may find a monochromatic copy of $F_{\lfloor n/R \rfloor}$. Removing the vertices of this copy leaves a 2-edge-coloring of a clique on $\lceil (1 - 1/R)n \rceil \leq (1 - 1/(2R))n$ vertices; iterating this procedure $a := 2R \log(e/\alpha)$ times gives a 2-edge coloring on at most $(1 - 1/(2R))^a n \leq e^{a/(2R)} n \leq \alpha n$ vertices, assuming that at each iteration the graph had at least $2R$ vertices. If at any point the graph failed to have at least $2R$ vertices, then we may remove each of the remaining

vertices as monochromatic copies of F_1; in either case, the total number of monochromatic copies of graphs from $\mathcal{F}$ used was at most $a + 2R = 2R\log(1/\alpha) + 2R = 2R\log(e/\alpha)$. □

Proof of Lemma 4.4.4.

By Theorem 4.3.2, either $n < 2^{2t}$ or we may find a monochromatic $(\varepsilon/2, 1/2, 0)$-super-regular, 0-balanced cylinder where each part is at least an $\alpha := \frac{1}{2(2^{2k})}(\varepsilon/2)^{2^{4k}(\varepsilon/2)^{-5}}$ fraction of the vertices. In the former case, we may simply take each vertex to be a monochromatic copy of F_1 and take the cylinder to be empty. In the latter case, we find a 0-balanced cylinder on vertex set $V_1, \ldots, V_k$ in which each part has size at least αn. Take $\varepsilon' := \varepsilon^2 \alpha/4$. Then, by Lemma 4.4.5, using at most $2R\log(e/\varepsilon') = R2^{O(k)}\varepsilon^{-O(1)}$ copies of graphs from $\mathcal{F}$, we may cover all but an ε' fraction of those vertices of the K_n that are outside of $V_1, \ldots, V_k$. There are n' vertices remaining outside of $V_1, \ldots, V_k$; we wish to assume n' is divisible by k, which we may do by covering up to k vertices by copies of F_1. Then, we simply add n'/k of the remaining vertices to each of $V_1, \ldots, V_k$; since we are adding fewer than $\varepsilon^2 \alpha n/4$ vertices to each part (each part has size αn), the new cylinder is $(\varepsilon, 1/2 - \varepsilon, 0)$-super-regular. □

Proof of Lemma 4.4.3. We proceed by induction on ℓ. If $\ell = 0$, then we may take the whole vertex set of the K_n to be a k-branching cylinder with 0 levels. Otherwise, if $\ell > 0$, by induction we may decompose the K_n into at most $\sum_{i=0}^{\ell-2} k^i R2^{Ck}\varepsilon^{-C}$ vertex-disjoint monochromatic copies from $\mathcal{F}$ where C is the constant from Lemma 4.4.4, into a k-branching $k\varepsilon$-scaling $(1 - \delta - k\varepsilon)$ degree cylinder with $\ell - 1$ levels with vertex sets V_v where v is a vertex of the corresponding branching tree and their corresponding parental neighborhoods S_v, and in each V_w for w not a leaf of the branching tree there is a $(2k\varepsilon, 1/2 - 2k\varepsilon, \delta - k\varepsilon)$-regular cylinder C_w satisfying that it does not intersect V_v for any descendant v of w, that it does not intersect any of the monochromatic copies of graphs from $\mathcal{F}$, and that it contains S_v for any v a child of w.

In order to prove Lemma 4.4.3 with ℓ levels rather than with $\ell - 1$ levels, we wish to extend the branching degree cylinder. To do so, for each of the $k^{\ell-1}$ leaves w of the

k-branching tree with ℓ-1 levels we apply Lemma 4.4.4 to decompose V_w into at most $R2^{Ck}\varepsilon^{-C}$ monochromatic copies of graphs from $\mathcal{F}$ and into vertex-disjoint 0-balanced sets $C'_1, \ldots, C'_k$ so that the induced cylinder C' in one of the colors is $(\varepsilon, 1/2 - \varepsilon, 0)$-super-regular. We now extract low-degree vertices from this cylinder. We define sets D_i recursively; D_i is the set of vertices in $C' \setminus C'_i$ that have degree at most $(\delta - \varepsilon)|C'_i|$ to C'_i, minus those vertices that appear in $D_1, \ldots, D_{i-1}$. We then take the sets corresponding to the children of w in the ℓ-level k-branching tree to be D_i with parental neighborhoods $C_i := C'_i \setminus D_i$ and take $C_w := C_1 \times \cdots \times C_k$ to be the cylinder corresponding to V_w. Note that, for each $j \neq i$, by ε-regularity at most an ε-fraction of the vertices of C'_j fail to have degree at least $(\delta - \varepsilon)|C'_i|$ to C'_i. Therefore, each D_i has size at most $(k-1)\varepsilon|C'_i|$. This implies the cylinder C_w is $(k-1)\varepsilon < k\varepsilon$-balanced. Furthermore, since each pair C'_i, C'_j was $(\varepsilon, 1/2 - \varepsilon, 0)$-super-regular and we removed all vertices that had degree to C'_i less than $(\delta - \varepsilon)|C'_i|$, in the process we removed at most $(k-1)\varepsilon|C'_j|$ vertices from C'_j and so C_v is $(2k\varepsilon, 1/2 - 2k\varepsilon, \delta - k\varepsilon)$-super-regular, as desired. □

4.4.2 Applying Lemma 4.4.3

We use the three structures found in Lemma 4.4.3, namely the monochromatic copies of graphs from $\mathcal{F}$, the super-regular cylinders, and the branching degree cylinder, to cover all of the vertices with monochromatic graphs from $\mathcal{F}$. We now explain how to use the latter two structures to do this.

To cover vertices in a super-regular cylinder, we will apply one of the blow-up lemmas. To that end, we wish to show how to embed few copies of graphs from $\mathcal{F}$ into a balanced complete multipartite graph. It is easier to embed graphs that are, in some sense, balanced.

Definition 4.4.6. *A graph G is k-chromatically equitable if it has a proper k-vertex-coloring (i.e. one in which no two adjacent vertices receive the same color) in which every color class has the same size.*

Note that, if a graph G is k-colorable, then the graph obtained by taking k disjoint

copies of G, which we denote by kG, is k-chromatically equitable. To see this, we may take any k-coloring of G and cyclically permute the colors throughout the k copies to get a chromatically equitable coloring.

Lemma 4.4.7. *Given a proper graph sequence $\mathcal{F} = \{F_1, F_2, \ldots\}$ with chromatic number at most k and given a complete $1/(k+1)$-balanced $k+1$-partite graph, we can cover the vertices using at most $(k+1)^2$ vertex-disjoint copies of graphs from $\mathcal{F}$.*

Proof. Let $C_1, \ldots, C_{k+1}$ be the parts of the complete graph. For each index $i \in [k+1]$, define $v_i = |C_i|$ and take $v = \max_i(v_i)$. Then, for each set $S \subseteq [k+1]$ of size k, define $w_S := v - v_i$ where i is the unique index not contained in S. For each such set S, we take a copy of kF_{w_S} (disjoint from all the copies chosen so far) that uses w_S vertices from each V_i with $i \in S$. We use $k+1$ such graphs, one for each S. Note that, because the structure is $1/(k+1)$-balanced, for every step corresponding to some S with $i \in S$ we remove at most a $1/(k+1)$ fraction of C_i, so there are always enough vertices in C_i to choose the next graph to be vertex-disjoint from the previous ones, and so the procedure does not fail.

After this procedure, the number of uncovered vertices in C_i is

$$v_i - \sum_{S:i\in S} w_S = v_i - \sum_S w_S + w_{[k+1]\setminus\{i\}} = v - \sum_S w_S.$$

That is, after this procedure, each C_i has the same number of uncovered vertices, say n. We cover these with a copy of $(k+1)F_n$. We use a total of $(k+1)^2$ copies of graphs from $\mathcal{F}$. $\square$

Finally, we wish to show how, given a branching degree cylinder, to cover the vertices of the V_v with v a leaf in the corresponding branching tree.

Lemma 4.4.8. *Given a 2-edge-coloring of a K_n, a d-degenerate k-chromatically equitable proper graph sequence $\mathcal{F}$, and a k-branching $(1/(32k^2d^2))$-scaling $(1-1/(32k^2d^2))$ degree cylinder with $2k-3$ levels which assigns the set V_v to a vertex v of the corresponding branching tree with S_v the parental neighborhood, we may cover all the V_v with v a leaf of the branching tree using at most k^{2k} monochromatic copies of graphs*

from $\mathcal{F}$ the vertices of which are contained entirely in the various S_v or in some V_v with v a leaf, and, in doing so, for any v with parent w we use either at most a ε fraction of the vertices of S_v or we use all of V_w.

Proof. For every path $P = (v_0, \ldots, v_{2k-3})$ from the root to a leaf in the branching tree, we define, for every vertex v_i with $0 \leq i < 2k-3$, a corresponding color: this color is red if the vertices of $V_{v_{i+1}}$ all have red-degree at least $(1 - 1/(32k^2d^2))|V_{v_i}|$ to V_{v_i}, and otherwise they all have blue-degree at least $(1 - 1/(32k^2d^2))|S_{v_i}|$ to S_{v_i} and the color is blue. This gives a sequence of $2k-3$ colors; exactly one of the two colors must occur at least $k-1$ times in this sequence. Then take, for $0 \leq j < k-1$, v_{P_j} to be the vertex of the path P corresponding to the jth occurrence of the more common color in the color sequence. Take $v_{P_{k-1}}$ to be v_{2k-3}. Take $V_{P_j} := V_{v_{P_j}}$.

Now, we claim that, given any vertex sets $V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{k-1}$ with $|V_i| \geq 4|V_{i+1}|$ and satisfying that, for every $i > 0$, we have that the vertices of V_i have degree at least $(1 - 1/(16kd^2))|V_{i-1}|$ to V_{i-1}, there is a k-partite subgraph with vertex-disjoint parts $V_i' \subseteq V_i$ with $V_{k-1}' = V_{k-1}$ where each part has size at least $|V_{k-1}|$ in which, for any pair $i \neq j$, we have that the vertices of V_i have degree larger than $(1 - 1/(2d))|V_j|$ to V_j. To see this, note that, for any i and for every $j > i$, the average degree of a vertex in V_i to V_j is at least $(1 - 1/(16kd^2))|V_j|$. By Markov's inequality, at most a $1/(4kd)$ fraction of the vertices of V_i fail to have degree at least $(1 - 1/(4d))|V_j|$ to V_j. We now proceed as follows: for each i, starting with $i = 0$ and proceeding in increasing order, we may remove at most a $1/(4d)$ fraction of the vertices of V_i to ensure that every vertex of V_i has degree more than $(1 - 1/(4d))|V_j|$ to V_j for every $j > i$. For every pair of distinct i, j, at some point in this process V_i has satisfied that every vertex has degree more than $(1 - 1/(4d))|V_j|$ to V_j. After this point, at most a $1/(4d)$ fraction of the vertices of V_j were removed by the process, so at the end of the process we must have that the vertices of V_i have degree more than $(1 - 1/(2d))|V_j|$ to (what remains of) V_j. Given this structure, we may simply greedily embed a copy of $kF_{|V_{k-1}|}$ into this structure, using all of V_{k-1} and at most $(k-i)|V_{k-1}|$ vertices of each V_i.

We now proceed as follows. Iterate over the various distinct paths P in an arbitrary

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order. We wish to cover the vertices of $v_{P_{k-1}}$ by few graphs from $\mathcal{F}$. By the previous argument, we may do so if, for each $j > i$, we have that each vertex of V_{P_j} has degree at least $(1 - 1/(16kd^2))|V_{P_i}|$ to V_{P_i}; this was true before we started iterating. If this is true, we will then use the previous argument to find a graph that covers $V_{P_{k-1}}$ and uses at most $(k - i)|V_{P_{k-1}}|$ vertices from V_{P_i}. Throughout this process, we remove at most a

$$\varepsilon^{k-j-1}\left(\sum_{i=0}^{k-j-1} k^i\right) \leq 2(\varepsilon k)^{k-j-1} \leq 2\varepsilon k < 1/(16kd^2)$$

fraction of any V_{P_j} for $j < k - 1$. Therefore, we will remove at most a $1/(16kd^2)$ fraction of the vertices from each V_{P_i} throughout the process, and so the necessary condition will hold and we will cover each $V_{P_{k-1}}$. $\quad\square$

We now prove Theorem 4.1.2 and Theorem 4.1.3.

Proof. Let a proper graph sequence $\mathcal{F}$ with maximum degree at most Δ and chromatic number at most k be given. We know that $\mathcal{F}$ is R-Ramsey linear for some R, and we know by [45] that R is at most $2^{C_1 k\Delta}$ and by [25] that R is at most $2^{C_2 \Delta \log \Delta}$. Take R to be the minimum of $2^{C_1 k\Delta}$ and $2^{C_2 \Delta \log \Delta}$. Take, for some sufficiently large constant C, $\varepsilon = R^{-C}$. Take $\ell = 2k - 3$. Take $\delta = 1/(64d^2 k^2)$. Apply Lemma 4.4.3 to obtain $k^{O(k)} R 2^{O(k)} \varepsilon^{-O(1)} = \varepsilon^{-O(1)}$ copies of graphs from $\mathcal{F}$, a k-branching $k\varepsilon$-scaling $(1 - \delta - k\varepsilon) \geq (1 - 2\delta)$ cylinder with ℓ levels, and, for each w in the underlying branching tree, a $(2k\varepsilon, 1/2 - 2k\varepsilon, \delta - k\varepsilon)$ super-regular 0-balanced cylinder C_w, where these structures have the properties described in Lemma 4.4.3. We now apply Lemma 4.4.8 to the branching cylinder; this allows us to cover all of the vertices of the V_v where v is leaf of the branching tree and V_v is the corresponding vertex set of the branching cylinder using at most k^{2k} monochromatic copies of graphs from $\mathcal{F}$. Furthermore, for any w in the branching tree, either V_w is entirely covered, or the only vertices from V_w that are used are from C_w and we use at most a $k\varepsilon$ fraction of C_w. At this point, the only vertices that we have not covered with monochromatic copies of graphs from $\mathcal{F}$ are those vertices in the various C_w from which we've used at most a $k\varepsilon$ fraction of the vertices. Notice that, because each C_w is a $(2k\varepsilon, 1/2 - 2k\varepsilon, \delta - k\varepsilon)$ super-regular 0-balanced $k + 1$-partite cylinder, after removing at most a $k\varepsilon$ fraction of C_w, the

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result is at least a $(1/(k+1))$-balanced $(R^{-\Omega(C)}, 1/4, \delta/2)$-cylinder, so if C is large enough we may apply Lemma 4.3.3 and Lemma 4.4.7 to cover the remainder of the cylinder using at most $(k+1)^2$ monochromatic copies of graphs from $\mathcal{F}$. We used a total of $R^{O(C)}$ monochromatic copies of graphs from $\mathcal{F}$ when applying Lemma 4.4.3, we used at most $k^{O(k)} \leq R^{O(1)}$ copies of graphs from $\mathcal{F}$ when applying Lemma 4.4.8, and we used at most $(k+1)^2$ monochromatic copies of graphs from $\mathcal{F}$ from each of the $k^{O(k)}$ vertices of the branching tree when applying Lemma 4.4.7, so we used a total of $R^{O(C)}$ monochromatic copies of graphs from $\mathcal{F}$. If $k \leq \log \Delta$, then this value is $2^{O(Ck\Delta)}$; in particular, when $k = 2$ as in the case of Theorem 4.1.2, we use $2^{O(C\Delta)}$ monochromatic copies from $\mathcal{F}$. For all values of k, we use $2^{O(C\Delta \log \Delta)}$ monochromatic copies of graphs from $\mathcal{F}$, giving the bound in Theorem 4.1.3. $\square$

The proof of Theorem 4.1.4 is very similar, though it has some additional complications.

Proof. Let a proper graph sequence $\mathcal{F}$ which is a-nicely-arrangeable with chromatic number at most k be given. We know that $\mathcal{F}$ is R-Ramsey linear for some $R = 2^{O(a \log^2 a)}$. Take, for some sufficiently large constant C, $\varepsilon = 2^{-Ca^4k^2}$. Take $\ell = 2k - 3$. Take $\delta = 1/(64d^2k^2)$. Apply Lemma 4.4.3 to obtain $k^{O(k)}R2^{O(k)}\varepsilon^{-O(1)} = \varepsilon^{-O(1)}$ copies of graphs from $\mathcal{F}$, a k-branching $k\varepsilon$-scaling $(1 - \delta - k\varepsilon) \geq (1 - 2\delta)$ cylinder with ℓ levels, and, for each w in the underlying branching tree, a $(2k\varepsilon, 1/2 - 2k\varepsilon, \delta - k\varepsilon)$ super-regular 0-balanced cylinder C_w, where these structures have the properties described in Lemma 4.4.3. We now apply Lemma 4.4.8 to the branching cylinder; this allows us to cover all of the vertices of the V_v where v is leaf of the branching tree and V_v is the corresponding vertex set of the branching cylinder using at most k^{2k} monochromatic copies of graphs from $\mathcal{F}$. Furthermore, for any w in the branching tree, either V_w is entirely covered, or the only vertices from V_w that are used are from C_w and we use at most a $k\varepsilon$ fraction of C_w. At this point, the only vertices that we have not covered with monochromatic copies of graphs from $\mathcal{F}$ are those vertices in the various C_w from which we've used at most a $k\varepsilon$ fraction of the vertices. Notice that, because each C_w is a $(2k\varepsilon, 1/2 - 2k\varepsilon, \delta - k\varepsilon)$ super-regular 0-balanced $k + 1$-partite cylinder, after removing at most a $k\varepsilon$ fraction of C_w, the result is at

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least a $(1/(k+1))$-balanced $(2^{-\Omega(Ca^4k^2)}, 1/4, \delta/2)$-cylinder, so if C is large enough and C_w has enough vertices, we may apply Lemma 4.3.4 and Lemma 4.4.7 to cover the remainder of the cylinder using at most $(k+1)^2$ monochromatic copies of graphs from $\mathcal{F}$. However, if C_w doesn't have enough vertices, that is it has $2^{2^{O(a^4)}}$ vertices, then we may cover it using $2^{O(a^4)}$ monochromatic copies of graphs from $\mathcal{F}$ using Lemma 4.4.5. We used a total of $2^{O(Ck^2a^4)} = 2^{O(Ca^6)}$ monochromatic copies of graphs from $\mathcal{F}$ when applying Lemma 4.4.3, we used at most $k^{O(k)}$ copies of graphs from $\mathcal{F}$ when applying Lemma 4.4.8, and we used at most $2^{O(a^4)}$ monochromatic copies of graphs from $\mathcal{F}$ from each of the $k^{O(k)}$ vertices of the branching tree when applying Lemma 4.4.7, so we used a total of $2^{O(Ca^6)}$ monochromatic copies of graphs from $\mathcal{F}$, giving the bound in Theorem 4.1.4.

4.5   Concluding Remarks

There are various interesting potential generalizations of Theorem 4.1.3. One may ask if the theorem holds for r colors for any positive integer r.

Conjecture 4.5.1. *For every positive integer r there exists a constant C_r (depending on r) such that, for every Δ-bounded sequence $\mathcal{F}$, every r-edge-colored complete graph can be partitioned into at most $2^{\Delta^{C_r}}$ vertex-disjoint monochromatic graphs from $\mathcal{F}$.*

Since bounds on Ramsey numbers were key in proving the theorem for $r = 2$, it is worth noting that Conlon, Fox, and Sudakov [25] proved that, for any fixed number of colors r, for any graph G on n vertices of maximum degree Δ the Ramsey number on r colors $R_r(G)$ is at most $2^{C_r\Delta^2}n$. The primary difficulty in adapting our proof to this setting is in constructing a $(1-\gamma)$ branching cylinder when there are more than 2-colors.

Finally, let us mention that since by now both the Regularity Lemma and the Blow-up Lemma have been generalized to hypergraphs (see [81] and [62], respectively), perhaps we can generalize our result to hypergraphs as well.

4.6 Lower Bound

We wish to show that there exists a Δ-bounded sequence $\mathcal{F} = \{F_1, F_2, \ldots\}$ and, for n sufficiently large, a two-edge-coloring of K_n that cannot be partitioned into fewer than $2^{\Omega(\Delta)}$ monochromatic copies of graphs from $\mathcal{F}$. To see this, for every n take G_n to be a graph on n vertices of degree at most Δ and, for n sufficiently large, with Ramsey number at least $2^{\Omega(\Delta)}n$, as given by the result of Graham, Rödl and Ruciński [49]. We define F_{2^i} recursively; take $F_{2^0} = G_1$. Then define F_{2^i} to be the disjoint union of $F_{2^{i-1}}$ with $G_{2^{i-1}}$. For integers of the form $2^i + j$ with $j < 2^i$, define F_{2^i+j} to be the disjoint union of F_{2^i} with an independent set on j vertices. Under this definition, each F_n is a graph on n vertices with maximum degree at most Δ. Furthermore, for $n_0 < n_1$, F_{n_0} is a subgraph of F_{n_1}. Finally, taking i to be the largest integer with $2^i \leq n$, F_n contains a copy of $G_{2^{i-1}}$ and so has Ramsey number at least $2^{\Omega(\Delta)}2^{i-1} = 2^{\Omega(\Delta)}n$ (for n sufficiently large). Take $\mathcal{F} = \{F_1, F_2, \ldots\}$. Now, for N sufficiently large, take a 2-edge-coloring of a complete graph on $2^{\Omega(\Delta)}N$ vertices without a monochromatic copy of F_N (this is possible by the condition on the Ramsey number). Since the sequence of graphs is increasing, this coloring also does not contain a monochromatic copy of any F_n for $n > N$. Therefore, any partition of the vertex set into monochromatic copies of graphs from $\mathcal{F}$ must use at least $2^{\Omega(\Delta)}$ such copies.

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