A STUDY OF FORMULATIONS FOR LARGE STRAIN ELASTO-PLASTIC FINITE ELEMENT ANALYSIS

by

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Abstract

This study addresses, within the framework of modern continuum mechanics, the existing "first order" constitutive models for finite deformation elasto-plasticity. Currently available theories are presented in a unified and comprehensive review and non-trivial consequences of the basic hypotheses are explored.

Extension of the classical infinitesimal theory of plasticity to the large deformation range requires three fundamental steps: the selection of the underlying finite strain elasticity theory, the choice of a pair of stress and strain measures and the appropriate characterization of plastic flow.

A comparison is made between related hyper-elastic and hypo-elastic stress-strain laws. Theoretical and practical advantages and disadvantages of the use of these theories in the formulation of elasto-plastic constitutive equations are discussed.

Large strain plastic flow can be modeled via the product decomposition of the deformation gradient or via the additive decomposition of the strain tensor. Constitutive equations based on the latter approach are shown to restrict the choice of strain measure to the Hencky strain and to present plastic-strain induced anisotropic elastic response.

Based on the previous considerations, a hyper-elastic based constitutive model for large deformation elasto-plasticity based on the product decomposition of the deformation gradient is presented. A time integration algorithm is developed and implemented. Comparison is made with an equivalent constitutive model based on the additive decomposition of the strain tensor. The two theories are shown to predict very similar results within the range of applicability of an isotropic hardening model.

Thesis Supervisor: Dr. Klaus-Jürgen Bathe
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To my loving parents

Vaša podpora je nadmašila granice Vaših snaga i mogućnosti.
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Introduction

Constitutive modeling of elasto-plastic materials in the finite deformation range has received considerable attention over the past years. The availability of powerful computers and efficient finite element techniques have made feasible the solution of large scale finite deformation problems, thus increasing the demand for more realistic and accurate models.

The formulation of constitutive equations with good predictive capabilities presents difficulties arising from many sources. Even in the small strain limit the phenomenological experimental database is so rich that no simple model accounts for the observed behavior. However, first-order approximations are well established, as in the case of the classical infinitesimal theory of plasticity.

Extensions of the isotropic and kinematic hardening theories to the large strain range are not straightforward, and a number of alternative choices, all having identical "small strain limit", can be made. Basic questions arise about the proper kinematic description of plastic flow, the characterization of the underlying elastic behavior, and the choice of adequate stress and strain measures among other relevant issues.

It is the purpose of this study to address, within the framework of modern continuum mechanics, the existing "first order" constitutive models for finite deformation elasto-plasticity. Currently available theories are presented in a unified and comprehensive review and non-trivial consequences of the underlying hypotheses are explored.

A time integration algorithm for a hyperelastic-based rate-independent isotropic hardening constitutive model with the assumption of product decomposition of the deformation gradient is presented, and results predicted by this theory are compared with an equivalent theory based on the assumption of additive decomposition of the strain tensor.

The thesis is divided in eleven chapters. Chapter 1 introduces the notation to be used, some of the required results of tensor algebra and basic kinematic concepts such as the material, spatial and referential descriptions of a physical quantity and related differential operators.

Chapter 2 summarizes the principles of mass conservation and balance of linear and angular momentum, and the theorem of power expended. With the definition of the first Piola-Kirchhoff stress tensor, the referential form of these principles and the associated field equations are obtained.

Chapter 3 summarizes the continuum version of the first and second thermodynamic laws and the reduced dissipation inequality. It presents the field equations in both the spatial and referential form.

Chapter 4 deals with generalized work-conjugate strain and stress measures. After some kinematic results concerning the stretching and strain rate tensors are obtained, explicit formulae for the stress measures are presented. A tensorial version of these formulae is obtained for the case of collinear stress and strain tensors.

Chapter 5 introduces the basics of constitutive modeling and the framework of thermodynamics with internal variables. Most of the Chapter is devoted to the principle of material frame indifference, and the concepts of objectivity and invariance under rigid body motions. The discussion on invariant constitutive equations motivates the use of the results of Chapter 4.

Chapter 6 deals with the derivation of constitutive equations for thermo-elasto-plasticity based on the assumption of additive decomposition of the strain tensor. The response functions are reduced by means of the Coleman-Noll methodology, material symmetry considerations and further simplifying assumptions. The structure of constitutive equations for temperature dependent and therefore also for isothermal processes is obtained. These equations allow in general for rate-dependency of plastic flow. The rate-independent limit of the models is referred to as well.

Chapter 7 analyses the consequences of the assumptions made in the constitutive equations of Chapter 6. It is shown that the only strain measure consistent with the condition of plastic deformation being isochoric is the Hencky strain. In general, the additive decomposition of the strain tensor is shown to predict changes in the initial elastic response moduli depending on the plastic state. As a consequence of previous plastic flow, initially isotropic elastic response becomes anisotropic, the higher the level of plastic stretching the lower the ratio of "modified" over initial shear moduli.

Chapter 8 introduces the alternative description of plastic flow based on the product decomposition of the deformation gradient. Paralleling the developments of Chapter 6, constitutive equations for large strain thermo-elasto-plasticity are derived. Using the reduced dissipation inequality, similar symmetry considerations and some simplifying assumptions, a reduced model is obtained. In particular, the isothermal case and the rate-independent limit are considered.

Chapter 9 refers to the elasticity description underlying an elasto-plastic model. After recalling the basic definitions of elastic, hyper-elastic and hypo-elastic response, a comparison is made between the simplest hypo-elastic stress-strain law and its related hyper-elastic counterpart. It is shown that for the hypo-elastic description to be a good approximation of the hyper-elastic (total) one, both elastic stretches and stretch rates have to be small. A hyper-elastic stress-strain law, however, seems to be the natural choice, presenting both theoretical and practical advantages.

Chapter 10 presents a time integration procedure for a hyperelastic-based rate independent isotropic hardening model of the type discussed in Chapter 8. This algorithm has been implemented and tested in the finite element program ADINA [1987]. Of special interest is the comparison with an equally hyperelastic-based rate independent isotropic hardening model of the type discussed in Chapter 6. The formulations based on the additive decomposition of the strain tensor and the product decomposition of the deformation gradient lead to the same results for problems where the elastic and plastic stretch tensors commute. When this is not the case the differences are shown to be very small for the range of moderate strains, where the isotropic hardening model is adequate.

Finally, Chapter 11 presents a final discussion, overview and conclusions.

Chapter 1

Basic concepts and notation

1.1 Tensor notation

First rank tensors—vectors—are denoted by lowercase letters in boldface, e. g. $\boldsymbol{x}$, $\boldsymbol{v}$. Second rank tensors—referred to simply by tensors—are denoted by uppercase letters in boldface, e. g. $\boldsymbol{U}$, $\boldsymbol{T}$. Fourth order tensors are also of interest, and will be denoted by bold calligraphic letters, e. g. $\boldsymbol{\mathcal{L}}$, $\boldsymbol{\mathcal{H}}$.

Let V be the space of all vectors $\boldsymbol{v}$. Given a basis $\{\mathbf{e}_i\}$ in V, the tensors $\boldsymbol{v}$, $\boldsymbol{T}$ and $\boldsymbol{\mathcal{H}}$ admit the representation,

$$\boldsymbol{v} = v_i\, \boldsymbol{e}_i\,, \tag{1}$$

$$\boldsymbol{T} = T_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j\,, \tag{2}$$

$$\boldsymbol{\mathcal{H}} = \mathcal{H}_{ijkl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l\,. \tag{3}$$

In these equations repeated indices indicate summation from 1 to 3 and the symbol '$\otimes$' denotes tensor product.

The dot product of vector $\mathbf{v}$ and vector $\boldsymbol{u}$ is the scalar $\mathbf{v} \cdot \boldsymbol{u}$. In coordinate representation,

$$\mathbf{v} \cdot \boldsymbol{u} = v_i u_i \,. \tag{4}$$

The product $\boldsymbol{AB}$ of the second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ is the second order tensor $\boldsymbol{AB}$ given by

$$(\boldsymbol{AB})\mathbf{v} = \boldsymbol{A}(\boldsymbol{B}\mathbf{v}) \,, \tag{5}$$

for all vectors $\mathbf{v}$. In coordinate representation,

$$\boldsymbol{AB} = A_{ik} B_{kj} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \,. \tag{6}$$

The dot product of the second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ is the scalar $\boldsymbol{A} \cdot \boldsymbol{B}$ given by

$$\boldsymbol{A} \cdot \boldsymbol{B} = \mathrm{tr}\,(\boldsymbol{A}^T \boldsymbol{B}) \,, \tag{7}$$

where ' tr ' is the trace operator. In coordinate representation,

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_{ij} B_{ij} \,. \tag{8}$$

The conmutator $[\boldsymbol{A}, \boldsymbol{B}]$ of tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ is the tensor

$$[\boldsymbol{A}, \boldsymbol{B}] = \boldsymbol{AB} - \boldsymbol{BA} \,. \tag{9}$$

For a second order tensor $\boldsymbol{T}$ operating on a vector $\boldsymbol{n}$ one defines $\boldsymbol{Tn}$ as the vector

$$\boldsymbol{Tn} = T_{ij} n_j \boldsymbol{e}_i \,. \tag{10}$$

For a fourth order tensor $\boldsymbol{\mathcal{L}}$ operating on a second order tensor $\boldsymbol{E}$ one defines $\boldsymbol{\mathcal{L}}[\boldsymbol{E}]$ as the second order tensor

$$\boldsymbol{\mathcal{L}}[\boldsymbol{E}] = \mathcal{L}_{ijkl}E_{kl}\mathbf{e}_i\otimes\mathbf{e}_j \,. \tag{11}$$

The second order identity tensor is denoted by $\boldsymbol{1}$. The fourth order symmetric identity tensor is denoted by $\boldsymbol{I}$. In coordinate representation,

$$\boldsymbol{1} = \delta_{ij}\mathbf{e}_i\otimes\mathbf{e}_j \,, \tag{12}$$

$$\boldsymbol{I} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_l \,, \tag{13}$$

where δ_{ij} is the Kronecker delta.

The eigenvalues λ_i of a second order tensor $\boldsymbol{U}$ are the roots of the characteristic polinomial of $\boldsymbol{U}$,

$$p(\lambda) = \det(\boldsymbol{U} - \lambda\boldsymbol{1}) = -\lambda^3 + \lambda^2 I_1(\boldsymbol{U}) - \lambda I_2(\boldsymbol{U}) + I_3(\boldsymbol{U}) \,, \tag{14}$$

where ' det ' is the determinant operator and $I_i(\boldsymbol{U})$ are the principal invariants of tensor $\boldsymbol{U}$. These can be written as

$$I_1(\boldsymbol{U}) = \operatorname{tr}\boldsymbol{U} \,, \tag{15}$$

$$I_2(\boldsymbol{U}) = \operatorname{tr}(\operatorname{adj}\boldsymbol{U}) \,, \tag{16}$$

$$I_3(\boldsymbol{U}) = \det\boldsymbol{U} \,, \tag{17}$$

where ' adj ' is the adjugate operator, given by

$$\operatorname{adj}\boldsymbol{U} = (\det\boldsymbol{U})\boldsymbol{U}^{-T} \,. \tag{18}$$

The Cayley-Hamilton theorem states that every tensor $\boldsymbol{U}$ satisfies its own characteristic equation,

$$-\boldsymbol{U}^3 + \boldsymbol{U}^2 I_1(\boldsymbol{U}) - \boldsymbol{U} I_2(\boldsymbol{U}) + \boldsymbol{1} I_3(\boldsymbol{U}) = \boldsymbol{0}. \tag{19}$$

We denote by $\mathcal{L}$ the space of all second order tensors. The following subsets of $\mathcal{L}$ will be referred to

$$\mathcal{L}_+ = \{\boldsymbol{X} \in \mathcal{L}, \quad \det \boldsymbol{X} > 0\}, \tag{20}$$

$$\mathcal{S} = \{\boldsymbol{T} \in \mathcal{L}, \quad \boldsymbol{T}^T = \boldsymbol{T}\}, \tag{21}$$

$$\mathcal{S}_+ = \{\boldsymbol{U} \in \mathcal{S}, \quad \mathbf{v} \cdot \boldsymbol{U}\mathbf{v} > 0 \quad \forall\, \mathbf{v} \neq \boldsymbol{0}\}, \tag{22}$$

$$\mathcal{O} = \{\boldsymbol{R} \in \mathcal{L}, \quad \boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{1}\}, \tag{23}$$

$$\mathcal{O}_+ = \{\boldsymbol{R} \in \mathcal{O}, \quad \det \boldsymbol{R} = 1\}, \tag{24}$$

$\mathcal{L}_+$ is the set of all second order tensors with positive determinant, $\mathcal{S}$ is the subspace of all symmetric second order tensors, $\mathcal{S}_+$ is the set of all symmetric positive definite second order tensors, $\mathcal{O}$ is the orthogonal group and $\mathcal{O}_+$ is the proper orthogonal group.

Let $\boldsymbol{X} \in \mathcal{L}_+$. Then the Polar Decomposition Theorem states that there exist tensors $\boldsymbol{U}, \boldsymbol{V} \in \mathcal{S}_+$ and $\boldsymbol{R} \in \mathcal{O}_+$ such that

$$\boldsymbol{X} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}. \tag{25}$$

Moreover, each of these decompositions is unique.

The exponential $\exp \boldsymbol{A}$ of a second order tensor $\boldsymbol{A}$ is the second order tensor defined by the series (see for example HIRSCH and SMALE [1974])

$$\exp \boldsymbol{A} = \sum_{n=0}^{\infty} \frac{\boldsymbol{A}^n}{n!} . \tag{26}$$

Let $\boldsymbol{A} \in \mathcal{L}$. Then the following proposition holds

$$\det(\exp \boldsymbol{A}) = \exp(\operatorname{tr} \boldsymbol{A}) . \tag{27}$$

1.2 Kinematics

1.2.1 Bodies, configurations and motions

A "body" $\mathcal{B}$ is a set of elements which, at some point in time, can be put into one-to-one correspondence with some region (an open connected subset) R of the Euclidean point space E. An element $\boldsymbol{p}$ of $\mathcal{B}$ is called a "particle" or "material point". A one-to-one mapping

$$\begin{gathered} \varphi : \mathcal{B} \rightarrow R; \boldsymbol{p} \mapsto \boldsymbol{x} , \\ \boldsymbol{x} = \varphi(\boldsymbol{p}) , \end{gathered} \tag{28}$$

is called a "configuration" of the body, $R = \varphi(\mathcal{B})$ is the region occupied by $\mathcal{B}$ in the configuration φ.

A "motion" χ of the body is a one-parameter family of configurations $\{\varphi_\tau\}$ where $\tau \in [\tau_0, \tau_1]$. So

$$\mathbf{x} = \varphi_\tau(\mathbf{p}) = \chi(\mathbf{p}, \tau), \tag{29}$$

is the position at time τ of particle $\mathbf{p}$, and $R = \varphi_\tau(\mathcal{B}) = \chi(\mathcal{B}, \tau)$ is the region occupied by the body at time τ.

The motion of a body can be described as relative to a "reference configuration". Let the configuration φ_t be chosen as reference configuration,

$$\mathbf{x}_t = \varphi_t(\mathbf{p}) = \chi(\mathbf{p}, t) \tag{30}$$

is the position at time t of particle $\mathbf{p}$, and $R_t = \varphi_t(\mathcal{B}) = \chi(\mathcal{B}, t)$ is the region occupied by the body at time t. The one-to-one mapping

$$\hat{\mathbf{x}} : R_t \to R_\tau;\, \mathbf{x}_t \mapsto \mathbf{x}\,, \tag{31}$$

defined by

$$\mathbf{x} = \hat{\mathbf{x}}(\mathbf{x}_t, \tau) = \varphi_\tau\big(\varphi_t^{-1}(\mathbf{x}_t)\big)\,, \tag{32}$$

is called a "deformation" of the body from the reference configuration φ_t to the configuration φ, and gives the position $\mathbf{x}$ at time τ of the particle $\mathbf{p}$ which at time t was at $\mathbf{x}_t$.

1.2.2 Spatial and referential descriptions

Let θ be any physical quantity associated with the particles of a body (e. g. temperature). The "material description" of θ relates the physical quantity directly with the material points $\boldsymbol{p}$, and is defined as the mapping

$$(\boldsymbol{p},\tau) \mapsto \theta = \tilde{\theta}(\boldsymbol{p},\tau). \tag{33}$$

The "spatial description" of θ relates the physical quantity with the particles by means of their position at the current time, and is defined as the mapping

$$(\mathbf{x},\tau) \mapsto \theta = \bar{\theta}(\mathbf{x},\tau). \tag{34}$$

Finally, the "referential description" of θ relates the physical quantity with the particles by means of their position in the reference configuration, and is defined by the mapping

$$(\mathbf{x}_t,\tau) \mapsto \theta = \hat{\theta}(\mathbf{x}_t,\tau). \tag{35}$$

The spatial and referential descriptions of quantity θ are related to the material description by proper composition with the configurations φ_τ and φ_t,

$$\theta = \tilde{\theta}(\boldsymbol{p},\tau), \tag{36}$$

$$\theta = \bar{\theta}(\mathbf{x},\tau) = \tilde{\theta}\big(\varphi_\tau^{-1}(\mathbf{x}),\tau\big), \tag{37}$$

$$\theta = \hat{\theta}(\mathbf{x}_t,\tau) = \tilde{\theta}\big(\varphi_t^{-1}(\mathbf{x}_t),\tau\big). \tag{38}$$

The spatial and referencial descriptions are also called "eulerian" and "lagrangian" descriptions respectively.

Unless otherwise specified, the "original configuration" φ_0, corresponding to time $t = 0$ will be used as the reference configuration for the lagrangian description.

1.2.3 Differential operators

The total time derivative of quantity θ, denoted by $\dot{\theta}$, is by definition the partial derivative of θ with respect to time for a given material point, i. e.

$$\dot{\theta} \equiv \frac{\partial \tilde{\theta}(\mathbf{p}, \tau)}{\partial \tau}. \tag{39}$$

Given a fixed reference frame $\{\mathbf{e}_i\}$, let the position vectors $\mathbf{x}$ and $\mathbf{x}_0$ be represented by $\mathbf{x} = x_i \mathbf{e}_i$ and $\mathbf{x}_0 = x_i^0 \mathbf{e}_i$ respectively. We define the differential operators ∇ and ∇_0 by

$$\nabla = \frac{\partial}{\partial \mathbf{x}} = \mathbf{e}_i \frac{\partial}{\partial x_i}, \tag{40}$$

$$\nabla_0 = \frac{\partial}{\partial \mathbf{x}_0} = \mathbf{e}_i \frac{\partial}{\partial x_i^0}, \tag{41}$$

The gradients of a scalar α in the current and reference configuration are the vector fields

$$\nabla \alpha = \frac{\partial \bar{\alpha}(\mathbf{x}, \tau)}{\partial x_i} \mathbf{e}_i, \tag{42}$$

$$\nabla_0 \alpha = \frac{\partial \hat{\alpha}(\mathbf{x}_0, \tau)}{\partial x_i^0} \mathbf{e}_i \,, \tag{43}$$

and the gradients of a vector $\boldsymbol{a}$ in the current and reference configuration are the tensor fields

$$\nabla \boldsymbol{a} = \frac{\partial \bar{a}_i(\mathbf{x}, \tau)}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j \,, \tag{44}$$

$$\nabla_0 \boldsymbol{a} = \frac{\partial \hat{a}_i(\mathbf{x}_0, \tau)}{\partial x_j^0} \mathbf{e}_i \otimes \mathbf{e}_j \,, \tag{45}$$

The divergences of a vector $\boldsymbol{a}$ in the current and reference configurations are the scalar fields

$$\nabla \cdot \boldsymbol{a} = \frac{\partial \bar{a}_i(\mathbf{x}, \tau)}{\partial x_i} \,, \tag{46}$$

$$\nabla_0 \cdot \boldsymbol{a} = \frac{\partial \hat{a}_i(\mathbf{x}_0, \tau)}{\partial x_i^0} \,, \tag{47}$$

and the divergences of a tensor $\boldsymbol{T}$ in the current and reference configurations are the vector fields

$$\nabla \cdot \boldsymbol{T} = \frac{\partial \bar{T}_{ij}(\mathbf{x}, \tau)}{\partial x_j} \mathbf{e}_i \,, \tag{48}$$

$$\nabla_0 \cdot \boldsymbol{T} = \frac{\partial \hat{T}_{ij}(\mathbf{x}_0, \tau)}{\partial x_j^0} \mathbf{e}_i \,, \tag{49}$$

We mention without proof the following forms of the divergence formula,

$$\int_{\partial P} \boldsymbol{T}\boldsymbol{n} \, dA = \int_{P} \nabla \cdot \boldsymbol{T} \, dV \,, \tag{50}$$

$$\int_{\partial P} \left[\mathbf{x} \otimes (\boldsymbol{T}\boldsymbol{n}) - (\boldsymbol{T}\boldsymbol{n}) \otimes \mathbf{x} \right] dA = \int_{P} \left[\mathbf{x} \otimes (\nabla \cdot \boldsymbol{T}) - (\nabla \cdot \boldsymbol{T}) \otimes \mathbf{x} + \boldsymbol{T}^T - \boldsymbol{T} \right] dV \,. \tag{51}$$

1.2.4 Fundamental kinematic quantities

The velocity vector $\mathbf{v}$ and acceleration vector $\boldsymbol{a}$ are obtained by differentiation of the position vector with respect to time for a given material point,

$$\mathbf{v} = \dot{\boldsymbol{x}} = \frac{\partial \boldsymbol{\chi}(\mathbf{p}, \tau)}{\partial \tau}, \tag{52}$$

$$\boldsymbol{a} = \ddot{\mathbf{x}} = \frac{\partial^2 \boldsymbol{\chi}(\mathbf{p}, \tau)}{\partial \tau^2}. \tag{53}$$

The "deformation gradient" is the second order tensor

$$\boldsymbol{X} = \nabla_0 \hat{\mathbf{x}}(\boldsymbol{x}_0, \tau) = \frac{\partial \hat{x}_i(\mathbf{x}_0, \tau)}{\partial x_j^0} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \,. \tag{54}$$

Note that $\boldsymbol{X}$ depends on the selection of the reference configuration. The deformation gradient maps "material fibers" from the reference to the current configuration,

$$\boldsymbol{dx} = \boldsymbol{X} \boldsymbol{dx}_0 \,. \tag{55}$$

Since by assumption, a deformation cannot change space orientation, the jacobian $J = \det \boldsymbol{X}$ has to be positive for all times, and the deformation gradient admits the rigth and left polar decompositions

$$\boldsymbol{X} = \boldsymbol{R}\boldsymbol{U}, \tag{56}$$

$$\boldsymbol{X} = \boldsymbol{V}\boldsymbol{R}, \tag{57}$$

where $\boldsymbol{R}$ is called the rotation tensor and $\boldsymbol{U}, \boldsymbol{V}$ are called the right and left stretch tensors, respectively. The rotation tensor is related to changes in direction of material fibers and the stretch tensors are related to changes in lengths and relative angles between material fibers.

The "velocity gradient" is the second order tensor$(\mathbf{x}, \tau) =$

$$\frac{\partial \bar{v}_i(\mathbf{x}, \tau)}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j \,. \tag{58}$$

Note that no reference configuration is involved in its definition. The symmetric and antisymmetric parts of the velocity gradient,

$$\boldsymbol{D} = \tfrac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T)\,, \tag{59}$$

$$\boldsymbol{W} = \tfrac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^T)\,, \tag{60}$$

are called respectively "stretching" and "spin" tensors. The stretching tensor is related to the rate of change in lengths and relative angles of the material fibers and the spin tensor is related to the angular velocities of the material fibers.

Finally, we mention that the time rate of change of the jacobian J is given by

$$\dot{J} = J \mathrm{tr}\,(\boldsymbol{D}) = J \nabla \cdot \mathbf{v} \,. \tag{61}$$

Chapter 2

The mechanical principles

2.1 Conservation of mass

Let $\mathcal{B}$ be a body. The mass function m gives the mass $m(\mathcal{P})$ of a part $\mathcal{P}$ of $\mathcal{B}$. For all pairs $\mathcal{P}_1, \mathcal{P}_2$ of disjoint parts of $\mathcal{B}$, the mass function satisfies

$$m(\mathcal{P}_1 \cup \mathcal{P}_2) = m(\mathcal{P}_1) + m(\mathcal{P}_2)\,, \tag{1}$$

and if φ is any configuration of $\mathcal{B}$,

$$m(\mathcal{P}) \to 0\,, \tag{2}$$

as the volume of $P = \varphi(\mathcal{P})$ tends to zero. Properties (1–2) imply the existence of a scalar field ρ defined over B such that

$$m(\mathcal{P}) = \int_P \rho\, dV\,, \tag{3}$$

$\rho = \bar{\rho}(\mathbf{x}, \tau)$ is called "mass density" or simply "density" of $\mathcal{B}$ in the configuration φ.

Let χ be a motion of the body. The principle of mass conservation states that the mass of any part of $\mathcal{B}$ remains constant over the motion, i. e.

$$\frac{dm}{d\tau}(\mathcal{P}) = 0 \qquad \forall \mathcal{P} \subset \mathcal{B}\,. \tag{4}$$

In terms of the mass density,

$$\frac{d}{d\tau}\int_P \rho\, dV = 0 \qquad \forall P \subset B\,. \tag{5}$$

Note that in this equation the region of integration $P = \boldsymbol{\chi}(\mathcal{P}, \tau)$ depends on τ. To perform the differentiation we change integration variables,

$$\frac{d}{d\tau}\int_P \rho\, dV = \frac{d}{d\tau}\int_{P_0} \rho J\, dV_0 = \int_{P_0} (\dot{\rho} J + \rho \dot{J})\, dV_0\,, \tag{6}$$

and recalling that $\dot{J} = J \nabla \cdot \mathbf{v}$ (equation 1.61) we obtain

$$\int_P (\dot{\rho} + \rho \nabla \cdot \mathbf{v})\, dV = 0 \qquad \forall P \subset B\,. \tag{7}$$

Under proper conditions, we obtain the "continuity equation"

$$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0 \qquad \forall \boldsymbol{x} \in B\,. \tag{8}$$

This is the field equation associated with the principle of mass conservation.

The continuity equation can be written in terms of the reference configuration. Integrating equation (5) from time 0 to time τ,

$$\int_P \rho\, dV = \int_{P_0} \rho_0\, dV_0\,, \tag{9}$$

where ρ_0 is the mass density in the original configuration. Changing integration variables on the left hand side and combining terms,

$$\int_{P_0} (\rho J - \rho_0)\, dV_0 = 0 \qquad \forall P_0 \subset B_0 \,. \tag{10}$$

Under proper conditions we obtain the continuity equation in terms of the reference configuration φ,

$$\rho J = \rho_0 \qquad \forall \mathbf{x}_0 \in B_0 \,. \tag{11}$$

Let $\mathbf{v}$ be any vector field associated with the motion, then

$$\frac{d}{d\tau} \int_P \rho \mathbf{v}\, dV = \int_P \rho \dot{\mathbf{v}}\, dV \,. \tag{12}$$

This result follows from a change in integration variable and equation (11),

$$\frac{d}{d\tau} \int_P \rho \mathbf{v}\, dV = \frac{d}{d\tau} \int_{P_0} \rho \mathbf{v} J\, dV_0 = \frac{d}{d\tau} \int_{P_0} \rho_0 \mathbf{v}\, dV_0 = \int_{P_0} \rho_0 \dot{\mathbf{v}}\, dV_0 = \int_P \rho \dot{\mathbf{v}}\, dV \,. \tag{13}$$

2.2 Linear and angular momentum principles

Let χ be a motion of body $\mathcal{B}$. Let $\mathcal{P} \subset \mathcal{B}$ and $P = \chi(\mathcal{P}, \tau)$. The linear momentum $\boldsymbol{l}$ of part $\mathcal{P}$ at time τ and the angular momentum $\boldsymbol{h}$ with respect to point $\boldsymbol{o}$ of part $\mathcal{P}$ at time τ are defined respectively by

$$\boldsymbol{l}(\mathcal{P}, \tau) = \int_P \rho \mathbf{v}\, dV \,, \tag{14}$$

$$\boldsymbol{h}(\mathcal{P}, \boldsymbol{o}, \tau) = \int_P \rho(\mathbf{x} - \boldsymbol{o}) \times \mathbf{v}\, dV\,, \tag{15}$$

where the symbol $\times$ indicates vector product.

We consider two kinds of forces acting on a part $\mathcal{P}$ of $\mathcal{B}$. Body forces $\boldsymbol{b}$ are forces per unit mass acting on every particle of $\mathcal{P}$. Surface forces $\boldsymbol{t}$ are forces per unit current area acting on the boundary ∂P of $\mathcal{P}$, $\boldsymbol{t}$ is called "traction vector". Body forces are supposed to depend on position and time, thus $\boldsymbol{b} = \bar{\boldsymbol{b}}(\mathbf{x}, \tau)$. Surface forces depend on position, time and the surface ∂P. The dependence with ∂P is supposed to be only through the unit normal $\boldsymbol{n}$, thus $\boldsymbol{t} = \bar{\boldsymbol{t}}(\mathbf{x}, \tau, \boldsymbol{n})$.

If distributed torques are not considered, the resultant force $\boldsymbol{f}$ acting on $\mathcal{P}$ at time τ and the resultant torque $\boldsymbol{m}$ with respect to point $\boldsymbol{o}$ acting on $\mathcal{P}$ at time τ are given respectively by

$$\boldsymbol{f}(\mathcal{P}, \tau) = \int_P \rho \mathbf{b}\, dV + \int_{\partial P} \boldsymbol{t}\, dA\,, \tag{16}$$

$$\boldsymbol{m}(\mathcal{P}, \boldsymbol{o}, \tau) = \int_P \rho(\mathbf{x} - \boldsymbol{o}) \times \mathbf{b}\, dV + \int_{\partial P} (\mathbf{x} - \boldsymbol{o}) \times \boldsymbol{t}\, dA\,. \tag{17}$$

The mechanical principles, also called laws of motion, state that given an inertial reference frame, the time rates of change of linear and angular momentum of any part $\mathcal{P} \subset \mathcal{B}$ are respectively equal to the total force and torque acting on $\mathcal{P}$, i. e.

$$\frac{d\boldsymbol{l}}{d\tau}(\mathcal{P}, \tau) = \boldsymbol{f}(\mathcal{P}, \tau) \qquad \forall \mathcal{P} \subset \mathcal{B}\,, \tag{18}$$

$$\frac{d\boldsymbol{h}}{d\tau}(\mathcal{P}, \boldsymbol{o}, \tau) = \boldsymbol{m}(\mathcal{P}, \boldsymbol{o}, \tau) \qquad \forall \mathcal{P} \subset \mathcal{B}, \forall \boldsymbol{o} \in V\,, \tag{19}$$

for all parts $\mathcal{P}$ of the body and for any point $\boldsymbol{o}$. In terms of the definitions (14–17), the laws of motion read

$$\frac{d}{d\tau}\int\limits_{P}\rho\mathbf{v}\,dV=\int\limits_{P}\rho\mathbf{b}\,dV+\int\limits_{\partial P}\boldsymbol{t}\,dA\qquad\forall P\subset B\,,\tag{20}$$

$$\frac{d}{d\tau}\int\limits_{P}\rho(\mathbf{x}-\mathbf{o})\times\mathbf{v}\,dV=\int\limits_{P}\rho(\mathbf{x}-\mathbf{o})\times\mathbf{b}\,dV+\int\limits_{\partial P}(\mathbf{x}-\mathbf{o})\times\boldsymbol{t}\,dA\qquad\forall P\subset B,\forall\boldsymbol{o}\in V\,.\tag{21}$$

In virtue of result (12), and considering the point $\boldsymbol{o}$ as fixed, the left hand sides can be written as

$$\frac{d}{d\tau}\int\limits_{P}\rho\mathbf{v}\,dV=\int\limits_{P}\rho\mathbf{a}\,dV\,,\tag{22}$$

$$\frac{d}{d\tau}\int\limits_{P}\rho(\mathbf{x}-\mathbf{o})\times\mathbf{v}\,dV=\int\limits_{P}\rho(\mathbf{x}-\mathbf{o})\times\mathbf{a}\,dV\,,\tag{23}$$

and substituting in equations (20–21) we obtain

$$\int\limits_{P}\rho(\mathbf{b}-\mathbf{a})\,dV+\int\limits_{\partial P}\boldsymbol{t}\,dA=0\qquad\forall P\subset B\,,\tag{24}$$

$$\int\limits_{P}\rho(\mathbf{x}-\mathbf{o})\times(\mathbf{b}-\mathbf{a})\,dV+\int\limits_{\partial P}(\mathbf{x}-\mathbf{o})\times\boldsymbol{t}\,dA=0\qquad\forall P\subset B,\forall\boldsymbol{o}\in V\,.\tag{25}$$

Each term in equation (25) is a polar vector. If the associated skew-symmetric tensors are used instead we have

$$\int\limits_{P}\rho\big[(\boldsymbol{x}-\boldsymbol{o})\otimes(\boldsymbol{b}-\boldsymbol{a})-(\boldsymbol{b}-\boldsymbol{a})\otimes(\boldsymbol{x}-\boldsymbol{o})\big]\,dV$$

$$+\int_{\partial P}\left[(\mathbf{x}-\boldsymbol{o})\otimes\boldsymbol{t}-\boldsymbol{t}\otimes(\mathbf{x}-\boldsymbol{o})\right]dA=0 \qquad \forall P\subset B, \forall \boldsymbol{o}\in V\,. \tag{26}$$

The main consequence of the linear momentum principle (20) is that the traction vector $\boldsymbol{t}$ depends linearly on the unit normal $\boldsymbol{n}$, i. e. there is a second order tensor $\boldsymbol{T}$, called "stress tensor" such that $\boldsymbol{t}=\boldsymbol{T}\boldsymbol{n}$ or

$$\bar{\boldsymbol{t}}(\mathbf{x},\tau,\boldsymbol{n})=\bar{\boldsymbol{T}}(\mathbf{x},\tau)\boldsymbol{n}\,. \tag{27}$$

This result is called Cauchy's theorem. The proof is based on the application of the linear momentum principle to an infinitesimal tetrahedron. The vector $\boldsymbol{t}$ is called Cauchy, or true, traction vector and $\boldsymbol{T}$ is called Cauchy, or true, stress tensor.

Substituting Cauchy's theorem in equations (24) and (26),

$$\int_P \rho(\boldsymbol{b}-\mathbf{a})\,dV+\int_{\partial P}\boldsymbol{T}\boldsymbol{n}\,dA=0\,, \tag{28}$$

$$\int_P \rho\left[(\mathbf{x}-\boldsymbol{o})\otimes(\boldsymbol{b}-\mathbf{a})-(\boldsymbol{b}-\mathbf{a})\otimes(\mathbf{x}-\boldsymbol{o})\right]dV$$
$$+\int_{\partial P}\left[(\mathbf{x}-\boldsymbol{o})\otimes(\boldsymbol{T}\boldsymbol{n})-(\boldsymbol{T}\boldsymbol{n})\otimes(\mathbf{x}-\boldsymbol{o})\right]dA=0\,. \tag{29}$$

and using the divergence formulae (1.50–1.51),

$$\int_P\left[\nabla\cdot\boldsymbol{T}+\rho(\boldsymbol{b}-\mathbf{a})\right]dV=0\,, \tag{30}$$

$$\int_P\Big\{(\mathbf{x}-\boldsymbol{o})\otimes\left[\nabla\cdot\boldsymbol{T}+\rho(\boldsymbol{b}-\mathbf{a})\right]$$
$$-\left[\nabla\cdot\boldsymbol{T}+\rho(\boldsymbol{b}-\mathbf{a})\right]\otimes(\mathbf{x}-\boldsymbol{o})+\boldsymbol{T}^T-\boldsymbol{T}\Big\}\,dV=0\,. \tag{31}$$

Under proper continuity conditions, we obtain the "equation of motion"

$$\nabla \cdot \boldsymbol{T} + \rho \boldsymbol{b} = \rho \boldsymbol{a} \qquad \forall \mathbf{x} \in B\,, \tag{32}$$

and the symmetry condition

$$\boldsymbol{T}^T = \boldsymbol{T} \qquad \forall \mathbf{x} \in B\,. \tag{33}$$

Note that the Cauchy stress tensor is symmetric only when no torques are present.

2.3 The theorem of power expended

The kinetic energy $T(\mathcal{P}, \tau)$ of part $\mathcal{P}$ at time τ is defined by

$$T(\mathcal{P}, \tau) = \int_P \tfrac{1}{2}\, \rho \mathbf{v} \cdot \mathbf{v}\, dV\,, \tag{34}$$

where $\mathbf{v}$ is the velocity vector.

The "stress power" $W(\mathcal{P}, \tau)$ of part $\mathcal{P}$ at time τ is defined by

$$W(\mathcal{P}, \tau) = \int_P \boldsymbol{T} \cdot \boldsymbol{D}\, dV\,, \tag{35}$$

where $\boldsymbol{T}$ is the stress tensor and $\boldsymbol{D}$ is the stretching tensor.

Let $R(\mathcal{P}, \tau)$ denote the rate at which work is done by the external forces acting on $\mathcal{P}$, i. e.

$$R(\mathcal{P}, \tau) = \int_P \rho \boldsymbol{b} \cdot \mathbf{v}\, dV + \int_{\partial P} \boldsymbol{t} \cdot \mathbf{v}\, dA\,, \tag{36}$$

where $\boldsymbol{b}$ is the body force per unit mass and $\boldsymbol{t}$ is the traction vector.

The theorem of power expended states that the sum of the rate of change of kinetic energy and the stress power of any part $\mathcal{P}$ of the body equals the rate of work done by external forces on $\mathcal{P}$, i. e.

$$\frac{dT}{d\tau}(\mathcal{P},\tau) + W(\mathcal{P},\tau) = R(\mathcal{P},\tau) \qquad \forall \mathcal{P} \subset \mathcal{B}\,. \tag{37}$$

In terms of the definitions (34–36) we have

$$\frac{d}{d\tau}\int\limits_{P} \tfrac{1}{2}\,\rho\mathbf{v}\cdot\mathbf{v}\,dV + \int\limits_{P} \boldsymbol{T}\cdot\boldsymbol{D}\,dV = \int\limits_{P} \rho\boldsymbol{b}\cdot\mathbf{v}\,dV + \int\limits_{\partial P} \boldsymbol{t}\cdot\mathbf{v}\,dA \qquad \forall P \subset B\,. \tag{38}$$

To prove this result, we start by multiplying the equation of motion by the velocity vector and integrate over P,

$$\int\limits_{P}(\nabla\cdot\boldsymbol{T} + \rho\boldsymbol{b} - \rho\mathbf{a})\cdot\mathbf{v}\,dV = 0\,. \tag{39}$$

The first term on the left hand side of this equation can be transformed by means of the divergence formula,

$$\begin{aligned}\int\limits_{P}(\nabla\cdot\boldsymbol{T})\cdot\mathbf{v}\,dV &= \int\limits_{\partial P}\boldsymbol{T}^{T}\mathbf{v}\cdot\boldsymbol{n}\,dA - \int\limits_{P}\boldsymbol{T}\cdot\nabla\mathbf{v}\,dV\\ &= \int\limits_{\partial P}\boldsymbol{T}\boldsymbol{n}\cdot\mathbf{v}\,dA - \int\limits_{P}\boldsymbol{T}\cdot\boldsymbol{L}\,dV\\ &= \int\limits_{\partial P}\boldsymbol{t}\cdot\mathbf{v}\,dA - \int\limits_{P}\boldsymbol{T}\cdot\boldsymbol{D}\,dV\,,\end{aligned} \tag{40}$$

where use was made of Cauchy's formula $\boldsymbol{t} = \boldsymbol{T}\boldsymbol{n}$. Note that $\nabla\mathbf{v} = \boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W}$ and $\boldsymbol{T}\cdot\boldsymbol{W} = 0$.

The third term of the left hand side of equation (39) represents the time rate of change of kinetic energy,

$$\int_P \rho \mathbf{a} \cdot \mathbf{v} \, dV = \int_{P_0} \rho_0 \frac{d\mathbf{v}}{d\tau} \cdot \mathbf{v} \, dV_0$$
$$= \frac{d}{d\tau} \int_{P_0} \tfrac{1}{2} \rho_0 \mathbf{v} \cdot \mathbf{v} \, dV_0$$
$$= \frac{d}{d\tau} \int_P \tfrac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \, dV \,. \tag{41}$$

Combining equations (39–41) we obtain the required result.

The theorem of power expended is a consequence of the balance law for linear momentum and therefore is valid for any continuum.

The stress power can be interpreted as the rate of internal work. We write it as

$$W = \int_P w \, dV \,, \tag{42}$$

where w is the stress power per unit reference volume, given by

$$w = \boldsymbol{T} \cdot \boldsymbol{D} \,. \tag{43}$$

2.4 The first Piola-Kirchhoff stress

Let $\mathcal{P}$ be a part of body $\mathcal{B}$. Let $\boldsymbol{\chi}$ be a motion of the body and φ_0 a reference configuration. Consider a surface $S_0 \subset \varphi_0(\mathcal{B})$ and let $S = \hat{\mathbf{x}}(S_0, \tau)$. The image $\boldsymbol{n}\, dA$

of the area element $\boldsymbol{n}_0 dA_0$ of S_0 in S is given by Nanson's formula

$$\boldsymbol{n}\, dA = J\boldsymbol{X}^{-T}\boldsymbol{n}_0\, dA_0 \,. \tag{44}$$

Recall that

$$\boldsymbol{t}\, dA = \boldsymbol{T}\boldsymbol{n}\, dA \,, \tag{45}$$

is the contact force acting at time τ on the area element $\boldsymbol{n}\, dA$. We use relation (44) to rewrite the contact force as

$$\boldsymbol{T}\boldsymbol{n}\, dA = J\boldsymbol{T}\boldsymbol{X}^{-T}\boldsymbol{n}_0\, dA_0 \,. \tag{46}$$

The first Piola-Kirchhoff stress tensor $\boldsymbol{P}$ is defined by

$$\boldsymbol{P} = J\boldsymbol{T}\boldsymbol{X}^{-T} \,, \tag{47}$$

then, in terms of the first Piola-Kirchhoff stress

$$\boldsymbol{T}\boldsymbol{n}\, dA = \boldsymbol{P}\boldsymbol{n}_0\, dA_0 \,. \tag{48}$$

The first Piola-Kirchhoff traction vector $\boldsymbol{t}_0$ is defined by

$$\boldsymbol{t}_0 = \boldsymbol{P}\boldsymbol{n}_0 \,, \tag{49}$$

it follows from equations (45), (48) and (49) that

$$\boldsymbol{t}\, dA = \boldsymbol{t}_0\, dA_0 \,, \tag{50}$$

consequently, the first Piola-Kirchhoff traction vector gives the contact force acting on $\boldsymbol{n}\,dA$ per unit reference area.

We use a referential description for $\boldsymbol{P}$ and $\boldsymbol{t}_0$,

$$\boldsymbol{P} = \hat{\boldsymbol{P}}(\mathbf{x}_0, \tau)\,, \tag{51}$$

$$\boldsymbol{t}_0 = \hat{\boldsymbol{t}}_0(\mathbf{x}_0, \tau, \boldsymbol{n}_0)\,. \tag{52}$$

Let $\{\mathbf{e}_i\}$ be the reference frame and choose $\boldsymbol{n}_0 = \boldsymbol{e}_j$, then

$$\hat{P}_{ij}(\mathbf{x}_0, \tau) = \boldsymbol{e}_i \cdot \hat{\boldsymbol{P}}(\mathbf{x}_0, \tau)\boldsymbol{e}_j = \boldsymbol{e}_i \cdot \hat{\boldsymbol{t}}_0(\mathbf{x}_0, \tau, \boldsymbol{e}_j)\,, \tag{53}$$

it follows that P_{ij} is the force per unit reference area in the direction $\mathbf{e}_i$ acting on the surface which was normal to the direction $\mathbf{e}_j$ in the reference configuration.

2.5 The mechanical principles in referential form

In terms of the reference configuration, the linear momentum $\boldsymbol{l}$ and the angular momentum $\boldsymbol{h}$ with respect to point $\boldsymbol{o}$ of part $\mathcal{P}$ at time τ can be written as

$$\boldsymbol{l}(\mathcal{P}, \tau) = \int\limits_{P_0} \rho_0 \mathbf{v}\, dV_0\,, \tag{54}$$

$$\boldsymbol{h}(\mathcal{P}, \boldsymbol{o}, \tau) = \int\limits_{P_0} \rho_0(\mathbf{x} - \boldsymbol{o}) \times \mathbf{v}\, dV_0\,. \tag{55}$$

Similarly, the resultant force $\boldsymbol{f}$ and torque $\boldsymbol{m}$ with respect to point $\boldsymbol{o}$ acting on $\mathcal{P}$ at time τ can be written as

$$\boldsymbol{f}(\mathcal{P},\tau) = \int_{P_0} \rho_0 \mathbf{b}\, dV_0 + \int_{\partial P_0} \boldsymbol{t}_0\, dA_0\,, \tag{56}$$

$$\boldsymbol{m}(\mathcal{P},\boldsymbol{o},\tau) = \int_{P_0} \rho_0(\mathbf{x}-\boldsymbol{o}) \times \mathbf{b}\, dV_0 + \int_{\partial P_0} (\mathbf{x}-\boldsymbol{o}) \times \boldsymbol{t}_0\, dA_0\,, \tag{57}$$

where use has been made of the first Piola-Kirchhoff traction vector $\boldsymbol{t}_0$ defined in (49). Therefore, in terms of the reference configuration, the mechanical principles read

$$\frac{d}{d\tau}\int_{P_0} \rho_0 \mathbf{v}\, dV_0 = \int_{P_0} \rho_0 \mathbf{b}\, dV_0 + \int_{\partial P_0} \boldsymbol{t}_0\, dA_0\,, \tag{58}$$

$$\frac{d}{d\tau}\int_{P_0} \rho_0(\mathbf{x}-\boldsymbol{o}) \times \mathbf{v}\, dV_0 = \int_{P_0} \rho_0(\mathbf{x}-\boldsymbol{o}) \times \mathbf{b}\, dV_0 + \int_{\partial P_0} (\mathbf{x}-\boldsymbol{o}) \times \boldsymbol{t}_0\, dA_0\,. \tag{59}$$

Taking the time derivatives inside the integrals, we obtain

$$\int_{P_0} \rho_0(\boldsymbol{b}-\mathbf{a})\, dV_0 + \int_{\partial P_0} \boldsymbol{t}_0\, dA_0 = \boldsymbol{0}\,, \tag{60}$$

$$\int_{P_0} \rho_0(\boldsymbol{x}-\boldsymbol{o}) \times (\boldsymbol{b}-\mathbf{a})\, dV_0 + \int_{\partial P_0} (\boldsymbol{x}-\boldsymbol{o}) \times \boldsymbol{t}_0\, dA_0 = \boldsymbol{0}\,. \tag{61}$$

Each term in equation (61) is a polar vector. If the associated skew-symmetric tensors are used instead we have

$$\begin{aligned} &\int_{P_0} \rho_0 \big[(\boldsymbol{x}-\boldsymbol{o})\otimes(\boldsymbol{b}-\boldsymbol{a}) - (\boldsymbol{b}-\boldsymbol{a})\otimes(\boldsymbol{x}-\boldsymbol{o})\big]\, dV_0 \\ &\qquad + \int_{\partial P_0} \big[(\boldsymbol{x}-\boldsymbol{o})\otimes\boldsymbol{t}_0 - \boldsymbol{t}_0\otimes(\boldsymbol{x}-\boldsymbol{o})\big]\, dA_0 = \boldsymbol{0}\,. \end{aligned} \tag{62}$$

Substituting for the first Piola-Kirchhoff traction vector in terms of the first Piola-Kirchhoff stress tensor in equations (60) and (62), and integrating by parts,

$$\int_{P_0} [\nabla_0 \cdot \boldsymbol{P} + \rho_0(\boldsymbol{b} - \mathbf{a})]\, dV_0 = \boldsymbol{0}\,, \tag{63}$$

$$\int_{P_0} \{(\mathbf{x} - \boldsymbol{o}) \otimes [\nabla_0 \cdot \boldsymbol{P} + \rho_0(\boldsymbol{b} - \mathbf{a})] \\ - [\nabla_0 \cdot \boldsymbol{P} + \rho_0(\boldsymbol{b} - \mathbf{a})] \otimes (\mathbf{x} - \boldsymbol{o}) + \boldsymbol{X}\boldsymbol{P}^T - \boldsymbol{P}\boldsymbol{X}^T\}\, dV_0 = \boldsymbol{0}\,, \tag{64}$$

from where we obtain the equation of motion in referential form

$$\nabla_0 \cdot \boldsymbol{P} + \rho_0 \boldsymbol{b} = \rho_0 \boldsymbol{a} \qquad \forall\, \mathbf{x}_0 \in B_0\,, \tag{65}$$

and the condition

$$\boldsymbol{X}\boldsymbol{P}^T = \boldsymbol{P}\boldsymbol{X}^T \qquad \forall\, \mathbf{x}_0 \in B_0\,. \tag{66}$$

Next, the kinetic energy can be written

$$T(\mathcal{P}, \tau) = \int_{P_0} \tfrac{1}{2}\, \rho_0 \mathbf{v} \cdot \mathbf{v}\, dV_0\,. \tag{67}$$

In virtue of the definition (47) of the first Piola-Kirchhoff stress,

$$J\boldsymbol{T} \cdot \mathbf{D} = J\boldsymbol{T} \cdot \mathbf{L} = J\boldsymbol{T} \cdot (\dot{\boldsymbol{X}}\boldsymbol{X}^{-1}) = J\boldsymbol{T}\boldsymbol{X}^{-T} \cdot \dot{\boldsymbol{X}} = \boldsymbol{P} \cdot \dot{\boldsymbol{X}}\,, \tag{68}$$

and the stress power W can been written as

$$W = \int_{P_0} w_0\, dV_0 = \int_{P_0} J\boldsymbol{T} \cdot \mathbf{D}\, dV_0 = \int_{P_0} \boldsymbol{P} \cdot \dot{\boldsymbol{X}}\, dV_0\,. \tag{69}$$

where $w_0 = J\boldsymbol{T} \cdot \boldsymbol{D} = \boldsymbol{P} \cdot \dot{\mathbf{X}}$ is the stress power per unit reference volume.

Finally, in terms of the reference configuration, the rate of external work reads,

$$R(\mathcal{P},\tau) = \int_{P_0} \rho_0 \boldsymbol{b} \cdot \mathbf{v}\, dV + \int_{\partial P_0} \boldsymbol{t}_0 \cdot \mathbf{v}\, dA\,, \tag{70}$$

where $\boldsymbol{t}_0$ is the first Piola-Kirchhoff traction vector.

Combining equations (67–70) we obtain the referential form of the theorem of power expended,

$$\frac{d}{d\tau} \int_{P_0} \tfrac{1}{2}\, \rho_0 \mathbf{v} \cdot \mathbf{v}\, dV_0 + \int_{P_0} \boldsymbol{P} \cdot \dot{\mathbf{X}}\, dV_0 = \int_{P_0} \rho_0 \boldsymbol{b} \cdot \mathbf{v}\, dV_0 + \int_{\partial P_0} \boldsymbol{t}_0 \cdot \mathbf{v}\, dA_0\,. \tag{71}$$

Chapter 3

The thermodynamical principles

3.1 First law of thermodynamics

3.1.1 Internal energy

Central to the first law of thermodynamics is the assumption of existence of an internal energy function U such that $U(\mathcal{P},\tau)$ is the energy stored in part $\mathcal{P}$ of the continuum at time τ, in addition to the kinetic energy of $\mathcal{P}$.

For all pairs $\mathcal{P}_1, \mathcal{P}_2$ of disjoint parts of $\mathcal{B}$ the internal energy function satisfies,

$$U(\mathcal{P}_1 \cup \mathcal{P}_2, \tau) = U(\mathcal{P}_1, \tau) + U(\mathcal{P}_2, \tau)\,, \tag{1}$$

and if φ is any configuration of $\mathcal{B}$,

$$U(\mathcal{P}, \tau) \to 0\,, \tag{2}$$

as the volume of $P = \varphi(\mathcal{P})$ tends to zero. Properties (1–2) imply the existence of a scalar field $\epsilon = \bar{\epsilon}(\mathbf{x}, \tau)$ defined over B such that

$$U(\mathcal{P}, \tau) = \int_P \rho\epsilon \, dV \,, \tag{3}$$

ϵ is called the specific internal energy, or internal energy per unit mass.

3.1.2 Heat

The rate of heating of a part $\mathcal{P}$ of a body $\mathcal{B}$ during a motion $\boldsymbol{\chi}$ is supposed to be due to two sources, heat generated within $\mathcal{P}$ and heat entering $\mathcal{P}$ through its boundary. The rate at which heat is generated within $\mathcal{P}$ is called "heat supply". The rate at which heat is entering $\mathcal{P}$ is called "heat flux". Let r be the heat supply per unit mass and let h be the heat flux per unit current area. Then the total rate of heating Q of part $\mathcal{P}$ is

$$Q(\mathcal{P}, \tau) = \int_P \rho r \, dV + \int_{\partial P} h \, dA \,. \tag{4}$$

The heat supply r depends at least on the particle $\boldsymbol{p}$ and time τ while the heat flux h depends at least on $\boldsymbol{p}$, τ and the surface ∂P. It will be further assumed that the dependence of h on ∂P is reduced to the normal $\boldsymbol{n}$ to the surface. Thus $r = \bar{r}(\mathbf{x}, \tau)$ and $h = \bar{h}(\mathbf{x}, \tau, \boldsymbol{n})$.

3.1.3 Energy balance

The first law of thermodynamics postulates that the sum of the rates of increase of kinetic and internal energy of any part $\mathcal{P}$ of a body $\mathcal{B}$ equals the sum of the rates of external work and heating, i. e.

$$\frac{dT}{d\tau}(\mathcal{P},\tau) + \frac{dU}{d\tau}(\mathcal{P},\tau) = R(\mathcal{P},\tau) + Q(\mathcal{P},\tau) \qquad \forall \mathcal{P} \subset \mathcal{B} . \tag{5}$$

In terms of the definitions (2.34), (2.36), (3) and (4),

$$\frac{d}{d\tau}\int\limits_{P} \tfrac{1}{2}\rho \mathbf{v}\cdot\mathbf{v}\, dV + \frac{d}{d\tau}\int\limits_{P} \rho\epsilon\, dV =$$
$$\int\limits_{P} \rho \boldsymbol{b}\cdot\mathbf{v}\, dV + \int\limits_{\partial P} \boldsymbol{t}\cdot\mathbf{v}\, dA + \int\limits_{P} \rho r\, dV + \int\limits_{\partial P} h\, dA \qquad \forall P \subset B . \tag{6}$$

The balance of energy equation (6), together with the equations of motion (2.32–2.33) and the continuity equation (2.8) imply that the heat flux $h = \bar{h}(\mathbf{x},\tau,\boldsymbol{n})$ depends linearly on the normal vector $\boldsymbol{n}$, i. e. there is a vector $\boldsymbol{q} = \bar{\mathbf{q}}(\mathbf{x},\tau)$ such that $h = -\boldsymbol{q}\cdot\boldsymbol{n}$ or

$$\bar{h}(\mathbf{x},\tau,\boldsymbol{n}) = -\bar{\mathbf{q}}(\mathbf{x},\tau)\cdot\boldsymbol{n} . \tag{7}$$

This result is called Fourier's theorem. The proof is based on the application of energy balance principle (6) to an infinitesimal tetrahedron. The vector $\boldsymbol{q}$ is called "heat flux vector". (The minus sign is chosen so that $\boldsymbol{q}$ pointing inwards corresponds to heat entering the body.)

Substituting the rate of external work in equation (6) in terms of the theorem of power expended (2.38), and using (7),

$$\int_P \rho\dot{\epsilon}\, dV = \int_P \boldsymbol{T}\cdot\boldsymbol{D}\, dV + \int_P \rho r\, dV - \int_{\partial P} \boldsymbol{q}\cdot\boldsymbol{n}\, dA\,, \tag{8}$$

which, on using the divergence theorem, leads to

$$\int_P \big(\rho\dot{\epsilon} - \boldsymbol{T}\cdot\boldsymbol{D} - \rho r + \nabla\cdot\boldsymbol{q}\big)\, dV = 0 \qquad \forall P \subset B\,. \tag{9}$$

Under proper continuity conditions we obtain the field equation associated with the first law of thermodynamics,

$$\rho\dot{\epsilon} = \boldsymbol{T}\cdot\boldsymbol{D} + \rho r - \nabla\cdot\boldsymbol{q}\,. \tag{10}$$

3.2 Second law of thermodynamics

3.2.1 Entropy

The entropy function S gives the entropy $S(\mathcal{P},\tau)$ of a part $\mathcal{P}$ of the body at time τ. For all pairs $\mathcal{P}_1, \mathcal{P}_2$ of disjoint parts of $\mathcal{B}$ the entropy function satisfies,

$$S(\mathcal{P}_1 \cup \mathcal{P}_2, \tau) = S(\mathcal{P}_1,\tau) + S(\mathcal{P}_2,\tau)\,, \tag{11}$$

and if φ is any configuration of $\mathcal{B}$,

$$S(\mathcal{P},\tau) \to 0, \tag{12}$$

as the volume of $P = \varphi(\mathcal{P})$ tends to zero. Properties (11) and (12) imply the existence of a scalar field $\eta = \bar{\eta}(\mathbf{x},\tau)$ defined over B such that

$$S(\mathcal{P},\tau) = \int_P \rho\eta \, dV, \tag{13}$$

η is called the specific entropy, or entropy per unit mass.

3.2.2 The Clausius-Duhem inequality

The rate of entropy supply in a part $\mathcal{P}$ of the body at time τ is given by

$$\int_P \rho\frac{r}{\theta} \, dV, \tag{14}$$

where $\theta = \bar{\theta}(\mathbf{x},\tau)$ is the absolute temperature of particle $\mathbf{p}$ at time τ.

The rate of entropy flux into $\mathcal{P}$ at time τ is given by

$$\int_{\partial P} \frac{h}{\theta} \, dA. \tag{15}$$

We denote by $M(\mathcal{P},\tau)$ the sum of the entropy supply in $\mathcal{P}$ and the rate of entropy flux into $\mathcal{P}$.

The second law of thermodynamics for a continuum, known as the Clausius-Duhem inequality, states that the rate of entropy increase of $\mathcal{P}$ is always greater than or equal to the sum of the rate of entropy flux and entropy supply, i. e.

$$\frac{dS}{d\tau}(\mathcal{P},\tau) \geq M(\mathcal{P},\tau) \qquad \forall \mathcal{P} \subset \mathcal{B} . \tag{16}$$

In terms of the definitions (13–15), the second law reads

$$\frac{d}{d\tau}\int\limits_{P} \rho\eta \, dV \geq \int\limits_{P} \rho\frac{r}{\theta}\, dV + \int\limits_{\partial P} \frac{h}{\theta}\, dA \qquad \forall P \subset B . \tag{17}$$

Substituting Fourier's theorem (7) in the second term on the right hand side of this inequality, and using the divergence formula (1.50),

$$\int\limits_{\partial P} \frac{h}{\theta}\, dA = -\int\limits_{\partial P} \left(\frac{\mathbf{q}}{\theta}\right)\cdot \mathbf{n}\, dA = -\int\limits_{P} \nabla\cdot\left(\frac{\mathbf{q}}{\theta}\right)\, dV , \tag{18}$$

and equation (17) may be written,

$$\int\limits_{P} \left[\rho\dot{\eta} - \rho\frac{r}{\theta} + \nabla\cdot\left(\frac{\mathbf{q}}{\theta}\right)\right]\, dV \geq 0 \qquad \forall P \subset B . \tag{19}$$

Under proper continuity conditions, we obtain the field equation associated with the second law of thermodynamics,

$$\rho\dot{\eta} \geq \rho\frac{r}{\theta} - \nabla\cdot\left(\frac{\mathbf{q}}{\theta}\right) \qquad \forall \mathbf{x} \in B . \tag{20}$$

3.3 The reduced dissipation inequality

Consider the local form of the thermodynamic laws that we have deduced so far

$$\rho\dot{\epsilon} = \boldsymbol{T}\cdot\boldsymbol{D} + \rho r - \nabla\cdot\boldsymbol{q}\,, \tag{21}$$

$$\rho\dot{\eta} \geq \rho\frac{r}{\theta} - \nabla\cdot\left(\frac{\boldsymbol{q}}{\theta}\right)\,. \tag{22}$$

The second term of the right hand side of (22) is equal to

$$\nabla\cdot\left(\frac{\boldsymbol{q}}{\theta}\right) = \theta^{-1}\nabla\cdot\boldsymbol{q} - \theta^{-2}\boldsymbol{q}\cdot\nabla\theta\,. \tag{23}$$

We denote by $\boldsymbol{g}$ the temperature gradient vector $\nabla\theta$,

$$\boldsymbol{g} = \nabla\theta\,. \tag{24}$$

Then, multiplying (22) by the temperature θ and using equations (23–24),

$$\rho\theta\dot{\eta} \geq \rho r - \nabla\cdot\boldsymbol{q} + \theta^{-1}\boldsymbol{q}\cdot\boldsymbol{g}\,. \tag{25}$$

and substituting for $\rho r - \nabla\cdot\boldsymbol{q}$ in terms of (21) we obtain,

$$\rho\dot{\epsilon} - \boldsymbol{T}\cdot\boldsymbol{D} - \rho\theta\dot{\eta} + \theta^{-1}\boldsymbol{q}\cdot\boldsymbol{g} \leq 0\,. \tag{26}$$

In many applications, it is useful to perform a Legendre transformation from entropy to temperature. For this purpose, the free-energy density ψ is defined by

$$\psi = \epsilon - \theta\eta\,. \tag{27}$$

Thus, the time rate of change of the free-energy is

$$\dot{\psi} = \dot{\epsilon} - \dot{\theta}\eta - \theta\dot{\eta}\,, \tag{28}$$

and substituting in equation (26) we arrive at the "reduced dissipation inequality"

$$\rho\dot{\psi} - \boldsymbol{T}\cdot\boldsymbol{D} + \rho\eta\dot{\theta} + \theta^{-1}\boldsymbol{q}\cdot\boldsymbol{g} \leq 0\,. \tag{29}$$

3.4 The thermodynamical principles in referential form

Let $\mathcal{P}$ be a part of body $\mathcal{B}$. Let $\boldsymbol{\chi}$ be a motion of the body and φ a reference configuration. Consider a material surface $S_0 \subset \varphi(\mathcal{B})$ and let $S = \hat{\mathbf{x}}(S_0, \tau)$. The image $\boldsymbol{n}\,dA$ of the area element $\boldsymbol{n}_0\,dA_0$ of S_0 in S is given by Nanson's formula (2.44)

$$\boldsymbol{n}\,dA = J\boldsymbol{X}^{-T}\boldsymbol{n}_0\,dA_0\,. \tag{30}$$

By Fourier's theorem (7), the heat flux through area element $\boldsymbol{n}\,dA$ at time τ can be written in terms of the heat flux vector as

$$h\,dA = -\boldsymbol{q}\cdot\boldsymbol{n}\,dA\,. \tag{31}$$

We use equation (30) to rewrite the heat flux as

$$h\,dA = -J\boldsymbol{q}\cdot\boldsymbol{X}^{-T}\boldsymbol{n}_0\,dA_0 = -J\boldsymbol{X}^{-1}\boldsymbol{q}\cdot\boldsymbol{n}_0\,dA_0\,. \tag{32}$$

We define the heat flux vector $\boldsymbol{q}_0 = \hat{\mathbf{q}}_0(\mathbf{x}_0, \tau, \boldsymbol{n}_0)$ with respect to the reference configuration by

$$\boldsymbol{q}_0 = J\mathbf{X}^{-1}\boldsymbol{q}\,, \tag{33}$$

then, in terms of $\boldsymbol{q}_0$,

$$\boldsymbol{q}\cdot\boldsymbol{n}\,dA = \boldsymbol{q}_0\cdot\boldsymbol{n}_0\,dA_0\,. \tag{34}$$

Let $\{\mathbf{e}_i\}$ be a reference frame and choose $\boldsymbol{n}_0 = \boldsymbol{e}_i$, then in virtue of equation (34), the components $\boldsymbol{q}_0\cdot\boldsymbol{e}_i$ of the heat flux vector with respect to the reference configuration can be interpreted as the heat flux through material surface elements that in the reference configuration where normal to the coordinate axes.

Having defined $\mathbf{q}_0$ we can now write the heat and entropy fluxes in terms of the reference configuration as

$$\int\limits_{\partial P} h\,dA = -\int\limits_{\partial P} \boldsymbol{q}\cdot\boldsymbol{n}\,dA = -\int\limits_{\partial P_0} \mathbf{q}_0\cdot\boldsymbol{n}_0\,dA_0\,, \tag{35}$$

$$\int\limits_{\partial P} \frac{h}{\theta}\,dA = -\int\limits_{\partial P} \theta^{-1}\mathbf{q}\cdot\boldsymbol{n}\,dA = -\int\limits_{\partial P_0} \theta^{-1}\mathbf{q}_0\cdot\boldsymbol{n}_0\,dA_0\,. \tag{36}$$

and the sums of heat and entropy supply and flux read

$$Q(\mathcal{P},\tau) = \int\limits_{P_0} \rho_0 r\,dV_0 - \int\limits_{\partial P_0} \mathbf{q}_0\cdot\boldsymbol{n}_0\,dA_0\,, \tag{37}$$

$$M(\mathcal{P},\tau) = \int\limits_{P_0} \rho_0\frac{r}{\theta}\,dV_0 - \int\limits_{\partial P_0} \theta^{-1}\mathbf{q}_0\cdot\boldsymbol{n}_0\,dA_0\,. \tag{38}$$

In terms of φ, the specific internal energy U and the entropy S can be written as

$$U(\mathcal{P},\tau) = \int\limits_{P_0} \rho_0\epsilon\,dV_0\,, \tag{39}$$

$$S(\mathcal{P},\tau) = \int\limits_{P_0} \rho_0 \eta \, dV_0 \,. \tag{40}$$

Combining equations (2.67), (2.70) and (37–40) we write the first and second laws with respect to configuration φ as follows

$$\frac{d}{d\tau}\int\limits_{P_0} \tfrac{1}{2}\rho_0 \mathbf{v}\cdot\mathbf{v}\, dV_0 + \frac{d}{d\tau}\int\limits_{P_0} \rho_0 \epsilon \, dV_0 = \int\limits_{P_0} \rho_0 \boldsymbol{b}\cdot\mathbf{v}\, dV + \int\limits_{\partial P_0} \boldsymbol{t}_0\cdot\mathbf{v}\, dA$$

$$+ \int\limits_{P_0} \rho_0 r \, dV_0 - \int\limits_{\partial P_0} \mathbf{q}_0\cdot\mathbf{n}_0 \, dA_0 \qquad \forall\, \mathbf{x}_0 \in B_0 \,, \tag{41}$$

$$\frac{d}{d\tau}\int\limits_{P_0} \rho_0 \eta \, dV_0 \geq \int\limits_{P_0} \rho_0 \frac{r}{\theta}\, dV_0 - \int\limits_{\partial P_0} \theta^{-1}\mathbf{q}_0\cdot\mathbf{n}_0 \, dA_0 \qquad \forall\, \mathbf{x}_0 \in B_0 \,. \tag{42}$$

Substituting for the rate of change of kinetic energy in equation (41) in terms of the theorem of power expended in referential form (2.71),

$$\frac{d}{d\tau}\int\limits_{P_0} \rho_0 \epsilon \, dV_0 = \int\limits_{P_0} \boldsymbol{P}\cdot\dot{\boldsymbol{X}}\, dV_0 + \int\limits_{P_0} \rho_0 r \, dV_0 - \int\limits_{\partial P_0} \mathbf{q}_0\cdot\mathbf{n}_0 \, dA_0 \,. \tag{43}$$

Using the divergence formula in equations (42) and (43),

$$\int\limits_{P_0} \left(\rho_0 \dot{\epsilon} - \boldsymbol{P}\cdot\dot{\boldsymbol{X}} - \rho_0 r + \nabla_0\cdot\mathbf{q}_0\right) dV = 0 \,, \tag{44}$$

$$\int\limits_{P_0} \left[\rho_0 \dot{\eta} - \rho_0 \frac{r}{\theta} + \nabla_0\cdot\left(\frac{\mathbf{q}_0}{\theta}\right)\right] dV_0 \geq 0 \qquad \forall\, \mathbf{x}_0 \in B_0 \,. \tag{45}$$

Under proper continuity conditions, we obtain the field equations associated with the thermodynamic laws in referential form,

$$\rho_0 \dot{\epsilon} = \boldsymbol{P}\cdot\dot{\boldsymbol{X}} + \rho_0 r - \nabla_0\cdot\mathbf{q}_0 \qquad \forall\, \mathbf{x}_0 \in B_0 \,, \tag{46}$$

$$\rho_0 \dot{\eta} \geq \rho_0 \frac{r}{\theta} - \nabla_0 \cdot \left(\frac{\boldsymbol{q}_0}{\theta} \right) \qquad \forall\, \mathbf{x}_0 \in B_0 \, . \tag{47}$$

Finally, to obtain the reduced dissipation inequality in referential form, we note that

$$\nabla_0 \cdot \left(\frac{\boldsymbol{q}_0}{\theta} \right) = \theta^{-1} \nabla_0 \cdot \boldsymbol{q}_0 - \theta^{-2} \boldsymbol{q}_0 \cdot \boldsymbol{g}_0 \, , \tag{48}$$

where we have denoted by $\boldsymbol{g}_0$ the gradient of the temperature with respect to the reference coordinates,

$$\boldsymbol{g}_0 = \nabla_0 \theta \, . \tag{49}$$

Then, multiplying (47) by the temperature θ and using equations (48–49),

$$\rho_0 \theta \dot{\eta} \geq \rho_0 r - \nabla_0 \cdot \boldsymbol{q}_0 + \theta^{-1} \boldsymbol{q}_0 \cdot \boldsymbol{g}_0 \, . \tag{50}$$

Substituting for $\rho_0 r - \nabla_0 \cdot \boldsymbol{q}_0$ in terms of (46), we obtain

$$\rho_0 \dot{\epsilon} - \boldsymbol{P} \cdot \dot{\boldsymbol{X}} - \rho_0 \theta \dot{\eta} + \theta^{-1} \boldsymbol{q}_0 \cdot \boldsymbol{g}_0 \leq 0 \, . \tag{51}$$

Performing the Legendre transformation as defined by the free-energy function given in (27), we obtain the desired result,

$$\rho_0 \dot{\psi} - \boldsymbol{P} \cdot \dot{\boldsymbol{X}} + \rho_0 \eta \dot{\theta} + \theta^{-1} \boldsymbol{q}_0 \cdot \boldsymbol{g}_0 \leq 0 \, . \tag{52}$$

Chapter 4

Stress and strain measures

4.1 Some basic kinematic results

4.1.1 Stretch and rotation tensors

Let $\mathcal{B}$ be a body. Consider a deformation $\hat{\mathbf{x}}$ of $\mathcal{B}$ from a reference configuration φ_0 to configuration φ. Since by definition the deformation is one-to-one and preserves orientation, its jacobian has to be positive, i. e., the deformation gradient $\boldsymbol{X}$ satisfies $J = \det \boldsymbol{X} > 0$ for all points in the region $\varphi_0(\mathcal{B})$ occupied by the body in the reference configuration. Thus $\boldsymbol{X} \in \mathcal{L}_+$ and we have the polar decompositions

$$\boldsymbol{X} = \boldsymbol{R}\boldsymbol{U}, \tag{1}$$

$$\boldsymbol{X} = \boldsymbol{V}\boldsymbol{R}, \tag{2}$$

where $\boldsymbol{U} \in \mathcal{S}_+$ is the "right stretch tensor", $\boldsymbol{V} \in \mathcal{S}_+$ is the "left stretch tensor" and

$\boldsymbol{R} \in \mathcal{O}_+$ is called "rotation tensor". Given any $\boldsymbol{X} \in \mathcal{L}_+$, the tensors $\boldsymbol{U}, \boldsymbol{V}$ and $\boldsymbol{R}$ are uniquely defined. It follows from (1) and (2) that the stretch tensors are related by

$$\boldsymbol{V} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T . \tag{3}$$

Since the right stretch tensor is positive definite, it admits the eigen-decomposition

$$\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i \boldsymbol{r}_i \otimes \boldsymbol{r}_i , \tag{4}$$

where the eigenvalues λ_i satisfy $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3$ and the eigenvectors $\boldsymbol{r}_i$ are taken to be unit vectors.

Substituting equation (4) into (3),

$$\boldsymbol{V} = \boldsymbol{R}\left(\sum_{i=1}^{3} \lambda_i \boldsymbol{r}_i \otimes \boldsymbol{r}_i\right)\boldsymbol{R}^T = \sum_{i=1}^{3} \lambda_i (\boldsymbol{R}\boldsymbol{r}_i) \otimes (\boldsymbol{R}\boldsymbol{r}_i) , \tag{5}$$

we conclude that λ_i are the eigenvalues of the left stretch tensor $\boldsymbol{V}$ and the unit vectors

$$\boldsymbol{l}_i = \boldsymbol{R}\boldsymbol{r}_i , \tag{6}$$

are corresponding eigenvectors of $\boldsymbol{V}$.

The common eigenvalues λ_i of $\boldsymbol{U}$ and $\boldsymbol{V}$ are called "principal stretches".

Given a reference frame $\{\mathbf{e}_i\}$, we define the "principal stretches tensor" $\boldsymbol{\Lambda}$ by

$$\boldsymbol{\Lambda} = \sum_{i=1}^{3} \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i . \tag{7}$$

Let $\{\boldsymbol{r}_i(\tau)\}$ be an orthonormal set such that for each τ, $\boldsymbol{r}_i(\tau)$ are eigenvectors of $\boldsymbol{U}(\tau)$. We call $\{\boldsymbol{r}_i(\tau)\}$ a "lagrangian frame". The orthonormal set $\{\boldsymbol{l}_i(\tau)\}$ obtained using equation (6) is the corresponding "eulerian frame".

Given a lagrangian frame, we construct the "lagrangian" and "eulerian" rotation tensors $\boldsymbol{R}_L, \boldsymbol{R}_E$ by

$$\boldsymbol{R}_L = \sum_{i=1}^{3} \boldsymbol{r}_i \otimes \boldsymbol{e}_i \,, \tag{8}$$

$$\boldsymbol{R}_E = \sum_{i=1}^{3} \boldsymbol{l}_i \otimes \boldsymbol{e}_i \,. \tag{9}$$

Thus $\boldsymbol{R}_L$ represents a rotation from the reference frame to the lagrangian frame, and $\boldsymbol{R}_E$ represents a rotation from the reference frame to the eulerian frame,

$$\boldsymbol{r}_i = \boldsymbol{R}_L \boldsymbol{e}_i \,, \tag{10}$$

$$\boldsymbol{l}_i = \boldsymbol{R}_E \boldsymbol{e}_i \,. \tag{11}$$

Substituting (6) into equation (9) we conclude that the rotation tensors are related by

$$\boldsymbol{R}_E = \boldsymbol{R}\boldsymbol{R}_L \,. \tag{12}$$

The definitions of the principal stretches tensor Λ and the lagrangian and eulerian rotation tensors $\boldsymbol{R}_L$, $\boldsymbol{R}_E$ depend on the choice of reference frame $\{\boldsymbol{e}_i\}$. They are introduced only to simplify the notation. In virtue of (7–9) the eigen-decompositions of the stretch tensors can be written as

$$\boldsymbol{U} = \boldsymbol{R}_L \Lambda \boldsymbol{R}_L^T \,, \tag{13}$$

$$V = R_E \Lambda R_E^T . \tag{14}$$

4.1.2 Rates of rotation

We define the spin tensor Ω_R by the equation

$$\dot{R} = \Omega_R R , \tag{15}$$

and, given a lagrangian frame, we define the spin tensors Ω_L and Ω_E by the equations

$$\dot{R}_L = R_L \Omega_L , \tag{16}$$

$$\dot{R}_E = R_E \Omega_E . \tag{17}$$

In view of equation (12) relating the rotations, the spin tensors are not independent. To see this, we solve (12) for R and differentiate with respect to time,

$$\dot{R} = \dot{R}_E R_L^T + R_E \dot{R}_L^T , \tag{18}$$

substitute the definitions (15–17),

$$\Omega_R R = (R_E \Omega_E) R_L^T + R_E (\Omega_L^T R_L^T) , \tag{19}$$

and rearrange,

$$\Omega_R R = R_E (\Omega_E - \Omega_L) R_L^T . \tag{20}$$

Left-multiplying by $\boldsymbol{R}^T$ and using again (12) we finally obtain,

$$\boldsymbol{\Omega}_R = \boldsymbol{R}_E(\boldsymbol{\Omega}_E - \boldsymbol{\Omega}_L)\boldsymbol{R}_E^T\,. \tag{21}$$

4.1.3 The velocity gradient

The velocity gradient $\boldsymbol{L}$, defined in (1.58), can be decomposed into the sum

$$\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W}\,, \tag{22}$$

of its symmetric part $\boldsymbol{D}$, called the "stretching tensor", and its skew-symmetric part $\boldsymbol{W}$, called the "spin tensor". The stretching tensor is related to the rate at which the material fibers are changing their length and relative angles, and the spin tensor is related to the average angular velocity of material fibers at a point.

The purpose of this section is to relate the stretching and spin tensors, $\boldsymbol{D}$ and $\boldsymbol{W}$, to the time rate of change of the principal stretches $\dot{\boldsymbol{\Lambda}}$ and the spins $\boldsymbol{\Omega}_L$ and $\boldsymbol{\Omega}_R$.

From equation (1) we have

$$\dot{\boldsymbol{X}} = \dot{\boldsymbol{R}}\boldsymbol{U} + \boldsymbol{R}\dot{\boldsymbol{U}}\,, \tag{23}$$

$$\boldsymbol{X}^{-1} = \boldsymbol{U}^{-1}\boldsymbol{R}^T\,, \tag{24}$$

which substituted into the definition of the velocity gradient lead to

$$\boldsymbol{L} = \dot{\boldsymbol{X}}\boldsymbol{X}^{-1} = \dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}\dot{\boldsymbol{U}}\boldsymbol{U}^{-1}\boldsymbol{R}^T\,. \tag{25}$$

In view of definition (15),

$$\boldsymbol{L} = \boldsymbol{\Omega}_R + \boldsymbol{R}\dot{\boldsymbol{U}}\boldsymbol{U}^{-1}\boldsymbol{R}^T , \tag{26}$$

and taking the transpose,

$$\boldsymbol{L}^T = -\boldsymbol{\Omega}_R + \boldsymbol{R}\boldsymbol{U}^{-1}\dot{\boldsymbol{U}}\boldsymbol{R}^T . \tag{27}$$

Adding and subtracting equations (26–27), we obtain for the stretching and spin tensors

$$\boldsymbol{D} = \tfrac{1}{2}\boldsymbol{R}(\dot{\boldsymbol{U}}\boldsymbol{U}^{-1} + \boldsymbol{U}^{-1}\dot{\boldsymbol{U}})\boldsymbol{R}^T , \tag{28}$$

$$\boldsymbol{W} = \boldsymbol{\Omega}_R + \tfrac{1}{2}\boldsymbol{R}(\dot{\boldsymbol{U}}\boldsymbol{U}^{-1} - \boldsymbol{U}^{-1}\dot{\boldsymbol{U}})\boldsymbol{R}^T . \tag{29}$$

Next, we differentiate equation (13) with respect to time,

$$\dot{\boldsymbol{U}} = \boldsymbol{R}_L\dot{\boldsymbol{\Lambda}}\boldsymbol{R}_L^T + \dot{\boldsymbol{R}}_L\boldsymbol{\Lambda}\boldsymbol{R}_L^T + \boldsymbol{R}_L\boldsymbol{\Lambda}\dot{\boldsymbol{R}}_L^T . \tag{30}$$

recall definition (16) and rearrange,

$$\dot{\boldsymbol{U}} = \boldsymbol{R}_L(\dot{\boldsymbol{\Lambda}} + \boldsymbol{\Omega}_L\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{\Omega}_L)\boldsymbol{R}_L^T . \tag{31}$$

Besides, taking the inverse of equation (13),

$$\boldsymbol{U}^{-1} = \boldsymbol{R}_L\boldsymbol{\Lambda}^{-1}\boldsymbol{R}_L^T , \tag{32}$$

and combining (31) and (32) we obtain

$$\dot{\boldsymbol{U}}\boldsymbol{U}^{-1} = \boldsymbol{R}_L(\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^{-1} + \boldsymbol{\Omega}_L - \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1})\boldsymbol{R}_L^T , \tag{33}$$

$$\boldsymbol{U}^{-1}\dot{\boldsymbol{U}} = \boldsymbol{R}_L(\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^{-1} - \boldsymbol{\Omega}_L + \boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda})\boldsymbol{R}_L^T . \tag{34}$$

Note that since $\dot{\boldsymbol{\Lambda}}$ and $\boldsymbol{\Lambda}^{-1}$ are collinear,

$$\boldsymbol{\Lambda}^{-1}\dot{\boldsymbol{\Lambda}} = \dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^{-1}\,. \tag{35}$$

The results obtained in equations (33–34) can now be substituted into equations (28–29), which in combination with equation (21) lead to the final result

$$\boldsymbol{D} = \boldsymbol{R}_E\big[\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^{-1} + \tfrac{1}{2}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1})\big]\boldsymbol{R}_E^T\,, \tag{36}$$

$$\boldsymbol{W} = \boldsymbol{R}_E\big[\boldsymbol{\Omega}_E - \tfrac{1}{2}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda} + \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1})\big]\boldsymbol{R}_E^T\,. \tag{37}$$

4.2 Strain measures

4.2.1 Scale functions

The fundamental measures of the deformation of a body are the right and left stretch tensors, $\boldsymbol{U}$ and $\boldsymbol{V}$. It is customary, however, to define a "strain tensor" as follows. A "scale function" is a function $g : (0,\infty) \to \mathcal{R}; \lambda \mapsto e = g(\lambda)$ with the conditions

$$g'(\lambda) > 0 \quad \text{for all} \quad \lambda > 0\,, \tag{38}$$

$$g(1) = 0\,, \tag{39}$$

$$g'(1) = 1\,. \tag{40}$$

Let the scale function acting on the principal stretches tensor $\mathbf{\Lambda}$ given in equation (7) be defined as the tensor-valued function

$$g(\mathbf{\Lambda}) = \sum_{i=1}^{3} g(\lambda_i)\boldsymbol{e}_i \otimes \boldsymbol{e}_i\,. \tag{41}$$

A "lagrangian strain measure" $\boldsymbol{E}$ is defined as the second order tensor

$$\boldsymbol{E} = \boldsymbol{R}_L g(\mathbf{\Lambda})\boldsymbol{R}_L^T\,. \tag{42}$$

The eigenvalues of $\boldsymbol{E}$ are equal to $e_i = g(\lambda_i)$ and are called "principal strains." The tensor $\boldsymbol{E}$ diagonalizes in the lagrangian frame, and therefore is collinear with the right stretch tensor $\boldsymbol{U}$.

It is apparent now that a scale function has to satisfy condition (38) for strain to be an increasing function of stretch, and condition (39) for strain to be zero when the deformation is the identity. Condition (40) is required so that the strain measures linearize properly to the small strain case.

4.2.2 Strain rates

The time rate of change of the strain measure defined above is to be obtained next.

Let

$$g'(\boldsymbol{\Lambda}) = \sum_{i=1}^{3} \frac{dg}{d\lambda}(\lambda_i)\mathbf{e}_i \otimes \mathbf{e}_i \,. \tag{43}$$

Then taking the time derivative in equation (42),

$$\dot{\boldsymbol{E}} = \boldsymbol{R}_L \dot{\boldsymbol{\Lambda}} g'(\boldsymbol{\Lambda}) \boldsymbol{R}_L^T + \dot{\boldsymbol{R}}_L g(\boldsymbol{\Lambda}) \boldsymbol{R}_L^T + \boldsymbol{R}_L g(\boldsymbol{\Lambda}) \dot{\boldsymbol{R}}_L^T \,, \tag{44}$$

and using the definition (16) and rearranging we obtain

$$\dot{\boldsymbol{E}} = \boldsymbol{R}_L \left[\dot{\boldsymbol{\Lambda}} g'(\boldsymbol{\Lambda}) + \boldsymbol{\Omega}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda}) \Omega_L \right] \boldsymbol{R}_L^T \,. \tag{45}$$

4.3 Stress measures

4.3.1 Work conjugacy

We recall the definition (2.69) of the stress power per unit reference volume w_0 ,

$$w_0 = J \boldsymbol{T} \cdot \boldsymbol{D} \,, \tag{46}$$

where J is the jacobian of the deformation, $\boldsymbol{T}$ is the Cauchy, or true, stress tensor and $\boldsymbol{D}$ is the stretching tensor. In terms of the first Piola-Kirchhoff stress tensor,

$$w_0 = \boldsymbol{P} \cdot \dot{\boldsymbol{X}} \,. \tag{47}$$

The stress power can be written in terms of the rate of strain $\dot{\boldsymbol{E}}$,

$$w_0 = \boldsymbol{S} \cdot \dot{\boldsymbol{E}}, \tag{48}$$

where $\boldsymbol{S}$ is a second order symmetric tensor called "stress measure". The stress measure $\boldsymbol{S}$ is said to be "work conjugate" to the strain measure $\boldsymbol{E}$. Given a scale function g, the associated stress measure $\boldsymbol{S}$ has to satisfy the equation

$$J\boldsymbol{T} \cdot \boldsymbol{D} = \boldsymbol{S} \cdot \dot{\boldsymbol{E}}, \tag{49}$$

for all possible motions $\boldsymbol{\chi}$.

4.3.2 An explicit formula for stress measures

The purpose of this section is to obtain an explicit expression for $\boldsymbol{S}$ in terms of the Cauchy stress tensor and the principal stretches λ_i.

We proceed by writing the strain rate $\dot{\boldsymbol{E}}$ and the stretching tensor $\boldsymbol{D}$ in terms of the independent rate quantities $\dot{\boldsymbol{\Lambda}}$ and $\boldsymbol{\Omega}_L$ as in equations (36) and (45),

$$\boldsymbol{D} = \boldsymbol{R}_E \left[\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^{-1} + \tfrac{1}{2}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1})\right]\boldsymbol{R}_E^T, \tag{50}$$

$$\dot{\boldsymbol{E}} = \boldsymbol{R}_L \left[\dot{\boldsymbol{\Lambda}} g'(\boldsymbol{\Lambda}) + \boldsymbol{\Omega}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda})\boldsymbol{\Omega}_L\right]\boldsymbol{R}_L^T. \tag{51}$$

Substituting these results in equation (49),

$$J\boldsymbol{T} \cdot \boldsymbol{R}_E \left[\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^{-1} + \tfrac{1}{2}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1})\right]\boldsymbol{R}_E^T$$

$$= \boldsymbol{S} \cdot \boldsymbol{R}_L \big[\dot{\boldsymbol{\Lambda}} g'(\boldsymbol{\Lambda}) + \boldsymbol{\Omega}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda}) \boldsymbol{\Omega}_L \big] \boldsymbol{R}_L^T \,, \tag{52}$$

and denoting by $\boldsymbol{S}_L$ and $\boldsymbol{T}_E$ the rotated tensors

$$\boldsymbol{T}_E = \boldsymbol{R}_E^T \boldsymbol{T} \boldsymbol{R}_E \,, \tag{53}$$

$$\boldsymbol{S}_L = \boldsymbol{R}_L^T \boldsymbol{S} \boldsymbol{R}_L \,, \tag{54}$$

we obtain

$$\begin{aligned} J\boldsymbol{T}_E \cdot \big[\dot{\boldsymbol{\Lambda}} \boldsymbol{\Lambda}^{-1} + \tfrac{1}{2} (\boldsymbol{\Lambda}^{-1} \boldsymbol{\Omega}_L \boldsymbol{\Lambda} - \boldsymbol{\Lambda} \boldsymbol{\Omega}_L \boldsymbol{\Lambda}^{-1}) \big] \\ = \boldsymbol{S}_L \cdot \big[\dot{\boldsymbol{\Lambda}} g'(\boldsymbol{\Lambda}) + \boldsymbol{\Omega}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda}) \boldsymbol{\Omega}_L \big] \,. \end{aligned} \tag{55}$$

The next step is to isolate the rate quantities on one side of the dot operator,

$$\begin{aligned} J\boldsymbol{T}_E \boldsymbol{\Lambda}^{-1} \cdot \dot{\boldsymbol{\Lambda}} + \tfrac{1}{2} J \big(\boldsymbol{\Lambda}^{-1} \boldsymbol{T}_E \boldsymbol{\Lambda} - \boldsymbol{\Lambda} \boldsymbol{T}_E \boldsymbol{\Lambda}^{-1} \big) \cdot \boldsymbol{\Omega}_L \\ = \boldsymbol{S}_L g'(\boldsymbol{\Lambda}) \cdot \dot{\boldsymbol{\Lambda}} + \big[\boldsymbol{S}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda}) \boldsymbol{S}_L \big] \cdot \boldsymbol{\Omega}_L \,, \end{aligned} \tag{56}$$

and group the corresponding terms,

$$\begin{aligned} \big[J\boldsymbol{T}_E \boldsymbol{\Lambda}^{-1} - g'(\boldsymbol{\Lambda}) \boldsymbol{S}_L \big] \cdot \dot{\boldsymbol{\Lambda}} \\ + \big\{ \tfrac{1}{2} J \big[\boldsymbol{\Lambda}^{-1} \boldsymbol{T}_E \boldsymbol{\Lambda} - \boldsymbol{\Lambda} \boldsymbol{T}_E \boldsymbol{\Lambda}^{-1} \big] - \big[\boldsymbol{S}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda}) \boldsymbol{S}_L \big] \big\} \cdot \boldsymbol{\Omega}_L = 0 \,. \end{aligned} \tag{57}$$

Equation (57) has to hold for any tensor $\dot{\boldsymbol{\Lambda}}$ that diagonalizes in the reference frame and any skew-symmetric tensor $\boldsymbol{\Omega}_L$. Pick first $\dot{\boldsymbol{\Lambda}} = \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\alpha$, where there is no sum on α, and $\boldsymbol{\Omega}_L = \boldsymbol{0}$, then

$$\big[J\boldsymbol{T}_E \boldsymbol{\Lambda}^{-1} - g'(\boldsymbol{\Lambda}) \boldsymbol{S}_L \big] \cdot \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\alpha = \boldsymbol{e}_\alpha \cdot \big[J\boldsymbol{T}_E \boldsymbol{\Lambda}^{-1} - g'(\boldsymbol{\Lambda}) \boldsymbol{S}_L \big] \boldsymbol{e}_\alpha \,, \tag{58}$$

and recalling that

$$\boldsymbol{\Lambda}^{-1}\boldsymbol{e}_\alpha = \lambda_\alpha^{-1}\boldsymbol{e}_\alpha\,, \tag{59}$$

$$g(\boldsymbol{\Lambda})\boldsymbol{e}_\alpha = g(\lambda_\alpha)\boldsymbol{e}_\alpha\,, \tag{60}$$

$$g'(\boldsymbol{\Lambda})\boldsymbol{e}_\alpha = g'(\lambda_\alpha)\boldsymbol{e}_\alpha\,, \tag{61}$$

we obtain

$$\boldsymbol{e}_\alpha \cdot \boldsymbol{S}_L\boldsymbol{e}_\alpha = \frac{J}{\lambda_\alpha g'(\lambda_\alpha)}(\boldsymbol{e}_\alpha \cdot \boldsymbol{T}_E\lambda_\alpha) \qquad (\text{no sum on } \alpha\,). \tag{62}$$

Substituting this result into equation (57), the condition

$$\left\{\tfrac{1}{2}\,J\big[\boldsymbol{\Lambda}^{-1}\boldsymbol{T}_E\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{T}_E\boldsymbol{\Lambda}^{-1}\big] - \big[\boldsymbol{S}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda})\boldsymbol{S}_L\big]\right\}\cdot\boldsymbol{\Omega}_L = 0\,, \tag{63}$$

has to be satisfied for every skew-symmetric tensor $\boldsymbol{\Omega}_L$. This implies that the quantity between braces in equation (63) has to be a symmetric tensor. Since, on the other hand,

$$\begin{aligned}&\left\{\tfrac{1}{2}J\big[\boldsymbol{\Lambda}^{-1}\boldsymbol{T}_E\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{T}_E\boldsymbol{\Lambda}^{-1}\big] - \big[\boldsymbol{S}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda})\boldsymbol{S}_L\big]\right\}^T \\ &\qquad = -\left\{\tfrac{1}{2}J\big[\boldsymbol{\Lambda}^{-1}\boldsymbol{T}_E\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{T}_E\boldsymbol{\Lambda}^{-1}\big] - \big[\boldsymbol{S}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda})\boldsymbol{S}_L\big]\right\},\end{aligned} \tag{64}$$

it necessarily follows that

$$\tfrac{1}{2}\,J\big[\boldsymbol{\Lambda}^{-1}\boldsymbol{T}_E\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{T}_E\boldsymbol{\Lambda}^{-1}\big] - \big[\boldsymbol{S}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda})\boldsymbol{S}_L\big] = \boldsymbol{0}\,. \tag{65}$$

Taking dot product with $\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta$,

$$\tfrac{1}{2}\,J\boldsymbol{e}_\alpha \cdot \big[\boldsymbol{\Lambda}^{-1}\boldsymbol{T}_E\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{T}_E\boldsymbol{\Lambda}^{-1}\big]\boldsymbol{e}_\beta = \boldsymbol{e}_\alpha \cdot \big[\boldsymbol{S}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda})\boldsymbol{S}_L\big]\boldsymbol{e}_\beta\,, \tag{66}$$

and using equations (59–60),

$$\tfrac{1}{2}\,J(\lambda_\beta\lambda_\alpha^{-1} - \lambda_\alpha\lambda_\beta^{-1})(\boldsymbol{e}_\alpha \cdot \boldsymbol{T}_E\boldsymbol{e}_\beta)\big[g(\lambda_\beta) - g(\lambda_\alpha)\big](\boldsymbol{e}_\alpha \cdot \boldsymbol{S}_L\boldsymbol{e}_\beta)\,, \tag{67}$$

We can now solve (67) for the components of $\boldsymbol{S}_L$,

$$\boldsymbol{e}_\alpha \cdot \boldsymbol{S}_L\boldsymbol{e}_\beta = \tfrac{1}{2}\,J\frac{\lambda_\beta\lambda_\alpha^{-1} - \lambda_\alpha\lambda_\beta^{-1}}{g(\lambda_\beta) - g(\lambda_\alpha)}(\boldsymbol{e}_\alpha \cdot \boldsymbol{T}_E\boldsymbol{e}_\beta) \qquad (\text{no sum on } \alpha,\beta\,)\,. \tag{68}$$

Note that equation (68) requires that $\lambda_\beta \neq \lambda_\alpha$. For $\lambda_\beta = \lambda_\alpha$ the right hand side is obtained by the limiting process

$$\lim_{\lambda_\beta \to \lambda_\alpha} \tfrac{1}{2}\,\frac{\lambda_\beta\lambda_\alpha^{-1} - \lambda_\alpha\lambda_\beta^{-1}}{g(\lambda_\beta) - g(\lambda_\alpha)} = \frac{1}{\lambda_\alpha g'(\lambda_\alpha)}\,. \tag{69}$$

Denoting by $[\boldsymbol{S}_L]_{\alpha\beta}$ and $[\boldsymbol{T}_E]_{\alpha\beta}$ the components of $\boldsymbol{S}_L$ and $\boldsymbol{T}_E$ in the reference frame,

$$[\boldsymbol{S}_L]_{\alpha\beta} = \boldsymbol{e}_\alpha \cdot \boldsymbol{S}_L\boldsymbol{e}_\beta\,, \tag{70}$$

$$[\boldsymbol{T}_E]_{\alpha\beta} = \boldsymbol{e}_\alpha \cdot \boldsymbol{T}_E\boldsymbol{e}_\beta\,, \tag{71}$$

we obtain from equations (62) and (68),

$$[\boldsymbol{S}_L]_{\alpha\beta} = \begin{cases} \dfrac{J}{\lambda_\alpha g'(\lambda_\alpha)}[\boldsymbol{T}_E]_{\alpha\alpha}\,, & \text{if } \beta = \alpha; \\[2ex] \tfrac{1}{2}J\dfrac{\lambda_\beta\lambda_\alpha^{-1} - \lambda_\alpha\lambda_\beta^{-1}}{g(\lambda_\beta) - g(\lambda_\alpha)}[\boldsymbol{T}_E]_{\alpha\beta}\,, & \text{otherwise.} \end{cases} \tag{72}$$

There is no sum over repeated indices in this expression. The final result, an expression for the stress measure in terms of the Cauchy stress and principal stretches, can now be written in terms of the components given in equation (72),

$$S = R_L\left(\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}[S_L]_{\alpha\beta}\mathbf{e}_\alpha\otimes\mathbf{e}_\beta\right)R_L^T . \tag{73}$$

Let g_* be a second scale function. The corresponding strain measure E^* can be written in terms of E as follows,

$$E^* = R_L g_*\big(g^{-1}(R_L^T E R_L)\big)R_L^T , \tag{74}$$

and in virtue of equation (72), the stress measures S^* and S are related by

$$[S_L^*]_{\alpha\beta} = \begin{cases} \dfrac{g'(\lambda_\alpha)}{g_*'(\lambda_\alpha)}[S_L]_{\alpha\alpha} , & \text{if } \beta = \alpha; \\[2ex] \dfrac{g(\lambda_\beta) - g(\lambda_\alpha)}{g_*(\lambda_\beta) - g_*(\lambda_\alpha)}[S_L]_{\alpha\beta} , & \text{otherwise.} \end{cases} \tag{75}$$

Note that the work conjugate pair (E, S) depends on the choice φ of reference configuration. When necessary, this dependence will be indicated by writing (E_φ, S_φ).

4.3.3 A particular class of measures

The most commonly used scale functions belong to the family

$$g_n(\lambda) = \frac{1}{2n}\left(\lambda^{2n} - 1\right) . \tag{76}$$

The associated strain and stress measures are denoted E^n and S^n. For this family $g_n'(\lambda) = \lambda^{2n-1}$ and the stress measures are obtained from equation (72),

$$[\boldsymbol{S}_L]_{\alpha\beta} = \begin{cases} \dfrac{J}{\lambda_\alpha^{2n}}[\boldsymbol{T}_E]_{\alpha\alpha}\,, & \text{if } \beta = \alpha; \\[2ex] nJ\dfrac{\lambda_\beta\lambda_\alpha^{-1} - \lambda_\alpha\lambda_\beta^{-1}}{\lambda_\beta^{2n} - \lambda_\alpha^{2n}}[\boldsymbol{T}_E]_{\alpha\beta}\,, & \text{otherwise.} \end{cases} \tag{77}$$

For $n = 1$, $\boldsymbol{E}^{(1)}$ is the Green-Lagrange strain tensor and $\boldsymbol{S}^{(1)}$ is the Second Piola-Kirchhoff stress tensor. These measures admit the explicit expressions

$$\boldsymbol{E}^{(1)} = \tfrac{1}{2}(\boldsymbol{X}^T\boldsymbol{X} - \boldsymbol{1})\,, \tag{78}$$

$$\boldsymbol{S}^{(1)} = J\boldsymbol{X}^{-1}\boldsymbol{T}\boldsymbol{X}^{-T}\,, \tag{79}$$

For $n = -1$, $\boldsymbol{E}^{(-1)}$ is the Almansi strain tensor. It can be easily shown that

$$\boldsymbol{E}^{(-1)} = \tfrac{1}{2}(\boldsymbol{1} - \boldsymbol{X}^{-1}\boldsymbol{X}^{-T})\,, \tag{80}$$

$$\boldsymbol{S}^{(-1)} = J\boldsymbol{X}\boldsymbol{T}\boldsymbol{X}^T\,. \tag{81}$$

4.3.4 The logarithmic strain and stress pair

As a limiting case, the scale function

$$g_0(\lambda) = \lim_{n\to 0} g_n(\lambda) = \ln\lambda\,, \tag{82}$$

defines the Hencky strain tensor

$$\boldsymbol{E}^{(0)} = \boldsymbol{R}_L \ln(\boldsymbol{\Lambda})\boldsymbol{R}_L^T\,. \tag{83}$$

Since $g'(\lambda) = \lambda^{-1}$, the associated stress tensor as obtained from equation (72) is

$$[\boldsymbol{S}_L]_{\alpha\beta} = \begin{cases} J[\boldsymbol{T}_E]_{\alpha\alpha}\,, & \text{if } \beta = \alpha; \\ \frac{1}{2} J \dfrac{\lambda_\beta \lambda_\alpha^{-1} - \lambda_\alpha \lambda_\beta^{-1}}{\ln(\lambda_\beta) - \ln(\lambda_\alpha)} [\boldsymbol{T}_E]_{\alpha\beta}\,, & \text{otherwise.} \end{cases} \tag{84}$$

4.4 Collinear stress and strain tensors

Given any scale function, equations (42) and (72) determine the associated strain and stress measures. It is not possible to write (72) in a tensorial form for the general case. There is, however, a particular case of considerable interest in which this can be achieved.

Assume that the stress measure is collinear with the strain measure, i. e. their principal axes coincide. It follows that

$$[\boldsymbol{S}, \boldsymbol{E}] = \boldsymbol{S}\boldsymbol{E} - \boldsymbol{E}\boldsymbol{S} = 0\,, \tag{85}$$

where $[\boldsymbol{S}, \boldsymbol{E}]$ denotes the commutator of $\boldsymbol{S}$ and $\boldsymbol{E}$.

We have seen that if a strain measure is given by

$$\boldsymbol{E} = \boldsymbol{R}_L g(\boldsymbol{\Lambda}) \boldsymbol{R}_L^T\,, \tag{86}$$

then the work conjugate stress measure defined by

$$w_0 = J\boldsymbol{T} \cdot \boldsymbol{D} = \boldsymbol{S} \cdot \dot{\boldsymbol{E}}\,, \tag{87}$$

has to satisfy

$$\left[J\boldsymbol{T}_E\boldsymbol{\Lambda}^{-1} - g'(\boldsymbol{\Lambda})\boldsymbol{S}_L\right] \cdot \dot{\boldsymbol{\Lambda}} = 0\,, \tag{88}$$

for all tensors $\dot{\boldsymbol{\Lambda}}$ that diagonalize in the reference frame, and

$$\left[\boldsymbol{S}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda})\boldsymbol{S}_L\right] - \tfrac{1}{2}J\left[\boldsymbol{\Lambda}^{-1}\boldsymbol{T}_E\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{T}_E\boldsymbol{\Lambda}^{-1}\right] = \boldsymbol{0}\,, \tag{89}$$

Recalling definition (54) and using (86) we have

$$[\boldsymbol{S}, \boldsymbol{E}] = \boldsymbol{R}_L[\boldsymbol{S}_L, g(\boldsymbol{\Lambda})]\boldsymbol{R}_L^T\,, \tag{90}$$

it follows from (85) that

$$[\boldsymbol{S}_L, g(\boldsymbol{\Lambda})] = \boldsymbol{S}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda})\boldsymbol{S}_L = \boldsymbol{0}\,, \tag{91}$$

and equation (89) reduces to

$$\boldsymbol{\Lambda}^{-1}\boldsymbol{T}_E\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{T}_E\boldsymbol{\Lambda}^{-1} = \boldsymbol{0}\,. \tag{92}$$

We conclude that assumption (85) implies

$$[\boldsymbol{T}_E, \boldsymbol{\Lambda}] = \boldsymbol{0}\,, \tag{93}$$

or

$$[\boldsymbol{T}, \boldsymbol{V}] = \boldsymbol{0}\,. \tag{94}$$

Conversely, if the Cauchy stress tensor diagonalizes in the eulerian triad, then the strain and stress measures are collinear.

In other words, tensors $\boldsymbol{T}_E$ and $\boldsymbol{S}_L$ diagonalize in the reference frame and equation (88) is thus equivalent to

$$J\boldsymbol{T}_E\boldsymbol{\Lambda}^{-1} - g'(\boldsymbol{\Lambda})\boldsymbol{S}_L = \boldsymbol{0}. \tag{95}$$

Solving for $\boldsymbol{S}_L$,

$$\boldsymbol{S}_L = J\boldsymbol{T}_E\{\boldsymbol{\Lambda}g'(\boldsymbol{\Lambda})\}^{-1}, \tag{96}$$

and using equations (12) and (53–54) we finally obtain

$$\boldsymbol{S} = J\boldsymbol{R}^T\boldsymbol{T}\boldsymbol{R}\{\boldsymbol{U}g'(\boldsymbol{U})\}^{-1}. \tag{97}$$

This formula is an explicit expression for any stress measure in terms of the stress tensor and the deformation gradient.

For the class (76) of scale functions, where $g'_n(\lambda) = \lambda^{2n-1}$ we have

$$\boldsymbol{S} = J\boldsymbol{R}^T\boldsymbol{T}\boldsymbol{R}\boldsymbol{U}^{-2n}, \tag{98}$$

and for the Hencky strain measure, where $g(\lambda) = \ln\lambda$ and $g'(\lambda) = 1/\lambda$, equation (97) simplifies to

$$\boldsymbol{S} = J\boldsymbol{R}^T\boldsymbol{T}\boldsymbol{R}. \tag{99}$$

Chapter 5

Constitutive equations

5.1 Basic assumptions

5.1.1 Thermo-mechanical processes

Let $\mathcal{B}$ be a body. A "thermo-mechanical process" of $\mathcal{B}$ is the ordered array

$$(\chi, \theta, \rho, \psi, \boldsymbol{T}, \eta, \boldsymbol{q}) \tag{1}$$

of functions defined over the body $\mathcal{B}$ and a time interval $[\tau_0, \tau_1]$. In this array, χ is a motion, θ is the temperature, ρ is the mass density, ψ is the free energy density, $\boldsymbol{T}$ is the Cauchy stress tensor, η is the entropy density, and $\boldsymbol{q}$ is the heat flux vector.

A thermo-mechanical process has to satisfy the continuity equation (2.8)

$$\dot{\rho} + \rho \nabla \cdot \boldsymbol{v} = 0 \tag{2}$$

the equations of motion (2.32–2.33),

$$\nabla \cdot \boldsymbol{T} + \rho \boldsymbol{b} = \rho \boldsymbol{a}\,, \tag{3}$$

$$\boldsymbol{T}^T = \boldsymbol{T}\,, \tag{4}$$

the first law of thermodynamics in the form (obtained from (3.21) and (3.28),)

$$\rho(\dot{\psi} + \dot{\theta}\eta + \theta\dot{\eta}) = \boldsymbol{T} \cdot \boldsymbol{D} + \rho r - \nabla \cdot \boldsymbol{q}\,, \tag{5}$$

and the Clausius-Duhem inequality

$$\rho\dot{\eta} \geq \rho\frac{r}{\theta} - \nabla \cdot \left(\frac{\boldsymbol{q}}{\theta}\right)\,. \tag{6}$$

These equations, which are valid for any continuum, are not sufficient to determine the list (1) of functions in a process. Roughly speaking, there are eight equations (and one inequality) and nineteen unknowns. The system (2–5) is supplemented by a set of (eleven) "constitutive equations" that describe the material of which the body is made.

One possible choice of constitutive equations is obtained by specifying the free energy density ψ, stress $\boldsymbol{T}$, entropy density η and heat flux vector $\boldsymbol{q}$ as functionals of the motion $\boldsymbol{\chi}$ and of the temperature field history θ,

$$\psi = \psi_{\mathcal{B}\times[\tau_0,\tau]}(\boldsymbol{\chi},\theta)\,, \tag{7}$$

$$\boldsymbol{T} = \boldsymbol{T}_{\mathcal{B}\times[\tau_0,\tau]}(\boldsymbol{\chi},\theta)\,, \tag{8}$$

$$\eta = \eta_{\mathcal{B}\times[\tau_0,\tau]}(\boldsymbol{\chi},\theta)\,, \tag{9}$$

$$\boldsymbol{q} = \boldsymbol{q}_{\mathcal{B}\times[\tau_0,\tau]}(\chi, \theta)\,. \tag{10}$$

The subindexes $\mathcal{B} \times [\tau_0, \tau]$ remind us that these are not functions of a vector and a scalar argument, but rather functionals of the functions χ, θ defined over $\mathcal{B} \times [\tau_0, \tau]$. Note that only the history of the motion and temperature up to time τ are involved, future does not affect present.

5.1.2 The principle of local action

We would expect that the response at a particle is affected only by the motion and temperature of a neighborhood of the particle. More specifically, let $\hat{\mathbf{x}}$ and $\hat{\theta}$ be the referential descriptions of motion and temperature, then

$$\hat{\mathbf{x}}(\mathbf{x}_0', \tau) = \hat{\mathbf{x}}(\mathbf{x}_0, \tau) + \nabla_0 \hat{\mathbf{x}}(\mathbf{x}_0, \tau)(\mathbf{x}_0' - \mathbf{x}_0) + \dots\,, \tag{11}$$

$$\hat{\theta}(\mathbf{x}_0', \tau) = \hat{\theta}(\mathbf{x}_0, \tau) + \nabla_0 \hat{\theta}(\mathbf{x}_0, \tau)(\mathbf{x}_0' - \mathbf{x}_0) + \dots\,, \tag{12}$$

are Taylor series expansions of the motion and temperature fields around $\mathbf{x}_0$ at time τ. In terms of the deformation gradient $\boldsymbol{X}$ and the temperature gradient $\boldsymbol{g}_0 = \boldsymbol{X}^T \boldsymbol{g}$,

$$\mathbf{x}' = \mathbf{x} + \boldsymbol{X}(\mathbf{x}_0' - \mathbf{x}_0) + \dots\,, \tag{13}$$

$$\theta' = \theta + (\boldsymbol{X}^T \boldsymbol{g}) \cdot (\mathbf{x}_0' - \mathbf{x}_0) + \dots\,. \tag{14}$$

The principle of local action states that material response at a particle $\boldsymbol{p}$ depends only on the history of the motion and temperature and their first spatial derivatives at $\boldsymbol{p}$,

$$\psi = \psi_{[\tau_0,\tau]}(\boldsymbol{x}, \boldsymbol{X}, \theta, \boldsymbol{g}\,; \boldsymbol{p}), \tag{15}$$

$$\boldsymbol{T} = \boldsymbol{T}_{[\tau_0,\tau]}(\boldsymbol{x}, \boldsymbol{X}, \theta, \boldsymbol{g}\,; \boldsymbol{p}), \tag{16}$$

$$\eta = \eta_{[\tau_0,\tau]}(\boldsymbol{x}, \boldsymbol{X}, \theta, \boldsymbol{g}\,; \boldsymbol{p}), \tag{17}$$

$$\boldsymbol{q} = \boldsymbol{q}_{[\tau_0,\tau]}(\boldsymbol{x}, \boldsymbol{X}, \theta, \boldsymbol{g}\,; \boldsymbol{p}), \tag{18}$$

where now the functionals depend on functions of time only. We include $\boldsymbol{p}$ as an argument to indicate that the dependence on $(\boldsymbol{x}, \boldsymbol{X}, \theta, \boldsymbol{g})$ can vary with the particle.

5.1.3 Thermodynamics with internal variables

Within the framework of themodynamics with internal variables (see COLEMAN and GURTIN [1967]), dependence on motion and temperature time history is achieved by means of an array of internal or "hidden" variables. To be specific, we consider a simple case where this array contains a second order tensor Φ and a scalar σ. Physically meaningful choices of Φ and σ will be discussed in Chapters 6 and 8. At this point we are interested in the structure of the constitutive equations and some preliminary results.

The variables Φ, σ are considered to be governed by a set of rate-type equations of the form

$$\dot{\Phi} = \bar{\dot{\Phi}}(\mathbf{x}, \boldsymbol{X}, \theta, \boldsymbol{g}, \Phi, \sigma; \boldsymbol{p}), \tag{19}$$

$$\dot{\sigma} = \bar{\dot{\sigma}}(\mathbf{x}, \boldsymbol{X}, \theta, \boldsymbol{g}, \Phi, \sigma; \boldsymbol{p}), \tag{20}$$

with initial conditions (Φ_0, σ_0). Equations (19–20) are called "evolution equations" for the array (Φ, σ). Substituting for the histories of motion and temperature up to time τ in (19–20), the evolution equations reduce to first order differential equations that determine Φ and σ up to time τ. The constitutive equations (15–18) are then written in terms of the motion, temperature, and internal variables at time τ,

$$\psi = \bar{\psi}(\mathbf{x}, \boldsymbol{X}, \theta, \boldsymbol{g}, \Phi, \sigma; \boldsymbol{p}), \tag{21}$$

$$\boldsymbol{T} = \bar{\boldsymbol{T}}(\mathbf{x}, \boldsymbol{X}, \theta, \boldsymbol{g}, \Phi, \sigma; \boldsymbol{p}), \tag{22}$$

$$\eta = \bar{\eta}(\mathbf{x}, \boldsymbol{X}, \theta, \boldsymbol{g}, \Phi, \sigma; \boldsymbol{p}), \tag{23}$$

$$\boldsymbol{q} = \bar{\boldsymbol{q}}(\mathbf{x}, \boldsymbol{X}, \theta, \boldsymbol{g}, \Phi, \sigma; \boldsymbol{p}), \tag{24}$$

where now $\bar{\psi}, \bar{\boldsymbol{T}}, \bar{\eta}$ and $\bar{\boldsymbol{q}}$ are functions of the scalars θ, σ, vectors $\mathbf{x}, \boldsymbol{q}$, and tensors $\boldsymbol{X}, \Phi$. Note that the presence of the deformation gradient in equations (19–24) is sufficient for the response functions to depend on the choice of reference configuration φ. When necessary, this dependence will be indicated by writing $\bar{\dot{\Phi}}_\varphi, \bar{\dot{\sigma}}_\varphi, \bar{\psi}_\varphi, \bar{\boldsymbol{T}}_\varphi, \bar{\eta}_\varphi$ and $\bar{\boldsymbol{q}}_\varphi$.

5.2 Material frame indifference

5.2.1 Rigid body motions

Roughly speaking, the principle of material frame indifference states that the response of a material is not altered by rigid body motions. Although this may sound trivial, it imposes important restrictions on the constitutive functions. We next formalize the principle.

Let χ^* and χ be two motions of a body $\mathcal{B}$ related by a time-dependent rigid body motion,

$$\chi^* = \boldsymbol{Q}\chi + \boldsymbol{c}\,, \tag{25}$$

where for each τ, $\boldsymbol{Q}(\tau)$ is a proper orthogonal tensor and $\boldsymbol{c}(\tau)$ is a vector. Let the associated temperature-field histories $\tilde{\theta}^*$ and $\tilde{\theta}$ be the related by

$$\tilde{\theta}^* = \tilde{\theta}\,, \tag{26}$$

i. e., each particle experiments the same temperature history in both motions.

Assuming that $\boldsymbol{Q}(0) = \boldsymbol{1}$ and $\boldsymbol{c}(0) = \boldsymbol{0}$, we have from (25)

$$\boldsymbol{x}_0 = \chi^*(\boldsymbol{p}, 0) = \chi(\boldsymbol{p}, 0)\,, \tag{27}$$

thus the original configurations of motions χ^* and χ coincide. Let this configuration be selected as reference configuration. Then in view of (25) the referential descriptions of motion are related by

$$\hat{\mathbf{x}}^*(\mathbf{x}_0, \tau) = \boldsymbol{Q}(\tau)\hat{\mathbf{x}}(\mathbf{x}_0, \tau) + \boldsymbol{c}(\tau)\,. \tag{28}$$

Combining equations (26) and (25) we see that the spatial descriptions of temperature are related by

$$\bar{\theta}^*(\boldsymbol{Q}\mathbf{x} + \boldsymbol{c}, \tau) = \bar{\theta}(\mathbf{x}, \tau)\,. \tag{29}$$

Differentiating equation (28) with respect to time, the velocity and acceleration fields satisfy

$$\mathbf{v}^* = \boldsymbol{Q}\mathbf{v} + \dot{\boldsymbol{Q}}\mathbf{x} + \dot{\boldsymbol{c}}\,, \tag{30}$$

$$\mathbf{a}^* = \boldsymbol{Q}\mathbf{a} + \ddot{\boldsymbol{Q}}\mathbf{x} + 2\dot{\boldsymbol{Q}}\mathbf{v}\,. \tag{31}$$

Differentiating equation (28) with respect to $\mathbf{x}_0$ and (29) with respect to $\mathbf{x}$ we obtain for the deformation and temperature gradients,

$$\boldsymbol{X}^* = \boldsymbol{Q}\boldsymbol{X}\,, \tag{32}$$

$$\boldsymbol{g}^* = \boldsymbol{Q}\boldsymbol{g}\,. \tag{33}$$

Since $\det \boldsymbol{Q} = 1$ it follows that $J^* = J$ and in virtue of the continuity equation (2.11), $\rho^* = \rho\,.$

Substituting respectively the right and left polar decompositions of $\boldsymbol{X}$ and $\boldsymbol{X}^*$ in equation (32)

$$\boldsymbol{R}^*\boldsymbol{U}^* = \boldsymbol{Q}\boldsymbol{R}\boldsymbol{U}, \tag{34}$$

$$\boldsymbol{V}^*\boldsymbol{R}^* = \boldsymbol{Q}\boldsymbol{V}\boldsymbol{R}, \tag{35}$$

since the polar decompositions of a tensor are unique, we must have

$$\boldsymbol{R}^* = \boldsymbol{Q}\boldsymbol{R} \tag{36}$$

$$\boldsymbol{U}^* = \boldsymbol{U}, \tag{37}$$

$$\boldsymbol{V}^* = \boldsymbol{Q}\boldsymbol{V}\boldsymbol{Q}^T. \tag{38}$$

Next, differentiating equation (32) with respect to time and recalling that the velocity gradient is given by $\boldsymbol{L} = \dot{\boldsymbol{X}}\boldsymbol{X}^{-1}$,

$$\boldsymbol{L}^* = \boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T + \dot{\boldsymbol{Q}}\boldsymbol{Q}^T, \tag{39}$$

from where the stretching and spin tensors are related by

$$\boldsymbol{D}^* = \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T, \tag{40}$$

$$\boldsymbol{W}^* = \boldsymbol{Q}\boldsymbol{W}\boldsymbol{Q}^T + \dot{\boldsymbol{Q}}\boldsymbol{Q}^T. \tag{41}$$

Note that the relationship between any other kinematical quantity in the two motions can be deduced.

We assume at this point that the internal variables Φ, σ are defined in such a way that they transform as

$$\Phi^* = \Phi\,, \tag{42}$$

$$\sigma^* = \sigma\,. \tag{43}$$

5.2.2 Objectivity

In general, a scalar field α, a vector field $\boldsymbol{a}$, and a tensor field $\boldsymbol{A}$ are said to be "objective" if for all rigid body motions $(\boldsymbol{Q}, \mathbf{c})$,

$$\bar{\alpha}^*(\boldsymbol{Q}\mathbf{x} + \mathbf{c}, \tau) = \bar{\alpha}(\mathbf{x}, \tau)\,, \tag{44}$$

$$\bar{\mathbf{a}}^*(\boldsymbol{Q}\mathbf{x} + \mathbf{c}, \tau) = \boldsymbol{Q}\bar{\mathbf{a}}(\mathbf{x}, \tau)\,, \tag{45}$$

$$\bar{\boldsymbol{A}}^*(\boldsymbol{Q}\mathbf{x} + \mathbf{c}, \tau) = \boldsymbol{Q}\bar{\boldsymbol{A}}(\mathbf{x}, \tau)\boldsymbol{Q}^T\,, \tag{46}$$

where α^*, $\boldsymbol{a}^*$ and $\boldsymbol{A}^*$ are the corresponding fields associated with the motion $\boldsymbol{\chi}^* = \boldsymbol{Q}\boldsymbol{\chi} + \mathbf{c}$.

Thus for example the density, the temperature gradient and the stretching tensors are objective, while the velocity vector, the deformation gradient, and the spin tensor are not objective.

Objectivity can be interpreted as follows. Let $\{\boldsymbol{e}_i\}$ be a fixed reference frame and let $\{\mathbf{e}_i^*\}$ be a frame that follows the rigid body motion, i. e.

$$\mathbf{e}_i^* = \boldsymbol{Q}\boldsymbol{e}_i \qquad \text{for} \quad i = 1,3\,, \tag{47}$$

then using equations (45–46),

$$\boldsymbol{a}^* \cdot \mathbf{e}_i^* = (\boldsymbol{Q}\mathbf{a}) \cdot (\boldsymbol{Q}\boldsymbol{e}_i) = \boldsymbol{a} \cdot \boldsymbol{e}_i\,, \tag{48}$$

$$\mathbf{e}_i^* \cdot \boldsymbol{A}^*\boldsymbol{e}_j^* = (\boldsymbol{Q}\boldsymbol{e}_i) \cdot (\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T)(\boldsymbol{Q}\boldsymbol{e}_j) = \boldsymbol{e}_i \cdot \boldsymbol{A}\boldsymbol{e}_j\,, \tag{49}$$

therefore the components of an objective vector or tensor in the motion χ^* with respect to a frame that follows the rigid body motion $\tilde{r}$ are equal to their components in the motion χ with respect to a fixed reference frame.

Note that the time rate of change of an objective vector or tensor are not objective. Let $\boldsymbol{a}$ and $\boldsymbol{A}$ be objective, then

$$\dot{\mathbf{a}}^* = \boldsymbol{Q}\dot{\mathbf{a}} + \dot{\boldsymbol{Q}}\boldsymbol{a}\,, \tag{50}$$

$$\dot{\boldsymbol{A}}^* = \boldsymbol{Q}\dot{\boldsymbol{A}}\boldsymbol{Q}^T + \dot{\boldsymbol{Q}}\boldsymbol{A}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{A}\dot{\boldsymbol{Q}}^T\,. \tag{51}$$

5.2.3 The principle of material frame indifference

The principle of material frame indifference postulates that the free energy density, the stress tensor, the entropy density and the heat flux vector are objective, i. e., for all time-dependent rigid body motions $(\boldsymbol{Q}, \boldsymbol{c})$,

$$\psi^* = \psi\,, \tag{52}$$

$$\boldsymbol{T}^* = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T\,, \tag{53}$$

$$\eta^* = \eta\,, \tag{54}$$

$$\boldsymbol{q}^* = \boldsymbol{Q}\boldsymbol{q}\,, \tag{55}$$

where ψ^*, ψ, $\boldsymbol{T}^*, \boldsymbol{T}$, η^*, η and $\boldsymbol{q}^*, \boldsymbol{q}$ are respectively the internal energy density, stress tensor, entropy density and heat flux vector associated with the motions χ^* and χ.

In virtue of the geometric relation

$$\boldsymbol{n}^* = \boldsymbol{Q}\boldsymbol{n}\,, \tag{56}$$

it follows from Cauchy's and Fourier's theorems (2.27) and (3.7) that the traction vector and the heat flux are objective,

$$\boldsymbol{t}^* = \boldsymbol{Q}\boldsymbol{t}\,, \tag{57}$$

$$h^* = h\,. \tag{58}$$

5.2.4 Invariance under rigid body motions

Having specified the transformation rules for the variables that define a thermomechanical process, we now turn to the constitutive equations (19–24). Material response is not altered by rigid body motions if the same functions $\bar{\psi}$, $\bar{\boldsymbol{T}}, \bar{\eta}, \bar{\boldsymbol{q}}$ and $\bar{\dot{\Phi}}, \bar{\dot{\sigma}}$ relate the quantities associated with the motion $\boldsymbol{\chi}^*$, i. e.,

$$\psi^* = \bar{\psi}(\boldsymbol{x}^*, \boldsymbol{X}^*, \theta^*, \boldsymbol{g}^*, \Phi^*, \sigma^* ; \boldsymbol{p}) , \tag{59}$$

$$\boldsymbol{T}^* = \bar{\boldsymbol{T}}(\boldsymbol{x}^*, \boldsymbol{X}^*, \theta^*, \boldsymbol{g}^*, \Phi^*, \sigma^* ; \boldsymbol{p}) , \tag{60}$$

$$\eta^* = \bar{\eta}(\boldsymbol{x}^*, \boldsymbol{X}^*, \theta^*, \boldsymbol{g}^*, \Phi^*, \sigma^* ; \boldsymbol{p}) , \tag{61}$$

$$\boldsymbol{q}^* = \bar{\boldsymbol{q}}(\boldsymbol{x}^*, \boldsymbol{X}^*, \theta^*, \boldsymbol{g}^*, \Phi^*, \sigma^* ; \boldsymbol{p}) , \tag{62}$$

$$\dot{\Phi}^* = \bar{\dot{\Phi}}(\boldsymbol{x}^*, \boldsymbol{X}^*, \theta^*, \boldsymbol{g}^*, \Phi^*, \sigma^* ; \boldsymbol{p}) . \tag{63}$$

$$\dot{\sigma}^* = \bar{\dot{\sigma}}(\boldsymbol{x}^*, \boldsymbol{X}^*, \theta^*, \boldsymbol{g}^*, \Phi^*, \sigma^* ; \boldsymbol{p}) . \tag{64}$$

Substituting in (59–64), equations (25–26), (32–33), (42–43) and (52–55) we obtain

$$\psi = \bar{\psi}(\boldsymbol{Qx} + \boldsymbol{c}, \boldsymbol{QX}, \theta, \boldsymbol{Qg}, \Phi, \sigma ; \boldsymbol{p}) , \tag{65}$$

$$\boldsymbol{QTQ}^T = \bar{\boldsymbol{T}}(\boldsymbol{Qx} + \boldsymbol{c}, \boldsymbol{QX}, \theta, \boldsymbol{Qg}, \Phi, \sigma ; \boldsymbol{p}) , \tag{66}$$

$$\eta = \bar{\eta}(\boldsymbol{Qx} + \boldsymbol{c}, \boldsymbol{QX}, \theta, \boldsymbol{Qg}, \Phi, \sigma ; \boldsymbol{p}) , \tag{67}$$

$$\boldsymbol{Qq} = \bar{\boldsymbol{q}}(\boldsymbol{Qx} + \boldsymbol{c}, \boldsymbol{QX}, \theta, \boldsymbol{Qg}, \Phi, \sigma ; \boldsymbol{p}) , \tag{68}$$

$$\dot{\Phi} = \bar{\dot{\Phi}}(\boldsymbol{Qx} + \boldsymbol{c}, \boldsymbol{QX}, \theta, \boldsymbol{Qg}, \Phi, \sigma ; \boldsymbol{p}) , \tag{69}$$

$$\dot{\sigma} = \bar{\dot{\sigma}}(\boldsymbol{Qx} + \boldsymbol{c}, \boldsymbol{QX}, \theta, \boldsymbol{Qg}, \Phi, \sigma ; \boldsymbol{p}) \qquad \forall \boldsymbol{Q} \in \mathcal{O}_+, \forall \boldsymbol{c} \in E . \tag{70}$$

These are the restrictions that the principle of material frame indifference imposes onto the constitutive equations.

As a first consequence, we see that since conditions (65–70) have to be valid for all vectors $\boldsymbol{c} \in E$, it follows that the response functions cannot depent on the position vector $\boldsymbol{x}$ and we write

$$\psi = \bar{\psi}(\boldsymbol{X}, \theta, \boldsymbol{g}, \Phi, \sigma; \boldsymbol{p}), \tag{71}$$

$$\boldsymbol{T} = \bar{\boldsymbol{T}}(\boldsymbol{X}, \theta, \boldsymbol{g}, \Phi, \sigma; \boldsymbol{p}), \tag{72}$$

$$\eta = \bar{\eta}(\boldsymbol{X}, \theta, \boldsymbol{g}, \Phi, \sigma; \boldsymbol{p}), \tag{73}$$

$$\boldsymbol{q} = \bar{\boldsymbol{q}}(\boldsymbol{X}, \theta, \boldsymbol{g}, \Phi, \sigma; \boldsymbol{p}), \tag{74}$$

$$\dot{\Phi} = \bar{\dot{\Phi}}(\boldsymbol{X}, \theta, \boldsymbol{g}, \Phi, \sigma; \boldsymbol{p}), \tag{75}$$

$$\dot{\sigma} = \bar{\dot{\sigma}}(\boldsymbol{X}, \theta, \boldsymbol{g}, \Phi, \sigma; \boldsymbol{p}). \tag{76}$$

5.3 Invariant constitutive equations

A scalar field β, a vector field $\boldsymbol{b}$, and a tensor field $\boldsymbol{B}$ are said to be "invariant under rigid body motions" if for all time-dependent rigid body motions $(\boldsymbol{Q}, \boldsymbol{c})$,

$$\bar{\beta}^*(\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}, \tau) = \bar{\beta}(\boldsymbol{x}, \tau), \tag{77}$$

$$\bar{\boldsymbol{b}}^*(\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}, \tau) = \bar{\boldsymbol{b}}(\boldsymbol{x}, \tau), \tag{78}$$

$$\bar{\boldsymbol{B}}^*(\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}, \tau) = \bar{\boldsymbol{B}}(\boldsymbol{x}, \tau), \tag{79}$$

where β^*, $\boldsymbol{b}^*$ and $\boldsymbol{B}^*$ are the corresponding fields associated with the motion $\boldsymbol{\chi}^* = \boldsymbol{Q}\boldsymbol{\chi} + \boldsymbol{c}$. We see that in the case of a scalar field, invariance and objectivity are equivalent. If, however, a vector $\boldsymbol{b}$ and a tensor $\boldsymbol{B}$ are invariant, then their time rates of change are also invariant,

$$\dot{\boldsymbol{b}}^* = \dot{\boldsymbol{b}}, \tag{80}$$

$$\dot{\boldsymbol{B}}^* = \dot{\boldsymbol{B}}. \tag{81}$$

Conditions (65–70) imposed by the principle of material frame indifference to the constitutive equations would be automatically satisfied if the variables characterizing the process were substituted for by an array of corresponding invariant variables.

Consider first the temperature gradient $\boldsymbol{g}$ and heat flux vector $\boldsymbol{q}$. We recall the definitions (3.49) and (3.33),

$$\boldsymbol{g}_0 = \boldsymbol{X}^T \boldsymbol{g}, \tag{82}$$

$$\boldsymbol{q}_0 = J\boldsymbol{X}^{-1}\boldsymbol{q}, \tag{83}$$

then using equations (32–33) and (55), we obtain

$$\boldsymbol{g}_0^* = \boldsymbol{g}_0, \tag{84}$$

$$\boldsymbol{q}_0^* = \boldsymbol{q}_0, \tag{85}$$

i. e., the temperature gradient and heat flux vector with respect to the reference configuration are invariant vectors.

We next note here that in virtue of equation (37), the right stretch tensor is invariant, it follows that

$$\boldsymbol{\Lambda}^* = \boldsymbol{\Lambda}, \tag{86}$$

$$\boldsymbol{R}_L^* = \boldsymbol{R}_L\,, \tag{87}$$

where $\boldsymbol{\Lambda}$ and $\boldsymbol{R}_L$ have been defined in equations (4.7) and (4.8). Any strain measure is consequently an invariant tensor,

$$\boldsymbol{E}^* = \boldsymbol{E}\,. \tag{88}$$

Recalling equation (4.12), and using (36) and (87), we see that

$$\boldsymbol{R}_E^* = \boldsymbol{R}^*\boldsymbol{R}_L^* = \boldsymbol{Q}\boldsymbol{R}\boldsymbol{R}_L = \boldsymbol{Q}\boldsymbol{R}_E\,, \tag{89}$$

and in virtue of equations (4.53), (53) and (89),

$$\boldsymbol{T}_E^* = (\boldsymbol{R}_E^*)^T\boldsymbol{T}^*\boldsymbol{R}_E^* = (\boldsymbol{R}_E^T\boldsymbol{Q}^T)(\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T)(\boldsymbol{Q}\boldsymbol{R}_E) = \boldsymbol{T}_E\,. \tag{90}$$

This result, together with equations (86–87) and the stress measure formula (4.72) allows us to conclude that any stress measure is also an invariant tensor,

$$\boldsymbol{S}^* = \boldsymbol{S}\,. \tag{91}$$

Given the deformation gradient $\boldsymbol{X}$, equations (82–91) establish a one to one correspondence between the objective variables $\boldsymbol{g}, \boldsymbol{q}$ and $\boldsymbol{T}$ and the invariant variables $\boldsymbol{g}_0, \boldsymbol{q}_0$ and $\boldsymbol{S}$. Note that equations (42–43) indicate that we have assumed the internal variables Φ, σ to be invariant. Then we can substitute the constitutive equations (71–76) by the set

$$\epsilon = \bar{\epsilon}(\boldsymbol{X}, \theta, \boldsymbol{g}_0, \Phi, \sigma\,; \boldsymbol{p})\,, \tag{92}$$

$$S = \bar{S}(\boldsymbol{X}, \theta, \boldsymbol{g}_0, \Phi, \sigma; \boldsymbol{p}), \tag{93}$$

$$\eta = \bar{\eta}(\boldsymbol{X}, \theta, \boldsymbol{g}_0, \Phi, \sigma; \boldsymbol{p}), \tag{94}$$

$$\boldsymbol{q}_0 = \bar{\boldsymbol{q}}_0(\boldsymbol{X}, \theta, \boldsymbol{g}_0, \Phi, \sigma; \boldsymbol{p}), \tag{95}$$

$$\dot{\Phi} = \bar{\dot{\Phi}}(\boldsymbol{X}, \theta, \boldsymbol{g}_0, \Phi, \sigma; \boldsymbol{p}), \tag{96}$$

$$\dot{\sigma} = \bar{\dot{\sigma}}(\boldsymbol{X}, \theta, \boldsymbol{g}_0, \Phi, \sigma; \boldsymbol{p}). \tag{97}$$

Restrictions (65–70) now reduce to the condition that equations (92–97) remain invariant under a change $\boldsymbol{X} \to \boldsymbol{Q}\boldsymbol{X}$, for all $\boldsymbol{Q} \in \mathcal{O}_+$. By selecting $\boldsymbol{Q} = \boldsymbol{R}^T \in \mathcal{O}_+$ we conclude from $\boldsymbol{R}^T\boldsymbol{X} = \boldsymbol{U}$ that the response functions depend on the deformation gradient only through the right stretch tensor $\boldsymbol{U}$. Since given a scale function (4.38–4.40) there is a one-to-one correspondence between $\boldsymbol{U}$ and the strain measure $\boldsymbol{E}$, we can finally write

$$\epsilon = \bar{\epsilon}(\boldsymbol{E}, \theta, \boldsymbol{g}_0, \Phi, \sigma; \boldsymbol{p}), \tag{98}$$

$$\boldsymbol{S} = \bar{\boldsymbol{S}}(\boldsymbol{E}, \theta, \boldsymbol{g}_0, \Phi, \sigma; \boldsymbol{p}), \tag{99}$$

$$\eta = \bar{\eta}(\boldsymbol{E}, \theta, \boldsymbol{g}_0, \Phi, \sigma; \boldsymbol{p}), \tag{100}$$

$$\boldsymbol{q}_0 = \bar{\boldsymbol{q}}_0(\boldsymbol{E}, \theta, \boldsymbol{g}_0, \Phi, \sigma; \boldsymbol{p}), \tag{101}$$

$$\dot{\Phi} = \bar{\dot{\Phi}}(\boldsymbol{E}, \theta, \boldsymbol{g}_0, \Phi, \sigma; \boldsymbol{p}), \tag{102}$$

$$\dot{\sigma} = \bar{\dot{\sigma}}(\boldsymbol{E}, \theta, \boldsymbol{g}_0, \Phi, \sigma; \boldsymbol{p}). \tag{103}$$

This set of constitutive equations is entirely equivalent to the original (19–24) and satisfies automatically the principle of material frame indifference.

Note that the choice of invariant variables is not unique, and therefore other sets of invariant constitutive equations (see Chapters 6 and 8) can be developed. The purpose of this section has been to show the general procedure.

Chapter 6

Thermo-elasto-plasticity I

6.1 State variables

One of the simplest elasto-plastic constitutive models is based on the array $(\boldsymbol{E}^p, \sigma)$ of plastic (internal) variables. The scalar σ, called "deformation resistance", has the dimensions of stress and represents isotropic resistance to plastic flow. It allows for isotropic "hardening" of the material. The "plastic strain tensor" $\boldsymbol{E}^p$, in a sense to be made more precise, accounts for plastic deformation. The (total) strain tensor $\boldsymbol{E}$ is assumed to admit the following additive decomposition

$$\boldsymbol{E} = \boldsymbol{E}^e + \boldsymbol{E}^p , \tag{1}$$

where $\boldsymbol{E}^e = \boldsymbol{E} - \boldsymbol{E}^p$ is the "elastic" strain tensor. Among the earliest models that used this approach for a large strain formulation are those of GREEN and NAGHDI [1965], and PERZYNA and WOJNO [1966,1968].

In virtue of the discussion in Section 5.2 and with the selection $(\boldsymbol{E}^p, \sigma)$, the array of variables that characterize a thermo-elasto-plastic process is given by

$$\left(\boldsymbol{X}, \theta, \boldsymbol{g}, \rho, \psi, \boldsymbol{T}, \eta, \mathbf{q}, \boldsymbol{E}^p, \sigma\right). \tag{2}$$

Motivated by the results of Section 5.3, we next obtain a set of invariant variables corresponding to (2).

In view of equation (4.49), given a scale function g, the stress power per unit reference volume can be written as

$$w_0 = J\boldsymbol{T}\cdot\boldsymbol{D} = \boldsymbol{S}\cdot\dot{\boldsymbol{E}}, \tag{3}$$

where the strain and stress measures are given by

$$\boldsymbol{E} = \boldsymbol{R}_L g(\Lambda)\boldsymbol{R}_L^T, \tag{4}$$

$$\boldsymbol{S} = \boldsymbol{R}_L \boldsymbol{S}_L \boldsymbol{R}_L^T, \tag{5}$$

and from (4.72),

$$[\boldsymbol{S}_L]_{\alpha\beta} = \begin{cases} \dfrac{J}{\lambda_\alpha g'(\lambda_\alpha)}[\boldsymbol{R}_E\boldsymbol{T}\boldsymbol{R}_E^T]_{\alpha\alpha}, & \text{if } \beta=\alpha; \\ \frac{1}{2}J\dfrac{\lambda_\beta\lambda_\alpha^{-1}-\lambda_\alpha\lambda_\beta^{-1}}{g(\lambda_\beta)-g(\lambda_\alpha)}[\boldsymbol{R}_E\boldsymbol{T}\boldsymbol{R}_E^T]_{\alpha\beta}, & \text{otherwise.} \end{cases} \tag{6}$$

As stated in equations (5.88) and (5.91), the strain and stress measures are invariant,

$$\boldsymbol{E}^* = \boldsymbol{E}, \tag{7}$$

$$\boldsymbol{S}^* = \boldsymbol{S}. \tag{8}$$

The plastic strain tensor is assumed to be invariant under rigid body motions,

$$(\boldsymbol{E}^p)^* = \boldsymbol{E}^p \,. \tag{9}$$

As a consequence of (9) and equation (7) the elastic strain tensor $\boldsymbol{E}^e$ introduced in (1) inherits the invariance property,

$$(\boldsymbol{E}^e)^* = \boldsymbol{E}^e \,. \tag{10}$$

According to (5.82–5.83), we write the temperature gradient and heat flux vector with respect to the reference configuration as

$$\boldsymbol{g}_0 = \boldsymbol{X}^T \boldsymbol{g} \,, \tag{11}$$

$$\boldsymbol{q}_0 = J \boldsymbol{X}^{-1} \boldsymbol{q} \,, \tag{12}$$

these vectors being invariant under rigid body motions (5.84–5.85),

$$\boldsymbol{g}_0^* = \boldsymbol{g}_0 \,, \tag{13}$$

$$\boldsymbol{q}_0^* = \boldsymbol{q}_0 \,. \tag{14}$$

With these givens we take the array of invariant variables associated with (2) to be

$$\left(\boldsymbol{E}^e, \theta, \boldsymbol{g}_0, \rho, \psi, \boldsymbol{S}, \eta, \boldsymbol{q}_0, \boldsymbol{E}^p, \sigma\right) . \tag{15}$$

6.2 Consequences of the reduced dissipation inequality

Racall the reduced dissipation inequality written with respect to the reference configuration (3.52),

$$\rho_0 \dot{\psi} - w_0 + \rho_0 \eta \dot{\theta} + \theta^{-1} \boldsymbol{q}_0 \cdot \boldsymbol{g}_0 \leq 0 \,. \tag{16}$$

In view of equations (3) and (1) we write the stress power per unit reference volume as

$$w_0 = w_0^e + w_0^p \,, \tag{17}$$

$$w_0^e = \boldsymbol{S} \cdot \dot{\boldsymbol{E}}^e \,, \tag{18}$$

$$w_0^p = \boldsymbol{S} \cdot \dot{\boldsymbol{E}}^p \,, \tag{19}$$

and the reduced dissipation inequality reads,

$$\rho_0 \dot{\psi} - \boldsymbol{S} \cdot \dot{\boldsymbol{E}}^e + \rho_0 \eta \dot{\theta} - \boldsymbol{S} \cdot \dot{\boldsymbol{E}}^p + \theta^{-1} \boldsymbol{q}_0 \cdot \boldsymbol{g}_0 \leq 0 \,. \tag{20}$$

As in Section (5.3), we specify constitutive equations for the variables $(\psi, \boldsymbol{S}, \eta, \boldsymbol{q}_0)$ and evolution equations for $(\boldsymbol{E}^p, \sigma)$. The variables $(\boldsymbol{E}^e, \theta, \boldsymbol{g}_0, \boldsymbol{E}^p, \sigma)$ are taken to be independent.

At this point it is convenient to make the assumption that the plastic strain tensor is not a state variable, in the sense that plastic deformation, other variables held constant, does not modify the state of a material neighborhood. This simplifying assumption does not follow from basic principles, and its use is justified a posteriori.

Consequently we write the following set of material response functions,

$$\psi = \bar{\psi}(\boldsymbol{E}^e, \theta, \boldsymbol{g}_0, \sigma) \,, \tag{21}$$

$$\boldsymbol{S} = \bar{\boldsymbol{S}}(\boldsymbol{E}^e, \theta, \boldsymbol{g}_0, \sigma) \,, \tag{22}$$

$$\eta = \bar{\eta}(\boldsymbol{E}^e, \theta, \boldsymbol{g}_0, \sigma), \tag{23}$$

$$\boldsymbol{q}_0 = \bar{\boldsymbol{q}}_0(\boldsymbol{E}^e, \theta, \boldsymbol{g}_0, \sigma), \tag{24}$$

$$\dot{\boldsymbol{E}}^p = \bar{\dot{\boldsymbol{E}}}^p(\boldsymbol{E}^e, \theta, \boldsymbol{g}_0, \sigma), \tag{25}$$

$$\dot{\sigma} = \bar{\dot{\sigma}}(\boldsymbol{E}^e, \theta, \boldsymbol{g}_0, \sigma), \tag{26}$$

where mention of the dependence on the particle $\boldsymbol{p}$ has been supressed for simplicity. Equations (21–26) satisfy the principle of material frame indifference for any choice of response functions $\bar{\psi}, \bar{\boldsymbol{S}}, \bar{\eta}, \bar{\boldsymbol{q}}_0, \bar{\dot{\boldsymbol{E}}}^p$ and $\bar{\dot{\sigma}}$.

Differentiating equation (21) with respect to time,

$$\dot{\psi} = \frac{\partial \psi}{\partial \boldsymbol{E}^e} \cdot \dot{\boldsymbol{E}}^e + \frac{\partial \psi}{\partial \theta}\dot{\theta} + \frac{\partial \psi}{\partial \boldsymbol{g}_0} \cdot \dot{\boldsymbol{g}}_0 + \frac{\partial \psi}{\partial \sigma}\dot{\sigma}, \tag{27}$$

substituting in (20) and rearranging,

$$\begin{aligned}\left(\rho_0 \frac{\partial \psi}{\partial \boldsymbol{E}^e} - \boldsymbol{S}\right) \cdot \dot{\boldsymbol{E}}^e + \left(\frac{\partial \psi}{\partial \theta} + \eta\right)\dot{\theta} + \rho_0 \frac{\partial \psi}{\partial \boldsymbol{g}_0} \cdot \dot{\boldsymbol{g}}_0 \\ + \rho_0 \frac{\partial \psi}{\partial \sigma}\dot{\sigma} - \boldsymbol{S} \cdot \dot{\boldsymbol{E}}^p + \theta^{-1}\boldsymbol{q}_0 \cdot \boldsymbol{g}_0 \leq 0.\end{aligned} \tag{28}$$

There are two possible interpretations of inequality (28). We can consider the response functions $(\bar{\psi}, \bar{\boldsymbol{S}}, \bar{\eta}, \bar{\boldsymbol{q}}_0, \bar{\dot{\boldsymbol{E}}}^p, \bar{\dot{\sigma}})$ as arbitrary, and then (28) is a restriction of the processes that the body can undergo. Or, we can say that (28) must hold for all processes, and consider it as a restriction on $(\bar{\psi}, \bar{\boldsymbol{S}}, \bar{\eta}, \bar{\boldsymbol{q}}_0, \bar{\dot{\boldsymbol{E}}}^p, \bar{\dot{\sigma}})$.

Following Coleman and Noll [1963] we adopt the second interpretation, and assume that the form (28) of the reduced dissipation inequality is required to hold for all values of the variables $(\boldsymbol{E}^e, \theta, \boldsymbol{g}_0)$ and all values of the derivatives $(\dot{\boldsymbol{E}}^e, \dot{\theta}, \dot{\boldsymbol{g}}_0)$. The following results hold.

(a.1) Given the free-energy response function $\bar{\psi}$, the stress and entropy response functions are obtained from

$$\bar{\boldsymbol{S}} = \rho_0 \frac{\partial \bar{\psi}}{\partial \boldsymbol{E}^e}, \tag{29}$$

$$\bar{\eta} = -\frac{\partial \bar{\psi}}{\partial \theta}. \tag{30}$$

(a.2) The free-energy does not depend on the temperature gradient $\boldsymbol{g}_0$,

$$\frac{\partial \bar{\psi}}{\partial \boldsymbol{g}_0} = 0. \tag{31}$$

It follows form (a) that the stress and entropy response functions do not depend on the temperature gradient either.

(a.3) The following inequality must hold for all processes,

$$\rho_0 \frac{\partial \psi}{\partial \sigma} \dot{\sigma} - \boldsymbol{S} \cdot \dot{\boldsymbol{E}}^p + \theta^{-1} \boldsymbol{q}_0 \cdot \boldsymbol{g}_0 \leq 0. \tag{32}$$

If the plastic response functions $\bar{\dot{\boldsymbol{E}}}^p$ and $\bar{\dot{\sigma}}$ do not depend on the temperature gradient $\boldsymbol{g}_0$, **(a.3)** leads to

(a.4) the functions $\bar{\dot{\boldsymbol{E}}}^p$ and $\bar{\dot{\sigma}}$ satisfy the internal (plastic) dissipation inequality,

$$\rho_0 \frac{\partial \psi}{\partial \sigma} \bar{\dot{\sigma}} - \boldsymbol{S} \cdot \bar{\dot{\boldsymbol{E}}}^p \leq 0, \qquad \text{and} \tag{33}$$

(a.5) the function $\bar{\boldsymbol{q}}_0$ satisfies the heat conduction inequality,

$$\bar{\boldsymbol{q}}_0 \cdot \boldsymbol{g}_0 \leq 0. \tag{34}$$

Conditions **(a.4–a.5)** do not follow from our general assumptions, but it could be proved that with some symmetry restrictions, $\bar{\dot{\boldsymbol{E}}}^p$ and $\bar{\dot{\sigma}}$ depend on $\boldsymbol{g}_0$ only through higher order terms. Hereafter we consider that **(a.4–a.5)** hold.

We have therefore reduced the set (21–26) of constitutive equations to

$$\psi = \bar{\psi}(\boldsymbol{E}^e, \theta, \sigma)\,, \tag{35}$$

$$\boldsymbol{q}_0 = \bar{\boldsymbol{q}}_0(\boldsymbol{E}^e, \theta, \boldsymbol{g}_0, \sigma)\,, \tag{36}$$

$$\dot{\boldsymbol{E}}^p = \bar{\dot{\boldsymbol{E}}}^p(\boldsymbol{E}^e, \theta, \sigma)\,, \tag{37}$$

$$\dot{\sigma} = \bar{\dot{\sigma}}(\boldsymbol{E}^e, \theta, \sigma)\,, \tag{38}$$

with the stress and entropy response functions given by **(a.1)**, the plastic variables satisfying the internal dissipation inequality **(a.4)** and the heat flux vector satisfying the heat conduction inequality **(a.5)**.

6.3 Material symmetry

6.3.1 Change in reference configuration

Let φ_0 and φ_0^* be two reference configurations. The vectors

$$\mathbf{x}_0 = \varphi_0(\boldsymbol{p}), \tag{39}$$

$$\mathbf{x}_0^* = \varphi_0^*(\boldsymbol{p}), \tag{40}$$

give respectively the position of particle $\boldsymbol{p}$ in configurations φ_0 and φ_0^*. These two configurations are related by the deformation $\hat{\boldsymbol{h}}$ such that

$$\mathbf{x}_0^* = \hat{\boldsymbol{h}}(\mathbf{x}_0) = \varphi_0^*(\varphi_0^{-1}(\mathbf{x}_0)) \tag{41}$$

gives the position in configuration φ_0^* of the particle $\boldsymbol{p}$ that in configuration φ_0 is at position $\mathbf{x}_0$. We consider changes in reference configuration $\hat{\boldsymbol{h}}$ of the form

$$\mathbf{x}_0^* = \hat{\boldsymbol{h}}(\mathbf{x}_0) = \boldsymbol{Q}_0\mathbf{x}_0 + \boldsymbol{r}, \tag{42}$$

where $\boldsymbol{Q}_0$ is a proper orthogonal tensor and $\boldsymbol{r}$ is a vector. Equation (42) represents a time-independent rotation.

Consider next a configuration φ with $\mathbf{x} = \varphi(\boldsymbol{p})$. Let $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}_*$ denote respectively the deformations from φ_0 and φ_0^* to φ,

$$\mathbf{x} = \hat{\mathbf{x}}(\mathbf{x}_0), \tag{43}$$

$$\boldsymbol{x} = \hat{\boldsymbol{x}}_*(\boldsymbol{x}_0^*)\,. \tag{44}$$

These two deformations are related by

$$\boldsymbol{x} = \hat{\boldsymbol{x}}(\boldsymbol{x}_0) = \hat{\boldsymbol{x}}_*\big(\hat{\boldsymbol{h}}(\boldsymbol{x}_0)\big)\,, \tag{45}$$

Differentiating with respect to $\boldsymbol{x}_0$ we obtain for the deformation gradients $\boldsymbol{X}$ and $\boldsymbol{X}_*$

$$\boldsymbol{X} = \boldsymbol{X}_*\boldsymbol{Q}_0\,, \tag{46}$$

or, solving for $\boldsymbol{X}_*$,

$$\boldsymbol{X}_* = \boldsymbol{X}\boldsymbol{Q}_0^T\,, \tag{47}$$

and since $\det \boldsymbol{Q}_0 = 1$ we have for the Jacobians,

$$J_* = J\,. \tag{48}$$

Let ρ_0, ρ_0^* and ρ be the densities at particle $\boldsymbol{p}$ in the configurations φ_0, φ_0^* and φ respectively. By conservation of mass,

$$\rho = \frac{\rho_0}{J_0} = \frac{\rho_0^*}{J_*}\,, \tag{49}$$

and in virtue of (47),

$$\rho_0^* = \rho_0\,. \tag{50}$$

Substituting the right and left polar decompositions of $\boldsymbol{X}_*$ and $\boldsymbol{X}$ in (47),

$$\boldsymbol{R}_*\boldsymbol{U}_* = \boldsymbol{R}\boldsymbol{U}\boldsymbol{Q}_0^T\,, \tag{51}$$

$$\boldsymbol{V}_*\boldsymbol{R}_* = \boldsymbol{V}\boldsymbol{R}\boldsymbol{Q}_0^T \,. \tag{52}$$

Since the polar decompositions of a tensor are unique, we must have

$$\boldsymbol{R}_* = \boldsymbol{R}\boldsymbol{Q}_0^T \,, \tag{53}$$

$$\boldsymbol{U}_* = \boldsymbol{Q}_0\boldsymbol{U}\boldsymbol{Q}_0^T \,, \tag{54}$$

$$\boldsymbol{V}_* = \boldsymbol{V} \,. \tag{55}$$

In virtue of equations (53) and (55), and recalling equation (4.12)

$$\boldsymbol{\Lambda}_* = \boldsymbol{\Lambda} \,, \tag{56}$$

$$\boldsymbol{R}_{L*} = \boldsymbol{Q}_0\boldsymbol{R}_L \,, \tag{57}$$

$$\boldsymbol{R}_{E*} = \boldsymbol{R}_E \,, \tag{58}$$

we conclude from (56) and (57) that under a time independent rotation of the reference configuration a strain measure transforms as

$$\boldsymbol{E}_* = \boldsymbol{Q}_0\boldsymbol{E}\boldsymbol{Q}_0^T \,. \tag{59}$$

The elastic and plastic strain tensors $\boldsymbol{E}^e, \boldsymbol{E}^p$ are assumed to inherit the same transformation rule,

$$\boldsymbol{E}_*^e = \boldsymbol{Q}_0\boldsymbol{E}^e\boldsymbol{Q}_0^T \,. \tag{60}$$

$$\boldsymbol{E}_*^p = \boldsymbol{Q}_0\boldsymbol{E}^p\boldsymbol{Q}_0^T \,, \tag{61}$$

and similarly for the time rate of change of the plastic strain rate we have from (61),

$$\dot{\boldsymbol{E}}^p_* = \boldsymbol{Q}_0 \dot{\boldsymbol{E}}^p \boldsymbol{Q}_0^T \,. \tag{62}$$

Using equations (47–48) and the definitions (11–12) we have for the temperature gradients and the heat flux vectors with respect to φ_0^* and φ_0 ,

$$\boldsymbol{g}_0^* = \boldsymbol{X}_*^T \boldsymbol{g} = \boldsymbol{Q}_0 \boldsymbol{X}^T \boldsymbol{g} = \boldsymbol{Q}_0 \boldsymbol{g}_0 \,, \tag{63}$$

$$\boldsymbol{q}_0^* = J_* \boldsymbol{X}_*^{-1} \boldsymbol{q} = J \boldsymbol{Q}_0 \boldsymbol{X}^{-1} \boldsymbol{q} = \boldsymbol{Q}_0 \boldsymbol{q}_0 \,. \tag{64}$$

To summarize, under a time-independent rotation of the reference configuration the independent variables transform as

$$(\boldsymbol{E}^e, \theta, \boldsymbol{g}_0, \sigma) \quad \mapsto \quad (\boldsymbol{Q}_0 \boldsymbol{E}^e \boldsymbol{Q}_0^T, \theta, \boldsymbol{Q}_0 \boldsymbol{g}_0, \sigma) \,, \tag{65}$$

and the dependent variables transform as

$$(\psi, \boldsymbol{q}_0, \dot{\boldsymbol{E}}^p, \sigma) \quad \mapsto \quad (\psi, \boldsymbol{Q}_0 \boldsymbol{q}_0, \boldsymbol{Q}_0 \dot{\boldsymbol{E}}^p \boldsymbol{Q}_0^T, \sigma) \,. \tag{66}$$

Let $(\bar{\psi}, \bar{\boldsymbol{q}}_0, \bar{\dot{\boldsymbol{E}}}^p, \bar{\sigma})$ and $(\bar{\psi}_*, \bar{\boldsymbol{q}}_0^*, \bar{\dot{\boldsymbol{E}}}^p_*, \bar{\sigma}_*)$ be the material response functions defined by equations (35–38) with respect to configurations φ_0 and φ_0^* respectively, then in virtue of (65–66),

$$\bar{\psi}_*(\boldsymbol{Q}_0 \boldsymbol{E}^e \boldsymbol{Q}_0^T, \theta, \sigma) = \bar{\psi} \,, \tag{67}$$

$$\bar{\boldsymbol{q}}_0^*(\boldsymbol{Q}_0 \boldsymbol{E}^e \boldsymbol{Q}_0^T, \theta, \boldsymbol{Q}_0 \boldsymbol{g}_0, \sigma) = \boldsymbol{Q}_0 \bar{\boldsymbol{q}}_0(\boldsymbol{E}^e, \theta, \boldsymbol{g}_0, \sigma) \,, \tag{68}$$

$$\bar{\dot{\boldsymbol{E}}}^p_*(\boldsymbol{Q}_0\boldsymbol{E}^e\boldsymbol{Q}_0^T,\theta,\sigma)=\boldsymbol{Q}_0\bar{\dot{\boldsymbol{E}}}^p(\boldsymbol{E}^e,\theta,\sigma)\boldsymbol{Q}_0^T\,, \tag{69}$$

$$\bar{\dot{\sigma}}_*(\boldsymbol{Q}_0\boldsymbol{E}^e\boldsymbol{Q}_0^T,\theta,\sigma)=\bar{\dot{\sigma}}(\boldsymbol{E}^e,\theta,\sigma)\,. \tag{70}$$

If the material response functions with respect to one configuration is given, equations (67–70) can be used to derive the material response fuctions with respect to any other configuration.

6.3.2 Material symmetry group

Let φ_0^* and φ_0 be two reference configurations. We say that φ_0^* and φ_0 are "thermo-mechanically equivalent" at point $\boldsymbol{p}$ if

$$\bar{\psi}_*(\boldsymbol{E}^e,\theta,\sigma)=\bar{\psi}(\boldsymbol{E}^e,\theta,\sigma)\,, \tag{71}$$

$$\bar{\mathbf{q}}_0^*(\boldsymbol{E}^e,\theta,\boldsymbol{g}_0,\sigma)=\mathbf{Q}_0\bar{\mathbf{q}}_0(\boldsymbol{E}^e,\theta,\boldsymbol{g}_0,\sigma)\,, \tag{72}$$

$$\bar{\dot{\boldsymbol{E}}}^p_*(\boldsymbol{E}^e,\theta,\sigma)=\mathbf{Q}_0\bar{\dot{\boldsymbol{E}}}^p(\boldsymbol{E}^e,\theta,\sigma)\boldsymbol{Q}_0^T\,, \tag{73}$$

$$\bar{\dot{\sigma}}_*(\boldsymbol{E}^e,\theta,\sigma)=\bar{\dot{\sigma}}(\boldsymbol{E}^e,\theta,\sigma)\,, \tag{74}$$

for all possible values of the variables $(\boldsymbol{E}^e,\theta,\boldsymbol{g}_0,\sigma)$.

In virtue of the change in configuration formulas (67–70), configuration φ_0^*, defined by the rotation tensor $\boldsymbol{Q}_0$ is thermo-mechanically equivalent to configuration φ at point $\boldsymbol{p}$ if and only if

$$\bar{\psi}(\boldsymbol{Q}_0\boldsymbol{E}^e\boldsymbol{Q}_0^T,\theta,\sigma)=\bar{\psi}(\boldsymbol{E}^e,\theta,\sigma)\,, \tag{75}$$

$$\bar{\mathbf{q}}_0(\boldsymbol{Q}_0\boldsymbol{E}^e\boldsymbol{Q}_0^T,\theta,\boldsymbol{Q}_0\boldsymbol{g}_0,\sigma)=\boldsymbol{Q}_0\bar{\mathbf{q}}_0(\boldsymbol{E}^e,\theta,\boldsymbol{g}_0,\sigma)\,, \tag{76}$$

$$\bar{\dot{\boldsymbol{E}}}^p(\boldsymbol{Q}_0\boldsymbol{E}^e\boldsymbol{Q}_0^T,\theta,\sigma)=\boldsymbol{Q}_0\bar{\dot{\boldsymbol{E}}}^p(\boldsymbol{E}^e,\theta,\sigma)\boldsymbol{Q}_0^T\,, \tag{77}$$

$$\bar{\dot{\sigma}}(\boldsymbol{Q}_0\boldsymbol{E}^e\boldsymbol{Q}_0^T,\theta,\sigma)=\bar{\dot{\sigma}}(\boldsymbol{E}^e,\theta,\sigma)\,. \tag{78}$$

for all possible values of the variables $(\boldsymbol{E}^e,\theta,\boldsymbol{g}_0,\sigma)$.

Given a reference configuration φ of body $\mathcal{B}$ and given the material response functions $(\bar{\psi},\bar{\mathbf{q}}_0,\bar{\dot{\boldsymbol{E}}}^p,\bar{\dot{\sigma}})$ with respect to φ, we define $\mathcal{G}_\varphi(\boldsymbol{p})$ as the collection of all tensors $\boldsymbol{Q}_0\in\mathcal{O}_+$ which satisfy equations (75–78) for all possible values of the variables $(\boldsymbol{E}^e,\theta,\boldsymbol{g}_0,\sigma)$.

It can be easily proved that if $\boldsymbol{Q}_0\in\mathcal{G}_\varphi(\boldsymbol{p})$ then $\boldsymbol{Q}_0^{-1}\in\mathcal{G}_\varphi(\boldsymbol{p})$ and if $\boldsymbol{Q}_1,\boldsymbol{Q}_2\in\mathcal{G}_\varphi(\boldsymbol{p})$ then $(\boldsymbol{Q}_1\boldsymbol{Q}_2)\in\mathcal{G}_\varphi(\boldsymbol{p})$. Therefore $\mathcal{G}_\varphi(\boldsymbol{p})$ has the structure of a group and is called "material symmetry group".

If $\boldsymbol{Q}_0\in\mathcal{G}_\varphi(\boldsymbol{p})$, then $\boldsymbol{Q}_0$ is the gradient of a deformation that maps φ onto another configuration which is thermo-mechanically equivalent to φ.

We say that a material is isotropic at $\boldsymbol{p}$ in configuration φ if $\mathcal{G}_\varphi(\boldsymbol{p})=\mathcal{O}_+$.

6.3.3 Material response functions for an isotropic solid

If the material is assumed to be isotropic, then conditions (75–78) must hold for every $\boldsymbol{Q}_0\in\mathcal{O}$, and using well known representation theorems (TRUESDELL and NOLL

[1965]) we have the following results.

(b.1) The free-energy response function is given by

$$\psi = \bar{\psi}(\ell)\,, \tag{79}$$

where ℓ is the list

$$\ell = \{I_1(\boldsymbol{E}^e), I_2(\boldsymbol{E}^e), I_3(\boldsymbol{E}^e), \theta, \sigma\}\,, \tag{80}$$

and $I_i(\boldsymbol{E}^e)$ are the invariants of tensor $\boldsymbol{E}^e$. In view of equations (29) and (79–80), and considering that

$$\frac{\partial I_1}{\partial \boldsymbol{E}^e} = \boldsymbol{1}\,, \tag{81}$$

$$\frac{\partial I_2}{\partial \boldsymbol{E}^e} = I_1 \boldsymbol{1} - \boldsymbol{E}^e\,, \tag{82}$$

$$\frac{\partial I_3}{\partial \boldsymbol{E}^e} = I_2 \boldsymbol{1} - I_1 \boldsymbol{E}^e + (\boldsymbol{E}^e)^2\,, \tag{83}$$

we obtain for the stress response function

$$\boldsymbol{S} = a_0 \boldsymbol{1} + a_1 \boldsymbol{E}^e + a_2 (\boldsymbol{E}^e)^2\,, \tag{84}$$

where

$$a_0 = \rho_r \left(\frac{\partial \bar{\psi}}{\partial I_1} + I_1 \frac{\partial \bar{\psi}}{\partial I_2} + I_2 \frac{\partial \bar{\psi}}{\partial I_3} \right)\,, \tag{85}$$

$$a_1 = -\rho_r \left(\frac{\partial \bar{\psi}}{\partial I_2} + I_1 \frac{\partial \bar{\psi}}{\partial I_3} \right)\,, \tag{86}$$

$$a_2 = \rho_r \frac{\partial \bar{\psi}}{\partial I_3}\,. \tag{87}$$

The entropy response function follows from equation (30),

$$\eta = \bar{\eta}(\ell) = -\frac{\partial \bar{\psi}(\ell)}{\partial \theta} \,. \tag{88}$$

(b.2) The heat flux vector response fuction is given by

$$\boldsymbol{q}_0 = \left[b_0 \boldsymbol{1} + b_1 \boldsymbol{E}^e + b_2 (\boldsymbol{E}^e)^2 \right] \boldsymbol{g}_0 \,, \tag{89}$$

where the coefficients b_i are functions of the augmented list

$$\{\ell, (\boldsymbol{g}_0 \cdot \boldsymbol{g}_0), (\boldsymbol{E}^e \boldsymbol{g}_0 \cdot \boldsymbol{g}_0), (\boldsymbol{E}^e \boldsymbol{g}_0 \cdot \boldsymbol{E}^e \boldsymbol{g}_0)\} \,. \tag{90}$$

(b.3) The time rate of change of the plastic strain tensor has to be symmetric, and it is given by

$$\dot{\boldsymbol{E}}^p = c_0 \boldsymbol{1} + c_1 \boldsymbol{E}^e + c_2 (\boldsymbol{E}^e)^2 \,, \tag{91}$$

where the coefficients c_i are functions of the list ℓ.

Finally the deformation resistence response function is given by

$$\dot{\sigma} = \bar{\dot{\sigma}}(\ell) \,. \tag{92}$$

In this way, by enforcing the second law of thermodynamics and assuming isotropic response, the twenty-one functions of eleven scalar arguments (21-26) have been reduced to the eight functions $\bar{\psi}, b_i, c_i$ and $\bar{\dot{\sigma}}$ of the lists (80) and (90).

To summarize, the most general set of constitutive equations for large-strain, isotropic thermo-elasto-plasticity with the assumption of additive decomposition of the strain tensor is given by

$$\psi = \bar{\psi}(\ell)\,, \tag{93}$$

$$\boldsymbol{S} = a_0\boldsymbol{1} + a_1\boldsymbol{E}^e + a_2(\boldsymbol{E}^e)^2\,, \tag{94}$$

$$\eta = \bar{\eta}(\ell)\,, \tag{95}$$

$$\boldsymbol{q}_0 = \left[b_0\boldsymbol{1} + b_1\boldsymbol{E}^e + b_2(\boldsymbol{E}^e)^2\right]\boldsymbol{g}_0\,, \tag{96}$$

$$\dot{\boldsymbol{E}}^p = c_0\boldsymbol{1} + c_1\boldsymbol{E}^e + c_2(\boldsymbol{E}^e)^2\,, \tag{97}$$

$$\dot{\sigma} = \bar{\dot{\sigma}}(\ell)\,, \tag{98}$$

where $\ell = \{I_i(\boldsymbol{E}^e), \theta, \sigma\}$, a_i and $\bar{\eta}$ are given in (85–88), b_i depend on the list (90) and c_i depend on ℓ. Furthermore, the heat flux response function has to satisfy inequality **(a.5)** and the plastic response functions have to satisfy inequality **(a.4)**.

6.4 Reduced constitutive equations

As a first approximation to (93–98), we consider the system

$$\psi = \frac{1}{\rho_r}\left[(\tfrac{1}{2}\lambda + \mu)I_1^2(\boldsymbol{E}^e) - 2\mu I_2(\boldsymbol{E}^e)\right], \tag{99}$$

$$\boldsymbol{S} = \lambda I_1(\boldsymbol{E}^e)\boldsymbol{1} + 2\mu\boldsymbol{E}^e\,, \tag{100}$$

$$\eta = \frac{1}{\rho_r}\left[-\left(\tfrac{1}{2}\frac{\partial\lambda}{\partial\theta} + \frac{\partial\mu}{\partial\theta}\right)I_1^2(\boldsymbol{E}^e) + 2\frac{\partial\mu}{\partial\theta}I_2(\boldsymbol{E}^e)\right], \tag{101}$$

$$\boldsymbol{q}_0 = -k\,\boldsymbol{g}_0\,, \tag{102}$$

$$\dot{\boldsymbol{E}}^p = \gamma_0 I_1(\boldsymbol{E}^e)\boldsymbol{1} + \gamma_1 \boldsymbol{E}^e , \tag{103}$$

$$\dot{\sigma} = \bar{\bar{\sigma}}(\ell) , \tag{104}$$

$$\ell = \{I_1, I_2, \theta, \sigma\} , \tag{105}$$

where λ, μ, k depend on θ and γ_0, γ_1 depend on the list ℓ.

The stress-strain law (100) can be written as

$$\boldsymbol{S} = \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e] , \tag{106}$$

where $\mathcal{L}$ is the elastic moduli tensor, given by

$$\boldsymbol{\mathcal{L}} = \lambda \boldsymbol{1} \otimes \boldsymbol{1} + 2\mu \boldsymbol{I} . \tag{107}$$

Note that $\boldsymbol{\mathcal{L}}^T = \boldsymbol{\mathcal{L}}$. Using the elastic moduli tensor the free-energy function can be written as

$$\psi = \frac{1}{2\rho_0} \boldsymbol{E}^e \cdot \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e] . \tag{108}$$

In equation (102), k is the thermal conductivity coefficient. If $k > 0$, the heat conduction inequality (34) is automatically satisfied.

Taking the trace in equations (100) and (103),

$$\operatorname{tr} \boldsymbol{S} = (3\lambda + 2\mu) \operatorname{tr} \boldsymbol{E}^e , \tag{109}$$

$$\operatorname{tr} \dot{\boldsymbol{E}}^p = (3\gamma_0 + \gamma_1) \operatorname{tr} \boldsymbol{E}^e , \tag{110}$$

and combining results,

$$\operatorname{tr} \dot{\boldsymbol{E}}^p = \frac{(3\gamma_0 + \gamma_1)}{(3\lambda + 2\mu)} \operatorname{tr} \boldsymbol{S} , \tag{111}$$

If we require that hydrostatic (pressure) loading does not produce plastic flow in the range of strains considered, we must have

$$3\gamma_0 + \gamma_1 = 0\,, \tag{112}$$

and (103) reduces to

$$\dot{\boldsymbol{E}}^p = \gamma_1\, \boldsymbol{E}^{e\prime}\,, \tag{113}$$

where the prime stands for the deviatoric part of the strain tensor,

$$\boldsymbol{E}^{e\prime} = \boldsymbol{E}^e - \tfrac{1}{3}\,(\mathrm{tr}\,\boldsymbol{E}^e)\boldsymbol{1}\,. \tag{114}$$

Taking the deviatoric part of the stress tensor in equation (100)

$$\boldsymbol{S}' = 2\mu \boldsymbol{E}^{e\prime}\,, \tag{115}$$

we use this equation to write the evolution equation for the plastic stretching tensor in the more familiar form

$$\dot{\boldsymbol{E}}^p = \gamma\, \boldsymbol{S}'\,, \tag{116}$$

where $\gamma = \gamma_1/(2\mu)$.

Defining the equivalent tensile stress s and the equivalent plastic strain rate $\dot{e}^p$ by

$$s = \sqrt{\tfrac{3}{2}\boldsymbol{S}' \cdot \boldsymbol{S}'}\,, \tag{117}$$

$$\dot{e}^p = \sqrt{\tfrac{2}{3}\dot{\boldsymbol{E}}^p \cdot \dot{\boldsymbol{E}}^p}\,, \tag{118}$$

we obtain from (116),

$$\gamma = \frac{3\dot{e}^p}{2s} \,. \tag{119}$$

It is customary to write the evolution equation for the deformation resistance (104) in the form

$$\dot{\sigma} = \bar{\bar{\sigma}}(\ell) = h(\ell)\dot{e}^p(\ell) \,, \tag{120}$$

where $h(\ell)$ is the "hardening function". (Note that the same letter h has been used to denote heat flux.)

Plastic flow is observed to depend only on s and to be independent of the first invariant of stress (pressure).

With these considerations, the reduced constitutive equations for isotropic thermo-elasto-plasticity with the assumption of additive decomposition of the strain tensor can be summarized as

$$\psi = \frac{1}{2\rho_0}\boldsymbol{E}^e \cdot \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e] \,, \tag{121}$$

$$\boldsymbol{S} = \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e] \,, \tag{122}$$

$$\eta = -\frac{1}{2\rho_0}\boldsymbol{E}^e \cdot \frac{\partial \boldsymbol{\mathcal{L}}}{\partial \theta}[\boldsymbol{E}^e] \,, \tag{123}$$

$$\boldsymbol{q}_0 = -k\,\boldsymbol{g}_0 \,, \tag{124}$$

$$\dot{\boldsymbol{E}}^p = \frac{3\dot{e}^p(\ell)}{2s}\boldsymbol{S}' \,, \tag{125}$$

$$\dot{\sigma} = h(\ell)\dot{e}^p(\ell) \,, \tag{126}$$

where

$$\boldsymbol{\mathcal{L}} = \lambda \boldsymbol{1} \otimes \boldsymbol{1} + 2\mu \boldsymbol{I}\,, \tag{127}$$

$$\ell = \{s, \theta, \sigma\}\,, \tag{128}$$

and λ, μ and k depend on θ.

6.5 Isothermal processes

An isothermal process is such that the temperature field is uniform and constant over time. In this case

$$\hat{\theta}(\mathbf{x}_0, \tau) = \theta_0\,, \tag{129}$$

$$\hat{\boldsymbol{g}}_0(\mathbf{x}_0, \tau) = \boldsymbol{0}\,, \tag{130}$$

for all $\mathbf{x}_0$ and for all τ, and it follows from equations (123–124) that

$$\hat{\eta}(\mathbf{x}_0, \tau) = 0\,, \tag{131}$$

$$\hat{\mathbf{q}}_0(\mathbf{x}_0, \tau) = \boldsymbol{0}\,, \tag{132}$$

throughout the process.

A consistent set of reduced constitutive equations for isothermal elasto-plastic processes with additive decomposition of strain can be obtained from (121–128) by using equations (129–130),

$$\psi = \frac{1}{2\rho_0} \boldsymbol{E}^e \cdot \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e] , \tag{133}$$

$$\boldsymbol{S} = \mathcal{L}[\boldsymbol{E}^e] , \tag{134}$$

$$\dot{\boldsymbol{E}}^p = \frac{3\dot{e}^p(s,\sigma)}{2s} \boldsymbol{S}' , \tag{135}$$

$$\dot{\sigma} = h(s,\sigma)\dot{e}^p(s,\sigma) , \tag{136}$$

where

$$\boldsymbol{\mathcal{L}} = \lambda \boldsymbol{1} \otimes \boldsymbol{1} + 2\mu \boldsymbol{I} , \tag{137}$$

and λ, μ are the Lame constants.

6.6 Rate-independent plasticity

Implicit in the preceeding constitutive model for thermo-elasto-plasticity is the fact that plastic flow occurs a any level of stress, (see equations (97–98) or (135–136).) A large class of materials, however, have negligible plastic stretching at stress levels below a specific limit.

The rate-independent theory of plasticity assumes the existence of an elastic range in the stress-temperature space characterized by a "loading function"

$$Y = \bar{Y}(\boldsymbol{E}^e, \theta, \sigma) \tag{138}$$

For a given value of σ, the elastic range E_σ is given by

$$E_\sigma = \{(\boldsymbol{E}^e, \theta)/\bar{Y}(\boldsymbol{E}^e, \theta, \sigma) < 0\}. \tag{139}$$

For any pair $(\boldsymbol{E}^e, \theta) \in E_\sigma$, we have $\dot{\boldsymbol{E}}^p = \boldsymbol{0}$ and $\dot{\sigma} = 0$, i. e. there is no plastic flow and the process is said to be purely elastic.

When a process passes through a point on the "yield surface"

$$\bar{Y}(\boldsymbol{E}^e, \theta, \sigma) = 0, \tag{140}$$

three cases are possible. To describe them we first define the "trial elastic" rate of change of Y by

$$\dot{Y}^{tr} = \frac{\partial Y}{\partial \boldsymbol{E}^e} \cdot \dot{\boldsymbol{E}} + \frac{\partial Y}{\partial \theta}\dot{\theta}. \tag{141}$$

When $\dot{Y}^{tr} > 0$, $\dot{\sigma} \neq 0$ and we have "loading". When $\dot{Y}^{tr} = 0$, $\dot{\sigma} = 0$ and we have "neutral loading". Finally, when $\dot{Y}^{tr} < 0$, $\dot{\sigma} = 0$ and we have "unloading". Plastic flow is assumed to occur only under loading condition.

Differentiating with respect to time equation (140), we obtain the "consistency condition" that has to be satisfied during plastic flow,

$$\frac{\partial Y}{\partial \boldsymbol{E}^e} \cdot \dot{\boldsymbol{E}}^e + \frac{\partial Y}{\partial \theta}\dot{\theta} + \frac{\partial Y}{\partial \sigma}\dot{\sigma} = 0. \tag{142}$$

For an isotropic material, the loading function (138) reduces to

$$Y = \bar{Y}(\ell), \tag{143}$$

where

$$\ell = \{I_1(\boldsymbol{E}^e), I_2(\boldsymbol{E}^e), I_3(\boldsymbol{E}^e), \theta, \sigma\}\,. \tag{144}$$

To summarize, we write the evolution equations for the plastic variables in the case of isotropic, rate independent plasticity as

$$\dot{\boldsymbol{E}}^p = \alpha[c_0\boldsymbol{1} + c_1\boldsymbol{E}^e + c_2(\boldsymbol{E}^e)^2]\,, \tag{145}$$

$$\dot{\sigma} = \alpha\bar{\sigma}(\ell)\,, \tag{146}$$

where the coefficients c_i depend on ℓ and α is a switching parameter, given by

$$\alpha = \begin{cases} 1 & \text{if } Y = 0 \text{ and } \dot{Y}^{tr} > 0, \\ 0 & \text{if } Y < 0 \text{ or } (Y = 0 \text{ and } \dot{Y}^{tr} \leq 0). \end{cases} \tag{147}$$

A frequently used yield surface is given by the Von Mises loading function,

$$Y = s - \sigma\,, \tag{148}$$

The trial elastic rate of change of the Von Mises loading function for isothermal processes is given by

$$\dot{Y}^{tr} = \frac{\partial Y}{\partial \boldsymbol{E}^e} \cdot \dot{\boldsymbol{E}} = \frac{3\mu}{s}\boldsymbol{S}' \cdot \dot{\boldsymbol{E}}\,. \tag{149}$$

Hence, the rate-independent counterpart of equations (133–137) is given by

$$\psi = \frac{1}{2\rho_0}\boldsymbol{E}^e \cdot \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e]\,, \tag{150}$$

$$\boldsymbol{S} = \mathcal{L}[\boldsymbol{E}^e]\,, \tag{151}$$

$$\dot{\boldsymbol{E}}^p = \alpha\frac{3\dot{e}^p(\sigma)}{2s}\boldsymbol{S}'\,, \tag{152}$$

$$\dot{\sigma} = \alpha h(\sigma)\dot{e}^p(\sigma)\,, \tag{153}$$

with the switching parameter defined as

$$\alpha = \begin{cases} 1 & \text{if } s = \sigma \text{ and } \boldsymbol{S}' \cdot \dot{\boldsymbol{E}} > 0, \\ 0 & \text{if } s < \sigma \text{ or } (s = \sigma \text{ and } \boldsymbol{S}' \cdot \dot{\boldsymbol{E}} \leq 0), \end{cases} \tag{154}$$

and the consistency condition

$$\dot{\sigma} = \alpha \dot{s}\,. \tag{155}$$

Chapter 7

Comments on the additive decomposition

7.1 Incompressibility of plastic flow

A deformation is volume-preserving, or isochoric, if the volume of any material element is preserved. Since

$$dV = J\,dV_0\,, \tag{1}$$

where $J = \det \boldsymbol{X}$ is the jacobian of the deformation, a necessary and sufficient condition for a deformation to be isochoric is that

$$J = \det \boldsymbol{X} = 1 \qquad \forall\, \mathbf{x}_0 \in B_0\,. \tag{2}$$

Micromechanical and experimental considerations lead to the conclusion that plastic flow is approximately isochoric. We explore the restrictions that this assumption imposes on the material model defined in (6.121–6.128).

Let $\mathcal{B}$ be a body. Let $\hat{\mathbf{x}}$ be an elastoplastic deformation such that the current configuration is stress-free,

$$S = T = 0\,, \tag{3}$$

it follows from the stress-strain law (6.122)

$$S = \mathcal{L}[E^e] = \mathcal{L}[E - E^p]\,, \tag{4}$$

that the elastic strain vanishes and that total strain equals plastic strain,

$$E^e = 0\,, \tag{5}$$

$$E = E^p\,. \tag{6}$$

Note that the motion that took place between the reference and the current configurations had in general both elastic and plastic strain histories, we only assume that elastic strain (and stress) is zero in the current configuration. Such deformation is called "purely plastic" in that total strain equals plastic strain. It follows from our preceeding statement that the deformation should be isochoric, i. e.,

$$J = \lambda_1 \lambda_2 \lambda_3 = 1\,. \tag{7}$$

Using equations (6.125) and (6) we have

$$\operatorname{tr} E = \operatorname{tr} E^p = 0\,, \tag{8}$$

or, equivalently

$$\operatorname{tr} E = \operatorname{tr}[R_L g(\Lambda) R_L^T] = g(\lambda_1) + g(\lambda_2) + g(\lambda_3) = 0\,. \tag{9}$$

Solving for λ_3 in (7) and substituting in (9),

$$g(\lambda_1) + g(\lambda_2) + g\left(\frac{1}{\lambda_1\lambda_2}\right) = 0\,. \tag{10}$$

This condition has to be satisfied for all λ_1, λ_2. Differentiating with respect to λ_1 and λ_2 we obtain respectively

$$g'(\lambda_1) - \frac{1}{\lambda_1^2\lambda_2}g'\left(\frac{1}{\lambda_1\lambda_2}\right) = 0\,, \tag{11}$$

$$g'(\lambda_2) - \frac{1}{\lambda_1\lambda_2^2}g'\left(\frac{1}{\lambda_1\lambda_2}\right) = 0\,, \tag{12}$$

eliminating $g'(\frac{1}{\lambda_1\lambda_2})$ and rearranging, we obtain

$$\lambda_1 g'(\lambda_1) = \lambda_2 g'(\lambda_2)\,. \tag{13}$$

Since λ_1 and λ_2 are independent we must have

$$\lambda g'(\lambda) = c\,, \tag{14}$$

$$g(\lambda) = c\ln\lambda\,, \tag{15}$$

and in virtue of the condition $g'(1) = 1$ the constant $c = 1$. We conclude that the only scale function that satisfies

$$g(\lambda_1) + g(\lambda_2) + g(\lambda_3) = 0\,, \tag{16}$$

for all $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1\lambda_2\lambda_3 = 1\,, \tag{17}$$

is the logarithmic function

$$g(\lambda) = \ln \lambda \,. \tag{18}$$

Hence, in the framework of the constitutive model (6.121–6.128) incompressibility of plastic flow restricts the choice of scale function, the Hencky strain measure and its conjugate stress are the only strain-stress pair "physically admissible".

7.2 Effects of plastic history on elastic moduli

7.2.1 Small strains limit

Let $\mathcal{B}$ be a body and let $\hat{\boldsymbol{x}}$ be a motion of $\mathcal{B}$ referred to the configuration φ_0,

$$\boldsymbol{x} = \hat{\boldsymbol{x}}(\boldsymbol{x}_0, \tau) \tag{19}$$

is the position at time τ of the particle $\boldsymbol{p} \in \mathcal{B}$ that at time 0 was at $\boldsymbol{x}_0$.

At time 0 we have for the kinematic variables,

$$\boldsymbol{X}_{(0)} = \boldsymbol{1} \,, \tag{20}$$

$$J_{(0)} = 1 \,, \tag{21}$$

$$\boldsymbol{U}_{(0)} = \boldsymbol{1} \,, \tag{22}$$

$$\boldsymbol{R}(0) = \boldsymbol{1}\,, \tag{23}$$

$$\boldsymbol{\Lambda}(0) = \boldsymbol{1}\,, \tag{24}$$

$$g(\boldsymbol{\Lambda}(0)) = \boldsymbol{0}\,, \tag{25}$$

$$g'(\boldsymbol{\Lambda}(0)) = \boldsymbol{1}\,, \tag{26}$$

$$\boldsymbol{E}(0) = \boldsymbol{0}\,. \tag{27}$$

Recalling equations (4.50–4.51) for the stretching and strain rate tensors,

$$\boldsymbol{D} = \boldsymbol{R}_E\left[\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^{-1} + \tfrac{1}{2}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1})\right]\boldsymbol{R}_E^T\,, \tag{28}$$

$$\dot{\boldsymbol{E}} = \boldsymbol{R}_L\left[\dot{\boldsymbol{\Lambda}}g'(\boldsymbol{\Lambda}) + \boldsymbol{\Omega}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda})\boldsymbol{\Omega}_L\right]\boldsymbol{R}_L^T\,. \tag{29}$$

and using (24–26) we have at time 0,

$$\dot{\boldsymbol{E}}(0) = \boldsymbol{D}(0) = \boldsymbol{R}_L(0)\dot{\boldsymbol{\Lambda}}(0)\boldsymbol{R}_L^T(0)\,. \tag{30}$$

We assume that comfiguration φ_0 is unstressed, i. e. that

$$\boldsymbol{T}(0) = \boldsymbol{0}\,. \tag{31}$$

From the definition (4.49) of the stress measure $\boldsymbol{S}$,

$$\boldsymbol{S}\cdot\dot{\boldsymbol{E}} = J\boldsymbol{T}\cdot\boldsymbol{D}\,, \tag{32}$$

and (31), we have

$$\boldsymbol{S}(0) = \boldsymbol{0}\,. \tag{33}$$

Differentiating equation (32) with respect to time,

$$\dot{\boldsymbol{S}} \cdot \dot{\boldsymbol{E}} + \boldsymbol{S} \cdot \ddot{\boldsymbol{E}} = \dot{J}\boldsymbol{T} \cdot \boldsymbol{D} + J\dot{\boldsymbol{T}} \cdot \boldsymbol{D} + J\boldsymbol{T} \cdot \dot{\boldsymbol{D}}, \tag{34}$$

and using (21), (27), (30), (31) and (33),

$$\dot{\boldsymbol{S}}(0) \cdot \boldsymbol{D}(0) = \dot{\boldsymbol{T}}(0) \cdot \boldsymbol{D}(0), \tag{35}$$

we conclude that

$$\dot{\boldsymbol{S}}(0) = \dot{\boldsymbol{T}}(0). \tag{36}$$

We finally assume that there is no plastic strain in the configuration φ_0 ,

$$\boldsymbol{E}^p(0) = \boldsymbol{0}. \tag{37}$$

With these givens, we consider the Taylor series expansions around $\tau = 0$,

$$\boldsymbol{E}(\tau) = \boldsymbol{E}(0) + \dot{\boldsymbol{E}}(0)\tau + \vartheta(\tau^2), \tag{38}$$

$$\boldsymbol{S}(\tau) = \boldsymbol{S}(0) + \dot{\boldsymbol{S}}(0)\tau + \vartheta(\tau^2), \tag{39}$$

$$\boldsymbol{E}^p(\tau) = \boldsymbol{E}^p(0) + \dot{\boldsymbol{E}}^p(0)\tau + \vartheta(\tau^2), \tag{40}$$

in view of equations (27), (30), (33), (36) and (37),

$$\boldsymbol{E}(\tau) = \boldsymbol{D}(0)\tau + \vartheta(\tau^2), \tag{41}$$

$$\boldsymbol{S}(\tau) = \dot{\boldsymbol{T}}(0)\tau + \vartheta(\tau^2). \tag{42}$$

$$\boldsymbol{E}^p(\tau) = \dot{\boldsymbol{E}}^p(0)\tau + \vartheta(\tau^2), \tag{43}$$

We recall at this point the stress-strain law (6.122)

$$\boldsymbol{S} = \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e] = \boldsymbol{\mathcal{L}}[\boldsymbol{E} - \boldsymbol{E}^p]\,, \tag{44}$$

At time τ, (44) reads

$$\boldsymbol{S}(\tau) = \boldsymbol{\mathcal{L}}[\boldsymbol{E}(\tau) - \boldsymbol{E}^p(\tau)]\,, \tag{45}$$

substituting (41–43) in (45),

$$\dot{\boldsymbol{T}}(0)\tau = \boldsymbol{\mathcal{L}}[\boldsymbol{D}(0) - \dot{\boldsymbol{E}}^p(0)]\tau + \vartheta(\tau^2)\,, \tag{46}$$

dividing by τ and taking limit as $\tau \to 0$,

$$\dot{\boldsymbol{T}}(0) = \boldsymbol{\mathcal{L}}[\boldsymbol{D}(0) - \dot{\boldsymbol{E}}^p(0)]\,. \tag{47}$$

Equation (47) governs the initial stress-strain response.

An "initially elastic" process is such that

$$\dot{\boldsymbol{E}}^p(0) = \boldsymbol{0}\,, \tag{48}$$

thus, for an initially elastic process the stress-strain response is governed by

$$\dot{\boldsymbol{T}}(0) = \boldsymbol{\mathcal{L}}[\boldsymbol{D}(0)]\,. \tag{49}$$

Note that (49) is independent of the scale function.

7.2.2 Motion referred to an intermediate configuration

Let t be any time in the interval $[0, \tau]$ and let φ_t be the configuration of $\mathcal{B}$ at time t,

$$\mathbf{x}_t = \hat{\mathbf{x}}(\mathbf{x}_0, t) \tag{50}$$

is the position at time t of the particle $\boldsymbol{p} \in \mathcal{B}$ that at time 0 was at $\boldsymbol{x}_0$.

The motion $\hat{x}$ can be referred to configuration φ_t as follows,

$$\mathbf{x} = \hat{\mathbf{x}}\big(\hat{\mathbf{x}}^{-1}(\mathbf{x}_t, t), \tau\big) \tag{51}$$

is the position at time τ of the particle $\mathbf{p} \in \mathcal{B}$ that at time t was at $\boldsymbol{x}_t$. The "relative" deformation gradient $\boldsymbol{X}_t$ is defined by

$$\boldsymbol{X}_t = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_t}, \tag{52}$$

and it follows from equation (51) that

$$\boldsymbol{X}_{t(\tau)} = \boldsymbol{X}_{(\tau)} \boldsymbol{X}^{-1}{}_{(t)}. \tag{53}$$

Differentiating equation (53) with respect to τ and taking the inverse of (53) we obtain respectively

$$\dot{\boldsymbol{X}}_{t(\tau)} = \dot{\boldsymbol{X}}_{(\tau)} \boldsymbol{X}^{-1}{}_{(t)}, \tag{54}$$

$$\boldsymbol{X}_t^{-1}{}_{(\tau)} = \boldsymbol{X}_{(t)} \boldsymbol{X}^{-1}{}_{(\tau)}. \tag{55}$$

from where we have for the velocity gradient

$$\boldsymbol{L} = \dot{\boldsymbol{X}}\boldsymbol{X}^{-1} = \dot{\boldsymbol{X}}_t\boldsymbol{X}_t^{-1}\,. \tag{56}$$

The relative deformation gradient admits the polar decomposition

$$\boldsymbol{X}_t = \boldsymbol{R}_t\boldsymbol{U}_t\,, \tag{57}$$

$$\boldsymbol{X}_t = \boldsymbol{V}_t\boldsymbol{R}_t\,. \tag{58}$$

Based on the eigen-decomposition of the relative right stretch tensor

$$\boldsymbol{U}_t = \boldsymbol{R}_L^t\boldsymbol{\Lambda}_t(\boldsymbol{R}_L^t)^T\,, \tag{59}$$

we write the relative strain tensor as

$$\boldsymbol{E}_t = \boldsymbol{R}_L^t g(\boldsymbol{\Lambda}_t)(\boldsymbol{R}_L^t)^T\,. \tag{60}$$

By an analogous procedure to the one used in Chapter 4, we obtain the following expressions for the relative stretching tensor and the strain rate tensor, (see equations (28–29),)

$$\boldsymbol{D} = \boldsymbol{R}_E^t\left[\dot{\boldsymbol{\Lambda}}_t\boldsymbol{\Lambda}_t^{-1} + \tfrac{1}{2}(\boldsymbol{\Lambda}_t^{-1}\boldsymbol{\Omega}_L^t\boldsymbol{\Lambda}_t - \boldsymbol{\Lambda}_t\boldsymbol{\Omega}_L^t\boldsymbol{\Lambda}_t^{-1})\right](\boldsymbol{R}_E^t)^T\,, \tag{61}$$

$$\dot{\boldsymbol{E}}_t = \boldsymbol{R}_L^t[\dot{\boldsymbol{\Lambda}}_t g'(\boldsymbol{\Lambda}_t) + \boldsymbol{\Omega}_L^t g(\boldsymbol{\Lambda}_t) - g(\boldsymbol{\Lambda}_t)\boldsymbol{\Omega}_L^t](\boldsymbol{R}_L^t)^T\,, \tag{62}$$

At time t we have from (53) and (57–60),

$$\boldsymbol{X}_{t(t)} = \boldsymbol{1}\,, \tag{63}$$

$$J_t(t) = 1\,, \tag{64}$$

$$\boldsymbol{U}_t(t) = \boldsymbol{1}\,, \tag{65}$$

$$\boldsymbol{R}_t(t) = \boldsymbol{1}\,, \tag{66}$$

$$\boldsymbol{\Lambda}_t(t) = \boldsymbol{1}\,, \tag{67}$$

$$g(\boldsymbol{\Lambda}_t(t)) = \boldsymbol{0}\,, \tag{68}$$

$$g'(\boldsymbol{\Lambda}_t(t)) = \boldsymbol{1}\,, \tag{69}$$

$$\boldsymbol{E}_t(t) = \boldsymbol{0}\,, \tag{70}$$

and from (61–62),

$$\dot{\boldsymbol{E}}_t(t) = \boldsymbol{D}(t) = \boldsymbol{R}^t_E \dot{\boldsymbol{\Lambda}}_t (\boldsymbol{R}^t_E)^T\,. \tag{71}$$

The relative stress measure $\boldsymbol{S}_t$ is defined by the equation

$$\boldsymbol{S}_t \cdot \dot{\boldsymbol{E}}_t = J_t \boldsymbol{T} \cdot \boldsymbol{D}\,, \tag{72}$$

and may be written in terms of $\boldsymbol{T}$, the relative rotations $\boldsymbol{R}^t_L$, $\boldsymbol{R}^t_E$ and the relative stretches $\boldsymbol{\Lambda}_t$ by means of an equation analogous to (4.72).

In what follows we assume that configuration φ_t is unstressed, i. e.

$$\boldsymbol{T}(t) = \boldsymbol{0}\,, \tag{73}$$

then from (72) we have

$$\boldsymbol{S}_t(t) = \boldsymbol{0}\,. \tag{74}$$

Differentiating (72) with respect to τ,

$$\dot{\boldsymbol{S}}_t \cdot \dot{\boldsymbol{E}}_t + \boldsymbol{S}_t \cdot \ddot{\boldsymbol{E}}_t = \dot{J}_t \boldsymbol{T} \cdot \boldsymbol{D} + J_t \dot{\boldsymbol{T}} \cdot \boldsymbol{D} + J_t \boldsymbol{T} \cdot \dot{\boldsymbol{D}}, \tag{75}$$

and evaluating at $\tau = t$, we obtain by means of (64), (70), (71), (73) and (74)

$$\dot{\boldsymbol{S}}_{t(t)} \cdot \boldsymbol{D}_{(t)} = \dot{\boldsymbol{T}}_{(t)} \cdot \boldsymbol{D}_{(t)}, \tag{76}$$

from where it follows that

$$\dot{\boldsymbol{S}}_{t(t)} = \dot{\boldsymbol{T}}_{(t)}, \tag{77}$$

Consider times τ in a neighborhood of t. We write a Taylor series expansion of the relative strain and stress tensors,

$$\boldsymbol{E}_{t(\tau)} = \boldsymbol{E}_{t(t)} + \dot{\boldsymbol{E}}_{t(t)}(\tau - t) + \vartheta[(\tau - t)^2], \tag{78}$$

$$\boldsymbol{S}_{t(\tau)} = \boldsymbol{S}_{t(t)} + \dot{\boldsymbol{S}}_{t(t)}(\tau - t) + \vartheta[(\tau - t)^2], \tag{79}$$

and using equations (70), (71), (74) and (77),

$$\boldsymbol{E}_{t(\tau)} = \boldsymbol{D}_{(t)}(\tau - t) + \vartheta[(\tau - t)^2], \tag{80}$$

$$\boldsymbol{S}_{t(\tau)} = \dot{\boldsymbol{T}}_{(t)}(\tau - t) + \vartheta[(\tau - t)^2]. \tag{81}$$

Motivated by (49) and (80–81), we define the initial elastic response moduli tensor $\boldsymbol{\mathcal{C}}_{(t)}$ at time t from the unstressed configuration φ_t by the equation

$$\dot{\boldsymbol{T}}_{(t)} = \boldsymbol{\mathcal{C}}_{(t)}[\boldsymbol{D}_{(t)}]. \tag{82}$$

Note that in virtue of (49),

$$\boldsymbol{C}_{(0)} = \boldsymbol{\mathcal{L}}\,, \tag{83}$$

but, unlike the case of φ_0, the plastic strain tensor $\boldsymbol{E}^p(t)$ at φ_t is in general nonzero. For the initial elastic response to be independent of the plastic strain, we should have

$$\boldsymbol{C}_{(t)} = \boldsymbol{\mathcal{L}} \tag{84}$$

for all unstressed configurations φ_t.

7.2.3 Initial elastic response moduli

Let configuration φ_t be unstressed,

$$\boldsymbol{T}_{(t)} = \boldsymbol{0}\,, \tag{85}$$

and consider an initially elastic process from φ_t,

$$\dot{\boldsymbol{E}}^p(t) = \boldsymbol{0}\,. \tag{86}$$

Then, in virtue of the stress strain law (44) we have at time t,

$$\boldsymbol{S}_{(t)} = \boldsymbol{\mathcal{L}}[\boldsymbol{E}_{(t)} - \boldsymbol{E}^p(t)]\,, \tag{87}$$

$$\dot{\boldsymbol{S}}_{(t)} = \boldsymbol{\mathcal{L}}[\dot{\boldsymbol{E}}_{(t)}]\,. \tag{88}$$

Equation (32), combined with (85) leads to

$$\boldsymbol{S}(t) = \boldsymbol{0}\,, \tag{89}$$

and we obtain from (87) that

$$\boldsymbol{E}(t) = \boldsymbol{E}^p(t)\,. \tag{90}$$

In order to obtain an expression for the initial elastic response moduli tensor $\boldsymbol{C}$ defined in (82), we start from (88) and seek expressions to relate $\dot{\boldsymbol{S}}(t)$ to $\dot{\boldsymbol{T}}(t)$ and $\dot{\boldsymbol{E}}(t)$ to $\boldsymbol{D}(t)$.

Consider first equations (28–29) for the stretching and strain rate tensors,

$$\boldsymbol{D} = \boldsymbol{R}_E\big[\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^{-1} + \tfrac{1}{2}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1})\big]\boldsymbol{R}_E^T\,, \tag{91}$$

$$\dot{\boldsymbol{E}} = \boldsymbol{R}_L\big[\dot{\boldsymbol{\Lambda}}g'(\boldsymbol{\Lambda}) + \boldsymbol{\Omega}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda})\boldsymbol{\Omega}_L\big]\boldsymbol{R}_L^T\,. \tag{92}$$

which are equivalent to

$$\boldsymbol{R}_E^T\boldsymbol{D}\boldsymbol{R}_E = \dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^{-1} + \tfrac{1}{2}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1})\,, \tag{93}$$

$$\boldsymbol{R}_L^T\dot{\boldsymbol{E}}\boldsymbol{R}_L = \dot{\boldsymbol{\Lambda}}g'(\boldsymbol{\Lambda}) + \boldsymbol{\Omega}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda})\boldsymbol{\Omega}_L\,. \tag{94}$$

Taking dot product with $\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$,

$$\begin{aligned}
[\boldsymbol{R}_E^T\boldsymbol{D}\boldsymbol{R}_E]_{\alpha\beta} &= \mathbf{e}_\alpha \cdot \big[\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^{-1} + \tfrac{1}{2}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1})\big]\mathbf{e}_\beta \\
&= \lambda_\beta^{-1}[\dot{\boldsymbol{\Lambda}}]_{\alpha\beta} + \tfrac{1}{2}(\lambda_\alpha^{-1}\lambda_\beta - \lambda_\alpha\lambda_\beta^{-1})[\boldsymbol{\Omega}_L]_{\alpha\beta}\,,
\end{aligned} \tag{95}$$

$$[\boldsymbol{R}_L^T\dot{\boldsymbol{E}}\boldsymbol{R}_L]_{\alpha\beta} = \mathbf{e}_\alpha \cdot \big[\dot{\boldsymbol{\Lambda}}g'(\boldsymbol{\Lambda}) + \boldsymbol{\Omega}_L g(\boldsymbol{\Lambda}) - g(\boldsymbol{\Lambda})\boldsymbol{\Omega}_L\big]\mathbf{e}_\beta$$

$$= g'(\lambda_\beta)[\dot{\mathbf{\Lambda}}]_{\alpha\beta} + \big(g(\lambda_\alpha) - g(\lambda_\beta)\big)[\mathbf{\Omega}_L]_{\alpha\beta} \, . \tag{96}$$

Considering that the matrix $[\dot{\mathbf{\Lambda}}]$ is diagonal and $[\mathbf{\Omega}_L]$ is skew-symmetric, we use (95) to solve for the components of stretch rate and lagrangian spin,

$$[\dot{\mathbf{\Lambda}}]_{\alpha\alpha} = \lambda_\alpha[\boldsymbol{R}_E^T\boldsymbol{D}\boldsymbol{R}_E]_{\alpha\alpha} \qquad \text{(no sum on } \alpha) \, , \tag{97}$$

$$[\mathbf{\Omega}_L]_{\alpha\beta} = 2(\lambda_\alpha^{-1}\lambda_\beta - \lambda_\alpha\lambda_\beta^{-1})^{-1}[\boldsymbol{R}_E^T\boldsymbol{D}\boldsymbol{R}_E]_{\alpha\beta} \, , \tag{98}$$

and substituting in (96) we obtain the desired result

$$[\boldsymbol{R}_L^T\dot{\boldsymbol{E}}\boldsymbol{R}_L]_{\alpha\beta} = \begin{cases} \lambda_\alpha g'(\lambda_\alpha)[\boldsymbol{R}_E^T\boldsymbol{D}\boldsymbol{R}_E]_{\alpha\alpha} \, , & \text{if } \beta = \alpha; \\ 2\dfrac{g(\lambda_\beta) - g(\lambda_\alpha)}{\lambda_\beta\lambda_\alpha^{-1} - \lambda_\alpha\lambda_\beta^{-1}}[\boldsymbol{R}_E^T\boldsymbol{D}\boldsymbol{R}_E]_{\alpha\beta} \, , & \text{otherwise.} \end{cases} \tag{99}$$

For an expression of the stress measure rate $\dot{\boldsymbol{S}}(t)$ in terms of $\dot{\boldsymbol{T}}(t)$ we consider equation (34), which, together with (85) and (89) gives

$$\dot{\boldsymbol{S}}(t) \cdot \dot{\boldsymbol{E}}(t) = J(t)\dot{\boldsymbol{T}}(t) \cdot \boldsymbol{D}(t) \, . \tag{100}$$

and using a similar procedure to that of Section 4.3, (see equation (4.72),) we obtain

$$[\boldsymbol{R}_L^T\dot{\boldsymbol{S}}\boldsymbol{R}_L]_{\alpha\beta} = \begin{cases} \dfrac{J}{\lambda_\alpha g'(\lambda_\alpha)}[\boldsymbol{R}_E^T\dot{\boldsymbol{T}}\boldsymbol{R}_E]_{\alpha\alpha} \, , & \text{if } \beta = \alpha; \\ \frac{1}{2}J\dfrac{\lambda_\beta\lambda_\alpha^{-1} - \lambda_\alpha\lambda_\beta^{-1}}{g(\lambda_\beta) - g(\lambda_\alpha)}[\boldsymbol{R}_E^T\dot{\boldsymbol{T}}\boldsymbol{R}_E]_{\alpha\beta} \, , & \text{otherwise.} \end{cases} \tag{101}$$

Since the tensor $\boldsymbol{\mathcal{L}}$ is isotropic, equation (88) is equivalent to

$$\boldsymbol{R}_L^T\dot{\boldsymbol{S}}\boldsymbol{R}_L = \boldsymbol{\mathcal{L}}[\boldsymbol{R}_L^T\dot{\boldsymbol{E}}\boldsymbol{R}_L] \, . \tag{102}$$

Recalling that

$$\boldsymbol{\mathcal{L}} = \lambda \boldsymbol{1} \otimes \boldsymbol{1} + 2\mu \boldsymbol{I}\,, \tag{103}$$

or equivalently

$$\boldsymbol{\mathcal{L}} = (\kappa - \tfrac{2}{3}\mu) \boldsymbol{1} \otimes \boldsymbol{1} + 2\mu \boldsymbol{I}\,, \tag{104}$$

where

$$\kappa = \lambda + \tfrac{2}{3}\,\mu\,, \tag{105}$$

is the bulk modulus, we have from (102),

$$[\boldsymbol{R}_L^T \dot{\boldsymbol{S}} \boldsymbol{R}_L]_{\alpha\beta} = \begin{cases} 3\kappa\,\mathrm{tr}\,[\boldsymbol{R}_L^T \dot{\boldsymbol{E}} \boldsymbol{R}_L]_{\alpha\alpha}\,, & \text{if } \beta = \alpha; \\ 2\mu\,[\boldsymbol{R}_L^T \dot{\boldsymbol{E}} \boldsymbol{R}_L]_{\alpha\beta}\,, & \text{otherwise.} \end{cases} \tag{106}$$

Substituting (99) and (101) into (106) and solving for the components of $[\boldsymbol{R}_E^T \dot{\boldsymbol{T}} \boldsymbol{R}_E]$ we obtain the final result,

$$[\boldsymbol{R}_E^T \dot{\boldsymbol{T}} \boldsymbol{R}_E]_{\alpha\beta} = \begin{cases} 3\kappa J^{-1}[\lambda_\alpha g'(\lambda_\alpha)]^2 [\boldsymbol{R}_E^T \boldsymbol{D} \boldsymbol{R}_E]_{\alpha\alpha}\,, & \text{if } \beta = \alpha; \\ 2\mu J^{-1} \left[2\dfrac{g(\lambda_\beta) - g(\lambda_\alpha)}{\lambda_\beta \lambda_\alpha^{-1} - \lambda_\alpha \lambda_\beta^{-1}} \right]^2 [\boldsymbol{R}_E^T \boldsymbol{D} \boldsymbol{R}_E]_{\alpha\beta}\,, & \text{otherwise.} \end{cases} \tag{107}$$

At the unstressed configuration φ_t the total strain tensor equals the plastic strain tensor, as indicated by equation (90). It follows that $\lambda(t) = \lambda_\alpha^p$ where λ_α^p are the plastic stretches from configuration φ_0 to φ_t. Thus, evaluating equation (106) at $\tau = t$

$$[\boldsymbol{R}_E^T(t) \dot{\boldsymbol{T}}(t) \boldsymbol{R}_E(t)]_{\alpha\beta}$$

$$= \begin{cases} 3\kappa J_p^{-1}[\lambda_\alpha^p g'(\lambda_\alpha^p)]^2[\boldsymbol{R}_E^T(t)\boldsymbol{D}(t)\boldsymbol{R}_E(t)]_{\alpha\alpha}\,, & \text{if } \beta=\alpha; \\ 2\mu J_p^{-1}\left[2\dfrac{g(\lambda_\beta^p)-g(\lambda_\alpha^p)}{\lambda_\beta^p\lambda_\alpha^{p\,-1}-\lambda_\alpha^p\lambda_\beta^{p\,-1}}\right]^2[\boldsymbol{R}_E^T(t)\boldsymbol{D}(t)\boldsymbol{R}_E(t)]_{\alpha\beta}\,, & \text{otherwise.} \end{cases} \tag{108}$$

where $J_p = J(t)$ is the jacobian of the plastic deformation.

Comparing equation (108) with the equation defining the initial elastic response moduli tensor $\boldsymbol{\mathcal{C}}$,

$$\dot{\boldsymbol{T}}(t) = \boldsymbol{\mathcal{C}}(t)[\boldsymbol{D}(t)]\,, \tag{109}$$

we observe that the initial elastic response from an unstressed configuration as predicted by the constitutive model (6.121–6.128) is anisotropic, and is characterized by the "modified" bulk and shear moduli,

$$\kappa'_\alpha = \kappa J_p^{-1}[\lambda_\alpha^p g'(\lambda_\alpha^p)]^2\,, \tag{110}$$

$$\mu'_{\alpha\beta} = \mu J_p^{-1}\left[2\frac{g(\lambda_\beta^p)-g(\lambda_\alpha^p)}{\lambda_\beta^p\lambda_\alpha^{p\,-1}-\lambda_\alpha^p\lambda_\beta^{p\,-1}}\right]^2\,. \tag{111}$$

For the class g_n of scale functions defined in (4.76) we have

$$g_n(\lambda) = \frac{1}{2n}(\lambda^{2n}-1)\,, \tag{112}$$

$$g'_n(\lambda) = \lambda^{2n-1}\,, \tag{113}$$

substituting in (110–111),

$$\kappa'_\alpha = \kappa J_p(\lambda_\alpha^p)^{4n}\,, \tag{114}$$

$$\mu'_{\alpha\beta} = \mu J_p\left[n^{-1}\frac{(\lambda_\beta^p)^{2n}-(\lambda_\alpha^p)^{2n}}{\lambda_\beta^p\lambda_\alpha^{p\,-1}-\lambda_\alpha^p\lambda_\beta^{p\,-1}}\right]^2\,. \tag{115}$$

If the logarithmic scale function is selected, then $g(\lambda) = \ln \lambda$ and $g'(\lambda) = 1/\lambda$. Furthermore, in virtue of the discussion in Section 1 we have $J_p = 1$. The modified moduli are in this case

$$\kappa'_\alpha = \kappa\,, \tag{116}$$

$$\mu'_{\alpha\beta} = \mu \left[2 \frac{\ln(\lambda^p_\beta) - \ln(\lambda^p_\alpha)}{\lambda^p_\beta \lambda^{p\,-1}_\alpha - \lambda^p_\alpha \lambda^{p\,-1}_\beta} \right]^2 . \tag{117}$$

Note that only in this case the bulk modulus is not affected by plastic history.

7.2.4 An example

To illustrate the results of the previous Subsection, we select a particular purely plastic deformation, and compute the modified shear moduli predicted by equation (117).

As a representative example we consider a case of two-dimensional plastic flow. A specimen is subjected to the displacement controlled isochoric deformation given by

$$\lambda^p_1 = 1\,, \tag{118}$$

$$\lambda^p_2 = 1/\lambda\,, \tag{119}$$

$$\lambda^p_3 = \lambda\,, \tag{120}$$

and $\boldsymbol{R} = \boldsymbol{1}$.

The initial elastic response of the specimen after the deformation (118–120) is characterized by "modified" shear moduli. These moduli are obtained by substituting (118–120) in (117),

$$\frac{\mu'_{23}}{\mu} = \left[\frac{4\ln\lambda}{\lambda^2 - \lambda^{-2}}\right]^2 , \tag{121}$$

$$\frac{\mu'_{31}}{\mu} = \left[\frac{2\ln\lambda}{\lambda - \lambda^{-1}}\right]^2 , \tag{122}$$

$$\frac{\mu'_{12}}{\mu} = \left[\frac{2\ln\lambda}{\lambda - \lambda^{-1}}\right]^2 , \tag{123}$$

Note that these moduli have the symmetry property

$$\mu'_{\alpha\beta}(1/\lambda) = \mu'_{\alpha\beta}(\lambda) , \tag{124}$$

thus is enough to show the dependence on plastic stretch λ for $\lambda > 1$.

Figure 7.1 shows schematically the deformation stages, and the plot of μ'_{23}/μ as a function of λ, for values of λ up to 2. Note that for this level of plastic stretch (100% engineering strain) the in-plane shear modulus decays to 54.7% of its original value.

Figure 7.2 shows the corresponding plot for $\mu'_{12} = \mu'_{13}$. For $\lambda = 2$ (100% engineering strain,) the out-of-plane shear moduli decay to 85.4% of their original value.

We mention in this context that the range of application of a large strain elasto-plastic model with isotropic hardening does not extend beyond 30–40% engineering strains. Figures 7.1–7.2 show the modified shear moduli in a 0–100% engineering shear strain range only for illustrative purposes.

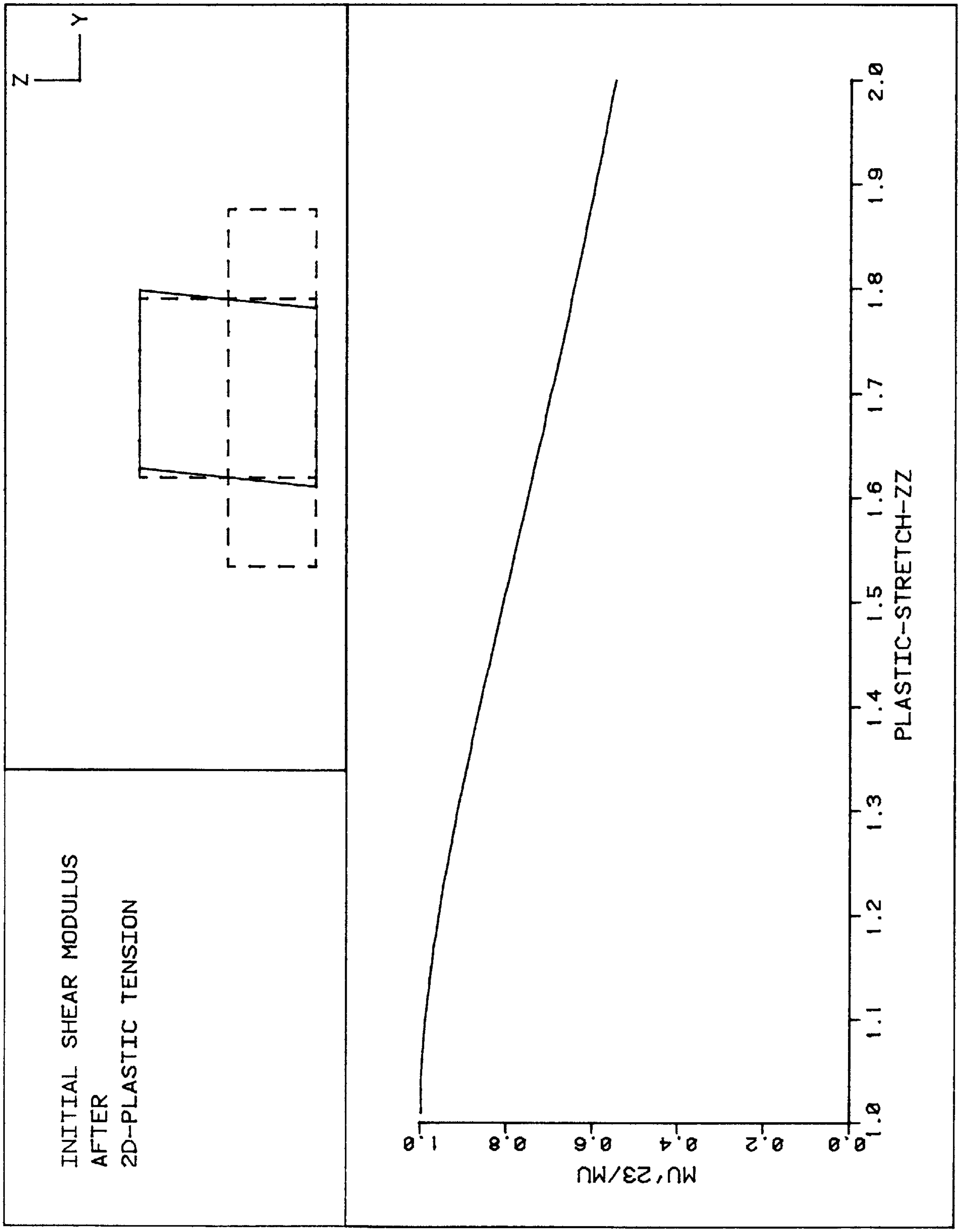

Figure 7.1 In-plane initial shear modulus as a function of previous plastic stretch for the elasto-plastic constitutive equations based on the additive decomposition of the strain tensor.

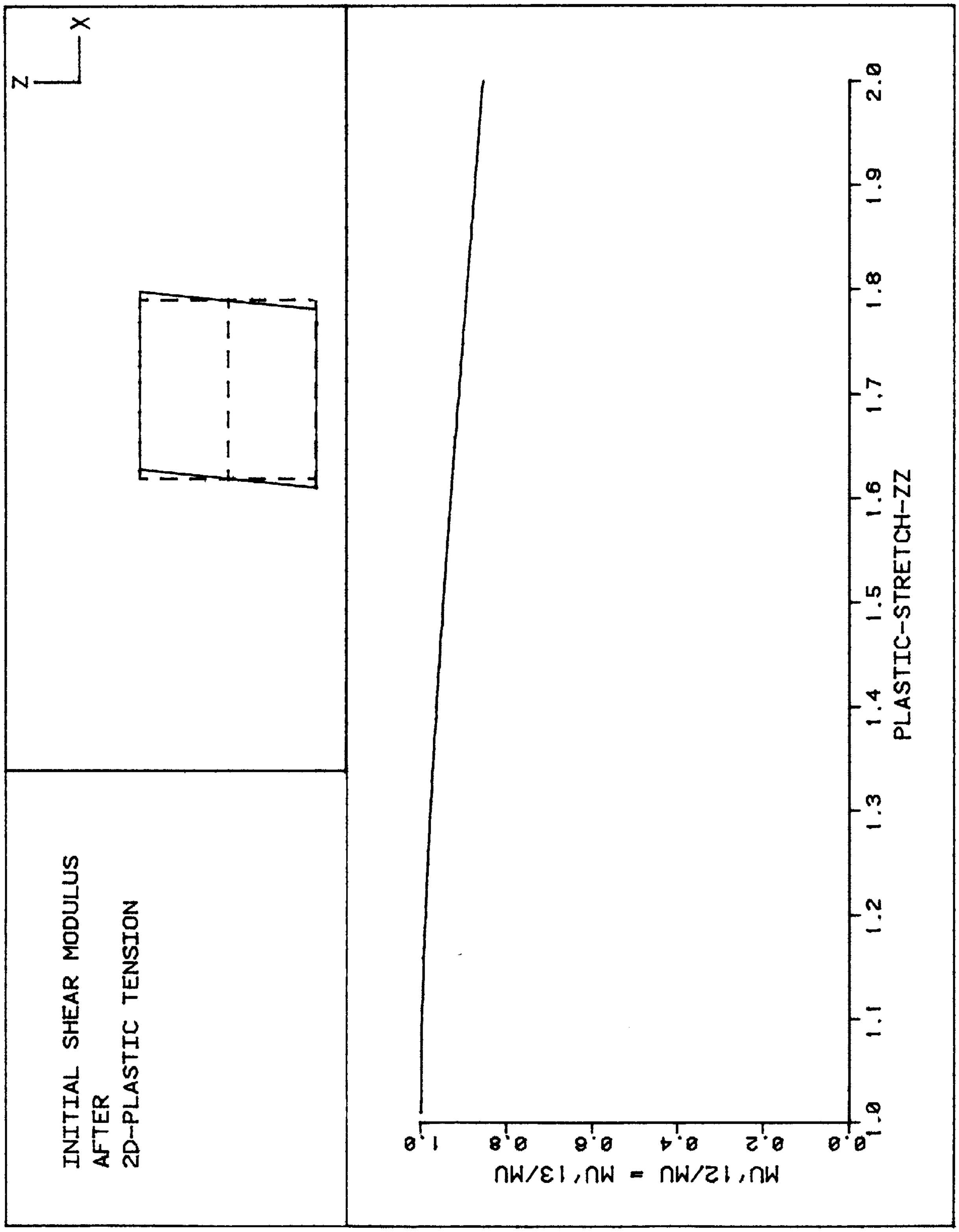

Figure 7.2 Out-of-plane initial shear moduli as a function of previous plastic stretch for the elasto-plastic constitutive equations based on the additive decomposition of the strain tensor.

Chapter 8

Thermo-elasto-plasticity II

8.1 Plastic variables

A class of elasto-plastic constitutive models that has received much attention is based on the array $(\boldsymbol{X}^p, \sigma)$ of plastic (internal) variables. The scalar σ, called "deformation resistence", has the dimensions of stress and represents isotropic resistence to plastic flow. It provides a scalar dependence of the history of plastic flow.

An elastoplastic deformation of a body from an original configuration φ_0 to configuration φ may be characterized by the well known multiplicative decomposition of the deformation gradient (LEE [1969])

$$\boldsymbol{X} = \boldsymbol{X}^e \boldsymbol{X}^p , \tag{1}$$

where $\boldsymbol{X}^e$ and $\boldsymbol{X}^p$ represent the elastic and plastic deformation gradients, respectively. Central to this assumption is the concept of a relaxed intermediate configuration φ_r, which is obtained conceptually at each particle by unloading a material neighborhood from the current configuration to a state of zero stress in such a way that no unelastic process takes place during the deformation. Note that this defines φ_r except for a rigid body motion. To resolve this difficulty, MANDEL [1973a, 1973b] introduced the concept of a director triad to determine the orientation of a material neighborhood. In the case of a single cristal, for example, the director triad is related to the atomic lattice of the cristal. Following MANDEL [1973b] we call isoclinic configuration the relaxed intermediate configuration that preserves the orientation that the director triad had in the original configuration. The plastic deformation gradient $\boldsymbol{X}^p$ is therefore uniquely defined as mapping the original configuration onto the isoclinic configuration. Since plastic deformation is assumed to be incompressible, $J^p = \det \boldsymbol{X}^p = 1$.

Having specified $\boldsymbol{X}^p$, the elastic deformation gradient $\boldsymbol{X}^e$ is given by

$$\boldsymbol{X}^e = \boldsymbol{X}(\boldsymbol{X}^p)^{-1}\,, \tag{2}$$

and the multiplicative decomposition (1) follows.

Note that since $J = \det \boldsymbol{X} > 0$ we must have $J^e = \det \boldsymbol{X}^e = J > 0$ and therefore the elastic deformation gradient admits the polar decomposition

$$\boldsymbol{X}^e = \boldsymbol{R}^e \boldsymbol{U}^e\,, \tag{3}$$

$$\boldsymbol{X}^e = \boldsymbol{V}^e \boldsymbol{R}^e\,, \tag{4}$$

where $\boldsymbol{R}^e$ is the elastic rotation tensor and $\boldsymbol{U}^e$ and $\boldsymbol{V}^e$ are the elastic rigth and left stretch tensors, respectively.

Next we recall the definition of velocity gradient

$$\boldsymbol{L} = \dot{\boldsymbol{X}}\boldsymbol{X}^{-1}, \tag{5}$$

and of stretching and spin tensors, $\boldsymbol{D}$ and $\boldsymbol{W}$ respectively,

$$\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W}, \tag{6}$$

$$\boldsymbol{D} = \mathrm{sym}\,(\boldsymbol{L}), \tag{7}$$

$$\boldsymbol{W} = \mathrm{skw}\,(\boldsymbol{L}). \tag{8}$$

From equation (2) we readily obtain

$$\dot{\boldsymbol{X}} = \dot{\boldsymbol{X}}^e \boldsymbol{X}^p + \boldsymbol{X}^e \dot{\boldsymbol{X}}^p, \tag{9}$$

$$\boldsymbol{X}^{-1} = (\boldsymbol{X}^p)^{-1}(\boldsymbol{X}^e)^{-1}, \tag{10}$$

hence the velocity gradient can be decomposed in the sum of two terms,

$$\boldsymbol{L} = \boldsymbol{L}^e + \boldsymbol{L}^p, \tag{11}$$

where

$$\boldsymbol{L}^e = \dot{\boldsymbol{X}}^e(\boldsymbol{X}^e)^{-1}, \tag{12}$$

$$\boldsymbol{L}^p = \boldsymbol{X}^e \dot{\boldsymbol{X}}^p(\boldsymbol{X}^p)^{-1}(\boldsymbol{X}^e)^{-1}, \tag{13}$$

$\boldsymbol{L}^e$ and $\boldsymbol{L}^p$ are called elastic and plastic velocity gradients.

Furthermore, the elastic stretching and spin tensors are given by

$$\boldsymbol{L}^e = \boldsymbol{D}^e + \boldsymbol{W}^e\,, \tag{14}$$

$$\boldsymbol{D}^e = \mathrm{sym}\,(\boldsymbol{L}^e)\,, \tag{15}$$

$$\boldsymbol{W}^e = \mathrm{skw}\,(\boldsymbol{L}^e)\,, \tag{16}$$

and similarly, the plastic stretching and spin tensors are given by

$$\boldsymbol{L}^p = \boldsymbol{D}^p + \boldsymbol{W}^p\,, \tag{17}$$

$$\boldsymbol{D}^p = \mathrm{sym}\,(\boldsymbol{L}^p)\,, \tag{18}$$

$$\boldsymbol{W}^p = \mathrm{skw}\,(\boldsymbol{L}^p)\,. \tag{19}$$

In virtue of equations (6), (11), (14) and (17) the stretching and spin tensors can be written as

$$\boldsymbol{D} = \boldsymbol{D}^e + \boldsymbol{D}^p\,, \tag{20}$$

$$\boldsymbol{W} = \boldsymbol{W}^e + \boldsymbol{W}^p\,. \tag{21}$$

8.2 Elastic-plastic processes

Given the plastic variables $\boldsymbol{X}^p, \sigma$, it follows from the discussion in Section 5.2 that the array of variables that characterize an elasto-plastic process is given by

$$(\boldsymbol{X}, \theta, \boldsymbol{g}, \rho, \psi, \boldsymbol{T}, \eta, \boldsymbol{q}, \boldsymbol{X}^p, \sigma)\,. \tag{22}$$

In the spirit of Section 5.3 we now turn to the selection of an array of invariant variables corresponding to (22). This can be accomplished by refering all vector and tensor variables to a reference configuration.

We select the isoclinic configuration φ_r as reference configuration. With this choice, the deformation gradient equals its elastic part and the array (22) reads

$$(\boldsymbol{X}^e, \theta, \mathbf{g}, \rho_r, \psi, \boldsymbol{T}, \eta, \mathbf{q}, \boldsymbol{X}^p, \sigma)\,. \tag{23}$$

It has been argued (see for example ANAND [1985]) that the plastic deformation gradient $\boldsymbol{X}^p$ is not a state variable, because it represents a deformation such that the neighborhood of a particle before and after plastic deformation is essentially the same. Consequently, (23) reduces to

$$(\boldsymbol{X}^e, \theta, \mathbf{g}, \rho_r, \epsilon, \boldsymbol{T}, \eta, \mathbf{q}, \sigma)\,. \tag{24}$$

The definition of the isoclinic configuration implies that the plastic deformation is invariant under rigid body motions,

$$(\boldsymbol{X}^p)^* = \boldsymbol{X}^p\,. \tag{25}$$

It follows from the transformation rule (5.32) and the multiplicative decomposition (1) that the elastic deformation gradient transforms according to

$$(\boldsymbol{X}^e)^* = \boldsymbol{Q}\boldsymbol{X}^e\,. \tag{26}$$

According to (5.82–5.83), we write the temperature gradient and heat flux vector with respect to the isoclinic configuration as

$$\mathbf{g}_r = (\mathbf{X}^e)^T \mathbf{g}\,, \tag{27}$$

$$\mathbf{q}_r = J^e (\mathbf{X}^e)^{-1} \mathbf{q}\,, \tag{28}$$

it follows from equations (5.33),(5.55) and (26) that

$$\mathbf{g}_r^* = \mathbf{g}_r\,, \tag{29}$$

$$\mathbf{q}_r^* = \mathbf{q}_r\,. \tag{30}$$

Next, according to the definition (2.69), the stress power per unit volume in the isoclinic configuration is given by

$$w_r = J^e \mathbf{T} \cdot \mathbf{D}\,. \tag{31}$$

Substituting for $\mathbf{D}$ in terms of equation (20), the stress power can be decomposed as

$$w_r = w_r^e + w_r^p\,, \tag{32}$$

where the elastic and plastic stress powers w_r^e and w_r^p are given respectively by

$$w_r^e = J^e \mathbf{T} \cdot \mathbf{D}^e\,, \tag{33}$$

$$w_r^p = J^e \mathbf{T} \cdot \mathbf{D}^p\,. \tag{34}$$

Let the elastic right stretch tensor defined in (3) be decomposed as

$$\mathbf{U}^e = \mathbf{R}_L^e \mathbf{\Lambda}^e (\mathbf{R}_L^e)^T\,, \tag{35}$$

where $\boldsymbol{R}^e_L$ is the elastic lagrangian rotation matrix and $\boldsymbol{\Lambda}^e$ stores the elastic principal stretches λ^e_i, then an "elastic strain measure" is a tensor

$$\boldsymbol{E}^e = \boldsymbol{R}^e_L g(\boldsymbol{\Lambda}^e)(\boldsymbol{R}^e_L)^T, \tag{36}$$

where g is a scale function as defined in (4.38–4.40).

The tensor pair $(\boldsymbol{S}^e, \boldsymbol{E}^e)$ is said to be "elastic work conjugate" if for any elasto-plastic process,

$$\boldsymbol{S}^e \cdot \dot{\boldsymbol{E}}^e = J^e \boldsymbol{T} \cdot \boldsymbol{D}^e, \tag{37}$$

$\boldsymbol{S}^e$ is the elastic stress measure associated with the elastic strain measure $\boldsymbol{E}^e$.

As described in Section 4.3 (for the case of total work conjugacy), given the elastic deformation gradient $\boldsymbol{X}^e$, there is an one-to-one correspondence between the elastic stress measure and the Cauchy stress. In fact,

$$[\boldsymbol{S}^e_L]_{\alpha\beta} = \begin{cases} \dfrac{J^e}{\lambda^e_\alpha g'(\lambda^e_\alpha)}[\boldsymbol{T}^e_E]_{\alpha\alpha}, & \text{if } \beta = \alpha; \\ \frac{1}{2} J^e \dfrac{\lambda^e_\beta(\lambda^e_\alpha)^{-1} - \lambda^e_\alpha(\lambda^e_\beta)^{-1}}{g(\lambda^e_\beta) - g(\lambda^e_\alpha)}[\boldsymbol{T}^e_E]_{\alpha\beta}, & \text{otherwise,} \end{cases} \tag{38}$$

where $\boldsymbol{S}^e_L$ and $\boldsymbol{T}^e_E$ are given by

$$\boldsymbol{T}^e_E = (\boldsymbol{R}^e_E)^T \boldsymbol{T} \boldsymbol{R}^e_E, \tag{39}$$

$$\boldsymbol{S}^e_L = (\boldsymbol{R}^e_L)^T \boldsymbol{S}^e \boldsymbol{R}^e_L, \tag{40}$$

and

$$\boldsymbol{R}^e_E = \boldsymbol{R}^e \boldsymbol{R}^e_L. \tag{41}$$

By an argument entirely similar to that of equations (5.86–5.91), we conclude that the elastic strain and stress measures are invariant under rigid body motions, i. e.,

$$(\boldsymbol{E}^e)^* = \boldsymbol{E}^e , \tag{42}$$

$$(\boldsymbol{S}^e)^* = \boldsymbol{S}^e , \tag{43}$$

In view of (29–30) and (42–43), we select the following array of invariant variables correponding to (24),

$$(\boldsymbol{E}^e, \theta, \boldsymbol{g}_r, \rho_r, \psi, \boldsymbol{S}^e, \boldsymbol{q}_r, \sigma) . \tag{44}$$

8.3 Consequences of the reduced dissipation inequality

We recall at this point the referential form of the reduced dissipation inequality (3.52),

$$\rho_r \dot{\psi} - w_r + \rho_r \eta \dot{\theta} + \theta^{-1} \boldsymbol{q}_r \cdot \boldsymbol{g}_r \leq 0 . \tag{45}$$

According to equations (32), (34) and (37), we have for the stress power per unit reference volume w_r

$$w_r = w_r^e + w_r^p , \tag{46}$$

$$w_r^e = \boldsymbol{S}^e \cdot \dot{\boldsymbol{E}}^e , \tag{47}$$

$$w_r^p = J^e \boldsymbol{T} \cdot \boldsymbol{D}^p . \tag{48}$$

In virtue of the symmetry of the stress tensor and equation (13), the plastic power expended can be written in terms of invariant variables as follows,

$$\begin{aligned} w_r^p &= J^e \boldsymbol{T} \cdot [\boldsymbol{X}^e \dot{\boldsymbol{X}}^p (\boldsymbol{X}^p)^{-1} (\boldsymbol{X}^e)^{-1}] \\ &= [J^e (\boldsymbol{X}^e)^T \boldsymbol{T} (\boldsymbol{X}^e)^{-T}] \cdot [\dot{\boldsymbol{X}}^p (\boldsymbol{X}^p)^{-1}] \\ &= \{[(\boldsymbol{X}^e)^T \boldsymbol{X}^e][J^e (\boldsymbol{X}^e)^{-1} \boldsymbol{T} (\boldsymbol{X}^e)^{-T}]\} \cdot [\dot{\boldsymbol{X}}^p (\boldsymbol{X}^p)^{-1}] \\ &= 2(\boldsymbol{E}_{(1)}^e + \boldsymbol{1}) \boldsymbol{S}_{(1)}^e \cdot \boldsymbol{L}_r^p , \end{aligned} \tag{49}$$

where $\boldsymbol{E}_{(1)}^e$ is the elastic Green-Lagrange strain tensor and $\boldsymbol{S}_{(1)}^e$ is the elastic second Piola-Kirchhoff stress tensor (see equations (4.78–4.79)), and $\boldsymbol{L}_r^p$ is defined by

$$\boldsymbol{L}_r^p = \dot{\boldsymbol{X}}^p (\boldsymbol{X}^p)^{-1} . \tag{50}$$

Note that given $\boldsymbol{X}^e$, the tensors $\boldsymbol{E}_{(1)}^e$ and $\boldsymbol{S}_{(1)}^e$ can be written in terms of $\boldsymbol{E}^e$ and $\boldsymbol{S}^e$ respectively by means of equations (4.74–4.75).

Substituting equations (47) and (49) in (45), we obtain for the referential form of the reduced dissipation inequality,

$$\rho_r \dot{\psi} - \boldsymbol{S}^e \cdot \dot{\boldsymbol{E}}^e + \rho_r \eta \dot{\theta} - 2(\boldsymbol{E}_{(1)}^e + \boldsymbol{1}) \boldsymbol{S}_{(1)}^e \cdot \boldsymbol{L}_r^p + \theta^{-1} \boldsymbol{q}_r \cdot \boldsymbol{g}_r \leq 0 . \tag{51}$$

We therefore select $(\boldsymbol{E}^e, \theta, \boldsymbol{g}_r, \sigma)$ as the array of governing variables. The variables $(\psi, \boldsymbol{S}^e, \eta, \boldsymbol{q}_r)$ are considered to be dependend. The corresponding set of constitutive equations for elastic-plastic processes is given by

$$\psi = \bar{\psi}(\boldsymbol{E}^e, \theta, \boldsymbol{g}_r, \sigma) , \tag{52}$$

$$\boldsymbol{S}^e = \bar{\boldsymbol{S}}^e(\boldsymbol{E}^e, \theta, \boldsymbol{g}_r, \sigma)\,, \tag{53}$$

$$\eta = \bar{\eta}(\boldsymbol{E}^e, \theta, \boldsymbol{g}_r, \sigma)\,, \tag{54}$$

$$\boldsymbol{q}_r = \bar{\boldsymbol{q}}_r(\boldsymbol{E}^e, \theta, \boldsymbol{g}_r, \sigma)\,, \tag{55}$$

$$\boldsymbol{L}_r^p = \bar{\boldsymbol{L}}_r^p(\boldsymbol{E}^e, \theta, \boldsymbol{g}_r, \sigma)\,, \tag{56}$$

$$\dot{\sigma} = \bar{\dot{\sigma}}(\boldsymbol{E}^e, \theta, \boldsymbol{g}_r, \sigma)\,, \tag{57}$$

where mention of the dependence on the particle $\boldsymbol{p}$ has been supressed for simplicity. Equations (52–57) satisfy the principle of material frame indifference for any choice of response functions $\bar{\psi}, \bar{\boldsymbol{S}}^e, \bar{\eta}, \bar{\boldsymbol{q}}_r, \bar{\boldsymbol{L}}_r^p$ and $\bar{\dot{\sigma}}$.

Taking the total time derivative in (52),

$$\dot{\psi} = \frac{\partial \psi}{\partial \boldsymbol{E}^e} \cdot \dot{\boldsymbol{E}}^e + \frac{\partial \psi}{\partial \theta}\dot{\theta} + \frac{\partial \psi}{\partial \boldsymbol{g}_r} \cdot \dot{\boldsymbol{g}}_r + \frac{\partial \psi}{\partial \sigma}\dot{\sigma}\,, \tag{58}$$

substituting in (51) and rearranging terms,

$$\left(\rho_r \frac{\partial \psi}{\partial \boldsymbol{E}^e} - \boldsymbol{S}^e\right) \cdot \dot{\boldsymbol{E}}^e + \rho_r \left(\frac{\partial \psi}{\partial \theta} + \eta\right)\dot{\theta}$$

$$+ \rho_r \frac{\partial \psi}{\partial \boldsymbol{g}_r} \cdot \dot{\boldsymbol{g}}_r + \rho_r \frac{\partial \psi}{\partial \sigma}\dot{\sigma} - 2(\boldsymbol{E}^e_{(1)} + \boldsymbol{1})\boldsymbol{S}^e_{(1)} \cdot \boldsymbol{L}_r^p + \theta^{-1}\boldsymbol{q}_r \cdot \boldsymbol{g}_r \leq 0\,. \tag{59}$$

There are two possible interpretations of inequality (59). We can consider the response functions $(\bar{\psi}, \bar{\boldsymbol{S}}^e, \bar{\eta}, \bar{\boldsymbol{q}}_r, \bar{\boldsymbol{L}}_r^p, \bar{\dot{\sigma}})$ as arbitrary, and then (59) is a restriction of the processes that the body can undergo. Or, we can say that (59) must hold for all processes, and consider it as a restriction on $(\bar{\psi}, \bar{\boldsymbol{S}}^e, \bar{\eta}, \bar{\boldsymbol{q}}_r, \bar{\boldsymbol{L}}_r^p, \bar{\dot{\sigma}})$.

Following COLEMAN and NOLL [1963] we adopt the second interpretation, and assume that the form (59) of the reduced dissipation inequality is required to hold for all values of the variables $(\boldsymbol{E}^e, \theta, \boldsymbol{g}_r)$ and all values of the derivatives $(\dot{\boldsymbol{E}}^e, \dot{\theta}, \dot{\boldsymbol{g}}_r)$. The following results hold.

(a.1) Given the free-energy response function $\bar{\psi}$, the stress and entropy response functions are obtained from

$$\bar{\boldsymbol{S}}^e = \rho_r \frac{\partial \bar{\psi}}{\partial \boldsymbol{E}^e}, \tag{60}$$

$$\bar{\eta} = -\frac{\partial \bar{\psi}}{\partial \theta}. \tag{61}$$

(a.2) The free-energy does not depend on the temperature gradient $\boldsymbol{g}_r$,

$$\frac{\partial \bar{\psi}}{\partial \boldsymbol{g}_r} = 0. \tag{62}$$

It follows form **(a.1)** that the stress and entropy response functions do not depend on the temperature gradient either.

(a.3) The following inequality must hold for all processes,

$$\rho_r \frac{\partial \psi}{\partial \sigma}\dot{\sigma} - 2(\boldsymbol{E}^e_{(1)} + \boldsymbol{1})\boldsymbol{S}^e_{(1)} \cdot \boldsymbol{L}^p_r + \theta^{-1}\boldsymbol{q}_r \cdot \boldsymbol{g}_r \leq 0. \tag{63}$$

If the plastic variables $\bar{\boldsymbol{L}}^p_r$ and $\bar{\dot{\sigma}}$ do not depend on the temperature gradient $\boldsymbol{g}_r$, **(a.3)** leads to

(a.4) the functions $\bar{\boldsymbol{L}}^p_r$ and $\bar{\dot{\sigma}}$ satisfy the internal (plastic) dissipation inequality,

$$\rho_r \frac{\partial \psi}{\partial \sigma}\bar{\dot{\sigma}} - 2(\boldsymbol{E}^e_{(1)} + \boldsymbol{1})\boldsymbol{S}^e_{(1)} \cdot \bar{\boldsymbol{L}}^p_r \leq 0, \qquad \text{and} \tag{64}$$

(a.5) the function $\bar{\boldsymbol{q}}_r$ satisfies the heat conduction inequality,

$$\bar{\boldsymbol{q}}_r \cdot \boldsymbol{g}_r \leq 0 . \tag{65}$$

Conditions **(a.4–a.5)** do not follow from our general assumptions, but it could be proved that with some symmetry restrictions, $\bar{\boldsymbol{L}}_r^p$ and $\bar{\sigma}$ depend on $\boldsymbol{g}_r$ only through higher order terms. Hereafter we consider that **(a.4–a.5)** hold.

We have therefore reduced the set (52–57) of constitutive equations to

$$\psi = \bar{\psi}(\boldsymbol{E}^e, \theta, \sigma) , \tag{66}$$

$$\boldsymbol{q}_r = \bar{\boldsymbol{q}}_r(\boldsymbol{E}^e, \theta, \boldsymbol{g}_r, \sigma) , \tag{67}$$

$$\boldsymbol{L}_r^p = \bar{\boldsymbol{L}}_r^p(\boldsymbol{E}^e, \theta, \sigma) , \tag{68}$$

$$\dot{\sigma} = \bar{\sigma}(\boldsymbol{E}^e, \theta, \sigma) , \tag{69}$$

with the stress and entropy response functions given by **(a.1)**, the plastic variables satisfying the internal dissipation inequality **(a.4)** and the heat flux vector satisfying the heat conduction inequality **(a.5)**.

8.4 Material symmetry

Motivated by our discussion in Section 6.3 and following HAHN [1974] we define the instantaneous symmetry group $\mathcal{S}_{(t)}$ at $\boldsymbol{X}^p, \sigma$ to be the group of all unimodular tensors $\boldsymbol{H}$ that satisfy

$$\psi = \bar{\psi}(\boldsymbol{H}\boldsymbol{E}^e\boldsymbol{H}^T, \theta, \sigma), \tag{70}$$

$$\boldsymbol{H}\boldsymbol{q}_r = \bar{\mathbf{q}}_r(\boldsymbol{H}\boldsymbol{E}^e\boldsymbol{H}^T, \theta, \boldsymbol{H}\boldsymbol{g}_r, \sigma), \tag{71}$$

$$\boldsymbol{H}\boldsymbol{L}_r^p\boldsymbol{H}^T = \bar{\boldsymbol{L}}_r^p(\boldsymbol{H}\boldsymbol{E}^e\boldsymbol{H}^T, \theta, \sigma), \tag{72}$$

$$\dot{\sigma} = \bar{\dot{\sigma}}(\boldsymbol{H}\boldsymbol{E}^e\boldsymbol{H}^T, \theta, \sigma), \tag{73}$$

An elastic-plastic material is called isotropic at $\boldsymbol{X}^p, \sigma$ if $\mathcal{S}_{(t)}$ contains the orthogonal group $\mathcal{O}$. If the material is isotropic at every $\boldsymbol{X}^p, \sigma$, then it is said to be isotropic.

If the material is assumed to be isotropic, then conditions (70–73) must hold for every $\boldsymbol{H} \in \mathcal{O}$, and using well known representation theorems (TRUESDELL and NOLL [1965]) we have the following results.

(b.1) The free-energy response function is given by

$$\psi = \bar{\psi}(\ell), \tag{74}$$

where ℓ is the list

$$\ell = \{I_1(\boldsymbol{E}^e), I_2(\boldsymbol{E}^e), I_3(\boldsymbol{E}^e), \theta, \sigma\}, \tag{75}$$

and $I_i(\boldsymbol{E}^e)$ are the invariants of tensor $\boldsymbol{E}^e$. In view of equations (60) and (74–75), and considering that

$$\frac{\partial I_1}{\partial \boldsymbol{E}^e} = \boldsymbol{1}, \tag{76}$$

$$\frac{\partial I_2}{\partial \mathbf{E}^e} = I_1 \mathbf{1} - \mathbf{E}^e \,, \tag{77}$$

$$\frac{\partial I_3}{\partial \mathbf{E}^e} = I_2 \mathbf{1} - I_1 \mathbf{E}^e + (\mathbf{E}^e)^2 \,, \tag{78}$$

we obtain for the stress response function

$$\boldsymbol{S}^e = a_0 \mathbf{1} + a_1 \mathbf{E}^e + a_2 (\mathbf{E}^e)^2 \,, \tag{79}$$

where

$$a_0 = \rho_r \left(\frac{\partial \bar{\psi}}{\partial I_1} + I_1 \frac{\partial \bar{\psi}}{\partial I_2} + I_2 \frac{\partial \bar{\psi}}{\partial I_3} \right) \,, \tag{80}$$

$$a_1 = -\rho_r \left(\frac{\partial \bar{\psi}}{\partial I_2} + I_1 \frac{\partial \bar{\psi}}{\partial I_3} \right) \,, \tag{81}$$

$$a_2 = \rho_r \frac{\partial \bar{\psi}}{\partial I_3} \,. \tag{82}$$

The entropy response function follows from equation (61),

$$\eta = \bar{\eta}(\ell) = -\frac{\partial \bar{\psi}(\ell)}{\partial \theta} \,. \tag{83}$$

(b.2) The heat flux vector response function is given by

$$\boldsymbol{q}_r = \left[b_0 \mathbf{1} + b_1 \mathbf{E}^e + b_2 (\mathbf{E}^e)^2 \right] \boldsymbol{g}_r \,, \tag{84}$$

where the coefficients b_i are functions of the augmented list

$$\{ \ell, (\boldsymbol{g}_r \cdot \boldsymbol{g}_r), (\mathbf{E}^e \boldsymbol{g}_r \cdot \boldsymbol{g}_r), (\mathbf{E}^e \boldsymbol{g}_r \cdot \mathbf{E}^e \boldsymbol{g}_r) \} \,. \tag{85}$$

(b.3) If the plastic velocity gradient is written in terms of its symmetric and skewsymmetric parts,

$$\boldsymbol{L}_r^p = \mathbf{D}_r^p + \mathbf{W}_r^p \,, \tag{86}$$

then we have

$$\boldsymbol{D}^p_r = c_0\,\boldsymbol{1} + c_1\boldsymbol{E}^e + c_2(\boldsymbol{E}^e)^2\,, \tag{87}$$

$$\boldsymbol{W}^p_r = \boldsymbol{0}\,, \tag{88}$$

where the coefficients c_i are functions of the list ℓ.

Finally the deformation resistance response function is given by

$$\dot{\sigma} = \bar{\dot{\sigma}}(\ell)\,. \tag{89}$$

In this way, by enforcing the second law of thermodynamics and assuming isotropic response, the twenty-one functions of eleven scalar arguments (52-57) have been reduced to the eight functions $\bar{\psi}, b_i, c_i$ and $\bar{\sigma}$ of the lists (75) and (85).

To summarize, the most general set of constitutive equations for large-strain, isotropic thermo-elasto-plasticity with the assumption of product decomposition of the deformation gradient is given by

$$\psi = \bar{\psi}(\ell)\,, \tag{90}$$

$$\boldsymbol{S}^e = a_0\,\boldsymbol{1} + a_1\boldsymbol{E}^e + a_2(\boldsymbol{E}^e)^2\,, \tag{91}$$

$$\eta = \bar{\eta}(\ell)\,, \tag{92}$$

$$\boldsymbol{q}_r = \left[b_0\,\boldsymbol{1} + b_1\boldsymbol{E}^e + b_2(\boldsymbol{E}^e)^2\right]\boldsymbol{g}_r\,, \tag{93}$$

$$\boldsymbol{D}^p_r = c_0\,\boldsymbol{1} + c_1\boldsymbol{E}^e + c_2(\boldsymbol{E}^e)^2\,, \tag{94}$$

$$\boldsymbol{W}^p_r = \boldsymbol{0}\,, \tag{95}$$

$$\dot{\sigma} = \bar{\sigma}(\ell) , \tag{96}$$

where $\ell = \{I_i(\boldsymbol{E}^e), \theta, \sigma\}$, a_i and $\bar{\eta}$ are given in (80–83), b_i depend on the list (85) and c_i depend on ℓ . Furthermore, the heat flux response function has to satisfy inequality **(a.5)** and the plastic response functions have to satisfy inequality **(a.4)**.

8.5 Reduced constitutive equations

As a first approximation to (90–96), we consider the system

$$\psi = \frac{1}{\rho_r}\left[(\tfrac{1}{2}\lambda + \mu)I_1^2(\boldsymbol{E}^e) - 2\mu I_2(\boldsymbol{E}^e)\right] , \tag{97}$$

$$\boldsymbol{S}^e = \lambda I_1(\boldsymbol{E}^e)\boldsymbol{1} + 2\mu \boldsymbol{E}^e , \tag{98}$$

$$\eta = \frac{1}{\rho_r}\left[-\left(\frac{1}{2}\frac{\partial\lambda}{\partial\theta} + \frac{\partial\mu}{\partial\theta}\right) I_1^2(\boldsymbol{E}^e) + 2\frac{\partial\mu}{\partial\theta} I_2(\boldsymbol{E}^e)\right] , \tag{99}$$

$$\boldsymbol{q}_r = -k\,\boldsymbol{g}_r , \tag{100}$$

$$\boldsymbol{D}_r^p = \gamma_0 I_1(\boldsymbol{E}^e)\boldsymbol{1} + \gamma_1 \boldsymbol{E}^e , \tag{101}$$

$$\boldsymbol{W}_r^p = \boldsymbol{0} , \tag{102}$$

$$\dot{\sigma} = \bar{\sigma}(\ell) , \tag{103}$$

$$\ell = \{I_1, I_2, \theta, \sigma\} , \tag{104}$$

where λ, μ, k depend on θ and γ_0, γ_1 depend on the list ℓ .

The stress-strain law (98) can be written as

$$\boldsymbol{S}^e = \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e], \tag{105}$$

where $\mathcal{L}$ is the elastic moduli tensor, given by

$$\boldsymbol{\mathcal{L}} = \lambda \boldsymbol{1} \otimes \boldsymbol{1} + 2\mu \boldsymbol{I}. \tag{106}$$

Note that $\boldsymbol{\mathcal{L}}^T = \boldsymbol{\mathcal{L}}$. Using the elastic moduli tensor the free-energy function can be written as

$$\psi = \frac{1}{2\rho_r} \boldsymbol{E}^e \cdot \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e]. \tag{107}$$

In equation (100), k is the thermal conductivity coefficient. If $k > 0$, the heat conduction inequality (65) is automatically satisfied.

Since plastic flow is assumed to be incompressible, $J^p = 1$ and

$$\dot{J}^p = J^p \mathrm{tr}\,[\dot{\boldsymbol{X}}^p (\boldsymbol{X}^p)^{-1}] = 0, \tag{108}$$

which, in virtue of (50) and (102) imply $\mathrm{tr}\,\boldsymbol{D}^p_r = 0$. Imposing this condition on (101) we must have $3\gamma_0 + \gamma_1 = 0$ and (101) reduces to

$$\boldsymbol{D}^p_r = \gamma_1 \, \boldsymbol{E}^{e\prime}, \tag{109}$$

where the prime stands for the deviatoric part of the strain tensor,

$$\boldsymbol{E}^{e\prime} = \boldsymbol{E}^e - \tfrac{1}{3}(\mathrm{tr}\,\boldsymbol{E}^e)\boldsymbol{1}. \tag{110}$$

Taking the deviatoric part of the stress tensor in equation (98)

$$\boldsymbol{S}^{e\prime} = 2\mu \boldsymbol{E}^{e\prime}, \tag{111}$$

we use this equation to write the evolution equation for the plastic stretching tensor in the more familiar form

$$\boldsymbol{D}_r^p = \gamma\, \boldsymbol{S}^{e\prime}, \tag{112}$$

where $\gamma = \gamma_1/(2\mu)$.

Define the "equivalent tensile stress" s an the "equivalent plastic stretching" d^p by

$$s = \sqrt{\tfrac{3}{2}\boldsymbol{S}^{e\prime} \cdot \boldsymbol{S}^{e\prime}}, \tag{113}$$

$$d^p = \sqrt{\tfrac{2}{3}\boldsymbol{D}_r^p \cdot \boldsymbol{D}_r^p}, \tag{114}$$

we obtain from (112),

$$\gamma = \frac{3d^p}{2s}. \tag{115}$$

Paralleling definition (6.120), we write the evolution equation for the deformation resistance (103) in the form

$$\dot{\sigma} = \bar{\sigma}(\ell) = h(\ell) d^p(\ell), \tag{116}$$

where $h(\ell)$ is the "hardening function". (Note that the same letter h has been used to denote heat flux.)

Plastic flow is observed to depend only on s and to be independent of the first invariant of stress (pressure).

The reduced constitutive equations for isotropic thermo-elasto-plasticity can be summarized as

$$\psi = \frac{1}{2\rho_r}\boldsymbol{E}^e \cdot \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e]\,, \tag{117}$$

$$\boldsymbol{S}^e = \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e]\,, \tag{118}$$

$$\eta = -\frac{1}{2\rho_r}\boldsymbol{E}^e \cdot \frac{\partial \boldsymbol{\mathcal{L}}}{\partial \theta}[\boldsymbol{E}^e]\,, \tag{119}$$

$$\mathbf{q}_r = -k\,\mathbf{g}_r\,, \tag{120}$$

$$\boldsymbol{D}^p_r = \frac{3d^p(\ell)}{2s}\boldsymbol{S}^{e\prime}\,, \tag{121}$$

$$\boldsymbol{W}^p_r = \boldsymbol{0}\,, \tag{122}$$

$$\dot{\sigma} = h(\ell)d^p(\ell)\,, \tag{123}$$

where

$$\boldsymbol{\mathcal{L}} = \lambda \boldsymbol{1}\otimes\boldsymbol{1} + 2\mu\boldsymbol{I}\,, \tag{124}$$

$$\ell = \{s, \theta, \sigma\}\,, \tag{125}$$

and λ, μ and k depend on θ.

8.6 Isothermal processes

An isothermal process is such that the temperature field is uniform and constant over time. In this case

$$\hat{\theta}(\mathbf{x}_r, \tau) = \theta_0\,, \tag{126}$$

$$\hat{\boldsymbol{g}}_r(\mathbf{x}_r, \tau) = \boldsymbol{0}\,, \tag{127}$$

and it follows from equations (119–120) that

$$\hat{\eta}(\mathbf{x}_r, \tau) = 0\,, \tag{128}$$

$$\hat{\boldsymbol{q}}_r(\mathbf{x}_r, \tau) = \boldsymbol{0}\,, \tag{129}$$

throughout the process.

A consistent set of reduced constitutive equations for isothermal elasto-plastic processes can be obtained from (117–125) by using equations (128–129),

$$\psi = \frac{1}{2\rho_r}\boldsymbol{E}^e \cdot \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e]\,, \tag{130}$$

$$\boldsymbol{S}^e = \mathcal{L}[\boldsymbol{E}^e]\,, \tag{131}$$

$$\boldsymbol{D}^p_r = \frac{3d^p(s,\sigma)}{2s}\boldsymbol{S}^{e\prime}\,, \tag{132}$$

$$\boldsymbol{W}^p_r = \boldsymbol{0}\,, \tag{133}$$

$$\dot{\sigma} = h(s,\sigma)d^p(s,\sigma)\,, \tag{134}$$

where

$$\boldsymbol{\mathcal{L}} = \lambda \boldsymbol{1} \otimes \boldsymbol{1} + 2\mu \boldsymbol{I}\,, \tag{135}$$

and λ, μ are the Lame constants.

8.7 Rate-independent case

In view of the discussion in Section 6.6 on rate-independent plasticity and equations (6.138–6.142), the rate-independent counterpart of the constitutive equations (130–135) is given by

$$\psi = \frac{1}{2\rho_r}\boldsymbol{E}^e \cdot \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e]\,, \tag{136}$$

$$\boldsymbol{S}^e = \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e]\,, \tag{137}$$

$$\boldsymbol{D}^p_r = \alpha\frac{3d^p(\sigma)}{2s}\boldsymbol{S}^{e\prime}\,, \tag{138}$$

$$\boldsymbol{W}^p_r = \boldsymbol{0}\,, \tag{139}$$

$$\dot{\sigma} = \alpha h(\sigma) d^p(\sigma)\,, \tag{140}$$

with the switching parameter α defined as

$$\alpha = \begin{cases} 1 & \text{if } s = \sigma \text{ and } \boldsymbol{S}^{e\prime}\cdot\dot{\boldsymbol{E}} > 0, \\ 0 & \text{if } s < \sigma \text{ or } (s = \sigma \text{ and } \boldsymbol{S}^{e\prime}\cdot\dot{\boldsymbol{E}} \leq 0), \end{cases} \tag{141}$$

and the consistency condition

$$\dot{\sigma} = \alpha\dot{s}\,. \tag{142}$$

Chapter 9

Hyper and hypo-elasticity

9.1 Introduction

A set of plasticity constitutive equations is said to be "based" on the elasticity model to which it reduces when no plastic flow occurs (i.e., when none of the plastic variables is changing with time). There are two classes of elasticity constitutive equations on which large strain plasticity models have been based, namely that of the hyper-elastic type and that of the hypo-elastic type.

The derivations of constitutive equations for large strain thermo-elasto-plasticity of Chapters 6 and 8 lead naturally to hyper-elastic based models. However, since in this context hypo-elastic based models have received much attention, we consider appropriate to compare the two approaches and draw some conclusions.

We first define properly the concepts of hyper and hypo-elasticity, and then establish a connection between the most commonly used hyper and hypo-elastic stress-strain laws in plasticity formulations.

9.2 Definitions

9.2.1 Elasticity

A material is "elastic" if the stress response function depends only on the deformation gradient,

$$\boldsymbol{T} = \bar{\boldsymbol{T}}(\boldsymbol{X})\,. \tag{1}$$

The most general constitutive equation for an elastic material, consistent with the principle of material frame indifference is given by

$$\boldsymbol{T} = \bar{\boldsymbol{T}}(\boldsymbol{X}) = \boldsymbol{R}\hat{\boldsymbol{T}}(\boldsymbol{U})\boldsymbol{R}^{T}\,, \tag{2}$$

where $\hat{\boldsymbol{T}}$ is any function $\hat{\boldsymbol{T}} : \mathcal{S}_{+} \to \mathcal{S}$. Note that $\hat{\boldsymbol{T}}$ is arbitrary, while $\bar{\boldsymbol{T}}$ is given by equation (2).

As has been discussed in Chapter 4, an equivalent way of writing (2) using a pair of work conjugate stress and strain measures is

$$\boldsymbol{S} = \bar{\boldsymbol{S}}(\boldsymbol{E})\,. \tag{3}$$

9.2.2 Hyper-elasticity

A "smooth cyclic motion" on a time interval $[t_0, t_1]$ is a motion $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{p}, \tau)$ such that

$$\boldsymbol{\chi}(\boldsymbol{p}, t_0) = \boldsymbol{\chi}(\boldsymbol{p}, t_1) \qquad \forall \boldsymbol{p} \in \mathcal{B}, \tag{4}$$

$$\tilde{\mathbf{v}}(\boldsymbol{p}, t_0) = \tilde{\mathbf{v}}(\boldsymbol{p}, t_1) \qquad \forall \boldsymbol{p} \in \mathcal{B}. \tag{5}$$

Recall that the rate at which work is done by external forces acting on a part $\mathcal{P} \subset \mathcal{B}$ at time τ can be written as (equation (2.70),)

$$R(\mathcal{P}, \tau) = \int_{P_0} \rho_0 \boldsymbol{b} \cdot \mathbf{v}\, dV_0 + \int_{\partial P_0} \boldsymbol{t}_0 \cdot \mathbf{v}\, dA_0 \,. \tag{6}$$

An elastic material is said to be "hyperelastic" (or conservative or Green elastic) if

$$\int_{t_0}^{t_1} R(\mathcal{P}, \tau)\, d\tau = 0\,, \tag{7}$$

for all parts $\mathcal{P}$ of $\mathcal{B}$ and all smooth cyclic motions on $[t_0, t_1]$.

In virtue of the referential form of the Theorem of Power Expended (2.71) and the definition (4.48) of stress measure, condition (7) can be written as

$$\int_{t_0}^{t_1} \int_{P_0} \boldsymbol{S} \cdot \dot{\boldsymbol{E}}\, dV_0\, d\tau = 0\,. \tag{8}$$

Using this equation it can be proved that an elastic material is hyperelastic if and only if there exists a function $\psi = \bar{\psi}(\boldsymbol{E})$ such that

$$\boldsymbol{S} = \frac{\partial \psi}{\partial \boldsymbol{E}}\,. \tag{9}$$

In the absence of plastic flow, the stress strain law derived in Chapters 6 and 8 reduce to

$$\boldsymbol{S} = \boldsymbol{\mathcal{L}}[\boldsymbol{E}]\,, \tag{10}$$

which is of the hyper-elastic type, with the free energy function playing the role of "elastic potential".

9.2.3 Hypo-elasticity

A material is said to be "hypo-elastic" if the stress tensor is determined from the constitutive equation (TRUESDELL and NOLL [1965], p. 404)

$$\overset{\triangledown}{\boldsymbol{T}} = \boldsymbol{\mathcal{H}}(\boldsymbol{T})[\boldsymbol{D}]\,, \tag{11}$$

where

$$\overset{\triangledown}{\boldsymbol{T}} = \dot{\boldsymbol{T}} - \boldsymbol{W}\boldsymbol{T} + \boldsymbol{T}\boldsymbol{W}\,, \tag{12}$$

is the Jaumann stress rate, $\boldsymbol{W}$ is the spin tensor and $\boldsymbol{D}$ is the stretching tensor. The tensor function $\boldsymbol{\mathcal{H}}(\boldsymbol{T})[\boldsymbol{D}]$ is linear in $\boldsymbol{D}$ and isotropic in $\boldsymbol{T}$ and $\boldsymbol{D}$. The tensor $\overset{\triangledown}{\boldsymbol{T}}$ in (11) can be replaced by any of the infinitely many objective stress rates of the form

$$\overset{\triangledown}{\boldsymbol{T}} + \boldsymbol{B}[\boldsymbol{T},\boldsymbol{D}]\,, \tag{13}$$

where $\boldsymbol{B}[\boldsymbol{T},\boldsymbol{D}]$ is an arbitrary bilinear isotropic tensor function of $\boldsymbol{T}$ and $\boldsymbol{D}$.

The most commonly used stress-strain law for hypo-elastic based plasticity constitutive equations is given by

$$\overset{\nabla}{\boldsymbol{T}} = \boldsymbol{\mathcal{L}}[\boldsymbol{D}]\,, \tag{14}$$

where $\boldsymbol{\mathcal{L}}$ is the elastic moduli tensor defined in (6.107).

9.3 Hyper and hypo-elasticity, a comparison

Consider a hyper-elastic material, with stress strain law given by

$$\boldsymbol{S} = \boldsymbol{\mathcal{L}}[\boldsymbol{E}]\,, \tag{15}$$

where $\boldsymbol{E}$ and $\boldsymbol{S}$ are based on the logarithmic scale function. It follows from (15) that the stress and strain tensors are collinear, and the stress measure can be written in terms of the Cauchy stress as in equation (4.99),

$$\boldsymbol{S} = J\boldsymbol{R}^T\boldsymbol{T}\boldsymbol{R}\,. \tag{16}$$

We consider isothermal processes, for which the tensor $\boldsymbol{\mathcal{L}}$ is constant. Consequently, the rate form of (15) is given by

$$\dot{\boldsymbol{S}} = \boldsymbol{\mathcal{L}}[\dot{\boldsymbol{E}}]\,. \tag{17}$$

Our purpose is to relate the stress and strain rates $\dot{\boldsymbol{S}}$ and $\dot{\boldsymbol{E}}$ to the Jaumann derivative $\overset{\nabla}{\boldsymbol{T}}$ and the stretching tensor $\boldsymbol{D}$ respectively.

Taking time derivative in (16) we obtain for the rate of change of the stress measure

$$\dot{\boldsymbol{S}} = \dot{J}\boldsymbol{R}^T\boldsymbol{T}\boldsymbol{R} + J\boldsymbol{R}^T\dot{\boldsymbol{T}}\boldsymbol{R} + J\dot{\boldsymbol{R}}^T\boldsymbol{T}\boldsymbol{R} + J\boldsymbol{R}^T\boldsymbol{T}\dot{\boldsymbol{R}}, \tag{18}$$

and using (1.61), and (4.15)

$$\dot{\boldsymbol{S}} = J\boldsymbol{R}^T\{\,\dot{\boldsymbol{T}} + \boldsymbol{T}\operatorname{tr}\boldsymbol{D} - \boldsymbol{\Omega}_R\boldsymbol{T} + \boldsymbol{T}\boldsymbol{\Omega}_R\}\boldsymbol{R}. \tag{19}$$

From the definition (12) of Jaumann stress rate we have

$$\dot{\boldsymbol{T}} = \overset{\triangledown}{\boldsymbol{T}} + \boldsymbol{W}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{W}, \tag{20}$$

and substituting in (19)

$$\dot{\boldsymbol{S}} = J\boldsymbol{R}^T\{\overset{\triangledown}{\boldsymbol{T}} + \boldsymbol{T}\operatorname{tr}\boldsymbol{D} - (\boldsymbol{\Omega}_R - \boldsymbol{W})\boldsymbol{T} + \boldsymbol{T}(\boldsymbol{\Omega}_R - \boldsymbol{W})\}\boldsymbol{R}, \tag{21}$$

or equivalently

$$\boldsymbol{R}\dot{\boldsymbol{S}}\boldsymbol{R}^T = \overset{\triangledown}{\boldsymbol{T}} + (J-1)\overset{\triangledown}{\boldsymbol{T}} + J\{\boldsymbol{T}\operatorname{tr}\boldsymbol{D} - (\boldsymbol{\Omega}_R - \boldsymbol{W})\boldsymbol{T} + \boldsymbol{T}(\boldsymbol{\Omega}_R - \boldsymbol{W})\}. \tag{22}$$

Note that from equations (4.29) and (4.33–4.34) we have

$$\boldsymbol{\Omega}_R - \boldsymbol{W} = \boldsymbol{R}_E\left[-\boldsymbol{\Omega}_L + \tfrac{1}{2}\left(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda} + \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1}\right)\right]\boldsymbol{R}_E^T, \tag{23}$$

and

$$\boldsymbol{T}(\boldsymbol{\Omega}_R - \boldsymbol{W}) - (\boldsymbol{\Omega}_R - \boldsymbol{W})\boldsymbol{T}$$

$$= \boldsymbol{R}_E\{\boldsymbol{T}_E\left[-\boldsymbol{\Omega}_L + \tfrac{1}{2}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda} + \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1})\right]$$

$$- \left[-\boldsymbol{\Omega}_L + \tfrac{1}{2}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda} + \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1})\right]\boldsymbol{T}_E\big\}\boldsymbol{R}_E^T\,, \tag{24}$$

where $\boldsymbol{T}_E = \boldsymbol{R}_E^T\boldsymbol{T}\boldsymbol{R}_E$. Equation (24) can be simplified to

$$\begin{aligned}&\boldsymbol{T}(\boldsymbol{\Omega}_R - \boldsymbol{W}) - (\boldsymbol{\Omega}_R - \boldsymbol{W})\boldsymbol{T}\\ &\qquad = \boldsymbol{R}_E\big\{-[\boldsymbol{T}_E,\boldsymbol{\Omega}_L] + \tfrac{1}{2}(\boldsymbol{\Lambda}^{-1}[\boldsymbol{T}_E,\boldsymbol{\Omega}_L]\boldsymbol{\Lambda} + \boldsymbol{\Lambda}[\boldsymbol{T}_E,\boldsymbol{\Omega}_L]\boldsymbol{\Lambda}^{-1})\big\}\boldsymbol{R}_E^T\,,\end{aligned} \tag{25}$$

where $[\boldsymbol{T}_E,\boldsymbol{\Omega}_L] = \boldsymbol{T}_E\boldsymbol{\Omega}_L - \boldsymbol{\Omega}_L\boldsymbol{T}_E$.

Also, combining equations (15–16) and the definition (4.83),

$$J\boldsymbol{R}^T\boldsymbol{T}\boldsymbol{R} = \boldsymbol{\mathcal{L}}[\boldsymbol{R}_L\ln(\Lambda)\boldsymbol{R}_L^T]\,, \tag{26}$$

and rearranging we obtain

$$\boldsymbol{T}_E = J^{-1}\boldsymbol{\mathcal{L}}[\ln(\Lambda)]\,, \tag{27}$$

$$\boldsymbol{T} = J^{-1}\boldsymbol{\mathcal{L}}[\boldsymbol{R}_E\ln(\Lambda)\boldsymbol{R}_E^T]\,, \tag{28}$$

Recalling expressions (4.50–4.51) for the stretching and strain rate tensors,

$$\boldsymbol{D} = \boldsymbol{R}_E\big\{\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^{-1} + \tfrac{1}{2}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1})\big\}\boldsymbol{R}_E^T\,, \tag{29}$$

$$\dot{\boldsymbol{E}} = \boldsymbol{R}_L\big\{\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^{-1} + \boldsymbol{\Omega}_L\ln\boldsymbol{\Lambda} - \ln\boldsymbol{\Lambda}\,\boldsymbol{\Omega}_L\big\}\boldsymbol{R}_L^T\,, \tag{30}$$

we have using (4.12)

$$\boldsymbol{R}\dot{\boldsymbol{E}}\boldsymbol{R}^T = \boldsymbol{R}_E\big\{\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^{-1} + \boldsymbol{\Omega}_L\ln\boldsymbol{\Lambda} - \ln\boldsymbol{\Lambda}\,\boldsymbol{\Omega}_L\big\}\boldsymbol{R}_E^T\,, \tag{31}$$

and combining this expression with (29)

$$\boldsymbol{R}\dot{\boldsymbol{E}}\boldsymbol{R}^T = \boldsymbol{D} + \boldsymbol{R}_E\{\,\boldsymbol{\Omega}_L \ln\boldsymbol{\Lambda} - \ln\boldsymbol{\Lambda}\,\boldsymbol{\Omega}_L - \tfrac{1}{2}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1})\}\boldsymbol{R}_E^T\,. \tag{32}$$

Since the elastic moduli tensor $\boldsymbol{\mathcal{L}}$ is isotropic, we have from (17)

$$\boldsymbol{R}\dot{\boldsymbol{S}}\boldsymbol{R}^T = \boldsymbol{\mathcal{L}}[\boldsymbol{R}\dot{\boldsymbol{E}}\boldsymbol{R}^T]\,, \tag{33}$$

substituting in this expression the results (22) and (32),

$$\overset{\triangledown}{\boldsymbol{T}} + (J-1)\overset{\triangledown}{\boldsymbol{T}} + J\{\boldsymbol{T}\,\mathrm{tr}\,\boldsymbol{D} - (\boldsymbol{\Omega}_R - \boldsymbol{W})\boldsymbol{T} + \boldsymbol{T}(\boldsymbol{\Omega}_R - \boldsymbol{W})\}$$
$$= \boldsymbol{\mathcal{L}}[\boldsymbol{D} + \boldsymbol{R}_E\{\boldsymbol{\Omega}_L \ln\boldsymbol{\Lambda} - \ln\boldsymbol{\Lambda}\,\boldsymbol{\Omega}_L - \tfrac{1}{2}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1})\}\boldsymbol{R}_E^T]\,. \tag{34}$$

and rearranging,

$$\overset{\triangledown}{\boldsymbol{T}} - \boldsymbol{\mathcal{L}}[\boldsymbol{D}] = (1-J)\overset{\triangledown}{\boldsymbol{T}} - J\{\boldsymbol{T}\,\mathrm{tr}\,\boldsymbol{D} - (\boldsymbol{\Omega}_R - \boldsymbol{W})\boldsymbol{T} + \boldsymbol{T}(\boldsymbol{\Omega}_R - \boldsymbol{W})\}$$
$$+ \boldsymbol{\mathcal{L}}[\boldsymbol{R}_E\{\boldsymbol{\Omega}_L \ln\boldsymbol{\Lambda} - \ln\boldsymbol{\Lambda}\,\boldsymbol{\Omega}_L - \tfrac{1}{2}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1})\}\boldsymbol{R}_E^T]\,. \tag{35}$$

Finally, using (25) and (30) we obtain

$$\overset{\triangledown}{\boldsymbol{T}} - \boldsymbol{\mathcal{L}}[\boldsymbol{D}] = (1-J)\overset{\triangledown}{\boldsymbol{T}} - J\boldsymbol{T}\,\mathrm{tr}\,(\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^{-1})$$
$$+ \boldsymbol{R}_E\{J[\boldsymbol{T}_E, \boldsymbol{\Omega}_L] - \tfrac{1}{2}J(\boldsymbol{\Lambda}^{-1}[\boldsymbol{T}_E, \boldsymbol{\Omega}_L]\,\boldsymbol{\Lambda} + \boldsymbol{\Lambda}\,[\boldsymbol{T}_E, \boldsymbol{\Omega}_L]\,\boldsymbol{\Lambda}^{-1})$$
$$+ \boldsymbol{\mathcal{L}}[\boldsymbol{\Omega}_L \ln\boldsymbol{\Lambda} - \ln\boldsymbol{\Lambda}\,\boldsymbol{\Omega}_L - \tfrac{1}{2}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}_L\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\boldsymbol{\Omega}_L\boldsymbol{\Lambda}^{-1})]\}\boldsymbol{R}_E^T\,. \tag{36}$$

We have therefore written the rate form of the hyper-elastic stress-strain law (17) in a way that is convenient for the comparison with the hypo-elastic relation (14). The right hand side of equation (34) represents the "error" of the hypo-elastic approximation to (15). We explore in the next section conditions under which this error is small.

9.4 Small strains limit

Let ϵ be the tensor

$$\epsilon = \Lambda - \mathbf{1}\,, \tag{37}$$

with

$$\dot{\epsilon} = \dot{\Lambda}\,. \tag{38}$$

In what follows, we assume that the components of ϵ are small. The following expansions hold,

$$J = \det(\mathbf{1} + \epsilon) = 1 + \operatorname{tr}\epsilon + \mathcal{O}(\epsilon^2)\,, \tag{39}$$

$$1 - J = -\operatorname{tr}\epsilon + \mathcal{O}(\epsilon^2)\,, \tag{40}$$

$$\ln\Lambda = \epsilon + \mathcal{O}(\epsilon^2)\,, \tag{41}$$

$$\Lambda^{-1} = \mathbf{1} - \epsilon + \mathcal{O}(\epsilon^2)\,. \tag{42}$$

In virtue of (27–28), (30), (17), (19) and these expansions,

$$\boldsymbol{T}_E = \boldsymbol{\mathcal{L}}[\epsilon] + \mathcal{O}(\epsilon^2)\,, \tag{43}$$

$$\boldsymbol{T} = \boldsymbol{\mathcal{L}}[\boldsymbol{R}_E \epsilon \boldsymbol{R}_E^T] + \mathcal{O}(\epsilon^2)\,, \tag{44}$$

$$\dot{\boldsymbol{E}} = \boldsymbol{R}_L \dot{\epsilon} \boldsymbol{R}_L^T + \mathcal{O}(\epsilon)\,, \tag{45}$$

$$\dot{\boldsymbol{S}} = \boldsymbol{\mathcal{L}}[\boldsymbol{R}_L \dot{\epsilon} \boldsymbol{R}_L^T] + \mathcal{O}(\epsilon)\,, \tag{46}$$

$$\dot{\boldsymbol{T}} = \boldsymbol{\mathcal{L}}[\boldsymbol{R}_L \dot{\epsilon} \boldsymbol{R}_L^T] + \mathcal{O}(\epsilon)\,, \tag{47}$$

$$\overset{\triangledown}{\boldsymbol{T}} = \boldsymbol{\mathcal{L}}[\boldsymbol{R}_L \dot{\epsilon} \boldsymbol{R}_L^T] + \mathcal{O}(\epsilon)\,, \tag{48}$$

from where we have

$$(1 - J)\overset{\triangledown}{\boldsymbol{T}} = -\mathrm{tr}\,(\epsilon)\boldsymbol{\mathcal{L}}[\boldsymbol{R}_E \dot{\epsilon} \boldsymbol{R}_E^T] + \mathcal{O}(\epsilon^2)\,, \tag{49}$$

$$J\boldsymbol{T}\mathrm{tr}\,(\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^{-1}) = \mathrm{tr}\,(\dot{\epsilon})\boldsymbol{\mathcal{L}}[\boldsymbol{R}_E \epsilon \boldsymbol{R}_E^T] + \mathcal{O}(\epsilon^2)\,. \tag{50}$$

On the other hand,

$$[\boldsymbol{T}_E, \boldsymbol{\Omega}_L] = \big[\boldsymbol{\mathcal{L}}[\epsilon], \boldsymbol{\Omega}_L\big] + \mathcal{O}(\epsilon^2)\,, \tag{51}$$

$$\boldsymbol{\Lambda}^{-1}[\boldsymbol{T}_E, \boldsymbol{\Omega}_L]\,\boldsymbol{\Lambda} = \big[\boldsymbol{\mathcal{L}}[\epsilon], \boldsymbol{\Omega}_L\big] + \mathcal{O}(\epsilon^2)\,, \tag{52}$$

$$\boldsymbol{\Lambda}\,[\boldsymbol{T}_E, \boldsymbol{\Omega}_L]\,\boldsymbol{\Lambda}^{-1} = \big[\boldsymbol{\mathcal{L}}[\epsilon], \boldsymbol{\Omega}_L\big] + \mathcal{O}(\epsilon^2)\,, \tag{53}$$

therefore

$$J[\boldsymbol{T}_E, \boldsymbol{\Omega}_L] - \tfrac{1}{2}\,J(\boldsymbol{\Lambda}^{-1}[\boldsymbol{T}_E, \boldsymbol{\Omega}_L]\,\boldsymbol{\Lambda} + \boldsymbol{\Lambda}\,[\boldsymbol{T}_E, \boldsymbol{\Omega}_L]\,\boldsymbol{\Lambda}^{-1}) = \mathcal{O}(\epsilon^2)\,. \tag{54}$$

Also, we have

$$\boldsymbol{\Omega}_L \ln \boldsymbol{\Lambda} = \boldsymbol{\Omega}_L \epsilon + \mathcal{O}(\epsilon^2)\,, \tag{55}$$

$$\ln \boldsymbol{\Lambda}\, \boldsymbol{\Omega}_L = \epsilon \boldsymbol{\Omega}_L + \mathcal{O}(\epsilon^2)\,, \tag{56}$$

$$\mathbf{\Lambda}^{-1}\mathbf{\Omega}_L\mathbf{\Lambda} = \mathbf{\Omega}_L - \boldsymbol{\epsilon}\mathbf{\Omega}_L + \mathbf{\Omega}_L\boldsymbol{\epsilon} + \mathcal{O}(\epsilon^2)\,, \tag{57}$$

$$\mathbf{\Lambda}\mathbf{\Omega}_L\mathbf{\Lambda}^{-1} = \mathbf{\Omega}_L + \boldsymbol{\epsilon}\mathbf{\Omega}_L - \mathbf{\Omega}_L\boldsymbol{\epsilon} + \mathcal{O}(\epsilon^2)\,, \tag{58}$$

from where,

$$\mathbf{\Omega}_L \ln\mathbf{\Lambda} - \ln\mathbf{\Lambda}\,\mathbf{\Omega}_L - \tfrac{1}{2}(\mathbf{\Lambda}^{-1}\mathbf{\Omega}_L\mathbf{\Lambda} - \mathbf{\Lambda}\mathbf{\Omega}_L\mathbf{\Lambda}^{-1}) = \mathcal{O}(\epsilon^2)\,. \tag{59}$$

Combining (36) with (49–50), (54) and (59) we finally obtain

$$\overset{\nabla}{\boldsymbol{T}} - \boldsymbol{\mathcal{L}}[\boldsymbol{D}] = -\boldsymbol{R}_E\boldsymbol{\mathcal{L}}\big[\dot{\boldsymbol{\epsilon}}\,\mathrm{tr}\,\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\,\mathrm{tr}\,\dot{\boldsymbol{\epsilon}}\big]\boldsymbol{R}_E^T + \mathcal{O}(\epsilon^2)\,. \tag{60}$$

9.5 Summary

We conclude from (60) that for the hypo-elastic stress-strain law to be a good approximation of the hyper-elastic (non-dissipative) law, both the stretches $\mathbf{\Lambda}$ and the stretch rates $\dot{\mathbf{\Lambda}}$ have to be small. That is the case of a quasistatic elasto-plastic process where the elastic stretches are small.

However, we mention the following disadvantages of a hypo-elastic approach

1. In a purely elastic process, such law is in general not conservative, i.e. it predicts energy dissipation (or energy generation!) for smooth cyclic processes.
2. For a time integration procedure of a hypo-elastic based formulation, special care is necessary to ensure numerical objectivity (see for example WEBER [1988]). A

hyper-elastic based formulation, on the contrary, has numerical objectivity "built in".

3. A hypo-elastic material is said to be of grade n if $\mathcal{H}(\boldsymbol{T})$ is a polinomial of degree n in the components of $\boldsymbol{T}$. For $n = 0$ (as is the case of the stress-strain law (14)), definition (11) is not invariant under change of objective stress rate, since from (13) we see that two objective stress rates differ from one another in terms linear in $\boldsymbol{T}$. This fact has led to the proposal of many alternative stress rates, with none of them having a clear advantage over any other. (see TRUESDELL and NOLL [1965], p. 404, footnote 1 and p. 405, footnote 5.)

Chapter 10

Time integration algorithm

10.1 Constitutive equations

We consider the set of constitutive equations for large strain elasto-plastic isothermal processes derived in Chapter 8. The purpose of this Chapter is to present an integration algorithm for the rate independent model of Section 8.6. We first summarize the governing equations.

Constitutive equation for stress.

The stress-strain law is taken to be

$$\boldsymbol{S}^e = \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e]\,, \tag{1}$$

where $\boldsymbol{\mathcal{L}}$ is the fourth-order isotropic elastic moduli tensor, given by

$$\boldsymbol{\mathcal{L}} = \lambda \boldsymbol{1} \otimes \boldsymbol{1} + 2\mu \boldsymbol{I}\,, \tag{2}$$

where λ and μ are the Lame constants.

We adopt the Hencky strain measure

$$\boldsymbol{E}^e = \boldsymbol{R}^e_L \ln \boldsymbol{\Lambda}^e (\boldsymbol{R}^e_L)^T \,, \tag{3}$$

and its elastic work conjugate stress measure

$$\boldsymbol{S}^e = J^e (\boldsymbol{R}^e)^T \boldsymbol{T} \boldsymbol{R}^e \,. \tag{4}$$

Evolution equation for $\boldsymbol{X}^p$.

The evolution equation for the plastic deformation gradient is

$$\dot{\boldsymbol{X}}^p = \boldsymbol{L}^p_r \boldsymbol{X}^p \,, \tag{5}$$

where the plastic stretching tensor is specified by the flow rule

$$\boldsymbol{D}^p_r = \alpha \frac{3d^p}{2s} \boldsymbol{S}^{e\prime} \,, \tag{6}$$

and the plastic spin tensor is taken to be zero,

$$\boldsymbol{W}^p_r = 0 \,. \tag{7}$$

In these equations the equivalent tensile stress s and the equivalent plastic strain rate d^p are given by

$$s = \sqrt{\tfrac{3}{2} \boldsymbol{S}^{e\prime} \cdot \boldsymbol{S}^{e\prime}} \,, \tag{8}$$

$$d^p = \sqrt{\tfrac{2}{3} \boldsymbol{D}^p_r \cdot \boldsymbol{D}^p_r} \,, \tag{9}$$

and the loading parameter α is defined by

$$\alpha = \begin{cases} 1 & \text{if } s = \sigma \text{ and } \boldsymbol{S}^{e\prime} \cdot \dot{\boldsymbol{E}} > 0, \\ 0 & \text{if } s < \sigma \text{ or } (s = \sigma \text{ and } \boldsymbol{S}^{e\prime} \cdot \dot{\boldsymbol{E}} \leq 0). \end{cases} \tag{10}$$

In virtue of equation (7) we rewrite (5) as

$$\dot{\boldsymbol{X}}^p = \boldsymbol{D}^p_r \boldsymbol{X}^p . \tag{11}$$

The evolution equation for σ.

The evolution equation for the deformation resistance is given by the hardening rule

$$\dot{\sigma} = \alpha h \, d^p , \tag{12}$$

where h is the hardening function

$$h = \bar{h}(\sigma) . \tag{13}$$

Finally the consistency condition requires that

$$\dot{\sigma} = \alpha \dot{s} . \tag{14}$$

Combining (12) and (14) we solve for αd^p ,

$$\alpha d^p = \alpha \frac{\dot{s}}{h} , \tag{15}$$

and substitute in the flow rule to obtain

$$\boldsymbol{D}^p_r = \alpha \frac{3\dot{s}}{2hs} \boldsymbol{S}^{e\prime} , \tag{16}$$

10.2 Computational procedure

In a displacement-based finite element procedure for nonlinear problems, the solution of the discretized equilibrium equations is obtained for each time step by an iterative technique. The result of each iteration is an estimate of the incremental displacement that is used to compute the stresses and other field variables at the integration points. If these stresses do not satisfy equilibrium to within given tolerance, then the estimate of the incremental displacement is revised and the process repeated until convergence is achieved.

We assume therefore that we are given the deformation gradient $\boldsymbol{X}_n$ and the list

$$\{\boldsymbol{S}^e_n, \boldsymbol{X}^p_n, \sigma_n\}, \tag{17}$$

at time $\tau = t_n$, with the Cauchy stress $\boldsymbol{T}_n$ satisfying equilibrium, and the deformation gradient $\boldsymbol{X}_{n+1}$ at time $\tau = t_{n+1} = t_n + \Delta t$.

Our purpose is to develop a time integration algorithm to obtain

$$\{\boldsymbol{S}^e_{n+1}, \boldsymbol{X}^p_{n+1}, \sigma_{n+1}\}, \tag{18}$$

and the Cauchy stress $\boldsymbol{T}_{n+1}$ at time t_{n+1} . A similar time integration algorithm for the case of a rate-dependent model has been developed by WEBER and ANAND [1988]. SIMO [1988a] and [1988b] formulates an equivalent theory based on maximum plastic dissipation but with more restrictive assumptions concerning the stress-strain pair.

The integration procedure for the present rate-independent model corresponds to the well-known "radial return" algorithm of WILKINS [1964] (see also KRIEG and KEY [1976], KRIEG and KRIEG [1977] and SCHREYER et al. [1979]).

10.2.1 Trial elastic state

Consider the evolution equation for $\boldsymbol{X}^p$,

$$\dot{\boldsymbol{X}}^p = \boldsymbol{D}_r^p \boldsymbol{X}^p \,, \tag{19}$$

we select the one-step, implicit integration operator given by

$$\boldsymbol{X}_{n+1}^p = \exp(\Delta t\, \boldsymbol{D}_{r,n+1}^p)\boldsymbol{X}_n^p \,. \tag{20}$$

We note that this operator satisfies the consistency conditions

$$\lim_{\Delta t \to 0} \boldsymbol{X}_{n+1}^p = \boldsymbol{X}_n^p \,, \tag{21}$$

and

$$\lim_{\Delta t \to 0} \frac{d\boldsymbol{X}_{n+1}^p}{dt_{n+1}} = \dot{\boldsymbol{X}}_n^p = \boldsymbol{D}_{r,n}^p \boldsymbol{X}_n^p \,, \tag{22}$$

i. e. is first order accurate in t.

The time rate of change of the equivalent tensile stress is approximated using an Euler backward operator

$$\dot{s}_{n+1} = \frac{s_{n+1} - s_n}{\Delta t} \,, \tag{23}$$

and combining this with (16) we obtain

$$\Delta t\, \boldsymbol{D}^p_{r,n+1} = \alpha \frac{3(s_{n+1} - s_n)}{2h_{n+1}s_{n+1}} \boldsymbol{S}^{e\prime}_{n+1}, \tag{24}$$

where

$$h_{n+1} = \bar{h}(s_{n+1}). \tag{25}$$

Taking the inverse in equation (20),

$$(\boldsymbol{X}^p_{n+1})^{-1} = (\boldsymbol{X}^p_n)^{-1} \exp(-\Delta t\, \boldsymbol{D}^p_{r,n+1}), \tag{26}$$

premultiplying by $\boldsymbol{X}_{n+1}$ and recalling that $\boldsymbol{X}^e = \boldsymbol{X}(\boldsymbol{X}^p)^{-1}$, we obtain

$$\boldsymbol{X}^e_{n+1} = \boldsymbol{X}^e_* \exp(-\Delta t\, \boldsymbol{D}^p_{r,n+1}), \tag{27}$$

where

$$\boldsymbol{X}^e_* = \boldsymbol{X}_{n+1}(\boldsymbol{X}^p_n)^{-1}, \tag{28}$$

is the “trial” deformation gradient. Solving equation (27) for $\boldsymbol{X}^e_*$,

$$\boldsymbol{X}^e_* = \boldsymbol{X}^e_{n+1} \exp(\Delta t\, \boldsymbol{D}^p_{r,n+1}), \tag{29}$$

and substituting the polar decompositions

$$\boldsymbol{X}^e_{n+1} = \boldsymbol{R}^e_{n+1} \boldsymbol{U}^e_{n+1}, \tag{30}$$

$$\boldsymbol{X}^e_* = \boldsymbol{R}^e_* \boldsymbol{U}^e_*, \tag{31}$$

we obtain

$$\boldsymbol{R}^e_* \boldsymbol{U}^e_* = \boldsymbol{R}_{n+1} \boldsymbol{U}_{n+1} \exp(\Delta t \, \boldsymbol{D}^p_{r,n+1}) \, . \tag{32}$$

In virtue of the flow rule (6), the stress-strain rule (1) and the definition of the strain measure (3),

$$[\boldsymbol{D}^p_r, \boldsymbol{S}^e] = 0 \, , \tag{33}$$

$$[\boldsymbol{S}^e, \boldsymbol{E}^e] = 0 \, , \tag{34}$$

$$[\boldsymbol{E}^e, \boldsymbol{U}^e] = 0 \, , \tag{35}$$

it follows that the elastic stretch tensor and the plastic stretching tensor are collinear,

$$[\boldsymbol{U}^e, \boldsymbol{D}^p_r] = 0 \, , \tag{36}$$

and in particular

$$[\boldsymbol{U}^e_{n+1}, \exp(\Delta t \, \boldsymbol{D}^p_{r,n+1})] = 0 \, . \tag{37}$$

Since each of the tensors $\boldsymbol{U}^e_{n+1}$ and $\exp(\Delta t \, \boldsymbol{D}^p_{r,n+1})$ are symmetric positive definite, and they commute, the product $\boldsymbol{U}_{n+1} \exp(\Delta t \, \boldsymbol{D}^p_{r,n+1})$ is positive definite. It follows from equation (32) and the uniqueness of the polar decomposition that

$$\boldsymbol{U}^e_* = \boldsymbol{U}^e_{n+1} \exp(\Delta t \, \boldsymbol{D}^p_{r,n+1}) \, , \tag{38}$$

$$\boldsymbol{R}^e_* = \boldsymbol{R}^e_{n+1} \, . \tag{39}$$

Taking logarithms in equation (38), and using the property $\ln(\boldsymbol{A}\boldsymbol{B}) = \ln \boldsymbol{A} + \ln \boldsymbol{B}$ whenever $[\boldsymbol{A}, \boldsymbol{B}] = \boldsymbol{0}$,

$$\boldsymbol{E}^e_* = \boldsymbol{E}^e_{n+1} + \Delta t \, \boldsymbol{D}^p_{r,n+1} \, , \tag{40}$$

where

$$\boldsymbol{E}^e_* = \ln \boldsymbol{U}^e_* , \tag{41}$$

is the "trial elastic strain". Using the stress-strain law (1) in (40) we obtain

$$\boldsymbol{S}^e_* = \boldsymbol{S}^e_{n+1} + \boldsymbol{\mathcal{L}}[\Delta t\, \boldsymbol{D}^p_{r,n+1}] , \tag{42}$$

where $\boldsymbol{S}^e_* = \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e_*]$ is the "trial elastic stress".

10.2.2 The effective stress function

Now, using equation (24),

$$\Delta t\, \boldsymbol{D}^p_{r,n+1} = \alpha \frac{3(s_{n+1} - s_n)}{2h_{n+1}s_{n+1}} \boldsymbol{S}^{e\,\prime}_{n+1} , \tag{43}$$

and recalling the definition of the elastic moduli tensor (2)

$$\boldsymbol{\mathcal{L}}[\Delta t\, \boldsymbol{D}^p_{r,n+1}] = 2\mu \Delta t\, \boldsymbol{D}^p_{r,n+1} = 3\mu\alpha \frac{s_{n+1} - s_n}{h_{n+1}s_{n+1}} \boldsymbol{S}^{e\,\prime}_{n+1} , \tag{44}$$

from where equation (42) is equivalent to

$$\boldsymbol{S}^e_* = \boldsymbol{S}^e_{n+1} + 3\mu\alpha \frac{s_{n+1} - s_n}{h_{n+1}s_{n+1}} \boldsymbol{S}^{e\,\prime}_{n+1} . \tag{45}$$

Equation (45) implies that

$$\mathrm{tr}\,(\boldsymbol{S}^e_*) = \mathrm{tr}\,(\boldsymbol{S}^e_{n+1}) , \tag{46}$$

and for the deviatoric parts,

$$\boldsymbol{S}_*^{e\,\prime} = \left[1 + 3\mu\alpha\frac{s_{n+1} - s_n}{h_{n+1}s_{n+1}}\right] \boldsymbol{S}_{n+1}^{e\,\prime} \,. \tag{47}$$

Taking dot product of equation (47) with itself,

$$\boldsymbol{S}_*^{e\,\prime} \cdot \boldsymbol{S}_*^{e\,\prime} = \left[1 + 3\mu\alpha\frac{s_{n+1} - s_n}{h_{n+1}s_{n+1}}\right]^2 \boldsymbol{S}_{n+1}^{e\,\prime} \cdot \boldsymbol{S}_{n+1}^{e\,\prime} \,, \tag{48}$$

recalling definition (8) of the equivalent tensile stress, and defining an "trial equivalent tensile stress" by

$$s_* = \sqrt{\tfrac{3}{2}\, \boldsymbol{S}_*^{e\,\prime} \cdot \boldsymbol{S}_*^{e\,\prime}} \,, \tag{49}$$

we can rewrite (48) as

$$s_*^2 = \left[1 + 3\mu\alpha\frac{s_{n+1} - s_n}{h_{n+1}s_{n+1}}\right]^2 s_{n+1}^2 \,, \tag{50}$$

or equivalently,

$$f(s_{n+1}) = (s_{n+1} - s_*)\bar{h}(s_{n+1}) + 3\mu\alpha(s_{n+1} - s_n) = 0 \,, \tag{51}$$

where $f(s_{n+1})$ is the "effective stress function" (KOJIĆ and BATHE [1987]).

Note that if $s_* < \sigma_n$, then the process is elastic, $\alpha = 0$ and $s_{n+1} = s_*$. The updated stress-strain state is equal to the trial elastic state.

10.2.3 Stress and strain updates

If $s_* \geq \sigma_n$ we consider the process plastic, $\alpha = 1$ and s_{n+1} is obtained by solving

$$f(s_{n+1}) = 0 \,. \tag{52}$$

Once s_{n+1} is obtained, we compute

$$\sigma_{n+1} = s_{n+1}\,, \tag{53}$$

$$\beta_{n+1} = 3\mu \frac{s_{n+1} - s_n}{s_{n+1}\bar{h}(s_{n+1})}\,, \tag{54}$$

and from (47),

$$\boldsymbol{S}^{e\,\prime}_{n+1} = (1 + \beta_{n+1})^{-1} \boldsymbol{S}^{e\,\prime}_{*}\,. \tag{55}$$

Combining (46) with (56) we obtain for the stress update

$$\boldsymbol{S}^{e}_{n+1} = \boldsymbol{S}^{e\,\prime}_{n+1} + \tfrac{1}{3}\,\mathrm{tr}\,(\boldsymbol{S}^{e}_{*})\boldsymbol{1}\,, \tag{56}$$

and for the elastic strain update

$$\boldsymbol{E}^{e}_{n+1} = \boldsymbol{\mathcal{L}}^{-1}[\boldsymbol{S}^{e}_{n+1}]\,. \tag{57}$$

Taking the determinant in equation (20) and recalling the identity

$$\det[\exp(\boldsymbol{A})] = \exp[\mathrm{tr}\,(\boldsymbol{A})] \qquad \forall\, \boldsymbol{A} \in \mathcal{S}\,, \tag{58}$$

we have

$$J^{p}_{n+1} = J^{p}_{n}\,. \tag{59}$$

Since $J^p_0 = 1$, equation (60) ensures that plastic incompressibility is preserved by the integration algorithm and

$$J^{e}_{n+1} = J_{n+1}\,. \tag{60}$$

The updated Cauchy stress tensor is obtained by inverting equation (4),

$$\boldsymbol{T}_{n+1} = (J^e_{n+1})^{-1}\boldsymbol{R}^e_{n+1}\boldsymbol{S}^e_{n+1}(\boldsymbol{R}^e_{n+1})^T\,, \tag{61}$$

and using (39) and (61),

$$\boldsymbol{T}_{n+1} = J^{-1}_{n+1}\boldsymbol{R}^e_*\boldsymbol{S}^e_{n+1}(\boldsymbol{R}^e_*)^T\,, \tag{62}$$

where

$$J_{n+1} = \det(\boldsymbol{X}_{n+1}) = \det(\boldsymbol{X}^e_*)\,. \tag{63}$$

10.2.4 Plastic deformation gradient update

The update for the plastic deformation gradient is obtained from (20),

$$\boldsymbol{X}^p_{n+1} = \exp(\Delta t\,\boldsymbol{D}^p_{r,n+1})\boldsymbol{X}^p_n\,. \tag{64}$$

Using (54) and (55), we rewrite equation (24) as

$$\begin{aligned}\Delta t\,\boldsymbol{D}^p_{r,n+1} &= \frac{3(s_{n+1}-s_n)}{2h_{n+1}s_{n+1}}\boldsymbol{S}^{e\prime}_{n+1}\\ &= \frac{\beta_{n+1}}{2\mu}\boldsymbol{S}^{e\prime}_{n+1}\\ &= \frac{\beta_{n+1}}{2\mu(1+\beta_{n+1})}\boldsymbol{S}^{e\prime}_*\\ &= \frac{\beta_{n+1}}{1+\beta_{n+1}}\boldsymbol{E}^{e\prime}_*\\ &= \gamma_{n+1}\boldsymbol{E}^{e\prime}_*\,,\end{aligned} \tag{65}$$

where we have defined for convenience

$$\gamma_{n+1} = \frac{\beta_{n+1}}{1+\beta_{n+1}} . \tag{66}$$

From the eigen-decomposition of the trial elastic strain,

$$\boldsymbol{E}^e_* = (\boldsymbol{R}^e_L)_* \ln \boldsymbol{\Lambda}^e_* (\boldsymbol{R}^e_L)^T_* , \tag{67}$$

we have for the trace

$$\begin{aligned} \operatorname{tr} \boldsymbol{E}^e_* &= \operatorname{tr}(\ln \boldsymbol{\Lambda}^e_*) \\ &= \ln \lambda^e_{*1} + \ln \lambda^e_{*2} + \ln \lambda^e_{*3} \\ &= \ln(\lambda^e_{*1}\lambda^e_{*2}\lambda^e_{*3}) \\ &= \ln J^e_* \\ &= \ln J_{n+1} , \end{aligned} \tag{68}$$

and for the deviatoric part

$$\begin{aligned} \boldsymbol{E}^{e\prime}_* &= (\boldsymbol{R}^e_L)_* \ln \boldsymbol{\Lambda}^e_* (\boldsymbol{R}^e_L)^T_* - \tfrac{1}{3}(\ln J_{n+1})\boldsymbol{1} \\ &= (\boldsymbol{R}^e_L)_* \big[\ln \boldsymbol{\Lambda}^e_* - (\ln J^{1/3}_{n+1})\boldsymbol{1}\big] (\boldsymbol{R}^e_L)^T_* \\ &= (\boldsymbol{R}^e_L)_* \ln\big(J^{-1/3}_{n+1} \boldsymbol{\Lambda}^e_*\big) (\boldsymbol{R}^e_L)^T_* . \end{aligned} \tag{69}$$

Combining equations (65) and (69)

$$\Delta t\, \boldsymbol{D}^p_{r,n+1} = \gamma_{n+1} (\boldsymbol{R}^e_L)_* \ln\big(J^{-1/3}_{n+1} \boldsymbol{\Lambda}^e_*\big) (\boldsymbol{R}^e_L)^T_* , \tag{70}$$

and taking the tensor exponential,

$$\begin{aligned} \exp(\Delta t\, \boldsymbol{D}^p_{r,n+1}) &= (\boldsymbol{R}^e_L)_* \exp\left[\gamma_{n+1} \ln(J^{-1/3}_{n+1} \boldsymbol{\Lambda}^e_*)\right](\boldsymbol{R}^e_L)^T_* \\ &= (\boldsymbol{R}^e_L)_* \left[J^{-1/3}_{n+1} \boldsymbol{\Lambda}^e_*\right]^{\gamma_{n+1}} (\boldsymbol{R}^e_L)^T_* , \end{aligned} \tag{71}$$

and substituting in (64) we obtain for the plastic deformation gradient update

$$\boldsymbol{X}^p_{n+1} = (\boldsymbol{R}^e_L)_* \left[J^{-1/3}_{n+1} \boldsymbol{\Lambda}^e_*\right]^{\gamma_{n+1}} (\boldsymbol{R}^e_L)^T_* \boldsymbol{X}^p_n . \tag{72}$$

10.2.5 Summary

The time integration algorithm presented in this section requires the following steps.

1. Obtain the trial deformation gradient

$$\boldsymbol{X}^e_* = \boldsymbol{X}_{n+1} (\boldsymbol{X}^p_n)^{-1} . \tag{73}$$

2. Perform the polar and eigen decompositions

$$\boldsymbol{X}^e_* = \boldsymbol{R}^e_* (\boldsymbol{R}^e_L)_* \boldsymbol{\Lambda}^e_* (\boldsymbol{R}^e_L)^T_* . \tag{74}$$

3. Obtain the trial elastic strain and stress tensors

$$\boldsymbol{E}^e_* = (\boldsymbol{R}^e_L)_* \ln \boldsymbol{\Lambda}^e_* (\boldsymbol{R}^e_L)^T_* , \tag{75}$$

$$\boldsymbol{S}^e_* = \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e_*]\,. \tag{76}$$

4. Decompose the trial elastic stress into its volumetric and deviatoric parts

$$\boldsymbol{S}^e_* = \boldsymbol{S}^{e\,\prime}_* + \tfrac{1}{3}\,(\mathrm{tr}\,\boldsymbol{S}^e_*)\boldsymbol{1}\,. \tag{77}$$

5. Obtain the trial equivalent tensile stress

$$s_* = \sqrt{\tfrac{3}{2}\,\boldsymbol{S}^{e\,\prime}_* \cdot \boldsymbol{S}^{e\,\prime}_*}\,. \tag{78}$$

6. If $s_* < \sigma_n$ then the process is elastic and

$$s_{n+1} = s_*\,, \tag{79}$$

$$\sigma_{n+1} = \sigma_n\,, \tag{80}$$

$$\boldsymbol{E}^e_{n+1} = \boldsymbol{E}^e_*\,, \tag{81}$$

$$\boldsymbol{S}^e_{n+1} = \boldsymbol{S}^e_*\,, \tag{82}$$

$$\boldsymbol{T}_{n+1} = J^{-1}_{n+1}\boldsymbol{R}^e_*\boldsymbol{S}^e_*(\boldsymbol{R}^e_*)^T\,, \tag{83}$$

$$\text{EXIT}\,. \tag{84}$$

Else, the process is elasto-plastic. Obtain s_{n+1} by solving

$$(s_{n+1} - s_*)\bar{h}(s_{n+1}) + 3\mu(s_{n+1} - s_n) = 0\,. \tag{85}$$

7. Update the deformation resistance

$$\sigma_{n+1} = s_{n+1}\,. \tag{86}$$

8. Calculate the constants β_{n+1} and γ_{n+1},

$$\beta_{n+1} = 3\mu \frac{s_{n+1} - s_n}{s_{n+1}\bar{h}(s_{n+1})}, \tag{87}$$

$$\gamma_{n+1} = \frac{\beta_{n+1}}{1 + \beta_{n+1}}. \tag{88}$$

9. Update the stress and elastic strain tensors

$$\boldsymbol{S}^e_{n+1} = \frac{\gamma_{n+1}}{\beta_{n+1}} \boldsymbol{S}^{e\prime}_* + \tfrac{1}{3}\mathrm{tr}\,(\boldsymbol{S}^e_*)\boldsymbol{1}, \tag{89}$$

$$\boldsymbol{E}^e_{n+1} = \boldsymbol{\mathcal{L}}^{-1}[\boldsymbol{S}^e_{n+1}]. \tag{90}$$

9. Update the Cauchy stress tensor

$$J_{n+1} = \det(\boldsymbol{\Lambda}^e_*), \tag{91}$$

$$\boldsymbol{T}_{n+1} = J^{-1}_{n+1} \boldsymbol{R}^e_{n+1} \boldsymbol{S}^e_{n+1} (\boldsymbol{R}^e_{n+1})^T. \tag{92}$$

10. Update the plastic deformation gradient

$$\boldsymbol{X}^p_{n+1} = (\boldsymbol{R}^e_L)_* \left[J^{-1/3}_{n+1} \boldsymbol{\Lambda}^e_*\right]^{\gamma_{n+1}} (\boldsymbol{R}^e_L)^T_* \boldsymbol{X}^p_n. \tag{93}$$

10.3 Numerical results

The algorithm presented in the last Section has been implemented in the finite element program ADINA [1987] and compared with an algorithm for a similar elasto-plastic model based on the additive decomposition of the strain tensor (ROLPH and

BATHE [1984]). For completeness, this model and its corresponding time integration algorithm is described in the Appendix.

Two basic displacements-driven experiments were performed.

First, stress predictions were obtained with both theories for an in-plane isochoric deformation, characterized by the deformation gradient

$$[\boldsymbol{X}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix}$$

where $\lambda \in [1,2]$. The results for the Cauchy stress components $[\boldsymbol{T}]_{33}$ and $[\boldsymbol{T}]_{22}$ are compared in Figures 10.1 and 10.2. It is seen that for this particular case (where no rotations are involved), both theories coincide.

Second, stress predictions were obtained with both theories for a simple shear deformation, characterized by the deformation gradient

$$[\boldsymbol{X}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix}$$

where $\gamma \in [0,1]$. The results for the Cauchy stress components $[\boldsymbol{T}]_{23}$, $[\boldsymbol{T}]_{22}$ and $[\boldsymbol{T}]_{33}$ are compared in Figures 10.3, 10.4 and 10.5. In this cases the stresses predicted are not the same, but the differences are extremely small over the range of applicability of an isotropic hardening theory. A broader range of engineering strain γ (0–100%) has been plotted only for illustrative purposes.

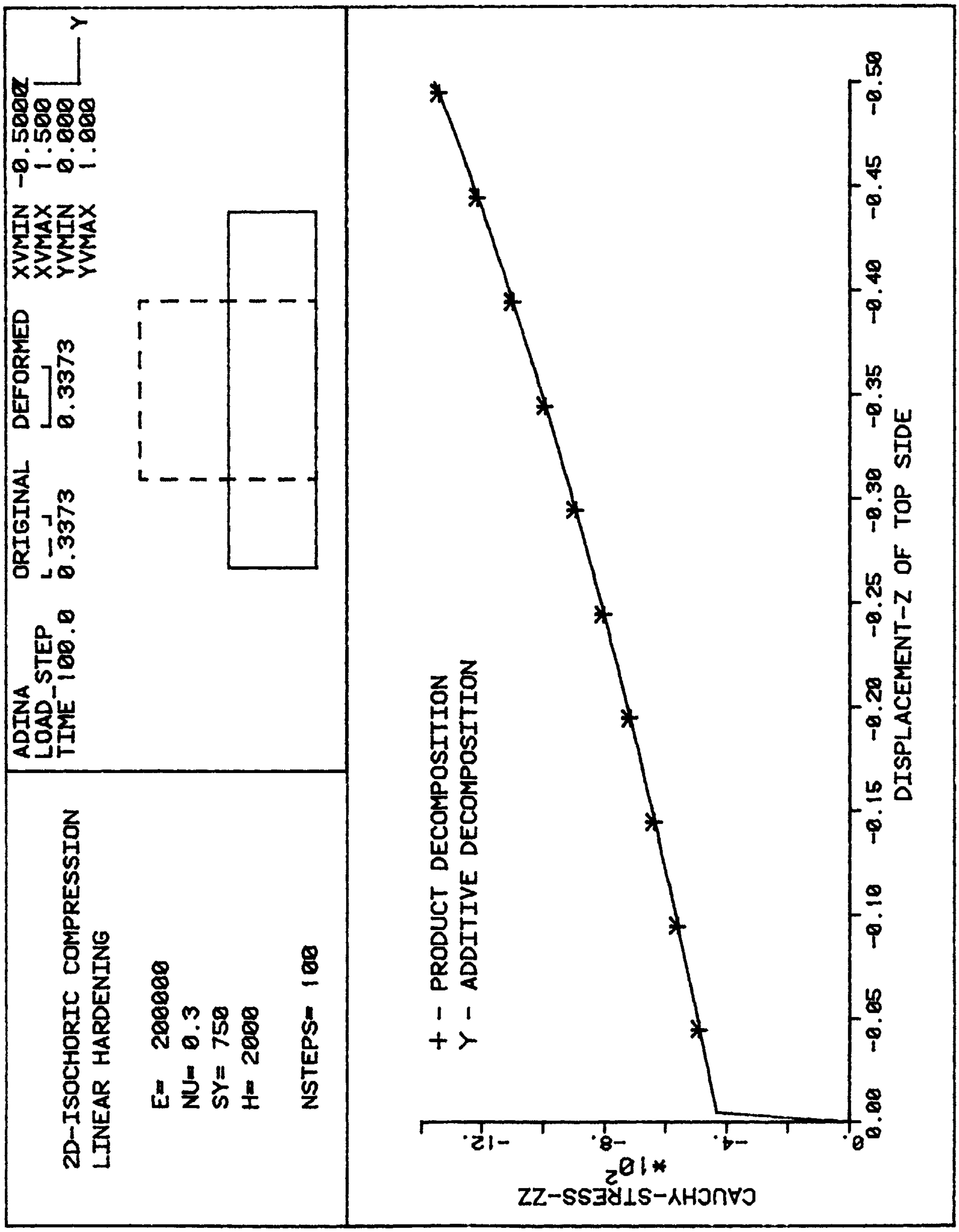

Figure 10.1 2-D isochoric compression. Cauchy stress component $[\boldsymbol{T}]_{33}$ as predicted by both the formulation based on the product decomposition of the deformation gradient and the formulation based on the additive decomposition of the strain tensor.

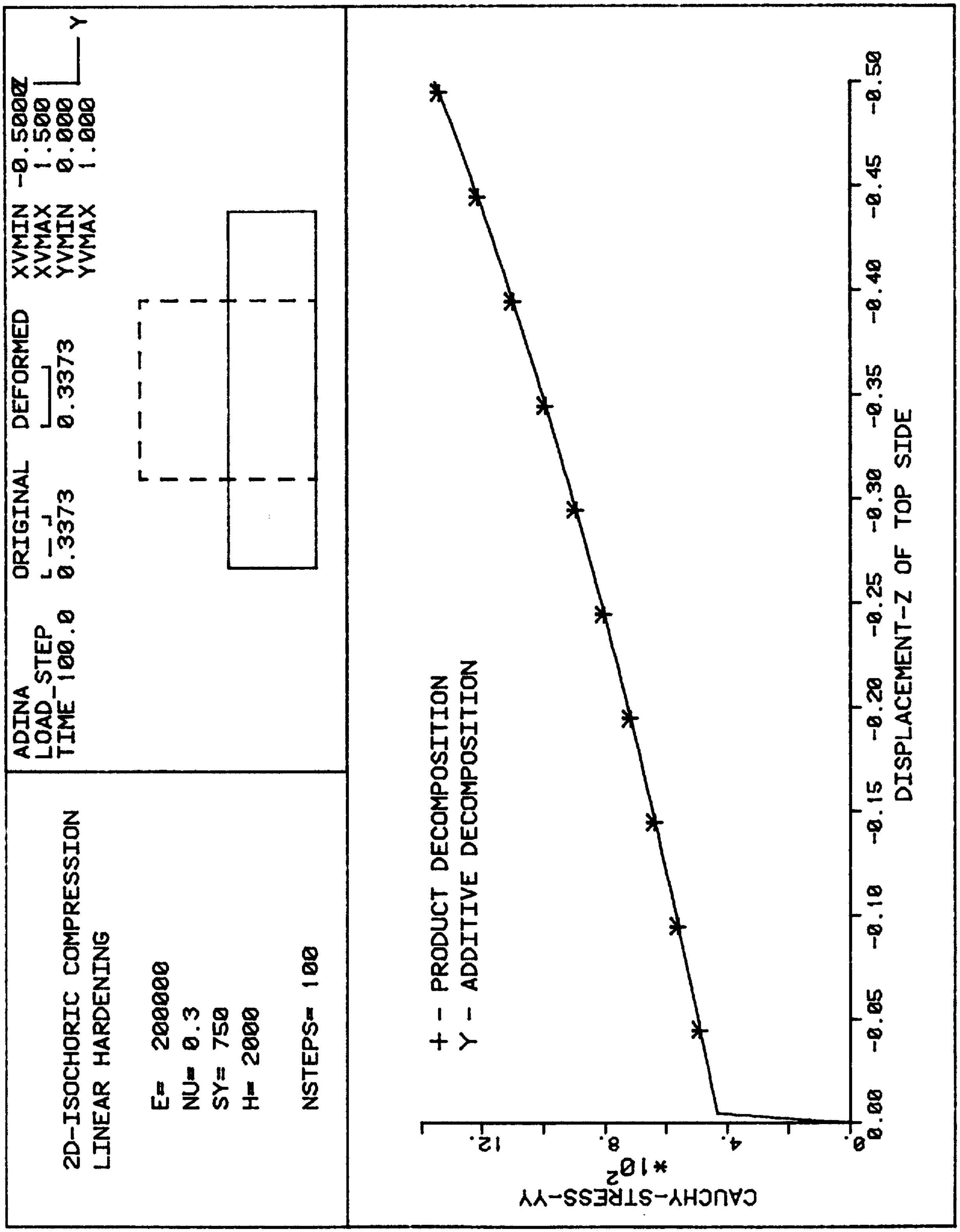

Figure 10.2 2-D isochoric compression. Cauchy stress component $[T]_{22}$ as predicted by both the formulation based on the product decomposition of the deformation gradient and the formulation based on the additive decomposition of the strain tensor.

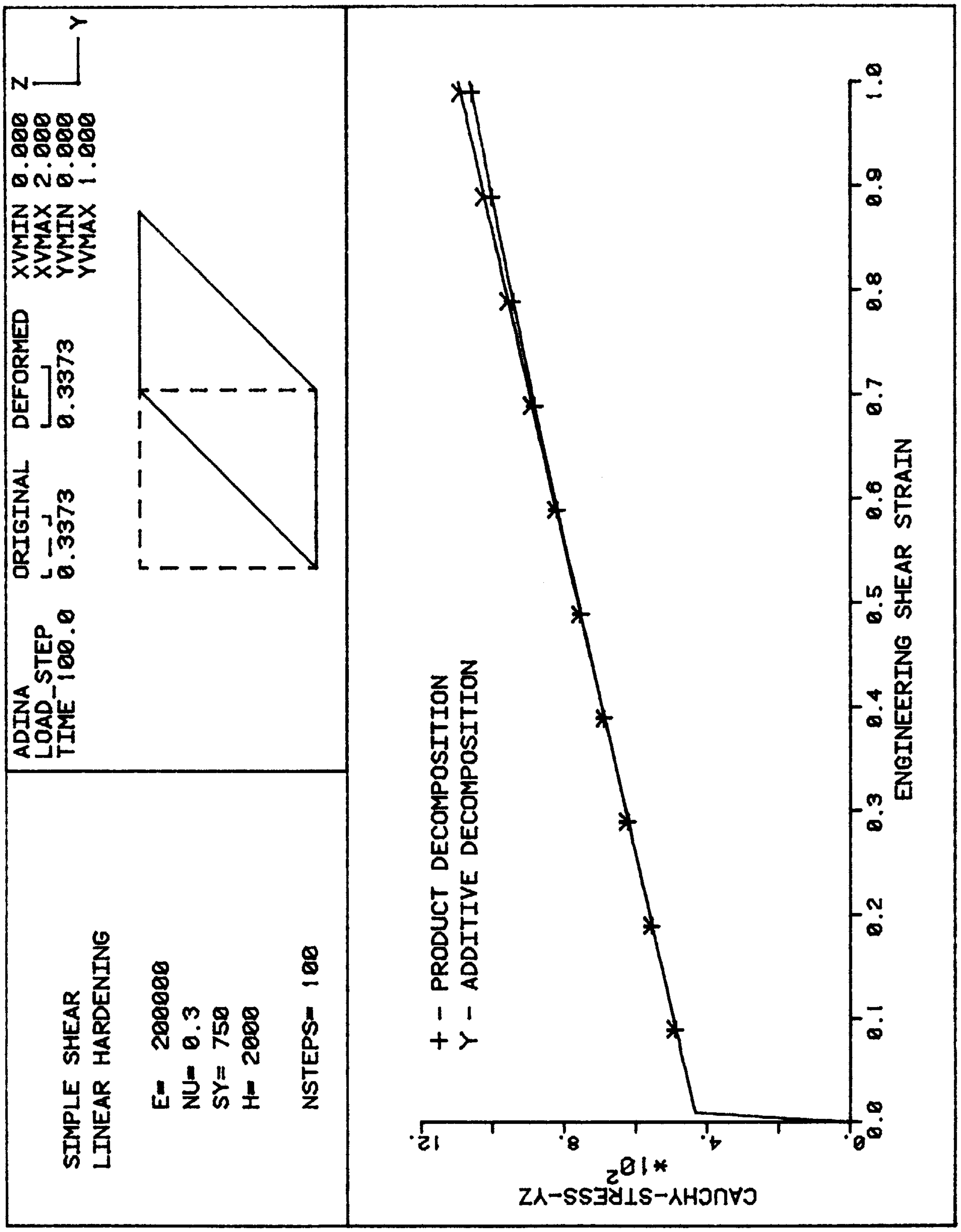

Figure 10.3 Simple shear. Cauchy stress component $[T]_{23}$ as predicted by both the formulation based on the product decomposition of the deformation gradient and the formulation based on the additive decomposition of the strain tensor.

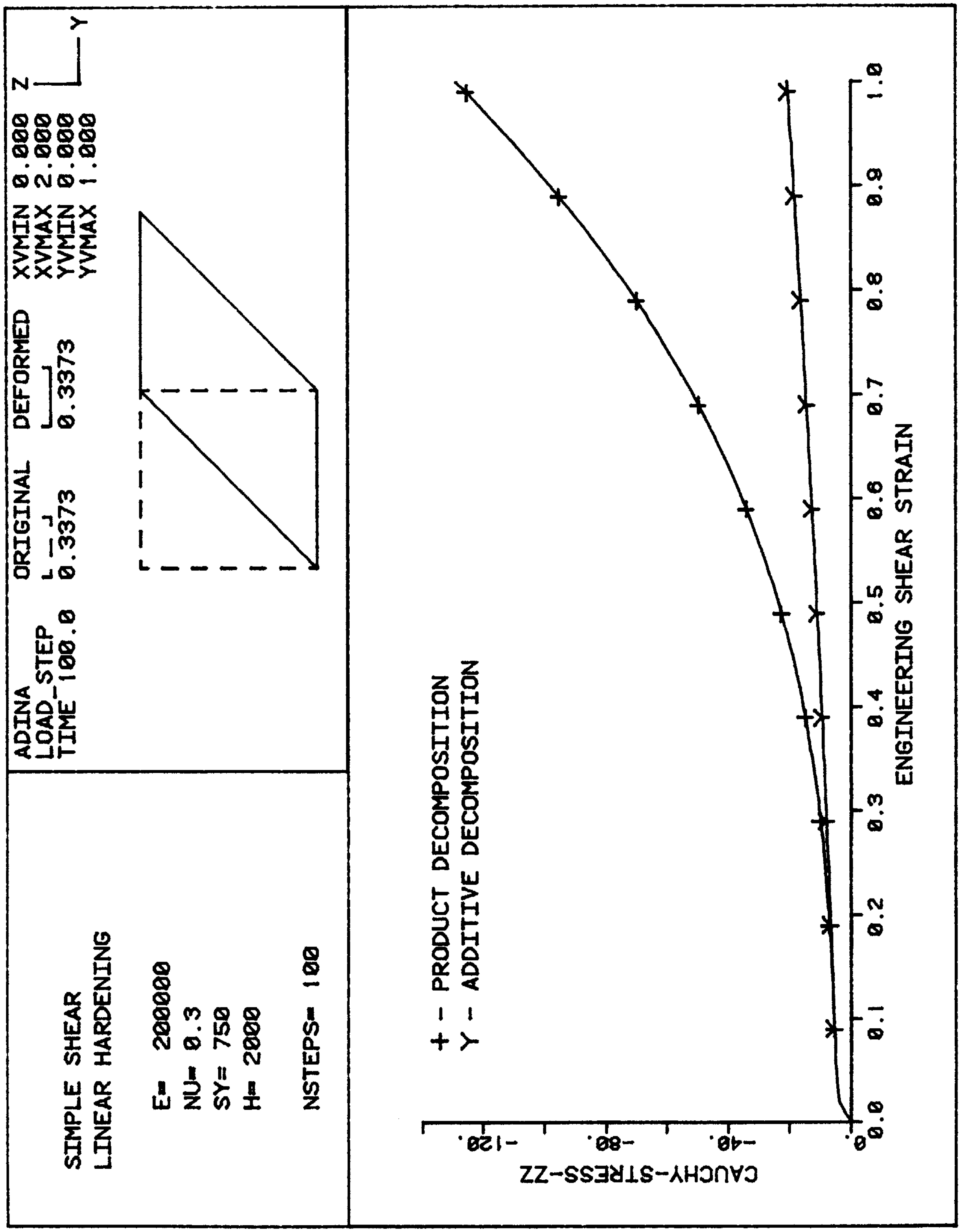

Figure 10.4 Simple shear. Cauchy stress component $[\boldsymbol{T}]_{33}$ as predicted by both the formulation based on the product decomposition of the deformation gradient and the formulation based on the additive decomposition of the strain tensor.

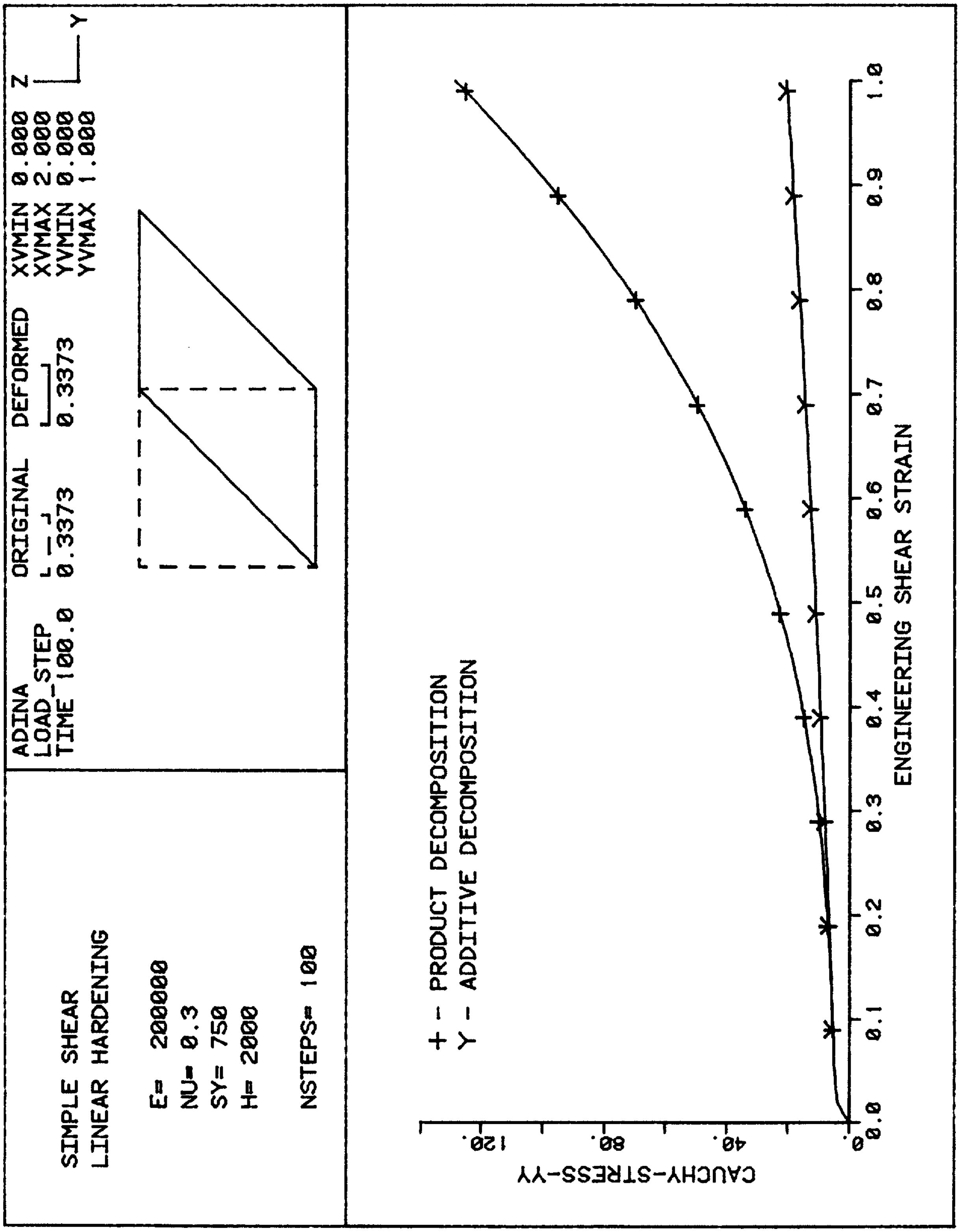

Figure 10.5 Simple shear. Cauchy stress component $[T]_{22}$ as predicted by both the formulation based on the product decomposition of the deformation gradient and the formulation based on the additive decomposition of the strain tensor.

Summary and Conclusions

This study has considered the extension of the classical infinitesimal theory of plasticity with isotropic hardening to the large deformation range.

Within the framework of modern continuum mechanics, this objective is achieved by using consequences of basic principles and some additional assumptions. This thesis focuses in both these aspects.

Three fundamental topics have been adressed: (a) the characterization of plastic flow, (b) the selection of stress and strain measures and (c) the underlying elasticity description.

Constitutive equations based on the additive decomposition of the strain tensor into an elastic and a plastic part, as developed in Chapter 6, are shown to present some undesirable effects. As discussed in Chapter 7, the only strain measure consistent with the condition of plastic deformation being isochoric is the Hencky strain. More important, the initial elastic response moduli depend on the plastic state. As a consequence of previous plastic flow, initially isotropic elastic response becomes anisotropic. The bulk modulus remains unmodified only in the case of logarithmic stress and strain measures. The shearing modulus presents a "decay" for any choice of scale function. The higher the level of plastic stretching, the lower the ratio of "modified" over initial shear moduli.

In the case of the constitutive equations based on the product decomposition of the deformation gradient into an elastic and a plastic part, as developed in Chapter 8, the elastic strain tensor is obtained from the elastic deformation gradient, thus ensuring independence of initial elastic response with plastic flow. The elastic moduli do not present any spurious decay in the sense of Chapter 7. In addition, the corresponding flow rule ensures that plastic deformation is isochoric independently of the choice of strain measure.

Chapter 9 compares hyper-elastic with hypo-elastic response. Hyper-elastic based elasto-plastic constitutive equations follow naturally from basic principles as exemplified in Chapters 6 and 8. The hypo-elastic approach is shown to give a good approximation when both the elastic stretches and the elastic stretch rates are small. The hypo-elastic stress-strain law of grade 0 used in the comparison (see equation 9.14) is the most commonly used for large deformation plasticity. This model, however, has the theoretical disadvantage of not being invariant under change of objective stress rates. A consequence of this feature has been the proposal of many alternative stress rates, with none of them having a clear advantage over any other one.

Hypo-elastic response, in general, is not conservative (in the hyper-elastic sense), purely elastic smooth cyclic motions dissipate or generate energy. Even is these effects are small, they are in contradiction with the first law of thermodynamics for a continuum.

From a practical point of view, with the specification of an appropriate elastic potential, hyper-elasticity allows for the automatic extension of an elasto-plastic model to the finite elastic deformation range.

In general, time integration algorithms for hypo-elastic based constitutive equations do not preserve the objectivity property of the continuum theory, i.e. they are not numerically objective. Additional modifications to the governing equations have to be made to ensure numerical objectivity. On the contrary, hyper-elastic based constitutive equations lead naturally to numerically objective time integration algorithms. This is so because no objective tensor rate is used in the formulation, but total invariant tensors are used instead.

Based on these considerations, a hyper-elastic based, rate-independent constitutive model for large deformation elasto-plasticity based on the product decomposition of the deformation gradient is presented. This model contains the features which have been found most appropriate in the preceeding discussion. A time integration algorithm is given in detail in Chapter 10 and its implementation tested against a similar model based on the additive decomposition of the strain tensor. The two theories are shown to predict the same results for a case where no rotations are involved and the elastic and plastic stretch tensors commute. When this is not the case, as in the simple shear experiment considered, shear stresses as predicted by both theories are basically identical within the range of applicability of an isotropic hardening model. Also, the predictions for normal stresses are close in this range.

Finally, we mention that a natural extension of this study is the development of a hyper-elastic based, combined isotropic-kinematic hardening constitutive model for large deformation plasticity.

Appendix

A.1 Constitutive equations

We consider the set of constitutive equations for large strain elasto-plastic isothermal processes of the type derived in Chapter 6. A solution procedure for this constitutive model has been introduced by ROLPH and BATHE [1984]. We first summarize the governing equations.

<u>Constitutive equation for stress.</u>

The stress-strain law is taken to be

$$\boldsymbol{S} = \boldsymbol{\mathcal{L}}[\boldsymbol{E} - \boldsymbol{E}^p]\,, \tag{1}$$

where $\boldsymbol{\mathcal{L}}$ is the fourth-order isotropic elastic moduli tensor, given by

$$\boldsymbol{\mathcal{L}} = \lambda \boldsymbol{1} \otimes \boldsymbol{1} + 2\mu \boldsymbol{I}\,, \tag{2}$$

where λ and μ are the Lame constants.

We adopt the Hencky strain measure

$$\boldsymbol{E} = \boldsymbol{R}_L \ln \Lambda (\boldsymbol{R}_L)^T\,, \tag{3}$$

and its total work conjugate stress measure $\boldsymbol{S}$, given by equation (4.72).

Evolution equation for $\boldsymbol{E}^p$.

The evolution equation for the plastic strain tensor is given by the flow rule

$$\dot{\boldsymbol{E}}^p = \alpha \frac{3\dot{e}^p}{2s} \boldsymbol{S}', \tag{4}$$

In this equation the equivalent tensile stress s and the equivalent plastic strain rate $\dot{e}^p$ are given by

$$s = \sqrt{\tfrac{3}{2} \boldsymbol{S}' \cdot \boldsymbol{S}'}, \tag{5}$$

$$\dot{e}^p = \sqrt{\tfrac{2}{3} \dot{\boldsymbol{E}}^p \cdot \dot{\boldsymbol{E}}^p}, \tag{6}$$

and the loading parameter α is defined by

$$\alpha = \begin{cases} 1 & \text{if } s = \sigma \text{ and } \boldsymbol{S}' \cdot \dot{\boldsymbol{E}} > 0, \\ 0 & \text{if } s < \sigma \text{ or } (s = \sigma \text{ and } \boldsymbol{S}' \cdot \dot{\boldsymbol{E}} \leq 0). \end{cases} \tag{7}$$

The evolution equation for σ.

The evolution equation for the deformation resistance is given by the hardening rule

$$\dot{\sigma} = \alpha h \, \dot{e}^p, \tag{8}$$

where h is the hardening function

$$h = \bar{h}(\sigma). \tag{9}$$

Finally the consistency condition requires that

$$\dot{\sigma} = \alpha \dot{s} \,. \tag{10}$$

Combining (8) and (10) we solve for $\alpha \dot{e}^p$,

$$\alpha \dot{e}^p = \alpha \frac{\dot{s}}{h} \,, \tag{11}$$

and substitute in the flow rule (4) to obtain

$$\dot{\boldsymbol{E}}^p = \alpha \frac{3\dot{s}}{2hs} \boldsymbol{S}' \,, \tag{12}$$

A.2 Computational procedure

We assume that we are given the deformation gradient $\boldsymbol{X}_n$ and the list

$$\{\boldsymbol{S}_n, \boldsymbol{E}^p_n, \sigma_n\} \,, \tag{13}$$

at time $\tau = t_n$, with the Cauchy stress $\boldsymbol{T}_n$ satisfying equilibrium, and the deformation gradient $\boldsymbol{X}_{n+1}$ at time $\tau = t_{n+1} = t_n + \Delta t$.

The time integration algorithm gives the updates

$$\{\boldsymbol{S}_{n+1}, \boldsymbol{E}^p_{n+1}, \sigma_{n+1}\} \,, \tag{14}$$

and the Cauchy stress $\boldsymbol{T}_{n+1}$ at time t_{n+1} .

A.2.1 Trial elastic state

The time rate of change of the plastic strain tensor and the equivalent tensile stress are approximated using the Euler backward operators

$$\dot{\boldsymbol{E}}^p_{n+1} = \frac{\boldsymbol{E}^p_{n+1} - \boldsymbol{E}^p_n}{\Delta t}\,, \tag{15}$$

$$\dot{s}_{n+1} = \frac{s_{n+1} - s_n}{\Delta t}\,, \tag{16}$$

Given the deformation gradient update $\boldsymbol{X}_{n+1}$, we perform the polar and eigen-decompositions

$$\boldsymbol{X}_{n+1} = \boldsymbol{R}_{n+1}\boldsymbol{R}_{L,n+1}\boldsymbol{\Lambda}_{n+1}\boldsymbol{R}^T_{L,n+1}\,, \tag{17}$$

and evaluate the total strain update $\boldsymbol{E}_{n+1}$,

$$\boldsymbol{E}_{n+1} = \boldsymbol{R}_{L,n+1}\ln\boldsymbol{\Lambda}_{n+1}\boldsymbol{R}^T_{L,n+1}\,. \tag{18}$$

Then we define a "trial elastic strain" by

$$\boldsymbol{E}^e_* = \boldsymbol{E}_{n+1} - \boldsymbol{E}^p_n\,, \tag{19}$$

solving for $\boldsymbol{E}^p_n$ and substituting in (15) we have

$$\boldsymbol{E}^e_* - \boldsymbol{E}^e_{n+1} = \Delta t\,\dot{\boldsymbol{E}}^p_{n+1}\,, \tag{20}$$

where

$$\boldsymbol{E}^e_{n+1} = \boldsymbol{E}_{n+1} - \boldsymbol{E}^p_{n+1}\,. \tag{21}$$

Using the stress-strain law (1) in (20),

$$\boldsymbol{S}_* - \boldsymbol{S}_{n+1} = \boldsymbol{\mathcal{L}}[\Delta t\, \dot{\boldsymbol{E}}^p_{n+1}]\,, \tag{22}$$

where $\boldsymbol{S}_* = \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e_*]$ is the "trial elastic stress".

A.2.2 The effective stress function

Substituting (16) into equation (12),

$$\Delta t \dot{\boldsymbol{E}}^p_{n+1} = \alpha \frac{3(s_{n+1} - s_n)}{2h_{n+1} s_{n+1}} \boldsymbol{S}'_{n+1}\,, \tag{23}$$

where

$$h_{n+1} = \bar{h}(s_{n+1})\,. \tag{24}$$

Using equation (23), and recalling the definition of the elastic moduli tensor (2) we obtain

$$\boldsymbol{\mathcal{L}}[\Delta t\, \dot{\boldsymbol{E}}^p_{n+1}] = 2\mu \Delta t\, \dot{\boldsymbol{E}}^p_{n+1} = 3\mu\alpha \frac{s_{n+1} - s_n}{h_{n+1} s_{n+1}} \boldsymbol{S}'_{n+1}\,, \tag{25}$$

from where equation (22) is equivalent to

$$\boldsymbol{S}_* = \boldsymbol{S}_{n+1} + 3\mu\alpha \frac{s_{n+1} - s_n}{h_{n+1} s_{n+1}} \boldsymbol{S}'_{n+1}\,. \tag{26}$$

Equation (26) implies that

$$\mathrm{tr}\,(\boldsymbol{S}_*) = \mathrm{tr}\,(\boldsymbol{S}_{n+1})\,, \tag{27}$$

and for the deviatoric parts,

$$\boldsymbol{S}'_* = \left[1 + 3\mu\alpha\frac{s_{n+1} - s_n}{h_{n+1}s_{n+1}}\right]\boldsymbol{S}'_{n+1}\,. \tag{28}$$

Taking dot product of equation (28) with itself,

$$\boldsymbol{S}'_* \cdot \boldsymbol{S}'_* = \left[1 + 3\mu\alpha\frac{s_{n+1} - s_n}{h_{n+1}s_{n+1}}\right]^2 \boldsymbol{S}'_{n+1} \cdot \boldsymbol{S}'_{n+1}\,, \tag{29}$$

recalling definition (5) of the equivalent tensile stress, and defining an "trial equivalent tensile stress" by

$$s_* = \sqrt{\tfrac{3}{2}\,\boldsymbol{S}'_* \cdot \boldsymbol{S}'_*}\,, \tag{30}$$

we can rewrite (29) as

$$s_*^2 = \left[1 + 3\mu\alpha\frac{s_{n+1} - s_n}{h_{n+1}s_{n+1}}\right]^2 s_{n+1}^2\,, \tag{31}$$

or equivalently,

$$f(s_{n+1}) = (s_{n+1} - s_*)\bar{h}(s_{n+1}) + 3\mu\alpha(s_{n+1} - s_n) = 0\,, \tag{32}$$

where $f(s_{n+1})$ is the "effective stress function" (Kojić and Bathe [1987]).

Note that if $s_* < \sigma_n$, then the process is elastic, $\alpha = 0$ and $s_{n+1} = s_*$. The updated stress-strain state is equal to the trial elastic state.

A.2.3 Stress and strain updates

If $s_* \geq \sigma_n$ we consider the process plastic, $\alpha = 1$ and s_{n+1} is obtained by solving

$$f(s_{n+1}) = 0\,. \tag{33}$$

Once s_{n+1} is obtained, we compute

$$\sigma_{n+1} = s_{n+1}\,, \tag{34}$$

$$\beta_{n+1} = 3\mu \frac{s_{n+1} - s_n}{s_{n+1}\bar{h}(s_{n+1})}\,, \tag{35}$$

and from (28),

$$\boldsymbol{S}'_{n+1} = (1 + \beta_{n+1})^{-1}\boldsymbol{S}'_{*}\,. \tag{36}$$

Combining (27) with (36) we obtain for the stress update

$$\boldsymbol{S}_{n+1} = \boldsymbol{S}'_{n+1} + \tfrac{1}{3}\,\mathrm{tr}\,(\boldsymbol{S}_{*})\boldsymbol{1}\,. \tag{37}$$

The elastic strain update is then given by

$$\boldsymbol{E}^{e}_{n+1} = \boldsymbol{\mathcal{L}}^{-1}[\boldsymbol{S}_{n+1}]\,. \tag{38}$$

and the plastic strain tensor update is obtained by taking the difference

$$\boldsymbol{E}^{p}_{n+1} = \boldsymbol{E}_{n+1} - \boldsymbol{E}^{e}_{n+1}\,. \tag{39}$$

To obtain the Cauchy stress update, we compute

$$J_{n+1} = \det(\boldsymbol{X}_{n+1}) = \det(\boldsymbol{\Lambda}_{n+1})\,, \tag{40}$$

and from (4.54)

$$\boldsymbol{S}_{L,n+1} = \boldsymbol{R}^{T}_{L,n+1}\boldsymbol{S}_{n+1}\boldsymbol{R}_{L,n+1}\,. \tag{41}$$

Use can be made now of equation (4.84),

$$[\boldsymbol{T}_E]_{\alpha\beta} = \begin{cases} J_{n+1}^{-1}[\boldsymbol{S}_{L,n+1}]_{\alpha\alpha}\,, & \text{if } \beta = \alpha; \\ 2J_{n+1}^{-1}\dfrac{\ln(\lambda_{\beta,n+1}) - \ln(\lambda_{\alpha,n+1})}{\lambda_{\beta,n+1}\lambda_{\alpha,n+1}^{-1} - \lambda_{\alpha,n+1}\lambda_{\beta,n+1}^{-1}}[\boldsymbol{S}_{L,n+1}]_{\alpha\beta}\,, & \text{otherwise,} \end{cases}$$

from where

$$\boldsymbol{T}_{n+1} = \boldsymbol{R}_{E,n+1}\boldsymbol{T}_{E,n+1}\boldsymbol{R}_{E,n+1}^T\,. \tag{42}$$

In this equation,

$$\boldsymbol{R}_{E,n+1} = \boldsymbol{R}_{n+1}\boldsymbol{R}_{L,n+1}\,. \tag{43}$$

A.2.4 Summary

The time integration algorithm presented in this section requires the following steps.

1. Perform the polar and eigen decompositions

$$\boldsymbol{X}_{n+1} = \boldsymbol{R}_{n+1}\boldsymbol{R}_{L,n+1}\boldsymbol{\Lambda}_{n+1}\boldsymbol{R}_{L,n+1}^T\,. \tag{44}$$

2. Obtain the total strain update

$$\boldsymbol{E}_{n+1} = \boldsymbol{R}_{L,n+1}\ln\boldsymbol{\Lambda}_{n+1}\boldsymbol{R}_{L,n+1}^T\,. \tag{45}$$

3. Obtain the trial elastic strain and stress tensors

$$\boldsymbol{E}_*^e = \boldsymbol{E}_{n+1} - \boldsymbol{E}_n^p\,, \tag{46}$$

$$\boldsymbol{S}_* = \boldsymbol{\mathcal{L}}[\boldsymbol{E}^e_*]\,. \tag{47}$$

4. Decompose the trial elastic stress into its volumetric and deviatoric parts

$$\boldsymbol{S}_* = \boldsymbol{S}_*{}' + \tfrac{1}{3}\,(\operatorname{tr}\boldsymbol{S}_*)\boldsymbol{1}\,. \tag{48}$$

5. Obtain the trial equivalent tensile stress

$$s_* = \sqrt{\tfrac{3}{2}\,\boldsymbol{S}'_* \cdot \boldsymbol{S}'_*}\,. \tag{49}$$

6. If $s_* < \sigma_n$ then the process is elastic and

$$s_{n+1} = s_*\,, \tag{50}$$

$$\sigma_{n+1} = \sigma_n\,, \tag{51}$$

$$\boldsymbol{E}^e_{n+1} = \boldsymbol{E}^e_*\,, \tag{52}$$

$$\boldsymbol{S}_{n+1} = \boldsymbol{S}_*\,, \tag{53}$$

$$\boldsymbol{T}_{n+1} = J^{-1}_{n+1}\boldsymbol{R}^e_*\boldsymbol{S}_*(\boldsymbol{R}^e_*)^T\,, \tag{54}$$

$$\text{EXIT}\,. \tag{55}$$

Else, the process is elasto-plastic. Obtain s_{n+1} by solving

$$(s_{n+1} - s_*)\bar{h}(s_{n+1}) + 3\mu(s_{n+1} - s_n) = 0\,. \tag{56}$$

7. Update the deformation resistance

$$\sigma_{n+1} = s_{n+1}\,. \tag{57}$$

8. Calculate the constant β_{n+1} ,

$$\beta_{n+1} = 3\mu \frac{s_{n+1} - s_n}{s_{n+1}\bar{h}(s_{n+1})} . \tag{58}$$

9. Update the stress and elastic and plastic strain tensors

$$\boldsymbol{S}_{n+1} = (1 + \beta_{n+1})^{-1}\boldsymbol{S}_*{}' + \tfrac{1}{3}\mathrm{tr}\,(\boldsymbol{S}_*)\boldsymbol{1} , \tag{59}$$

$$\boldsymbol{E}^e_{n+1} = \boldsymbol{\mathcal{L}}^{-1}[\boldsymbol{S}_{n+1}] , \tag{60}$$

$$\boldsymbol{E}^p_{n+1} = \boldsymbol{E}_{n+1} - \boldsymbol{E}^e_{n+1} . \tag{61}$$

10. Update the Cauchy stress tensor

$$\boldsymbol{S}_{L,n+1} = \boldsymbol{R}^T_{L,n+1}\boldsymbol{S}_{n+1}\boldsymbol{R}_{L,n+1} , \tag{62}$$

$$[\boldsymbol{T}_E]_{\alpha\beta} = \begin{cases} J^{-1}_{n+1}[\boldsymbol{S}_{L,n+1}]_{\alpha\alpha} , & \text{if } \beta = \alpha; \\ 2J^{-1}_{n+1}\dfrac{\ln(\lambda_{\beta,n+1}) - \ln(\lambda_{\alpha,n+1})}{\lambda_{\beta,n+1}\lambda^{-1}_{\alpha,n+1} - \lambda_{\alpha,n+1}\lambda^{-1}_{\beta,n+1}}[\boldsymbol{S}_{L,n+1}]_{\alpha\beta} , & \text{otherwise,} \end{cases} \tag{63}$$

$$\boldsymbol{R}_{E,n+1} = \boldsymbol{R}_{n+1}\boldsymbol{R}_{L,n+1} , \tag{64}$$

$$\boldsymbol{T}_{n+1} = \boldsymbol{R}_{E,n+1}\boldsymbol{T}_{E,n+1}\boldsymbol{R}^T_{E,n+1} . \tag{65}$$

References

1963 COLEMAN, B.D. and NOLL, W., "The thermodynamics of elastic materials with heat conduction and viscosity," *Arch. Rat. Mech. Anal.*, Vol. 13, pp. 245–261.

1964 WILKINS, M.L., "Calculation of Elastic-Plastic Flow," in ALDER *et al.* (eds.), Methods in Computational Physics, 3, Academic Press, New York.

1965 GREEN, A.E. and NAGHDI, P.M., "A general theory of an elastic-plastic continuum," *Arch. Rat. Mech. Anal.*, Vol. 18, pp. 251–281.

1965 TRUESDEL, C. and NOLL, W., "The Non-Linear Field Theories of Mechanics", Encyclopedia of Physics, Vol. III/3, Springer, Berlin.

1966 PERZYNA, P. and WOJNO, W., "On the constitutive equations elastic/visco-plastic materials at finite strain," *Archwm. Mech. Stosow.*, Vol. 18, pp 85–100.

1967 COLEMAN, B.D. and GURTIN, M.E., "Thermodynamics With Internal State Variables," *J. Chem. Phys.*, Vol. 47, pp. 597–613.

1968 HILL, R., "On constitutive inequalities for simple materials–II," *J. Mech. Phys. Solids*, Vol. 16, pp. 315–322.

1968 PERZYNA, P. and WOJNO, W., "Thermodynamics of a rate sensitive plastic material," *Archwm. Mech. Stosow.*, Vol. 20, pp. 499–510.

1969 LEE, E.H., "Elastic-Plastic Deformations at Finite Strains," *J. Appl. Mech.*, Vol. 36, pp. 1–6.

1971 RICE, J.R., "Inelastic Constitutive Relations for Solids: An Internal-Variable Theory and Its Applications to Metal Plasticity," *J. Mech. Phys. Solids*, Vol. 19, pp. 433–455.

1973a MANDEL, J., "Équations Constitutives et Directeurs dans les Milieux Plastiques et Viscoplastiques," *Int. J. Solids and Structures*, Vol. 9, pp. 725–740.

1973b MANDEL, J., "Thermodynamics and Plasticity," in J. J. DELGADO DOMINGOS, M.N.R. NINA, AND J.H. WHITELAW (eds.), Foundations of Continuum Thermodynamics, Wiley, New York, pp. 283–304.

1974 HAHN, H.T., "A Finite Deformation Theory of Plasticity," *Int. J. Solids Struct.*, Vol. 10, pp. 111–121.

1974 HIRSCH, M.W. and SMALE, S., "Differential Equations, Dynamical Systems and Linear Algebra," New York: Academic Press.

1976 KRIEG, R.D., and KEY, S.W., "Implementation of a Time-Independent Plasticity Theory into Structural Computer Programs," in STRICKLIN, J.A., and SACZALSKI, K.J. (eds.), Constitutive Equations in Viscoplasticity: Computational and Engineering Aspects, ASME AMD-20, pp. 125–137.

1977 KRIEG, R.D., and KRIEG, D.B., "Accuracies of Numerical Solution Methods for Elastic-Perfectly Plastic Model," ASME Journal of Pressure Vessel Technology, 99, 510.

1980 ANAND, L., "Constitutive Equations for Rate-Independent, Isotropic, Elastic-Plastic Solids Exhibiting Pressure-Sensitive Yielding and Plastic Dilatancy," *J. Appl. Mech.,* Vol. 47, 439.

1981 GURTIN, M.E., "An Introduction to Continuum Mechanics," Mathematics in Science and Engineering, Vol. 158, 1981, Academic Press.

1984 ROLPH III, D.W., and BATHE, K.J., "On a Large Strain Finite Element Formulation for Elasto-Plastic Analysis," in WILLAM, K. (ed.), Constitutive Equations: Macro and Computational Aspects, ASME, AMD.

1985 ANAND, L., "Constitutive Equations for Hot-Working of Metals," *Int. J. Plasticity*, Vol. 1, pp. 213.

1985 SIMO, J.C., and ORTIZ, M., "A unified approach to finite deformation elastoplasticity based on the use of hyperelastic constitutive relations," *Comp. Meth. Appl. Mech. Engng.,* Vol. 49, pp. 221.

1987 ADINA—A finite element program for automatic dynamic incremental nonlinear analysis. Report ARD 87-1, ADINA R&D, Inc., Watertown, MA.

1987 KOJIĆ, M., and BATHE, K.J., "The 'Effective-Stress-Function' Algorithm for Thermo-Elasto-Plasticity and Creep," *Int. J. Numer. Meths. Eng.*, Vol. 24, pp. 1509–1532.

1988 WEBER, G., Ph.D. thesis, Massachusetts Institute of Technology.

1988 WEBER, G. and ANAND, L., personal communication.

1988a SIMO, J.C., "A Framework For Finite Strain Elastoplasticity Based on Maximum Plastic Dissipation and the Multiplicative Decomposition: Part I. Continuum Formulation," *Comput. Meths. Appl. Mech. Engrg.*, Vol. 66, pp. 199–219.

1988b SIMO, J.C., "A Framework For Finite Strain Elastoplasticity Based on Maximum Plastic Dissipation and the Multiplicative Decomposition: Part II. Computational Aspects," *Comput. Meths. Appl. Mech. Engrg.*, Vol. 68, pp. 1–31.