NONLINEAR FEEDBACK SOLUTION FOR MINIMUM-TIME INTERCEPT WITH CONSTANT THRUST ACCELERATION

by

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ABSTRACT

The instantaneous thrust-direction for a missile to carry out a minimum-time intercept with a non-maneuvering target is determined as a function of instantaneous relative velocity and position. The magnitude of the thrust acceleration is assumed constant and the acceleration due to external forces is neglected.

This six-coordinate problem (three relative position coordinates and three relative velocity coordinates) can be reduced to a problem in two coordinates, namely $V^2/2ar$ and γ, where a is the magnitude of the thrust acceleration, V is the magnitude of the relative velocity, r is the distance between the missile and the target, and γ is the angle between the relative velocity vector and the line-of-sight between the missile and the target.

Natural quantities to measure during the intercept maneuver are r, $\dot{r}$, and $\dot{\sigma}$, where $\dot{\sigma}$ is the rate of rotation of the line of sight relative to a fixed reference axis. The two dimensionless coordinates in terms of these measured quantities are:

$$\frac{V^2}{2ar} = \frac{(\dot{r}) \;\; + (r\dot{\sigma})^2}{2ar} \qquad \tan\gamma = \frac{r\dot{\sigma}}{(-\dot{r})}$$

Let θ be the angle between the thrust vector and the line-of-sight, and let T be the time-to-intercept. Θ and $\frac{aT}{V}$ for minimum-time intercept are given, both analytically and graphically, as functions of $V^2/2ar$ and γ. The analytic solutions are transcendental equations

$$\frac{V^2}{2ar} = \frac{\sin^2\theta}{4\sin\gamma\,\sin(\theta+\gamma)}$$

and

$$\frac{aT}{V} = \frac{2\sin\gamma}{\sin\theta}$$

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References

1 Bryson, A.E., "Nonlinear Feedback Solution for Minimum-time Rendezvous with Constant Thrust Acceleration," Cruft Laboratory Technical Report No. 478, July 15, 1965.

CHAPTER 1

INTRODUCTION

The intercept maneuver consists of bringing the relative position of a pursuer with respect to a target to zero. Feedback control will almost certainly be required to do it. This thesis considers the intercept feedback control for the case where the target is not maneuvering and the pursuing vehicle has a thrust acceleration of constant magnitude, a, but a controllable direction. External forces are neglected, or equivalently the external forces per unit mass (such as gravity) are assumed to be constant in magnitude and direction during the maneuver. This latter assumption is reasonably good for nearly-circular satellite orbits if the maneuver time is short enough that the angular distance travelled around the attracting center is smaller than 30° to 40°[1].

CHAPTER 2

MINIMUM-TIME INTERCEPT USING CONTINUALLY VARYING THRUST DIRECTION

Consider the origin in the target, which is assumed to be moving with constant velocity with respect to an inertial coordinate system. The intercepting vehicle must then bring its position to zero in minimum time. The problem is two-dimensional since the target, the intercepting vehicle, and the relative velocity vector determine a maneuvering plane. The equations of motion for the intercepting vehicle are:

$$\dot{u} = -a\cos\theta \tag{2.1}$$

$$\dot{v} = -a\sin\theta \tag{2.2}$$

$$\dot{x} = u \tag{2.3}$$

$$\dot{y} = v \tag{2.4}$$

where u and v are velocity components, x and y are position components, and the magnitude of the thrust acceleration, a, is assumed constant. (See Figure 1)

The Hamiltonian of the system is

$$H = 1 - \lambda_u a\cos\theta - \lambda_v a\sin\theta + \lambda_x u + \lambda_y v \tag{2.5}$$

Thus the Euler-Lagrange equations are

$$\dot{\lambda}_u = -\frac{\partial H}{\partial u} = -\lambda_x \tag{2.6}$$

$$\dot{\lambda}_v = -\frac{\partial H}{\partial v} = -\lambda_y \tag{2.7}$$

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0 \tag{2.8}$$

$$\dot{\lambda}_y = -\frac{\partial H}{\partial y} = 0 \tag{2.9}$$

$$\frac{\partial H}{\partial \theta} = 0 = \lambda_u a\sin\theta - \lambda_v a\cos\theta \tag{2.10}$$

Since u and v are not specified at time $t = T$ when the intercept occurs, the influence functions λ_u and λ_v at this time are zero. Equations (2.6) to (2.9) may be integrated to yield

$$\lambda_u = \lambda_x(T-t) \tag{2.11}$$

$$\lambda_v = \lambda_y(T-t) \tag{2.12}$$

$$\lambda_x = \text{constant} \tag{2.13}$$

$$\lambda_y = \text{constant} \tag{2.14}$$

Combining equations (2.11) to (2.14) with equation (2.10) yields

$$\tan\theta = \text{constant} \tag{2.15}$$

or

$$\theta = \text{constant} \tag{2.16}$$

CHAPTER 3

REDUCTION OF MINIMUM-TIME INTERCEPT TO A TWO COORDINATE PROBLEM

Recalling that $x = 0 = y$ at $t + T$, the time of intercept, and that equation (2.16) states that the control angle θ is a constant, equations (2.1) to (2.4) may be integrated to yield

$$0 = r - VT \cos\gamma - 1/2\ (a \cos\theta)\ T^2 \quad (3.1)$$

$$0 = VT \sin\gamma - 1/2\ (a \sin\theta)\ T^2 \quad (3.2)$$

Equation (3.2) yields

$$T = \frac{2V \sin\gamma}{a \sin\theta} \quad (3.3)$$

or

$$T = 0 \quad (3.4)$$

Substituting equation (3.3) into (3.1) yields

$$r = \frac{2V^2 \cos\gamma \sin\gamma}{a \sin\theta} + \frac{2V^2 \sin\gamma \cos\theta}{a \sin^2\theta} \quad (3.5)$$

which may be simplified to

$$1 = \frac{V^2\ 4\sin\gamma\ \sin(\theta+\gamma)}{2ar\ \sin^2\theta} \quad (3.6)$$

Thus the transcendental equation in terms of the two coordinates V^2/ar and γ for the optimal control angle θ, which produces a minimum-time intercept is

$$\frac{V^2}{2ar} = \frac{\sin^2\theta}{4\sin\gamma\ \sin(\theta+\gamma)} \quad (3.7)$$

Figure 2 and Figure 3 show curves of constant thrust angle plotted versus the dimensionless parameters $V^2/2ar$ and γ.

If a dimensionless time to go

$$X = \frac{aT}{V} \quad (3.8)$$

is designated, then substitution of equation (3.8) into (3.3) yields

$$X = \frac{2\sin\gamma}{\sin\theta} \quad (3.9)$$

and substitution of equation (3.9) into (3.7) yields

$$\frac{V^2}{2ar} = \frac{\sin\theta}{2X\sin(\theta+\gamma)} \tag{3.10}$$

which may be further simplified to

$$\frac{V^2}{2ar} = \frac{1}{X\,[2\cos\gamma + \sqrt{X^2 - 4\sin^2\gamma}]} \tag{3.11}$$

Equation (3.11) is a transcendental equation for the dimensionless time to go $\frac{aT}{V}$ in terms of the two dimensionless parameters $\frac{V^2}{2ar}$ and γ. Figure 4 shows some curves of constant $\frac{aT}{V}$ plotted versus $\frac{V^2}{2ar}$ and γ. Figure 5 shows some optimal paths plotted versus $\frac{V^2}{2ar}$ and γ.

Figures 2, 3, 4 and 5 all show a barrier which is discussed in detail in Chapter 4.

CHAPTER 4

DEVELOPMENT OF THE BARRIER EQUATION

In Chapter 2, it was shown that a constant thrust angle with respect to an inertial reference line is required to achieve a minimum-time intercept. In some cases there exist two control angles resulting in two paths which accomplish the intercept, with one path being a local minimum and the other being the global minimum. Figure 6 shows an example of two possible paths with constant thrust angle from the same initial point.

As can be seen in Figure 6, there is one path where the interceptor thrusts in the general direction of the target in order to rotate its velocity vector directly towards the target and thus accomplish the intercept. The other path is one where the interceptor uses its thrust to first slow down, then turns to come back to hit the target. This second path takes a longer period of time than the first. It is easily seen that if the initial conditions in Figure 6 were altered sufficiently, the interceptor would find there is a set of initial conditions where it can no longer thrust in the general direction of the target in order to hit it, but must thrust to first slow down, then reverse direction and come back to hit the target.

This set of initial conditions determines a barrier, the equation of which will now be developed. Let

$$y = \frac{V^2}{2ar} = f(\gamma,\theta) \tag{4.1}$$

where

$$f = \frac{\sin^2\theta}{4\sin\gamma\ \sin(\theta+\gamma)} \tag{4.2}$$

and $$dy = \frac{\partial f}{\partial \theta} d\gamma + \frac{\partial f}{\partial \theta} d\theta \tag{4.3}$$

For the barrier

$$\frac{\partial f}{\partial \theta} = 0 = \frac{1}{4\sin\gamma} \quad [\sin(\theta+\gamma)\ 2\sin\theta\cos\theta - \sin^2\theta\cos(\theta+\gamma)] \tag{4.4}$$

therefore

$$\sin\theta\cos\theta\cos\gamma + \cos^2\theta\sin\gamma + \sin\gamma = 0 \tag{4.5}$$

which leads to

$$\cos\theta = \pm\sqrt{1/2\ [1 - 3\sin^2\gamma \pm \cos\gamma\sqrt{1 - 9\sin^2\gamma}\ ]} \tag{4.6}$$

If Figure 2 is initially plotted without the barrier, it can be seen that for $\gamma \geq 0$ the barrier exists at values of $\theta > 0$ since the value $\theta = 0$ is also the $\gamma = 0$ line. Equation (4.6) may then be reduced to

$$\cos\theta = -\sqrt{1/2[1 - 3\sin^2\gamma - \cos\gamma\sqrt{1 - 9\sin^2\gamma}\ ]} \tag{4.7}$$

It can then be shown that the barrier starts at $\gamma = 0°$, $\theta = 90°$, $\frac{V^2}{2ar} = \infty$ and follows equation (4.7) to its limit at $\gamma = 19.5°$, $\theta = 125.2°$ and $\frac{V^2}{2ar} = .866$.

CHAPTER 5

THE MINIMUM THRUSTING-TIME INTERCEPT PROBLEM

If the interceptor is only able to start its engine once, and if the engine is to be kept on until intercept, then the minimum thrusting-time solution will be of interest. Figure 7 shows the situation where the interceptor is coasting towards the target with constant relative velocity V_o, and, if no thrust were applied, the interceptor would miss the target by a distance y_o. This and other straight line coasting paths will show as sine curves on figures plotting $V^2/2ar$ against γ, with the paths commencing at $V^2/2ar = 0$, $\gamma = 0$ and going to $V^2/2ar = 0$, $\gamma = 180°$. This is easily shown since along the straight line paths

$$\frac{V^2}{2ar} = \frac{Vo^2}{2ay_o} \sin\gamma \qquad (5.1)$$

The minimum thrusting-time problem is the determination of the position at which the engine should be turned on, and the direction in which to thrust.

The solution to this problem is to thrust normal to the velocity vector V_o and to commence thrusting at that position which leads to intercept if the thrust is applied normal to the velocity vector V_o. Figure 1 shows that the angle between the velocity and the acceleration is the sum of the angles θ and γ. Thus at the starting point

$$\theta + \gamma = 90° \qquad (5.2)$$

The locus of starting points described by equation 5.2 is shown on

Figure 8 with minimum thrusting-time paths added. Figure 8 shows that the interceptor must immediately start its engine and follow a minimum time path if the initial conditions place the point $V_o^2/2ar$ and γ to the right of the starting locus, or otherwise the interceptor must coast to the starting locus.

CHAPTER 6

IMPLEMENTATION OF RESULTS

For a missile to utilize the results obtained in the previous chapters, Figures 2, 3, 4 and 8 could be stored digitally in the memory of a missile computer. The missile would then measure r, $\dot{r}$ and $\dot{\sigma}$, compute $V^2/2ar$ and γ, and then refer to its memory to obtain the optimum control angle θ. This cycle would then be repeated as often as the missile could perform the necessary measurements and computations.

With a digital storage of the nonlinear feedback law, it would be necessary to use an interpolation scheme to obtain the control angle θ. The total differential of equation (3.7) yields

$$\Delta\theta = \frac{\dfrac{\Delta V^2/2ar}{V^2/2ar} + [\cot\gamma + \cot\,(\theta+\gamma)]\Delta\gamma}{2\cot\theta - \cot(\theta+\gamma)} \tag{6.1}$$

Even if the missile does not apply the exact optimal control angle θ, it will nevertheless thrust in a direction such as to create a near miss. Using an interpolation scheme based on equation (6.1), it can be seen that as the missile gets closer and closer to the target, the amount of interpolation required becomes less and less.

The method described to effect a minimum-time intercept appears feasible; with the amount of information about the optimal control law stored in the missile computer being dependent upon the missile characteristics.

CHAPTER 7

CONCLUSIONS

Continuous nonlinear feedback laws have been obtained for controlling the thrust direction to effect a minimum-time intercept and a minimum thrusting-time intercept of a pursuer with a non-maneuvering target. These feedback laws depend on only two dimensionless quantities which can be determined by measurements of three physical quantities: (1) distance to the target, (2) closing velocity along the line of sight, and (3) rate of rotation of the line-of-sight with respect to an inertial axis in the maneuver plane.

A method of implementing these feedback laws was also presented.

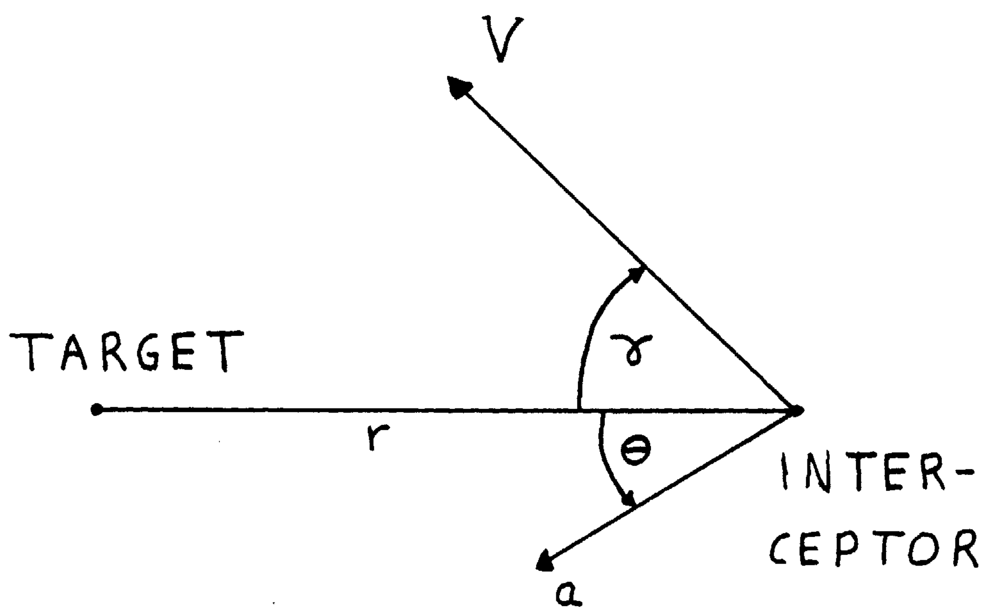


FIGURE 1

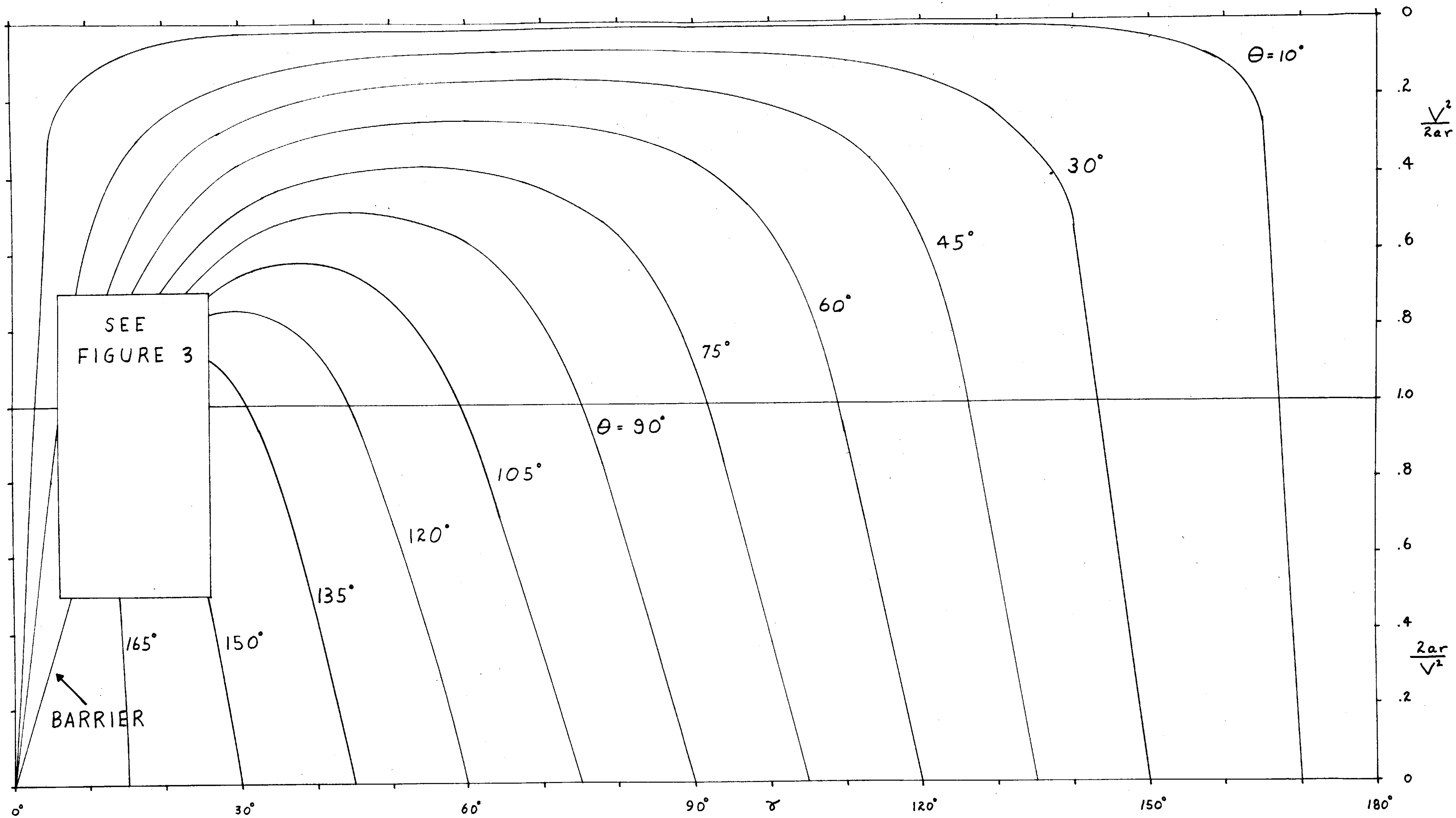
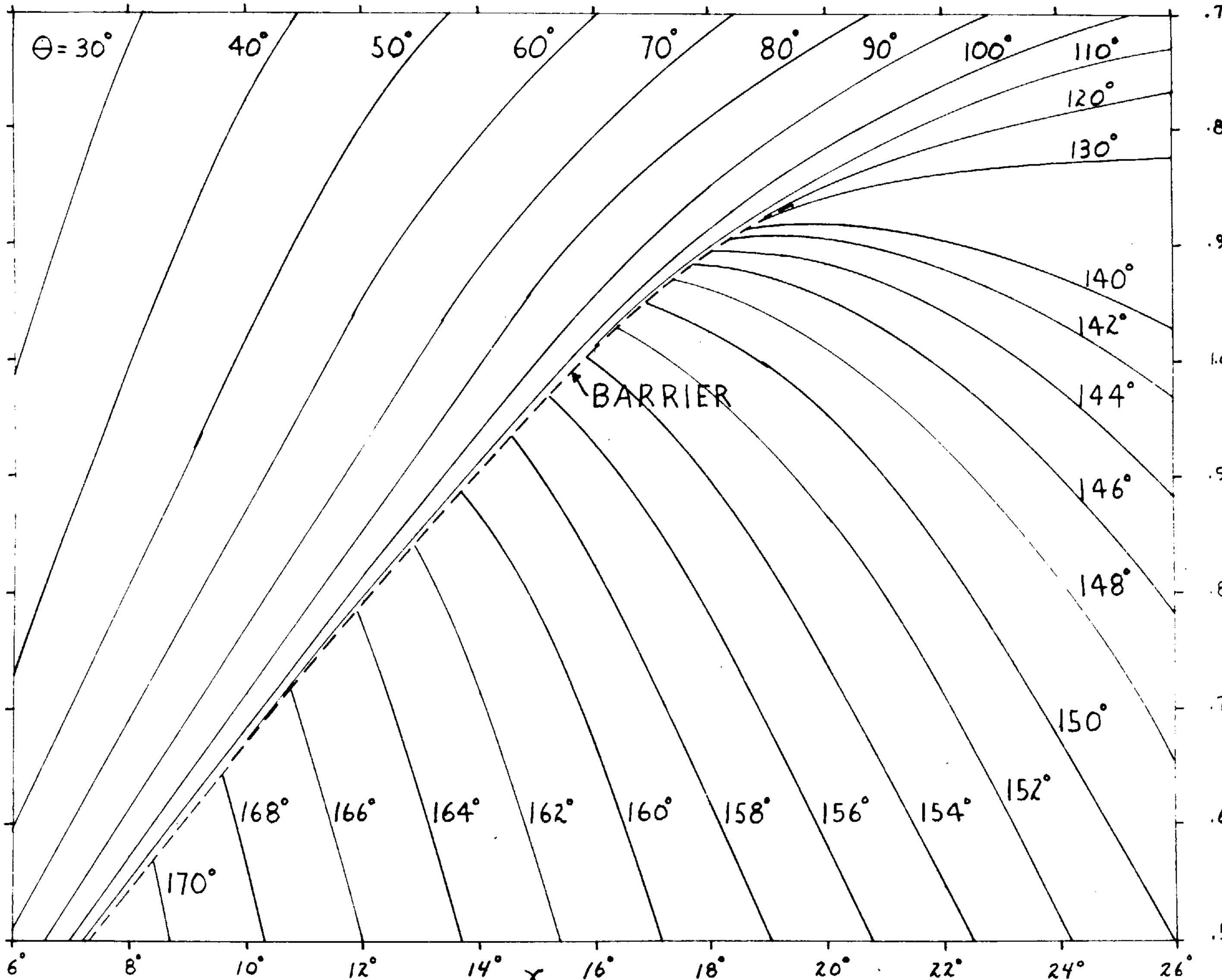


FIGURE 2 OPTIMAL THRUST DIRECTION, θ, AS A FUNCTION OF $V^2/2ar$ AND γ



Θ = 30°
40°
50°
60°
70°
80°
90°
100°
110°
120°
130°
140°
142°
144°
146°
148°
150°
152°
154°
156°
158°
160°
162°
164°
166°
168°
170°
BARRIER
6°
8°
10°
12°
14°
16°
18°
20°
22°
24°
26°
γ
.7
.8
.9
1.0
.9
.8
.7
.6
.5
V²/2ar
2ar/V²

FIGURE 3

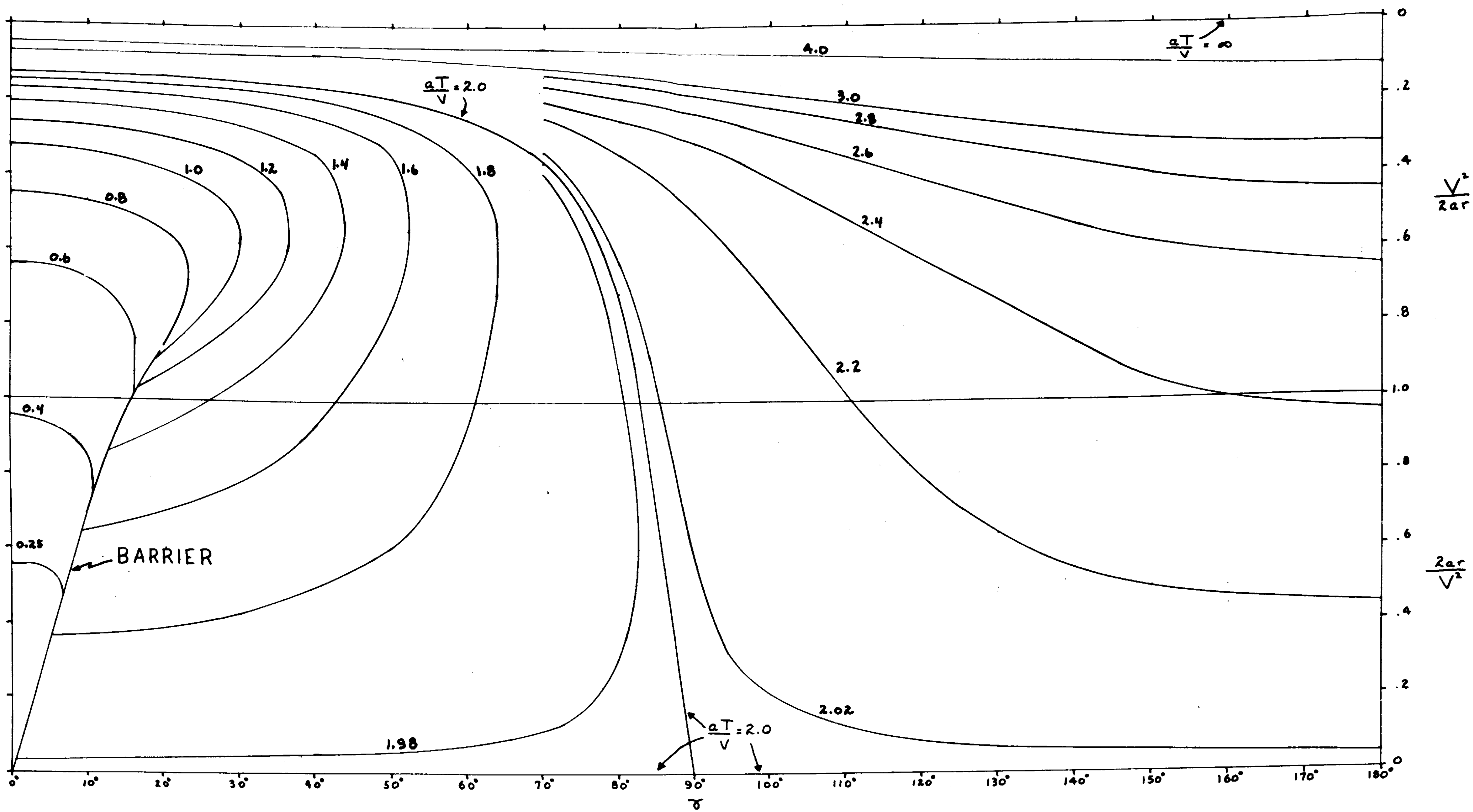


FIGURE 4 $\frac{aT}{V}$ AS A FUNCTION OF $\frac{V^2}{2ar}$ AND γ

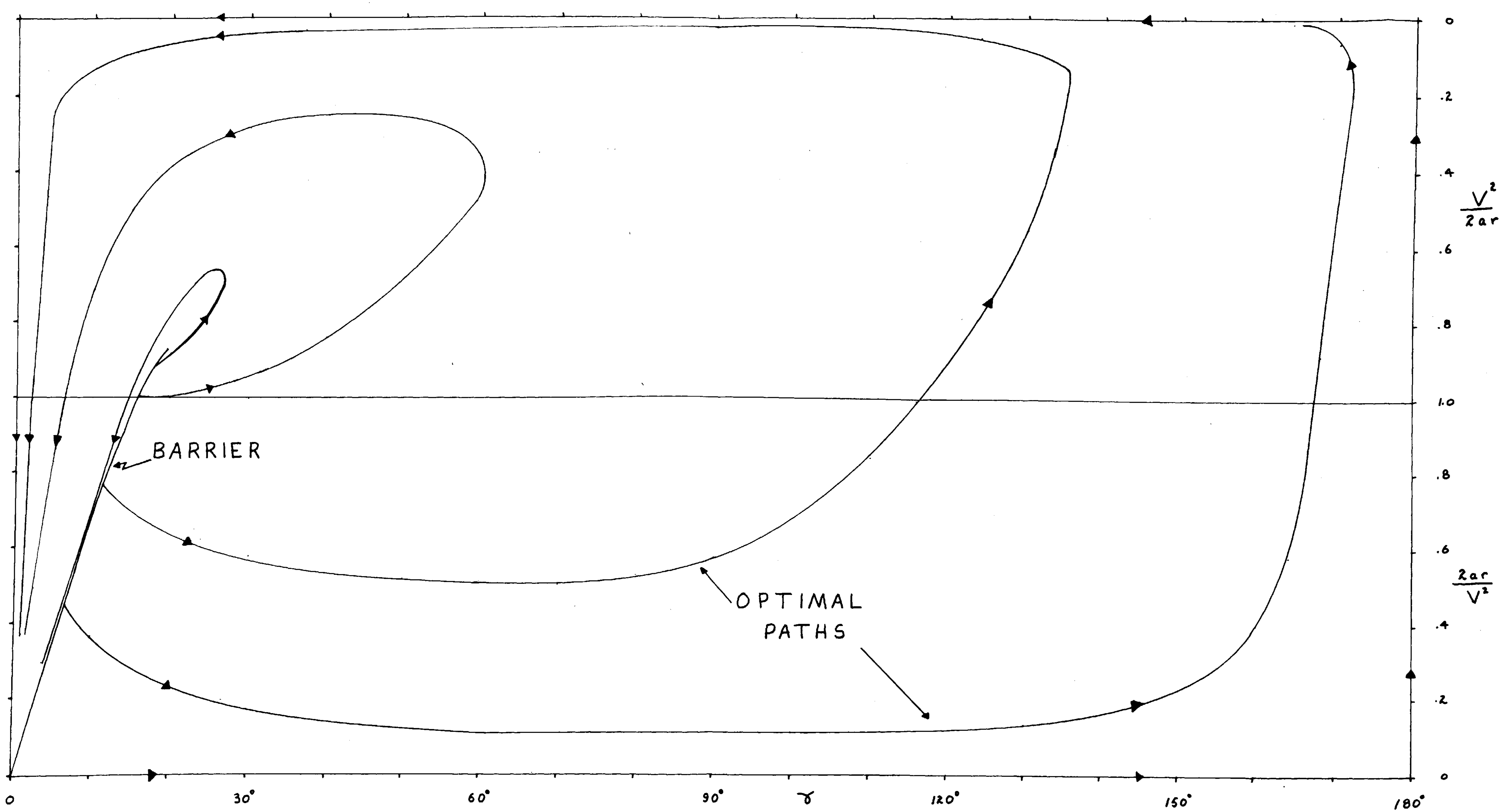
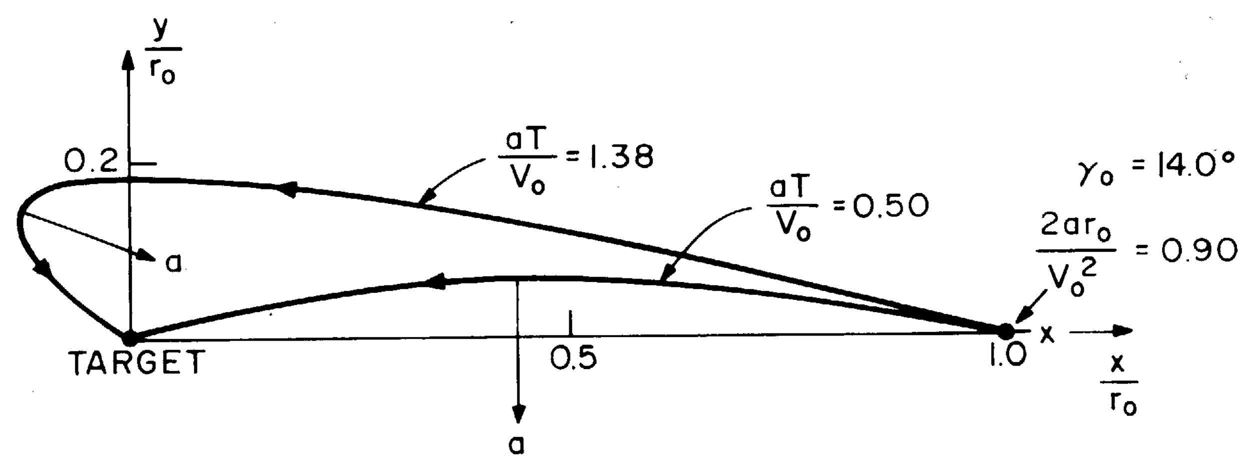


FIGURE 5 OPTIMAL PATHS FOR MINIMUM-TIME INTERCEPT AS A FUNCTION OF $V^2/2ar$ AND γ

FIGURE 6

MINIMUM TIME INTERCEPT—
TWO STATIONARY PATHS FROM SAME
INITIAL POINT



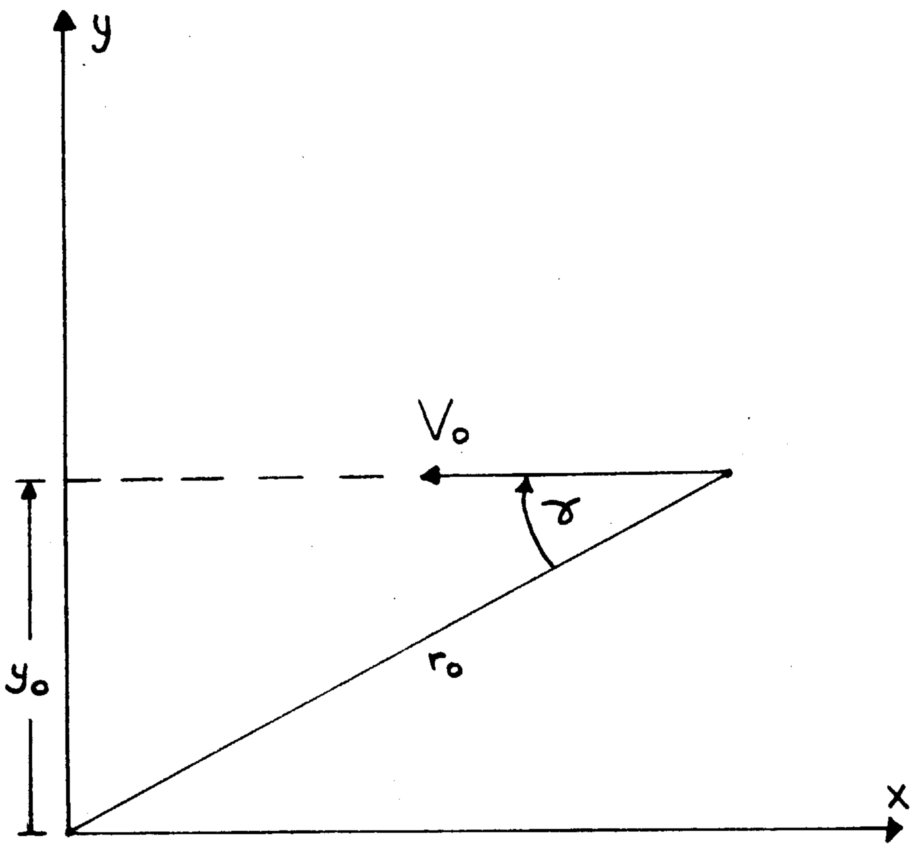


FIGURE 7

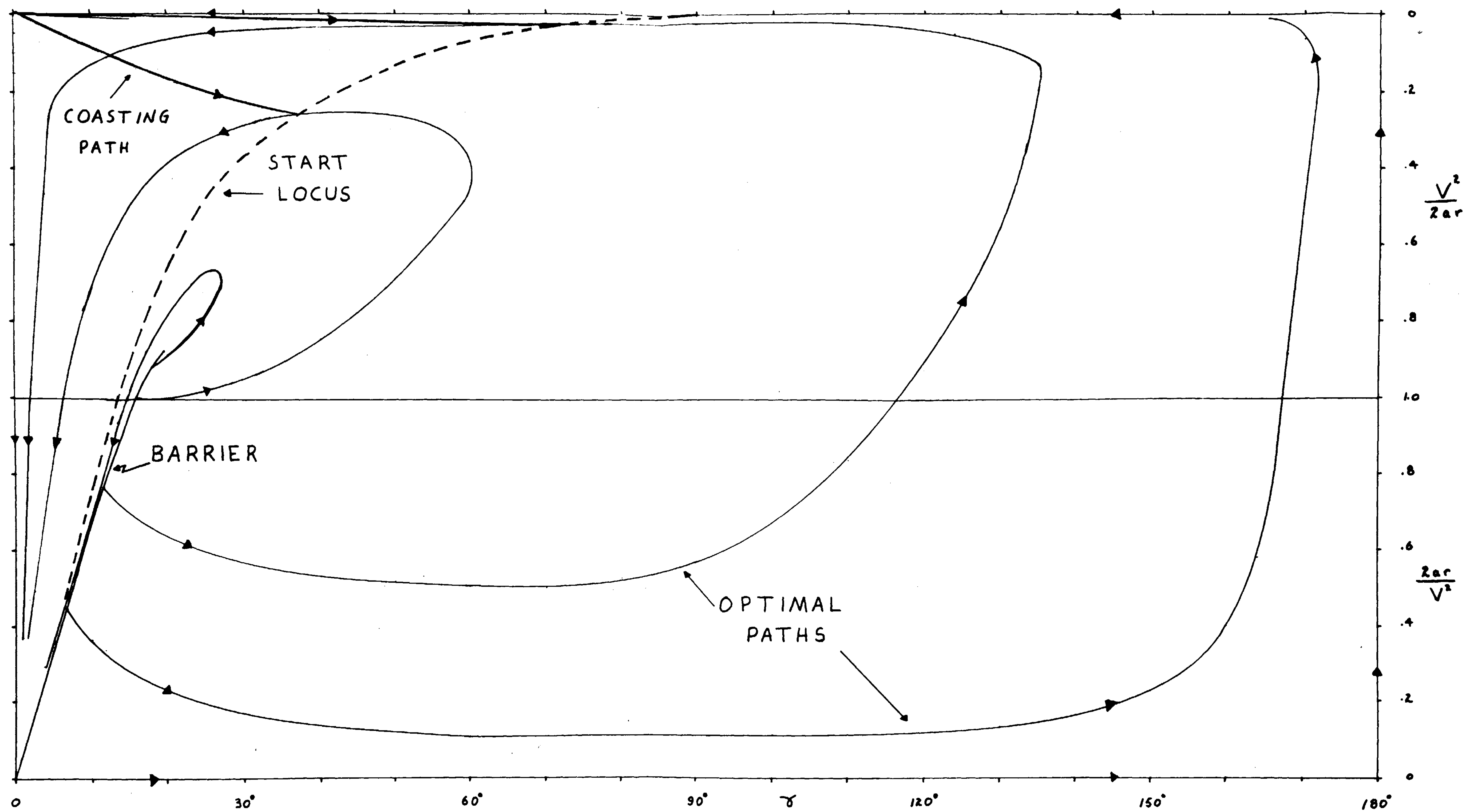


FIGURE 8 OPTIMAL PATHS FOR MINIMUM THRUSTING-TIME INTERCEPT AS A FUNCTION OF $V^2/2ar$ AND γ