PHYSICAL REVIEW E **92**, 022136 (2015)

Lower-critical spin-glass dimension from 23 sequenced hierarchical models

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(Received 19 December 2014; revised manuscript received 11 July 2015; published 24 August 2015)

The lower-critical dimension for the existence of the Ising spin-glass phase is calculated, numerically exactly, as $d_L = 2.520$ for a family of hierarchical lattices, from an essentially exact (correlation coefficent $R^2 = 0.999\,999$) near-linear fit to 23 different diminishing fractional dimensions. To obtain this result, the phase transition temperature between the disordered and spin-glass phases, the corresponding critical exponent y_T, and the runaway exponent y_R of the spin-glass phase are calculated for consecutive hierarchical lattices as dimension is lowered.

DOI: 10.1103/PhysRevE.92.022136 PACS number(s): 64.60.De, 75.10.Nr, 05.10.Cc, 75.50.Lk

I. INTRODUCTION

Singular phase diagram behavior as a function of spatial dimensionality d compounds the interest and challenge of the phase transitions problems, as effectively posing the "phase transition of phase transitions" problem. Most visible are the lower-critical dimensions, which are the spatial dimensional thresholds for different types of orderings. For example, the lower-critical threshold for ferromagnetic ordering in magnetic systems is $d_L = 1$ for one-component (Ising) spins and $d_L = 2$ for spins with more than one component. Similarly, the lower-critical dimensions for ferromagnetic ordering under quenched random fields [1–7] are respectively $d_L = 2$ and $d_L = 4$ for one-component spins and for spins with more than one component. The method that we use in this study gives correctly the lower-critical dimensions of the Ising and q-state Potts models ($d_L = 1$), of the ($n > 1$)-component vector spin models ($d_L = 2$), of the Ising model with quenched random fields ($d_L = 2$), as well as the algebraic order of the XY model at its lower-critical dimension $d_L = 2$ [7–13].

On the question of the spin-glass lower-critical dimension, $d_L = 2.5$ was obtained from replica symmetry-breaking mean-field theory[14]. Renormalization-group work, on a family of hierarchical lattices different from ours below, has found d_L close to 2.5 [15]. Extrapolation to lowest d of a high-d expansion on this family of hierarchical lattices has yielded $d_L = 2.504$ [16]. Detailed numerical fit to the spin-glass critical temperatures for integer dimensions has also suggested $d_L = 2.5$ [17].

In other theory, early renormalization-group work [18], on in effect two hierarchical lattices again different from ours below, has obtained $2 < d_L < 3$. Other theoretical works have claimed $d_L = 4$ from ordered-phase stability studies [19–21], $2 < d_L < 3$ from transfer-matrix studies [22], and $d_L = 2$ from Monte Carlo [23,24] and ground-state studies [25]. A very recent experimental study [26] on Ge:Mn films has shown the spin-glass lower-critical dimension to be $2 < d_L < 3$.

As seen above, the lower-critical dimension need not be integer, in view of physical fractal systems, hierarchical lattices, and algebraic manipulations that analytically continue. To our knowledge, spin-glass ordering is the only system that exhibits this behavior. In fact, it is of interest to find the exact value of the lower-critical dimension. Our current work does this for the case of the family of hierarchical lattices studied here, with $d_L = 2.520$. We obtain this result from a remarkably good fit (correlation coefficient $R^2 = 0.999\,999$) to the renormalization-group runaway exponent y_R from the numerically exact renormalization-group solution of a family of 23 hierarchical models with noninteger dimensions $d =$ 2.46, 2.63, 2.77, 2.89, 3.00, 3.10, 3.18, 3.26, 3.33, 3.40, 3.46, 3.52, 3.58, 3.63, 3.68, 3.72, 3.77, 3.81, 3.85, 3.89, 3.93, 3.97, 4.00. Our result is also consistent with the results that are graphically displayed in Ref. [15] and with the extrapolation to lowest d of a high-d expansion in Ref. [16] for a *different* family of hierarchical lattices. In fact, the comparison and coincidence of spin-glass lower-critical dimensions from different families of hierarchical lattices, started here, is of continuing interest.

II. LOWER-CRITICAL DIMENSION FROM SEQUENCED HIERARCHICAL MODELS

Hierarchical models are constructed [27–31] by imbedding a graph into a bond, as examplified in Fig. 1, and repeating this procedure by self-imbedding infinitely many times. This procedure can also be done on units with more than two external vertices, e.g., the layered Sierpinski gasket in Ref. [32]. When interacting systems are placed on hierarchical lattices, their renormalization-group solution proceeds in the reverse direction than the lattice build-up just described, each eliminated elementary graph generating a renormalized interaction strength for the ensuing elementary bond. Hierarchical lattices were originally introduced [27] as presenting exactly soluble models with renormalization-group recursion relations that are identical to those found in approximate position-space renormalization-group treatments of Euclidian lattices [8,9], identifying the latter as physically realizable approximations. However, from the above, it is clear that any graph (or graphs [31]) may be chosen in the self-imbedding procedure and one need not be faithful to any approximate renormalization-group solution. Hierarchical lattices [33–70] have been used to study a variety of spin-glass [71] and other statistical mechanics problems.

The length rescaling factor b in a hierarchical lattice is the number of bonds on the shortest distance between the external vertices of the elementary graph which is replaced by a single

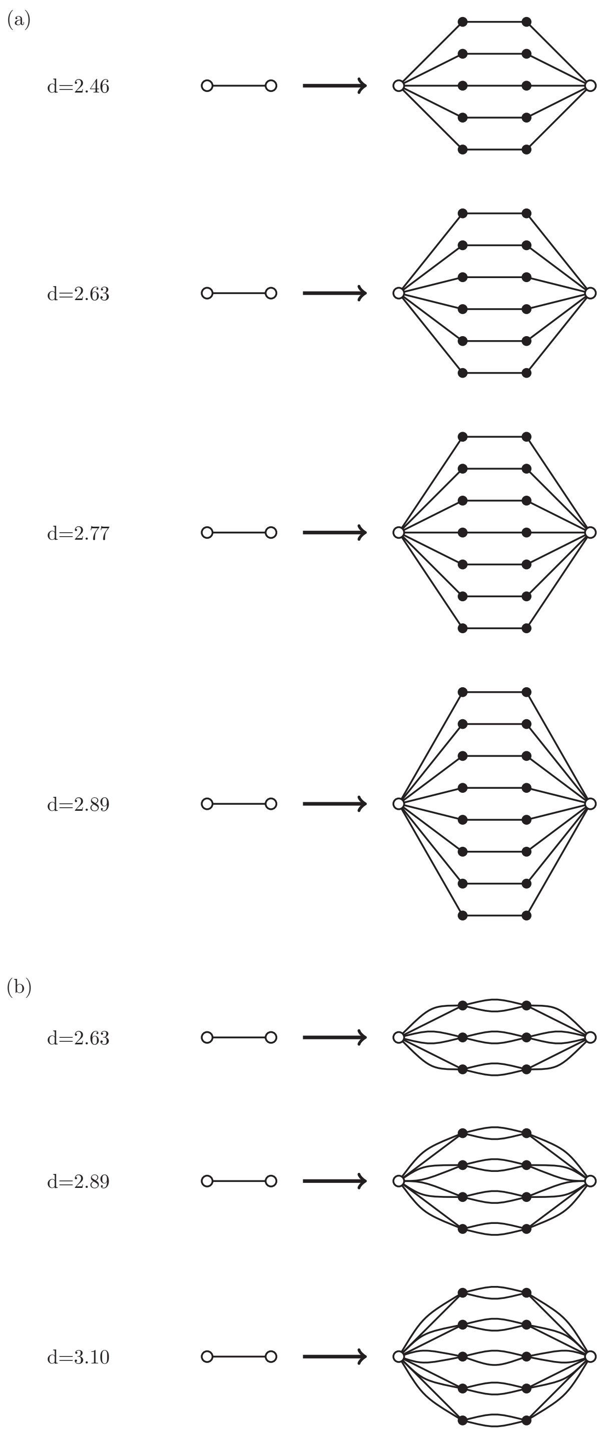

FIG. 1. (a) The construction of the family of hierarchical lattices used in this study. Each lattice is constructed by repeatedly self-imbedding the graph. The graphs here are n parallel series of $b = 3$ bonds. The dimension $d = 1 + \ln n / \ln b$ of each lattice is given. The renormalization-group solution consists in implementing this process in the reverse direction for the derivation of the recursion relations of the local interactions. The lattices shown here and 19 other lattices with nearby fractional dimensions are used in our calculations. (b) The family of hierarchical lattices with n_1 parallel $b = 3$ series of n_2 parallel bonds. The resulting hierarchical models are equivalent to the family in (a) with $n = n_1 n_2$, with respect to identical critical exponents and phase diagram topology including the occurrence or nonoccurrence of a spin-glass phase.

bond in one scale change. The volume rescaling factor b^d is the number of bonds inside the elementary graph. From these two rescaling factors, the dimensionality d is extracted, as exemplified in Fig. 1. In our study, $b = 3$ is used in order to treate the ferromagnetic and antiferromagnetic correlations on equal footing. The lower-critical dimension of spin-glass systems is studied here by considering a systematic family of hierarchical lattices in all its possible decreasing dimensions.

III. THE SPIN-GLASS SYSTEM AND THE RENORMALIZATION-GROUP METHOD

The Ising spin-glass system is defined by the Hamiltonian

$$-\beta\mathcal{H} = \sum_{\langle ij\rangle} J_{ij} s_i s_j, \tag{1}$$

where $\beta = 1/kT$, at each site i of a lattice the spin $s_i = \pm 1$, and $\langle ij\rangle$ denotes that the sum runs over all nearest-neighbor pairs of sites. The bond strengths J_{ij} are $+J > 0$ (ferromagnetic) with probability $1 - p$ and $-J$ (antiferromagnetic) with probability p.

The renormalization-group transformation is achieved by a decimation,

$$e^{J_{im}^{(\mathrm{dec})} s_i s_m + G_{im}} = \sum_{s_j, s_k} e^{J_{ij} s_i s_j + J_{jk} s_j s_k + J_{km} s_k s_m}, \tag{2}$$

where the additive constants G_{ij} are unavoidably generated, followed by n bond movings,

$$J_{ij}^{(bm)} = \sum_{k=1}^{n} J_{ij}^{(k)}. \tag{3}$$

The starting bimodal quenched probability distribution of the interactions, characterized by p and described above, is not conserved under rescaling. The renormalized quenched probability distribution of the interactions is obtained by the convolution [72]

$$P'(J'_{i'j'}) = \int \left[\prod_{ij}^{i'j'} dJ_{ij} P(J_{ij}) \right] \delta(J'_{i'j'} - R(\{J_{ij}\})), \tag{4}$$

where $R(\{J_{ij}\})$ represents the decimation and bond moving given in Eqs. (2) and (3). For numerical practicality, the bond moving and decimation of Eqs. (2) and (3) are achieved by a sequence of pairwise combinations of interactions, each pairwise combination leading to an intermediate probability distribution resulting from a pairwise convolution as in Eq. (4). The probability distribution is represented by 200 histograms [34,36,37,39,40,42], which are apportioned in $J \gtrless 0$ according to total probability weight. The histograms are distributed in the interval $J_+ \pm 2.5\sigma_+$, where J_+ and σ_+ are the average and standard deviation of the $J > 0$ interactions, and similarly for the $J < 0$ interactions.

The different thermodynamic phases of the system are identified by the different asymptotic renormalization-group flows of the quenched probability distributions. For all renormalization-group flows, inside the phases and on the phase boundaries, Eq. (4) is iterated until asymptotic behavior is reached. Thus, we are able to calculate phase transition temperatures and, by linearization around the unstable asymptotic fixed distribution of the phase boundaries, critical exponents.

Similar previous studies, on other spin-glass systems, are in Refs. [33–42].

IV. DIMINISHING CRITICAL, RUNAWAY EXPONENTS, CRITICAL TEMPERATURES AND THE LOWER-CRITICAL DIMENSION OF THE SEQUENCE

For our chosen sequence of hierarchical systems (Fig. 1), we have calculated, at antiferromagnetic bond concentration $p = 0.5$, the phase transition temperature $1/J_C$ where the renormalization-group flows bifurcate between the disordered-phase and the spin-glass-phase attractor sinks. The spin-glass sink is characterized by an interaction probability distribution $P(J_{ij})$ that is symmetric in ferromagnetism-antiferromagnetism ($J_{ij} \gtrless 0$) and that diverges in interaction absolute value: The average interaction strength $\langle |J| \rangle$ across the system diverges as $b^{n y_R}$ where n is the number of renormalization-group iterations and $y_R > 0$ is the runaway exponent. The spin-glass sink and simultaneously the spin-glass phase disappears when the runaway exponent y_R reaches 0 [42]. The calculated spin-glass phase transition temperatures and critical and runaway exponents are given in Fig. 2 and in Table I as a function of spatial dimension d. The lattice with $d = 2.46$, not having a spin-glass phase, is below the lower-critical dimension. For the 22 other consecutive lattices with a spin-glass phase, we have chosen to fit the runaway exponent values, since they gives an excellent, near-linear fit with

$$y_R = -1.309\,08 + 0.528\,513d - 0.003\,548\,05d^2, \qquad (5)$$

with an amazingly satisfactory correlation coefficient of $R^2 = 0.999\,999$. This fit gives, with a small extrapolation, $y_R = 0$ for $d = 2.520$. Note the near linearity, namely, the smallness of the quadratic coefficient in Eq. (5). (In fact, a linear fit gives $y_R = 0$ for $d = 2.516$, with a little less amazingly satisfactory correlation coefficient of $R^2 = 0.999\,992$.)

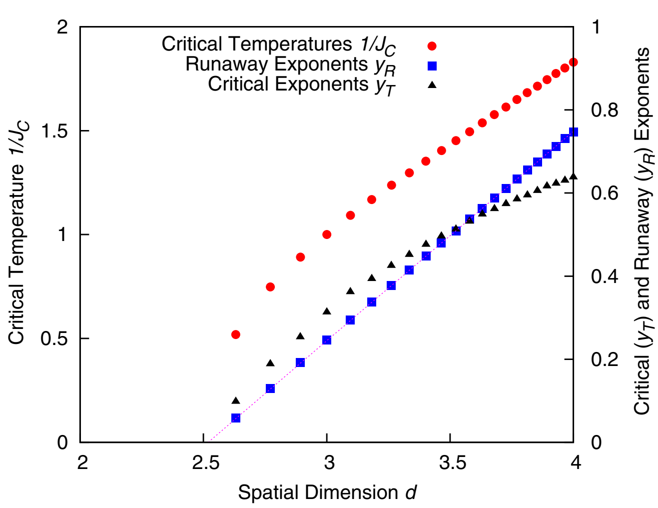

FIG. 2. (Color online) Critical temperatures $1/J_C$ and critical exponents y_T of the phase transitions between the spin-glass and paramagnetic phases as a function of dimension d, for the hierarchical models with antiferromagnetic bond concentration $p = 0.5$. The runaway exponents y_R of the spin-glass phase are also shown and give a perfect fit to $y_R = -1.309\,08 + 0.528\,513d - 0.003\,548\,05d^2$, leading with a small extrapolation to the lower-critical dimension $d = 2.520$ for $y_R = 0$, with a very satisfactory correlation coefficient of $R^2 = 0.999\,999$.

TABLE I. Critical temperatures $1/J_C$ and critical exponents y_T of the phase transitions between the spin-glass and paramagnetic phases as a function of dimension d, for the hierarchical models with antiferromagnetic bond concentration $p = 0.5$. The runaway exponents y_R of the spin-glass phase are also shown and give a perfect fit to $y_R = -1.309\,08 + 0.528\,513d - 0.003\,548\,05d^2$, leading with a small extrapolation to the lower-critical dimension $d = 2.520$ for $y_R = 0$, with a very satisfactory correlation coefficient of $R^2 = 0.999\,999$.

Spatial dimension d	Critical temperatures $1/J_C$	Critical exponents y_T	Runaway exponents y_R
2.630 930	0.519 268	0.098 077	0.058 731
2.771 244	0.747 982	0.188 596	0.129 983
2.892 789	0.890 503	0.253 690	0.191 904
3.000 000	1.001 319	0.313 414	0.246 144
3.095 903	1.091 770	0.361 975	0.294 649
3.182 658	1.168 653	0.393 837	0.338 155
3.261 860	1.237 723	0.425 397	0.377 881
3.334 718	1.298 225	0.451 743	0.414 440
3.402 174	1.354 258	0.476 199	0.448 214
3.464 974	1.404 661	0.495 999	0.479 850
3.523 719	1.452 817	0.513 016	0.509 181
3.578 902	1.496 452	0.531 699	0.536 880
3.630 930	1.538 271	0.549 022	0.563 149
3.680 144	1.577 300	0.562 079	0.587 707
3.726 833	1.613 844	0.573 932	0.610 941
3.771 244	1.649 036	0.585 283	0.633 434
3.813 588	1.682 659	0.594 959	0.654 932
3.854 050	1.714 417	0.605 789	0.675 080
3.892 789	1.745 469	0.616 496	0.693 914
3.929 947	1.774 176	0.623 179	0.712 461
3.965 647	1.802 906	0.630 527	0.730 927
4.000 000	1.829 792	0.638 313	0.747 294

Our calculated lower-critical dimension d_L, where the spin-glass phase disappears at zero temperature, is thus seen to be $d_L = 2.520$, for the sequence of hierarchical lattices studied here. It is noteworthy that d_L is not an integer, contrary to previous examples of lower-critical dimensions (and even contrary to upper-critical dimensions, where mean-field behavior sets in) for other systems.

Another important quantity is the critical exponent $y_T = 1/\nu > 0$ of the phase transition between the disordered and spin-glass phases. This exponent is calculated from the scaling behavior of small deviations of the average interaction strength from its fixed finite value at the unstable fixed distribution of the phase transition. The calculated critical exponents are also given in Fig. 2. As the spatial dimension is lowered, y_T also approaches 0. At the lower-critical dimension, y_T reaches 0. The disordered-spin-glass phase transition disappears at d_L, where the spin-glass phase disappears.

V. CONCLUSION

Our family of hierarchical lattices (Fig. 1) yields smooth and systematic behavior in all three quantities: the critical temperatures $1/J_C$, the critical exponents y_C, and, eminently fitably, the runaway exponents y_R. All three quantities yield the lower-critical temperature of $d_L = 2.520$. It is noteworthy that d_L is not an integer, contrary to previous examples of lower-critical dimensions (and even contrary to upper-critical dimensions, where mean-field behavior sets in) for other systems.

ACKNOWLEDGMENTS

We thank Prof. H. Nishimori for suggesting this calculation to us. Support by the Alexander von Humboldt Foundation, the Scientific and Technological Research Council of Turkey (TÜBITAK), and the Academy of Sciences of Turkey (TÜBA) is gratefully acknowledged.

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