HYDRODYNAMIC APPROXIMATION TO

TIME-DEPENDENT HARTREE-BOGOLYUBOV

by

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ABSTRACT

By means of a varitional method the TDHB equations have
been derived. The Wigner representation of TDHB equations has
been used for a hydrodynamic description of a nuclear system.
Restricted dynamical parametrization of the TDHB theory has been
considered. Various fluid models are developed and their small
density oscillations have been discussed. For a comparison between
hydrodynamic approach and microscopic consideration, a model for
neutron matter is introduced. The numerical calculation of QPRPA
and the hydrodynamical approach for phonon energies are compared.

Thesis Supervisor: Arthur Kerman

Title: Professor of Physics

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Chapter I

INTRODUCTION

Usually, a physicist's job is to explain naturally occurring

or man-made physical phenomena by as simple means as possible. In

nuclear physics, however, we are dealing with a system of a few

hundred particles interacting via a strong interaction potential.

Solving this system is quite complex in general. Many branches of

physics help us to understand some behaviour of this system. Due

to the short range of the interaction potential, many classical

ideas can be applied to describe the gross property of the nuclei.

Among successful models the liquid drop model should be mentioned.

In the static limit, the Weizsacker-Bethe[1][2] formula gives the overall

trend of the binding energy of a nucleus with mass and charge numbers.

The liquid model also served to explain the dynamics of the nuclei.

Here one introduces collective deformation coordinates and uses

classical equations of motion to treat the dynamics of the system.

The choice of the collective coordinates is more or less arbitrary

and is only guided by physical intuition and by the anticipation

that the effect of intrinsic motion become small compared to the

collective effects. Specific examples of collective coordinates

are, a) Bohr's surface parameters used in the collective model of
vibration states [3], b) elongation, necking-in, and asymmetry of a
strongly deformed nucleus in the hydrodynamical model of fission [4-5],
c) the displacements of neutrons and protons in the hydrodynamic
model of giant resonance [6-7] and, d) the relative distance and angle
between two nuclei in the classical description of heavy ion
reactions [8].

Behind the introduction of collective coordinates, there is
an assumption that many nucleons participate in the motion, and
the collective coordinate is an idealization of a general displacement.
With this point of view in mind we may conclude that intrinsic
motions are negligible. The dynamics of such a nuclear system follow
then from the classical equations of motion, where the collective
coordinates are empolyed as generalized coordinates. The potential
energy consists in general of volume, coulomb, and surface energy
contribution, now depending on generalized coordinates. The kinetic
energy is derived from similar arguments. For a small change from
equilibrium, it is assumed to be a quadratic expression in the
generalized velocities, where the masses or intertia with respect
to the coordinates have a generalized meaning and may themselves
be functions of the coordinates.

Although many classical ideas are successful in describing
the gross behaviour of nuclei, certainly our system for most of
the considerations of nuclear physics is a non-relativistic quantal
system. Many successful phenomenological models were developed
over the last thirty years, the shell models[9] and the collective
models[10] being among them. The shell model is based on the assumption
of a large mean free nucleon path, and it describes nuclei as a
collection of nucleons moving independently in well defind orbits.
There are actually so many phenomenological theories of collective
motion that a complete list of them would be difficult. It would
be safe to describe them as quantal versions of classical collective
motion

In fact the quantal system under consideration is truly a
many body system, and eventually one must find a microscopic
foundation for each of the successful phenomenological and hydro-
dynamical models. The motivation for this work is an attempt
in that direction. It has been known for decades[11], that the time
dependent schrodinger equation for a single particle can be cast
into fluid dynamical form with the phase and the square of the
modulus taking the roles of velocity potential and density, respectively.

Recently, with the advent of heavy ion accelerators, evidence of

hydrodynamical behaviour in nuclei has been enhanced, generating

renewed interest in the advancement and developement of the

hydrodynamic method. Recent formulations and applications of

fluid-dynamical methods are derived by different approaches.

One approach is based upon the direct use of the Schrodinger

equation,[12] another method utilizes the Wigner transformation,[13]

and a third approach is based on a classical interpretation of

the "Lagrangian".[14-15] All three approaches are based on assumption

that the time-dependent Hartree-Fock (TDHF) equation is a

valid description of nuclear motion. The present work is a

continuation of the second and third approaches, and here we

assume that the time dependent Hartree-Bogolyubov (TDHB)[16]

equations as a reasonable description of nuclear motion.

As mentioned above, the aim of this work is the exploration

of the hydrodynamic approximation in a many body system. In

the second chapter, a dynamical theory of pair correlations is

developed. Utilizing various methods, we derive the time dependent

Hartree-Bogolyubov equations in an arbitrary representation for a

general two body interaction. Different limits of TDHB equations,

notably the time dependent Bardeen-Cooper-Schrieffer[17] and the time

dependent Hartree-Fock[18] are discussed. A brief derivation of the

linearized TDHB equations is also included, in anticipation of the

Quasi-Particle Random-Phase-Approximation (QPRPA).

Chapter three contains the Koonin approach to the hydrodynamic interpretation in which we use the Wigner representation of the TDHB equations. We have a semiclassical interpretation for one of the TDHB equations of motion for the phase distribution function. It is easily recognizable as a quantal version of a modified Vlasov equation,[19] which approaches the expected classical result in the limit $\hbar \longrightarrow 0$ and where the number of particles is fixed. Also due to abandonment of a definite number of particles, we have an equation of motion for the deviation distribution function which identically vanishes for a system with a fixed number of particles.

In the fourth chapter, we discuss hydrodynamics of the system following the line of reasoning of Kerman-Koonin in the Lagrangian approach. The parametrization of the trial wave function enabled us to develop various fluid models. As a specific example, derivation of the Landau's theory of two fluid model from TDHB theory is one of our goals in this chapter. In the first method, we empoly a diagonalized form of TDHB density matrix, then include dynamics of the system through time dependence of the single particle wave function. We have a two fluid model Lagrangian, and from Hamilton's procedure the equations of motion has been derived. With approprite definition of the velocities for Irrotational and normal fluids, we derive a set of equations which has some similarity with equations of motion in Landau's theory. We

have an Irrotational superfluid and a normal fluid which contains the

vortex motion. The second and third models are also two fluid models:

in the second model we used a general TDHB trial wave function, and in

the third model our attention is focussed on coherent excitation of

two particles and two holes in the trial wave function. The resemblance

between these two model with Landau's theory is poor. The second

approach gives a simple set of equations of motion and in the third

approach, the equations of motion describing this model are a set of

integro-differential equations with no resemblance with Landau's equations

of motion. For each of these three models, density oscillation of fluids

near their equilibrium are described, various limits of the dispersion

relations are discussed, and dispersion relations for each of the models

relative to the others are compared. Finally, as an application, an

Irrotational fluid model similar to the Kerman-Koonin is developed.

A study of neutron matter is the subject of our discussion in the

fifth chapter of this work. We assume that neutrons are interacting via

soft-core ptentials.[20] The static part of the equations of motion are

solved. The energy gaps are evalauated for various densities and as a

function of wave number. For evaluation of the velocity of sound in

the system, we utilize the Irrotational fluid model of chapter four.

The energy density as a functional of the density is approximated by

its static solutions. Our numerical results for the velocity of sound

in the neutron matter are reasonable, considering the results of other

studies of nuclear matter.[21] In the last section of chapter five, we

derived the QPRPA utilizing the Generalized Hartree-Fock method,[22] and

applied it to our model of neutron matter. The numerical calculation of the QPRPA and the hydrodynamical approach for phonon energies are compared. The agreement between two approaches is poor, for exmple the hydrodynamic dispersion relation was linear in terms of wave number while its corresponding in QPRPA is a hyperbola. Also, there is disagreement for the phonon energy in terms of density between two approach. These disagreements between QPRPA and the corresponding hydrodynamics results enhanced the uncertainty in applicability of hydrodynamic approximations for the nuclear systems.

Chapter II

THE DYNAMICAL THEORY OF PAIR CORRELATIONS

In this chapter, we derive the time dependent Hartree-Bogolyubov
(TDHB) equation for an arbitary two body interaction. It is also
shown that the TDHB, in two different limits, is identical with the
time dependent Hartree-Fock (TDHF), and time dependent Bardeen-
Cooper-Schrieffer (TDBCS). The TDHB approximation furnishes a comput-
ationaly possible scheme for treating a system of interacting fermions,
reducing the many-body problem to a set of coupled one body problems.
The TDHB equations may be derived from at least two different
methods. The first method is the time evolution of the expectation
value of the operators $a_{\lambda}^{T} a_{\beta}$, $a_{\beta}^{T} a_{\lambda}^{T}$ and $a_{\lambda} a_{\beta}$, using the Schrodinger
equation for the evolution of the Hartree-Bogolyubov (HB) trial
wave function. In the second approach, one could have a classical
interpretation for the description of the system. Then the equation
of motion are determined by a least action principle. Alternatively,
the real and imaginary part of Bogolyubov matrix elements can be
viewed as momentums and coordinates of a constrained system of
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particles and their motions are derived by Hamilton's procedure

In section (2-1) we summarize beriefly the results and
discussion of the static theory of pair correlation. [24] Section
(2-2) deals with the derivation of the TDHB equations. In
section (2-3) we discuss some properties of the TDHB theory.
A brief derivation of the linearized TDHB equations is the
subject of the last section in this chapter.

2.1 The static theory of pair correlations.

The procedure of this section follows from Baranger's[24]
treatment of the theory of pair correlations. To treat the quantum
mechanics of a many-body system, it is convenient to use the
techniques of second quantiz-on.[25] For fermions, one introduces
a creation operator a_β^T which creates a particle in the single
particle state denoted by β (β constitues a complete label
for a state). The hermitian adjoint of the creation operator
is written as a_α (annihilation operator),which when acting to
the right, destroys a particle in the state α. The a_α^T and
a_β satisfy the following anticommutation relation:

$$\{ a_\alpha^T , a_\beta^T \} = \{ a_\alpha , a_\beta \} = 0 , \qquad (2-1)$$

$$\{ a_\alpha^T \ a_\beta \} = \delta_{\alpha\beta} . \qquad (2-2)$$

The particle vacuum state, $|0\rangle$, is defined by the property, that
it is annihilated by all the a_χ :

$$a_\chi |0\rangle = 0 \qquad\qquad (2\text{-}3)$$

i.e. it contains no particles. The state of the many-body system is
defined with respect to the vacuum state; for example a system with
a definite number of particles may be represented as a linear
combination of Slater determinant kets of the form:

$$a_\chi^T a_\beta^T - - - a_\ell^T |0\rangle. \qquad\qquad (2\text{-}4)$$

The second example is the BCS wave function:

$$|BCS\rangle = \prod_\alpha \left(u_\alpha + v_\alpha a_\alpha^T a_{\bar\alpha}^T \right) |0\rangle.$$

The product is over half the total number of states, the index $\bar\alpha$
represent the state which is paired with α .The third example
is the Blatt's [26] wave function:

$$|\phi_N\rangle = \frac{1}{N!} \left(\sum_{\alpha\beta} g_{\alpha\beta} \, a_\alpha^T a_\beta^T \right)^N |0\rangle \qquad\qquad (2\text{-}6)$$

where $g_{\alpha\beta}$ is an antisymmetric second order state tensor.

To elucidate some points in the future discussion of HB
and TDHB theory, it is necessary to repeat Baranger's observation
about the equivalance of Blatt's and BCS trial wave functions.

We start with the Bloch and Messiah[27] theorem, which states that

for any antisymmetric second order tensor, such as $\varphi_{\alpha\beta}$, there

exists a unitary transformation, U, such that the transformed

φ in canonical representation has the simple form shown in figure

[1]. In the new representation, $\varphi_{\alpha\beta}$ is non zero only if α and β

are paired. Therefore, Blatt's wave function in the new representation

can be written as

$$| \phi_N' \rangle = \frac{1}{N!} \left(\sum_{\alpha} \varphi_\alpha \, a_\alpha^T a_{\bar{\alpha}}^T \right)^N | 0 \rangle. \qquad (2\text{-}7)$$

Utilizing the Pauli principle, the above equation reduces to the

following form:

$$| \phi_N' \rangle = \prod_\alpha^N \phi_\alpha \, a_\alpha^T a_{\bar{\alpha}}^T \, | 0 \rangle , \qquad (2\text{-}8)$$

which is a single Slater determinate ket of N pairs. With a

change of normalization, one may rewrite the BCS wave function

in the following form:

$$| BCS \rangle = \prod_\alpha \left(1 + \frac{V_\alpha}{U_\alpha} \, a_\alpha^T a_{\bar{\alpha}}^T \right) | 0 \rangle , \qquad (2\text{-}9)$$

The projection of the BCS wave function, equation (2-9), on the

subspace of 2N (N pairs) is exactly

$$\prod_\alpha^N \frac{V_\alpha}{U_\alpha} \, a_\alpha^T a_{\bar{\alpha}}^T \, | 0 \rangle . \qquad (2\text{-}10)$$

Therefore from the start one sees the equivalence of the two wave functions

in the following sense. For a very large number of particles $(N \gg 1)$

the projected BCS wave function on the subspace of N pairs is equivlent

to the corresponding canonical Blatt's wave function, provided one sets

$$\phi_\eta = \frac{v_\eta}{v_\eta} . \qquad (2\text{-}11)$$

Let us consider a general Hamiltonian with one body and two

body terms

$$H = \sum_{\eta\gamma} T_{\eta\gamma} \, a_\eta^T a_\gamma + \sum_{\eta\beta\gamma\delta} \frac{1}{4} V_{\eta\beta\gamma\delta} \, a_\eta^T a_\beta^T a_\delta a_\gamma \qquad (2\text{-}12)$$

where, the coefficients T, V have the following symmetry properties:

$$T_{\eta\gamma} = T_{\gamma\eta}^{\kappa} , \qquad (2\text{-}13)$$

$$V_{\eta\beta\gamma\delta} = V_{\gamma\delta\eta\beta}^{\kappa} , \qquad (2\text{-}14)$$

$$V_{\eta\beta\gamma\delta} = -V_{\beta\eta\gamma\delta} = -V_{\eta\beta\delta\gamma} = V_{\beta\eta\delta\gamma} . \qquad (2\text{-}15)$$

The antisymmetric choice of the coefficients V means that the

exchange term is already included together with the direct term in

the interaction. In the usual derivation of HB equations, the ground

state of HB wave function $|\psi\rangle$ is defined by the property that it is

annihilated by all quasi-particle annihlation operators. Where the

quasi-particle operators are defined in the most general case as

linear combination of particle and hole operators. In this case from the start one no longer has a definite number of particles for the ground state, and the following expectation values are all nonzero:

$\langle \psi | a_\alpha^T a_\beta | \psi \rangle$, $\langle \psi | a_\alpha a_\beta | \psi \rangle$, and $\langle \psi | a_\alpha^T a_\beta^T | \psi \rangle$. But in Baranger's reformulation of HB theory the number of particles is kept fixed as long as possible and it is the result of mathematical approximations that things look at the end as though one had mixed the number of particles.

For a derivation of the HB equations from a variational point of view, one has to calculate the expectation value of Hamiltonian. Therefore it is necessary to calculate the expectation value of the matrix elements. Since the Blatt's trial wave function is not normalized, one has to calculate the norm, the one body and the two body matrix elements, given by

$$\langle \phi_N | \phi_N \rangle = \frac{1}{(N!)^2} \langle 0 | \left(\sum_{\alpha\beta} q_{\alpha\beta}^* a_\beta a_\alpha \right)^N \left(\sum_{\gamma\delta} q_{\gamma\delta} a_\gamma^T a_\delta^T \right)^N | 0 \rangle \qquad (2\text{-}16)$$

$$\langle \phi_N | a_\alpha^T a_\beta | \phi_N \rangle = \frac{1}{(N!)^2} \langle 0 | \left(\sum_{\gamma\delta} q_{\gamma\delta}^* a_\gamma a_\delta \right)^N a_\alpha^T a_\beta \left(\sum_{\mu\nu} q_{\mu\nu} a_\mu^T a_\nu^T \right)^N | 0 \rangle \qquad (2\text{-}17)$$

$$\langle \phi_N | a_\alpha^T a_\beta^T a_\delta a_\gamma | \phi_N \rangle = \frac{1}{(N!)^2} \langle 0 | \left(\sum_{\mu\nu} q_{\mu\nu}^* a_\nu a_\mu \right)^N \times$$

$$a_\alpha^T a_\beta^T a_\delta a_\gamma \left(\sum_{\eta\lambda} q_{\eta\lambda} a_\eta^T a_\lambda^T \right)^N | 0 \rangle. \qquad (2\text{-}18)$$

Baranger calculated the matrix element with a diagrammatic method, de
fin'ing a closed chain (figure [2]) and open chains (figure [3]and
figure [4]). Each chain has even number of lines, which are repre-
sentative of the contraction, and their label represents the state
involved. The wave function φ represented by white vertices, φ^*
stand for the black vertices. For example, the contribution of the
closed hexagonal chain in figure [2] is

$$- \varphi_{12} \, \varphi^*_{23} \, \varphi_{34} \, \varphi^*_{45} \, \varphi_{56} \, \varphi^*_{61} \, . \tag{2-19}$$

He defined R_c as an independent closed chain C, n_c as its order,
and m_c as the number of times which C occurs in the set. Then

$$\langle \phi_N | \phi_N \rangle = \sum \prod_C \left[\frac{(R_c)^{m_c}}{m_c!} \right] = F(2N) \, . \tag{2-20}$$

This being the definition of function F, the sum is over all possible
set of the closed chain such that

$$\sum_C m_c \, n_c = 2N \, . \tag{2-21}$$

$R^{(i)}_{\beta\alpha}$ is defined as the contribution of an open chain i with n_i as its
order. For example, the contribution of the chain in figure [3] would
be

$$R^{(i)}_{\beta\alpha} = - \varphi_{\beta 1} \, \varphi^*_{12} \, \varphi_{23} \, \varphi^*_{34} \, \varphi_{45} \, \varphi^*_{5\alpha} \, . \tag{2-22}$$

Then, the one body matrix element is

$$\langle \phi_N | a^T_\alpha a_\beta | \phi_N \rangle = \sum_i R^{(i)}_{\beta\alpha} \, F(2N - n_i) \, , \tag{2-23}$$

where the sum is over all possible open chains starting in β with a

$\mathcal{G}$ and ending in α with a $\mathcal{G}^{\alpha}$. For the two body matrix element, there

are two open chains, with three possible ways of forming them. there

are two possibilities of even chains like that of figure [3] , the

third possibility is that it may also be two odd chains as in figure [4] .

He defined $K_{ss}^{(i)}$ as the contribution of an odd chain i starting and ending

with a $\mathcal{G}$, the complex conjuate of k would be the contribution of

odd chain starting and ending with a $\mathcal{G}^{*}$. Thus, the two body matrix

element is

$$\langle \phi_N | a_\gamma^T a_\beta^T a_\delta a_\alpha | \phi_N \rangle = \sum_{i,j} \left(R_{\gamma\alpha}^i R_{\delta\beta}^j - R_{\delta\alpha}^i R_{\gamma\beta}^j + K_{\beta\alpha}^{*i} K_{\delta\gamma}^j \right) F(2N - n_i - n_j).$$

$$(2-24)$$

The main point in Baranger's argument is that the normalization

of $\mathcal{G}$ can be chosen in such a way that $F(2N)$ is approximately

independent of N for large N. With this approximation in mind

one may conclude that the F's in equations (2-20,2-23,2-24) can

be considered equal. Defining,

$$\rho_{\beta\alpha} = \sum_i R_{\beta\alpha}^{(i)},$$

$$(2-25)$$

$$K_{\beta\alpha} = \sum_i K_{\beta\alpha}^{(i)},$$

$$(2-26)$$

assuming the convergence of these series, one may derive the expectation

values of one and two body operators, the result being

$$\langle \phi_N | a_\alpha^T a_\beta | \phi_N \rangle / \langle \phi_N | \phi_N \rangle = \rho_{\beta\alpha} , \qquad (2-27)$$

$$\langle \phi_N | a_\alpha^T a_\beta^T a_\delta a_\gamma | \phi_N \rangle / \langle \phi_N | \phi_N \rangle = \rho_{\gamma\alpha} \rho_{\delta\beta} - \rho_{\delta\alpha} \rho_{\gamma\beta} + K_{\beta\alpha}^* K_{\delta\gamma} . \qquad (2-28)$$

This is identical to what one gets from other methods of HB theory.

Finally to complete the equivalence of digrammatic method with other

methods, one should derive supplementary condition diagrammatically,

and one obtains:

$$\rho_{\beta\alpha}^2 = - \sum_i \left(\frac{n_i}{2} - 1 \right) R_{\beta\alpha}^i , \qquad (2-29)$$

$$\left(K K^* \right)_{\beta\alpha} = - \sum_i \frac{n_i}{2} R_{\beta\alpha}^i , \qquad (2-30)$$

$$\left(\rho K \right)_{\beta\alpha} = - \sum_i \frac{1}{2}(n_i - 1) K_{\beta\alpha}^i , \qquad (2-31)$$

$$\left(K \rho^* \right)_{\beta\alpha} = - \sum_i \frac{1}{2}(n_i - 1) K_{\beta\alpha}^i . \qquad (2-32)$$

The above equations (2-29)-(2-32) are identical with the supplementary

conditions.

$$\rho_{\alpha\beta} = \rho_{\alpha\gamma} \rho_{\gamma\beta} - K_{\alpha\gamma} K_{\gamma\beta}^* , \qquad (2-33)$$

$$\rho_{\alpha\gamma} K_{\delta\beta} = K_{\alpha\delta} \rho_{\gamma\beta}^* , \qquad (2-34)$$

where we used the convention that repeated indices should be summed.

To prove the main point he rewrote the expression (2-20)

for $F(2N)$ in the following form:

$$F(2N) = \left(\frac{1}{2\pi i}\right) \oint z^{-2N-1} dz \sum_{c} \prod_{c} \left(z^{k_c} R_c\right)^{m_c}/m_c! \qquad (2-35)$$

The integral is on a contour enclosing the origin. Now one may

withraw the restriction of equation (2-21), that is to say the

sum includes all possible sets of closed chains. F(2N) may also

be written as

$$F(2N) = (2\pi i)^{-1} \oint z^{-2N-1} dz \exp\left(\sum_c z^{k_c} R_c\right)$$

$$= (2\pi i)^{-1} \oint z^{-1} dz \exp\left(\sum_c z^{k_c} R_c - 2N \ln z\right) \qquad (2-36)$$

For a slightly differnt number of particle we shall write the integral

as

$$F(2N-n) = (2\pi i)^{-1} \oint z^{n-1} dz \exp\left(\sum_c z^{k_c} R_c - 2N \ln z\right) \qquad (2-37)$$

Since z^{n-1} is a slow varing function of z relative to the exponential,

one may calculate F(2N-n) by the steepest descent method. The

saddle point z_s can be found by setting the derivative of the exponent

equal to zero

$$\sum_c k_c z^{k_c-1} R_c - 2N/z_s = 0 \qquad (2-38)$$

We may assume the normalization of $\oint$ is such that the saddle point

does occur at $\zeta=1$. Then, one obtains

$$\sum_c n_c R_c = 2N \qquad\qquad (2\text{-}39)$$

which is the supplementary condition equation (2-21).

For HB theory he improved the derivation along the line of Bayman's argument in reference 28. Define

$$\phi(z) = \sum_N z^N \mid \phi_N \rangle , \qquad\qquad (2\text{-}40)$$

one can then write

$$\langle \phi(z) \mid \phi(z) \rangle = \sum_N z^{2N} \langle \phi_N \mid \phi_N \rangle = \sum_N z^{2N} F(2N)$$

$$= \exp \sum_c z^n R_c \equiv F(z) . \qquad\qquad (2\text{-}41)$$

Similarly,

$$\langle \phi(z) \mid a_\alpha^T a_\beta \mid \phi(z) \rangle = \sum_N z^{2N} \langle \phi_N \mid a_\alpha^T a_\beta \mid \phi_N \rangle = \sum_N z^{2N} R_{\beta\alpha}^i F(2N-n_i)$$

$$= \sum_{m,n_i} z^m F(m) R_{\beta\alpha}^i z^{n_i}$$

$$= R_{\beta\alpha}(z) F(z) \qquad\qquad (2\text{-}42)$$

where

$$R_{\beta\alpha}(z) = \sum_{n_i} z^{n_i} R_{\beta\alpha}^{(i)} . \qquad\qquad (2\text{-}43)$$

Similar expression holds for the two body matrix element,

$$\langle \phi(z) \mid a_\alpha^T a_\beta^T a_\delta a_\gamma \mid \phi(z) \rangle = \sum_N z^{2N} \langle \phi_N \mid a_\alpha^T a_\beta^T a_\delta a_\gamma \mid \phi_N \rangle$$

$$= \sum_{N} z^{2N} F(3N - n_i - n_j) \left[R^i_{\gamma\alpha} R^j_{\delta\beta} - R^i_{\delta\alpha} R^j_{\gamma\beta} + k^i_{\beta\alpha} k^j_{\delta\gamma} \right]$$

$$= F(z) \left(R_{\gamma\alpha}(z) R_{\delta\beta}(z) - R_{\delta\alpha}(z) R_{\gamma\beta}(z) + K_{\beta\alpha}(z) K_{\delta\gamma}(z) \right)$$

$$(2-44)$$

where

$$k_{\beta\alpha}(z) = \sum_{n_i} z^{n_i} k^i_{\beta\alpha} .$$

$$(2-45)$$

By introducing $\phi(z)$ in equation (2-40), we abandon the requirement of equation (2-20) for evaluating the norm and the matrix elements. In other words the total number of different chains (closed or open) is infinite. One may assume the convergence of the above series for small $|z|$, and the analytic continuation to the saddle point without difficutly. Define

$$\chi_{\alpha\beta} = - \phi_{\delta\gamma} \phi_{\gamma\beta} ,$$

$$(2-46)$$

then for the equation (2-46) one obtains

$$R(z) = z^2 \chi - z^4 \chi^2 + z^6 \chi^3 - \underline{\quad} .$$

$$(2-47)$$

Because, the sum involves all possible even chains built up of all possible intermediate states, equation (2-47) can also be written in matrix form:

$$R(z) = z^2 \chi \left(1 + z^2 \chi \right)^{-1} .$$

$$(2-48)$$

And for the saddle point z=z.=1, we get

$$\mathcal{S} = \chi(1+\chi)^{-1} \qquad\qquad (2\text{-}49)$$

From the definition of κ and $\mathcal{S}$ as sums of odd and even chains, respectively, one finds that κ is related to $\mathcal{P}$ and $\mathcal{S}$ by

$$\kappa = (\mathcal{P}-1)\,\mathcal{S} = \mathcal{P}(\mathcal{P}^{\times}-1) \qquad\qquad (2\text{-}50a)$$

$$\kappa = -(1+\chi)^{-1}\mathcal{P} = -\mathcal{P}(1+\chi^{\times})^{-1}. \qquad\qquad (2\text{-}50b)$$

Where equation (2-50-b) derived by inserting the equivalent of $\mathcal{P}$ from equation (2-49).

Let us take HB trial wave function as

$$|\phi_{HB}\rangle = \frac{|\phi(z)\rangle}{(\langle\phi(z)|\phi(z)\rangle)^{1/2}}\Bigg|_{z=z_{*}=1} \qquad\qquad (2\text{-}51)$$

Then, one could easily see that, for HB theory, $\mathcal{P}_{\alpha\beta}$, $\kappa^{\alpha}_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ are the expectation values of the density operator, the two particle creation operator, and the two particle annihilation operator, respectively,

$$\mathcal{P}_{\beta\alpha} = \langle\phi_{HB}|\,a^{T}_{\alpha}\,a_{\beta}\,|\phi_{HB}\rangle \qquad\qquad (2\text{-}52)$$

$$\kappa^{\alpha}_{\alpha\beta} = \langle\phi_{HB}|\,a^{T}_{\beta}\,a^{T}_{\alpha}\,|\phi_{HB}\rangle \qquad\qquad (2\text{-}53)$$

$$\kappa_{\alpha\beta} = \langle\phi_{HB}|\,a_{\alpha}\,a_{\beta}\,|\phi_{HB}\rangle \qquad\qquad (2\text{-}54)$$

with their properties

$$\rho_{\alpha\beta} = \rho_{\beta\alpha}^{\alpha} \quad , \qquad\qquad (2-55)$$

$$k_{\alpha\beta} = -k_{\beta\alpha} \quad , \qquad\qquad k_{\alpha\beta}^{\alpha} = -k_{\beta\alpha}^{\alpha} \, . \qquad (2-56)$$

Utilizing the expectation value of one body, and two body matrix elements, we may evaluate the expectation value of the Hamiltonian:

$$H_o = \langle \phi_{HB} | H | \phi_{HB} \rangle = T_{\alpha\delta}\rho_{\delta\alpha} + \frac{1}{2} V_{\alpha\beta\delta\delta}\rho_{\delta\alpha}\rho_{\delta\beta} + \frac{1}{4} V_{\alpha\beta\delta\delta} k_{\beta\alpha}^{\alpha} k_{\delta\delta} \, . \qquad (2-57)$$

For derivation of HB equations one may set up a variational procedure. Then one introduces a Lagrange multiplier λ to make sure that the physical requirement [equation (2-39)] is satisfied and tries to minimize

$$H^{'} = H_o - \lambda \, T_r \rho \, . \qquad\qquad (2-58)$$

It is more convenient and elegant to utilize the Bogolyubov matrices in which each index takes twice as many values as there are states in the orginal formulation. In particular, define

$$R = \begin{pmatrix} \rho & -k \\ k^{\alpha} & 1-\rho^{\alpha} \end{pmatrix} \qquad\qquad (2-59)$$

which can also be written as

$$R_{\alpha\beta}^{11} = \rho_{\alpha\beta} \quad , \quad R_{\alpha\beta}^{12} = -k_{\alpha\beta} \quad , \quad R_{\alpha\beta}^{21} = k_{\alpha\beta}^{\alpha} \quad , \quad R_{\alpha\beta}^{22} = \delta_{\alpha\beta} - \rho_{\beta\alpha} \, .$$

$$(2-60)$$

This martix has the properties

$$R = R^T,$$

<div align="right">(2-61)</div>

$$f R f = 1 - R^*,$$

<div align="right">(2-62)</div>

where

$$f_{\alpha\beta}^{11} = f_{\alpha\beta}^{22} = 0, \qquad\qquad f_{\alpha\beta}^{12} = f_{\alpha\beta}^{21} = \delta_{\alpha\beta}, \qquad (2-63)$$

The supplementary conditions can be easily expressed in terms of R ;

they reduce to

$$R^2 = R.$$

<div align="right">(2-64)</div>

In similar notation, define a matrix ϑ by

$$\vartheta^{1111}_{\alpha\beta\gamma\delta} = V_{\alpha\beta\gamma\delta} , \qquad \vartheta^{1212}_{\alpha\beta\gamma\delta} = -V_{\alpha\delta\gamma\beta} , \qquad \vartheta^{1221}_{\alpha\beta\gamma\delta} = -V_{\alpha\delta\beta\gamma} ,$$

$$\vartheta^{2222}_{\alpha\beta\gamma\delta} = V_{\gamma\delta\alpha\beta} , \qquad \vartheta^{2121}_{\alpha\beta\gamma\delta} = -V_{\gamma\beta\alpha\delta} , \qquad \vartheta^{2112}_{\alpha\beta\gamma\delta} = -V_{\beta\delta\alpha\gamma} . \qquad (2-65)$$

Submatrices corresponding to the ten other possible contributions of

the superscripts are all set to zero. Also, define

$$T = T^T = \begin{pmatrix} T - \lambda + \frac{1}{\lambda} U & 0 \\ 0 & -T^* + \lambda - \frac{1}{\lambda} U^* \end{pmatrix}$$

<div align="right">(2-66)</div>

where

$$U_{\alpha\delta} = U^*_{\delta\alpha} = V_{\alpha\beta\gamma\beta} \qquad\qquad (2-67)$$

Then, one can find the Hamiltonian in a simple form:

$$H_o' = \sum_{ac} \frac{1}{2} T_{ac} R_{ca} + \sum_{abcd} \frac{1}{8} \mathcal{V}_{abcd} R_{ca} R_{db} + \frac{1}{2} \sum_{\alpha} (T_{\alpha\alpha} - \lambda) + \frac{1}{8} \sum_{\alpha\beta} V_{\alpha\beta\alpha\beta} ,$$

$$(2-68)$$

where Roman subscripts assume twice as many values as Greek

subscripts. The problem reduces to minimization of the Hamiltonian

H_o' , the variable R being restricted by supplementary condition (2-64)

and the properties of matrix R (2-61), (2-62). The final result is

that R must be constructed in such a way as to commute with $\mathcal{W}$

$$[R, \mathcal{W}] = 0 \qquad\qquad (2-69)$$

where $\mathcal{W}$ is defined by:

$$\mathcal{W}_{ac} = T_{ac} + \frac{1}{2} \sum_{bd} \mathcal{V}_{abcd} R_{db} . \qquad\qquad (2-70)$$

2.2 Time-Dependent Hartree-Bogolyubov (TDHB).

Imagine now that the coefficient of Blatt's trial wave function is time dependent, and consequently the wave function is time dependent. However the diagrammatical method still holds as wellas the supplementary conditions (2-33), (2-34). The equations of motion for ρ and k can be derived as follows. Consider the matrix elements of the operator $a_\alpha^T a_\delta$ and $a_\alpha a_\beta$ for a non-stationary wave function $|\psi\rangle$. By virtue of the Schrodinger equation

$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \quad , \qquad (2\text{-}71)$$

and its hermitian conjugate, we have

$$i\frac{\partial}{\partial t}\rho_{\alpha\beta} = \langle\psi|[a_\beta^T a_\alpha , H]|\psi\rangle \qquad (2\text{-}72)$$

$$i\frac{\partial}{\partial t}k_{\alpha\beta} = \langle\psi|[a_\alpha a_\beta , H]|\psi\rangle \quad . \qquad (2\text{-}73)$$

The right hand side of equations (2-72), (2-73) can be easily calculated using Wick's theorem, and the result is

$$i\frac{\partial}{\partial t}\rho_{\alpha\beta} = (T-\lambda+\Gamma)_{\alpha\delta}\rho_{\delta\beta} - \rho_{\alpha\delta}(T-\lambda+\Gamma)_{\delta\beta} + \Delta_{\alpha\delta}k_{\gamma\beta}^* - k_{\alpha\delta}\Delta_{\gamma\beta}^* \qquad (2\text{-}74)$$

$$i\frac{\partial}{\partial t}k_{\alpha\beta} = (T-\lambda+\Gamma)_{\alpha\delta}k_{\delta\beta} - (T-\lambda+\Gamma)_{\beta\gamma}k_{\gamma\alpha} - \Delta_{\alpha\beta} + \rho_{\alpha\delta}\Delta_{\gamma\beta} + \Delta_{\alpha\delta}\rho_{\gamma\beta}^* \quad . \qquad (2\text{-}75)$$

Where Hartree potential Γ and pairing potential Δ are given by

$$\Gamma_{\alpha\gamma} = v_{\alpha\beta\gamma\delta}\rho_{\delta\beta} \qquad (2\text{-}76)$$

and

$$\Delta_{\gamma\beta} = \frac{1}{2} V_{\gamma\beta\delta\zeta} \, k_{\zeta\delta}. \qquad\qquad (2\text{-}77)$$

The equations (2-74) and (2-75) are compatible with supplementary
conditions (2-33) and (2-34).

Kerman and Koonin showed that the time dependent Hartree-Fock
can be derived from a variational procedure. Then, it is natural to
ask whether the TDHB equations can be derived from a variational
method. To answer this question, we would follow the diagrammatic
method of the last section, and define the " Lagrangian "

$$\mathcal{L} = \left(\langle \phi_N | i\frac{\partial}{\partial t} - H | \phi_N \rangle \right) \Big/ \langle \phi_N | \phi_N \rangle. \qquad\qquad (2\text{-}78)$$

A more appropriate choice of the wave function would be a normalized
one such as

$$|\psi\rangle = \frac{|\phi_N\rangle}{\left(\langle \phi_N | \phi_N \rangle\right)^{1/2}} \qquad\qquad (2\text{-}79)$$

and a corresponding Lagrangian can be written as

$$\mathcal{L} = \langle \psi | i\frac{\partial}{\partial t} - H | \psi \rangle. \qquad\qquad (2\text{-}80)$$

The difference between two definition of the Lagrangian equations
(2-78) and (2-80) is a total time derivitive term. Since, we accept
the classical interpretation of the Lagrangian due to Kerman-
Koonin, a total time derivitive term in the Lagrangian will not
cause any change in the equation of motion, and it may be disregarded.

For the derivation of the time dependent part of Lagrangian:

$$\langle \phi_N | i \frac{\partial}{\partial t} | \phi_N \rangle / \langle \phi_N | \phi_N \rangle \quad , \qquad (2-81)$$

we utilize the diagrammatic method of the section 2.1. One may write

$$\langle \phi_N | i \frac{\partial}{\partial t} | \phi_N \rangle = \langle o | \left(\sum_{\alpha\beta} \varphi_{\alpha\beta}^{\alpha} a_\beta a_\alpha \right)^N i \frac{\partial}{\partial t} \left(\sum_{\delta\delta} \varphi_{\delta\delta} a_\delta^T a_\delta^T \right)^N | o \rangle . \qquad (2-82)$$

We assume that only the coeficient, φ , are time dependent, and it is easily seen that this term can be written as

$$\langle \phi_N | i \frac{\partial}{\partial t} | \phi_N \rangle = \sum_{\alpha\beta} i \frac{\partial \varphi}{\partial t}_{\alpha\beta} \frac{\delta}{\delta \varphi_{\alpha\beta}} \langle \phi_N | \phi_N \rangle \qquad (2-83)$$

using equation (2-20), we get

or $$\langle \phi_N | i \frac{\partial}{\partial t} | \phi_N \rangle = \sum_{\alpha\beta} i \frac{\partial}{\partial t} \varphi_{\alpha\beta} \frac{\delta}{\delta \varphi_{\alpha\beta}} \left(\sum_c \prod_c R_c^{n_c} /_{n_c !} \right) , \qquad (2-84)$$

$$\langle \phi_N | i \frac{\partial}{\partial t} | \phi_N \rangle = \sum_{\alpha\beta} \sum_i i \frac{\partial}{\partial t} \varphi_{\alpha\beta} \left[(n_i + 1)/2 \right] K_{\beta\alpha}^{\alpha i} F(2N - n_i - 1) . \qquad (2-85)$$

Where n_i is the order of open chain $K_{\beta\alpha}^{\alpha i}$, and the number of ways in which one could break the closed chain R^c is

$$\frac{n_c}{2} = \frac{n_i + 1}{2} .$$

Similar to the static formulation, one may improve the derivation of the time dependent part of Lagrangian. Using the Hartree-Bogolyubov tri wave function in equation (2-51). We obtain

$$\langle \phi_{HB} | i \frac{\partial}{\partial t} | \phi_{HB} \rangle = \sum_{\alpha\beta,i} i \frac{\partial}{\partial t} \varphi_{\alpha\beta} \left[(n_i + 1)/2 \right] K_{\beta\alpha}^{\alpha i} . \qquad (2-86)$$

Utilizing the complex conjugate of equation (2-32),the equation (2-88) reduces to the following form

$$\langle \phi_{HB} | \, i\frac{\partial}{\partial t} \, | \phi_{HB} \rangle = i \, \frac{\partial}{\partial t} \rho_{\alpha\beta} \, K_{\beta\gamma}^{*} (1-\rho)_{\gamma\alpha} \, , \qquad (2\text{-}89)$$

where we used the convention that repeated indices should be summed. As we observed in section 2.1, the expectation value of the Hamiltonian has no explicit dependence on ϕ . Then, one may expect similar behaviour for the time dependent part of the Lagrangian. Therefore, one may try to eliminate ϕ in favour of K and ρ in the expression (2-89). This is done by using equations (2-50 a,b). With some algebra, one obtains

$$\langle \phi_{HB} | \, i\frac{\partial}{\partial t} | \phi_{HB} \rangle = i \, \frac{\partial}{\partial t} \rho_{\alpha\beta} \, K_{\beta\delta}^{*} (1-\rho)_{\delta\alpha} = \rho_{\alpha\beta}^{*} \, i\frac{\partial}{\partial t} \rho_{\alpha\beta} + K_{\alpha\beta}^{*} \, i\frac{\partial}{\partial t} K_{\alpha\beta} \qquad (2\text{-}90)$$

Finally, the most appropriate form for the time dependent part of Lagrangian can be given in terms of Bogolyubov matrices.

$$\langle \phi_{HB} | \, i\frac{\partial}{\partial t} | \phi_{HB} \rangle = \frac{1}{2} R_{ab} \, i\frac{\partial}{\partial t} R_{ba} + \sum_{\alpha} \, i\frac{\partial}{\partial t} \rho_{\alpha\alpha} \, . \qquad (2\text{-}91)$$

One may utilize the classical Hamiltonian with HB trial wave function (2-68) for the Lagrangian:

$$\mathcal{L} = \langle \phi_{HB} | \, i\frac{\partial}{\partial t} - H \, | \phi_{HB} \rangle . \qquad (2\text{-}92)$$

Although, the dynamics of TDHB equations are independent of chemical potential λ , and the inclusion of λ in the classical Hamiltonian is not necessary; we include λ in order to be consistent in the static

limit of the theory. Then in section 2.3 we will prove λ independence

of the TDHB equations by introduces a new phase for κ . The final

result in terms of Bogolyubov matrices is

$$\mathcal{L} = \tfrac{1}{\lambda} R_{ab}^{T} \, i\tfrac{\partial}{\partial t} R_{ba} - \tfrac{1}{\lambda} R_{ab} \, T_{ba} - \tfrac{1}{8} R_{ab} R_{dc} \mathcal{V}_{bcad}^{+} - \tfrac{1}{\lambda} \sum_{a}(T_{aa} - \lambda)$$

$$- \sum_{\alpha\beta} \tfrac{1}{8} V_{\alpha\beta\alpha\beta} + i \sum_{\alpha} \tfrac{\partial}{\partial t} f_{\alpha\alpha} . \qquad (2\text{-}93)$$

The problem reduces to finding the equation for the above Lagrangian

subject to the constraint equation (2-64). Introducing $\mu_{\alpha\beta}$ as Lagrange

multiplier for the supplementary condition, one obtains a constraint

Lagrangian:

$$\mathcal{L} = \tfrac{1}{\lambda} R_{ab}^{T} \, i\tfrac{\partial}{\partial t} R_{ba} - \tfrac{1}{\lambda} R_{ab}^{T} \, T_{bc} R_{ca} - \tfrac{1}{8} R_{ab}^{T} R_{dc}^{T} \mathcal{V}_{bcfg}^{+} R_{fa} R_{gd} - \tfrac{1}{\lambda}(R_{ab} - R_{ac}^{T} R_{cb}) \mu_{ba}$$

where, we omitted the unnecessary terms. The equations of motion $\qquad (2\text{-}94)$

can be found by requiring the restricted $\mathcal{S}$

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L} \, dt \qquad (2\text{-}95)$$

to be stationary with respect to variation of the path of motion

between fixed end points t_1 and t_2. The results for the equations

of motion are:

$$i\tfrac{\partial}{\partial t} R_{ia} = W_{ib} R_{ba} - R_{ib} \mu_{ba} , \qquad (2\text{-}96)$$

and

$$-i\tfrac{\partial}{\partial t} R_{\alpha f}^{T} = R_{ab} V_{bf} - \mu_{a b} R_{bf} . \qquad (2\text{-}97)$$

Multiplying equation (2-96) from the right by R_{af} and summing over a

and then multiplying equation (2-97) from the left by R_{ia} and summing

over a , we obtain the following equations

$$i \left(\frac{\partial}{\partial t} R_{ia}\right) R_{a\beta} = W_{ia} R_{a\beta} - R_{ic} \mu_{ca} R_{a\beta} , \qquad (2\text{-}98)$$

and

$$-i R_{ia} \left(\frac{\partial}{\partial t} R_{a\beta}\right) = R_{ia} W_{a\beta} - R_{ia} \mu_{ac} R_{c\beta} . \qquad (2\text{-}99)$$

where we used the definition of ν in equation (2-70),and supplementary

condition (2-64). Subtracting equation (2-99) from (2-98), and using

the supplementary condition (2-64), one obtains

$$i \frac{\partial}{\partial t} R_{i\beta} = W_{ia} R_{a\beta} - R_{ia} W_{a\beta} , \qquad (2\text{-}100)$$

or, in matrix form

$$i \frac{\partial}{\partial t} R = [W, R] \qquad (2\text{-}101)$$

Comparing equation (2-100) and (2-96), the evolution of the Lagrange

multiplier $\mu_{\alpha\beta}$ is equal to $\nu_{\alpha\beta}$. Since, the Hermition matrices form

a complete set, the inclusion of equation (2-61) as a constraint

in the Lagrangian was not necessary. The similar argument holds

for the matrices satisfying (2-62). One may prove the compatiblity

of the above line of reasoning. From the equation (2-100) one obtains

$$-i \frac{\partial}{\partial t} R^T = - [W^T, R^T] , \qquad (2\text{-}101)$$

using

$$V = W^T ,$$

(2-102)

then, one obtains

$$i \frac{\partial}{\partial t} (R - R^T) = [W, R - R^T] .$$

(2-103)

Similar argument goes through for the second restriction (2-62). Utilizing

$$f W f = - W^{\kappa} ,$$

(2-104)

one obtains

$$i \frac{\partial}{\partial t} \left(f R f - 1 + R^{\kappa} \right) = [W, \quad f R f - 1 + R^{\kappa}] .$$

(2-105)

one may insert $R - R^T = 0$,and $f R f - 1 + R^{\kappa} = 0$ as initial conditions into (2-103) and (2-105) respectively. Then, $R - R^T$, and $f R f - 1 + R^{\kappa}$ will be zero at all succeeding times. Finally it is intersting to rewrite the equations of motion (2-100) with Greek indces.

$$i \frac{\partial}{\partial t} \begin{pmatrix} f_{\alpha\beta} & -\kappa_{\alpha\beta} \\ \kappa^{\kappa}_{\alpha\beta} & \delta_{\alpha\beta} - f^{\kappa}_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} (T - \lambda + \Gamma)_{\alpha\gamma} & \Delta_{\alpha\delta} \\ -\Delta^{\kappa}_{\alpha\delta} & -(T - \lambda + \Gamma)^{\kappa}_{\alpha\delta} \end{pmatrix} \begin{pmatrix} f_{\gamma\beta} & -\kappa_{\delta\beta} \\ \kappa^{\kappa}_{\gamma\beta} & (1 - f^{\kappa})_{\delta\beta} \end{pmatrix}$$

$$- \begin{pmatrix} f_{\alpha\gamma} & -\kappa_{\alpha\delta} \\ \kappa^{\kappa}_{\alpha\delta} & (1 - f^{\kappa})_{\alpha\delta} \end{pmatrix} \begin{pmatrix} (T - \lambda + \Gamma)_{\gamma\beta} & \Delta_{\delta\beta} \\ -\Delta^{\kappa}_{\gamma\beta} & -(T - \lambda + \Gamma)^{\kappa}_{\delta\beta} \end{pmatrix} .$$

(2-106)

As one may expect the matrix equation (2-106) reduces to the equations (2-74) and (2-75).

2.3 Properties of the TDHB theory

With a glance at the TDHB equations, a number of ideas come to our mind for the discussion of the properties of TDHB theory. These are a) conservation laws in this theory, b) various limits of the TDHBequations, c) time evolution for the chemical potential, d) possibility of a new derivation method for the TDHB equations, e) utilization of the TDHB theory for the derivation of a collective Lagrangian, f) utilization of the TDHB equations for a hydrodynamic description of a nuclear system similar to the one Koonin has for TDHF equations.

a) Conservation laws: There are three quantities conserved by equation (2-100), and hence three corresponding constants of motion. These are:

1- Energy conservation: One defines the total energy ,E, as the expectatation value of the Hamiltonian (2-57)

$$E = H_o = \langle \phi_{HB} | H | \phi_{HB} \rangle = T_{4\epsilon} \rho_{\delta\beta} + \frac{1}{2} V_{\alpha\beta\delta\delta} \rho_{\delta\alpha} \rho_{\delta\beta} + \frac{1}{4} V_{\alpha\beta\delta\delta} K^*_{\beta\alpha} K_{\delta\delta}.$$

$$(2-57)$$

The time derivative of the energy is

$$\frac{dE}{dt} = \langle (\frac{\partial}{\partial t} \phi_{HB}) | \hat{H} | \phi_{HB} \rangle + \langle \phi_{HB} | \hat{H} \frac{\partial}{\partial t} | \phi_{HB} \rangle$$

$$(2-107)$$

and by virtue of the Schrodinger Equation (2-18)

$$\frac{dE}{dt} = -i \langle \phi_{HB} | [\hat{H}, \hat{H}] | \phi_{HB} \rangle = 0 \implies \frac{dE}{dt} = 0.$$

$$(2-108)$$

Thus, the total engery , E, is a constant of motion. Also, one could prove the conservation of energy (2-108) directly by using the equations of motion (2-74),(2-75)and equation (2-57), but this method involoves a little more algebra with the same result.

2) Particle number: The average number of particles is conserved. Using the equation of motion (2-74), taking α equal to β , and summing over α , we get

$$i \frac{d}{dt} tr\rho = Tr \left[(T+\Gamma+\lambda)\rho - \rho(T+\Gamma+\lambda) + \Delta k^* - k\Delta^* \right].$$

Using the definition of pairing energy (2-77), one obtains

$$i \frac{d}{dt} tr\rho = 0 \qquad\qquad (2-109)$$

Thus, the average number of particles is also a constant of motion Therefore, one should not expect any time evolution for the chemical potential

3) Conservation of the "form":

In section 2.2 we proved that if at time $t_o = 0$ the Bogolyubov matrix R has the properties $R=R^T$ and $\rho R \rho = 1-R^*$ then it will keep these properties at all succeeding times. Now we will prove with similar method the same result for the supplementary condition (2-64). Let us evaluate the time derivitive of R^2

$$i \frac{d}{dt} R^2 = i \left(\dot{R}R + R\dot{R} \right). \qquad\qquad (2-110)$$

Using the equation of motion (2-100), one obtains

$$i \frac{\partial}{\partial t} R^2 = [W, R] R + R [W, R] \qquad (2-111a)$$

or

$$i \frac{\partial}{\partial t} R^2 = [W, R^2] . \qquad (2-111b)$$

Hence

$$i \frac{\partial}{\partial t} (R^2 - R) = [W, (R^2 - R)] \qquad (2-112)$$

Thus, if $R^2 - R = 0$ is inserted as an intial condition into equation (2-112) $R^2 - R$ will be zero at all succeeding times. Indeed this supplementary condition was included as a constraint in the variational derivation of the TDHB equation.

b) Various limits of the TDHB equations: First, as one may expect, the static limit of the TDHB equation coincides with Baranger's static theory of pairing. Secondly, imagine one wants to utilize the equivalnce of TDHB and TDBCS. The Bloch-Messiah theorm requires

$$\rho_{\alpha\beta} = \rho_{\alpha\alpha} \delta_{\alpha\beta} \qquad (2-113)$$

and

$$K_{\alpha\beta} = K_{\alpha\bar{\alpha}} \delta_{\beta\bar{\alpha}} . \qquad (2-114)$$

Where, $\bar{\alpha}$ corresponds to the pair state of α. The simplified version

of the equations of motion reduces to

$$i \frac{\partial}{\partial t} f_{\alpha\alpha} = \Delta^*_{\alpha\bar{\alpha}} k_{\alpha\bar{\alpha}} - \Delta_{\alpha\bar{\alpha}} k^*_{\alpha\bar{\alpha}} \quad , \qquad (2\text{-}115)$$

and

$$i \frac{\partial}{\partial t} k_{\alpha\bar{\alpha}} = (2\epsilon_\alpha - \lambda) k_{\alpha\bar{\alpha}} + (2 f_{\alpha\alpha} - 1) \Delta_{\alpha\bar{\alpha}} \quad , \qquad (2\text{-}116)$$

where

$$\epsilon_\alpha = T_{\alpha\alpha} + \Gamma_{\alpha\alpha} \qquad (2\text{-}117)$$

and, it can be understood as single particle energy. Here, in the equations (2-105), (2-116) and (2-117) the repeated indeces do not constitue the summation. These equation (2-115) and (2-116) are similar to the TDBCS equations of motion according to the reference 29. Also in the static limit equation (2-116) gives the BCS gap equation. Finally, as we already mentioned many times the TDHB theory is not based on a definite number of particles. Now, let us limit TDHB equations to those terms which conserve the number of particles. In other words, taking $k_{\alpha\beta} = 0$ for any state of α and β, one obtains

$$i \frac{\partial}{\partial t} f_{\alpha\beta} = (T + \Gamma)_{\alpha\delta} f_{\delta\beta} - f_{\alpha\delta} (T + \Gamma)_{\delta\beta} . \qquad (2\text{-}118)$$

With proper choice of filled and unfilled states, one could write

$$i \frac{\partial}{\partial t} f_{mi} = (\epsilon_m - \epsilon_i) f_{mi} + \sum_{n,j} V_{mn\,ij} f^*_{nj} + \sum_{nj} V_{jm\,in} f_{nj} . \qquad (2\text{-}119)$$

where ϵ_m and ϵ_i are the single particle energy of the diagonalized

Hamiltonian, (m,n stand for unfilled states and i,j stand for filled ones). Equation (2-119) is exactly the TDHF equations according to the reference [25].

c) Time evolution of the chemical potential: Let us recall the TDHB equations (2-74) and (2-75)

$$i\frac{\partial}{\partial t}\rho_{\alpha\beta} = (T-\lambda+\Gamma)_{\alpha\gamma}\rho_{\gamma\beta} - \rho_{\alpha\delta}(T-\lambda+\Gamma)_{\gamma\beta} + \Delta_{\alpha\delta}k^{*}_{\delta\beta} - k_{\alpha\delta}\Delta^{*}_{\delta\beta},$$

$$(2\text{-}74)$$

$$i\frac{\partial}{\partial t}k_{\alpha\beta} = (T-\lambda+\Gamma)_{\alpha\delta}k_{\delta\beta} - (T-\lambda+\Gamma)_{\beta\delta}k_{\delta\alpha} - \Delta_{\alpha\beta} + \rho_{\alpha\delta}\Delta_{\delta\beta} + \Delta_{\alpha\delta}\rho^{*}_{\delta\beta}.$$

$$(2\text{-}75)$$

With a glance at the equation (2-74) one realizes that the λ dependent terms cancel each other, and thus equation (2-74) reduces to

$$i\frac{\partial}{\partial t}\rho_{\alpha\beta} = (T+\Gamma)_{\alpha\delta}\rho_{\delta\beta} - \rho_{\alpha\delta}(T+\Gamma)_{\delta\beta} + \Delta_{\alpha\delta}k^{*}_{\delta\beta} - k_{\alpha\gamma}\Delta^{*}_{\gamma\beta} \qquad (2\text{-}120)$$

which is independent of λ explicitly. For the second equation (2-75), one may choose a specific time dependent phase for $k_{\alpha\beta}$ as

$$k_{\alpha\beta} \equiv e^{2i\int_0^t \lambda dt'} k_{\alpha\beta}. \qquad (2\text{-}121)$$

With this chioce for $k_{\alpha\beta}$, the equations of motion reduce to

$$i\frac{\partial}{\partial t}\rho_{\alpha\beta} = (T+\Gamma)_{\alpha\delta}\rho_{\gamma\beta} - \rho_{\alpha\delta}(T+\Gamma)_{\delta\beta} + \delta_{\alpha\delta}k^{*}_{\delta\beta} - k_{\alpha\delta}\delta^{*}_{\delta\beta}, \qquad (2\text{-}122)$$

and

$$i\; \dot{k}_{\alpha\beta} = (T+\Gamma)_{\alpha\delta}\, k_{\delta\beta} - (T+\Gamma)_{\beta\delta}\, k_{\delta\alpha} - \delta_{\alpha\beta} + \int_{\alpha\delta} \delta_{\delta\beta} + \delta_{\alpha\delta}\, f^{\alpha}_{\delta\beta} \tag{2-123}$$

where

$$\delta_{\alpha\beta} = \tfrac{1}{2}\, V_{\alpha\beta\delta\delta}\, k_{\delta\gamma}. \tag{2-124}$$

Equations (2-122) and (2-123) show the independence of TDHB theory

in the chemical potential. Considering the equations of motion (2-74)

and (2-75). In fact, any arbitrary function of time can be added to

the chemical potential , λ, it would cause only a change in the phase

of κ . It means that at any given time one has the choice of the

chemical potential for the system, as one expect physically. Similar

result has been indicated by Blocki and Flocard for the TDBCS

equations. Therefore one does not expect any time evolution for

the Lagrange multiplier λ ,and may take λ as a time independent

chemical potential.

d) Possibility of a new derivation method for the TDHB equations:

Let us consider the time derivitive part of the Lagrangian (2-91)

$$\tfrac{1}{2}\, i\, R_{ba}\frac{\partial}{\partial t} R_{ab} = \tfrac{1}{2}\, i\, R^{*}_{ab}\frac{\partial}{\partial t} R_{ab}. \tag{2-125}$$

Define each Bogolyubov matrix element R_{ab} as

$$R_{ab} = P_i - i q_i \tag{2-126}$$

then, the expression (2-125) can be written as

$$\sum_{i} P_i\, \dot{q}_i \tag{2-127}$$

P_i and q_i can be considered as canonical variables for a classical

system with q_i as a coordinate and P_i the momentum conjugate to the

q_i coordinate. The equations (2-61) and (2-62) can be viewed as

holonomic constraints. They will cause a reduction in the number of

Generalized coordinates. The supplementary conditions (2-64) are

nonholonomic constraints. Therefore one should introduce the Lagrange

multipilers to include these constraints. Then the equation of motion

can be derived by utilizing the Hamilton's principle. The final

result is equal to one obtained by variational method.

e) Utilization of TDHB theory for derivation of a collective

Lagrangian: One major reason for the derivation of TDHB equations

from a variational point of view was the derivation of a Lagrangian

which describes the system and can be utilized for a reduction in

number of coordinates. In this way, we hope the reduced coordinates

are good approximations to the collective coordinates in the system.

In chapter 4 we will discuss this property in more detail.

f) Utilization of TDHB equations for a hydrodynamic description

of a nuclear system : Koonin developed a hydrodynamic approximation

for a nuclear system from TDHF equations and by utilizing the Wigner

transformation. Similar procedure can be employed for the derivation

of the hydrodynamic approximation from TDHB equations. In chapter 3

we will discuss this approach to the hydrodynamic approximation in

more detail.

2.4 The linearized solution to the TDHB equation.

The TDHB equations (2-100) with their constraints (2-64),(2-61)

and (2-62) are formally similar to the TDHF ones. The difference lies

on the much bigger dimension of the Bogolyubov matrices, and the lack

of a fixed number of particles. In statistical sense, the TDHB equations,

like TDHF equations, can be viewed as a deterministic theory in the

sense that a given intial condition gives rise to a specific final state.

The TDHB theory in comparison to the TDHF theory has some advantages,

notably, its relaxation of a single determinate, and the inclusion

of superconducting solutions. But the larger dimension of its matrices

certainly will cause technical difficulty for a comprehensive solution

to the TDHB equations.

Although there are several attempts to solve TDHF equations in

the literature,[30] there is not a precise behaviour of solutions to

the TDHF equations for a given single determinate. Therefore it would

be reasonable to assume that a general attack for solution of the TDHB

equations will be for future. In the following pages we will try to

find an analogue to RPA solution in the TDHF theory.

Static solution: As we already mentioned in section 2.1, in order to find the ground state energy and the static solution to the system, one should find W and R in such a way that they commute with each other. This can be achieved by diagonalizing W and R simultaneously, and satisfying selfconsistency requirement and the supplementary conditions. Condition (2-64) says that all eigenvalues of R are zero or one, and the condition Equation (2-62) tells us that the number of eigenvectors with eigenvalue one is equal to the number of eigenvectors with zero eigenvalue. Let us recall the equation (2-104)

$$ f W f = - W^{\alpha} \qquad\qquad (2-104) $$

this equation says that, if W has an eigenvector a_i for eigenvalue ε_i, it also has an eigenvector $f a_i^{\chi}$ for eigenvalue $-\varepsilon_i$. In this procedure the eigenvalue ε_i has no physical significance, but with a choice of R with eigenvalue zero for each a_i eigenvector, and eigenvalue one for eigenvector $f a_i^{\chi}$, the ε_i is seen to be the energy of an elementary excitation or quasi-particle excitation energy, according to reference [32]. Supperficially, in analogy with Hartree-Fock theory one could interpret ground state of the HB theory as a state in which all negative

energy of the quasi-particle states are filled. The eigenvalue equations

for W are the HB equations, and they can be written as

$$E_i A_{\alpha i} = \sum_{\gamma} \left(T_{\alpha\gamma} - \lambda \delta_{\alpha\gamma} + \Gamma_{\alpha\gamma} \right) A_{\gamma i} + \sum_{\beta} \Delta_{\alpha\beta} B_{\beta i} \quad , \qquad (2\text{-}128)$$

$$-E_i B_{\alpha i} = \sum_{\gamma} \left(T_{\alpha\delta}^{\kappa} - \lambda \delta_{\alpha\delta} + \Gamma_{\alpha\delta}^{\kappa} \right) B_{\gamma i} + \sum_{\beta} \Delta_{\alpha\beta}^{\kappa} A_{\beta i} . \qquad (2\text{-}129)$$

where, we used the notation

$$\left(a_i \right)'_{\alpha} = A_{\alpha i} \qquad , \qquad \left(a_i \right)^2_{\alpha} = B_{\alpha i} . \qquad (2\text{-}130)$$

Now let us seek a solution to the TDHB equations by linearizing in

small deviation of R about the equilibrium point R^o. Then, one

could write

$$R = R^o + \delta R = R^o + \delta R^+ e^{i\omega t} + \delta R^- e^{-i\omega^\times t} \qquad (2\text{-}131)$$

with a (complex) frequency and small transition density in quasi-

particle densities δR^+ to be determined as an eigenvalue problem.

Hermitian condition restricts excursions to the two independent ones.

$$\left(\delta R^+ \right)^T = \delta R^- . \qquad (2\text{-}132)$$

The expansion of the condition (2-64) to the first order in small

deviation requires that

$$\delta R^{(i)} R^o + R^o \delta R^{(i)} = \delta R^{(i)} . \qquad (2\text{-}133)$$

Due to the choice of R^o as a diagonal matrix, equation (2-133) requires

that $\delta R^{(1)}$ should have only quasi-particle quasi-hole matrix elements i. e.

$$\delta R^{(1)}_{ij} = \delta R^{(1)}_{mn} = 0 , \qquad\qquad (2-134)$$

$$\delta R^{+}_{ni} = \left(\delta R^{\mp}\right)^{\alpha}_{in} \neq 0 . \qquad\qquad (2-135)$$

Where i, j are refered to occupied quasi-particle states and m, n as unoccupied ones. Inserting equation (2-131) into equation (2-100) keeping only first order term in excursions, one obtains

$$i \frac{\lambda}{\partial t} \delta R = \left[W^{0}, \delta R\right] + \left[\frac{\delta W}{\delta R} \delta R, R^{0}\right] . \qquad\qquad (2-136)$$

Writing the linearized equation for each of two independent components

$$-\omega \, \delta R^{+} = \left[W^{0}, \delta R^{+}\right] + \left[\frac{\delta W}{\delta R} \delta R^{+}, R^{0}\right] \qquad\qquad (2-137)$$

$$\omega^{\alpha} \, \delta R^{-} = \left[W^{0}, \delta R^{-}\right] + \left[\frac{\delta W}{\delta R} \delta R^{-}, R^{0}\right] . \qquad\qquad (2-138)$$

Using equation (2-70) for evaluation of $\frac{\delta W}{\delta R}$, taking the (i,n) and (n,i) matrix element of these equations, and using the fact that R^{0} is diagonal and vanishes for any subscript of an unoccupied state, one could obtain

$$-\omega \left(\delta R^{+}\right)_{in} = \left(E_{i} - E_{n}\right) \delta R^{+}_{in} - \frac{1}{2} v^{-}_{i \, bnd} \, \delta R^{+}_{db} \qquad (2-139a)$$

$$-\omega \left(\delta R^{+}\right)_{ni} = \left(E_{n} - E_{i}\right) \delta R^{+}_{ni} - \frac{1}{2} v^{-}_{n \, bid} \, \delta R^{T}_{db} \qquad (2-139b)$$

$$\omega^* \left(\delta R^-\right)_{in} = \left(E_i - E_n\right) \delta \bar{R}_{in} - \frac{1}{2} \mathcal{V}_{ibnd}^* \delta \bar{R}_{db} \qquad (2\text{-}14\,a)$$

$$\omega^* \left(\delta R^-\right)_{ni} = \left(E_n - E_i\right) \delta \bar{R}_{ni} - \frac{1}{2} \mathcal{V}_{nbid} \delta \bar{R}_{db} . \qquad (2\text{-}14\,b)$$

Using equation (2-135), then one could rewrite the linearized equation of motion in the more familiar form:

$$-\omega X_{in} = \left(E_i - E_n\right) X_{in} + \frac{1}{2} \mathcal{V}_{injn} X_{jm} - \frac{1}{2} \mathcal{V}_{nmij}^* Y_{jm} , \qquad (2\text{-}141)$$

$$\omega Y_{in} = \left(E_i - E_n\right) Y_{in} + \frac{1}{2} \mathcal{V}_{imjn}^* Y_{jm} - \frac{1}{2} \mathcal{V}_{nmij} X_{jm} , \qquad (2\text{-}142)$$

where

$$X_{in} = \delta R_{in}^+ , \qquad Y_{in} = \delta R_{ni}^+ . \qquad (2\text{-}143)$$

The equations (2-141) and (2-142) can be viewed as generalized

form of RPA or quasi-particle RPA. In analogy to RPA, one can introduce

two matrices

$$A_{in,jm} = \left(E_n - E_i\right) \delta_{ij} \delta_{nm} + \frac{1}{2} \mathcal{V}_{imjn}^+ , \qquad (2\text{-}144)$$

$$B_{in,jm} = \frac{1}{2} \mathcal{V}_{ijmn}^+ . \qquad (2\text{-}145)$$

In matrix notation, equations (2-141) and (2-142) can be written as

$$\omega \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} A & B \\ -B^* & -A^* \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} . \qquad (2\text{-}146)$$

Equation (2-146) can be considered as an eigenvalue problem for

the normal modes δR , and normal frequency " ω ". If we used the

equation (2-140 a,b) instead of the equation (2-139 a, b) we would have

$$-\omega' \begin{bmatrix} Y^* \\ X^* \end{bmatrix} = \begin{bmatrix} A & B \\ -B^* & -A^* \end{bmatrix} \begin{bmatrix} Y^* \\ X^* \end{bmatrix},$$

$$(2-147)$$

That suggests ω and $-\omega^*$ are both solutions of the eigenvalue equations. The expansion of R would be stable only if ω is real, otherwise the excurisons will grow exponentially, suggesting that the stationary state is not a ground state. Diagonalization of the linearized TDHB equations (2-146), (2-147) can be viewed as equivalent to solving the problem of coupled oscillators, and ω as normal mode of the frequency in the space of quasi particle, quasi hole excitations. In analogy to the Hartree-Fock theory one could quantize these modes. Thus $E_0 + \hbar\omega$, $E + 2\hbar\omega$ ···· etc. can be viewed as collective vibrational states of the system.

Chapter III

Time dependent Hartree-Bogolyubov in the Wigner representation.

In this chapter we explore the Wigner representation of the TDHB equations. We show that, in this representation, TDHB equations reduce to a form easily recognizable as a quantal version of a modified Vlasov equation, which approaches the expected classical result in the limit $\hbar \to 0$ and a definite number of particles. Also, due to the abandonment of a definite number of particles, we have an equation of motion for the deviation distribution function, which identically vanishes for a system of fixed number of particles. In section 3.1, we will discuss the properties of the distribution function, and the study of TDHB equations in Wigner reoresentation is our goal in section 3.2.

3.1 Properties of the distribution functions.

Density matrices in the coordinate representation can be written as

$$\rho(x\,x') = \langle \phi_{HB} | a^{\dagger}(x') a(x) | \phi_{HB} \rangle , \qquad (3-1)$$

$$\kappa(xx') = \langle \phi_{HB} | a(x) a(x') | \phi_{HB} \rangle , \qquad (3-2)$$

$$\hat{K}(x\,x') = \langle \phi_{HB} \mid a^{T}_{(x')} a^{T}_{(x)} \mid \phi_{HB} \rangle \;,\qquad (3-3)$$

where, as usual, X constitutes spatial $\vec{r}$, spin σ_s and isospin σ_I coordinates, and $a^{T}_{(x)}$ and $a_{(x)}$ are the field operators. For a semi-classical interpretation of TDHB equations, we use Winger's[33] suggestion. Let us define "phase space distribution functions" $f(R,\ell \, \S\S')$ and $g(R,\ell,\S\S')$ as

$$f(R,\ell \, \S\S') = \int d\vec{s}\; e^{-i\vec{k}\cdot\vec{s}}\; \rho\left(\vec{R}+\tfrac{\vec{s}}{2}\,,\vec{R}-\tfrac{\vec{s}}{2}\,,\S\S'\right)\;, \qquad (3-4)$$

$$g(R,\ell \, \S\S') = \int d\vec{s}\; e^{-i\vec{k}\cdot\vec{s}}\; \mathcal{K}\left(\vec{R}+\tfrac{\vec{s}}{2}\,,\vec{R}-\tfrac{\vec{s}}{2}\,\S\S'\right)\;, \qquad (3-5)$$

where the "center of mass" and "relative" coordinates are defind by

$$\vec{R} = \tfrac{1}{2}\left(\vec{r}+\vec{r}'\right)\;,\qquad (3-6)$$

$$\vec{s} = \left(\vec{r}-\vec{r}'\right)\;,\qquad (3-7)$$

respecttively, and $\S$ constitutes spin and isospin coordinates.

Equations (3-4) and (3-5) simply define f and $g(R,\ell\,\S\S')$ as the ℓth component of the fourier transform of $\rho(x\,x')$ and $\mathcal{K}(x\,x')$ respectively. For a saturated system such as nuclear matter or an even-even nuclei with equal number of proton and neutron, one could take $\rho(x\,x')$ as digonal

in spin and isospin coordinates, and has many of the properties one
would expectof a classical phase distribution. Therefore, from a
semiclassical point of view, f is the probability density for finding
a nucleon at postion $\vec{R}$ with momentum $\vec{k}$ and spin-isospin label ζ .

The existance of $K(x,x')$ and in turn g is due to the inclusion of
the pair correlation in interaction, describing the many body system.
Therefore, we do not expect a semiclassical interpretation for $g(k,k',\zeta\zeta')$
as we had for $f(k,\zeta\zeta')$. Although there is not a semiclassical interpretation
for $g(k,\zeta\zeta')$, we may consider the fourier transform of the deviation function
which is defind by

$$D(x,x') = -2 \; K(x,y) K^{\alpha}(y,x') . \qquad (3-8)$$

Again, we use the convention that repeating variables are summed
(integrated over continuous variables). Then, one may define "deviation
distribution function" $d(R,k,\zeta\zeta')$ as

$$d(R,k,\zeta\zeta') = \int d^3s \; e^{-i\vec{k}\cdot\vec{s}} D\left(\vec{R}+\tfrac{\vec{s}}{2} , \vec{R}-\tfrac{\vec{s}}{2} , \zeta\zeta'\right) \qquad (3-9)$$

Similar to the phase distribution function f , the deviation distri-
bution function d is real and digonal in spin-isospin coordinates.
From a semiclassical point of view, d is the probability of deviation

density, which is the square of fluctuation density at postion $\vec{R}$ with momentum $\vec{\ell}$ and spin-isospin label $\int$.

The properties of distribution functions are: a) The distribution function f and d are real. This property of f can be seen by taking the conjugate of equation (3-4), utilizing the hermiticity of the density matrix, and defining a new variable $\vec{s} \rightarrow -\vec{s}$. Then

$$f^{*}(R,\ell \int \int') = \int d\vec{s}\ e^{i\vec{\ell}\cdot\vec{s}}\ \int^{*}(\vec{R}+\tfrac{\vec{s}}{2},\vec{R}-\tfrac{\vec{s}}{2}\ \int\int')$$

$$= \int d\vec{s}\ e^{i\vec{\ell}\cdot\vec{s}}\ \int(\vec{R}-\tfrac{\vec{s}}{2},\vec{R}+\tfrac{\vec{s}}{2}\ \int\int)$$

$$= \int d\vec{s}\ e^{-i\vec{\ell}\cdot\vec{s}}\ \int(\vec{R}+\tfrac{\vec{s}}{2},\vec{R}-\tfrac{\vec{s}}{2}\ \int\int)$$

$$= f(R\ \ell \int\int')$$

$$= f(R\ \ell \int\int') \tag{3-10}$$

where we have used the assumption that $\int$ is diagonal in spin and isospin coordinates. This property for d can also be seen by subtituting in the equation (3-9) from equation (3-8), taking the conjugate of the result, utilizing antisymmetric property of κ and defining a new variable $\vec{s} \rightarrow -\vec{s}$

$$d^{*}(R,\ell \int\int') = \int d\vec{s}\ e^{i\vec{\ell}\cdot\vec{s}}\ D^{*}(\vec{R}+\tfrac{\vec{s}}{2},\vec{R}-\tfrac{\vec{s}}{2}\ \int\int')$$

$$= -2 \int d\vec{s}\ e^{i\vec{\ell}\cdot\vec{s}}\ \kappa^{*}(\vec{R}+\tfrac{\vec{s}}{2},\int\int\gamma)\ \kappa(\gamma\ \vec{R}-\tfrac{\vec{s}}{2}\ \int')$$

$$= -2 \int ds \; e^{i\vec{k}\cdot\vec{s}} \; K(\vec{R}-\tfrac{\vec{s}}{2} \; \dot{j} \; y) \; k^x(y \; \vec{R}+\tfrac{\vec{s}}{2} \; \dot{j})$$

$$= -2 \int ds \; e^{-i\vec{k}\cdot\vec{s}} \; K(\vec{R}+\tfrac{\vec{s}}{2} \; \dot{j} \; y) \; k^x(y \; \vec{R}-\tfrac{\vec{s}}{2} \; \dot{j})$$

$$= d \; (R, k, \dot{j}\dot{j})$$

$$= d(R, k, \dot{j}\dot{j}) \hspace{4cm} (3-11)$$

Where the last line follows from the fact that $D(\chi\chi')$ is a diagonal matrix with respect to $\dot{j}\dot{j}$ spin and isospin coordinates, which itself can be realized from assumption that the density matrix $\mathcal{P}$ is diagonal with respect to $\dot{j}$ and $\dot{j}$, using the supplementary condition (3-12)

$$\mathcal{P}(\chi\chi') = \mathcal{P}(\chi y) \mathcal{P}(\dot{j}x') - k(\chi y) k^{\dagger}(\dot{j}x') . \hspace{2cm} (3-12)$$

b) The function g has the property

$$g(R k \dot{j}\dot{j}') = - g(R-k, \dot{j}\dot{j}) . \hspace{3cm} (3-13)$$

Using the definition of $g(R,k\dot{j}\dot{j}')$ from equation (3-5), utilizing the anti-symmetric property of K and defining a new variable $\vec{s}\rightarrow-\vec{s}$, one may write

$$g(R k, \dot{j}\dot{j}') = \int ds \; e^{-i\vec{k}\cdot\vec{s}} \; k(\vec{R}+\tfrac{\vec{s}}{2} \; \vec{R}-\tfrac{\vec{s}}{2} \; \dot{j}\dot{j})$$

$$= - \int ds \; e^{-i\vec{k}\cdot\vec{s}} \; k(\vec{R}-\tfrac{\vec{s}}{2} \; \vec{R}+\tfrac{\vec{s}}{2} \; \dot{j}\dot{j})$$

$$= - \int ds \; e^{-i\vec{k}\cdot(-s)} \; k(\vec{R}+\tfrac{\vec{s}}{2}, \vec{R}-\tfrac{\vec{s}}{2} \; \dot{j}\dot{j})$$

$$= - g(R-k \; \dot{j}\dot{j}) . \hspace{4cm} (3-14)$$

c) Various expectation values of one body operator are given in terms of f by their expected classical from. For a one body operator O , we may write

$$\langle \phi_{HB} | O | \phi_{HB} \rangle = O(xx') f(x'x)$$

$$= \int d^3r \, d^3r' \, O(\vec{r}\vec{r}, \{\}) \, f(\vec{r}'\vec{r} \{\})$$

$$= \int d^3s \, d^3R \quad O(\vec{R}+\vec{s}_{/2} , \vec{R}-\vec{s}_{/2} \{\}) \, f(\vec{R}-\vec{s}_{/2}, \vec{R}+\vec{s}_{/2} \{\})$$

$$= \int d^3s \, d^3R \quad O(\vec{R}+\vec{s}_{/2} , \vec{R}-\vec{s}_{/2} \{\}) \, f(R \, k \, \{\}) \, e^{-i\vec{k}\cdot\vec{s}}$$

$$= \int d^3R \, \frac{d^3k}{(2\pi)^3} \, O(R \, k \, \{\}) \, f(R \, k \, \{'\}) . \tag{3-15}$$

Where in the second line, we used $f(xx)$ as the inverse fourier transform of $f(Rk\{\})$ and in the fifth line the operator expressed in the Wigner representation is

$$O(R, k \, \{\}) = \int d^3s \, e^{-i\vec{k}\cdot\vec{s}} \, O(\vec{R}+\vec{s}_{/2}, \vec{R}-\vec{s}_{/2} \, \{\}') \tag{3-16}$$

From the last line in equation (3-15) we see that $O(R\{\})$ provides the approprite weighting factor for the distribution function in phase space needed to compute the required expectation value. For example, the corresponding operator to the total nuclear density at the point Q is

$$O(\vec{R}, \vec{s}) = \delta(\vec{R}-\vec{Q}) \delta(s). \tag{3-17}$$

In the Wigner representation, it takes the form:

$$O(R\ell) = \delta(\vec{q}-\vec{R})$$ (3.-18)

Thus from equation (3-15), we get

$$f(Q\,jj') = \int d^3R \frac{d^3k}{(2\pi)^3}\,\delta(Q-R)\,f(R,\ell jj') = \int \frac{d^3k}{(2\pi)^3}\,f(Q\,\ell jj')$$ (3-19)

which is the classical result. Similarly, one would get the expected classical result for the quantum mechanical current

$$\vec{J}(Q\,jj') = \int \frac{d^3k}{(2\pi)^3}\,\vec{k}\,f(\vec{Q},\ell jj')$$ (3-20)

d) The distribution function f and deviation distribution function d also have the expected form for simple system. For example, taking the BCS theory with plane wave as the single particle wave function, we obtain

$$f(R\,\ell jj') = \frac{(2\pi)^3}{V}\,\delta_{jj'}\left(\sum_{\rho}(\ell-\ell_\rho)\,f_{\rho\rho}\right)$$ (3-21)

and utilizing equation (-19), we get

$$f(R\,jj') = \frac{\delta_{jj'}}{V}\left(\sum_{\beta}f_{\rho\rho}\right)$$ (3-22)

The total number of particles with spin-isospin j is given by the integral of f over space. One would get the total number of particle as sum of the occupation probabilities, as expected

$$2N = \sum_{\beta}f_{\rho\rho}$$ (3-23)

For the deviation distribution function d, we have

$$d(R\ell jj') = -i \int d^3s\, e^{-i\vec{\ell}\cdot\vec{s}}\, K(\vec{R}+\tfrac{\vec{s}}{2};jy)\, k^*(y\; \vec{R}-\tfrac{\vec{s}}{2};j')$$

$$= \sum_\beta (-2)\, \frac{(2\pi)^3}{V}\, \delta_{jj'}\; \delta(\vec{\ell}-\vec{\lambda}_\beta)\; k_{\beta\vec{\beta}}\, k_{\vec{\beta}\beta} \; . \qquad (3\text{-}24)$$

One may get for the total deviation

$$\langle \phi_{HB}|\,(2\hat{N}\, -\, \langle 2\hat{N}\rangle)^2\,|\phi_{HB}\rangle = \int d^3R\, \frac{d\ell}{(2\pi)^3}\, d(R\ell jj')\Big|_{jj'}$$

$$= \sum_\beta -2 K_{\beta\vec{\beta}}\, k_{\vec{\beta}\beta} \;\; = \sum_\beta 2\,(v_\beta\, u_\beta)^2 . \qquad (3\text{-}25)$$

Where we have used the BCS analog of $k_{\uparrow\downarrow} = u_q v_q$ and $k_{\downarrow\uparrow} = -u_q v_q$.

Before ending this section, it would be interesting to derive the supplementary conditions in the Wigner representation. The equation (3-12) in the Wigner representation can be written as (we may suppress the spin and isospin coordinates for convenience).

$$f(R,\ell) = \int d^3s\, d^3r''\, e^{-i\vec{\ell}\cdot\vec{s}}\Big[\, \rho(\vec{R}+\tfrac{\vec{s}}{2};\vec{r}'')\,\rho(\vec{r}'',\vec{R}-\tfrac{\vec{s}}{2}) - K(\vec{R}+\tfrac{\vec{s}}{2};\vec{r}'')\, K^*(\vec{r}'',(\vec{R}-\tfrac{\vec{s}}{2}))\Big]$$

$$= \tfrac{1}{2} d(R,\ell) + \int d^3s\, d^3r''\, e^{-i\vec{\ell}\cdot\vec{s}}\, \rho(\vec{R}+\tfrac{\vec{s}}{2},\vec{r}'')\, \rho(\vec{r}'';\vec{R}-\tfrac{\vec{s}}{2})$$

$$= \tfrac{1}{2} d(R,\ell) + \int d^3s\, d^3r''\, \frac{d^3q}{(2\pi)^3}\, \frac{d^3\tau}{(2\pi)^3}\, e^{i\vec{q}\cdot(\vec{R}+\tfrac{\vec{s}}{2}-\vec{r}'')}\, f[(\vec{R}+\tfrac{\vec{s}}{2}+\vec{r}'')/2,\, q] \times$$

$$e^{i\vec{\tau}\cdot(\vec{r}''+\tfrac{\vec{s}}{2}-\vec{R})}\, f\,[\,(\vec{R}+\vec{r}''-\tfrac{\vec{s}}{2})/2\,,\, \tau)]\; e^{-i\vec{\ell}\cdot\vec{s}}$$

$$= \tfrac{1}{2} d(R,\ell) + \int d^3s\, d^3r''\, \frac{d^3q}{(2\pi)^3}\, \frac{d^3\tau}{(2\pi)^3}\, \Big[e^{-i\vec{\ell}\cdot\vec{s}}\, e^{i(\frac{R}{2}-\frac{r'}{2}-s/4)\cdot\vec{D}_R^{(1)}}\, e^{i(\vec{q}-\vec{\tau})\cdot\vec{D}_q^{(1)}}$$

$$e^{i(\vec{q}-\vec{\tau})\cdot\vec{D}_q^{(2)}}\, e^{i\vec{q}\cdot(\vec{R}+\tfrac{\vec{s}}{2}-\vec{r}'')}\, e^{i\vec{\tau}\cdot(\vec{r}''+\tfrac{\vec{s}}{2}-\vec{R})}\,\Big]\, f^{(1)}(R,q)\, f^{(2)}(q,\ell) .$$

$$(3\text{-}26)$$

Where in the third line we used $\mathcal{P}$ as the inverse fourier transform of f, and in the fourth line we have introduced the spatial and momentum shift operators

$$\vec{D}_R = -i\vec{\nabla}_R \tag{3-27}$$

$$\vec{D}_{\vec{z}} = -i\vec{\nabla}_{\ell}. \tag{3-28}$$

In the last line the superscripts (1) and (2) indicate which distribution function the shift operators apply to. All the integrals in the equation (3-26) are simple, so may easily done to yield:

$$f(R,t) = \frac{1}{\lambda} d(R,t) + e^{-\frac{i}{2}\left(\vec{D}_R^{(1)}\cdot\vec{D}_{\vec{z}}^{(2)} - \vec{D}_{\vec{z}}^{(1)}\vec{D}_R^{(2)}\right)} f^{(1)}_{(R,t)} f^{(2)}_{(R,t)} \tag{3-29}$$

The second supplementary condition in coordinate space is

$$f(x\,x'') \; K(x''x') = K(x\,x'') f(x''\,x'). \tag{3-30}$$

In Wigner representation, this has the form:

$$e^{-\frac{i}{2}\left(\vec{D}_R^{(1)}\cdot\vec{D}_{\vec{z}}^{(2)} - \vec{D}_{\vec{z}}^{(1)}\cdot\vec{D}_{\vec{z}}^{(1)}\right)} f_{(R,t)}^{(1)} g^{(2)}_{(R,t)} = e^{\frac{i}{2}\left(\vec{D}_{\vec{z}}^{(2)}\cdot\vec{D}_R^{(1)} + \vec{D}_{\vec{z}}^{(1)}\cdot\vec{D}_R^{(2)}\right)} g^{(1)}_{(R,t)} f_{(R,t)}^{(2)}. \tag{3-31}$$

For the derivation of equation (3-31) one could use the same method as the first supplementary condition. It is also interesting to work out $d(R,t)$ in terms of $g_{(R,t)}$ and the manipulations are almost equivlent to those in equation (3-29). Therefore, without repeating the manipulations, we

give the result as

$$d(R,\lambda) = -\lambda\, e^{\frac{i}{2}\left(\vec{D}_R^{(1)}\cdot\vec{D}_\lambda^{(2)} + \vec{D}_\lambda^{(1)}\cdot\vec{D}_R^{(2)}\right)} g^{(1)}(R,\lambda)\, g^{(2)}(R-\lambda).\qquad (3\text{-}32)$$

Thus, equation (3-29) can also be written in the form:

$$f(R,\lambda) = e^{-\frac{i}{2}\left(\vec{D}_R^{(1)}\,\vec{D}_\lambda^{(2)} - \vec{D}_\lambda\cdot\vec{D}_R^{(2)}\right)} f^{(1)}(R,\lambda)\, f^{(2)}(R,\lambda)$$
$$- e^{\frac{i}{2}\left(\vec{D}_R^{(1)}\cdot\vec{D}_\lambda^{(2)} + \vec{D}_\lambda^{(1)}\cdot\vec{D}_R^{(2)}\right)} g^{(1)}(R,\lambda)\, g^{(2)}(R-\lambda)\qquad (3\text{-}33)$$

For BCS with plane wave for the single particle wave function equation (3-33) is equivalent to the simple relation

$$V_k^2 = V_k^4 + V_k^2 U_k^2 .\qquad (3\text{-}34)$$

Where we used the analogy of $f(R,\lambda) = f_{kk} = V_k^2$, $g(R-\lambda)|_k = k_{k\bar{k}} = U_k V_k$ and $g(R-\lambda)|_k) = k_{\bar{k}\lambda} = -U_k V_k$. The second supplementary condition (3-32) reduces to another identity

$$V_k^2 U_k V_k = U_k V_k V_k^2 \qquad (3\text{-}35)$$

for the simple Bcs system.

3.2 TDHB equations in the Wigner representation.

The TDHB equations (2-74) and (2-75) in the coordinate representation assume the form

$$i\frac{\partial}{\partial t} \rho(x x') = \eta(x\,x'') \rho(x''x') - \rho(x x'')\eta(x''x') + \Delta(x x'') k(x''x') - k(x x'')\Delta(x''x')$$
$$(3\text{-}36)$$

and

$$i \frac{\partial}{\partial t} \, \kappa(x \, x') = \zeta(x \, x'') \, k(x''x') + \kappa(x \, x'') \, h_\zeta^*(x''x') \; - \Delta(x \, x') + \mathcal{S}(x \, x'') \, \Delta(x''x') + \Delta(x \, x'') \mathcal{S}_\zeta^*(x''x').$$

$$(3\text{-}37)$$

Again, we used the convention that repeating variables are summed. We define one body thermodynamic potential density ζ and pairing density potential Δ as

$$\zeta(x \, x') = \psi_\alpha^*(x) \left(-\frac{\nabla^2}{2} - \lambda \right) \psi_\alpha(x') + V(x \, \gamma, x' \, \beta) \mathcal{S}(\gamma' \gamma),$$

$$(3\text{-}38a)$$

$$\Delta(x \, x') = \frac{1}{2} \, V(x \, x' \, \gamma \, \gamma') \, \kappa(\gamma' \gamma),$$

$$(3\text{-}39)$$

where

$$V(x \, \gamma, x' \, \gamma') = \frac{1}{2} \left(\psi_\alpha^*(x) \, \psi_\beta(\gamma) - \psi_\alpha^*(\gamma) \, \psi_\beta(x) \right) V(x \, \gamma, x' \, \gamma') \left(\psi_\alpha(x') \, \psi_\beta(\gamma') - \psi_\alpha(\gamma') \, \psi_\beta(x') \right).$$

$$(3\text{-}40)$$

As we discussed in section 2.3, the inclusion of the chemical potential in equations (3-36) and (3-37) is not necessary. These equations can be replaced by equations (2-122) and (2-123). We keep the TDHB equations (3-36) and (3-38a) in this form in order to have appropriate limit of the static solutions. But in the following equations in this chapter the thermodynamic potential ζ and pairing potential Δ can be replaced by λ and $\mathcal{S}$ respectively, without any change in the physical content of the equations. ζ , Δ and $\mathcal{S}$ are defined by equations (3-38a), (3-39) and (2-124) respectively, and h can be written as

$$h(x \, x') = \psi_\alpha^*(x) \left(-\frac{\nabla^2}{2} \right) \psi_\alpha(x') + V(x \, \gamma \, x' \, \gamma') \, \mathcal{S}(\gamma' \gamma).$$

$$(3\text{-}38b)$$

Before recasting TDHB equations in the Wigner representation, let us derive
the equation of motion for the deviation distribution function in the
coordinate space. Utili zing equation (3-37) and its conjugate, we obtain

$$i \frac{\partial}{\partial t} \left(K(xx') \overset{*}{K}(x''x') \right) = \eta(x y) k(y y')k^{+}_{}(y'x') - k(xy) \overset{\alpha}{k}(y y') \eta(y'x') - \Delta(xy) \overset{*}{k}(y x')$$

$$+ k(x y) \Delta^{\alpha}(y x') + f(xy) \Delta(y y') k^{*}(y x') + \Delta(xy) f(y y') k^{*}(y'x') - k(xy) f^{+}(y y') \Delta(y'x')$$

$$- k(xy) \overset{\alpha}{\Delta}(y y') f(y'x') \tag{3-41}$$

or

$$i \frac{\partial}{\partial t} D(xx') = \eta(xy) D(yx') - D(xy) \eta(yx') + 2 \Delta(xy) \overset{*}{k}(yx') - 2 k(xy) \overset{*}{\Delta}(yx')$$

$$- 2 f(xy) \Delta(y y') k^{*}(y'x') - 2 \Delta(xy) f(y y') k^{*}(y'x') + 2 k(xy) f^{0}(y y') \Delta(y'x')$$

$$+ 2 k(xy) \Delta^{\alpha}(y y') f(y'x') \tag{3-42}$$

Comparing equations (3-36) and (3-42), there are some similarities between
them, which we will discuss in their Wigner representation from at end
of this section. We now recast the TDHB equations in coordinate space and
Winger representation. Let us take the Fourier transform of equation
(3-36)

$$\int d\vec{s} \, e^{-i\vec{k}\cdot\vec{s}} \left[i \frac{\partial}{\partial t} f(\vec{R}+\frac{\vec{s}}{2}, \vec{R}-\frac{\vec{s}}{2}) \right] = \int d\vec{s} \, e^{-i\vec{k}\cdot\vec{s}} \left[\eta(\vec{R}+\frac{\vec{s}}{2}, \vec{r}'') f(\vec{r}'', \vec{R}-\frac{\vec{s}}{2}) \right.$$

$$\left. - f(\vec{R}+\frac{\vec{s}}{2}, \vec{r}'') \eta(\vec{r}'', \vec{R}-\frac{\vec{s}}{2}) + \Delta(\vec{R}+\frac{\vec{s}}{2}, \vec{r}'') k^{*}(\vec{r}'', \vec{R}-\frac{\vec{s}}{2}) - k(\vec{R}+\frac{\vec{s}}{2}, \vec{r}'') \Delta^{*}(\vec{r}''; \vec{R}-\frac{\vec{s}}{2}) \right] \tag{3-43}$$

We may write f in terms of its fourier transform $\underline{f}$, and define $\eta(R, \lambda)$

and Δ in analogy with equation (3-16)

$$\eta(R,\ell) = \int d^3s \; e^{-i\vec{\ell}\cdot\vec{s}} \eta(\vec{R}+\tfrac{\vec{s}}{2}, \vec{R}-\tfrac{\vec{s}}{2}) , \qquad (3\text{-}44)$$

$$\Delta(R,\ell) = \int d^3s \; e^{-i\vec{\ell}\cdot\vec{s}} \Delta(\vec{R}+\tfrac{\vec{s}}{2}, \vec{R}-\tfrac{\vec{s}}{2}) . \qquad (3\text{-}45)$$

With analogous manipulation as we had in recasting supplementary conditions, we obtain

$$i\frac{\partial}{\partial t} f(R,\ell) = \left[e^{\frac{i}{2}(\vec{D}_R^{(1)}\cdot\vec{D}_\ell^{(2)} - \vec{D}_\ell^{(1)}\cdot\vec{D}_R^{(2)})} - e^{-\frac{i}{2}(\vec{D}_R^{(1)}\cdot\vec{D}_\ell^{(2)} - \vec{D}_\ell^{(1)} D_R^{(1)})}\right] f^{(1)}(R,\ell) \, \eta^{(2)}(R,\ell)$$

$$+ e^{\frac{i}{2}(\vec{D}_\ell^{(2)}\cdot\vec{D}_R^{(1)} + \vec{D}_\ell^{(1)}\cdot\vec{D}_R^{(2)})} \left[\Delta^{(1)}(R,\ell) \, g^{(2)}(R,\ell) - g^{(1)}(R,\ell) \, \Delta^{*(2)}(R,\ell)\right] . \qquad (3\text{-}46)$$

Similar analysis can be applied for the second equations of motion (3-37), the result is

$$i\frac{\partial}{\partial t} g(R,\ell) = -\Delta(R,\ell) + e^{\frac{i}{2}(\vec{D}_R^{(2)}\cdot\vec{D}_\ell^{(1)} - \vec{D}_\ell^{(2)}\cdot\vec{D}_R^{(1)})} \left[\eta^{(1)}(R,\ell) \, g^{(2)}(R,\ell) + f^{(1)}(R,\ell)\, \Delta^{(2)}(R,\ell)\right]$$

$$+ e^{\frac{i}{2}(\vec{D}_\ell^{(2)}\cdot\vec{D}_R^{(1)} + \vec{D}_\ell^{(1)}\cdot\vec{D}_R^{(2)})} \left[g^{(1)}(R,\ell) \, \eta^{*(2)}(R,\ell) + \Delta^{(1)}(R,\ell) \, f^{(2)}(R,\ell)\right] . \qquad (3\text{-}47)$$

where the shift operator are defined by equations (3-27) and (3-28).

Recasting the deviation equation (3-42) in coordinate space in the Wigner representation and utilizing equation (3-41), we may write

$$
\int d\vec{s}\, e^{-i\vec{k}\cdot\vec{s}}\,\tfrac{i\partial}{\partial t}\big[K(\vec{R}+\tfrac{\vec{s}}{2},\vec{r}'')\,k(\vec{r}'',\vec{R}-\tfrac{\vec{s}}{2})\big] = \int d\vec{s}\, e^{-i\vec{k}\cdot\vec{s}}\Big[\, \eta(\vec{R}+\tfrac{\vec{s}}{2},\vec{r})\,K(\vec{r}\,\vec{r}')\,k(\vec{r}',\vec{R}-\tfrac{\vec{s}}{2})
$$

$$
- k(\vec{R}+\tfrac{\vec{s}}{2},r)\,k(\vec{r},\vec{r}')\,\eta(\vec{r}'',\vec{R}-\tfrac{\vec{s}}{2}) - \Delta(\vec{R}+\tfrac{\vec{s}}{2},r)\,k(\vec{r},R-\tfrac{\vec{s}}{2}) + k(\vec{R}+\tfrac{\vec{s}}{2},r)\,\Delta(\vec{r},\vec{R}-\tfrac{\vec{s}}{2})
$$

$$
+ \rho(\vec{R}+\tfrac{\vec{s}}{2},r)\,\Delta(\vec{r},\vec{r}'')\,k(\vec{r}'',\vec{R}-\tfrac{\vec{s}}{2}) + \Delta(\vec{R}+\tfrac{\vec{s}}{2},r)\,\rho(\vec{r}\,\vec{r}')\,k(\vec{r}',\vec{R}-\tfrac{\vec{s}}{2})
$$

$$
- k(\vec{R}+\tfrac{\vec{s}}{2},r)\,\rho(\vec{r}\,\vec{r}')\,\Delta(\vec{r}',\vec{R}-\tfrac{\vec{s}}{2}) - k(\vec{R}+\tfrac{\vec{s}}{2},\vec{r})\,\Delta(\vec{r},\vec{r}'')\,\rho(\vec{r}'',\vec{R}-\tfrac{\vec{s}}{2})\Big].
$$

$$(3\text{-}48)$$

The left hand side of the above equation is just half of the partial time derivative of the deviation distribution function $d(R,k)$. In the right hand side of the equation, the first four terms are similar to the ones in the equations of motion. Thus they are easy to be recasted in the Wigner representation. For example, let us recast one of the remaining terms, say,

$$
\int d\vec{s}\, e^{-i\vec{k}\cdot\vec{s}}\, \rho(\vec{R}+\tfrac{\vec{s}}{2},r)\,\Delta(\vec{r},\vec{r}'')\,k(\vec{r}'',\vec{R}-\tfrac{\vec{s}}{2})
$$

$$(3\text{-}49)$$

writting ρ, Δ and k in terms of their fourier transform $\rho(R,k)$, $\Delta(R,k)$ and $g(R,k)$, and using the shift operators (3-27) and (3-28), we obtain

$$
\int d\vec{s}\, d\vec{r}\, d\vec{r}''\cdot \frac{d\vec{q}}{(2\pi)^3}\frac{d\vec{t}}{(2\pi)^3}\frac{d\vec{p}}{(2\pi)^3}\, e^{i\vec{q}\cdot(\vec{R}+\frac{\vec{s}}{2}-\vec{r})}\, e^{-\frac{i}{2}(\vec{r}+\frac{\vec{s}}{2}-\vec{R})\cdot\vec{D}_{q}^{(4)}}\, e^{i(\vec{t}-\vec{q})\cdot\vec{D}_{k}^{(4)}}\, \rho(R,k)\times
$$

$$e^{i\,\vec{k}\cdot(\vec{r}-\vec{r}\,')}\quad e^{-\frac{i}{2}(\vec{r}+\vec{r}\,'-2\vec{R})\cdot\vec{D}_{\vec{R}}^{(2)}}\quad e^{i(\vec{k}-\vec{c})\cdot\vec{D}_{\vec{k}}^{(2)}}\quad \Delta^{(2)}(R,k)\quad e^{i\vec{k}\cdot(\vec{r}-\vec{R}+\vec{s}_{\vec{k}})}\times$$

$$e^{-\frac{i}{2}(\vec{r}\,'-\vec{s}_{\vec{k}}-\vec{R})\cdot\vec{D}_{\vec{R}}^{(3)}}\quad e^{i(\vec{k}-\vec{p})\cdot\vec{D}_{\vec{k}}^{(3)}}\quad g^{(3)\times}(R,k)\quad . \tag{3.-50}$$

The integration are easy to carry out and the result is

$$e^{\frac{i}{2}\left[\,(\vec{D}_{\vec{k}}^{(3)}-\vec{D}_{\vec{k}}^{(1)})\cdot\vec{D}_{\vec{R}}^{(1)}+(\vec{D}_{\vec{k}}^{(1)}+\vec{D}_{\vec{k}}^{(3)})\vec{D}_{\vec{R}}^{(2)}+(\vec{D}_{\vec{k}}^{(1)}+\vec{D}_{\vec{k}}^{(2)})\vec{D}_{\vec{R}}^{(3)}\right]}\;f^{(1)}(R,k)\;\Delta^{(2)}(R,k)\;g^{(3)\times}(R,k) \tag{3-51}$$

with some manipulations similar to the sample term (3-50), the remaining terms can be derived. The result are

$$e^{\frac{i}{2}\left[\,(\vec{D}_{\vec{k}}^{(2)}+\vec{D}_{\vec{k}}^{(3)})\cdot\vec{D}_{\vec{R}}^{(1)}+(\vec{D}_{\vec{k}}^{(1)}+\vec{D}_{\vec{k}}^{(3)})\vec{D}_{\vec{R}}^{(2)}+(\vec{D}_{\vec{k}}^{(1)}+\vec{D}_{\vec{k}}^{(2)})\cdot\vec{D}_{\vec{R}}^{(3)}\right]}\;\Delta^{(1)}(R,k)\;f^{(2)}(R,-k)\;g^{(3)}(R-k) \tag{3.-52}$$

$$e^{\frac{i}{2}\left[\,(\vec{D}_{\vec{k}}^{(2)}+\vec{D}_{\vec{k}}^{(3)})\vec{D}_{\vec{R}}^{(1)}+(\vec{D}_{\vec{k}}^{(1)}+\vec{D}_{\vec{k}}^{(3)})\cdot\vec{D}_{\vec{R}}^{(2)}+(\vec{D}_{\vec{k}}^{(1)}-\vec{D}_{\vec{k}}^{(2)})\vec{D}_{\vec{R}}^{(3)}\right]}\;g^{(1)}(R,k)\;f^{(2)}(R,-k)\;\Delta^{(3)}(R-k) \tag{3-53}$$

$$e^{\frac{i}{2}\left[\,(\vec{D}_{\vec{k}}^{(2)}-\vec{D}_{\vec{k}}^{(3)})\cdot\vec{D}_{\vec{R}}^{(1)}+(\vec{D}_{\vec{k}}^{(1)}-\vec{D}_{\vec{k}}^{(3)})\cdot\vec{D}_{\vec{R}}^{(2)}+(\vec{D}_{\vec{k}}^{(1)}-\vec{D}_{\vec{k}}^{(2)})\cdot\vec{D}_{\vec{R}}^{(3)}\right]}\;g^{(1)}(R,k)\;\Delta^{(2)}(R-k)\;f^{(3)}(R,k) \;. \tag{3-54}$$

Collecting the various terms, the equation of motion for the deviation distribution function d in the Wigner representation is

$$i\frac{\partial}{\partial t}d(R,k) = \left[\,e^{\frac{i}{2}(\vec{D}_{\vec{R}}^{(1)}\cdot\vec{D}_{\vec{k}}^{(2)}-\vec{D}_{\vec{k}}^{(1)}\cdot\vec{D}_{\vec{R}}^{(2)})}-e^{-\frac{i}{2}(\vec{D}_{\vec{R}}^{(1)}\cdot\vec{D}_{\vec{k}}^{(2)}-\vec{D}_{\vec{k}}^{(1)}\cdot\vec{D}_{\vec{R}}^{(2)})}\right]d^{(1)}(R,k)\,\eta^{(2)}(R,k)$$

$$-2\,e^{\frac{i}{2}(\vec{D}_{\vec{k}}^{(2)}\cdot\vec{D}_{\vec{R}}^{(1)}+\vec{D}_{\vec{k}}^{(1)}\cdot\vec{D}_{\vec{R}}^{(2)})}\left[\,g^{(1)}(R,k)\,\Delta^{(2)\times}(R-k)-\Delta^{(1)}(R,k)\,g^{(2)}(R-k)\right]$$

$$-2\,e^{\frac{i}{2}\left[\,(\vec{D}_{\vec{k}}^{(3)}-\vec{D}_{\vec{k}}^{(2)})\cdot\vec{D}_{\vec{R}}^{(1)}+(\vec{D}_{\vec{k}}^{(1)}+\vec{D}_{\vec{k}}^{(3)})\cdot\vec{D}_{\vec{R}}^{(2)}+(\vec{D}_{\vec{k}}^{(1)}+\vec{D}_{\vec{R}}^{(2)})\cdot\vec{D}_{\vec{R}}^{(3)}\right]}$$

$$f^{(1)}(R,k)\,\Delta^{(2)}(R,k)\,g^{(3)\times}(R-k)$$

$$-i \, e^{\frac{i}{2}\left[\, (\vec{D}_{\vec{z}}^{(3)} - \vec{D}_{\vec{z}}^{(2)}) \cdot \vec{D}_{\vec{R}}^{(1)} + (\vec{D}_{\vec{z}}^{(1)} + \vec{D}_{\vec{z}}^{(3)}) \cdot \vec{D}_{\vec{R}}^{(2)} + (\vec{D}_{\vec{z}}^{(1)} - \vec{D}_{\vec{z}}^{(2)}) \cdot \vec{D}_{\vec{R}}^{(3)} \, \right]} \Delta^{(1)}(R,k) \, f^{(2)}(R-k) \, g^{(3)}(R-k)$$

$$+ i \, e^{\frac{i}{2}\left[\, (\vec{D}_{\vec{z}}^{(2)} + \vec{D}_{\vec{z}}^{(3)}) \cdot \vec{D}_{\vec{R}}^{(1)} + (\vec{D}_{\vec{z}}^{(1)} + \vec{D}_{\vec{z}}^{(3)}) \cdot \vec{D}_{R}^{(2)} + (\vec{D}_{\vec{z}}^{(1)} - \vec{D}_{\vec{z}}^{(2)}) \cdot \vec{D}_{\vec{R}}^{(3)} \, \right]} g^{(1)}(R,k) \, f^{(2)}(R-k) \, \Delta^{(3)}(R-k)$$

$$+ i \, e^{\frac{i}{2}\left[\, (\vec{D}_{\vec{z}}^{(2)} - \vec{D}_{\vec{z}}^{(3)}) \cdot \vec{D}_{R}^{(1)} + (\vec{D}_{\vec{z}}^{(1)} - \vec{D}_{\vec{z}}^{(3)}) \cdot \vec{D}_{\vec{z}}^{(2)} + (\vec{D}_{\vec{z}}^{(1)} - \vec{D}_{\vec{z}}^{(2)}) \cdot \vec{D}_{\vec{R}}^{(3)} \, \right]} g^{(1)}(R,k) \, \Delta^{(2)}(R-k) \, f^{(3)}(R,k) .$$

$$(3-55)$$

Some observations about the equations of motion: First of all, as already mentioned in the begining, we may disregard the Lagrange multiplier λ in the dynamics. Secondly, for the one-body thermodynamic potential density in the Wigner representation (3-44), we obtain

$$\eta(R,k) \;=\; \tfrac{1}{2} k^2 - \lambda + W(R,k) \, ,$$
$$(3-56)$$

where

$$W(R,k) \;=\; \int d\vec{s} \; e^{-i\vec{k}\cdot\vec{s}} \; V\!\left(\vec{R}+\tfrac{\vec{s}}{2}, y \;\; \vec{s}-\tfrac{\vec{s}}{2}, y'\right) \, f(y,y) .$$
$$(3-57)$$

As mentioned above, we may disregard the Lagrange multiplier λ , and hence $\eta(R,k)$ can be replaced by

$$h(R,k) \;=\; \tfrac{1}{2} k^2 + W(R,k) \, .$$
$$(3-58)$$

Substituting equation (3-56) or (3-58) in equation (3-46) and carring out some algebra, the equation of motion for the distribution function f reduces to

$$\frac{\partial f}{\partial t} + \vec{k}\cdot\vec{\nabla}_{\vec{z}} \, f(R,k) + 2 \sin \tfrac{1}{2}\left(\vec{\nabla}_{\vec{R}}^{(1)} \, \vec{\nabla}_{\vec{k}}^{(2)} - \vec{\nabla}_{\vec{z}}^{(1)} \cdot \vec{\nabla}_{\vec{R}}^{(2)} \right) f^{(1)}(R,k) \, W^{(2)}(R,k)$$

$$+i\, e^{\frac{i}{2}\left(\vec{D}_{\vec{L}}^{(2)}\cdot\vec{D}_{\vec{R}}^{(I)}+\vec{D}_{\vec{L}}^{(I)}\cdot\vec{D}_{\vec{R}}^{(2)}\right)}\left[\Delta^{(I)}(R,\lambda)\, g^{(2)*}(R-\lambda) - g^{(I)}(R,\lambda)\,\Delta^{(2)*}(R,\lambda)\right]=0.$$

<div align="right">(3-59)</div>

Note the similarity between the first three terms of equation (3-59) and the collisionless Boltzman equation for a system in an external potential

$$\frac{\partial f}{\partial t} + \vec{k}\cdot\vec{\nabla}_{\vec{R}}\, f(R,\lambda) - \vec{\nabla} U\cdot\vec{\nabla}_{\vec{L}}\, f(R,\lambda) = 0.$$

<div align="right">(3-60)</div>

The last two terms are certainly a modification due to the abondonment of a fixed number of particles, and can be considered as the collison term in Boltzman equation. Similarly, the equation of motion for the deviation distribution function reduces to

$$\frac{\partial}{\partial t} d(R,\lambda) + \vec{k}\cdot\vec{\nabla}_{\vec{R}}\, d(R,\lambda) + 2\sin\frac{1}{2}\left(\vec{\nabla}_{\vec{R}}^{(I)}\cdot\vec{\nabla}_{\vec{L}}^{(2)} - \vec{\nabla}_{\vec{L}}^{(I)}\,\vec{\nabla}^{(2)}\right) d(R,\lambda)\, W(R,\lambda)$$

$$+2i\, e^{\frac{i}{2}\left(\vec{D}_{\vec{L}}^{(2)}\cdot\vec{D}_{\vec{R}}^{(I)}+\vec{D}_{\vec{L}}^{(I)}\cdot\vec{D}_{\vec{L}}^{(2)}\right)}\left[g^{(I)}(R,\lambda)\,\Delta^{(2)*}(R-\lambda) - \Delta^{(I)}(R,\lambda)\, g^{(2)*}(R-\lambda)\right]$$

$$+2i\, e^{\frac{i}{2}\left[(\vec{D}_{\vec{L}}^{(3)}-\vec{D}_{\vec{L}}^{(2)})\vec{D}_{\vec{R}}^{(I)} + (\vec{D}_{\vec{L}}+\vec{D}_{\vec{L}}^{(3)})\cdot\vec{D}_{\vec{R}}^{(2)} + (\vec{D}_{\vec{L}}^{(3)}+\vec{D}_{\vec{L}}^{(2)})\cdot\vec{D}_{\vec{R}}^{(3)}\right]} f^{(I)}(R,\lambda)\,\Delta^{(2)}(R\lambda)\, g^{(3)*}(R\lambda)$$

$$+2i\, e^{\frac{i}{2}\left[(\vec{D}_{\vec{L}}^{(2)}+\vec{D}_{\vec{L}}^{(3)})\vec{D}_{\vec{R}}^{(I)} + (\vec{D}_{\vec{L}}^{(I)}+\vec{D}_{\vec{L}}^{(3)})\cdot\vec{D}_{\vec{R}}^{(2)} + (\vec{D}_{\vec{L}}^{(I)}-\vec{D}_{\vec{L}}^{(2)})\cdot\vec{D}_{\vec{R}}^{(3)}\right]} \Delta^{(I)}(R,\lambda)\, f^{(2)}(R-\lambda)\, g^{(3)*}(R-\lambda)$$

$$-2i\, e^{\frac{i}{2}\left[(\vec{D}_{\vec{L}}^{(2)}+\vec{D}_{\vec{L}}^{(3)})\cdot\vec{D}_{\vec{L}}^{(I)} + (\vec{D}_{\vec{L}}^{(I)}+\vec{D}_{\vec{L}}^{(2)})\vec{D}_{\vec{R}}^{(2)} + (\vec{D}_{\vec{L}}^{(I)}-\vec{D}_{\vec{L}}^{(3)})\vec{D}_{\vec{R}}^{(3)}\right]} g^{(I)}(R,\lambda)\, f^{(2)}(R-\lambda)\,\Delta^{*}(R-\lambda)$$

$$-2i\, e^{\frac{i}{2}\left[(\vec{D}_{\vec{L}}^{(2)}-\vec{D}_{\vec{L}}^{(3)})\cdot\vec{D}_{\vec{R}}^{(I)} + (\vec{D}_{\vec{L}}^{(I)}-\vec{D}_{\vec{L}}^{(3)})\cdot\vec{D}_{\vec{R}}^{(2)} + (\vec{D}_{\vec{L}}^{(I)}-\vec{D}_{\vec{L}}^{(2)})\vec{D}_{\vec{R}}^{(3)}\right]} g^{(I)}(R,\lambda)\,\Delta^{(2)}(R,\lambda)\, f^{(3)}(R,\lambda) = 0$$

<div align="right">(3-61)</div>

This equation (3-61) is also similar to the Boltzman equation; and hence the distribution name for $\mathcal{J}$ can be justified. Thirdly, equation (3-59) and (3-61) together with equation (3-47) are a complete set of equations which describe the system. From this set of equations, it is apparent that a solution to the dynamical problem expressed in terms of the Wigner representation involves as much complexity, and therefore as much information, as does a solution in more abstract representation (2-74) and (2-75). Finally one may utilize these equations for a derivation of the semiclassical hydrodynamic equations by taking various moments of the distribution functions, similar to Koonin method for the TDHF approxmation.

Chapter IV

4 Restricted dynamical parametrization.

In section 2.2, we derived the TDHB equations from a varitional
point of view. It offered a classical interpretation for the equations
of motion as a system of equations for an infinite number of classical
particles. Although the TDHB equations give a complete description of
the system, the complexity of solving these equations is a problem.
Also, the interest of many physicists is more restricted to the time
evolution of a few macroscopic variables describing the system (such
as its quadruple or rms radius) than the fine details of motion of
each microscopic variables. Therefore a reduction in number of variables
is desirable and various methods are devised to do so. We briefly
review these methods (a) intuitive parmetrization: One assumes, the
time dependence of $R_{\alpha\beta}$ is through a few number of variables u_i , v_j i.e.

$$R_{\alpha\beta}(t) \longrightarrow R_{\alpha\beta}\left(u(t)\right) . \tag{4-1}$$

Then the Lagrangian reduces to

$$\mathcal{L} = \sum_j \dot{u}_j \; R^*_{\alpha\beta}(u) \frac{\partial}{\partial u_j} R_{\alpha\beta}(u) - H(u) . \tag{4-2}$$

One may obtain the equations of motion by using the Lagrange bracket.
The result is

$$\{u_i \; u_j\} \dot{u}_j = \frac{\partial H}{\partial u_i} . \tag{4-3}$$

Where the Lagrange bracket [23] is defined by

$$\{u_i, u_j\} = i\left[\frac{\partial}{\partial u_i} R_{ip}^{*} \frac{\partial}{\partial u_j} R_{ip} - \frac{\partial}{\partial u_j} R_{ip}^{*} \frac{\partial}{\partial u_i} R_{ip}\right].$$ (9-4)

The number of parameters is restricted only by the number of variables, and one may think of a complete description of the system by introducing as many parmeters as the number of degrees of freedom. (b) The parameters in the above procedure may not be canonical in general. The choice of a canonical parametrization for u_i , u_j - - - would result in a classical Hamiltonian $H(P_i, q_i)$ and the equations of motion reduce to Hamilton's equations. For a quantal description of these collective degrees of freedom, one may utilize this classical Hamiltonian in a " second quantization " to evaluate the spectrum of the collective energy. Although the justification of this procedure is an open question, one may obtain some useful information by comparison of this method's result with other methods describing the collective motion of the system. (c)In cases (a) and (b) the dependence on the parameters should be known a priori. One may modify these methods by introducing parameters through a constrained static calculation for the system. This method has been used in an adiabatic approach to the TDHF from a variational point of view.[23] In the following, after a brief review of the two fluid model, we

discuss the derivation of the two fluid model from a variational point of view. Various trial wave function has been used in the derivation of the Lagrangians describing the systems. The parameterization of the Lagrangian density is canonical for two of the models and for the third it is adiabaticaly canonical. Finally in the last section of this chapter, as an application an Irrotational fluid model similar to the Kerman-Koonin model is developed.

4.1 A two fluid model: First approach.

One of the successful theories for describing the peculiar behaviour of liquid helium is the two fluid model. This model, originally proposed and developed by Tisza[34], is an analogy to the structure of a degenerate ideal Bose gas. Landau,[35] with quantization of the hydrodynamic equations advanced the theory. In his paper, with a classical analogy, he reached the conclusion that there is not a continuous transition between the Irrotational portion of the quantum liquid state $\vec{\nabla} \wedge \vec{v} = 0$, and the vortex $\vec{\nabla} \wedge \vec{v} \neq 0$ part. Thus the energy spectrum of liquid can be divided into

two spectra, with a certain gap between the states of the Irrotational

(phonons) and the vortex motion (roton). Feynman[36] with his interpretation

of liquid helium as a system which exhibits quantum mechanical behaviour

on a large scale, and with his approach of finding some explanation for

phonon and roton excitation from first principles, put some light on the

theory. BCS theory also helped us to understand some aspect of super-

fluidty. On the microscopic level, there is a derivation of the hydro-

dynamic equation starting with a time dependent BCS trial wave function.[37]

In nuclear physics, as we already know, pair correlations are impor-

tant. Thus, in analogy to liquid helium, one might expect some aspect of

microscopic phenomena in nuclear phyics experimently. In astrophysics

this phenomena may be important in the dynamics of neutron stars. In

section 2.2 we derived the Lagrangian for a many body system with HB

trial wave function, it can be written as

$$\mathcal{L} = i \, \rho^{\alpha}_{\gamma\beta} \, \frac{\partial}{\partial t} \, \rho_{\alpha\beta} + i \, K^{\nu}_{\gamma\beta} \, \frac{\partial}{\partial t} \, K_{\gamma\beta} - T_{\alpha\gamma} \rho_{\gamma\alpha} - \frac{1}{2} \, V_{\alpha\beta\gamma\delta} \, \rho_{\gamma\alpha} \rho_{\delta\beta} - \frac{1}{4} \, V_{\alpha\beta\gamma\delta} \, K^{\nu}_{\beta\alpha} K_{\delta\gamma} .$$

$$(4-5)$$

As we discussed in section 2.3 the total number of particles is conserved

in TDHB theory, and the TDHB equations are independent of chemical

potential. Therefore the Lagrange multiplier λ need not be included in

the equations (4-5). Also in section 2.1, we discussed Baranger's argument

about the equivalence between Blatt's wave function and the BCS wave function. Now let us take $\varphi_{\gamma\beta}$ in its canonical representation, i. e. taking φ as a real, paired and antisymmertic second order tensor:

$$\varphi_{\gamma\beta} = \varphi_{\gamma\bar{\alpha}}\,\delta_{\beta\bar{\alpha}},\qquad (4\text{-}6)$$

where $\bar{\alpha}$ represents the state which is paired with α . Thus f and k will have simpler forms. Utilizing equations (2-46), (2-49) and (2-50), we obtain

$$f_{\alpha\beta} = f_{\alpha\alpha}\,\delta_{\alpha\beta},\qquad (4\text{-}7)$$

$$k_{\alpha\beta} = k_{\gamma\bar{\alpha}}\,\delta_{\beta\bar{\alpha}},\qquad (4\text{-}8)$$

where

$$f_{\gamma\alpha} = \frac{\varphi_{\alpha\bar{\alpha}}^{2}}{1+\varphi_{\alpha\bar{\alpha}}^{2}}\qquad (4\text{-}9)$$

and

$$k_{\gamma\bar{\alpha}} = \frac{\varphi_{\alpha\bar{\alpha}}}{1+\varphi_{\alpha\bar{\alpha}}^{2}}\qquad (4\text{-}10)$$

Now, let us assume the single particle wave function be time dependent, then the simplified version of the Lagrangian reduces to

$$\mathcal{L} = \int d^3r\ \psi_\alpha^*(rt)\,i\frac{\partial}{\partial t}\psi_\alpha(rt)\ f_{\alpha\alpha} - T_{\alpha\alpha}f_{\alpha\alpha} - \tfrac{1}{2}V_{\alpha\beta\alpha\beta}f_{\alpha\alpha}f_{\beta\beta} - \tfrac{1}{4}V_{\gamma\bar{\alpha}\,\beta\bar{\beta}}k_{\alpha\bar{\alpha}}k_{\bar{\beta}\beta}$$

$$(4\text{-}11)$$

Let us write the single particle wave function as

$$\psi_\alpha(\vec{r}t) = e^{i\chi_\alpha(\vec{r}t)} |\psi_\alpha(\vec{r}t)| , \qquad (4\text{-}12)$$

where χ_α and $|\psi_\alpha|$ are real functions. One could decompose the single particle wave function in the following manner: $|\psi_\alpha|$ is the real wave function which corresponds to the single particle state α with respect to the rest frame of the many body system. In the classical interpretation, χ_α can be viewed as a momentum density of a classical field and $|\psi_\alpha|^2 f_{\alpha\alpha}$ as a density field corresponding to the state . One can easily see that the momentum density χ_α and the density field $|\psi_\alpha|^2 f_{\alpha\alpha}$ are canonically conjugate to each other.

The corresponding Lagrangian is

$$\mathcal{L} = \int d\vec{r} \sum_\alpha \left[\chi_\alpha(\vec{r}) \dot{f}_\alpha(\vec{r}) - \tfrac{1}{2}(\nabla \chi_\alpha(\vec{r}))^2 f_{\alpha\alpha} \right] - \mathcal{E}\left(\{f_\alpha\}, \chi_\alpha\right). \qquad (4\text{-}13)$$

Which may be obtained by adding the total time derivative

$$\sum_\alpha \int d\vec{r} \; \frac{d}{dt} \chi_\alpha(\vec{r}t) \, f_\alpha(\vec{r}t) \qquad (4\text{-}14)$$

to equation (4-11), and defining functions

$$f_\alpha(\vec{r}t) = |\psi_\alpha(\vec{r}t)|^2 f_{\alpha\alpha} , \qquad (4\text{-}15)$$

and

$$\mathcal{E}\left(\{f_\alpha\}, \chi_\alpha\right) = \sum_\alpha \int d\vec{r} \, |\psi_\alpha(\vec{r}t)| \left(-\frac{\nabla^2}{2}\right) |\psi_\alpha(\vec{r}t)| \, f_{\alpha\alpha} + \sum_{\alpha\kappa} \tfrac{1}{2} V_{\alpha\kappa\alpha\kappa} f_{\alpha\alpha} f_{\kappa\kappa}$$
$$+ \sum_{\alpha\kappa} \tfrac{1}{4} V_{\alpha\bar\kappa\bar\alpha\kappa} \kappa_{\alpha\alpha} \kappa_{\kappa\kappa} . \qquad (4\text{-}16)$$

In this Lagrangian (4-13), we have a large number of degrees of
freedom. Thus, as we mentioned in the introduction of this chapter,
a reduction in the number of canonical variables is desriable.
Kerman-Koonin derived the continuty and the Euler's equation for
an Irrotational fluid, by taking a coherent phase for the single particle
wave function in TDHF theory. Encouraged by this derivation, let us
keep two phase in the Lagrangian. We may use the time independent
chemical potential λ for distingushing the two phasees. We define
$\varphi(\vec{r}t)$ and $\zeta(\vec{r}t)$ such that

$$\chi_\alpha(rt) = \begin{cases} \varphi(\vec{r}t) & \epsilon_\alpha > \lambda \\ \zeta(rt) & \epsilon_\alpha \leq \lambda \end{cases} \qquad (4\text{-}17)$$

where ϵ_α is the single particle energy. Utilizing equation (4-17)
in the Lagrangian (4-16) one obtains

$$\mathcal{L} = \int d\vec{r} \left[\varphi(\vec{r}) \dot{\rho}_u(\vec{r}t) + \zeta(\vec{r}t) \dot{\rho}_\ell(\vec{r}t) - \frac{1}{\lambda}(\nabla\varphi)^2 \rho_u(\vec{r}t) - \frac{1}{2}(\nabla\zeta)^2\rho_\ell(\vec{r}t) \right] - E\left(\rho_u, \rho_\ell, \varphi, \zeta \right)$$

$$(4\text{-}18)$$

where

$$\rho_u(\vec{r}t) = \sum_{\alpha > \lambda} \rho_\alpha(\vec{r}t) \qquad , \qquad (4\text{-}19)$$

and

$$\rho_\ell(\vec{r}t) = \sum_{\alpha \leq \lambda} \rho_\alpha(\vec{r}t) . \qquad (4\text{-}20)$$

In the derivation of the Lagrangian, we used an arbitrary local two body interaction potential. Physically the choice of taking coherent phases corresponds to freezing out a large number of degrees of freedom. Thus $E(\{\eta\},\{\psi\}, \varphi\text{-}\xi)$ can be viewed as the minimized energy as functionals of f_u, f_ℓ and an explicit function of $\varphi\text{-}\xi$. In other words, that means all other degrees of freedom are chosen to minimize E under the constraints of fixed $f_\ell(\vec{r},t)$, $f_u(\vec{r},t)$ and $\varphi\text{-}\xi$. One could derive the equations of motion for the Lagrangian $(4\text{-}18)$, but it is more interesting to define new variables

$$\psi = \tfrac{1}{\lambda}\left(\xi - \varphi\right) , \qquad (4\text{-}21)$$

$$\eta = \tfrac{1}{\lambda}\left(\xi + \varphi\right) , \qquad (4\text{-}22)$$

and

$$f = f_u + f_\ell , \qquad (4\text{-}23)$$

$$f_s = f_\ell - f_u . \qquad (4\text{-}24)$$

Utilizing equations $(4\text{-}21)\text{-}(4\text{-}24)$ in the Lagrangian $(4\text{-}18)$, we obtain

$$\mathcal{L} = \int d^3r \left[\eta \dot{f} + \psi \dot{f_s} - \tfrac{1}{\lambda}\left((\nabla\eta)^2 + (\nabla\psi)^2\right) f - (\nabla\eta \cdot \nabla\psi) f_s \right] - E\left(\{f\},\{f_s\}, \psi\right).$$

$$(4\text{-}25)$$

Thus, one can derive the equations of motion for the Lagrangian (4-25)

$$\frac{\delta \mathcal{L}}{\delta \eta} = 0 \quad \Rightarrow \quad \dot{\rho} + \vec{\nabla}\cdot(\rho\,\vec{\nabla}\eta + \rho_s\,\vec{\nabla}\psi) = 0, \qquad (4-26)$$

$$\frac{\delta \mathcal{L}}{\delta \rho} = 0 \quad \Rightarrow \quad \dot{\eta} + \frac{1}{2}\Big[(\nabla\eta)^2 + (\nabla\psi)^2\Big] + \frac{\delta E}{\delta \rho} = 0, \qquad (4-27)$$

$$\frac{\delta \mathcal{L}}{\delta \psi} = 0 \quad \Rightarrow \quad \dot{\rho}_s + \vec{\nabla}(\rho\,\vec{\nabla}\psi + \rho_s\,\nabla\eta) - \frac{\delta E}{\delta \psi} = 0, \qquad (4-28)$$

$$\frac{\delta \mathcal{L}}{\delta \rho_s} = 0 \quad \Rightarrow \quad \dot{\psi} + \vec{\nabla}\eta\cdot\vec{\nabla}\psi + \frac{\delta E}{\delta \rho_s} = 0. \qquad (4-29)$$

Define

$$\vec{v}_s = \nabla\psi, \qquad (4-30)$$

$$\rho_n = \rho - \rho_s, \qquad (4-31)$$

$$\vec{v}_n = \frac{\rho}{\rho_n}\vec{\nabla}\eta = \left(1 + \frac{\rho_s}{\rho_n}\right)\vec{\nabla}\eta. \qquad (4-32)$$

Taking the gradint of the equations (4-27), (4-29), and utilizing equations (4-30), (4-31) and (4-32) we obtain the equations of motion for the two fluid model as

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho_n \vec{v}_n + \rho_s \vec{v}_s) = 0 \ , \tag{4-33}$$

$$\frac{\partial \rho_s}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{v}_s + \frac{\rho_s \rho_n}{\rho} \vec{v}_n \right) - \frac{\delta \varepsilon}{\delta \psi} = 0 \ , \tag{4-34}$$

$$\frac{\partial \vec{v}_s}{\partial t} + \vec{\nabla} \left[\frac{\rho_n}{\rho} (\vec{v}_n \cdot \vec{v}_s) + \frac{\delta \varepsilon}{\delta \rho_s} \right] = 0 \ , \tag{4-35}$$

$$\frac{\partial \vec{v}_n}{\partial t} + \vec{v}_n \frac{\partial}{\partial t} \left(\ln \frac{\rho_n}{\rho} \right) + \frac{1}{2} \frac{\rho_s}{\rho_n} \vec{\nabla} \left[\left(\frac{\rho_s}{\rho} \right)^2 v_n^2 + \vec{v}_s^2 \right] + \frac{\rho}{\rho_n} \vec{\nabla} \frac{\delta \varepsilon}{\delta \rho} = 0 \ , \tag{4-36}$$

We observe that, $\vec{\nabla} \wedge \vec{v}_s = 0$ (superfluid is Irrotational), $\vec{\nabla} \wedge \vec{v}_n = \left(\vec{\nabla} \frac{\rho_s}{\rho_n} \right) \wedge \vec{\nabla} \psi$, (normal fluid contains the vortex motion). Due to the canonical behaviour of the varibles, the total energy would be conserved and thus one should not expect any vicosity term in the equations of motion. We will discuss small oscillations of the densities near thier equilibria in section (4-5).

4.2 A two fluid model: Second approach.

In section 4.1 we derived and discussed the two fluid model for a special case of the HB trial wave function. Therefore it would be interesting to derive the two fluid model with a more general HB trial wave function, which is our aim in this section. Let us introduce a new wave function:

$$| \phi \rangle = e^{i \psi(x) a^T(x) a(x)} | \phi_{HB} \rangle \tag{4-37}$$

Where $a^T(x)$($a(x)$) is the creation (destruction) field operator, X[33]

represents the spatial $\vec{r}$, spin σ_s and isospin σ_I coordinates; and

is the usual HB trial wave function in the coordinate representation.

The time dependent part of the Lagrangian can be written as

$$\langle \dot{\Phi} \mid i \frac{\partial}{\partial t} \mid \Phi \rangle = -\frac{\partial}{\partial t}\sigma(x) \langle \phi_{HB} \mid a^T(x)\, a(x) \mid \phi_{HB} \rangle + \langle \phi_{HB} \mid i\frac{\partial}{\partial t} \mid \phi_{HB} \rangle .$$

(4-38)

From equation (2-90) one may write

$$\langle \phi_{HB} \mid i\frac{\partial}{\partial t} \mid \phi_{HB} \rangle = \rho^{*}(x,y)\, i\frac{\partial}{\partial t}\, \rho(x,y) + k^{*}(x,y)\, i\frac{\partial}{\partial t}\, k(x,y) .$$

(4-39)

We assume that the two body density matrix elements have the properties

$$\rho(x,y) = e^{i\left(\varphi(x) - \varphi(y)\right)}\, \rho_r(x,y)$$

(4-40)

$$k(x,y) = e^{i\left(\varphi(x) + \varphi(y)\right)}\, k_r(x,y) ,$$

(4-41)

and

$$\rho_r(x,y) = \rho_r(y,x)$$

(4-42)

$$k_r(x,y) = - k_r(y,x) .$$

(4-43)

This form of the two body density matrix elements satisfies the
supplementary conditions equations (2-33) and (2-34). Employing
equations (4-40) and (4-41) for deriving the right hand side of
of equation (4-39), the final result reduces to

$$\langle \phi_{HB} | i\frac{\partial}{\partial t} \phi_{HB} \rangle = -\int \partial r \, \dot{\phi}(r) \, k(r) \qquad (4-44)$$

where

$$k(r) = 2 \sum_{\sigma_s \sigma_I} \int dr' \, |k_s(x,y)|^2 \qquad (4-45)$$

And for the time dependent part of the Lagrangian (4-38), we have

$$\langle \Phi | i\frac{\partial}{\partial t} | \Phi \rangle = -\int dr \left[\dot{\vartheta}(r) f(r) + \dot{\phi}(r) k(r) \right] \qquad (4-46)$$

where

$$f(r) = \sum_{\sigma_s \sigma_I} f(x x) \qquad (4-47)$$

The kinetic part of the classical Hamiltonian can be written as

$$\langle \Phi | \hat{T} | \Phi \rangle = T(x,y) \langle \phi_{HB} | e^{-i\vartheta(x) a(x)^T a(x')} a(x) a(y) e^{i\vartheta(y') a(y)^T a(y)} | \phi_{HB} \rangle \qquad (4-48)$$

We may expand the expoential coeficient of the trial wave function (4-37), then evaluate the right hand side of equation (4-48); and next expand the expectation value of the density operator (4-40). The final result for the kinetic energy (4-48) up to second order in terms ϕ and ϑ is

$$\langle \Phi | \hat{T} | \Phi \rangle = T(x,y) \left[1 - \frac{1}{2} \left(\phi(x) - \phi(y) + \vartheta(x) - \vartheta(y) \right)^2 \right] f(x,y) \qquad (4-49)$$

Now, let us use the explicit form of the kinetic energy operator

$$T(xx') = -\frac{1}{2} \nabla_r^2 \, \delta(r-r') \, \delta_{\sigma_I \sigma_{I'}} \, \delta_{\sigma_S \sigma_{S'}} \qquad (4-50)$$

in evaluating kinetic energy. We obtain

$$\langle \phi | \hat{T} | \phi \rangle = - \int d r \, d r' \, \delta(r-r') \, \nabla_{r'}^2 \, \int_r (x,x')/_2 + \frac{1}{2} \int d r \, (\vec{\nabla} \mathcal{G}(r) + \vec{\nabla} \psi(r))^2 \rho(r) \qquad (4-51)$$

Although we expand the kinetic energy to second order in terms of $\mathcal{G}$ and ψ, the final result is correct to all order of them. Finally, the potential energy term of the Hamiltonian can be obtained with a similar expansion as we had in the kinetic energy term. The result (to second oreder in terms of $\mathcal{G}$ and ψ) is

$$\langle \phi | \hat{V} | \phi \rangle = \frac{1}{4} \, V(x y x' y') \left[2 \rho_r(x x) \rho_\ell(y' y') + K(y x) K(y'x') \right] \Big\{ 1 -$$

$$\frac{1}{2} \left[(\psi(x) + \psi(y)) - (\psi(x') + \psi(y')) + (\mathcal{G}(x) + \mathcal{G}(y)) - (\mathcal{G}(x') + \mathcal{G}(y')) \right]^2 \Big\} \qquad (4-52)$$

In case of a local potential, the potential energy (4-52) is independent of $\mathcal{G}$ and ψ. Now putting together three terms of the Lagrangian, utilizing equations (4-51), (4-52) and (4-46); the Lagrangian reduces to the form

$$\mathcal{L} = - \int d r \, \left[\dot{\mathcal{G}}(r) k(r) + \dot{\psi}(r) \rho(r) + \frac{1}{2} (\nabla \mathcal{G} + \nabla \psi)^2 \rho(r) \right] - E \left(\{ \rho \}, \{ k \} \right) \qquad (4-53)$$

Where we assume a local potential for V , and as usual $E(\rho,\kappa)$ is the minimized energy under constraints of fixed ρ and κ. The equations of motion may be derived by Hamilton's principle.

$$\frac{\delta\mathcal{L}}{\delta\vartheta} = 0 \;\Rightarrow\; \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\,\vec{\nabla}\vartheta + \rho\,\vec{\nabla}\varphi\right) = 0 \;, \qquad (4\text{-}54)$$

$$\frac{\delta\mathcal{L}}{\delta\rho} = 0 \;\Rightarrow\; \dot{\vartheta} + \frac{1}{2}\left(\nabla\varphi + \nabla\vartheta\right)^{2} + \frac{\delta E}{\delta\rho} = 0 \;, \qquad (4\text{-}55)$$

$$\frac{\delta\mathcal{L}}{\delta\varphi} = 0 \;\Rightarrow\; \frac{\partial\kappa}{\partial t} + \vec{\nabla}\cdot\rho\left(\vec{\nabla}\varphi + \vec{\nabla}\vartheta\right) = 0 \;, \qquad (4\text{-}56)$$

$$\frac{\delta\mathcal{L}}{\delta\kappa} = 0 \;\Rightarrow\; \dot{\varphi} + \frac{\delta E}{\delta\kappa} = 0 \;. \qquad (4\text{-}57)$$

The equations (4-54) through (4-57) are a simple set of equations describing the motion of the two fluid model. One of the special properties of this model is having zero roots for dispersion relation relation, and we will discuss that property in more detial in section 4.5

4.3 A two fluid model: Third approach.

The motivation of this section is to derive a two fluid model from

direct use of the TDBCS wave function. Let us consider the trial wave function:

$$| \Phi \rangle = e^{i\hat{A}} \, e^{i\hat{\phi}} \, |BCS\rangle \qquad (4\text{-}58)$$

where

$$\hat{A} = \int d^3r \sum_k \chi(r t) \frac{1}{k} \left[\psi_{k\downarrow}^*(r) \, a(r\uparrow) \, a_{k\downarrow}^T + \psi_{k\uparrow}^*(r) \, a_{k\uparrow}^T \, a^T(r\downarrow) + \psi_{k\downarrow}(r) \, a_{k\downarrow} \, a(r\uparrow) \right.$$
$$\left. + \psi_{k\uparrow}(r) \, a(r\downarrow) a_{k\uparrow} \right], \qquad (4\text{-}59)$$

$$\hat{\phi} = \int d^3r \, g(r t) \left[a^T(r\uparrow) \, a(r\uparrow) + a^T(r\downarrow) a(r\downarrow) \right], \qquad (4\text{-}60)$$

and BCS is the usual TDBCS wave function:

$$|BCS\rangle = \prod_k \left(U_k + V_k \, a_{k\uparrow}^T a_{k\downarrow}^T \right) |0\rangle, \qquad (4\text{-}61)$$

with u_k and v_k assumed to be real and time dependent.

In the adiabatic approximation to the Lagrangian, the potential velocities g and χ are assumed to be small enough to permit an expansion of these phase operators, retaining only terms up to second order in the phases. The time dependent part of the Lagrangian can be written as

$$\langle \Phi | i \frac{\partial}{\partial t} | \Phi \rangle = - \int d^3r \left[\dot{g}(r) \, \rho(r) + \dot{\chi}(r) \, \kappa(r) \right] \qquad (4\text{-}62)$$

where $\rho(r)$ and $\kappa(r)$ are defined by

$$\rho(r) = \sum_k \left(|\psi_{k\uparrow}(r)|^2 + |\psi_{k\downarrow}(r)|^2 \right) V_k^2, \qquad (4\text{-}63)$$

and

$$K(r) = \sum_{k} 2 \left(|\psi_{k\uparrow}(r)|^2 + |\psi_{k\downarrow}(r)|^2 \right) u_k v_k f_k .$$

$$(4\text{-}64)$$

χ and K are adiabaticly canonical variables as one may notice from equation (4-62), by retaining terms higher than second order in the phases. For a simple potential such as

$$\hat{V} = \sum_{kk'} V_{kk'} \, a_{k\uparrow}^{T} a_{k\downarrow}^{T} \, a_{k'\downarrow} a_{k'\uparrow}$$

$$(4\text{-}65)$$

the Hamiltonian operator can be written as

$$H = T(rr') \left(a^{T}(r\uparrow) a(r'\uparrow) + a^{T}(r\downarrow) a(r'\downarrow) \right) + \sum_{kk'} V_{kk'} \, a_{k\uparrow}^{T} a_{k\downarrow}^{T} a_{k'\downarrow} a_{k'\uparrow}.$$

$$(4\text{-}66)$$

Again, we expand the phase operator in the trial wave function for evaluation of the classical Hamiltonian; the result to the second order in terms of $\int$ and χ can be written as

$$\langle \Phi | \hat{H} | \Phi \rangle = T(r,r') \int (r'r) - \tfrac{1}{2} \langle BCS | [\hat{A} , [\hat{\lambda}, \hat{H}]] | BCS \rangle$$

$$- \tfrac{1}{2} \langle BCS | [\hat{\Phi} [\hat{\Phi}, \hat{H}]] | BCS \rangle - \langle BCS | [\hat{\Phi} , [\hat{A}, \hat{H}]] | BCS \rangle$$

$$(4\text{-}67)$$

where

$$\int (r'r) = \sum_{k} \left(\psi_{k\uparrow}(r') \psi_{k\uparrow}^{*}(r) + \psi_{k\downarrow}(r') \psi_{k\downarrow}^{*}(r) \right) v_k^2 .$$

$$(4\text{-}68)$$

The Lagrangian can be obtained, by evaluating the classical Hamiltonian (4-67) and subtracting it from equation (4-62). The final result is

$$\mathcal{L} = - \int dr \left[\dot{\varphi}(r)\varphi(r) + \dot{\chi}(r)\chi(r) + \frac{1}{2}(\nabla\varphi)^2 f(r) + \frac{1}{2}\varphi(r)^2 A(\{\rho\},\{\kappa\}) + \frac{1}{2}(\nabla\chi)^2 B(\{\rho\},\{\kappa\}) \right.$$

$$\left. + \frac{1}{2}(\nabla\varphi\nabla\chi)\,\kappa(r) + \frac{1}{2}(\chi(r))^2 C(\{\rho\},\{\kappa\}) + \frac{1}{2}\varphi(r)\chi(r)\,D(\{\rho\},\{\kappa\}) \right] - \int dr\, dr' \left[\right.$$

$$\left. \chi(r)\chi(r')\,F(\{\rho\},\{\kappa\}) + \varphi(r)\varphi(r')\,H(\{\rho\},\{\kappa\}) + \varphi(r)\chi(r')\,G(\{\rho\},\{\kappa\}) \right] - E(\{\rho\},\{\kappa\})$$

$$(4-69)$$

Where ρ and κ are defined by equations (4-63) and (4-64) respectively. Other coeficient are defined by equation (4-70) through equation (4-76)

$$A(\{\rho\},\{\kappa\}) = -4 \sum_{kk'} V_{kk'}\, u_k v_k\, u_{k'} v_{k'} |\psi_k(r)|^2 \qquad (4-70)$$

$$B(\{\rho\},\{\kappa\}) = 2 \sum_{k} \hbar_k^2 v_k^2 |\psi_k(r)|^2 \qquad (4-71)$$

$$C(\{\rho\},\{\kappa\}) = -2 \sum_{k} \left[\hbar_k^2 u_k^2 \psi_k^*(r)\nabla^2\psi_k(r) - \sum_{k'} \hbar_k \hbar_{k'} v_k^2 \psi_k(\omega)\psi_k^*(\omega) T(\omega,\kappa) \times \right.$$

$$\left. \psi_k(r)\psi_{k'}(r) + 2\hbar_k^2 V_{kk'} u_k v_k u_{k'} v_{k'} |\psi_k(r)|^2 \right] \qquad (4-72)$$

$$D(\{\rho\},\{\kappa\}) = 2 \sum_{k} \left[\hbar_k u_k v_k \psi_k^*(r)\nabla^2\psi_k(r) + \sum_{k'} \hbar_k u_{k'} v_{k'} \psi_k(r)\psi_k(r)\psi_k(\omega)\nabla\psi_k(\omega) \right.$$

$$\left. - 4\hbar_k V_{kk'} v_k^2 u_{k'} v_{k'} |\psi_k(r)|^2 \right] \qquad (4-73)$$

$$F(\{\rho\},\{\kappa\}) = 2 \sum_{kk'} \left[-2\hbar_k^2 v_k^2 \psi_k(r)\psi_k(r') T(r',\omega) \psi_k^*(\omega)\psi_k(r) + 4\hbar_k \hbar_{k'} u_k^2 \psi_k(r)\psi_k(r') T(r',\omega)\psi_k^*(\omega)\psi_k(r) \right.$$

$$- 2\hbar_k \hbar_{k'} v_k^2 \psi_k(r)\psi_k^*(r') T(\omega,r')\psi_k(\omega)\psi_k(r) v_{k'}^2 - 2\hbar_k \hbar_{k'} |\psi_k(r)|\psi_k(r')|^2 V_{kk'}(u_{k'}^2 - v_{k'}^2)$$

$$\left. + 8\hbar_k \hbar_{k'} |\psi_k(r)|^2 |\psi_k(r')|^2 V_{kk'} v_{k'}^2 (2v_{k'}^2 - 1) \right]$$

$$+ \sum_{\ell} 2 f_\ell^2 \, \psi_\ell(r) \, \psi_\ell(r') \left(V_{kk'} \, u_k v_{k'} \, v_\ell v_\ell \, \psi_\ell(r) \psi_\ell(r') + V_{\ell\ell'} \, \psi_\ell(r) \psi_\ell(r') \, u_k v_k \, u_{k'} v_{k'} \right.$$

$$\left. - V_{\ell\ell'} \, \psi_\ell(r) \psi_\ell(r') \, v_\ell v_\ell \, u_k v_{k'} \right) - 4 f_\ell \, f_{k'} \, \psi_\ell(r) \psi_\ell(r') \, T(\omega,u) \, \psi_k^*(u) \, \psi_k(u) \psi_\ell(r') \, \psi_\ell(r) \, v_\ell^2 \Big] \tag{4-74}$$

$$G(\{f\},\{k\}) = \sum_{k\ell'} 8 \Big[f_k \, \psi_\ell(r) \psi_\ell(r') \, \psi_k^*(u) \, \psi_k(r') \, T(r,u) + f_k \, \psi_\ell(r') \psi_\ell(u) \, \psi_k^*(r) \, \psi_k(r') u_k v_{k'} \, T(u,r)$$

$$+ 2 f_k \, |\psi_\ell(r)|^2 \, |\psi_k(r')|^2 \, V_{kk'} \, u_k v_{k'} - 4 f_k \, |\psi_k(r)|^2 |\psi_k(r')|^2 V_{kk'} \, v_k^2 \, u_{k'} v_{k'} + \sum_\ell f_k \, \psi_\ell(r) \psi_\ell(r')$$

$$\psi_\ell(r) \psi_\ell(r') \, V_{kk'} \, v_k^2 \, u_k v_{k'} + f_k \, \psi_\ell(r) \psi_\ell(r') \, \psi_\ell(r) \psi_\ell(r') \, V_{\ell k} \, v_k^2 \, u_k v_{k'} - f_k \, \psi_\ell(r) \psi_\ell(r') \, \psi_\ell(r) \times$$

$$\psi_\ell(r') \, V_{\ell\ell'} \, v_\ell^2 \, u_k v_{k'} \Big] \tag{4-75}$$

$$H(\{f\},\{k\}) = \sum_{kk'} \Big[8 \, V_{kk'} \, u_k v_k \, u_{k'} v_{k'} \, |\psi_k(r)|^2 |\psi_k(r')|^2 - \sum_\ell 4 \, V_{kk'} \, u_k v_{k'} \, v_\ell v_\ell \, \psi_\ell(r) \psi_\ell(r') \times$$

$$\psi_k(r) \psi_k(r') \Big] \tag{4-76}$$

where the spin coordinates label is suppressed for convenience. In deriving the classical Lagrangian we would freeze all the degrees of freedom except the two collective ones, namely f and χ . That mean that all the dynamical variables in the trial wave function are chosen to minimize the classical Hamiltonian under the constriant of fixed q , χ , f and k . The equations of motion can be derived by Hamilton's procedure from the Lagrangian (4- 69), and we obtain

$$\frac{\delta \mathcal{L}}{\delta f} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial t} + \vec{\nabla}\cdot(f \vec{\nabla} f) - f A + \frac{1}{2} \nabla(k \, \nabla \chi) - \frac{1}{2} \chi \, D$$

$$- \frac{1}{2} \int d\vec{r}' \, \big[2 \, f(r') \, H + \chi(r') \, G \big] = 0 \tag{4-77}$$

$$\frac{\delta \mathcal{L}}{\delta \mathcal{P}} = 0 \implies \dot{\mathcal{P}} + \frac{1}{2}(\nabla \mathcal{P})^2 + \frac{1}{2}(\mathcal{P})^2 \frac{\delta A}{\delta \mathcal{P}} + \frac{1}{2}(\nabla \chi)^2 \frac{\delta G}{\delta \mathcal{P}} + \frac{1}{2} \mathcal{P}(r) \chi(r) \frac{\delta D}{\delta \mathcal{P}}$$

$$+ \frac{1}{2}(\chi(r))^2 \frac{\delta G}{\delta \mathcal{P}} + \frac{1}{2}\int dr' \left[\chi(r)\chi(r') \frac{\delta F}{\delta \mathcal{P}} + \mathcal{P}(r)\chi(r') \frac{\delta G}{\delta \mathcal{P}} \right.$$

$$+ \left. \mathcal{P}(r)\mathcal{P}(r') \frac{\delta H}{\delta \mathcal{P}} \right] + \frac{\delta E}{\delta \mathcal{P}} = 0 \quad , \qquad (4-78)$$

$$\frac{\delta \mathcal{L}}{\delta \chi} = 0 \implies \frac{\partial \kappa}{\partial t} + \vec{\nabla}\cdot(B\vec{\nabla}\chi) + \frac{1}{2}\vec{\nabla}(\kappa \vec{\nabla}\mathcal{P}) - \frac{1}{2}\mathcal{P}(r) D - \chi(r) C$$

$$- \frac{1}{2}\int dr' \left[2\chi(r') F + \mathcal{P}(r') G \right] = 0, \qquad (4-79)$$

$$\frac{\delta \mathcal{L}}{\delta \kappa} = 0 \implies \dot{\chi}(r) + \frac{1}{2}(\mathcal{P}(r))^2 \frac{\delta A}{\delta \kappa} + \frac{1}{2}(\nabla \chi(r))^2 \frac{\delta B}{\delta \kappa} + \frac{1}{2}\nabla \mathcal{P}\nabla\chi$$

$$+ \frac{1}{2}(\chi)^2 \frac{\delta G}{\delta \kappa} + \frac{1}{2}\mathcal{P}(r)\chi(r) \frac{\delta D}{\delta \kappa} + \frac{1}{2}\int dr' \left[\chi(r)\chi(r') \frac{\delta F}{\delta \kappa} \right.$$

$$+ \left. \mathcal{P}(r)\chi(r') \frac{\delta G}{\delta \kappa} + \mathcal{P}(r)\mathcal{P}(r') \frac{\delta H}{\delta \kappa} \right] + \frac{\delta E}{\delta \kappa} = 0$$

$$(4-80)$$

The equations of motion (4-77) through (4-80) describing the fluid
model are complex and we will consider only small oscillations of the
variables in an extended system near its equilibrium in the next section.

4.4 Dispersion relations for the two fluid models.

For each of the three two fluid models described in section 4.1, 4.2, and 43, we may consider small oscillations of the densities near their equilibria. In the first model, the equations (4-33) through (4-36) can be applied to the propagation of sound in the system. We assume the velocities are small and the densities are almost equal to the their constant equilbrium values. Thus terms which are quadratic in excursion can be neglected. By differentiating equations (4-33) and (4-34) with respect to time, using equation (4-35) and (4-36) and eliminating $\vec{v}_n$ and $\vec{v}_s$ in fovour of ρ and ρ_s , one gets

$$- \frac{\partial^2 \rho}{\partial t^2} + \rho \nabla^2 I + \rho_s \nabla^2 g = 0 \quad , \tag{4-81}$$

$$- \frac{\partial^2 \rho_s}{\partial t^2} + \rho \nabla^2 g + \rho_s \nabla^2 I - g F \quad , \tag{4-82}$$

where

$$I \equiv \frac{\delta E}{\delta \rho} \quad , \tag{4-83}$$

$$g \equiv \frac{\delta E}{\delta \rho_s} \quad , \tag{4-84}$$

and

$$F = \frac{\partial}{\partial t} \frac{\delta E}{\delta q}$$

$$(4-85)$$

One could write f and f_s as functions of I and g, then equation (4-81) and (4-82) can be written as

$$\frac{\partial f}{\partial I}\bigg|_g \frac{\partial^2 I}{\partial t^2} + \frac{\partial f}{\partial g}\bigg|_I \frac{\partial^2 g}{\partial t^2} = f \nabla^2 I + f_s \nabla^2 g \quad ,$$

$$(4-86)$$

$$\frac{\partial f_s}{\partial I}\bigg|_g \frac{\partial^2 I}{\partial t^2} + \frac{\partial f_s}{\partial g} \frac{\partial^2 g}{\partial t^2} = f_s \nabla^2 I + f \nabla^2 g - F_g \quad .$$

$$(4-87)$$

For small oscillations, one may write

$$I = I_o + \delta I \; e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$(4-88)$$

$$q = \delta_g \; e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$(4-89)$$

and obtain (in matrix form)

$$\begin{pmatrix} \omega^2 \frac{\partial f}{\partial I}\bigg|_g - f k^2 & \omega^2 \frac{\partial f}{\partial g}\bigg|_I - f_s k^2 \\ \omega^2 \frac{\partial f_s}{\partial I}\bigg|_g - f_s k^2 & \omega^2 \frac{\partial f_s}{\partial g}\bigg|_I - f k^2 - F \end{pmatrix} \begin{pmatrix} \delta f \\ \delta g \end{pmatrix} = 0$$

$$(4-90)$$

then, the dispersion relation can be written as

$$\omega^4\left[\frac{\partial f}{\partial I}\Big|_g \frac{\partial f_s}{\partial g}\Big|_I - \frac{\partial f}{\partial g}\Big|_I \frac{\partial f_s}{\partial I}\Big|_g\right] - \omega^2\left[\frac{\partial f}{\partial I}\Big|_g\left(f k^2 + F\right) + f k^2 \frac{\partial f}{\partial g}\Big|_f\right.$$

$$\left.-f_s k^2\left(\frac{\partial f}{\partial g}\Big|_I + \frac{\partial f_s}{\partial I}\Big|_g\right)\right] + f k^2\left(f k^2 + F\right) - \left(f_s k^2\right)^2 = 0 \qquad (4\text{-}91)$$

With some algebric manipluation, the dispersion relation (4-91) can be written in a more simple form:

$$\omega^4 - \omega^2\left[\ F + (f - f_s) k^2 \frac{\partial g}{\partial f}\Big|_f + k^2\left(f \frac{\partial I}{\partial f}\Big|_{f_s} + f \frac{\partial g}{\partial f}\Big|_{f_s}\right)\right]$$

$$+ \left[f k^2\left(F + f k^2\right) - \left(f_s k^2\right)^2\right] \frac{\partial I}{\partial f}\Big|_g \frac{\partial g}{\partial f_s}\Big|_f = 0 . \qquad (4\text{-}92)$$

This dispersion relation (4-92) has the appropriate limits. First: For normal fluid $f_s = 0$ and $\frac{\partial g}{\partial f_s}\Big|_f = 0$ one gets

$$\omega^2 = 0 \qquad\qquad\qquad (4\text{-}93)$$

$$\omega^2 = f k^2 \frac{\partial I}{\partial f} \qquad\qquad (4\text{-}94)$$

This equation (4-94) is eqivalent to the ordinary sound dispersion relation in the normal fluid. Second, for small f_s and $\frac{\partial g}{\partial f_s}\Big|_f$, one could make the approximation

$$\left.\frac{\partial I}{\partial f}\right)_{f_s} \simeq \left.\frac{\partial I}{\partial f}\right)_{\eta} \quad , \tag{4-95}$$

which is similar to the

$$\left.\frac{\partial P}{\partial f}\right)_s \simeq \left.\frac{\partial P}{\partial f}\right)_T \tag{4-96}$$

approximation in the Landau theory of the two fluid model. Then the dispersion relation can be factorized to

$$\left(\omega^2 - f k^2 \left.\frac{\partial I}{\partial f}\right)_{f_s}\right)\left[\omega^2 - \left(F + f k^2\right)\left.\frac{\partial g}{\partial f}\right)_f\right] \approx 0 \tag{4-97}$$

or

$$\omega^2 \simeq f k^2 \left.\frac{\partial I}{\partial f}\right)_{f_s} \quad , \tag{4-98}$$

and

$$\omega^2 \simeq \left(F + f k^2\right)\left.\frac{\partial g}{\partial f}\right)_f \cdot \tag{4-99}$$

The dispersion relation (4-98) corresponds to normal sound in the fluids; and for the other relation (4-99), it is similar to the electron plasma oscillation dispersion relation. For a general solution to the equation (4-92), we would have

$$\omega^2 = \left(\alpha + \beta k^2 \pm \sqrt{\left(\alpha + \beta k^2\right)^2 - 4 k^2 \left(\delta + \delta k^2\right)}\right)/2 \quad , \tag{4-100}$$

where

$$\alpha = F \left.\frac{\partial g}{\partial \rho_s}\right)_\rho , \tag{4-101}$$

$$\beta = (\rho - \rho_s) \left.\frac{\partial g}{\partial \rho_s}\right)_\rho + \rho \left.\frac{\partial I}{\partial \rho}\right)_{\rho_s} + \rho \left.\frac{\partial g}{\partial \rho}\right)_{\rho_s} , \tag{4-102}$$

$$\gamma = \rho F \left.\frac{\partial I}{\partial \rho}\right)_g \left.\frac{\partial g}{\partial \rho_s}\right)_\rho , \tag{4-103}$$

$$\delta = (\rho^2 - \rho_s^2) \left.\frac{\partial I}{\partial \rho}\right)_g \left.\frac{\partial g}{\partial \rho_s}\right)_\rho . \tag{4-104}$$

Equations (4-100) describe the general form of the dispersion relation for the two fluid model. For small k the square-root can be expanded and one of the branches contains the effective mass, while the other would be similar to the normal sound dispersion relation.

The propagation of sound in the second fluid model can be considered similar to the first model. To avoid repetition of similar arguments, we may derive the dispersion relation by direct use of equations of motion (4-54) through (4-57). For small oscillations of the variables near their equlibria, one may write

$$\rho = \rho_e + \delta\rho \; e^{i(\vec{k}\cdot\vec{r} - \omega t)} \tag{4-105}$$

$$K = K_e + \delta k \; e^{i(\vec{k}\cdot\vec{r} - \omega t)} \tag{4-106}$$

$$\mathcal{N} = \delta\mathcal{N}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \tag{4-107}$$

$$\mathcal{S} = \delta\mathcal{S}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \tag{4-108}$$

Substituting equations (4-105)-(4-108) in equations (4-54)-(4-57), then the equations of motion reduce to (in matrix form)

$$
\begin{pmatrix}
-i\omega & 0 & -k^2\rho & k^2\rho \\
0 & -i\omega & k^2\mathcal{S} & -k^2\rho \\
\frac{\partial}{\partial\mathcal{S}}\frac{\delta E}{\delta\mathcal{S}} & \frac{\partial}{\partial k}\frac{\delta E}{\delta\mathcal{S}} & -i\omega & 0 \\
\frac{\partial}{\partial\mathcal{S}}\frac{\delta E}{\delta\kappa} & \frac{\partial}{\partial k}\frac{\delta E}{\delta\kappa} & 0 & -i\omega
\end{pmatrix}
\begin{pmatrix}
\delta\mathcal{S} \\
\delta k \\
\delta\mathcal{N} \\
\delta\mathcal{S}
\end{pmatrix}
= 0 \tag{4-109}
$$

The dispersion relation has two branches

$$\omega^2 = 0\, , \tag{4-110}$$

and

$$\omega^2 = -k^2\mathcal{S}\left[\frac{\partial}{\partial\mathcal{S}}\frac{\delta E}{\delta\mathcal{S}} + \frac{\partial}{\partial k}\frac{\delta E}{\delta\kappa} - \frac{\partial}{\partial k}\frac{\delta F}{\delta\mathcal{S}} - \frac{\partial}{\partial\mathcal{S}}\frac{\delta E}{\delta\kappa}\right]. \tag{3-111}$$

The simple form of the dispersion relation is due to our choice of local potential interaction. If we use a general form of potential, the dispersion relation will be modified to

$$\omega^4 - \omega^2 \left[\left(\hbar^2 \rho - \eta \right) \left(\frac{\partial}{\partial \rho} \frac{\delta E}{\delta \rho} + \frac{\partial}{\partial \kappa} \frac{\delta E}{\delta \kappa} \right) - \left(\hbar^2 \rho + \eta \right) \left(\frac{\partial}{\partial \rho} \frac{\delta E}{\delta \kappa} + \frac{\partial}{\partial \kappa} \frac{\delta E}{\delta \rho} \right) \right]$$

$$+ 4 \hbar^2 \rho \eta \left(\frac{\partial}{\partial \kappa} \frac{\delta E}{\delta \rho} \frac{\partial}{\partial \rho} \frac{\delta E}{\delta \kappa} - \frac{\partial}{\partial \rho} \frac{\delta E}{\delta \rho} \frac{\partial}{\partial \kappa} \frac{\delta E}{\delta \kappa} \right) = 0 ,$$

$$\qquad (4\text{-}112)$$

where

$$\eta = \frac{\partial}{\partial \rho} \frac{\delta V}{\delta \rho} .$$

$$\qquad (4\text{-}113)$$

The dispersion relation equation (4-112) has the approprate limit. for $\frac{\delta E}{\delta \kappa} = 0$ we obtain

$$\omega^2 = 0 ,$$

$$\qquad (4\text{-}114)$$

and

$$\omega^2 = \left(\hbar^2 - \eta \right) \frac{\partial}{\partial \rho} \frac{\delta E}{\delta \rho} .$$

$$\qquad (4\text{-}115)$$

For $\eta = 0$ the equation (4-112) reduces to the equation (4-110) and (4-111).

For the third model, let us consider the linearized form of equations of motion

$$\frac{\partial \rho^{(1)}}{\partial t} + \rho \nabla^2 \phi^{(1)} - A(\rho, \kappa) \phi^{(1)} + \frac{1}{2} \kappa \nabla^2 \chi^{(1)} - \frac{1}{2} D(\rho, \kappa) \chi^{(1)}$$

$$- \frac{1}{2} \int dr' \left[2 \phi^{(1)}(r') H(\rho, \kappa) + \chi^{(1)}(r') G(\rho, \kappa) \right] = 0 ,$$

$$\qquad (4\text{-}116)$$

$$\frac{\partial \phi^{(1)}}{\partial t} + \frac{\partial}{\partial \rho} \frac{\delta E}{\delta \rho} \rho^{(1)} + \frac{\partial}{\partial \kappa} \frac{\delta E}{\delta \rho} \kappa^{(1)} = 0 ,$$

$$\qquad (4\text{-}117)$$

$$\frac{\partial \kappa^{(1)}}{\partial t} + B(\rho, \kappa) \nabla^2 \chi^{(1)} + \frac{1}{2} \kappa \nabla^2 \phi^{(1)} - \frac{1}{2} D(\rho, \kappa) \phi^{(1)} - C(\rho, \kappa) \chi^{(1)}$$

$$- \frac{1}{2} \int dr' \left[2 \chi^{(1)}(r') F(\rho, \kappa) + \phi^{(1)}(r') G(\rho, \kappa) \right] = 0 ,$$

$$\qquad (4\text{-}118)$$

$$\frac{\partial \chi^{(1)}}{\partial t} + \frac{\partial}{\partial \rho}\frac{\delta E}{\delta \rho}\rho^{(1)} + \frac{\partial}{\partial k}\frac{\delta E}{\delta \rho}\rho^{(1)} + \frac{\partial}{\partial k}\frac{\delta E}{\delta k}\kappa^{(1)} = 0 .$$

$$(4-119)$$

For small oscillations, we may write

$$\rho^{(1)} = \delta\rho\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$(4-120)$$

$$\phi^{(1)} = \delta\phi\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$(4-121)$$

$$\kappa^{(1)} = \delta k\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} ,$$

$$(4-122)$$

$$\chi^{(1)} = \delta\chi\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} .$$

$$(4-123)$$

Substituting equations (4-120) through (4-123) in the equations of motion (4-116)-(4-119), writting the results in the matrix form, then the dispersion relation can be obtained by evaluating the determinate of the matrix. The final result is

$$\omega^4 - \omega^2 \left[\frac{\partial}{\partial k}\frac{\delta E}{\delta k}\left(B(\rho,\kappa)k^2 + C(\rho,\kappa) + FF(\rho,\kappa,\ell) + \frac{\partial}{\partial \rho}\frac{\delta E}{\delta \rho}\left(\rho k^2 + A(\rho,\kappa) \right. \right.$$

$$+ \frac{1}{2}FH(\rho,\kappa,\ell) \right] + \left(\frac{\partial}{\partial \rho}\frac{\delta E}{\delta \rho}\cdot\frac{\partial}{\partial k}\frac{\delta E}{\delta k} + \frac{\partial}{\partial \rho}\frac{\delta E}{\delta k}\cdot\frac{\partial}{\partial k}\frac{\delta E}{\delta \rho} \right) \left[\left(\rho k^2 + A(\rho,\kappa) + \frac{1}{2}FH(\rho,\kappa,\ell) \right) \right.$$

$$\left(B(\rho,\kappa)k^2 + C(\rho,\kappa) + FF(\rho,\kappa,\ell) - \frac{1}{4}\left(\kappa k^2 + D(\rho,\kappa) + FG(\rho,\kappa,\ell) \right)^2 \right] = 0$$

$$(4-124)$$

where

$$FF(\rho,\kappa,\ell) = \int d r'\, e^{-i\,k(r-r')}\, F(\rho,\kappa)$$

$$(4-125)$$

$$F_G(\mathfrak{f}, k, \ell) = \int dr' \, e^{-i\vec{\ell}(\vec{r}-\vec{r}')} \, G(\mathfrak{f}, k) \, , \qquad (4\text{-}126)$$

$$F_H(\mathfrak{f}, k, \ell) = \int dr' \, e^{-i\vec{\ell}\cdot(\vec{r}-\vec{r}')} \, H(\mathfrak{f}, k) \, . \qquad (4\text{-}127)$$

Let us consider the limits of the dispersion relation (4-124). For $\delta F/\delta k$ we would get the usual

$$\omega^2 = 0 \, , \qquad (4\text{-}128)$$

and

$$\omega^2 = \frac{\partial}{\partial \mathfrak{f}} \frac{\delta E}{\delta \mathfrak{f}} \left[\mathfrak{f} \, k^2 + A(\mathfrak{f}, k) + \frac{1}{2} \, F_H(\mathfrak{f}, k, \ell) \right] \qquad (4\text{-}129)$$

for the dispersion relation equations. The equation (4-129) is the modified version of normal fluid dispersion relation, the extra term A and 1/2FH are due to nonlocality in the two body potential (4-65). In the region in which the term

$$\frac{1}{\mathfrak{f}} \frac{\delta E}{\delta k} \frac{\partial}{\partial k} \frac{\delta E}{\delta \mathfrak{f}} \left[\left(\mathfrak{f} \, k^2 + A(\mathfrak{f}, k) + \frac{1}{2} F_H(\mathfrak{f}, k, \ell) \right) \left(B(\mathfrak{f}, k) \, k^2 + C(\mathfrak{f}, k) + F_F(\mathfrak{f}, k, \ell) \right) \right.$$

$$\left. - \frac{1}{4} \left(k \, k^2 + D(\mathfrak{f}, k) + F_G(\mathfrak{f}, k, \ell) \right)^2 \right] - \frac{1}{4} \frac{\partial}{\partial \mathfrak{f}} \frac{\delta E}{\delta \mathfrak{f}} \frac{\partial}{\partial k} \frac{\delta E}{\delta k} \left(k \, k^2 + D(\mathfrak{f}, k) + F_G(\mathfrak{f}, k, \ell) \right)^2$$

$$(4\text{-}130)$$

is small, one may factorize the dispersion relation (4-124), the results being

$$\omega^2 = \frac{\partial}{\partial \rho} \frac{\delta E}{\delta \rho} \left(\rho k^2 + A(\rho,k) + \frac{1}{2} FH(\rho,k,\ell) \right) \tag{4-131}$$

and

$$\omega^2 = \frac{\partial}{\partial k} \frac{\delta E}{\delta k} \left(B(\rho,k) k^2 + C(\rho,k) + FF(\rho,k,\ell) \right). \tag{4-132}$$

For small k one may expand FH and FF in equations (4-131) and (4-132), respectively. Then the result reduces to a familar form, except in this case both branches of the dispersion relation have a finte mass. In general we could slove the quadratic expression of the dispersion relation, the result is

$$\omega^2 = \alpha \pm \sqrt{\alpha^2 - \beta} \tag{4-133}$$

where

$$\alpha = \frac{1}{2} \left[\frac{\partial}{\partial k} \left(\frac{\delta E}{\delta k} \right) \left(B(\rho,k) k^2 + C(\rho,k) + FF(\rho,k,\ell) \right) + \frac{\partial}{\partial \rho} \frac{\delta E}{\delta \rho} \left(\rho k^2 \right. \right. $$
$$\left. \left. + A(\rho,k) + \frac{1}{2} FH(\rho,k,\ell) \right) \right. \tag{4-134}$$

and

$$\beta = \left[\frac{\partial}{\partial \rho} \frac{\delta E}{\delta \rho} \frac{\partial}{\partial k} \frac{\delta E}{\delta k} + \frac{\partial}{\partial \rho} \frac{\delta E}{\delta k} \frac{\partial}{\partial k} \frac{\delta E}{\delta \rho} \right] \left[\left(\rho k^2 + A(\rho,k) + \frac{1}{2} FH(\rho,k,\ell) \right) \right($$
$$B(\rho,k) k^2 + C(\rho,k) + FF(\rho,k,\ell) \right) - \frac{1}{4} \left(k k^2 + B(\rho,k) + FG(\rho,k,\ell) \right)^2 \right]. \tag{4-135}$$

Let us summarize the discussion of the dispersion relations for these three different approaches. For each model we had a quadratic

expression in terms of ω^2 which has been solved. For the first model, we have two branches with approprite limits. For a reasonable approximation, one branch is similar to the normal sound dispersion relation and the other contains an extra zero effective mass. For the second approach one branch is $\omega^2 = 0$ and the second branch contains an extra effective mass in the normall sound dispersion relation. In the case of the third model, the dispersion relation are more complicated, and usually both of the branch have the effective mass. Also their dependence on the wave number is more complicated than in the normal sound dispersion relation.

4.5 An Irrotational fluid model.

In sections 4.1, 4.2 and 4.3 we developed the two fluid models first from a special HB, and secondly from a general HB, and a TDBCS trial wave function respectively. But the development and carring out of the calculations for those models was complex and time consuming. It would be more attractive to develop a simple model with a restricted trial wave function. Therefore in this section we begin with time dependent BCS trial wave function and construct a simple Irrotational fluid model. This model will be used in chapter five for the calculation

of sound in neutron stars.

Let us consider the trial wave function:

$$| \tilde{\phi} \rangle = e^{i \, q(r) \, [a^T(\vec{r}\uparrow) a(\vec{r}\uparrow) + a^T(\vec{r}\downarrow) a(\vec{r}\downarrow)]} | BCS \rangle \qquad (4-136)$$

where as usual $a^T(\vec{r}\uparrow)$ and $a(\vec{r}\downarrow)$ are creation and destruction field operators respectively and BCS correspond to the time dependent BCS wave function (4-61). The time dependent part of the Lagrangian reduces to

$$\langle \tilde{\phi} | i \frac{\partial}{\partial t} | \tilde{\phi} \rangle = - \int d\vec{r} \, \dot{q}(\vec{r}) \, \rho(\vec{r}) , \qquad (4-137)$$

where $\rho(\vec{r})$ is defined by equation (4-63). Corresponding to the Hamiltonian:

$$H = T(\vec{r}\vec{r}) \left(a^T(\vec{r}\uparrow) a(\vec{r}\uparrow) + a^T(\vec{r}\downarrow) a(\vec{r}\downarrow) \right) + V(xx' yy') \, a^T(x) \, a(x') \, a^T(y') a(y) \qquad (4-138)$$

we obtain the classical Hamiltonian,

$$\langle \tilde{\phi} | H | \tilde{\phi} \rangle = \int d\vec{r} \, \tfrac{1}{2} (\nabla q)^2 \rho(\vec{r}) + E(\{\rho\}) . \qquad (4-139)$$

Where, as usual, $E(\{\rho\})$ is the minimized energy as a functional of $\rho(\vec{r})$ and in the case of a local two body potential $E(\{\rho\})$ is independent of q.

The Lagrangian can be written as

$$\mathcal{L} = - \int d\vec{r} \left(\dot{q}(\vec{r}) + \tfrac{1}{2} (\nabla q)^2 \right) \rho(\vec{r}) - E(\{\rho\}) . \qquad (4-140)$$

This Lagrangian is similar to the Irrotational fluid model of Kerman-Koonin, the only difference being the inclusion of the pair approximation in deriving the classical Hamiltonian. Using Lagrangian (4-140) one may obtain equations of motion:

$$\frac{\partial f}{\partial t} + \vec{\nabla}\cdot(f\,\vec{\nabla}\varphi) = 0 ,$$

(4-141)

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + \frac{\delta E}{\delta f} = 0 .$$

(4-142)

Where (4-141) can be considered as the continuty equation and (4-142) as Euler's equation. For small oscillations near equilibrium we may write

$$f = f_o + \delta f \, e^{i(\vec{k}\cdot\vec{r} - \omega t)} ,$$

(4-143)

$$\varphi = \delta\varphi \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

(4-144)

Substituing equations (4-143) and (4-144) in equations (4-141) and (4-142), writting equations of motion in the matrix form, and evaluating the determinate of the matrix, we obtain the dispersion relation as:

$$\omega^2 = f\,\frac{\partial}{\partial f}\left(\frac{\delta E}{\delta f}\right)\,k^2$$

(4-145)

and the velocity of sound in the system will be

$$\frac{\omega^2}{k^2} = u^2 = \sum_{i} \frac{\partial}{\partial j} \frac{\delta}{\delta j} E(\{p_j\}) \qquad (4\text{-}146)$$

Chapter V

A study of neutron matter.

In this chapter, we study the effect of pairing on the dynamical properties of neutron matter. Neutron matter is a hypothetical system of neutrons in equal spin populations. It is assumed to fill all space with uniform density of neutron ρ . Although this system will not be bound by itself, the interior of a neutron star would be a very good approximation of the system in nature. Usually, parallel to the study of nuclear matter, the neutron matter is also discussed. Specifically, the superfluidty of neutrons in the neutron star has been reviewed by Baym and Pethick.[40]

In section 5.1, first a static study of pairing in the neutron matter has been considered. We assume that neutrons are interacting via soft core potentials. The energy gap equation has been solved. Then we treat the description of sound waves (small amplitude oscillations) in the system, utilizing the Irrotational fluid model of section 3.5. In section 5.2 we derive the QPRPA utilizing the Generalized Hartree-Fock method, and apply it to our model of neutron matter. The numerical calculations of the QPRPA and the hydrodynamical approach for phonon energies are compared.

5.1 Sound in neutron matter.

As mentioned above the study of sound waves in neutron matter is our aim in this section. One of our assumptions is that neutrons are interacting via soft core potentials. The potentials adopted in this section and the next are taken from reference [20]. The potential consists of three parts of gaussian shape

$$\mathcal{V} = \sum_{\alpha=1}^{3} V_{\alpha} \, \exp\left[-(r/a_{\alpha})^2\right] . \qquad (5\text{-}1)$$

The values of V_{α} and a_{α} are given in table [1]. The term ($\alpha=1$) is adjusted to the OPEP and the term ($\alpha=2$) to the strong attractive part in the intermediate region in the singlet even state. The third term ($\alpha=3$) represents the repulsive soft-core and is confined to the region $r \leqslant 0.7 f$. V_3 and a_3 are determined by the singlet scattering length and effective range.

Now, we solve the static part of the equation of motion. Utilizing equation (2-75) in its static form, we may write

$$2\left(\epsilon_{\alpha} - \lambda\right) k_{\alpha\bar{\alpha}} + \Delta_{\alpha\bar{\alpha}} \left(2 f_{\alpha\alpha} - 1\right) = 0 . \qquad (5\text{-}2)$$

In solving equation (5-2) we use the effective mass approximation for the single particle excitation energy ϵ_{α} (which is relative to the chemical potential λ)

$$\xi_{d} = (\epsilon_{d} - \lambda) \simeq \frac{1}{2m^{*}} \left(k_{d}^{2} - k_{f}^{2} \right) . \qquad (5-3)$$

Using equations (5-2),(5-3) and the supplementary condition (2-33), we obtain BCS gap equation which is a non linear integral equation for the gap equation

$$\Delta_{d} = -\frac{1}{2} \sum_{\beta} \langle \beta - \beta \mid \nu \mid d - d \rangle \frac{\Delta_{\beta}}{\left(\xi_{\beta}^{2} + \Delta_{\beta}^{2} \right)^{1/2}} . \qquad (5-4)$$

Integration of equation (5-4) over the angles is performed and the final result is

$$\Delta_{x} = -\sum_{d} \nu_{d} \int dx' \frac{x'}{2x} \left\{ exp -\mu_{d}^{2}(x-x')^{2} - exp -\mu_{d}^{2}(x+x')^{2} \right\} \times$$

$$\Delta x' / \left[\left(\frac{m}{m^{*}} \right)^{2} \left(x'^{2} - 1 \right)^{2} + \Delta_{x'}^{2} \right]^{1/2} , \qquad (5-5)$$

where

$$X = k/k_{f} \quad , \quad \Delta_{x} = \Delta_{x} / (k k_{f}^{2} / 2m) \quad , \quad \nu_{d}' = V_{d} \left(\frac{m a_{d}}{\sqrt{\pi} k_{f}} \right) , \quad \mu_{d}' = a_{d} k_{f} / 2 \quad (5-6)$$

The iteration method is used in solving equation (5-5); the input of

$\Delta x' / \left[\left(\frac{m}{m^{*}} \right)^{2} (x'^{2} - 1)^{2} + \Delta_{x'}^{2} \right]^{1/2}$ is $exp - (x-1)^{2}$, and the convergence condition is

$$Max \left| \frac{\Delta_{x}^{n} - \Delta_{x}^{n-1}}{\Delta_{x}^{n-1}} \right| < 0.01 \qquad (5-7)$$

where n is the number of times the iteration is carried out. The region and the mesh size of the integration are $[0.0, 7.5]$ and 0.15 respectively.

The ordinary iteration method is modified by relaxation method, which makes convergence more rapid. Numerical results are given graphically. Δ_k is positive and decreasing uniformly for $k < k_f$ but becomes negative for $k > k_f$ due to the repulsive core as shown in figures $[5, 6]$. Figures $[7, 8]$ show the dependence of the gap energy at the fermi momentum on the density of neutron matter for various effective masses. As we see from figures $[5-8]$, the calculated energy gap are reasonable when compared with the experimental values in heavy nuclei.

In the second part of this section, we use the formalism of the Irrotational fluid model of chapter three to evaluate the sound velocity in neutron matter. Besides the effective mass approximation we used in the above calculation, we assume that the introduction of the phase φ in the trial wave function (3-136) does not change $E\{\varphi\}$ from its ground state values. Equation (5-8) show $E\{\varphi\}$ as a function of density

$$E\{\varphi\} = \sum_k \frac{k^2}{m^*} v_k^2 + \sum_{kk'} \langle k - k | v | k' - k' \rangle u_k v_k u_{k'} v_{k'}$$

$$= \sum_k \frac{k^2}{m_k} v_k^2 + \sum_k \Delta_k u_k v_k$$

$$= \frac{k^2}{m^*} v_k^2 + \sum_k \frac{\Delta_k^2}{2(\xi_k^2 + \Delta_k^2)^{1/2}} \qquad (5-8)$$

where we have used the definition of gap energy (5-9) and the static value

of $U_k V_k$ (5-10)

$$A_k = \sum_{k'} \langle \vec{k} - \vec{k} | v | \vec{k}' - \vec{k}' \rangle \, U_k V_{k'} \quad , \qquad (5-9)$$

and

$$U_k V_k = \frac{A_k}{2(\zeta_k^2 + A_k^2)^{1/2}} \, . \qquad (5-10)$$

Equation (4-146) gives the velocity of sound in neutron matter

$$u^2 = \rho \, \frac{\partial}{\partial \rho} \frac{\delta}{\delta \rho} E(\{\rho\}) \, . \qquad (4-146)$$

Utilizing equation (5-8) one may obtain $\frac{\partial}{\partial \rho} \frac{\delta}{\delta \rho} E(\{\rho\})$:

$$\frac{\partial}{\partial \rho} \frac{\delta}{\delta \rho} E(\rho) = \int \frac{d^3 k}{(2\pi)^3} \left\{ \frac{k^2}{m^* (\zeta_k^2 + A_k^2)^{5/2}} \left[\frac{\partial^2 \lambda}{\partial \rho^2} A_k^2 \left(\zeta_k^2 + A_k^2 \right) + 3 \left(\frac{\partial \lambda}{\partial \rho} \right)^2 \zeta_k A_k^2 \right. \right.$$

$$+ \frac{\partial \lambda}{\partial \rho} \frac{\partial A_k}{\partial \rho} \left(2 A_k \zeta_k^2 - A_k^3 \right) + \frac{\partial^2 A_k}{\partial \rho^2} \left(\zeta_k^3 A_k + \zeta_k A_k^3 \right) + \left(\frac{\partial A_k}{\partial \rho} \right)^2 \left(\zeta_k^2 - 2 A_k^2 \zeta_k \right) \right]$$

$$+ \frac{1}{2(\zeta_k^2 + A_k^2)^{5/2}} \left[\frac{\partial^2 \lambda}{\partial \rho^2} \zeta_k A_k^2 \left(\zeta_k^2 + A_k^2 \right) + \left(\frac{\partial \lambda}{\partial \rho} \right)^2 \left(2 \zeta_k^2 A_k - A_k^4 \right) + 2 \frac{\partial \lambda}{\partial \rho} \frac{\partial A_k}{\partial \rho} \left(2 \zeta_k^3 A_k \right. \right.$$

$$\left. - A_k^3 \zeta_k \right) + \frac{\partial^2 A_k}{\partial \rho^2} \left(2 \zeta_k^2 A_k + A_k^3 \right) + \left(\frac{\partial A_k}{\partial \rho} \right)^2 \left(2 \zeta_k^4 - A_k^2 \zeta_k^2 \right) \right] \right\} \, . \qquad (5-11)$$

In our nummerical calculation, the region and the mesh size of differentiation are $[\, 0.50 \, \xi^{-1} \, , \, 0.95 \, \xi^{-1} \,]$ and $0.05 \, \xi^{-1}$ respectively (for the corresponding fermi momenta). The region and mesh size of the integration are $[\, 0.0 \, , \, 7.5 \,]$ and 0.15 respectively. Figures $[9, 10]$ show the dependence of velocity of sound in neutron matter on the density for various effective masses. The dependence of the velocity on the effective mass is shown in figure

[11] for various densities. As we can see from figures [9-11], the numerical results for the velocity of sound in neutron matter are reasonable, considering the result of other studies of nuclear matter.

5.2 Quasi Particle Random Phase Approximation.

The aim of this section is the derivation of QPRPA and its application to our model of neutron matter. Besides the variational derivation of Baranger, there are other methods for the derivation of Hartree-Bogolyubov equations. One among them is the Generalized Hartree Fock method developed by Kerman and Klein. This method has been devised to describe the excited states of system in equal footing with the ground state. It is based upon the assumption that off-diagonal matrix elements of certain one particle operators are of the same order of magnitude as the diagonal elements, and the two body matrix element can be factorized similar to the RPA, with a simple generalization to include the collective states. Although this method is used mainly for the description of excited states in Hartree-Fock approximation, it is equally capable to describe systtem in the Hartree-Bogolyubov approximation by retaining the expctation values of the pairing matrix elements. In the following we utilize this method for the derivation of HB equations. As usual, we consider the Hamiltonian:

$$H = T(x,y)\, a^T(x)\, a(y) + \frac{1}{4}\, V(xy, x'y')\, a^T(x)\, a^T(y)\, a(y')\, a(x').$$

(5-12)

We begin with the operator equations of motion which follow from the Hamilatonian equation (5-12)

$$[a(x), H] = T(x,y)\, a(y) + \frac{1}{2}\, V(xy, x'y')\, a^T(y)\, a(y')\, a(x'),$$

(5-13)

$$[a^T(x), H] = -T(x,y)\, a^T(y) - \frac{1}{2}\, V(xy, x'y')\, a^T(x')\, a^T(y')\, a(y).$$

(5-14)

We study the matrix element $\langle i | a(x) | BCS \rangle$ connecting the $|BCS\rangle$ state of the system with one of the eigenstates $|i\rangle$ of the system:

$$\psi_i(x) = \langle i | a(x) | BCS \rangle.$$

(5-15)

Then, the equation of motion can be derivied utilizing equation (5-13)

$$i\frac{\partial}{\partial t}\langle i | a(x) | BCS \rangle = i\frac{\partial}{\partial t}\psi_i(x) = \langle i | [a(x), H] | BCS \rangle$$

$$= T(x,y)\, \psi_i(y) + \frac{1}{2}\, V(xy, x'y')\, \langle i | a^T(y)\, a(y')\, a(x') | BCS \rangle$$

(5-16)

We may follow the same procedure for the matrix element $\langle i | a^T(x) | BCS \rangle$. Then we obtain:

$$i\frac{\partial}{\partial t}\phi_i^*(x) = -T(x,y)\, \phi_i^*(y) - \frac{1}{2}\, V(xy, x'y')\, \langle i | a^T(x)\, a^T(y')\, a(y) | BCS \rangle,$$

(5-17)

where

$$\phi_i^*(x) = \langle i | a^T(x) | BCS \rangle.$$

(5-18)

To arrive at the HB approximation, we set

$$V(xy,x'y') \langle i| a^T_{(y)} a_{(y')} a_{(x')} |BCS\rangle \cong V(xy,x'y') \left[2\langle BCS| a^T_{(y)} a_{(y')} |BCS\rangle \times \right.$$

$$\langle i| a_{(x')} |BCS\rangle + \langle i| a^T_{(y)} |BCS\rangle \langle BCS| a_{(y')} a_{(x')} |BCS\rangle \left. \right.$$

$$(5-19)$$

Substituting HB approximation (5-19) in the equation of motion (5-16)
and (5-17), the TDHB equations can be written as

$$i\frac{\partial}{\partial t} \psi_i(x) = T(x,y) \psi_i(y) + \frac{1}{2} V(xy,x'y') \left[2 \rho(y'y) \psi_i(x') + k(y'x')\psi^*_i(y) \right],$$

$$(5-20)$$

$$i\frac{\partial}{\partial t} \phi^*_i(x) = -T(x,y) \phi^*_i(y) - \frac{1}{2} V(xy,x'y') \left[2 \rho^*(y'y)\phi^*_i(x') + k^*(y'x') \phi_i(y) \right].$$

$$(5-21)$$

where $\rho(y'y)$ and $k(y'x')$ can be written as

$$\rho(y'y) = \sum_i \psi^*_i(y) \psi_i(y')$$

$$(5-22)$$

and

$$k(y'x') = \sum_i \phi_i(x') \psi_i(y')$$

$$(5-23)$$

For zero order solution to the equations of motion, we may write single
particle wave function as

and

$$\psi^{(o)}_i(x,t) = e^{iE_i t + i\vec{k}_i \cdot \vec{r}} v_i$$

$$(5-24)$$

$$\phi^{(o)}_i(x,t) = e^{iE_i t + i\vec{k}_i \cdot \vec{r}} u_i.$$

$$(5-25)$$

Using equation (5-20),(5-21),(5-24) and (5-25) we obtain for zero order
approximation to the equations of motion

$$-E_i V_i = \zeta_i V_i - \Delta_i U_i \quad , \tag{5-26}$$

$$E_i U_i = \zeta_i U_i + \Delta_i V_i \quad . \tag{5-27}$$

Where U_i, V_i, and ζ_i are the usual BCS values as defined in chapter three, and E_i is the single quasi-particle energy as defined by equation (5-28).

$$E_i = \left(\zeta_i^2 + \Delta_i^2 \right)^{1/2} \tag{5-28}$$

One may easily derive the gap equation from equations (5-26) and (5-27), the result is

$$2 \zeta_i U_i V_i = \Delta_i \left(U_i^2 - V_i^2 \right). \tag{5-29}$$

For the linearized equations of motion, let us consider the equations of motion and their conjugates

$$i \frac{\partial}{\partial t} \psi_i(x) = T(x,x) \psi_i(x) + \frac{1}{2} V(xy,x'y') \left[2 \, \rho(y'y) \, \psi_i(x') + k(y'x') \, \phi_i^*(y) \right], \tag{5-30}$$

$$i \frac{\partial}{\partial t} \phi_i^*(x) = T(x,x)^* \phi_i^*(x) + \frac{1}{2} V(xy,x'y')^* \left[2 \, \rho(y'y)^* \, \phi_i^*(x) + k(y'x') \, \psi_i(y) \right], \tag{5-31}$$

$$- i \frac{\partial}{\partial t} \psi_i^*(x) = T(x,x) \psi_i^*(x) + \frac{1}{2} V(xy,x'y')^* \left[2 \rho(y'y) \psi_i^*(x') + k(y'x) \phi_i(y) \right], \tag{5-32}$$

$$- i \frac{\partial}{\partial t} \phi_i(x) = -T(x,x) \phi_i(x) - \frac{1}{2} V(xy,x'y') \left[2 \rho(y'y) \phi_i(x) + k(y'x) \psi_i^*(y) \right]. \tag{5-33}$$

The linearized equations of motion follow directy from equations (5-30), (5-31), (5-32) and (5-33) upon neglect of the third and fourth order

terms:

$$\frac{i\partial}{\partial t} \delta \psi_i(x) = T(x,y) \delta \psi_i(y) + \frac{1}{2} V(x\,y,x'y') \left[2 f(y'y) \delta \psi_i(x') + k(y'x') \delta \phi_i^*(y) \right.$$
$$\left. + 2 \psi_i(x') \delta f(y'y) + \phi_i^*(y) \delta k(y'x') \right], \qquad (5-34)$$

$$i\frac{\partial}{\partial t} \phi_i^*(x) = -T(x,y)^* \delta \psi_i^*(y) - \frac{1}{2} V(x\,y,x'y')^* \left[2 f^*(y'y) \delta \phi_i^*(x') + k(y'x')^* \delta \psi_i(y) \right.$$
$$\left. + 2 \phi_i^*(x) \delta f^*(y'y) + \psi_i^*(y) \delta k(y'x') \right], \qquad (5-35)$$

$$-i\frac{\partial}{\partial t} \delta \psi_i^*(x) = T(x,y) \delta \psi_i^*(y) + \frac{1}{2} V(x\,y,x'y')^* \left[2 f^*(y'y) \delta \psi_i^*(x') + k^*(y'x') \delta \phi_i(y) \right.$$
$$\left. + 2 \psi_i^*(x) \delta f^*(y'y) + \phi_i(y) \delta k^*(y'x') \right], \qquad (5-36)$$

$$\frac{-i\partial}{\partial t} \delta \phi_i(x) = -T(x,y) \delta \phi_i(y) - \frac{1}{2} V(x\,y\, x'y') \left[2 f(y'y) \delta \phi_i(x') + k(y'x') \delta \psi_i^*(y) \right.$$
$$\left. + 2 \phi_i(x') \delta f(y'y) + \psi_i^*(y) \delta k(y'x') \right]. \qquad (5-37)$$

Let us define new varibles $\delta \tilde{\psi}_i(x)$ and $\delta \tilde{\phi}_i(x)$ in terms of old varibles $\delta \psi_i(x)$

and $\delta \phi_i(x)$:

$$\delta \psi_i(x) = e^{i\left(\varepsilon_i t + \vec{\ell}_i \cdot \vec{r}\right)} \delta \tilde{\psi}_i(r), \qquad (5-38)$$

$$\delta \phi_i^*(x) = e^{i\left(\varepsilon_i t + \vec{\ell}_i \cdot \vec{r}\right)} \delta \tilde{\phi}_i^*(r). \qquad (5-39)$$

We may use the effective mass approximation as it has been used in

section 5.1 for the single particle energy. Utilizing equation (5-38),

(5-39) and zero order approximation to the wave function, equations

(5-24) and(5-25) in the equations of motion; the final results are:

$$-\varepsilon_i \, \delta \tilde{\psi}_i(r) + i\frac{\partial}{\partial t} \delta \tilde{\psi}_i(r) = \xi_i \, \delta \tilde{\psi}_i(r) - i \vec{\ell}_i \frac{\nabla \delta \tilde{\psi}_i(r)}{m_x} - \frac{\nabla^2}{2m_x} \delta \tilde{\psi}_i(r) + 2 \sum_j \int dr' \, V(r-r') \times$$

$$u_i v_j \, \delta \widetilde{\psi}_j(\vec{r}') - \sum_j \int dr' \, e^{i(\vec{k}_i - \vec{k}_j)(\vec{r} - \vec{r}')} \, v(r-r') \, v_i v_j \, \delta \widetilde{\psi}_\ell(r) + \frac{1}{2} \sum_j \int dr' \, e^{i(\vec{k}_i - \vec{k}_j)(\vec{r} - \vec{r}')} \, v(r-r') \lambda$$

$$u_i v_j \, \delta \widetilde{\psi}_j(\vec{r}) - \frac{1}{2} \sum_j \int dr' \, e^{i(\vec{k}_i - \vec{k}_j)(r-v)} \, v(v-r') \, u_i u_j \, \delta \widetilde{\psi}_j(v) + \sum_j \int dr \, e^{i(\vec{k}_i - \vec{k}_j)(r-v)} \, v(v-r') \, u_j v_j \, \delta \widetilde{\psi}_\ell(v)$$

$$+ 2 \sum_j \int dr' \, v(v-r') \, v_i v_j \, \delta \widetilde{\psi}_j(r) - \sum_j \int dr' \, e^{i(\vec{k}_i - \vec{k}_j)(\vec{r} - \vec{r}')} \, v_i v_j \, \delta \widetilde{\psi}_\ell(r) + \sum_j \int dr' \, \frac{1}{\lambda} e^{i(\vec{k}_i - \vec{k}_j)(v-r')} \cdot$$

$$v(r-r') \, u_i v_j \, \delta \widetilde{\psi}_\ell(r) - \frac{1}{\lambda} \sum_j \int dr' \, e^{i(\vec{k}_i - \vec{k}_j)(\vec{r} - \vec{r}')} \, v(r-r') \, u_i v_j \, \delta \widetilde{\psi}_j(r) . \tag{5-40}$$

with three similar equations for $\delta\widetilde{\psi}_\ell(x)$, $\delta\widetilde{\psi}(x)$ and $\delta\widetilde{\psi}_\ell(x)$. Solution to the

equations of motion can be derived by introducing

$$\delta\widetilde{\psi}_\ell(r) = \delta\psi_{iq} \, e^{-i(\vec{q}\vec{r} - \omega_q t)} + \delta\psi^*_{qi} \, e^{i(\vec{q}\cdot\vec{r} - \omega_q t)} \tag{5-41}$$

and similar equation for $\delta\widetilde{\psi}(x)$. Then we may rewrite the equations of

motion as

$$\left(E_i - \omega_q + \gamma_i + \frac{\vec{k}_i \vec{q}}{m} + q^2/_{2m*} \right) \delta\psi_{iq} + \sum_j \int dr' \, v(r-r') \left\{ \left[\lambda e^{i\vec{q}(\vec{r} - \vec{r}')} v_i v_j - e^{i(\vec{k}_i - \vec{k}_j)(v-r')} v_i v_j \right. \right.$$

$$+ \frac{1}{\lambda} e^{i(\vec{k}_i - \vec{k}_j)(v-r')} u_i u_j - \frac{1}{2} e^{i(\vec{k}_i - \vec{k}_j + \vec{q})(\vec{r} - \vec{r}')} \right] \delta\psi_{jq} + e^{i(\vec{k}_i - \vec{k}_j + \vec{q})(\vec{r} - \vec{r}')} u_j v_j \, \delta\psi^*_{iq}$$

$$+ \left[e^{i\vec{q}(\vec{r} - \vec{r}')} v_i v_j - e^{i(\vec{k}_i - \vec{k}_j + \vec{q})(\vec{r} - \vec{r}')} v_i v_j \right] \delta\psi^*_{jq} + \left[\frac{1}{\lambda} e^{i(\vec{k}_i - \vec{k}_j + \vec{q})(\vec{r} - \vec{r}')} u_i v_j \right.$$

$$\left. - e^{i(\vec{k}_i - \vec{k}_j)(\vec{r} - \vec{r}')} u_i v_j \right] \delta\psi_{jq} = 0 \tag{5-42}$$

with three similar equations for $\delta\psi_{iq}$, $\delta\psi_i$ and $\delta\psi_{iq}$.

For solution to the equations of motion, we may expand the

variables in the Legendre polynomial, such as

$$\delta\psi_{iq} = \delta\psi^{(0)}_{iq} + P_1(\cos\theta_{iq}) \, \delta\psi^{(1)}_{iq} + \cdots \tag{5-43}$$

where θ_{iq} is the angle between $\vec{\ell_i}$ and $\vec{q}$. Substituting the expanded

varibles in the equation of motion, and keeping only the first term in

those expansions, and taking average over angle θ_{iq}; the final result

can be written as

$$
\begin{pmatrix}
A_{ij} & B_{ij} & C_{ij} & D_{ij} \\
-B_{ij} & -A_{ij} & -D_{ij} & -C_{ij} \\
E_{ij} & F_{ij} & G_{ij} & H_{ij} \\
-F_{ij} & -E_{ij} & -H_{ij} & -G_{ij}
\end{pmatrix}
\begin{pmatrix}
\delta\psi_{jq} \\
\delta\psi^{*}_{jq} \\
\delta\phi_{jq} \\
\delta\phi^{*}_{jq}
\end{pmatrix}
= \omega_q
\begin{pmatrix}
\delta\psi_{iq} \\
\delta\psi^{*}_{iq} \\
\delta\phi_{iq} \\
\delta\phi^{*}_{iq}
\end{pmatrix}
$$

where

$$(5\text{-}44)$$

$$
A_{ij} = \left(E_i + \xi_i + \dfrac{q^2}{2m^*} \right) \delta_{ij} + 2 \left(V(q) - V(\ell_i - \ell_j) \right) v_i v_j + \dfrac{1}{2} \left(2\, V(\ell_i - \ell_j) - V(\ell_i - \ell_j, q) \right) u_i v_j ,
$$

$$(5\text{-}45)$$

$$
B_{ij} = \left[2 V(q) - V(\ell_i - \ell_j, q) \right] v_i v_j ,
$$

$$(5\text{-}46)$$

$$
C_{ij} = \dfrac{1}{2} \left[V(\ell_i - \ell_j, q) - V(\ell_i - \ell_j) \right] u_i u_j ,
$$

$$(5\text{-}47)$$

$$
E_{ij} = \left(2 V(q) - V(\ell_i - \ell_j) \right) u_i v_j + \dfrac{1}{2} \left[V(\ell_i - \ell_j, q) - V(\ell_i - \ell_j) \right] u_i u_j ,
$$

$$(5\text{-}48)$$

$$
D_{ij} = \left[\sum_{\ell} V(\ell_i - \ell_\ell, q) u_\ell v_\ell \right] \delta_{ij} ,
$$

$$(5\text{-}49)$$

$$
F_{ij} = \left(2 V(q) - V(\ell_i - \ell_j, q) \right) u_i v_j - \left(\sum_{\ell} V(\ell_i - \ell_\ell, q) u_\ell v_\ell \right) \delta_{ij} ,
$$

$$(5\text{-}50)$$

$$
G_{ij} = \left(-E_i + \xi_i + \dfrac{q^2}{2m^*} \right) \delta_{ij} + \dfrac{1}{2} \left(V(\ell_i - \ell_j) - V(\ell_i - \ell_j, q) \right) v_i v_j ,
$$

$$(5\text{-}51)$$

$$H_{ij} = 0, \qquad\qquad (5\text{-}52)$$

and

$$V(q) = \int dr \; e^{i\vec{q}\cdot\vec{r}} \; v(\vec{r}) \qquad\qquad (5\text{-}53)$$

$$V(\vec{\ell}_i - \vec{\ell}_j , \vec{q}) = \int d\Omega_{\vec{\ell}_i\vec{\ell}_j} \int dr \; e^{i(\vec{\ell}_i - \vec{\ell}_j + \vec{q})\cdot\vec{r}} \; v(r) \qquad (5\text{-}54)$$

As one may expect, the equation (5-44) reduces to the ordinary RPA

when one sets $\delta f_{jq} = 0$ As in the RPA the solution to the equation (5-44)

has the property that if (ω , $\delta\psi$, $\delta\psi^*, \delta q, \delta f^*$) is a solution, ($-\omega^*$, $\delta\psi^*$,

$\delta\psi$, $\delta f^*, \delta q$) is also a solution.

For the derivation of the excited state energy for the neutron

matter, we use the potential (5-1) introduced in section 5.1. The

three dimension sum over j is simplified by the usual interchange to

the integral form:

$$\sum_j \Rightarrow \int \frac{d\vec{k}}{(2\pi)^3}$$

then integral over radial dimension $\int dk_j$ is interchanged by sum over $\vec{\ell}$.

The final result is similar to equation (5-44) with sum over radial

dimension in the momentum space ($\vec{\ell}$). In numerical calculations the

region and the mesh size are $[0.0, 3]$ and 0.15 respectively. The

corresponding matrices of 80X80 dimension are diagonalized, using

the EIPAC subroutines (Eigen system subroutine package). Figures [12-17]

show the dispersion relation of the system for a given density and

effective mass. For modes with large energy gap the excitiation energy

is a function of square of wave number as shown in figure [12]. Figures
[13-15] show some modes in which interaction traces are appreant in the
long wave length region. The upper and lower modes of a band of dispersion
are shown in figure [16]. The modes in this band are also hyperbolas in
terms of wave number. Finally the unstable mode of dispersion relation
is shown in figure [17]. These figures show similar dependence on wave
number between QPRPA and RPA in short wave length region, as one may
expect physically. The numerical calculation supports the hyperbola shape
for some of these modes in the long wave length region.

Now, consider the comparison between fluid model and the corresp-
onding QPRPA results. Figure [18] shows the dependence of phonon energy
in fluid model for a given effective mass and propagation wave number,
and dashed lines are the corresponding QPRPA values. This figure shows
the QPRPA excitation energy is an increasing function on density, and
the corresponding fluid model is smaller in the region $[0.5, 0.7\ f]^{-1}$,
higher in the region $[0.7, 0.9\ f]^{-1}$ and difference in the excitation energies
is sharply widening for high densities. The dependence in the phonon
energy on effective mass is shown in figure [19] for a given density
and propagation wave number, and dashed lines are the corresponding
QPRPA values. As we see from this figure, the QPRPA excitation energy is a
slow decreasing function on effective mass, and the corresponding fluid
model also is a decreasing function on it with a much higher slope.

Finally figure [20] compares the dispersion relation between the fluid
model and the QPRPA for a given density and effective mass.

Actually, the fact that for a contact interaction in the long wave
length region the RPA coincides with its corresponding hydrodynamic
model porvided the impetus for a similar comparison between QPRPA and
the fluid model in the TDHB theory. But as figures [18-20] show there
are some disagreements for the solution of the linearized TDHB equations
and the corresponding hydrodynamic approach. Figure [20] shows that the
hydrodynamic dispersion relation is a linear function of wave number
while the QPRPA dispersion relation is a hyperbola in terms of wave
number. Similarly, there is disagreement for the phonon energy in
terms of density between the two approaches. These disagreements between
QPRPA and the corresponding hydrodynamic results enhance the uncertainty
in applicablity of hydrodynamic approximations for a nuclear system.

Chapter VI

Conclusion.

This work has explored the hydrodynamic approximation to the
Hartree-Bogolyubov theory. From a varitional point of view TDHB equations
were derived, and various limits and properties of these equations
have been discussed. The TDHB equations were utilized for a hydrodynamic
description of a nuclear system. For this purpose, the Koonin approach
to the hydrodynamic interpretation has been employed in which the Wigner
representation of the TDHB equations is used. Similar to the Koonin
result, we also had a semiclassical interpretation for one of the TDHB
equations as the equation of motion for the phase distribution function.
It was shown that the distribution function satisfies a quantal version of
a modified Valasov equation, which approaches the classical result in
the limit $\hbar \rightarrow$ · and where the number of particles is fixed. Although,
the fourier transform of the expectation value of two particles (holes)
field did not have a semiclassical interpretation, we did find the
deviation distribution function with interesting semiclassical interpre-
tation. It was shown that the deviation function satisfies Valasov's
equation similar to the phase distribution function, and classicaly
it is the square of fluctuation density. It was possible to utilize the
Wigner representation of the equations of motion for a derivation of the
semiclassical hydrodynamics by taking various moments of the distribution
functions, but we did not expect any physical gain to follow in this path.

Actually, derivation of TDHB equations from a variational point
of view enabled us to have a Lagrangin which not only describes the
system in its microscopic level, it can also be utilized for description
of collective motion of the system. For this purpose, we assumed all
variables in the system are changing through a few variables, and hoped
these variables are good approximation to the collective coordinates in
the system. For a specific example in this line of reasoning, we consid-
red the two fluid model as our goal. With various form of parametrization
for the trial wave function, equation of motion for the two fluid model
were derived. In the first method, we empolyed a diagonalized form of
TDHB density matrix, then included dynamics of the system through time
dependence of the single particle wave function. The phases of single
particle wave functions are divided by two coherent groupes, and the
chemical potential was used for the division of each category. We had a
two fluid model Lagrangian, and from Hamilton's procedure the equations
of motion were derived with appropriate definition of velocities for Irr-
otational and normal fluids, we derived a set of equations which had some
resemblance to the equations of motion in Landau's theory of the two fluid
model. Small oscillations of the densities near their equilibria for this
model were considered. The dispersion relation were derived and its vari-
ous limits have been discussed. Similar approximation as one had in the
Landau theory was employed for factorization of the dispersion relation.
One branch of the dispersion relation corresponds to the normal sound,
and the second one was similar to the electron plasma oscillation disper-

sion relation. In the second approach, we utilized a general HB trial

wave function for the derivation of Lagrangian density. Of the two veloc-

ity potentials, one was derived from the expectation values of density

and two particle field operators and the other one was introduced through

a general phase factor in the trial wave function; the corresponding

conjugate variables to the these phase velocities were the deviation

density and density, respectively. Due to the fact that kinetic energy

is a one body operator, the result of the kinetic part of the classical

Lagrangian was independent of the deviation density. This independence

is the reason for the simplicity of the second approach, and subsequently

it was responsible for poor resemblance of the model with the Landau

theory. In the third approach our attention was focussed on coherent

excitation of the two particles and two holes in the trial wave function.

One of the phase velocities and its corresponding density were only adia-

batically canonical conjugates. The equations of motion describing this

model were a set of integro-differential equations which did not have

any similarty with Landau's equations. In both the second and third

model density oscillation was considered to derive the dispersion rela-

tions. The dispersion relation for the second approach was simple; one

of its branches had zero roots and the second branch had a simple normal

fluid dispersion relation. For the third approach the dispersion relation

was more complicated. And for small wave numbers, both branches of the

dispersion relation had a finite effective mass.

In   derivation of the restricted Lagrangian, the assumption
that the trial wave function changing through only a few variables at
best is questionable, the difficulty will not end at this point. One
must find the energy density as functional of densities and an honest
calculation of this energy density will be as cumbersome as solving the
whole microscopic equations of motion. However an intelligent guess for
the energy density as functional of densities may help to understand the
dynamics of the system. For this purpose, an Irrotational fluid model
was developed from a TDBCS trial wave function. For a contact interaction
with the TDHF equations as microscopic solution to a many body system
it can easily be seen that the microscopic energy for the phonon in the
small wave number region coincides with the corresponding hydrodynamic
approximation, where static solution of HF theory is approximated for
the energy density as a functional of density. This fact was an impetus
for studying similar comparison for TDHB theory results and their hydro-
dynamic counterpart. For this purpose, a study of sound in neutron matter
was considered, and a model has been developed. It was assumed that
neutrons are interacting via soft-core potentials. The velocity of sound
was calculated, and the numerical results for the velocity of sound
were reasonable considering the result of other studies of nuclear matter.
Finally in the last section, the time dependent BCS equations of motion
were derived, utilizing the Generlized Hartree-Fock method of Kerman-
Klein. These equations of motion were linearized in the anticipation of

QPRPA solutions. Due to the nonseparability of the linearized equations of motion, a reduction in number of excurison amplitudes could not be achieved. For example, it was not possible to find a schematic potential in which deviation of density could be the only arbitrary parameter. Therefore, the linearized Hartree-Bogolyubov equations were solved microscopically, and the closest mode in QPRPA to the hydrodynamic approximation has been chosen for comparsion with the fluid model dispersion relation. The numerical calculation for the neutron matter was carried out, and the phonon energies were compared with the corresponding hydrodynamical approach. The agreement between the two approaches was rather a poor one, for example the hydrodynamic dispersion relation was linear in terms of wave number, but in QPRPA it was a hyperbola. Also there was disagreement for the phonon energy in terms of density in the two approaches. These disagreements between QPRPA and the corresponding hydrodynamics results enhanced the uncertainty in the applicability of hydrodynamic approximations for the nuclear systems. In the other words, the nuclear system can not be told to choose only a few parameters for its time evolution, and a microscopic solution, often a complicated one, gives a better understanding for this system.

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FIGURE CAPTIONS

Figure 1

Canonical form of a two fermion wave function according to reference 24

Figure 2

A particular set of closed chains arising on the right-hand side of

equation (2-16), according to reference 24

Figure 3

An open chain, which arises together with a set of closed chain in each

term of the right-hand side of equation (2-17), according to reference 24

Figure 4

Two odd open chains together with a set of closed chains, form one of

three possibilties for a term in the expansion of a two body matrix

element according to reference 24.

Figures 5-6

These figures show the dependence of gap energy on wave number for the

neutron matter model.

Figure 7-8

These figures show the dependence of the gap energy at the fermi momentum

on the density of neutron matter for various effective masses.

Figures 9-10

These figures show the dependence of velocity of sound in neutron matter

on the density for various effective masses

Figure 11

This figure showes the dependence of the velocity on the effective mass for various densities.

Figures 12-17

These figures show various modes of dispersion relation of the system for a given density and effective mass.

Figure 18

This figure showes the dependence of phonon energy in fluid model for a given effective mass and propagation wave number, and dashed lines are the corresponding QPRPA values.

Figure 19

The dependence in the phonon energy on effective mass is shown in this figure for a given density and propagation wave number, and dashed lines are the corresponding QPRPA values.

Figure 20

This figure compares the dispersion relation between fluid model and the QPRPA for a given density and effective mass.

Table (1)

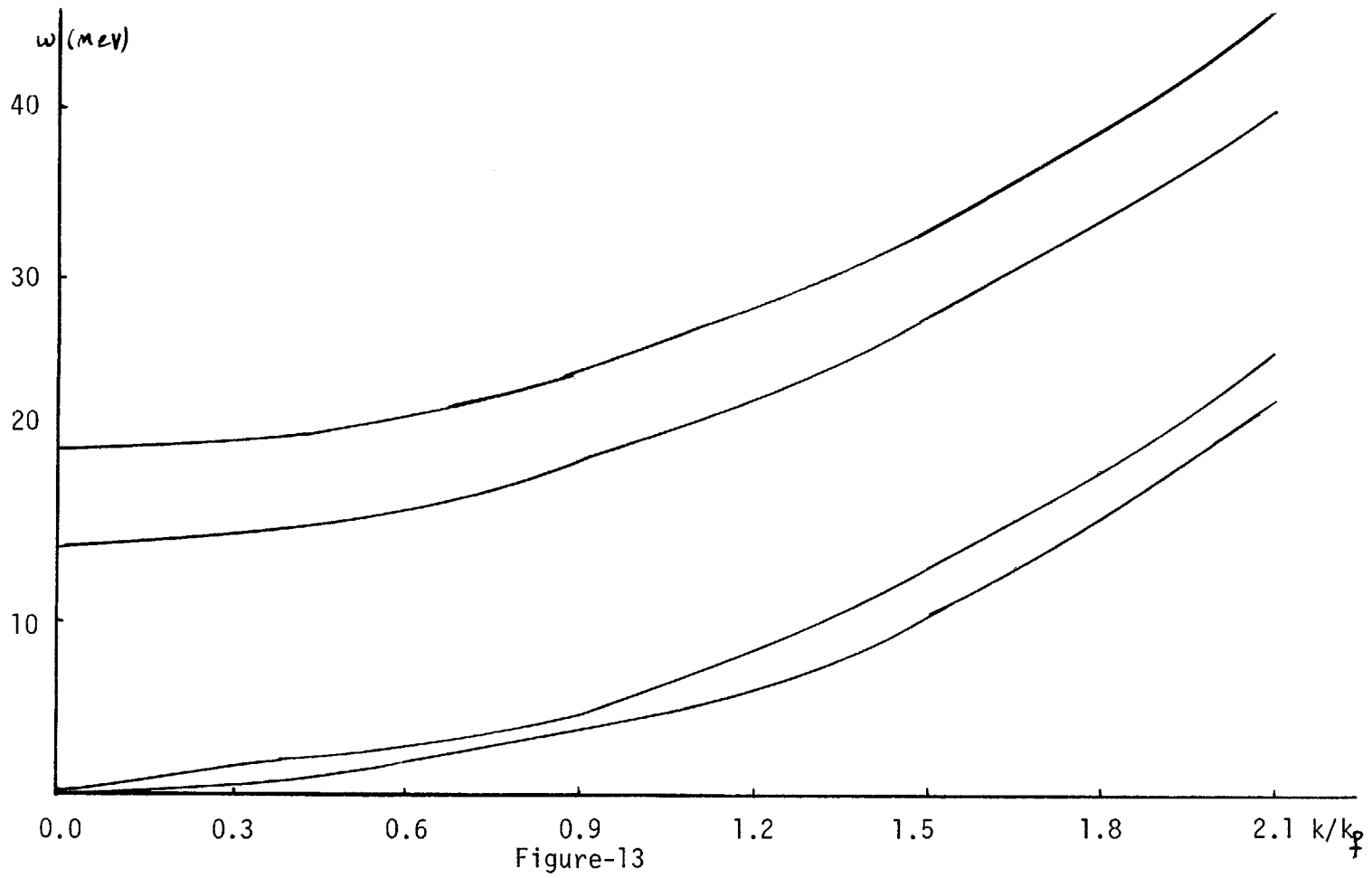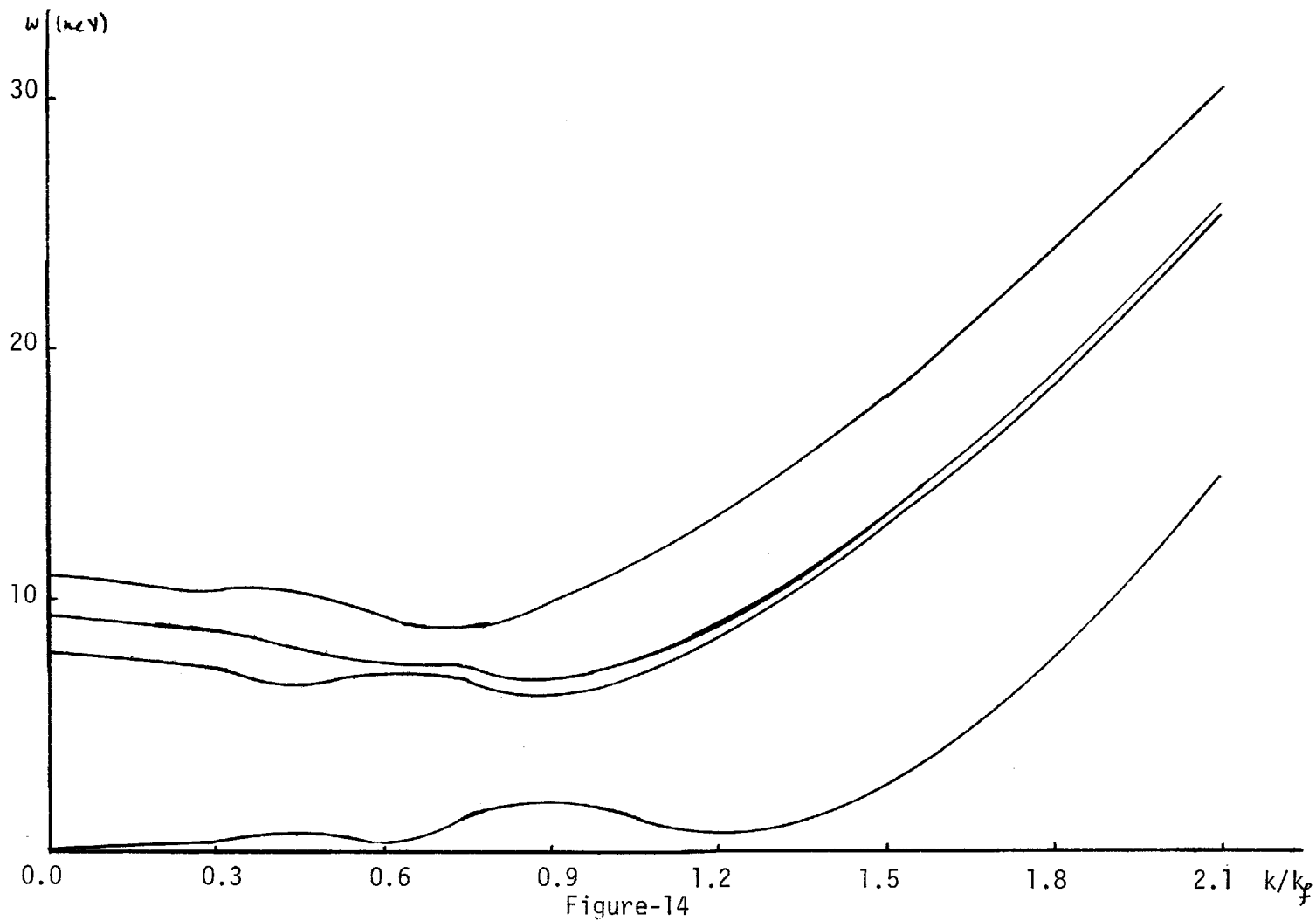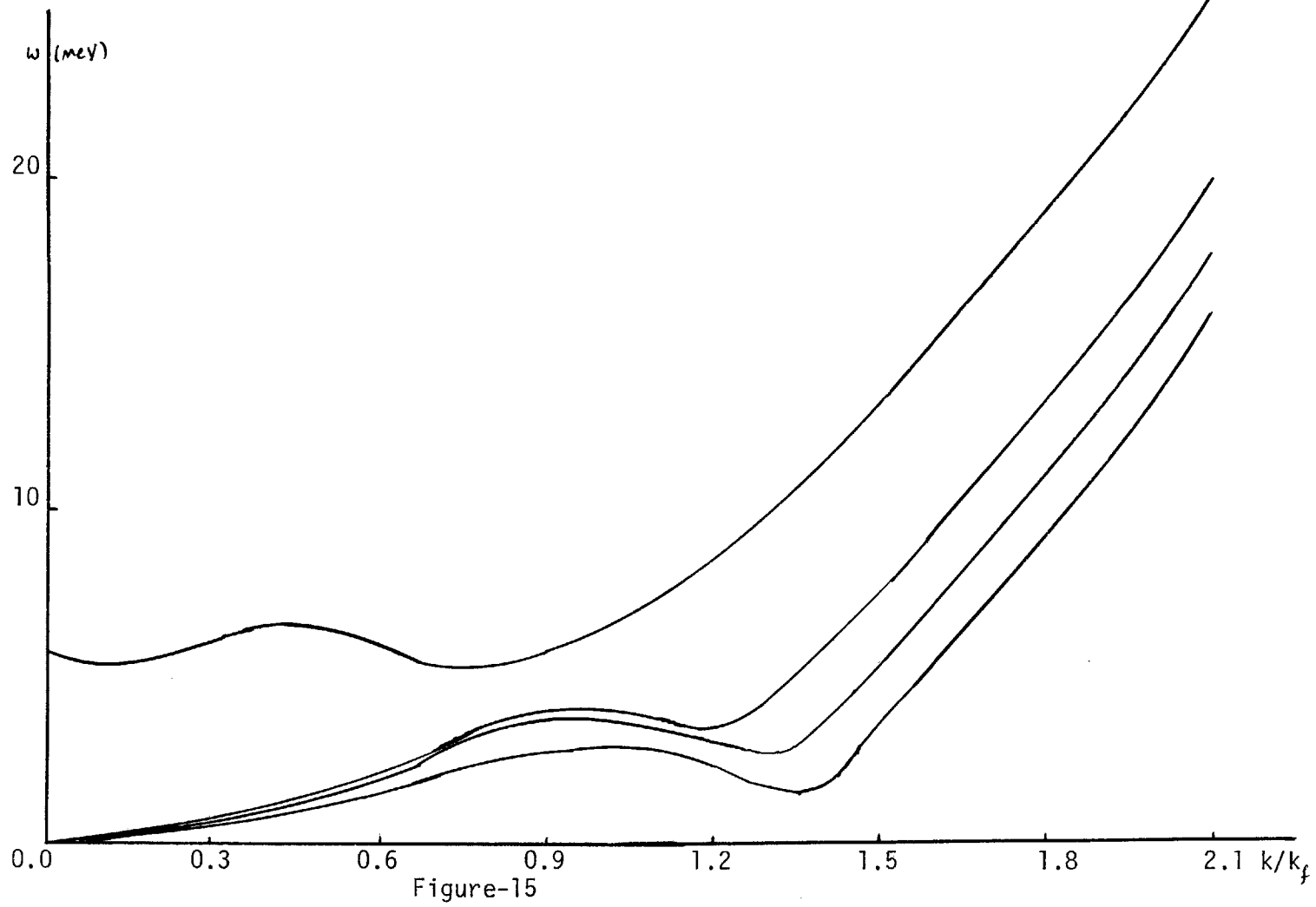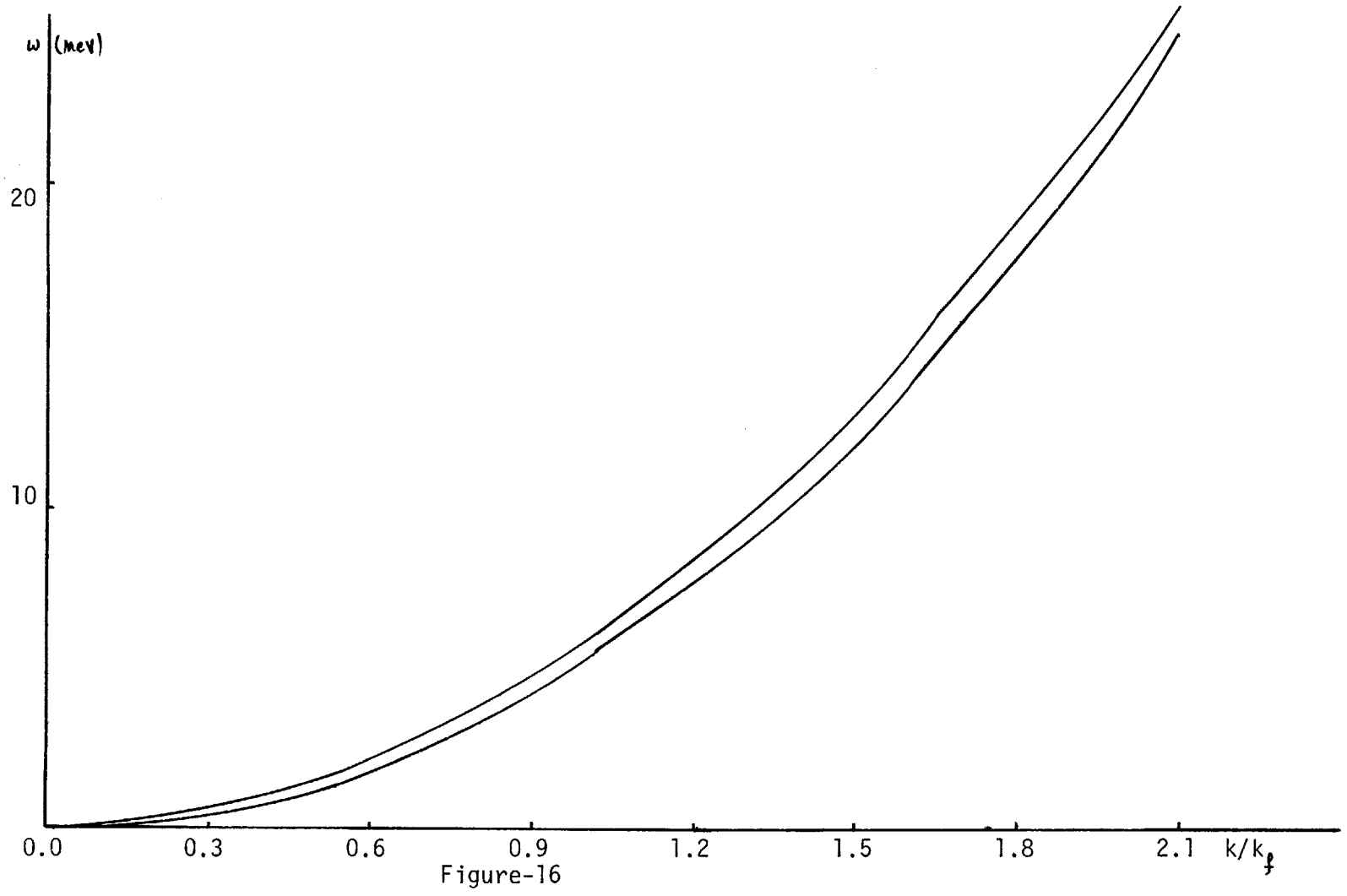| V_α in Mev	a_α in fm
$V_1 = -7.2$	$a_1 = 1.876$
$V_2 = -279$	$a_2 = 0.9427$
$V_3 = 1000$	$a_3 = 0.533$

0

1
ī
2
2̄
3
3̄

0

| 0 | θ₁ |
| −θ₁ | 0 |

0

| 0 | θ₂ |
| −θ₂ | 0 |

| 0 | θ₃ |
| −θ₃ | 0 |

0
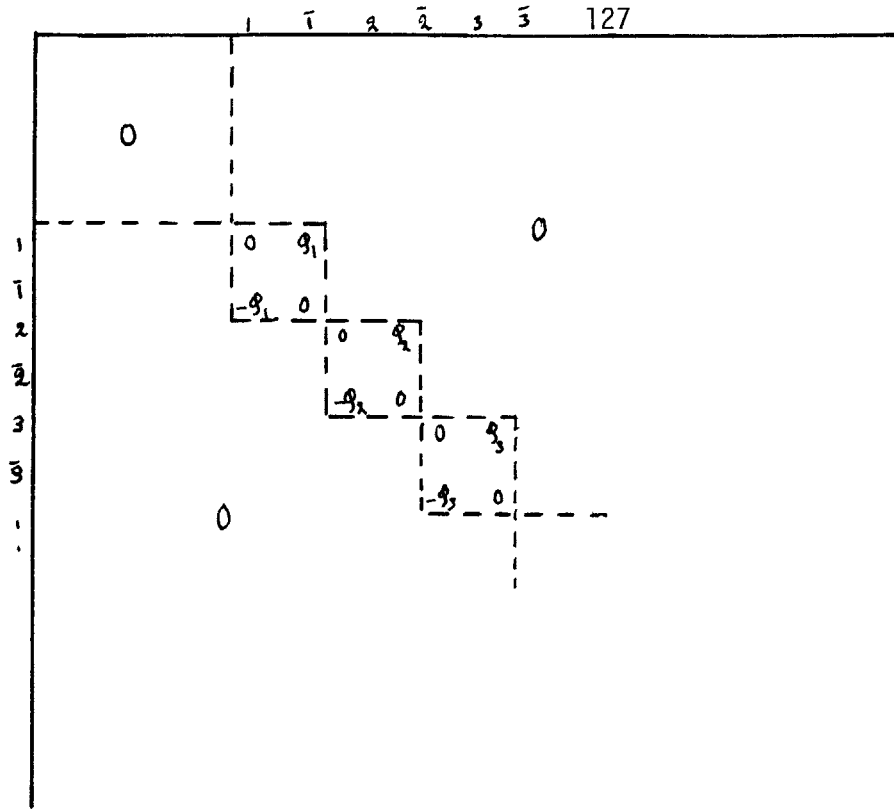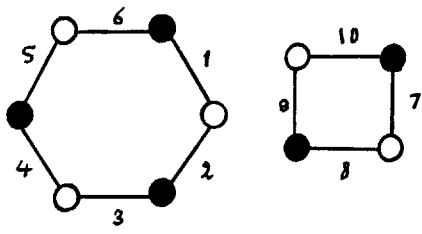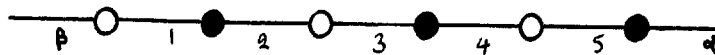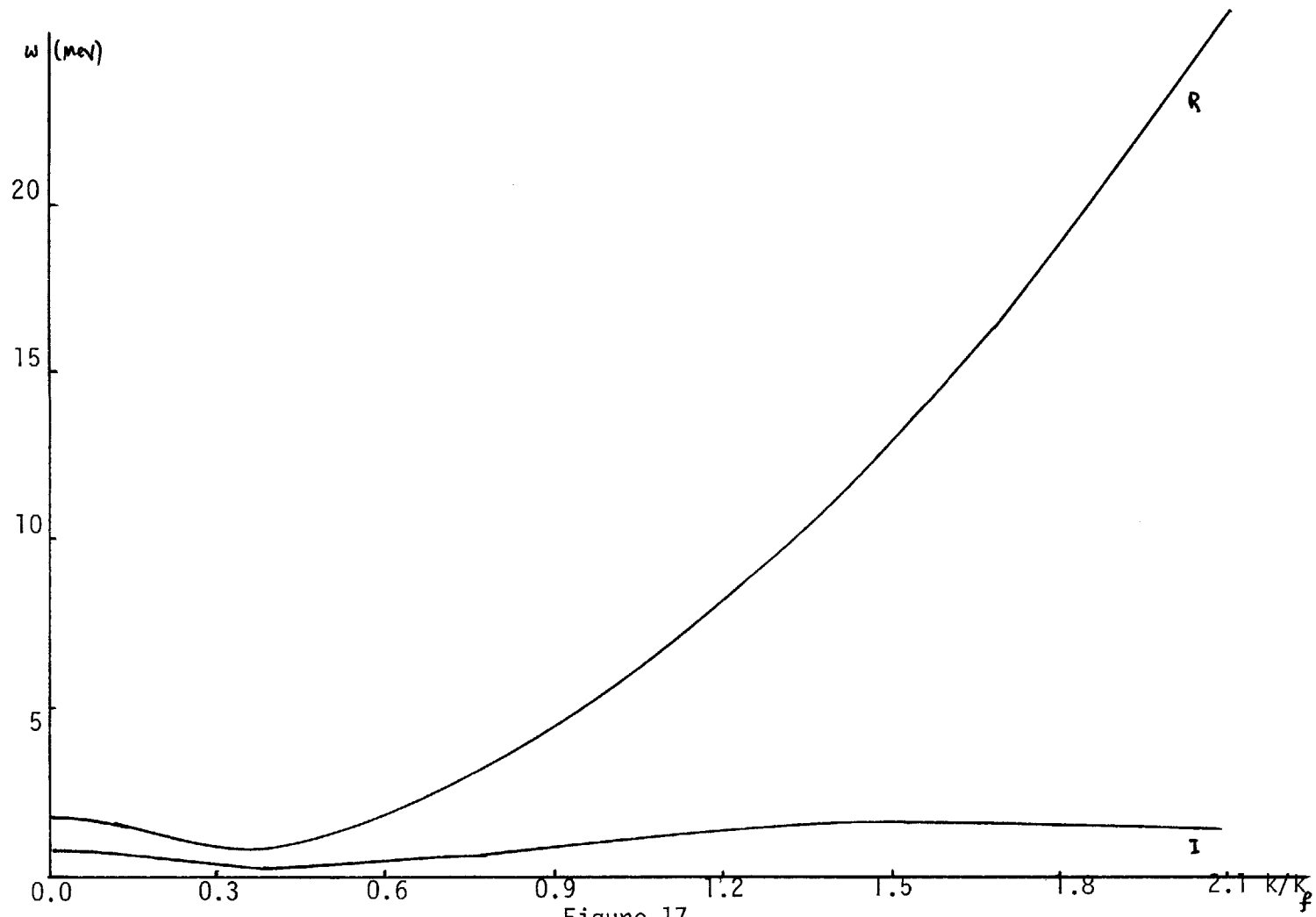
Figure-1

Figure-2

Figure-3

Figure-4

Figure-5

$m^*/m = 0.8$

$k_f = 0.7 \, fm^{-1}$

$\Delta_k \, (meV)$

k/k_f

Figure-6

Figure-7

Δ_{k_f} (MeV)

$m^*/m = 0.95$

$m^*/m = 0.85$

$m^*/m = 0.75$

k_f f^{-1}
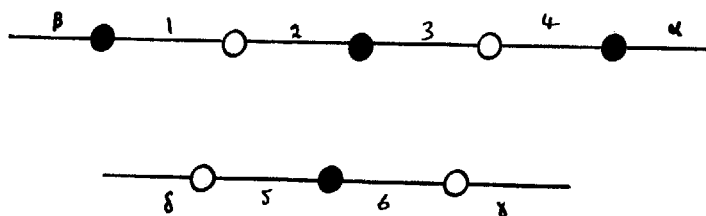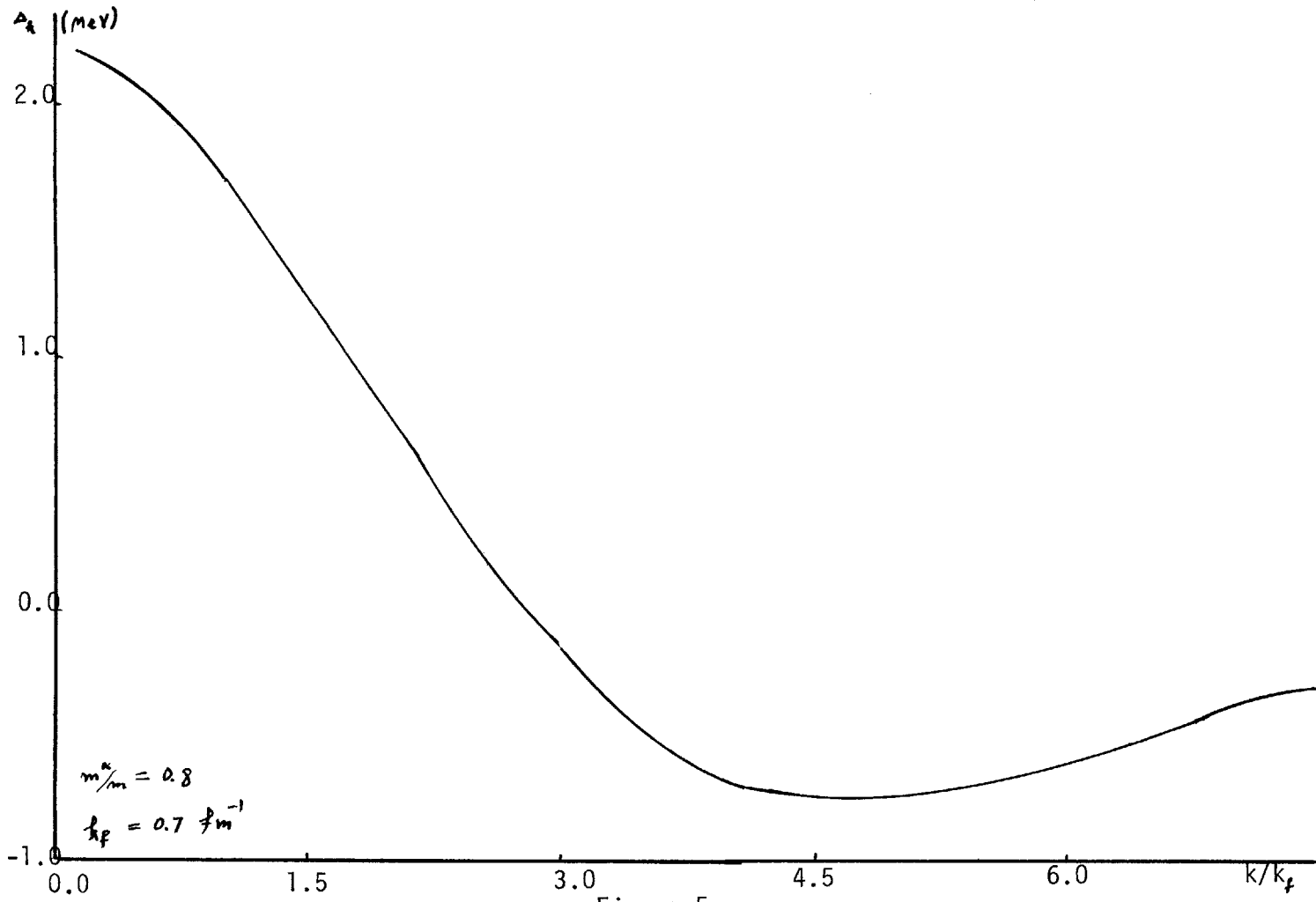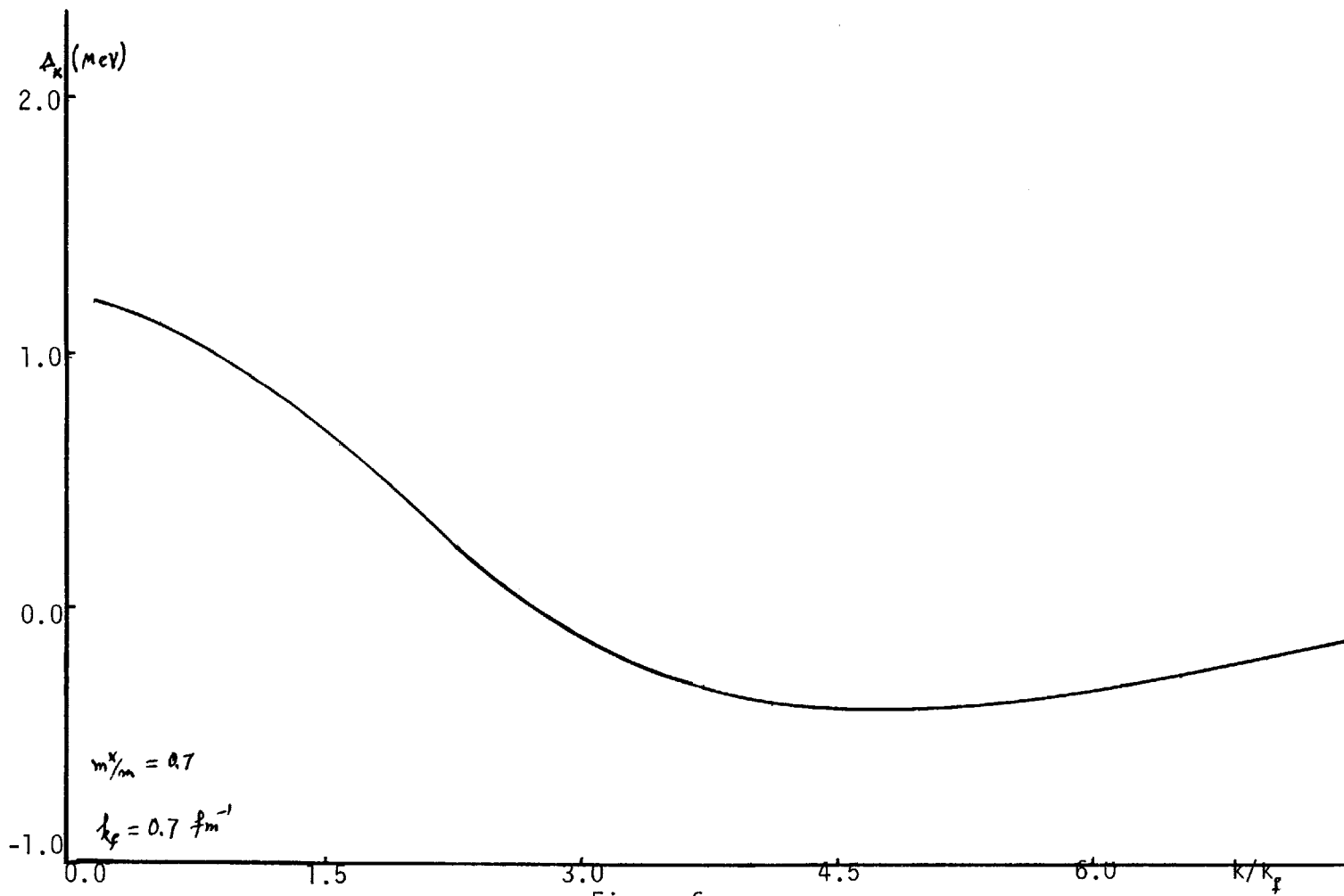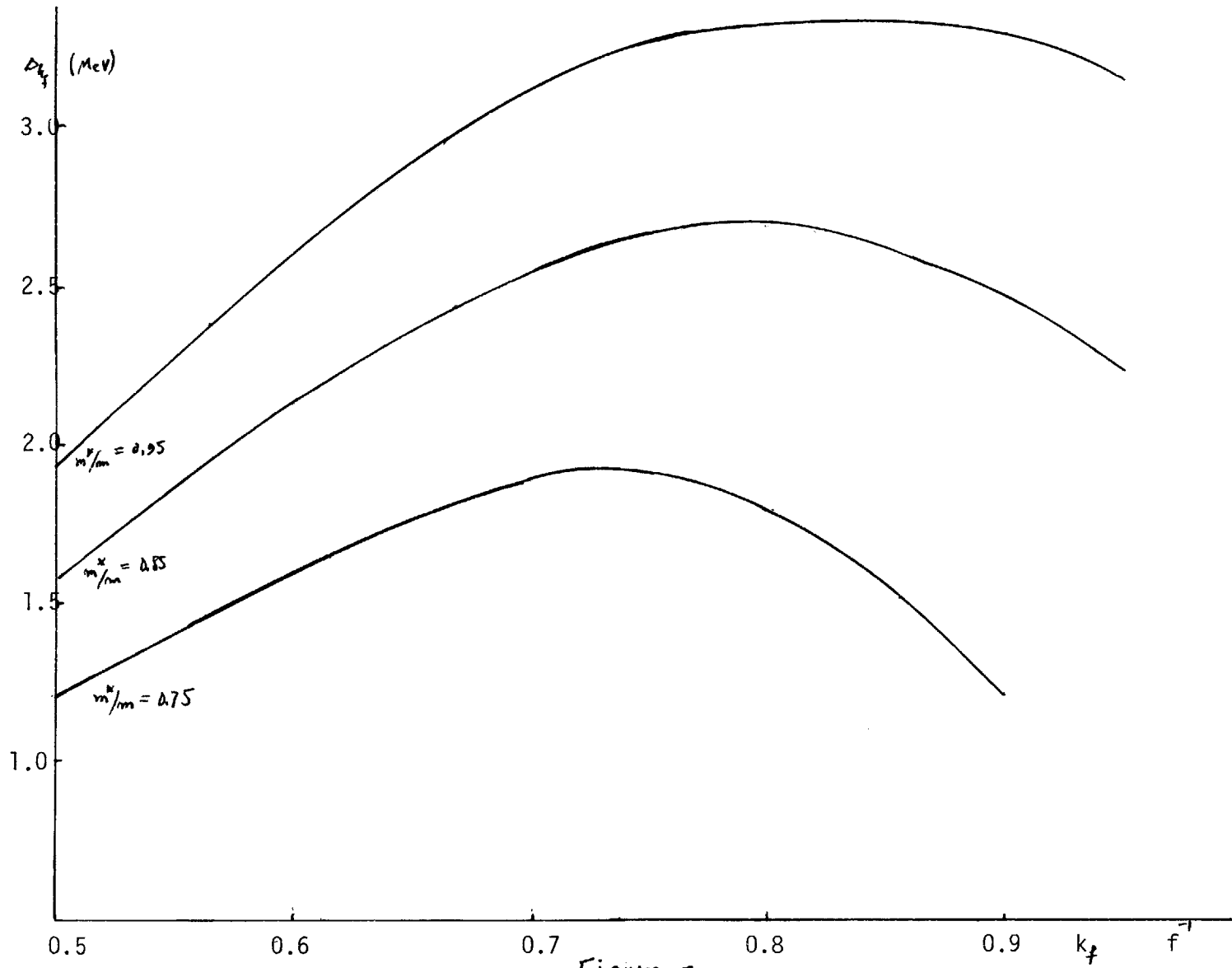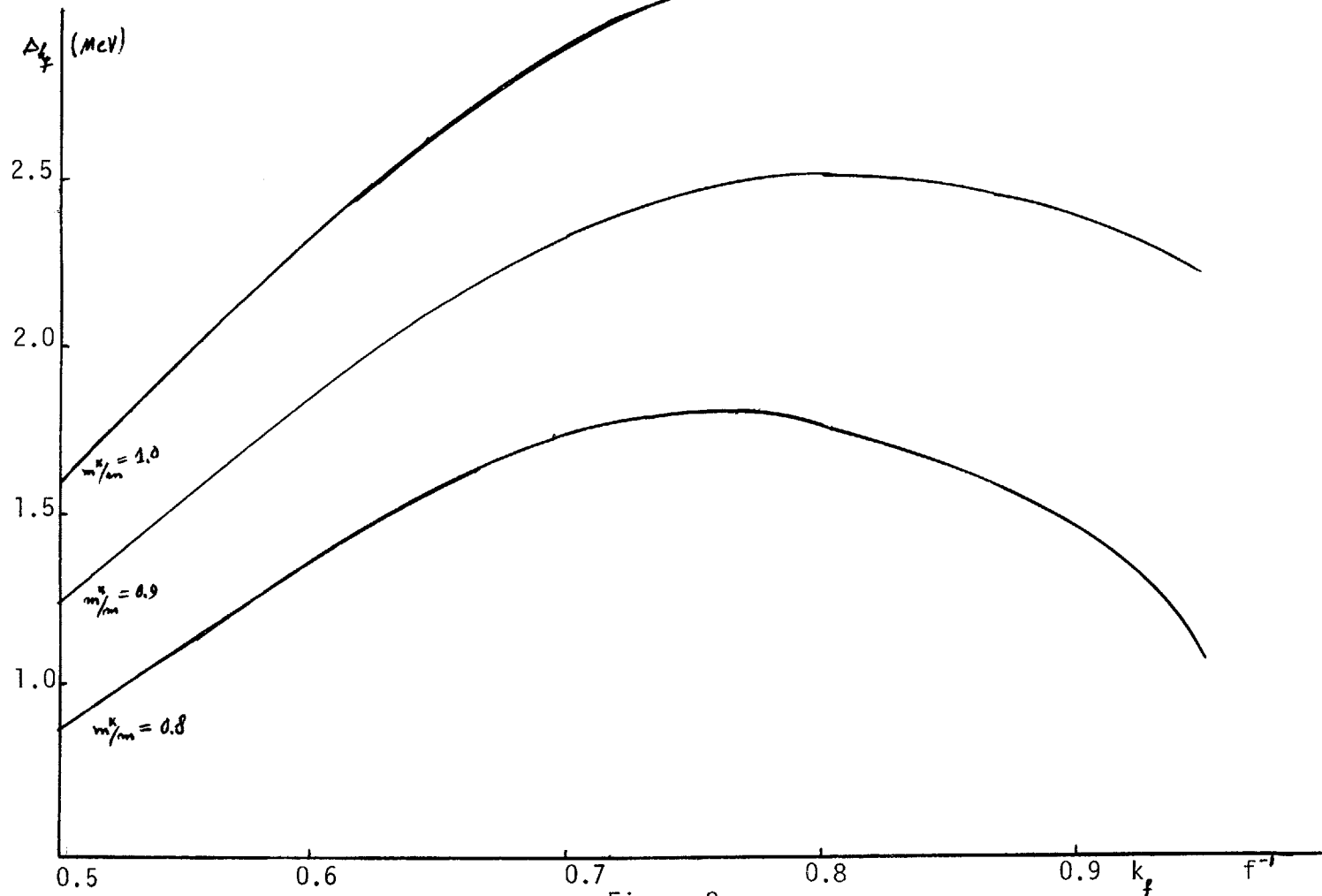
Figure-8

Figure-9

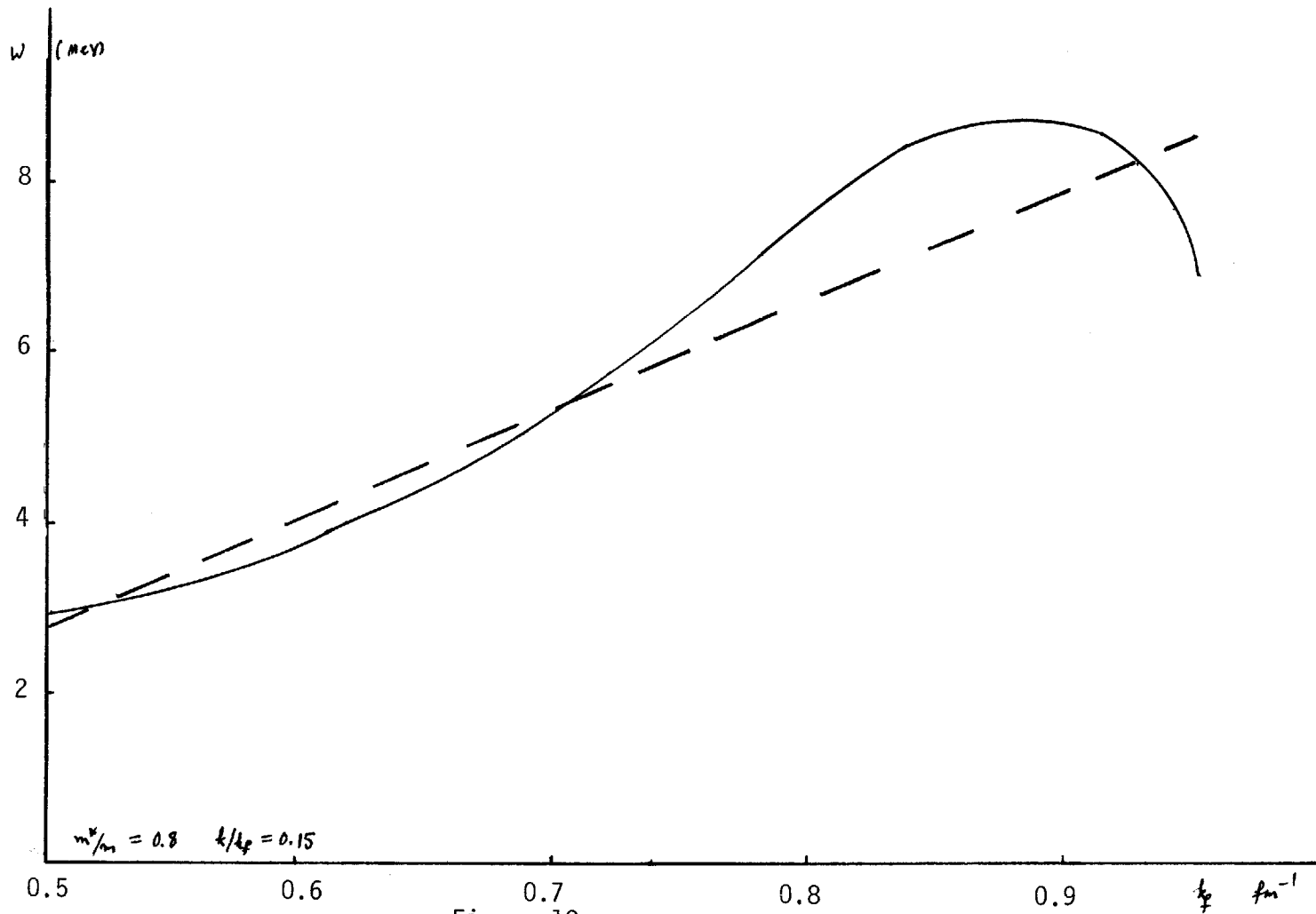Figure-10

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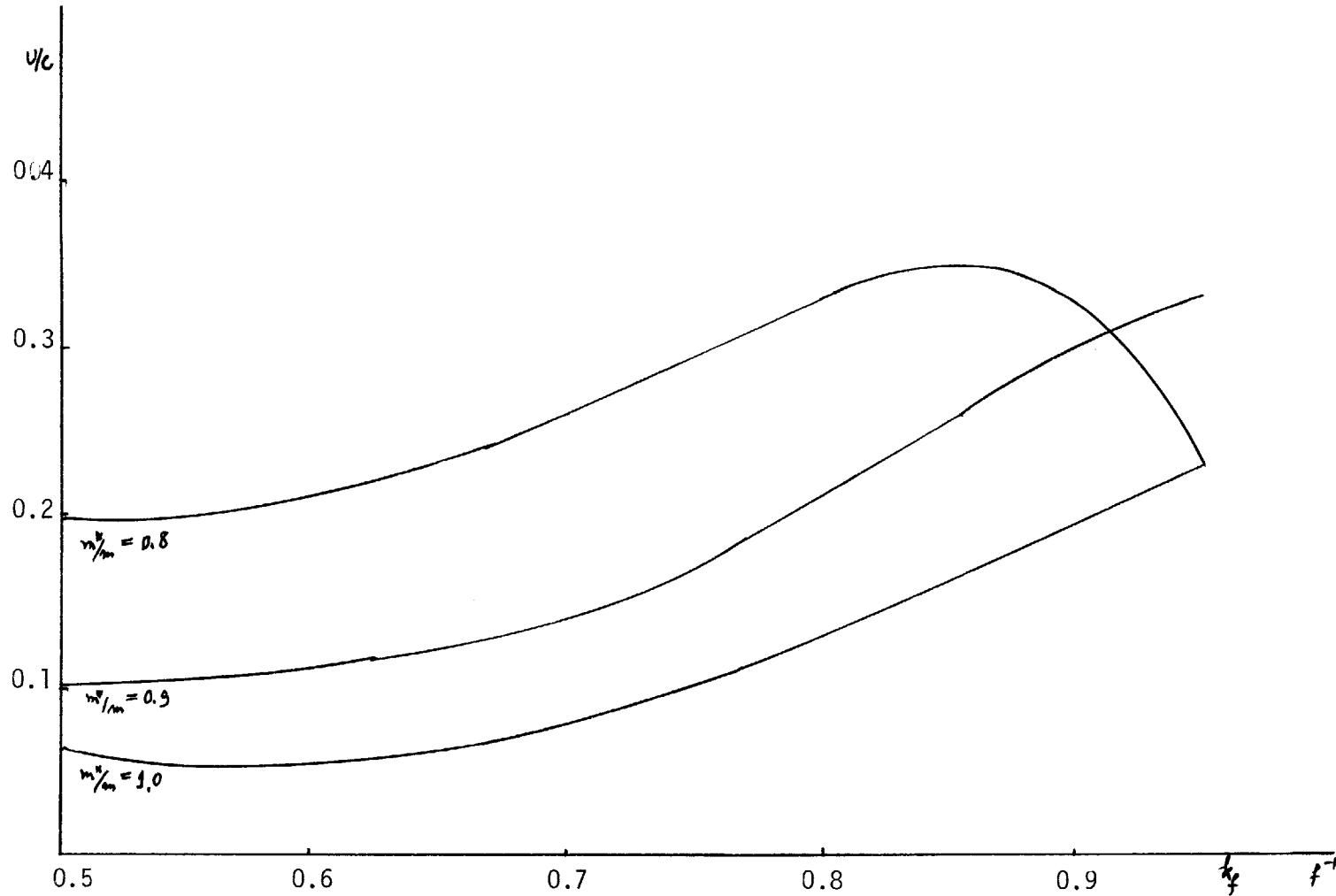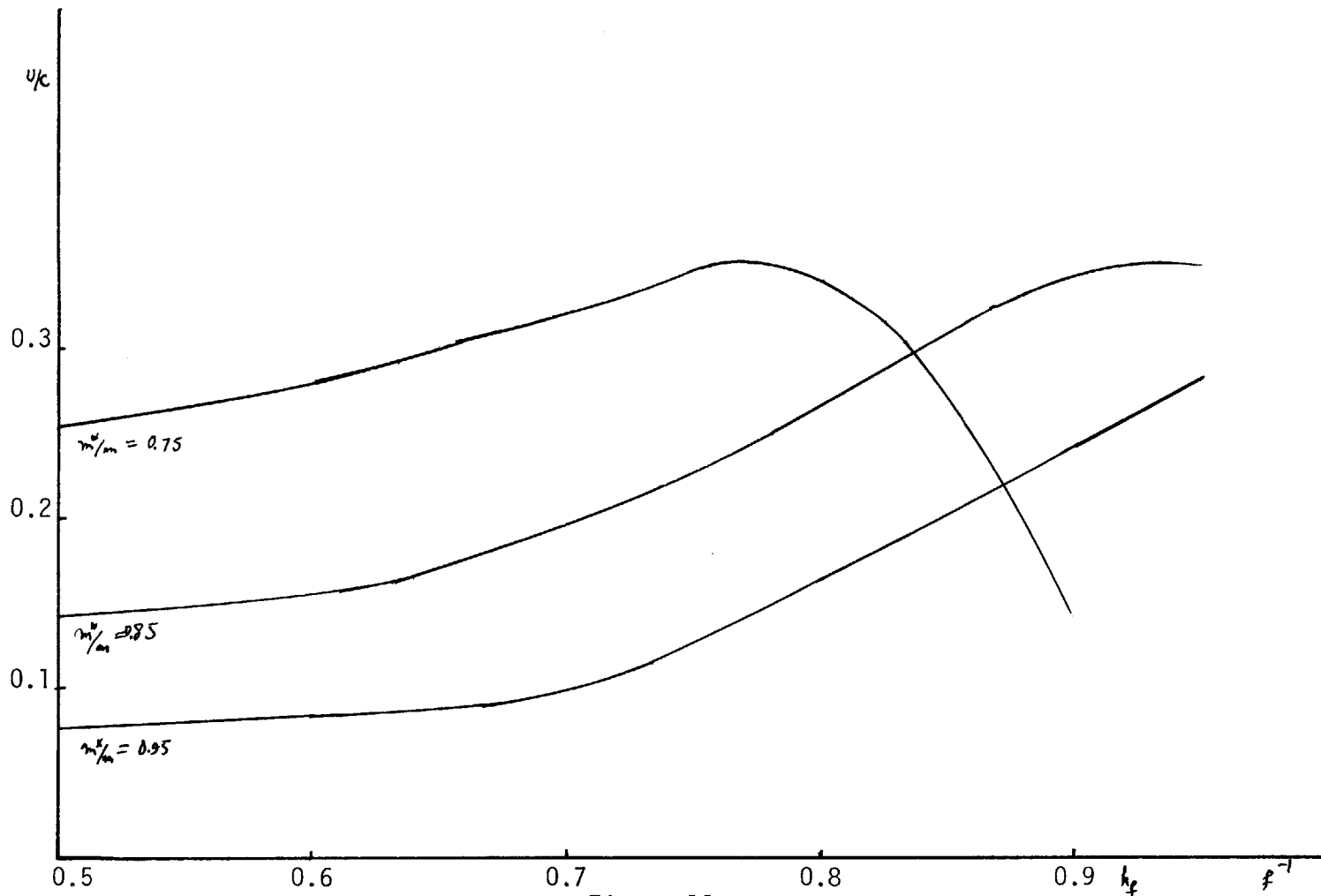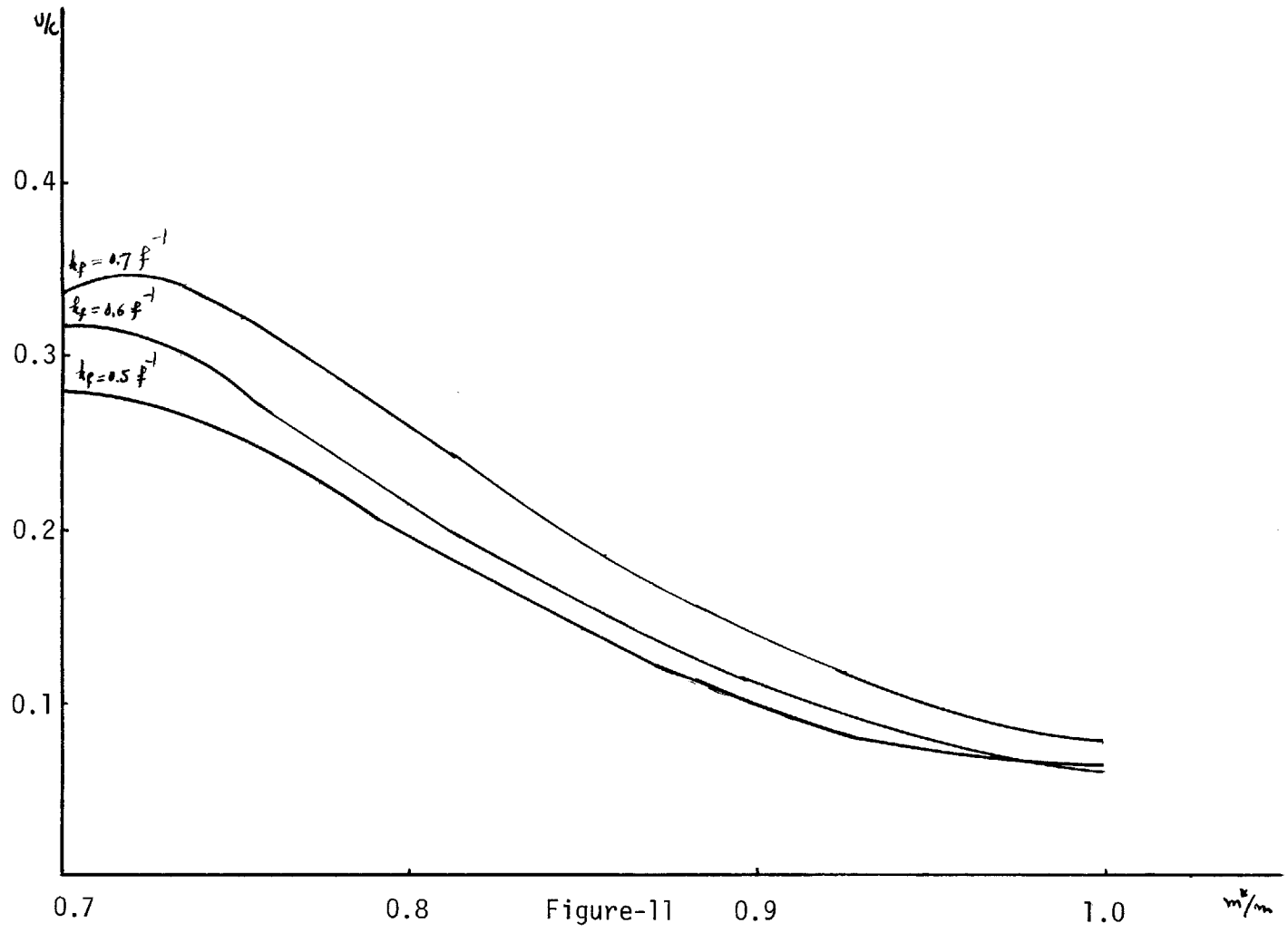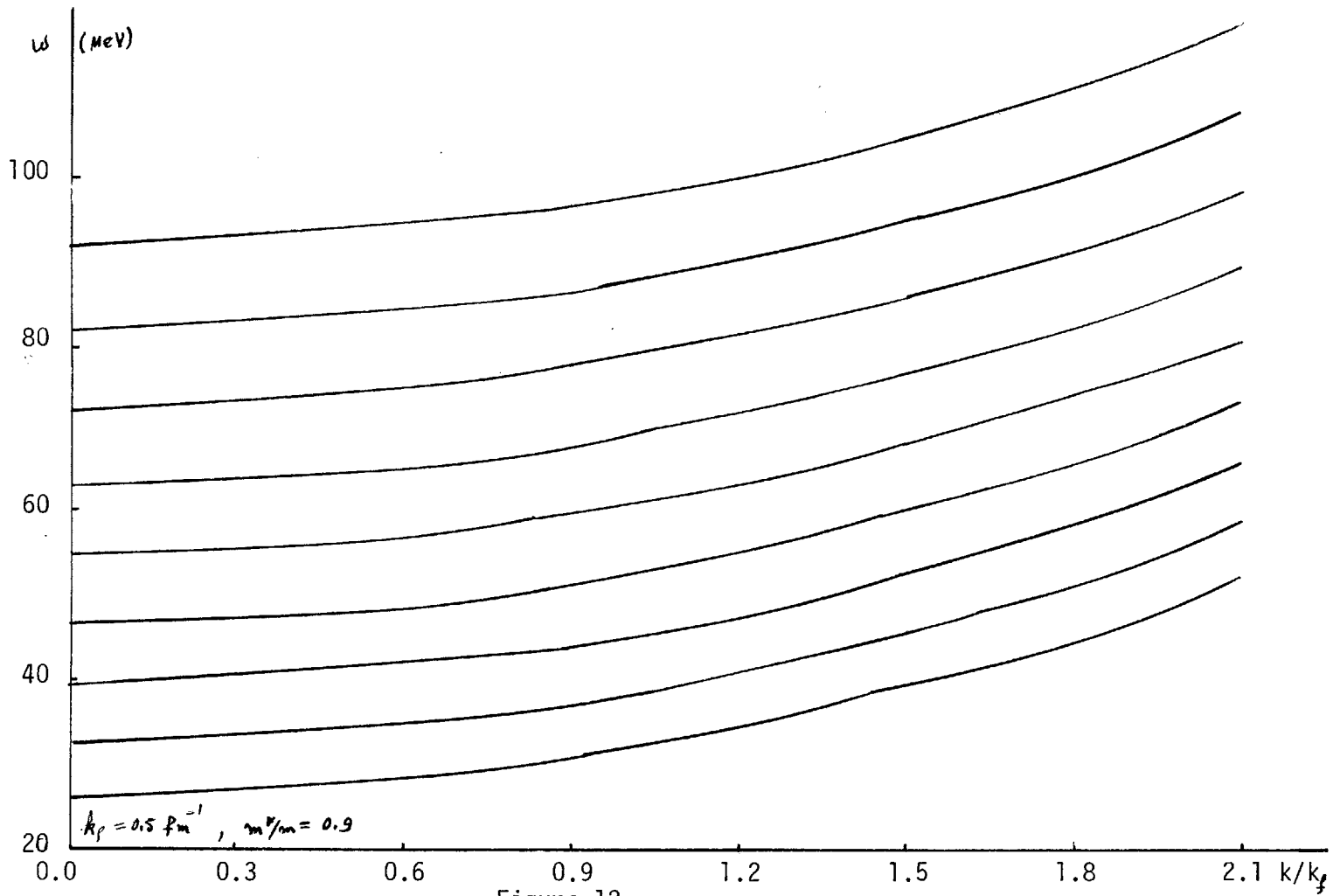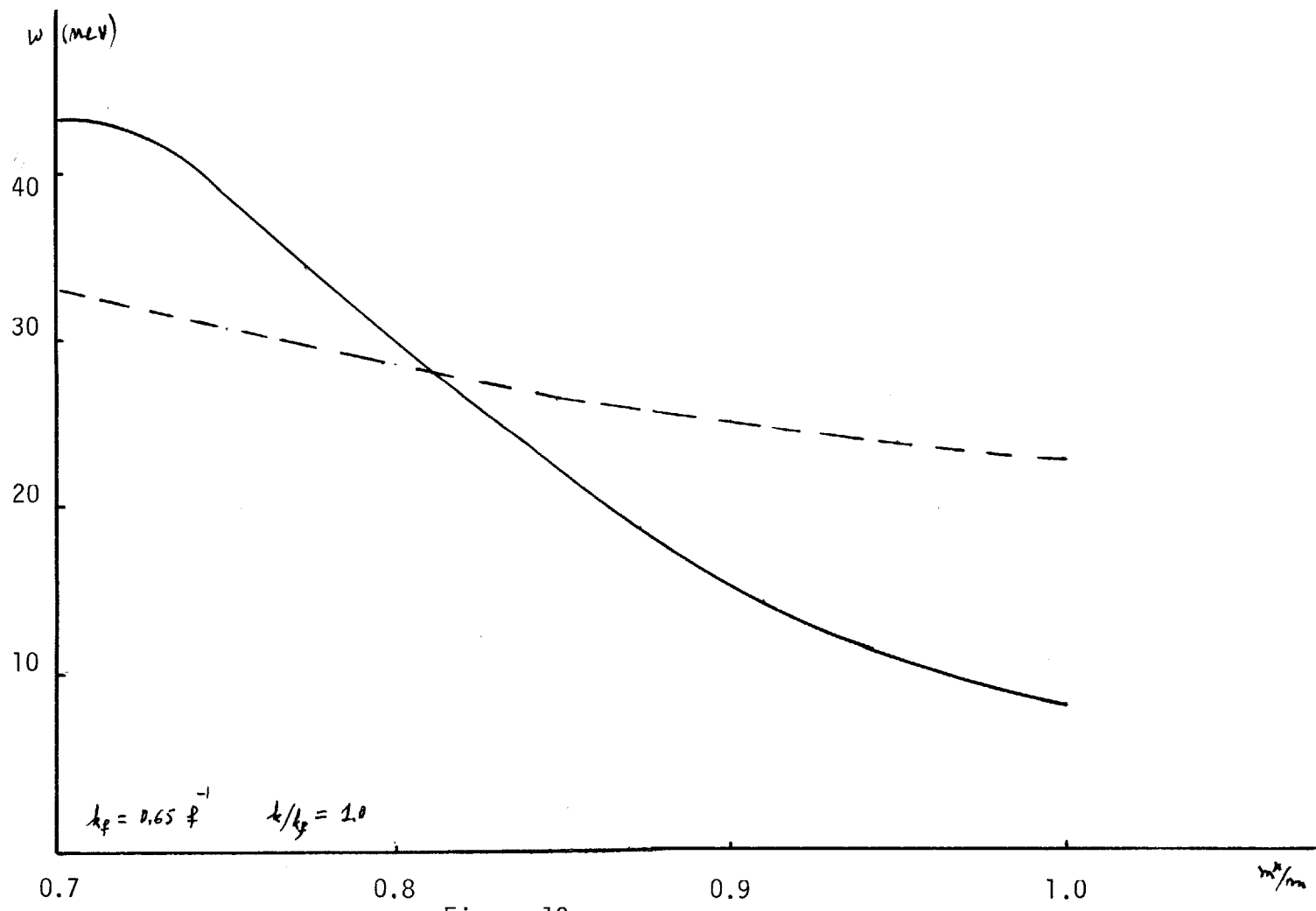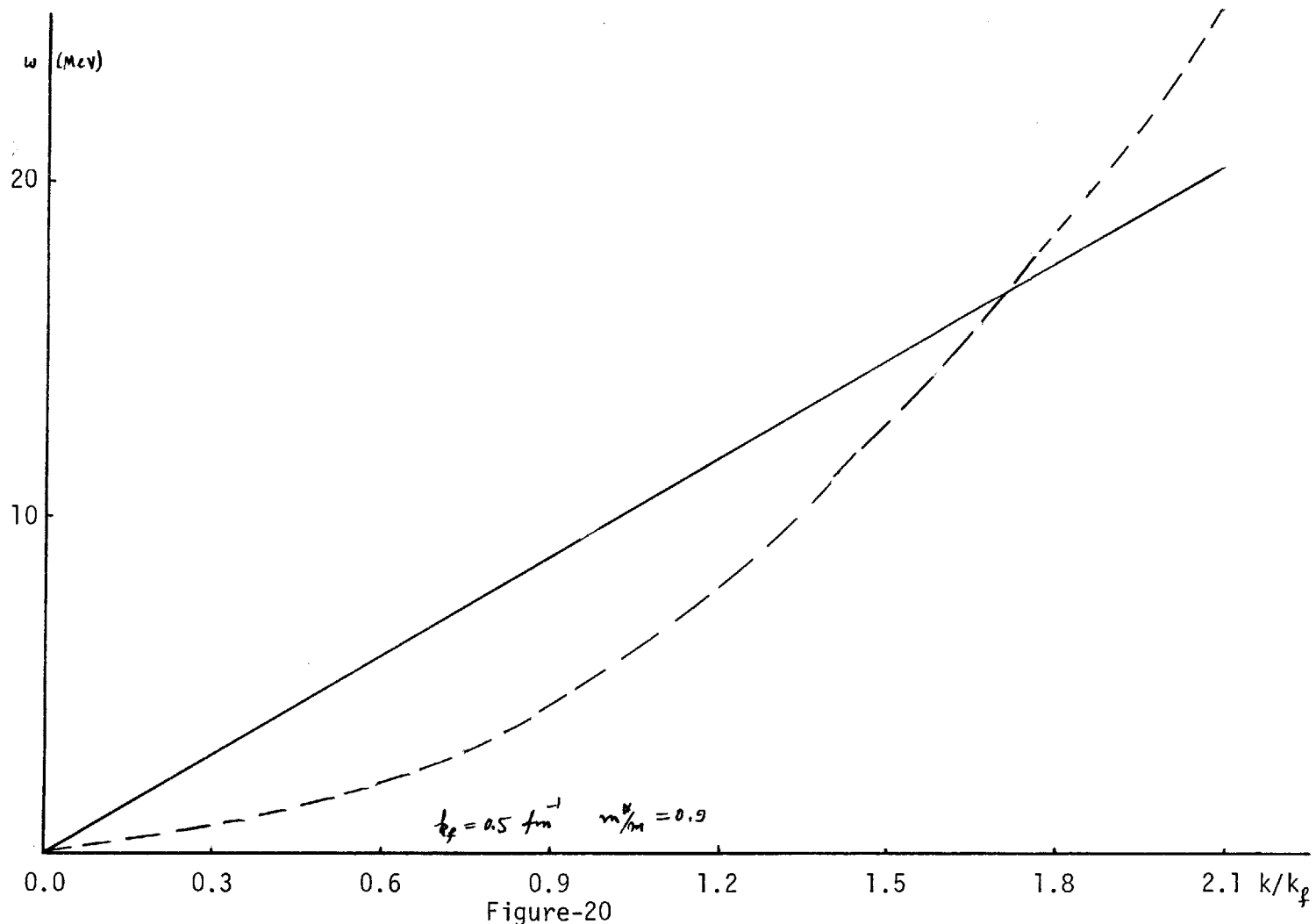
Figure-11

Figure-12

Figure-13

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Figure-14

Figure-15

Figure-16

Figure-17

Figure-18

W (MeV)

$m^*/m = 0.8$ $k/k_f = 0.15$

k_f fm^{-1}

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Figure-19

$k_f = 0.65 \, f^{-1}$ $k/k_f = 1.0$

Figure-20

ω (Mev)

$k_f = 0.5\ fm^{-1}$ $m^*/m = 0.9$