ELEMENTS OF EQUILIBRIUM METHODS

FOR SOCIAL ANALYSIS

by

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ABSTRACT

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The study is introduced by a first chapter.

The second chapter, "Formal Elements of Social Systems", first
gives set-theoretical definitions for abstract social systems and their
elements, establishing some terminology and notation as well. It is then
discussed how a social system is a generalization of an economy and how
the latter is, in turn, a generalization of a game.

Chapter 3, "Topological Foundations of Social Systems," develops some
continuity and convexity results for behavors in static and dynamic social
systems - all of which are defined in the previous chapter - after
presenting some mathematics which is of special use in social analysis.
This mathematics includes some facts concerning hyperspaces, a treatment
of semi-linear topological spaces and their fixed point properties as
investigated by Prakash and Sertel, and some further facts relating to
the continuity and convexity properties of objective functionals and
their associated infimum and supremum functionals, dealing with feasible
regions as points in suitable hyperspaces.

Chapter 4, "Evolution and Equilibrium in Social Systems", first
discusses some notions related to social equilibria, including Nash,
Pareto, and core points, and then demonstrates existence results for
social equilibrium for static and dynamic social systems. The contractual
set i.e., set of social equilibria, is proved to be non-vacuous for a type
of static social system and four types of dynamic social system. In the
static case, Fan's fixed point theorem is applied. In the four dynamic
types of social system, a more powerful theorem is needed, as a fixed
point is sought in a semi-linear space. A fixed point theorem of Prakash
and Sertel fits the specification and is applied. In all the cases where
the contractual set is shown to be non-empty, it is shown to be compact
as well. For certain social systems the contractual set is shown also

to be <u>convex</u>.

The fifth and final chapter discusses "<u>Extensions and Applications</u>"
of the framework and theory above.   The first extension indicated is that
of <u>probabilistic social systems</u>.   For these, a notion of a <u>behavior</u> as a
probability measure on a <u>sigma-field of actions</u> is offered, matters
pertaining to the <u>measurable numerical representability of preferences</u>
settled, and a notion of probabilistic social system formalized. Second, as an
application, a framework for the analysis of <u>power</u> is suggested, after a
certain <u>causal</u> relation of an event <u>inducing</u> another is formally introduced.
The resulting concept and <u>measure</u> of power is presented as a corrected
generalization and formalization of Dahl's concept and measure of power.
The importance of equilibrium methods for power analysis based on the above
is clarified.   Third, it is indicated how social systems may be viewed
as <u>evolutionary systems</u>, modifying the notion of <u>dynamical system</u>, so that
the <u>attraction</u> and <u>stability</u> of <u>contractual sets</u> and <u>cores</u> may be
investigated.   Finally, the large topic of the guidance of social systems
and <u>organizations</u> via <u>incentive schemes</u>, <u>information systems</u> and other
means is discussed as an area of application, suggesting also a number of
extensions which promise use in the area of <u>legislation</u> and the analysis
of <u>multi-level social systems</u>.

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TABLE OF CONTENTS

## 1. INTRODUCTION

This study is motivated by the belief that notions and methods of equilibrium can be developed and applied in social analysis. By social analysis is meant the positive and normative study of social systems and social phenomena.

The first thing to do was to clarify and reduce to a few the notions needed, assembling them in a consistent framework containing all that is needed and nothing that is not. The key notion here is that os a social system. For clarity and purity, it is defined set-theoretically. Next it is equipped with certain topological properties.

To do anything with the notion of equilibrium for social systems, it has to be demonstrated that such a thing exists, and that it exists for a wide enough class of social systems. This is done. Extensions and applications are discussed.

Since each chapter, except the present, begins with an introduction, the reader will be spared a lengthy introduction here.

## 2. <u>FORMAL ELEMENTS OF SOCIAL SYSTEMS</u>

The present chapter first gives definitions for social
systems and their elements, also introducing some basic terminology
and notation to be used in the sequel. The formal notion of a
'social system' as defined is then compared with the more familiar
'economy' and 'game', in both of which the existence and various
optimality and stability properties of equilibrium have been
investigated extensively and equilibrium methods of analysis
have long been used fruitfully. This comparison, from which
games emerge as restricted versions of economies and the latter
emerge as restricted versions of social systems, is intended to
provide perspective and context within which to evaluate the
framework and results presented in this study.

## 2.1 Preliminaries

This section is designed to introduce the basic notions and

some of the notation and terminology to be used in the sequel.

The central notion is that of a social system, defined set-

theoretically in 2.1.2 after some notation is established in 2.1.1.

Some terminology and further notation is established in 2.1.3

to refer to the elements of social systems and to some important

formal objects which are derived from these elements. The discussion

in 2.1.4 turns from matters of definition and denotation to the

intended connotation of the terminology introduced, so as to

provide some intuitive grounding for the reader's formal under-

standing. This is done by examining the typical manner in which

the elements of a social system operate. In this way it is

hoped also to communicate the motivation for the way in which

those elements are named. A motivation for the next definition,

2.1.5, distinguishing between static and dynamic social systems,

is also derived from that discussion. The consistency

of the definition of social systems is checked by 2.1.7, after

some requisite topological conventions and terminology are

established in 2.1.6. Finally, 2.1.8 adds a note to clarify

the important notion of 'incentive' in social systems.

2.1.1   Standing Notation:   The empty set will be denoted by $\phi$.

The set of real numbers will be denoted by $R$.

For any set $X$, $[X]$ will denote the set of non-empty subsets

of $X$. Whenever $[X]$ is endowed with a topology, that topology

will at least be as fine as the upper semi-finite (usf) topology.
(For these notions, see 3.1.2 or the classic work of Michael
(1951) concerned with topologies on spaces of subsets.)

The rather usual symbols 'Π' and 'π' will be used to
denote, respectively, products and projections. As regards
products, an important word of caution is due. Whenever the
product is the Cartesian product of mere sets, the product is to
be understood merely as such. Whenever it is a product of
topologized sets, the product is to be understood as equipped
with the product topology. Whenever it is the product of
sigma-fields, it is to be understood as the product sigma-field,
and similarly for measure spaces. When the index set is not
very crowded, the symbol '×' will also be used for products.
E.g., if $\{X_i | i \in I\}$ is a family of sets indexed by $i \in I$, and
if the index set $I = \{1,2\}$, then the product may be written
as $X_1 \times X_2$, rather than $\Pi_I\{X_i\}$ or $\Pi\{X_i | i \in I\}$ or $\underset{i \in I}{\Pi} X_i$, all
of which denote the same. As a special kind of product, $Y^X$
will denote the set of all functions $f:X \to Y$ mapping X into Y,
i.e. such that $f(x) \in Y (x \in X)$. (Of course, $Y^X$ is the same as
$\underset{X}{\Pi}\{Y_x | Y_x = Y\}$.) Finally, subscripts of $\pi$ will indicate the
range of the function, so that, e.g., $\pi_X$ denotes projection into X.

2.1.2    <u>Definition</u> :    A <u>social system</u> is an ordered seven-tuplet

$S = <W, U, H, G, I, T, A>$ , where

(2.1.2.1)

$$W = \{X_\alpha \neq \phi \,|\, \alpha \in A\} \neq \phi$$

is a non-empty family of non-empty sets $X_\alpha$,

from which we define $X = \Pi_A X_\alpha$, $X^\alpha = \Pi_{A\{x\}} X_\beta$:

(2.1.2.2)

$$U = \{u_\alpha : X \times R \to R \,|\, \alpha \in A\}$$

is an associated family of real-valued functions

$u_\alpha$ on $X \times R$; (see also 2.1.8);

(2.1.2.3)

$$H = \{h_\alpha : X^\alpha \to X^\alpha \,|\, \alpha \in A\}$$

is an associated family of transformations $h_\alpha$,

of $X^\alpha$;

(2.1.2.4)

$$G = \{g_\alpha : X \to R \,|\, \alpha \in A\}$$

is an associated family of real-valued functions

$g_\alpha$ on $X$:

(2.1.2.5)

$$I = \{i_\alpha : R^X \to R^X \,|\, \alpha \in A\}$$

is an associated family of function-valued

functions $i_\alpha$ assigning a real-valued function on

$X$ to each real-valued function on $X$:

(2.1.2.6)
$$T = \{t_\alpha : X \times \underset{A}{\Pi D}_\alpha \to D_\alpha \subset [X_\alpha] \mid D_\alpha \neq \phi, \alpha \in A\}$$

is an associated family of functions $t_\alpha$ assigning

a non-empty subset of $X_\alpha$ to each ordered pair

whose first element is a point $x \in X$ and second

element belongs to the product $\underset{A}{\Pi D}_\alpha$ of a certain

family $\{D_\alpha \mid \alpha \in A\}$ of non-empty collections

$D_\alpha \subset [X_\alpha]$;


(2.1.2.7)
$$A = \{\alpha : X^\alpha \times D_\alpha \to \underset{d_\alpha \in D_\alpha}{\cup} [d_\alpha] \mid \alpha \in A\}.$$
is a self-indexed family of mappings


(2.1.2.7')
$$\alpha(x^\alpha, d_\alpha) = \{x_\alpha \in d_\alpha \mid \ddot{w}_\alpha(x_\alpha, x^\alpha) \geq \underset{y_\alpha \in d_\alpha}{\mathrm{Sup}} \ \ddot{w}_\alpha(y_\alpha, x^\alpha)\},$$

$$(x^\alpha \in X^\alpha, \ d_\alpha \in D_\alpha),$$

where $\ddot{w}_\alpha$ is defined as in 2.1.3.10 below.


2.1.3   Standing Terminology and Notation:   Let S be as in 2.1.2.


(2.1.3.1)
$X_\alpha$ will be called the behavior space of $\alpha$,

and $x_\alpha$ will be called a behavior of $\alpha$

iff $x_\alpha \in X_\alpha$.

(2.1.3.2)    $X^\alpha$ will be called the $\underline{\alpha\text{-exclusive behavior}}$

$\underline{\text{space}}$ of S (or of A), and $x_\alpha$ will be called

an $\underline{\alpha\text{-exclusive behavior}}$ of S (or of A) iff

$x^\alpha \in X^\alpha$.


(2.1.3.3)    X will be called the $\underline{\text{(collective) behavior space}}$

of S (or of A), and x will be called a $\underline{\text{(collective)}}$

$\underline{\text{behavior}}$ of S (or of A) iff $x \in X$.


(2.1.3.4    $u_\alpha$ will be called the $\underline{\text{utility function}}$ of $\alpha$.

The set $R^A$ will sometimes be called the

$\underline{\text{distribution space}}$ of S (or of A), generic

elements $\Pi_A\{\rho_\alpha\} \in R^A$ being denoted by $\rho$, so

that $\pi_{R_\alpha}(\rho) = \rho_\alpha$ , with $R^A = \Pi_A\{R_\alpha | R_\alpha = R, \alpha \in A\}$.

In this case, a point $\rho \in R^A$ will be called a

$\underline{\text{distribution}}$ to A and $\rho_\alpha$ the $\underline{\text{share}}$ of $\alpha$ in $\rho$.

The function u: $X \times R^A \to R^A$, defined by $u(x,\rho)$

$= \Pi_A\{u_\alpha(x,\rho_\alpha)\}(x \in X, \rho \in R^A)$ will be called

the $\underline{\text{utility scheme}}$ of S (or of A).


(2.1.3.5)    $h_\alpha$ will be called the $\underline{\text{impression function}}$

of $\alpha$. The function h: $X \to \Pi_A X^\alpha$, defined by

$h(x) = \Pi_A\{h_\alpha(\pi_{X^\alpha}(x))\}$ $(x \in X)$ will be called

the $\underline{\text{impression scheme}}$ of S (or of A).

(2.1.3.6)    $g_\alpha$ will be called the <u>incentive function</u>

for $\alpha$. The function $g: X \rightarrow R^A$, defined by

$g(x) = \underset{A}{\Pi}\{g_\alpha(x)\}$ ($x \in X$) will be called the

<u>incentive scheme</u> for (or of) S ( or for A).

(2.1.3.7)    $i_\alpha$ will be called the <u>interpretation function</u>

of $\alpha$   The function $i: (R^X)^A \rightarrow (R^X)^A$, defined

by $i(g) = \underset{A}{\Pi}\{i_\alpha(g_\alpha)\}$ ($g \in (R^X)^A$), will be

called the <u>interpretation scheme</u> of S (or of A).

(2.1.3.8)    $t_\alpha$ will be called the <u>feasibility transformation</u>

for $\alpha$, $D_\alpha$ being called the <u>feasibility space of</u> $\alpha$

and a subset $d_\alpha$ being called a <u>feasibility</u>

for $\alpha$ iff $d_\alpha \in D_\alpha$. The function $t$, defined by

$t(x,d) = \underset{A}{\Pi}\{t_\alpha(x,d)\}$ ($x \in X$; $d = \underset{A}{\Pi}\{d_\alpha\}$, $d_\alpha \in D_\alpha$,

$\alpha \in A$), will be called the <u>collective feasibility</u>

<u>transformation</u> for (or of) S (or for A).

(2.1.3.9)    A will be called the <u>personnel</u> of S, each member

$\alpha \in A$ being called a <u>behavor</u> .

(2.1.3.10)   For various abbreviations, the following

alternate notations will be used:

$$i_\alpha(g_\alpha) = \ddot{g}_\alpha,$$

$$u_\alpha(x_\alpha, \; h_\alpha(y^\alpha), \; i_\alpha(g_\alpha) \; (x_\alpha, h_\alpha(x^\alpha)))$$

$$= \ddot{u}_\alpha(x_\alpha, h_\alpha(y^\alpha), \; \ddot{g}_\alpha(x_\alpha, h_\alpha(x^\alpha)))$$

$$= \ddot{v}_\alpha(x_\alpha, h_\alpha(x^\alpha))$$

$$= \ddot{w}_\alpha(x_\alpha, x^\alpha).$$

The derived function $\ddot{w}_\alpha$ will be called the effective utility function of $\alpha$.

2.1.4  <u>Discussion</u>:  Although the rigorous development of formal results may necessitate the use of uncommon terminology and symbolism, it is difficult to overemphasize the usefulness of being able to express the underlying postulates and emerging results of a theory in intuitively pleasing fashion, reasonably within common language.  The difficulties in achieving this sort of a restriction to the simple and plain, of course, are the food on which technical jargon and alienated scholarism thrive.

Realizing that there is no scarcity of jargon, especially in the class of disciplines concerned with social phenomena, the aim in term-coining here cannot be to expand the present glossary, as it would be foolish to wish to irrigate the sea. What the aim is can be expressed in two components: first, to convey the meaning of the formal framework and theory in

reasonably common and unartificial terminology, so that it is easily understood, at least in its broad outline; and, second to offer in this terminology some precision and conciseness for what are the essential elements of the framework and theory, thus hoping to bring attention to what is a small class of important elements, while giving that attention a clear focus by eliminating ambiguity and vagueness via the formalism of definition.

Now that the basic elements of a social system have been defined and a long list of terminology and notation has been introduced, some elucidation might be gained by turning to the connotation of the terms above. What the formalism of a social system S in 2.1.2-3 roughly amounts to can be expressed in plain language by describing the personnel A and the typical feasibility transformation $t_\alpha$.

The personnel A can be understood in terms of its typical member $\alpha$. The typical behavor $\alpha$ has its individual "tastes", which are in the form of a (complete) preference ordering represented (in order-preserving fashion) by the utility function $u_\alpha$. In general, the utility achieved by $\alpha$ depends on both its own behavior $x_\alpha$ and the ($\alpha$-exclusive) behavior $x^\alpha$ of all others in A. Furthermore, it may depend on a real number $\rho_\alpha \varepsilon$ R, where $\rho_\alpha$ is to be understood as either income or wealth, or status, prestige, power or any composite of things such as these which can be expressed suitably in real numbers. The

operation of $\alpha$ consists of choosing a behavior, i.e., behaving, so as to optimize subject to the constraint of its feasibility and subject to its "perception" of the circumstances. This "perception" is expressed summarily by the impression function $h_\alpha$ and the interpretation function $i_\alpha$. "Observing" a collective behavior $x \in X$, it is assumed that $\alpha$ "sees" the component $\pi_{X_\alpha}(x) = x_\alpha$ of $x$ pertaining to itself as is. Although the component pertaining to the rest of $A$ is $\pi_{X_\alpha}(x) = x^\alpha$, however, $\alpha$'s impression $h_\alpha(x^\alpha)$ may very well be different from $x^\alpha$. Just as the equality $h_\alpha(x^\alpha) = x^\alpha$ need not hold, neither need the equality $i_\alpha(g_\alpha) = g_\alpha$ be satisfied. Thus $\alpha$ may "interpret" the incentive function $g_\alpha$ to be some different incentive function $\ddot{g}_\alpha \neq g_\alpha$. Now $x^\alpha$, $u_\alpha$, $h_\alpha$, $g_\alpha$, and $i_\alpha$ all influence the choice $x_\alpha$ made by $\alpha$. Given an $\alpha$-exclusive behavior $x^\alpha$, $\alpha$ forms an impression $h_\alpha(x^\alpha) = y^\alpha$. Having interpreted $g_\alpha$ as $i_\alpha(g_\alpha) = \ddot{g}_\alpha$, the real number $r_\alpha = g_\alpha(x_\alpha, y^\alpha)$ is understood to be forthcoming as a function of the choice $x_\alpha$. Thus, $u_\alpha(x_\alpha, y^\alpha, r_\alpha)$ depends on this choice, both directly and through $r_\alpha$. This partly defines the optimization problem for which $\alpha$ is to compute a solution.

The problem to be solved by $\alpha$ is, in general, one of <u>constrained</u> optimization. That is to say, apart from the fact that $y^\alpha$ is now fixed, already imposing restraints on the values that can be taken by $u_\alpha$ in the present computations according

to that $y^{\alpha}$, the choice of behavior $x_{\alpha}$ is constrained to be within a certain set $d_{\alpha} \subset X_{\alpha}$. This set, called a feasibility, is determined by the feasibility transformation $t_{\alpha}$ . The operation of $\alpha$ is completely described by saying that it identifies the set $\alpha(x^{\alpha}, d_{\alpha})$ of behaviors $x_{\alpha} \varepsilon d_{\alpha}$ which maximize $u_{\alpha}(\cdot, y^{\alpha}, g_{\alpha}(\cdot, y^{\alpha}))$ on $d_{\alpha}$. (If S is well-defined, then $\alpha(x^{\alpha}, d_{\alpha}) \neq \phi$.) Exactly one of these "best" behaviors $x_{\alpha} \varepsilon \alpha(x^{\alpha}, d_{\alpha})$ is chosen, it being immaterial to $\alpha$ – and to us – which particular one it is.

The fashion in which the feasibility transformations operate can be seen by assuming that a feasibility $d_{\alpha}$ is given for each $\alpha \varepsilon A$ and that each $\alpha$ chooses a behavior $x_{\alpha} \varepsilon d_{\alpha}$ in the manner already described. The collective behavior $x = \Pi_{A}\{x_{\alpha}\}$ arising in this way will, in general, now alter each feasibility in the fashion described by $t_{\alpha}$ . Specifically, each $d_{\alpha}$ is now transformed into $t_{\alpha}(x, d)$, where $d = \Pi_{A}\{d_{\beta} | \beta \varepsilon A\} \varepsilon \Pi_{A} D_{\beta}$ represents the family of feasibilities, including that $(d_{\alpha})$ of $\alpha$, which , as constraints, had governed the choice of x. In general, the equality $t_{\alpha}(x,d) = d_{\alpha}$ does not hold, $t_{\alpha}$ yielding certain behaviors $x_{\alpha} \varepsilon d_{\alpha}$ no longer feasible for $\alpha$, while bringing some behaviors $z_{\alpha} \notin d_{\alpha}$ into the new feasibility as elements $z_{\alpha} \varepsilon t_{\alpha}(x,d)$.

The operation of all of the feasibility transformations is summarized in that of the collective feasibility transformation t. Given a collective feasibility $\Pi_{A}\{d_{\alpha} \varepsilon D_{\alpha} | \alpha \varepsilon A\} = d \varepsilon D = \Pi_{A} D_{\alpha}$

which consists of collective behaviors $z \epsilon X$ such that $\pi_{X_\alpha}(z) \epsilon d_\alpha$

$(\alpha \epsilon A)$, and given a collective behavior x such that

$\pi_{X_\alpha}(x) \epsilon d_\alpha (\alpha \epsilon A)$, $t(x,d) = \prod_A \{t_\alpha(x,d) | \alpha \epsilon A\}$.

Thus, to summarize in natural language, a social system

is a collection of behavors, each seeking its self-interest

subject to an incentive function and guided by its individual

preferences, by its interpretation of its incentive function

and by its impression of the others' behavior, and subject

also to a feasibility - which feasibility, in turn is

influenced by a history of past (collective) feasibilities and

(collective) behaviors chosen within these past feasibilities.

The remarks so far in explaining the operation of

feasibility transformations should yield the motivation for the

following definition, as well as the definition itself, rather

clear.


2.1.5  Definition:  Let S be a social system and t the collective

feasibility transformation of S. S will be called (a) static

(social system) iff t is a constant map, i.e., $t(x,d) = d$ for all

$x \epsilon X$ and $d \epsilon D = \prod_A D_\alpha$ .  S will be called (a) dynamic (social

system) iff S is not static.

It is important to know that 2.1.2 is not a self-contradiction,

so that there exists an ordered seven-tuplet S satisfying the

definition of social system.  Although that may be obvious, it

is also important to know a reasonably unrestrictive sufficient

condition for S to exist. Such a condition will be given immediatly the following conventions are agreed upon.

2.1.6    Standing Topological Conventions and Terminology: Whenever R is considered as a topological space, it will be assumed to have the order topology of the natural order of real numbers. (Recall that this is the same as the Euclidean topology for R.)

Following Bourbaki [1966], a topological space will be called quasi-compact iff the Borel-Lebesque condition is satisfied, i.e., every open cover has a finite subcover. A topological space will be compact if it is quasi-compact and Hausdorff.

A real-valued function u: $X \to R$ on a topological space X will be called upper semi-continuous (usc) iff $u^{-1}(\{r \in R | r \geq b\})$ is closed for all $b \in R$; it will be called lower semi-continuous (lsc) iff $u^{-1}(\{r \in R | r \leq b\})$ is closed for all $b \in R$.

A point-to-set mapping $F: X \to Y$ of a topological space X into a topological space Y will be called upper semi-continuous (usc) iff $F: X \to [Y]$ is continuous with the upper semi-finite topology on [Y] (see 3.1.2 or [Michael, 1951]). Thus, F is usc iff for each $x \in X$, and for each neighborhood (nbd) V of $F(x)$, there exists a nbd U of x such that $F(U) \subset V$.

2.1.7   Proposition (Existence of S): With reference to 2.1.2-3,

each $\alpha \ \epsilon$ A (hence, A; and hence, S itself) is well-defined (i.e.,

S exists as a social system), if for each $\alpha \ \epsilon$ A, $d_\alpha \ \epsilon \ D_\alpha$ and

$y^\alpha \ \epsilon \ h_\alpha(X^\alpha)$, $d_\alpha$ is quasi-compact and $\ddot{v}_\alpha$ is upper semi-continuous

on $\{y^\alpha\}x \ d_\alpha$.

Proof: Assume that the hypothesis is satisfied. Clearly, all

that needs to be shown is that $\alpha(x^\alpha, d_\alpha) \neq \phi$ ($x^\alpha \ \epsilon \ X^\alpha$, $d_\alpha \ \epsilon \ D_\alpha$,

$\alpha \ \epsilon$ A). Denote $y^\alpha = h_\alpha(x^\alpha)$. Since $d_\alpha$ is quasi-compact, so is

$\{y^\alpha\}x \ d_\alpha$. By upper semi-continuity of $\ddot{v}_\alpha$ on $\{y^\alpha\}x \ d_\alpha$, $\ddot{v}_\alpha$ attains

a supremum on $\{y^\alpha\}x \ d_\alpha$. Hence, $\ddot{w}_\alpha$ attains a supremum on $\{x^\alpha\}x \ d_\alpha$.

Thus, $\alpha(x^\alpha, d_\alpha) \neq \phi$, as to be shown.

#

2.1.8   Note (Preference and Incentives): As remarked in 2.1.4,

the utility functions $u_\alpha$ are meant to be order-preserving

representations of complete (preference) orders of the

behavors $\alpha$ on X x R. Meanwhile, the functions $g_\alpha$ have been

called "incentive" functions to the effect that the real

numbers r $\epsilon$ R serve to order the incentives, indicated as the

values taken by any $g_\alpha$. To justify this usage of "incentive"

it is assumed from here on that, for any x $\epsilon$ X and any $\alpha \ \epsilon$ A,

if r and s are real numbers such that r $\leq$ s, then $u_\alpha(x,r) \leq u_\alpha(x,s)$.

This is to say that, ceteris paribus, a behavor does not prefer

less of the real-valued variable (incentive) to more, without

implying that more is actually preferred to less.

## 2.2    Games, Economies and Social Systems

Having introduced the formal notion of a social system, it is
appropriate now to compare this with the more familiar notions
of an economy and a game. This will give a perspective within which
the place of the present study might better be judged.

The notion of an economy which will be used in this comparison
is that of Arrow and Debreu (1954), although it will be presented
within the present terminology and notation. This is proper
enough procedure, for it willturn out that an economy - or
an "abstract economy" as Arrow and Debreu called it - is a special
case of a social system and that a game is a special case yet
of an economy. All this will be very clear as soon as the
defintions are given.


2.2.1    Definition:    An economy is an ordered quadruplet

$$S = <W, \ddot{U}, \hat{T}, A>, \text{ where}$$


(2.2.1.1)        $W$ is as in 2.1.2.1; from which $X$ and $X^{\alpha}$ are

defined as there;

(2.2.1.2)        $\ddot{U} = \{\ddot{u}_{\alpha} : X \to R \mid \alpha \ \varepsilon \ A\}$

is an associated family of real-valued functions

$\ddot{u}_{\alpha}$ defined on $X$;

(2.2.1.3)        $\hat{T} = \{\hat{t}_{\alpha} : X^{\alpha} \to [X_{\alpha}] \mid \alpha \ \varepsilon \ A\}$

is an associated family of mappings $\hat{t}_\alpha$ assigning

a non-empty subset $\hat{t}_\alpha(x^\alpha)$ of $X_\alpha$ to each $x^\alpha \in X^\alpha$:

(2.2.1.4)    $A = \{\alpha : X^\alpha \times \hat{t}_\alpha(x^\alpha) \to [X_\alpha] \,|\, \alpha \in A\}$

is a self indexed family of mappings defined by

$$\alpha(x^\alpha, d_\alpha) = \{x_\alpha \in d_\alpha \,|\, \ddot{u}_\alpha(x_\alpha, x^\alpha) \geq \underset{y_\alpha \in d_\alpha}{\mathrm{Sup}}\ \ddot{u}_\alpha(v_\alpha, x^\alpha)\},$$

$$(x^\alpha \in X^\alpha, d_\alpha \in \hat{t}_\alpha(x^\alpha)).$$

2.2.2    <u>Definition</u>:  A <u>game</u> is an economy in which $\hat{t}_\alpha(x^\alpha) = X_\alpha$ for

all $x^\alpha \in X^\alpha$ and $\alpha \in A$.

2.2.3    <u>Note</u>:  The actual definitions of Arrow and Debreu (1954) from

which 2.2.1-2 is generalized actually has A as a finite set,

but given more recent developments (Aumann, 1964) in which

a continuum of traders (players) is considered, it is unreasonable

to stick to such a restriction - a restriction which is

unnecessary in the first place, except possibly to yield

economics understandable with the tools of Euclidean space.

A simple comparison of 2.2.1-2 with 2.1.2 yields that, indeed, an

economy is a special sort of social system and that a game is a

special case of an economy.  This notion of a game may or may not

be appropriate.  It does have the authorization of Arrow and Debreu,

however, and that should carry some weight. In any case the above

defintions 2.2.1-2 will not be used to derive any results of this

study, but are recorded merely for the comparison they allow.

Coming to that comparison, it is to be noted firstly that the

elements H, G and I of a social system have been suppressed in

2.2.1. Hence, the functions $\ddot{u}_\alpha$, as is intended to be suggested by

using the notation of umlaut ('¨'), are analogous to the effective

utility functions $\ddot{w}_\alpha$ of 2.1.3.10, but not necessarily such, since

they are not explicitly derived from functions $u_\alpha$, $h_\alpha$, $g_\alpha$ and $i_\alpha$ as are

the functions $\ddot{w}_\alpha$ . Finally, the mappings $\hat{t}_\alpha$ of 2.2.1.3 are restricted

versions of the feasibility transformations $t_\alpha$ of 2.1.2.6 and 2.1.3.8.

The $\hat{t}_\alpha$ 's depend only on $\alpha$-exclusive behaviors $x^\alpha$. The $t_\alpha$ 's

are allowed to depend, in addition, on the behavior $x_\alpha$ of $\alpha$ as well

as the collective feasibility d in $D = \Pi D_\alpha$.
$$A$$

All this being so, the restrictions of 2.2.1 may be viewed in

different lights. Accordingly, one view might be that the

economist is not interested in the details of the full-blown social

system and it is a useful simplification to suppress the perceptive-

cognitive and information-systemic elements for which H and I stand and

that the incentive scheme – which is associated with G – does not

matter. Before turning to the simplification of the feasibility

transformations $t_\alpha$ to the form $\hat{t}_\alpha$ , it is worth challenging the above

view. For to say that economists are not concerned with incentives

wouldbe to say that they are not concerned with prices – wages included –

or with taxes and subsidies – incentives for investment, etc.,
included. And to say that they are not interested in the effects
of imperfections in information or in its processing by the user,
e.g., in the "marketplace", would imply statements to the effects,
for instance, that a devaluation can be announced a week earlier
than it is consummated or that advertising has yet to be invented. It
is very difficult, therefore, to defend that ignorning the elements
H, G and I is a useful simplification or idealization in the genre
of the "ideal gas" or the "billiard ball" model of gasses.

The simplification of the feasibility transformations $t_\alpha$ to the
form $\hat{t}_\alpha$ is also a difficult one to defend. For one thing, what is
feasible for an economic agent obviously depends on the behavior
of that agent itself. It would require some rather strong metaphysics
otherwise to explain why people or firms or governments choose to save
and invest if tomorrow's vacations, factories and parks did not
depend on whether one saved a penny or built a factory or upkept a
park today. Secondly, the factories one has tomorrow depends on
what factories one has today. For instance, one allows the textile
industry to slowly depreciate its equipment and invests in the
electronics industry. The production possibilities set of tomorrow
depends on the present one and on the point now chosen in it. That
is to say, the feasibility for the agent $\alpha$ depends on $x_\alpha$ and $d_\alpha$.

To include certain external effects, it is here permitted to depend on the feasibilities $d_\beta$ of the other agents $\beta \in A$ as well. (To give a possibly odd example for where this may become relevant, it might be considered that the precedence set by <u>allowing</u> one boy to be a conscientious objector to war, <u>whether or not</u> the boy uses this priviledge, will probably make it easier for the next boy to gain this choice).

The result of the above discussion seems to be not really that an economy <u>is</u> a special case of a social system, but that this would be so if one went by the definitions which were compared. But the result is also that this would be a very artificial exercise of classification and that a social system, as defined by 2.1.2 is really something dear to the interests of economics. In its formal specification, nevertheless, it is more general than the economy for which Arrow and Debreu proved the existence of an equilibrium. It will be found then that the equilibria proved in 4.3. to exist in the case of dynamic social systems generalize the result of Debreu (1952) obtained for a certain special class of social systems. The mentioned work of Debreu is actually the main mathematical pillar on which the outstanding Arrow and Debreu study is based, the social systems treated in it being correspondingly specialized. (Typically, both works deal with a finite personnel and with behaviors in Euclidean space, and these constitute a further restriction on their results. Although a preference to work in such spaces is often thought

to be"realistic" (perhaps because it is "less abstract"), as far

as realism is concerned, any result which is true with weaker

assumptions is at least as realistic as the same result with stronger

assumptions. Furthermore, it is not possible to represent, for

example, an infinite-horizon plan naturally as a point in finite

dimensional space, its natural habitat being infinite dimensional.

Hence, for this and many other reasons, neither is it the case that

all economics can be reasoned in Eucliden space.)

The most summary comparison of a social system with a game,

to end this section, would be that the latter is static (see 2.1.5)

Thus, the existence result of 4.2 can be regarded as a generalization

of Nash's [1950, 1951] result for (again a restricted variety of)

finite-personnel games to the case of a certain class (type 0) of

static social systems.

### 3. TOPOLOGICAL FOUNDATIONS OF SOCIAL SYSTEMS

This chapter first presents some mathematics, mostly topology, which is especially useful in the analysis of social systems. In one way or another, all of this material is actually used in the present study, but much more can be expected from its use than would fit within the constraints of this investigation. Furthermore, most of the material is either new to the field of social analysis or plain new. As best as an amateur historian can do, the origins and intellectual history of the material are indicated.

Next, some fundamental topological properties of behavors – and, thus, indirectly of social systems – are demonstrated as deriving from various properties, if they pertain, of elements such as utility, impression, interpretation and incentive functions and behavior spaces. These are demonstrated first for static and then for dynamic social systems. They are used in the corresponding theories of existence for social equilibrium, presented in the next chapter.

## 3.1   Topological Preliminaries

This section collects some topological facts crucial to the later sections and chapters, so that they may be used freely without explicit reference once they are recorded. Most of these facts relate to hyperspaces and to real semi-linear topological spaces (rst spaces).

The idea of hyperspace, i.e., a topological space whose points are subsets of a topological space, dates at least as far back as the metric defined by Hausdorff [1937] on the set of non-empty closed subsets of a bounded metric space X. (see also [Kelley, 1942] for a study of the Hausdorff metric hyperspace when X is compact.)  Meanwhile, Vietoris [1923] defined the finite topology (see 3.1.2,10) for the set of non-empty closed subsets of an arbitrary topological space X. The standard reference adopted here, however, is the complete and unifying study of Michael [1951].

The importance of hyperspaces for optimization, economic theory and social analysis in general derives from at least two considerations. One of these in turn derives from the importance of point-to-set mappings in these fields. For a point-to-set mapping, such as an optimizing algorithm, a consumer choosing bundles of goods or a behavor choosing a set of behaviors, can be looked upon as a point-to-point mapping on the same domain

of definition into a suitable hyperset (set of subsets) of the range. If

the domain and range of the point-to-set mappings are topological spaces,

then matters relating to the (upper or lower semi-) continuity of this

mapping are often simplified by appropriate choice of a topology for the

hyperset serving as range for the associated point-to-point mapping.  This

is a $_{\text{primary}}$  use made of hyperspaces in this study, as 3.1.4.4-5 and the

application of these in 3.3.1, and thus in each of the results of 4.3,

constitute such a use.

These mentioned applications in the present study also illustrate

the second general use of hyperspaces for the mentioned fields of

inquiry.  The consideration here  is that "feasible regions" can be

regarded as points in a hyperspace, so that changes in these can be

analyzed by use of point-to-point mappings (and even, as in 4.3.3.3-5,

by use of point-to-set mappings) into that hyperspace.  The power of such

methods will probably be felt less in optimization constrained to

feasible regions in Euclidean space (especially when the constraints

are finitely parametrized, as in the case of linear constraints of budget,

etc.), but in dynamic optimization where decisions taken are allowed to alter

the very feasible regions within which they are taken - as in the case of

(dis-)investment - and especially when the feasible regions lie in some

abstract space, such as a function space, and the constraints are

not finitely parametrizable, these methods may be expected to bear fruit.

Even a restriction of the results in 4.3 to the case of a singleton

personnel might testify to the validity of such an expectation.

Semi-linear topological spaces, and, more generally, semi-linear spaces were first investigated, to the knowledge of this author, by Prakash and Sertel [1970, a,b]. As a generalization of linear (or vector) spaces, semi-linear (or semi-vector) spaces have an algebra which is satisfied, notably for present purposes, by the set of all non-empty subsets of a vector space. In the case where L is a linear topological space, the set of non-empty quasi-compact subsets of L form a semi-linear space. Among semi-linear topological spaces, those which are used in this study are the ones formed by the Hausdorff metric space of non-empty compact and convex subsets of a normed real linear topological space. Thus, the usual feasible regions in usual constrained optimization are typical points of such a (rst) space.

3.1.1   Standing Notation:  For any topological space X, C(X) will denote the set of all non-empty closed subsets of X, k(X) will denote the set of all non-empty quasi-compact subsets of X, and K(X) will denote the set of all non-empty compact subsets of X. If X is a convex set, Q(X) will denote the set of all non-empty convex subsets of X. Furthermore, CQ(X) = C(X) $\cap$ Q(X), kQ(X) = k(X) $\cap$ Q(X), etc., will also be used.

If f: X $\to$ Y is a mapping, then $\Gamma(f) = \{(x,y) \mid x \in X, y \in f(x)\} \subset X \times Y$ will be used as standard notation for the graph of f.

3.1.2 <u>Hyperspaces</u>:    This section follows [Michael, 1951], extracting

the bare minimum of information needed for the subsequent

development.


3.1.2.1    <u>Notation</u>:    Let $U \subset X$.   Then denote


$$<U>^{+} = \{Y\epsilon[X] \mid Y \subset U\},$$


$$<U>^{-} = \{Y\epsilon[X] \mid Y \cap U \neq \phi\}.$$


Let $\{U_i \mid i \ \epsilon \ I\}$ be a collection of subsets $U_i \subset X$.   Then

denote


$$<U_i \mid i \ \epsilon \ I> = \{Y\epsilon[X] \mid Y \subset \bigcup_I U_i, \ Y \cap U_i \neq \phi \text{ for all } i \ \epsilon \ I\}.$$


If $I$ above is finite, so that $\{U_i \mid i \ \epsilon \ I\} = \{U_1, \ldots, U_n\}$,

then also denote $<U_i \mid i \ \epsilon \ I>$ by $<U_1, \ldots, U_n>$.


3.1.2.2    <u>Definition</u>:    Let $X$ be a topological space with topology $\tau$.

The <u>upper semi-finite (usf) topology</u> on $[X]$ is the topology

generated by $\{<U>^{+} \mid U \ \epsilon \ \tau\}$ as a basis.   The <u>lower semi-finite</u>

<u>(lsf) topology</u> on $[X]$ is the topology generated by

$\{<U>^{-} \mid U \ \epsilon\tau\}$ as a sub-basis.   The <u>finite (f) topology</u> on

$[X]$ is the topology generated by $\{<U_1, \ldots, U_n> \mid \{U_1, \ldots, U_n\} \subset \tau\}$.

3.1.2.3   Remark:  Using 'c' as a superscript to denote complements,

the following equations follow easily from the definitions:

$$(i) \quad <U>^+ = <U> = [U] = (<U^c>^-)^c,$$

$$(ii) \quad <U>^- = <U^c>^c,$$

$$(iii) \quad <U>^c = <X, U^c>$$

The following spells out, for the benefit of the reader, a proof for

a proposition observed by Michael.

3.1.2.4   Proposition:  Let X be a topological space.  The finite

topology on [X] is the coarsest topology, in the lattice of

all topologies on [X], which is finer than both the usf and

the lsf topology on [X].  [Michael, 1951, p. 179].

Proof:   It suffices to show that the usf and the lsf topology

on [X] are contained in any topology containing the finite

topology on [X].  From the first equation of 3.1.2.3 it

follows that the usf topology is so contained.  Let $U \subset X$

be open.  Then, using the last two equations of 3.1.2.3,

$<U>^- = <U^c>^c = <X,U>$ , proving that the lsf topology is

also so contained.

#

The purpose of the following is, again, elucidation.

3.1.2.5  Proposition:  Let X be a topological space, and denote the
usf, lsf and f topologies, on [X], by $\tau^+, \tau^-, \tau^{+-}$, respectively.
Then

(i) $\tau^+$ is the coarsest topology on [X] for which (a)
$<U>^+$ is open if U is open in X and (b) $<M>^-$ is closed
if M is closed in X;

(ii) $\tau^-$ is the coarsest topology on [X] for which
(a) $<U>^-$ is open if U is open in X and (b) $<M>^+$ is closed
if M is closed in X;

(iii) $\tau^{+-}$ is the coarsest topology on [X] for which
(a)  $<U>$ is open if U is open in X and (b) $<M>$ is closed
if M is closed in X.

Proof:  It follows directly from the definitions that $\tau^+$ and
$\tau^-$ are the coarsest topologies in [X] satisfying parts (a)
of (i) and (ii), respectively.  In the following let $U \subset X$
be open, and w.l.g., let $M = U^c$.  It suffices to show that parts
(b) of (i) and (ii) hold for $\tau^+$ and $\tau^-$, respectively, for then
(iii) will follow by 3.1.2.4.  To see that (i) (b) holds for
$\tau^+$, just note that

$$<M>^- = (<U>^+)^c,$$

which is closed since $<U>^+$ is open. To see that (ii) (b)

holds for $\tau^-$, note, similarly, that $< M>^+ = (<U>^-)^c$ is closed.

#

The rest of this section following the present paragraph,
with the possible exception of 3.1.2.9 is merely paraphrased
from [Michael, 1951]. The first definition gives a useful
equivalent rewording of the usual definition of uppar and
lower semi-continuity for multi-valued binary relations
(point-to-set mappings). The remainder will be useful after
the next section introduces a special rst space which is a
hyperspace.

3.1.2.6 **Definition:** If X and Y are topological spaces, a mapping
F: X →[Y] is called **upper (lower) semi-continuous (u(l)sc)**
iff F is continuous with the u(l)sf topology on [Y].
[Michael, 1951; p. 179].

The last definition can be reworded also as follows.

3.1.2.7 **Proposition:** If X and Y are topological spaces, then a
function F: X → [Y] is u(l) sc iff $\{x \in X | F(x) \cap A \neq \phi\}$ is
closed (open) whenever A is closed (open) in Y. [Michael,
1951; Thm. 9.1]

3.1.2.8   <u>Definition</u>:  Define the ("layout...) <u>map (...of X")</u>

$L:[[X]] \to [X]$ by

$$L(\{Z_i \in [X] \mid i \in I\ ) = \bigcup_I Z_i.$$

[Michael, 1951; Def. 5.5.1]

3.1.2.9   <u>Proposition</u>:  If X is a topological space, then the layout

map of X is (i) usc, (ii) lsc and (iii) continuous accordingly

as [X] and [[X]] carry the (i) usf, (ii) lsf, and (iii) finite

topology (Cf. [Michael 1951 Thm. 5.7.2]).

<u>Proof</u>:  Denote generic elements of [[X]] by E, and defin·

$j \in J_{T}$ iff $Y_j \in$ E, so as to be able to write $E = \{Y_j \mid j \in J_E\}$.

The proofs of (i) (ii), (iii) are entirely set-theoretic.

<u>ad (i)</u>:  Let $V = <U>^+$ be a basic open nbd of $Y = L(E) \in [X]$.

It suffices to show that $L^{-1}(V) = <V>^+$:

$$E \in L^{-1} (V) \text{ iff } \bigcup_{J_E} Y_j = Y \in V$$

$$\text{iff } Y \subset U$$

$$\text{iff} \quad (j \in J_E \text{ only if } Y_j \subset U)$$

$$\text{iff} \quad (j \in J_E \text{ only if } Y_j \in V)$$

$$\text{iff} \quad E \subset V$$

$$\text{iff} \quad E \in \langle V \rangle^+ \,.$$

ad (ii): Let $V = \langle U \rangle^-$ be a basic open nbd of $Y = L(E) \in [X]$. It suffices to claim that $L^{-1}(V) = \langle V \rangle^-$, and to show it as follows:

$$E \in L^{-1}(V) \quad \text{iff} \quad Y \in V$$

$$\text{iff} \quad Y \cap U \neq \phi$$

$$\text{iff} \quad (\exists j_* \in J_E \text{ such that } Y_{j_*} \cap U \neq \phi)$$

$$\text{iff} \quad (\exists j_* \in J_E \text{ such that } Y_{j_*} \in V)$$

$$\text{iff} \quad E \cap V \neq \phi$$

$$\text{iff} \quad E \in \langle V \rangle^- \,.$$

ad (iii): Let $V = \langle U_1, \ldots, U_n \rangle$ be a basic open nbd of $Y = L(E) \in [X]$ and denote $N = \{1, \ldots, n\}$, $U = \underset{N}{U} U_i$ and $W = \langle \langle U \rangle^+ \rangle^+ \cap (\underset{N}{\cap} \langle \langle U_i \rangle^- \rangle^-)$. It suffices to show that $L^{-1}(V) = W$:

$$E \in L^{-1}(V) \quad \text{iff} \quad Y \in V$$

iff $Y \subset U$ and $Y \cap U_i \neq \phi$ for all $i \in N$

iff ($j \in J_E$ and $i \in N$ only if $Y_j \subset U$ and $Y \cap U_i \neq \phi$)

iff ($j \in J_E$ and $i \in N$ only if $Y_j \in <U>^+$ and $Y \in <U_i>^-$)

iff $E \in <<U>^+>^+$ and $E \in \bigcap_N <<U_i>^->^-$

iff $E \in W$.


3.1.2.10    <u>Notation-Definition-Remark-Proposition</u>:  Replace [X] in

3.1.2.1-9 by C(X) and modify 3.1.2.3 to state that $<U> = C(U)$

if U is closed in X.


3.1.2.11    <u>Proposition</u>:

1.  If X is a regular space and $E \in k(C(X))$ with the usf

topology on C(X), then $L(E) \in C(X)$;

2.  If X is a topological space and $E \in k(k(X))$ with the

usf topology on k(X), then $L(E) \in k(X)$.

[Michael, 1951; Thm. 2.5.1-2, Thm. 9.5].


3.1.2.12   <u>Proposition</u>:  Let X be a topological space, and let C(X)

be equipped with the finite topology.  Then X is quasi-compact,

locally quasi-compact, separable, compact iff C(X) has the

same property [Michael, 1951; Thm. 4.2., Thm. 4.4.1, Thm. 4.5.1,

Thm. 4.9.6].

3.1.2.13  Proposition:  Let X be a metric space.  Then the finite
topology on C(X) agrees with the Hausdorff metric topology
on C(X) iff X is (quasi-)compact. [Michael, 1951, Thm. 3.3,
Prop. 3.5].

3.1.3  Rst Spaces: Rather than lengthen the present chapter by para-
phrasing or reproducing, the original work by Prakash and
Sertel on semi-vector spaces, semi-linear topological spaces,
rst spaces and their fixed point properties is appended to this
study.

3.1.4  Special Facts Basic to Optimization:  This section collects some
facts relating to the continuity and convexity matters pertaining
to functionals typically playing the role of objective functional
in optimization problems.  All of these facts are well-known in
certain restricted instances, as when the functional is defined on
a subspace of Euclidean space.  The novelty in the more general
facts presented here derives from the treatment of the usual
feasible regions as points in suitable spaces.  The various
continuity and convexity properties of the objective functional
are related through this treatment to corresponding properties of
the optimal value attained on a feasible region, depending on the
abstract feasible region as a variable .  Thus, e.g., the
supremum attained by an objective functional on, say, a compact
feasible region is seen to share much of the continuity and

convexity properties of the objective functional itself, although

the supremum depends on the feasible region while the objective

functional depends on a generic element of the space in which

such feasible regions lie as sets. Such facts are essential

when the feasible regions become endogenous variables of the model,

as in dynamic optimization or dynamic social systems.

First considered are matters of continuity. The first lemma,

3.1.4.1, plays a key role here. While 3.1.4.1-3 are concerned with

the objective functional, the two simple propositions 3.1.4.4-5

are related in an obvious way to the feasibility. Then

considered are the convexity properties, relating, again, to the

objective functional.

The main results are all concerned with how continuity,

convexity  - and various weaker versions of these properties -

for the objective functional relate to corresponding properties of

the associated "supremum" or "infimum functional". The results

are presented in "disaggregated" form, that is, continuity questions

are split into questions of upper and lower semi-continuity,

and convexity or concavity matters are formulated in terms of

strict and non-strict versions and in terms of (strict and non-strict)

quasi-convexity or quasi-concavity.


3.1.4.1   LEMMA:  Let B be a closed set in a compact (Hausdorff)

space X × Y.  Define

$$D = \{(k,y) \; \epsilon \; C(X) \times Y \,|\, (k \times \{y\}) \subset B\} \; ,$$

$$E = \{(k,y) \; \epsilon \; C(X) \times Y \,|\, (k \times \{y\}) \cap B \neq \phi , \},$$

and assume that $C(X)$ has the finite topology.  Then

1.  $D$ is compact in $C(X) \times Y$ ; and

2.  $E$ is compact in $C(X) \times Y$.

**Proof:**  Denote $Y^* = \pi_Y(B)$, $B_y = B \cap (X \times \{y\})$, $B^*_y = \pi_X(B_y)$; it is clear that all of these are compact.  Furthermore, $C(X) \times Y$ is a compact (Hausdorff) space when $C(X)$ has the finite topology, as $C(X)$ is then compact (Hausdorff).  (See 3.1.2.12). Hence, it suffices to show that (1) $D$ and (2) $E$ are closed.

<u>ad (1)</u>: There is nothing to prove if $D = C(X) \times Y$, so let $A \; \epsilon \; (C(X) \times Y) \backslash D$.  Denote $D_y = <B^*_y> \times \{y\}$ and generic elements $d_y = (k,y) \; \epsilon \; D_y (y \; \epsilon \; Y^*)$.  Then $A$ and $d_y$ are distinct for each $d_y \; \epsilon \; D$.  Fixing $y \; \epsilon \; Y^*$, there thus exists an open cover $\{W(d_y) \,|\, d_y \; \epsilon \; D_y \}$ of $D_y$ with open boxes

$$W_y(d_y) = U_y(k) \times V_y(y), \qquad (d_y \; \epsilon \; D_y),$$

and a family $\{N_y(d_y) \,|\, d_y \; \epsilon \; D_y\}$ of nbds $N_y(d_y) \subset C(X) \times Y$ of $A$, such that for each $d_y \; \epsilon \; D_y$, $W_y(d_y) \cap N_y(d_y) = \phi$ (where $U_y(k) \subset C(X)$ is an open nbd of $k$ and $V_y(y) \subset Y$ is an open

nbd of y). Since $\langle B^*_y \rangle$ and $\{y\}$ are compact, so is $D_y$.

Hence, $\{W_y(d_y) \mid d_y \, \epsilon \, D_y\}$ affords a finite subcover

$$\{W_y(d^i_y) \; = \; U^i_y \times V^i_y \mid i = 1, \ldots, m(y)\} \; .$$

Define $V_y = V^1_y \cap \ldots \cap V^{m(y)}_y$ ,

and $N_y = N_y(d^1_y) \cap, \ldots \cap N_y(d^{m(y)}_y)$ and

$$W^i_y = U^i_y \times V_y \qquad (i = 1, \ldots, m(y)).$$

Then $\{W^i_y \mid i = 1, \ldots, m(y)\}$ is an open cover of $D_y$ such that

$$W^i_y \cap N_y = \phi \qquad (i = 1, \ldots, m(y)).$$

Now $\{V_y \mid y \, \epsilon \, Y^*\}$ is an open cover of the compact $Y^*$,

affording a finite subcover $\{V_{y_1}, \ldots, V_{v_n}\}$ . Define

$N = \bigcap_{j=1}^{n} N_{y_j}$ . Then $N$ is a nbd of A which is disjoint from

$$W = \bigcup \{W^i_{y_j} \mid i = 1, \ldots, m(y_j); \; j = 1, \ldots, n\} \; .$$

Since $D \subset W$, $N \cap D = \phi$ . Thus, D is closed, since its

complement in $C(X) \times Y$ is open.

ad(2): To show that E is closed, note that

$$E = \bigcup_{Y^*} (\langle (B^*_y)^c \rangle^+)^c \times \{y\}$$

and that $E_y = (\langle (B^*_y)^c \rangle^+)^c \times \{y\} = \overline{\langle B^*_y \rangle} \times \{y\}$ is closed, hence

compact, for each $y \, \epsilon \, Y^*$. Then the proof of (1) applies, merely

by replacing $\langle B^*_y \rangle$ by $\overline{\langle B^*_y \rangle}$, $D_y$ by $E_y$, D by E and $d_y$ by $e_y$.

#

3.1.4.2    <u>Lemma</u>: Let $X = X_\alpha \times X^\alpha$ be a compact (Hausdorff) space,

let $u: X \to R$ be continuous, and define the real-valued

functions $\bar{u}$, $\underline{u}$ on $C(X_\alpha) \times X^\alpha$ by

$$\bar{u}(k_\alpha, \ x^\alpha) = \underset{k_\alpha}{\mathrm{Sup}} \ \ u(\cdot, \ x^\alpha)$$

$$\underline{u}(k_\alpha, \ x^\alpha) = \underset{k_\alpha}{\mathrm{Inf}} \ u(\cdot, \ x^\alpha)$$

$$k_\alpha \, \varepsilon \ C(X_\alpha), \ x^\alpha \ \varepsilon \ X^\alpha \ .$$

Assume $C(X_\alpha)$ has the finite topology. Then

1.1. u is usc iff $\underline{u}$ is usc,

  2   u is lsc iff $\bar{u}$ is lsc;

2.1. u is usc iff $\bar{u}$ usc,

  2. u is lsc iff $\underline{u}$ is lsc.

<u>Proof</u>: Upon noting that $X_\alpha$ is $T_1$, it is obvious that continuity

properties of $\bar{u}$ or $\underline{u}$ hold also for u, since a continuity

property holding for $\bar{u}$ or $\underline{u}$ on the whole of $C(X_\alpha) \times X^\alpha$ also

holds on the subspace $\{(\{x_\alpha\}; \ x^\alpha)\}$ , while this subspace is

honeomorphic to X and $u = \bar{u} = \underline{u}$ on this subspace. All the

implications 'if' are thus proved.

To prove the rest, let $p \in R$ be arbitrary, and

denote $P^+ = \{r \in R | r \geq p\}$ , $P^- = \{r \in R | r \leq p\}$ , $u^{-1}(P^+)$

$= B^+$, $u^{-1}(P^-) = B^-$, $\bar{u}^{-1}(P^+) = E^+$, $\bar{u}^{-1}(P^-) = D^-$, $\underline{u}^{-1}(P^+) = D^+$,

$\underline{u}^{-1}(P^-) = E^-$. From this notation, it is clear how to use the

last lemma, observing that $B^+$ is closed (compact) if u is usc

and that $B^-$ is closed (compact) if u is lsc. Thus, in case

of 1.1., $D^+$ is closed, so that u is usc; in case of 1.2, $D^-$

is closed, so that $\bar{u}$ is lsc; in case of 2.1, $E^+$ is closed, so

that $\bar{u}$ is usc; in case of 2.12, $E^-$ is closed, so that $\underline{u}$ is lsc.

This completes the proof.

#

3.1.4.3   Corollary: Using the definanda and notation of the last

lemma, among the following statements i.a, i.b and i.c

are equivalent (i = 1 , 2, 3):

1.   a.   u  is usc.

b.   $\underline{u}$ is usc with the finite topology on $C(X_\alpha)$.

c   $\bar{u}$ is usc with the finite topology on $C(X_\alpha)$.

2.   a.   u is lsc.

b.   $\underline{u}$ is lsc with the finite topology on $C(X_\alpha)$.

c.   $\bar{u}$ us lsc with the finite bopology on $C(X_\alpha)$.

3.   a.   u is continuous

b.   $\underline{u}$ is continuous with the finite topology on $C(X_\alpha)$.

c. $\bar{u}$ is continuous with the finite topology on $C(X_\alpha)$.

Proof: All is plain as a rearrangement of the last lemma.

#

The form in which 3.1.2.9 will actually be used (together with 3.1.4.5 in proving 3.3.1) is actually the following simple proposition, needing no proof.

3.1.4.4   Proposition: Let X be a topological space, and define the map
$\ell : [X] \to X$ by

$$\ell(E) = \{x \mid x \in E\} \qquad (E \in [X]).$$

Then $\ell$ is usc (lsc) with the usf(lsf) topology on [X].

3.1.4.5   Proposition: The graph $\Gamma(\ell) = \{(E,x) \mid E \in C(X), x \in E\} \subset C(X) \times X$
is closed if X is regular and C(X) has the usf topology.

Proof: To see that the complement of $\Gamma(\ell)$ is open, let $F \in C(X)$
and $y \in X \setminus F$. Since F is closed and X is regular, there
exist disjoint open sets $U, V \subset X$ such that $F \subset U$ and $y \in V$.
Then $<U>$ is open with $F \in <U> \subset C(X)$, and $(<U> \times V) \cap \Gamma(\ell) = \phi$,
showing that $\Gamma(\ell)$ has open complement.

#

In the rest of the section attention is directed to matters
of convexity of u, $\bar{u}$, and $\underline{u}$.


3.1.4.6    <u>LEMMA</u>: Let $L_\alpha$ and $L^\alpha$ be real linear topological spaces

and, with reference to 3.1.4.2-3, let $X^\alpha \in KQ(L^\alpha)$, $X_\alpha \in KQ(L_\alpha)$,

and assume u to be continuous.  Then u has any of the properties

under (1) below iff $\bar{u}$ has, and u has any of the properties

under (2) below iff $\underline{u}$ has:


   1.  a (strict) quasi-concavity,

       b.(strict) concavity,

       c.linearity, i.e., concavity <u>and</u> convexity;


   2.  a.(strict) quasi-convexity,

       b.(strict) convexity,

       c. linearity.


<u>Proof</u>:   The 'if' parts of the proposition are all obvious

upon noting that singleton subsets of $X_\alpha$ are closed, and that the

collection of these is convex. The 'only if' parts are all

straightforward, so only (1.a) will be treated; imitation will

yield the remaining proofs.

Let $'\lambda = 1 - \lambda \in [0,1]$, let $(k_\alpha, x^\alpha)$, $('k_\alpha, 'x^\alpha) \in C(X_\alpha) \times X^\alpha$,

and let $x_\alpha \in k_\alpha$ and $'x_\alpha \in 'k_\alpha$ with $u(x_\alpha, x^\alpha) = \bar{u}(k_\alpha, x^\alpha)$ and

$u('x_\alpha, 'x^\alpha) = \bar{u}('k_\alpha, 'x^\alpha)$.  Finally, denote $\bar{k}_\alpha = \lambda k_\alpha + '\lambda 'k_\alpha$,

$$\bar{x}^{\alpha} = \lambda x^{\alpha} + '\lambda' x^{\alpha} \text{ and } \bar{x}_{\alpha} = \lambda x_{\alpha} + '\lambda' x_{\alpha}. \text{ Then } \bar{u}\ (\bar{k}_{\alpha},\ \bar{x}^{\alpha}) \geq$$

$u(\bar{x}_{\alpha}\ ,\ \bar{x}^{\alpha})$. If $u$ is quasi-concave, then

$$u(\bar{x}_{\alpha}\ ,\bar{x}^{\alpha}) \geq \text{Min } [u(x_{\alpha},\ x^{\alpha}),\ u('x_{\alpha},\ 'x^{\alpha})],$$

whereby $\bar{u}$ is quasi-concave also; similarly, if $u$ is strictly

quasi-concave (i.e., the last inequality is strict), then

$\bar{u}$, too, is so.

$\#$

3.1.4.7 <u>Remark</u>: From 3.1.4.2.-3, it is clear that 3.1.4.6 can be

strengthened by assuming only that $u$ is usc for part 1 and

lsc for part 2.

## 3.2 <u>Topological Properties of Behavors in Static Social Systems</u>

From a narrow viewpoint, the prime motivation for recording the

properties collected in this section is the existence theory for

equilibrium in static social systems, presented in 4.2. The end

of the proof for an existence theorem for social equilibirum, however,

marks just a beginning for social analysis, no matter how general

and powerful that existence theorem. The properties enjoyed (or

suffered) by behavors in a social system – static or dynamic –

deserve consideration, therefore, as main building blocks of social

analysis, rather than merely as stepping stones useful only for proving an equilibrium existence theorem. For this reason, the present section treats selected properties of behavors in some detail, restricting attention to the case of a static social system. Actually, the properties of a social system itself are expressed quite well, as a rule, in terms of the properties of its behavors. So the present section may be looked upon also as a treatment of selected properties of social systems in the static case.

To clarify what particular properties are gained for behavors from what properties of the behavior spaces, utility functions, impression functions, interpretation functions and incentive functions, the results of the present section are displayed in as "disaggregated" form as is feasible here.

3.2.1  THEOREM: Let $X = X_\alpha \times X^\alpha$ be a compact (Hausdorff) space, let $f: X \to R$ be continuous, and let $\bar{f}: X^\alpha \to R$ be defined by $\bar{f}(x^\alpha) = $ Sup $f(\cdot, x^\alpha)$. Define $\alpha: X^\alpha \to X$   by

$$\alpha(x^\alpha) = \{x_\alpha | f(x_\alpha, x^\alpha) \geq \bar{f}(x^\alpha)\}.$$

Then the graph $\Gamma(\alpha) \subseteq X$ of $\alpha$ is compact and, hence, $\alpha$ is upper semi-continuous with $\alpha(x^\alpha)$ non-empty and compact ($x^\alpha \in X^\alpha$).

Proof: From the continuity of f, it follows that $Y = f(X)$ is

compact and, by 3.1.4.3, that $\bar{f}$ is continuous. Since R is Hausdorff,

the graph $\Gamma(f) \subseteq X \times Y$ of f and the graph $\Gamma(\bar{f}) \subseteq X^\alpha \times Y$ of $\bar{f}$

are then both closed, hence compact. Thus, $X_\alpha \times \Gamma(\bar{f})$ is compact

and so is $(X_\alpha \times \Gamma(\bar{f})) \cap \Gamma(f) = B$. Hence, the projection $\pi_X(B)$

is compact. Obviously, $\Gamma(\alpha) = \pi_X(B)$. For each $x^\alpha \in X^\alpha$, $\alpha(x^\alpha)$ is

non-empty by the compactness of $X_\alpha \times \{ x^\alpha \}$ and the continuity of

f; it is compact, since it is the projection $\pi_{X_\alpha}((X_\alpha \times \{x^\alpha\}) \cap \Gamma(\alpha))$

of a compact set. Finally, $\alpha$ is upper semi-continuous by

closedness of $\Gamma(\alpha)$ and compactness of $X_\alpha$, using Lemma 2 of

[Fan, 1952].


3.2.2  COROLLARY: If the collective behavior space of a static social

system is compact and $\ddot{w}_\alpha$ is continuous for (each) behavor $\alpha$, then

(each) $\alpha$ has a compact graph and, hence is upper semi-continuous,

selecting a compact and non-empty choice set in its behavior

space in reaction to each $\alpha$-exclusive behavior.


Proof: Replace f in 3.2.1 by $\ddot{w}_\alpha$.

#


3.2.3  COROLLARY: The consequence in 3.2.2 holds if the collective

behavior space is compact and $u_\alpha$, $h_\alpha$ and $\ddot{g}_\alpha$ are continuous

Proof: If $u_\alpha$, $h_\alpha$ and $\ddot{g}_\alpha$ are all continuous,

#

3.2.4    <u>LEMMA</u>: Let Z be convex in a real vector space and let $g:Z \to R$

u:Z × R → R be two real-valued functions, such that u is non-

decreasing in $r \in R$, i.e., such that for all $z \in Z$, if r, $s \in R$,

and $r \leq s$, then $u(z,r) \leq u(z,s)$. Define $f:Z \to R$ by $f(z) = u(z,g(z))$.

Then f is quasi-concave if g is concave and u is quasi-concave.

Proof:   If Z is empty, then there is nothing to prove. So let

z, $z' \in Z$, $\lambda = 1 - \lambda' \in [o,1]$, and denote $\bar{z} = \lambda z + \lambda' z'$. Then

$$f(\bar{z}) = u(\bar{z}, g(\bar{z}))$$

$$\geq u(\bar{z}, \lambda g(z) + \lambda' g(z'))$$

$$\geq Min [u(z,g(z)), u(z',g(z'))]$$

$$= Min [f(z), f(z')].$$

#

3.2.5    <u>COROLLARY</u>: If the behavior space $X_\alpha$ is convex in a real vector

space and $Y^\alpha = h_\alpha(X^\alpha)$, then $\ddot{w}_\alpha$ is quasi-concave on $X_\alpha \times \{y^\alpha\}$ if

$\ddot{g}_\alpha$ is concave on $X_\alpha \times \{y^\alpha\}$ and $u_\alpha$ is quasi-concave on

$X_\alpha \times \{y^\alpha\} \times \ddot{g}_\alpha(X_\alpha \times \{y^\alpha\})$ $(v^\alpha = h_\alpha(x^\alpha)$, $x^\alpha \in X^\alpha)$.

Proof:   A direct application of 3.2.4.

#

3.2.6    <u>COROLLARY</u>: If, in addition to the hypothesis of 3.2.5,

$X_\alpha$ is compact and convex in a real linear topological space,

and for each $y^\alpha \in Y^\alpha$, $\ddot{g}_\alpha$ and $u_\alpha$ are, respectively, continuous

on $X_\alpha \times \{y^\alpha\}$

and upper semi-continuous on $X_\alpha \times \{y^\alpha\} \times \ddot{g}_\alpha$ $(X_\alpha \times \{y^\alpha\}$ ), then

$\alpha(x^\alpha)$ is non-empty, compact and convex for each $x^\alpha \in X^\alpha$ .

<u>Proof</u>: Since $X_\alpha$ is compact, so is $X_\alpha \times \{y^\alpha\}$. Since $\ddot{g}_\alpha$ is

continuous so is $\ddot{g}_\alpha$ $(X_\alpha \times \{y^\alpha\})$ compact. By the same reason

and the upper semi-continuity of $u_\alpha$ , $\ddot{w}_\alpha$ is upper semi-continuous

on $X_\alpha \times x^\alpha$, for any $x^\alpha \in X^\alpha$. Thus, for each $x^\alpha \in X^\alpha$ , $\ddot{w}_\alpha$ attains

a supremum $s*(x^\alpha)$ on some $(x*_\alpha, x^\alpha) \in X_\alpha \times \{x^\alpha\}$. Since $\ddot{w}_\alpha$ is

upper semi-continuous on $X_\alpha \times \{x^\alpha\}$ , $\alpha$ $(x^\alpha) = \{x_\alpha \in X_\alpha | \ddot{w}_\alpha$ $(x_\alpha, x^\alpha)$

$\geq s*(x^\alpha)\}$ is closed, hence compact, while obviously non-empty

from the previous sentence, $(x^\alpha \in X^\alpha)$. From 3.2.5, $\ddot{w}_\alpha$ is also

quasi-concave on $X_\alpha \times \{x^\alpha\}$, so that $\alpha(x^\alpha)$ is also convex $(x^\alpha \in X^\alpha)$.

*#*

### 3.3 <u>Topological Properties of Behavors in Dynamic Social Systems</u>

The present section is offered to serve the analysis of

dynamic social systems in a role analogous to that of the last

section for the case of static social systems. In a narrow sense,

the section is aimed at the existence theory of equilibrium for

dynamic social systems, presented in 4.3, but the facts recorded

are actually of wider interest.

As the last section spells out - albeit for the static case -

how various properties of its behavors derive from those of other

elements of a social system, the present section avoids the

corresponding exercise in the dynamic case. This is in the belief

that the last section provides an easy enough model to imitate.

The present section can, therefore, afford to be briefer.

3.3.1  <u>THEOREM</u>: Let $X_\alpha$ and $X^\alpha$ be compact (Hausdorff) spaces,

and let $\ddot{w}_\alpha: X_\alpha \times X^\alpha \to R$ be continuous.  For generic $k_\alpha \in C(X_\alpha)$

and $x^\alpha \in X^\alpha$, denote $\bar{\ddot{w}}_\alpha = \text{Sup}_{k_\alpha} \ddot{w}_\alpha(\cdot, x^\alpha)$, and define

$$\alpha(k_\alpha, x^\alpha) = \{x_\alpha \in k_\alpha | \ddot{w}_\alpha(x_\alpha, x^\alpha) \geq \bar{\ddot{w}}_\alpha(k_\alpha, \dot{x}^\alpha)\}$$

Then, taking the finite topology on $C(X_\alpha)$, the graph

$\Gamma(\alpha) \subseteq C(X_\alpha) \times X^\alpha \times X_\alpha$ is compact and, hence, $\alpha$ is usc,

with each $\alpha(k_\alpha x^\alpha)$ non-empty and compact.

<u>Proof</u>:  By continuity of $\ddot{w}_\alpha$ , the set

$$B = \{(k_\alpha, x^\alpha, x_\alpha, r) | r \leq \ddot{w}_\alpha(x_\alpha, x^\alpha)\}$$

is closed. By continuity of $\ddot{w}_\alpha$, $\bar{\ddot{w}}_\alpha$ is continuous, so that the

set

$$\Gamma_+(\bar{\ddot{w}}) = \{(k_\alpha, x^\alpha, r) | r \geq \bar{\ddot{w}}(k_\alpha, x^\alpha)\}$$

is closed also. Hence $B \cap (\Gamma_+(\bar{\ddot{w}}) \times X_\alpha)$ is closed. In fact, it is

compact as a subset of the compact $C(X_\alpha) \times X^\alpha \times X_\alpha \times \bar{\ddot{w}}_\alpha(X_\alpha \times X^\alpha)$,

so that its projection P into $C(X_\alpha) \times X^\alpha \times X_\alpha$ is compact.

Now P is simply the set

$$P = \{(k_\alpha, x^\alpha, x_\alpha) | \ddot{w}_\alpha(x_\alpha, x^\alpha) \geq \bar{\ddot{w}}_\alpha(k_\alpha, x^\alpha)\}.$$

x

4. <u>EVOLUTION AND EQUILIBRIUM IN SOCIAL SYSTEMS</u>

This chapter is concerned with equilibrium in social systems.
The notion of 'equilibrium' is briefly compared with those of
'Pareto point' and 'core point' as treated in economic theory and
the theory of games. Equilibrium points are then considered
as fixed points of certain mappings, called "evolutions", representing
the way in which certain adjustment processes operate for various
types of social system.

Of the five very general types of social system studied for
the existence of an equilibrium, the first (type 0) is static.
The remaining types (I-IV) are dynamic. All five types, classified
according to various topological properties which they satisfy, are
unrestricted in the cardinality of their personnel, so that the
personnel can be finite, countably infinite or uncountably infinite.
The behavior spaces are compact and convex in locally convex
real linear topological spaces in the case of type 0 social
systems; they are compact and convex in normed real linear
topological spaces in the case of types I-IV. For all five
types of social system, the set of equilibria is shown to be
non-vacuous, compact, and, in certain cases, also convex.

The existence result for equilibrium in the case of type 0
social systems is proved by use of (effectively a fixed point)
theorem of Fan (1952). In social systems of types I-IV, the existence
of social equilibirum is established by applying a fixed point
theorem of Prakash and Sertel (1970 a) for certain (real semi-

linear topological) spaces developed by these authors (1970 b).
These latter existence results are extensions - obtained by
encorporation of impression, incentive and interpretation
schemes - of results developed by Prakash and Sertel (1970 c).

## 4.1 Notions of Social Equilibrium

Of the many different notions of 'equilibrium' that are

relevant to the study of social systems, those which have attracted

the most attention in the study of games and economies are three

kinds. Only one of these really goes under the name of "equilibrium",

although the other two "Pareto point" and "core point", are also

equilibria in a real sense and sometimes recognized as such. While

the present study is concerned mainly with the first of these and

it is the existence of equilibrium in this particular sense that

is established in the other sections of this chapter for certain

types of social systems, the study might gain perspective if the

relation between the mentioned three kinds of equilibrium is

indicated. (To adhere to the most common usage, "equilibrium"

will hereafter be used to refer to the first mentioned of these.)

To define 'equilibrium', 'Pareto point' and 'core point' in

a fashion that will allow easy comparison, a few preparatory

definitions are in order. These become simpler in the case of

a game or, in general, a static social system, since the notion

of 'admissibility' loses its importance in that case, as the

reader will notice for himself immediately the definition is

given. The notion of 'blocking' by "coalitions" is pivotal for

the comparison between the three kinds of point (equilibrium,

Pareto and core).

4.1.1. <u>Definition</u>:  Let A be the personnel of a social system S.  Any

non-empty subset $B \subset A$ will be called a <u>coalition</u> of S, and

the notation $X_B = \Pi X_\alpha$, $X^B = \Pi X_\alpha$, $D_B = \Pi D_\alpha$, $D^B = \Pi D_\alpha$, will

be adopted with generic elements denoted as $x_B \in X_B$, $x^B \in X^B$,

$d_B \in D_B$, $d^B \in D^B$.


4.1.2. <u>Definition</u>:  Let x be a collective behavior and d a collective

feasibility of a social system S with personnel A.  The pair

(x,d) will be said to be <u>inadmissible for $\alpha$</u> ($\alpha \in A$) iff

$\pi_{X_\alpha}(x) = x_\alpha \notin d_\alpha = \pi_{D_\alpha}(d)$;  otherwise, it will be said to

be <u>admissible for $\alpha$</u>; (x,d) will be said to be <u>admissible for B</u>,

a coalition of S, iff (x,d) is admissible for each $\alpha \in B$;

otherwise, it will be said to be  <u>inadmissible for B.</u>  (x,d)

will be said to be <u>(in)admissible</u> iff it is (in)admissible for

A.


4.1.3 <u>Definition</u>:  Let S be a social system, Let B be a coalition

of S, and let $x = (x_B, x^B)$, $d = (d_B, d^B)$ be a collective

behavior and a collective feasibility, respectively.  If

the pair (x,d) is admissible for B, then it will be said

to be <u>blocked by B</u> iff there exists $y_B \in d_B$ such that

$$\ddot{w}_\beta(y_B, x^B) \geq \ddot{w}_\beta(x_B, x^B)$$

holds for all $\beta \in B$ and

$$\ddot{w}_{\beta_*}(y_B, x^B) > \tilde{w}_{\beta_*}(x_B, x^B)$$

holds for some $\beta \in B$. If $(x,d)$ is admissible for $B$, then it will be said to be <u>unblocked by B</u> iff it is not blocked by $B$.

The notions of 'equilibrium', 'Pareto point' and 'core point' can now be given precise meaning by the following defintion.

4.1.4    <u>Definition</u>:  Let **x** be a collective behavior and d a collective feasibility of a social system  S with personnel A, such that the pair $(x,d)$ is admissible.  The pair $(x,d)$ will be called an <u>equilibrium</u> (point) of S iff it is unblocked by each singleton coalition $\{\alpha\}$  A; it will be called a <u>Pareto point</u> of S iff it is unblocked by A; it will be called a <u>core point</u> of S iff it is unblocked by all coalitions of S.  The <u>equilibrium set</u> of S is the set of all equilibria of S.  The <u>Pareto set</u> of S is the set of all Pareto points of S; the <u>core</u> of S is the set of all core points of S.

The comparison between equilibrium, Pareto and core points of a social system is now absolutely clear.  Simple but important consequences of the definitions are that all core points are equilibria as well as Pareto points, so that the core is contained in the intersection of the equilibrium set with the Pareto set. Thus, the core is empty when, for instance, S does not have an equilibrium.  In the later

sections of this chapter, sufficient conditions are given

for S to have an equilibrium. Demonstrating sufficient

conditions for the Pareto set or the core to be

non-vacuous is a research problem not tackled in this study.

Although it certainly will not be a survey of the

literature, a brief sketch of the history of the above ideas

will now be given. The idea of an equilibrium point for a

competitive economy, in the sense of a price vector equating

supply to demand in each market, is commonly attributed

to Walras (1881). Wald (1933-4, 1934-5, 1951) proved,

under rather restricted conditions, that an equilibrium

exists for each of a pure production and a pure exchange

economy. Arrow and Debreu (1954), based on a study of Debreu

(1952), demonstrated for the first time the existence of an

equilibrium for a competitive economy in which production,

exchange and consumption all take place, using less

stringent assumptions than those of Wald. One very good

reason why a survey of the literature up to 1954 is not

given here is that the mentioned work of Arrow and Debreu

contains an excellent such survey. The existence theorem

of Debreu (1952) for a social equilibrium (where, however,

the "social system" is actually the same as that defined to

be an "economy" in Arrow and Debreu (1954) - see also section

2.2 of the present study) is used in establishing the result

of Arrow and Debreu (1954), and that theorem is itself based
on a corollary of the fixed point theorem of Eilenberg
and Montgomery (1946). Using Kakutani's (1941) fixed point
theorem, McKenzie (1955) obtained improvements on the Arrow
and Debreu study. All of these works used finite dimensional
methods. They succeeded in providing economics with the
equilibrium whose various optimality and stability aspects
had long been discussed and many worked out by a long and
formidable list of authors. Newman's (1968) excellent
collection is one good entry into the box of gems to which
all of the above belong.

The idea of an equilibrium for a game was first formally
introduced in some generality by Nash (1950) and proved to
exist by him (1950, 1951), first by use of Kakutani's and
then by use of Brouwer's fixed point theorem. The game dealt
with was a finite personnel non-cooperative one with behavior
spaces (compact and convex) in Euclidean spaces.

For information concerning the Pareto set, the core and
the relations between these and the equilibrium set, the
following short list might provide a reasonable means of
entry into the associated game-theoretic and economic
literature: (Edgeworth, 1881), (von Neumann and Morgenstem,
1944), (Arrow, 1950) ,(Gillies, 1953, 1959), (Bondareva,1962)
(Debreu and Scarf, 1963), (Vind, 1964), (Aumann, 1964),
(Shapley, 1965), Scarf, 1967). Of these (Aumann, 1964)

contains a short but illuminating discussion of the development

of the area.

4.2    Evolution and Equilibrium in Static Social Systems

      In this section an equilibrium of a static social system is defined as a fixed point of a certain transformation called an "evolution", and a sufficient condition is demonstrated for the existence of such an equilibrium. The "static contractual set", i.e., the set of equilibria of a static social system, is shown to be compact if the mentioned sufficient condition is satisfied. This contractual set is shown also to be convex if certain linearity conditions hold which yield the effective utility function$^s$ linear.

4.2.1    Notation and Convention: In the case of static social systems it is possible to somewhat simplify the notation adopted for dealing with social systems in general. This simplification is permitted by the fact that the feasibility transformations $t_\alpha$ of a static social system are all constant functions, the typical $t_\alpha$ assigning a fixed $d_\alpha \subset X_\alpha$ to every point in its domain. From this fact it is clear that no generality is lost by assuming $d_\alpha = X_\alpha$, i.e., $D_\alpha = \{X_\alpha\}$, for all the behavors $\alpha \in A$. Taking advantage of this, the set $T \in S$ can be fully specified and suppressed by representing a static social system $S$ in the form $<W, U, H, G, I\ A>$ . For, whenever a social system is specified in this way, it will be understood to be static and the constant collective feasibility will be taken to be

the collective behavior space X. In this case, since there

is always one constant feasibility $d_\alpha = X_\alpha$ for each behavor

$\alpha \in A$, the behavors become simply point-to-set mappings

$\alpha: X^\alpha \to X_\alpha$ defined by

$$\alpha(x^\alpha) = \{x_\alpha \in X_\alpha | \ddot{w}_\alpha(x_\alpha, x^\alpha) \geq \ddot{\bar{w}}_\alpha(x^\alpha)\} \ ,$$

where

$$. \quad \ddot{\bar{w}}_\alpha(x^\alpha) = \sup_{y_\alpha \in X_\alpha} \ddot{w}_\alpha(y_\alpha, \ x^\alpha).$$

4.2.2 <u>Definitions</u>: The <u>evolution</u> of a(static) social system

S = <W, U, H, G, I, A> is a transformation E: X → [X] defined

for each x ∈ X, by

$$E(x) = \prod_A \alpha(\pi_{X\alpha}(x)).$$

4.2.3 <u>Definition</u>: The <u>(static) contractual set</u> of a static social

system S is the set $C = \{x \in X | x \in E(x)\} \subset X$ of fixed points

of the evolution E of S. A collective behavior x ∈ X is called

a <u>(static)social contract</u> or <u>equilibrium</u> of S iff x ∈ C.

4.2.4. <u>Definition</u>: A static social system S will be classified as

<u>type 0</u> ("type zero") iff the following conditions are satisfied

for each behavor $\alpha \in A$:

(1)  $X_\alpha \in KQ(L_\alpha)$ for some locally convex real linear

topological space $L_\alpha$ :

(2)  $\ddot{w}_\alpha$ is continuous on X and quasi-concave on $X_\alpha \times \{x^\alpha\}$ for each

$x^\alpha \in X^\alpha$.

The main theorem, 4.2.6, of this section will be proved

by use of the following (fixed point) theorem.

4.2.5   THEOREM [Fan, 1952]:  Let $\{L_\alpha | \alpha \in A\}$ be a family of locally

convex real linear topological spaces.  For each $\alpha \in A$, let

$X_\alpha \in KQ(L_\alpha)$ and let $X^\alpha = \prod_{A\setminus\{\alpha\}} X_\beta$. Let $X = \prod_A X_\alpha$ and let

$\{\Gamma(\alpha) | \alpha \in A\}$ be a family of closed subsets of X.  If,

for any point $x \in X$, and for any $\alpha \in A$, the set $\alpha(x^\alpha) \in O(X_\alpha)$,

where $x^\alpha = \pi_{X\alpha}(x)$    and

$$\alpha(x^\alpha) = \pi_{X_\alpha} (\Gamma(\alpha) \cap (X_\alpha \times \{x^\alpha\})),$$

then $\cap_A \Gamma(\alpha) \neq \phi$.

4.2.6   THEOREM:  Every type 0 social system has an equilibrium.

Proof:  By 3.2.2, the graph $\Gamma(\alpha)$ of each behavor $\alpha$ in

the personnel A satisfies all but the requirement that

$\alpha(x^\alpha) \subseteq X_\alpha$ is convex for each $x^\alpha \in X^\alpha$. This requirement

is satisfied, however, since $\ddot{w}_\alpha$ is quasi-concave.  Thus,

$\cap_{A} \Gamma(\alpha) \neq \phi$ and, evidently any point x in this intersection is a fixed point of the evolution, i.e., an equilibrium.

#

4.2.7 **COROLLARY**: A static social system with a collective behavior space which is compact and convex in a locally convex real linear topological space is of type 0, hence has an equilibrium, if the following conditions are satisfied for each behavor α in the personnel:

(1) $u_\alpha$ is quasi-concave on $X_\alpha \times \{y^\alpha\}$ for each $y^\alpha \in h_\alpha(X^\alpha)$, and $u_\alpha$ is continuous;

(2) $h_\alpha$ is continuous;

(3) $\ddot{g}_\alpha$ is concave on $X_\alpha \times \{y^\alpha\}$ for each $y^\alpha \in h_\alpha(X^\alpha)$ and $\ddot{g}_\alpha$ is continuous.

**Proof**: From the quasi-concavity and concavity of $u_\alpha$, $\ddot{g}_\alpha$, respectively, on $X_\alpha \times \{y^\alpha\}$, it follows by 3.2.4 that $\ddot{w}_\alpha$ is quasi-concave on the same ($y^\alpha \in h_\alpha(X^\alpha)$; $\alpha \in A$). Hence $\alpha(x^\alpha)$ is convex for each $x^\alpha \in X^\alpha$ and $\alpha \in A$. From the continuity of $u_\alpha$, $h_\alpha$ and $\ddot{g}_\alpha$, it follows that $\ddot{w}_\alpha$ is continuous, so that the social system is of type 0. Now 4.2.6 applies.

#

4.2.8 **THEOREM**: The contractual set C of a type 0 social system is compact.

Proof: Clearly, C is precisely the intersection of the graphs $\Gamma(\alpha)$ of all the behavors, which is compact since each $\Gamma(\alpha)$ is compact by 3.2.2.

#

4.2.9 THEOREM: The contractual set of a type 0 social system S is convex if the effective utility function $\ddot{w}_\alpha$ of each behavor $\alpha$ in the personnel of S is linear (both concave and convex).

Proof: It suffices to show that the graph $\Gamma(\alpha)$ of an arbitrary behavor $\alpha$ in the personnel is convex. Let $\alpha$ be such a behavor, and let x, y, $\varepsilon$ $\Gamma(\alpha)$, where $x = (x^\alpha, x_\alpha)$ and $y = (y^\alpha, y_\alpha)$. Define $z = (z^\alpha, z_\alpha) = \lambda x + \lambda' y$ for any $\lambda = 1 - \lambda' \varepsilon [0,1]$. Then $\ddot{w}_\alpha(z) = \lambda \ddot{w}_\alpha(x) + \lambda' \ddot{w}_\alpha(y)$, by linearity of $\ddot{w}_\alpha$. Since x, y $\varepsilon \Gamma(\alpha)$, $\ddot{w}_\alpha(x) = \bar{w}_\alpha(x^\alpha)$ and $\ddot{w}_\alpha(y) = \bar{w}_\alpha(y^\alpha)$. But, by 3.1.4.6.1.c and the linearity of $\ddot{w}_\alpha$, $\bar{w}_\alpha$ is linear also. Hence, $\bar{\bar{w}}_\alpha(z^\alpha) = \lambda \bar{w}_\alpha(x^\alpha) + \lambda' \bar{\bar{w}}_\alpha(y^\alpha)$, so that $\bar{\bar{w}}_\alpha(z^\alpha) = \ddot{w}_\alpha(z)$, implying that $z_\alpha \varepsilon \alpha(z^\alpha)$, i.e., that z $\varepsilon \Gamma(\alpha)$. Thus, $\Gamma(\alpha)$ is convex, completing the proof.

4.2.10 COROLLARY: Let S be as in 4.2.9, and let A be the personnel of S. Denote $Y^\alpha = h_\alpha(X^\alpha)$ for each $\alpha \varepsilon$ A. Assume that, for each $\alpha \varepsilon$ A, $h_\alpha(\lambda x^\alpha + '\lambda' x^\alpha) = \lambda h_\alpha(x^\alpha) + '\lambda h_\alpha(' x^\alpha)$ if $x^\alpha$ and $'x^\alpha$ belong to $X^\alpha$. Assume that $\ddot{g}_\alpha$ is linear (concave and

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convex on $X_\alpha \times Y^\alpha$, for each $\alpha \in A$. If $u_\alpha$ is linear on

$X_\alpha \times Y^\alpha \times \ddot{g}_\alpha(X_\alpha \times Y^\alpha)$ for each $\alpha \in A$, then the contractual

set of S is convex.


<u>Proof</u>: It follows from the hypothesis concerning $h_\alpha$ and $\ddot{g}_\alpha$

that $\ddot{g}_\alpha (x^\alpha, h_\alpha(x^\alpha))$ is linear on X. It is easy to see

(though tedious to show) that the hypothesis concerning $u_\alpha$ then

ensures the linearity of $\ddot{w}_\alpha$ . Since $\ddot{w}_\alpha$ is thus linear from

each $\alpha \in A$, the desired result follows by 4.2.9.

#

## 4.3. Evolution and Equilibrium in Dynamic Social Systems

The primary aim of this section is to demonstrate reasonably unrestrictive sufficient conditions for the existence of what will be defined to be a _dynamic_ social equilibrium. Although only one type of evolution and social equilibrium needed consideration in the case of static social systems, in the case of dynamic social systems several types of evolution and several corresponding types of equilibrium deserve attention. For, in the dynamic case, the feasibility transformations are no longer restricted to be constant maps, so that various forms of relaxation of the constantness of these maps can be considered in conjunction with various assumptions governing the behavors, yielding a variety of conditions each of which affords an existence theorem for an associated type of social equilibrium. Furthermore, none of these sufficient conditions implies the other, so the existence theory for social equilibrium in dynamic social systems does not reduce to a single theorem as it did in the static case – at least, this author is not able to assert such a reduction at this time.

To compactify the statement of the existence theorems for dynamic social equilibrium, it is convenient earlier to have defined types of social systems accordingly as they satisfy certain conditions. In this way the sufficient conditions for existence of an equilibrium are collected under each "type", and the hypotheses of the existence theorems are shortened to become assumptions that the social system is of one type or another. Also, collecting

all of these types in one subsection is intended to facilitate their comparison with each other. The following section is thusly motivated.

### 4.3.1   Types of Dynamic Social System

It helps the exposition to collect at the outset all the common features of the dynamic social systems to be considered here. The first common feature of these is that the behavior space $X_\alpha$ of each behavior $\alpha$ is assumed to be non-empty compact and convex in some normed real linear topological space $L_\alpha$:

(4.3.1.1) $$X_\alpha \in kQ(L_\alpha) \quad (\alpha \in A).$$

Secondly, the effective utility function, $\ddot{w}_\alpha$, of each behavior $\alpha \in A$ is assumed to be continuous (on X) and (its restriction) quasi-concave on $X_\alpha \times \{x^\alpha\}$ for each $x^\alpha \in X^\alpha$ :

(i)   $\ddot{w}_\alpha : X \to R$ is continuous,

(4.3.1.2)                                                                $(\alpha \in A)$.

(ii)   $\ddot{w}_\alpha : X_\alpha \times \{x^\alpha\} \to R$ is quasi-concave

$$(x^\alpha \in X^\alpha)$$

Finally, the feasibility space $D_\alpha$ of each behavor $\alpha \in A$ is assumed to be the Hausdorff metric space (see 4.3.1.5) of all

non-empty closed (hence compact) and convex subsets of $X_\alpha$:

(4.3.1.3) $\qquad\qquad D_\alpha = CQ(X_\alpha) \qquad\qquad (\alpha \ \epsilon \ A).$

This gives simplicity to the following definition.

4.3.1.4  $\underline{\text{Definition}}$:  A social system satisfying 4.3.1.1-3 is classified

as $\underline{\text{type I}}$ iff the feasibility transformation

$$t_\alpha: \ \underset{A}{\Pi} \ CQ(X_\alpha) \ \times \ X \ \rightarrow \ CQ(X_\alpha)$$

is continuous for each behavor $\alpha \ \epsilon \ A$

4.3.1.5  $\underline{\text{LEMMA}}$:  If X is compact and convex in a metric linear space

and C(X) has the Hausdorff metric [or, equivalently (see 3.1.2.13),

the finite] topology, then C(X) and its subspace CQ(X) are both

compact and convex.

$\underline{\text{Proof}}$:  The compactness of C(X) follows by 3.1.2.12.  The

convexity of C(X) follows by the continuity of scalar multi-

plication and vector addition in the linear topological space

where X lies.  Since convex combinations of convex sets are

convex, it follows also that $CQ(X) \subset C(X)$ is convex, and the

following simple convergence argument establishes what is

needed.  Let $\{{}^i k\} \ {}_{i=1}^{\infty} \ \rightarrow \ {}^* k$ be a converging sequence of points

${}^i k \ \epsilon \ CQ(X)$.  Then ${}^* k \ \epsilon \ C(X)$, since C(X) is closed.  Let $*x$,

$*x' \ \epsilon \ *k$, and denote an arbitrary convex combination

$\lambda *x + \lambda' *x'$ by $*\bar{x}$ ($\lambda = 1 - \lambda' \in [0,1]$). Then there exist

sequences $\{^i x\}_{i=1}^{\infty} \to *x, \{^i x'\}_{i=1}^{\infty} \to *x'$ with $^i x, {}^i x' \in {}^i k$

($i = 1,2, \ldots$). Since $^i k$ is convex, $\lambda {}^i x + \lambda' {}^i x' = {}^i \bar{x} \in {}^i k$

($i = 1, 2, \ldots$), so that $\{^i \bar{x}\}_{i=1}^{\infty} \to *\bar{x}$. Hence, $*\bar{x} \in *k$, proving

that $*k$ is convex and, thus, that $CO(X)$ is closed.

#

In the next two definitions one is able to relax the

condition on the feasibility transformations by restricting the

effective utility functions in alternate ways.

4.3.1.6   Definition:  A social system satisfying 4.3.1.1-3 is classified

as type II  iff, for each behavor $\alpha \in A$, the effective utility

function $\ddot{w}_\alpha$ is "linear" (both concave and convex) and the

feasibility transformation $t_\alpha$ is usc as a point-to-set mapping

with $t_\alpha$ $(k, x) \subset CQ(X_\alpha)$ being closed and convex,

$(k \in \Pi_A CQ(X_\alpha), \quad x \in X)$.

4.3.1.7   Remark:   In this case, it is possible to view $t_\alpha$ as a

point-to-point mapping into $CQ(CQ[X_\alpha])$,   ($\alpha \in A$).

4.3.1.8   Definition: A social system satisfying 4.3.1.1-3 is classified

as type III  iff, for each behavor $\alpha \in$  A, the effective

utility function $\ddot{w}_\alpha$ is strictly quasi-concave on each $X_\alpha \times \{x^\alpha\}$

($x^\alpha \in X^\alpha$) and the feasibility transformation $t_\alpha$ is as in type II.

Finally, these last two restrictions on the effective

utility functions can be eliminated with appeal to an interesting

restriction on the feasibility transformations.

4.3.1.9    <u>Definition</u>:  A mapping $\xi$: X → Y of a convex set X into a

convex set Y is called a <u>convex process</u> iff its graph

$\Gamma(\xi) \subset$ X × Y  is convex.

4.3.1.10   <u>Definition</u>:  A social system satisfying 4.3.1.1-3 is classified

as <u>type IV</u> iff the collective feasibility transformation is an

usc convex process.

4.3.1.11   <u>Remark</u>:  Of course, a collective feasibility transformation

t is a convex process iff each coordinate feasibility

transformation is a convex process.

This completes the  present classification of social

systems into types. It is clear that, if the feasibility

transformations are all constant maps, a social system

of type I - IV is slightly more restricted than that

dealt with in section 4.2.  The restriction comes from the

fact that each $X_\alpha$ is now assumed to lie in a <u>normed</u> (real

linear topological) space, while before it was only

assumed to lie in a <u>locally convex</u> (real linear topological)

space.  Thus, by taking on this restriction, the constancy

assumption for the feasibility transformations has  been

eliminated.

Before turning in section 4.3.3 to prove the existence
of equilibrium for all four types of social system defined,
the next brief section will identify types of evolutions and
equilibria corresponding to these types of social system.

### 4.3.2    Types of Evolution and Equilibrium for Dynamic Social Systems

Corresponding to the types of social system     defined
in 4.3.1, a set of mappings will now be defined.  Corresponding
to each such mapping, called an "evolution", a type of social
equilibrium will be identified as a fixed point of that mapping.
It is such types of fixed point which will be shown in the next
section to exist.

4.3.2.1    <u>Notation:</u>  Let $k$, $\ell$ denote generic elements of $\underset{A}{\Pi} CO(X_\alpha)$ and let
$x$, $y$ denote generic elements of $X$. Denote $\pi_{CO(X_\alpha)}(k) = k_\alpha$ ,
$\pi_{CO(X_\alpha)}(\ell) = \ell_\alpha$, $\pi_{X_\alpha}(x) = x_\alpha$ , $\pi_{X^\alpha}(x) = x^\alpha$, $\pi_{X_\alpha}(y) = y_\alpha$.
Also denote $Z_\alpha = CO(X_\alpha) \times X_\alpha$ , $Z = \underset{A}{\Pi} Z_\alpha$, $\underline{A}(k,x) = \underset{A}{\Pi}\alpha(k_\alpha, x^\alpha)$.
Finally, denote generic elements of $Z$ also by $z$.

4.3.2.2    <u>Definition:</u>   The <u>evolution</u> of a type I social system is
a  transformation $E: Z \to Z$ defined for $z \in Z$ by

$$E_I(z) = \{t(z)\} \times \underline{A}(t(z), x):$$

it is classified as <u>type I</u>.  The <u>evolution</u> of a type II
social system is a transformation $F_{II}: Z \to Z$ defined, for

z $\epsilon$ Z, by

$$E_{II}(z) = t(z) \times \underline{A}(t(z),z);$$

it is classified as type II. The <u>evolutions</u> of a type

III social system are transformations $E_{III}:Z \to Z$ and

$E_{IV}: Z \to Z$ defined, for $z = (k,x) \epsilon Z$ by

$$E_{III}(k,x) = (\prod_{A} t_{\alpha}(k, \alpha(k_{\alpha},x^{\alpha}),x^{\alpha})) \times \underline{A}(k,x),$$

$$E_{IV}(k,x) = t(\{k\} \times \underline{A}(k,x)) \times \underline{A}(k,x),$$

respectively; $E_{III}$ is classified as <u>type III</u> and $E_{IV}$ as

<u>type IV</u> . The evolution of a type IV social svstem is a

transformation $E_{IV}:Z \to Z$ defined and classified as above.

4.3.2.3   <u>Definition</u>:   A fixed point of an evolution is called a <u>social</u>

<u>equilibrium</u> (or <u>social contract</u>) and classified according to the

type of evolution of which it is a fixed point.

So much preparation finally allows turning to the existence

of social equilibrium in the dynamic case.

## 4.3.3   Existence of Dynamic Social Equilibrium

In this section it is shown that social systems of types
I - IV have equilibria.  In particular, it is shown that the
(type i) evolution of a type i social system has a (type i)
social contract for i $\epsilon$ {I, II, III, IV }, and that a type III
social system has, in addition, a type IV social contract.

In each of the existence results, the following fixed
point theorem is used:

4.3.3.1   <u>THEOREM [Prakash and Sertel]</u>:  Let  $\{Z_\alpha | \alpha \epsilon$  A} be a family
of $2^o$ convex, compact and convex spaces, and let  $\{E_\alpha$ :

$Z \rightarrow Z_\alpha \mid \alpha \epsilon$  A}be a family of usc point-to-set mappings on

$Z = \underset{A}{\Pi Z_\alpha}$ such that $E_\alpha(z) \epsilon CQ(Z_\alpha)$ $(z \epsilon Z, \alpha \epsilon A)$.  Define

E: $Z \rightarrow Z$ by $E(z) = \underset{A}{\Pi} E_\alpha(z)$ $(z \epsilon Z)$.  Then there exists a

(fixed )point $z \epsilon Z$ such that $z \epsilon E(z)$.  [1970, 3.16 Theorem VI].

In each application of this theorem, $Z_\alpha$  will be taken
to be $CQ(X_\alpha) \times X_\alpha$, as indicated in 4.3.2.1.  In this case, both
$CQ(X_\alpha)$ and $X_\alpha$ are easily seen to be $3^o$ convex, so that
their product $Z_\alpha$ is $3^o$ convex, hence $2^o$ convex.

It is worth noting that the fixed point theorem of Fan, used
in 4.2, cannot be applied in this case, as $Z_\alpha = CQ(X_\alpha) \times X_\alpha$ lies in
a <u>semi</u>-linear topological space while it must be assumed to lie
in a linear topological space for Fan's mentioned theorem to apply.

Thus, an extension of Fan's theorem is needed, and 4.3.3.1

provides just the desired sort of extension. Actually, other

fixed point theorems of Prakash and Sertel [1970] can safely

be conjectured to suffice for some of the above types of

social system, but 4.3.3.1 works for all of these, as will be

seen. It is because 4.3.3.1 works for all of these cases that it is

possible to economize on the number of fixed point theorems to be

used (known).

4.3.3.2    $\underline{\text{THEOREM}}$:  Every type I social system has a type I social contract.

$\underline{\text{Proof}}$:  By 3.3.2, each behavor $\alpha \in A$ is usc in $(t_\alpha(k,x), x^\alpha)$,

with  $\alpha(t_\alpha(k,x), x^\alpha) \in KQ(X_\alpha) = CQ(X_\alpha)$ for each $(k,x) \in Z$.

Since each $t_\alpha$ and $\pi_{x^\alpha}(k,x) = x^\alpha$ is continuous

on Z, $\alpha(t_\alpha(k,x), x^\alpha)$ is usc on Z. Hence, $\{t_\alpha(k,x)\} \times \alpha(t_\alpha(k,x), x^\alpha)$

is usc on Z, while $\{t_\alpha(k,x)\} \times \alpha(t_\alpha(k,x), x^\alpha) \in CQ(Z_\alpha)$

$[(k,x) \in Z, \alpha \in A]$.  Also, $Z = \underset{A}{\Pi} Z_\alpha$ is the product of $2^O$ convex

compact and convex spaces $Z_\alpha$ , and

$$E_I(k,x) = \underset{A}{\Pi} (\{t_\alpha(k,x)\} \times \alpha(t_\alpha(k,x), x^\alpha))$$

Thus, by 4.3.3.1, there exists a type I social contract

$z \in E_I(z) \subset Z$.

#

4.3.3.3  <u>THEOREM</u>:  Every type II social system has a type II social

contract.

<u>Proof</u>:  Since each $t_\alpha$ is usc, each $\alpha\,(t_\alpha(k,x),\ x^\alpha)$, as a

composition of usc mappings, is usc on Z.  Thus, $t(k,x) \times_\alpha (t_\alpha$

$(k,x),\ x^\alpha)$, for each $\alpha \in A$, is usc on Z.  Since $F_{II}(k,x) =$

$\underset{A}{\Pi}E_{II}$ , where

$$E_{II\alpha}(k,x) = t_\alpha(k,x) \times \alpha(t_\alpha(k,x),x^\alpha),$$

to apply 4.3.3.1 it remains only to show that $E_{II}(k,x) \in CQ(Z\ )$

for each $\alpha \in A$.  Since $t_\alpha(k,x) \in CQ(CQ(X_\alpha))$ by hypothesis,

it suffices to show that $\alpha(t_\alpha(k,x),\ x^\alpha) \in CQ(X_\alpha)$.  In fact,

as the graph $\Gamma(\alpha) \subset CQ(X_\alpha) \times X^\alpha \times X_\alpha$ is compact by 3.3.1,

and $t_\alpha(k,x)$ is closed in the compact $CQ(X_\alpha)$, it follows that

$t_\alpha(k,x) \times \{x^\alpha\} \subset CQ(X_\alpha) \times X^\alpha$ is compact, implying that

$\alpha(t_\alpha(k,x),\ x^\alpha)$ is compact, hence closed.  This leaves only

the convexity of $\alpha(t_\alpha(k,x),\ x^\alpha)$ to prove.  Let $x_\alpha$ , $x'_\alpha \in$

$\alpha(t_\alpha(k,x),\ x^\alpha)$.  Then $\ddot{w}_\alpha(x_\alpha,x^\alpha) = \bar{\ddot{w}}_\alpha(\ell_\alpha,\ x^\alpha)$ and $\ddot{w}_\alpha(x'_\alpha,\ x^\alpha) =$

$\bar{\ddot{w}}_\alpha(\ell'_\alpha,\ x^\alpha)$, for some $\ell_\alpha$, $\ell'_\alpha \in t_\alpha(k,x)$.  By 3.1.4.6, linearity

of $\ddot{w}_\alpha$ implies that of $\bar{\ddot{w}}_\alpha$ , so that $\bar{\ddot{w}}_\alpha\ (\bar{\ell}_\alpha,\ x^\alpha) = \lambda\bar{w}(\ell_\alpha,x^\alpha) +$

$\lambda'\bar{w}(\ell'_\alpha,x^\alpha)$ for all $\lambda = 1 - \lambda' \in [o,1]$, where $\bar{\ell}_\alpha = \lambda\ell_\alpha + \lambda'\ell'_\alpha$ .  But,

by linearity of $\ddot{w}_\alpha$ , $\ddot{w}_\alpha(\bar{x}_\alpha,\ x^\alpha) = \lambda\ddot{w}_\alpha(x_\alpha,\ x^\alpha) + \lambda'\ \ddot{w}_\alpha(x'_\alpha,x^\alpha) =$

$\bar{\ddot{w}}\ (\bar{\ell}_\alpha,\ x^\alpha)$, for $\bar{x}_\alpha = \lambda x_\alpha + \lambda'x'_\alpha$ .  Hence, $\bar{x}_\alpha \in \alpha(t_\alpha(k,x),\ x^\alpha)$,

showing all that was required.

4.3.3.4   <u>THEOREM</u>:   Every type III social system has (1) a tvpe III

and (2) a type IV social contract.


<u>Proof</u>:


<u>ad(1)</u>: Since the effective utility function $\ddot{w}_\alpha$ is strictlv

concave on $X_\alpha \times \{x^\alpha\}$ for each $x^\alpha \in X^\alpha$, $\alpha(k_\alpha, x^\alpha)$ is singleton

for each $k_\alpha \in CQ(X_\alpha)$, $x^\alpha \in X^\alpha$, $\alpha \in A$. Then $t_\alpha(k, \alpha(k_\alpha, x^\alpha), x^\alpha)$

is the image under $t_\alpha$ of a point $(k, \alpha(k_\alpha, x^\alpha), x^\alpha) \in Z$, hence it

is non-empty, compact and convex in $CQ(X_\alpha)$, by hypothesis, for

each such point $z \in Z(\alpha \in A)$. Thus, for each $z = (k,x) \in Z$,

$$E_{III_\alpha}(z) \in CQ(CQ(X_\alpha)) \times \{\{x_\alpha\} | x_\alpha \in X_\alpha\} \subset CQ(CQ(X_\alpha) \times \{\{x_\alpha\} | x_\alpha \in X_\alpha\}),$$ which is homeomorphic to $CQ(Z_\alpha)$, where, for each $\alpha \in A$,

$$E_{III_\alpha}(z) = t_\alpha(k, \alpha(k_\alpha, x^\alpha), x^\alpha) \times \alpha(k_\alpha, x^\alpha).$$

But the map $E_{III_\alpha} : Z \to CO(Z_\alpha)$ is usc, as $t_\alpha$ is so and $\alpha$ and

all projections are continuous ($\alpha \in A$), and $E_{III}(z) = \prod_A E_{III_\alpha}(z)$

for any $z \in Z$. Hence, 4.3.3.1 applies, yielding that there

exists a type III social contract $z \in Z$ such that $z \in E_{III}(z)$.


<u>ad(2)</u>: Defining

$$E_{IV_\alpha}(k,x) = t_\alpha(\{k\} \times \underline{A}(k,x)) \times \alpha(k_\alpha, x^\alpha)$$

for each $\alpha \in A$ and $z = (k,x) \in Z$, $E_{IV}(z) = \prod_A E_{IV_\alpha}(z)$. Since

each $\alpha$ is usc, so is the map $\underline{A}$. As each $\alpha(k_\alpha, x^\alpha)$ is

singleton, so is $\underline{A}(z)$ $(z \in Z)$. Hence, $\{k\} \times \underline{A}(k,x) \in Z$

for each $(k,x) \in Z$. Thus, $E_{IV_\alpha} : Z \to CQ(CQ(X_\alpha)) \times X_\alpha$ has been

defined for each $\alpha \in A$, and the range is a subspace of

$CQ(Z_\alpha)$, so that $E_{IV}$ is the product of usc maps $E_{IV} : Z \to CQ(Z_\alpha)$.

By 4.3.3.1, there exists a type IV social contract $z \in Z$ such

that $z \in E_{IV}(z)$.

$\#$

4.3.3.5    THEOREM: Every type IV social system has a type IV social

contract.

Proof: For each $\alpha \in A$ define the map $E_{IV_\alpha}$ (as in the last

proof) by

$$E_{IV_\alpha}(k,x) = t_\alpha(\{k\} \times \underline{A}(k,x)) \times \alpha(k_\alpha, x^\alpha).$$

By the upper semi-continuity of each $\alpha$, $\underline{A}$ is usc, so that each

$E_{IV_\alpha}$ is usc, as each $t_\alpha$ is so by hypothesis. As $\alpha(k_\alpha, x^\alpha)$ is

non-empty, compact and convex for each $k_\alpha \in CQ(X_\alpha)$, $x^\alpha \in X^\alpha$,

$\alpha \in A$, it follows that $\underline{A}(k,x)$, hence, $\{k\} \times \underline{A}(k,x)$, has the

same properties for each $(k,x) \in Z$. Since $t$ is a convex

and usc process, it follows that each $t_\alpha$ has a convex and

compact graph $\Gamma(t_\alpha) \subset Z \times CO(X_\alpha)$, as both Z and $CQ(X_\alpha)$ are

compact (and convex). Thus, $t_\alpha(\{k\} \times \underline{A}(k,x))$ is non-empty,

compact and convex, for each $(k,x) \in Z$, as it is the set

$$\pi_{CQ(X\alpha)}(\Gamma(t_\alpha) \cap (\{k\} \times \underline{A}(k,x) \times CQ(X_\alpha))).$$

Hence, $E_{IV_\alpha}(z)$ is non-empty, compact and convex in Z, for

each $z \in Z$ and $\alpha \in A$. As $E_{IV}$ is nothing but the map defined

by $E_{IV}(z) = \bigcap_{A} E_{IV_{\alpha}}(z)$, 4.3.3.1 yields the result that there

exists a type IV social contract $z \in Z$ such that $z \in E_{IV}(z)$.

#

## 4.3.4    Compactness and Convexity Results for Dynamic Contractual Sets

In this section it is shown that the contractual sets,

i.e., sets of social contracts, shown to be non-empty in

the last section are all compact, and that the contracutal set

consisting of type IV social contracts is also convex for a social

system which is both type III and type IV.

4.3.4.1    **THEOREM:** Let C stand for the contractual set shown to be

non-vacuous in any one of the theorems 4.3.3.2-3, 4.3.3.4.1-2

and 4.3.3.5 above.  Then C is compact.

**Proof:** Let $E$ by any one of the evolutions $E_i$ (i.e.{ I, II, III,

IV }).  Then E is usc, as each $E_{\alpha}$ was shown to be so ($\alpha \in A$).

Then the graph

$$\Gamma(E) = \{(z, z') \mid z \in Z, \; z' \in E(z)\}$$

is compact as a closed subset of the compact $Z \times Z$.  Denoting

the diagonal $\{(z, z') \mid z = z' \in Z\}$ by $\Delta$, $\Delta$ is compact and C

is nothing but

$$C = \pi_Z(\Gamma(E) \cap \Delta),$$

hence compact.

#

4.3.4.2    <u>THEOREM</u>:   Let S be a social system which is both

type III and type IV, and let C be the set of type IV

social contracts of S.   Then C is convex.

Proof:   As Z is convex, so is the diagonal $\Delta$  of Z × Z. Hence

it suffices to show that the graph $\Gamma(E_{IV})$ is convex, as C =

$\pi_Z(\Gamma(E_{IV}) \cap \Delta)$.   It is obvious that $\Gamma(E_{IV})$ is convex if the

graph $\Gamma(\underline{A})$ is convex, for the graph $\Gamma(t)$ is convex by

hypothesis.   Furthermore, $\Gamma(\underline{A})$ is convex if the graph $\Gamma(\alpha)$ of

each behavor $\alpha \in A$ is convex.   It is tedious but

straightforward to show that $\Gamma(\alpha)$ is convex, using the

hypothesis that $\ddot{w}_\alpha$ is linear ( $\alpha \in A$).

#

5.  EXTENSIONS AND APPLICATIONS

This chapter will identify a selected number of directions

in which the framework and theory of the previous chapters can

fruitfully be extended, also indicating application areas.

One large extension is into probabilistic social systems,

occupying the first section. In that section a particular notion

of behavior as a probability measure on a sigma-field of actions is

developed, the numerical representability of preferences on sets

of such behaviors is discussed, and the notion of probabilistic

social system constructed.

The next section proposes a couple of axioms as necesaarily

satisfied by a causal relation, and applies the resulting notion,

of an event 'inducing' an event, by building on it a notion of

power in probabilistic social systems. The result is compared

with Dahl's [1957] concept of power and the importance of

equilibrium methods for power analysis is pointed out.

The attraction and stability properties of equilibrium sets

and cores is the topic of the next section. The required concepts

are presented, as borrowed and modified, from the theory of

dynamical systems.

Finally, the study is closed by a discussion of a number of

extensions and applications particularly relevent for the management

of organizations. Organizations are defined, optimal incentive

problems posed, the choosing of incentive schemes related to the

act of legislation, and the relevance of equilibrium methods for all of these clarified. Three main types of control, remunerational, socializational and informational, are illustrated. Extending the detail in defining impression functions, it is shown how information-systemic elements can be incorporated into the model. Finally, the direction of multi-level social systems is pointed to as an area into which the model can be extended.

## 5.1  Towards the Analysis of Probabilistic Social Systems

Along lines earlier suggested in [Sertel, 1969 a, b], probabilistic social systems will now be developed as an important extension of the social systems so far studied.  The next section, 5.2., illustrates one of the motivating reasons for studying probabilistic social systems.

In 5.1.1 the notion of a behavior is particularized to that of a probability measure on a sigma-field of actions.  5.1.2 settles matters pertaining to the measurable numerical representability of preferences on behavior spaces when the preferences are originally specified on a sure action ( a set of "sure prospects").  Then 5.1.3 finally assembles probabilistic social systems on the basis of this groundwork.

## 5.1.1   Action and Behavior

The notion of 'behavior' introduced in 2.1 and used up to

here is extremely general and abstract. In this section a more

particular and concrete version of that notion will be constructed,

founded on a certain notion of 'action'.  This construction will,

in turn, serve as a foundation for the treatment of "probabilistic

social systems", after certain questions relating to the rep-

resentation of preferences are dealt with in the next section.

This and the next two sections  will thus give  an extension

of part of the framework offered in (Sertel,1969 a).

The term 'individual' and phrases such as "things which

an individual can do" will be formally undefined here; they

are to be understood in the natural language sense.  This

prepares the ground for what follows.


5.1.1.1   Definition: The sure action $_o\theta_J$ of an individual j is

the set of all mutually exclusive things which j can do.

A sigma-field $\theta_j$ of subset of $_o\theta_j$ will be called a sigma-field

of actions (or, for short, an action field) of j iff it

contains the finest partition  of $_o\theta_j$ :


$$\{ \theta_j = \{\omega_j\} \ |\omega_j \ \varepsilon \ _o\theta_j\} \ \ \theta_j.$$


There will always be assumed to be a unique non-trivial

action field $\Theta_j$ with which an individual is associated.  A

subset $\theta_j \subset {}_o\theta_j$ will be called an <u>action</u> of j iff $\theta_j \in \Theta_j$.

5.1.1.2 <u>Definition</u>: A probability measure $p_j: \Theta_j \to [0,1]$ defined

on the action field $\Theta_j$ of an individual j will be called

a <u>behavior</u> of j. The set $P_j$ of all behaviors of j will be

called the <u>behavior space</u> of j.

5.1.1.3 <u>Definition</u>: A <u>collectivity</u> is an ordered pair $\langle W, J \rangle$, where

$W = \{P_j | j \in J\}$ is the family of behavior spaces $P_j$ of the

individuals $j \in J$, and where J is a non-empty collection

of individuals.

5.1.1.4 <u>Remark</u>: Since each action field is non-trivial, each behavior

space is non-empty. Compare 5.1.1.3 and 5.1.1.5 with 2.1.2-3.

5.1.1.5 <u>Definition</u>: The <u>sure joint action</u> of a collectivity

$\langle W, J \rangle$ is the product ${}_o\theta_J = \prod_J {}_o\theta_j$ of the sure actions

${}_o\theta_j (j \in J)$. The <u>sigma-field of joint actions</u> (or, for

short, <u>joint action field</u>) of $\langle W, J \rangle$ is the product

$\Theta_J = \prod_J \Theta_j$ of the action fields $\Theta_j (j \in J)$. A subset

$\theta_J \subset {}_o\theta_J$ is called a <u>joint action</u> of $\langle W, J \rangle$ iff $\theta_J \in \Theta_J$.

5.1.1.6 <u>Definition</u>: A (joint) probability measure $p_J: \Theta_J \to [0,1]$

defined on the joint action field $\Theta_J$ of a collectivity

$\langle W, J \rangle$ will be called a <u>joint behavior</u> or <u>state</u> of $\langle W, J \rangle$.

The set $P_J$ of all such measures will be called the <u>joint</u>

<u>behavior space</u> of <u>state space</u> of $<W,J>$.


5.1.1.7    <u>Definition</u>: The <u>collective behavior space</u> of a collectivity

$<W, J>$ is the product $P = \prod_J P_j$ of the behavior spaces

$P_j (j \in J)$. A function $p: J \to \bigcup_J P_j$ in P is called a <u>collective</u>

<u>behavior</u> of $<W,J>$ iff $p(j) \in P_j$ $(j \in J)$, i.e., iff $p \in P$. In

this case, $p(j)$ is also denoted by $p_j = p(j) \in P_j$. (Cf.2.1.2-3).


5.1.1.8    <u>Remark</u>: The distinction between <u>joint</u> behaviors (or states)

and <u>collective</u> behaviors of a collectivity is crucial. To

the probabilist it will already have been clear that a behavior

is simply a certain marginal of a joint behavior and a collective

behavior is simply a specification of all such marginals for

a joint behavior. Thus, there is a unique collective behavior

specifying these marginals of a joint behavior, whatever

joint behavior is given. Specifying a collective behavior,

however, determines either not more than one joint behavior

or not less than the cardinality of-the-continuum joint

behaviors. (This is so, for, if two joint behaviors have the

same specified marginals, then a continuum of convex

combinations also satisfy this condition, i.e., the set

of joint behaviors consistent with a collective behavior

is convex). That there exist a joint behavior given a

collective behavior is governed, of course, by the

satisfaction of the Kolmogoroff consistency conditions.

(Kolmogoroff, 1933). (See also (Kingman and Taylor, 1966)

and (Parthasarathy, 1967).)

## 5.1.2 The Representation of Preference

In the social systems dealt with up to the present chapter,
no mention was mde of how it was that the utility functions
used were guaranteed to exist as numerical representations of
(complete) preference orderings on the sets in question. This
is because all of the properties which were assumed, at one stage
or another, to hold for these functions were properties which
have been proved in the literature to be assumable without any
loss of generality when certain conditions are met, and these
sufficient conditions happened always to be satisfied whenever
needed.

Main reference in the literature is to the representation
theory of Debreu (1954) and of Herstein and Milnor (1953). For
all of the results developed so far, the utility functions were
assumed to have certain continuity and convexity properties. The
domains of definitions for these functions were always compact
and convex. Assuming the necessary and sufficient conditions
demonstrated by Herstein and Milnor and presented below,
convexity of the domain guarantees the existence of a real-valued
linear function preserving the order of the given preferences.
If the domain is topologized by the order of the preference
relation, this function is easily seen also to be continuous.

Thus, all that was ever assumed so far can be seen to be
assumable, using the convexity of the set ordered by preference

and the results of Herstein and Milnor. In certain cases,
Debreu's theory could have been used as an alternative,
e.g., when the domain was compact in a metric space, hence
satisfying the second axiom of countability (see Debreu's
Theorem 2).

In building toward probabilistic social systems, studied
in the next section, where the behavior spaces are of the special
kind introduced in the last section, matters of representation
of preferences are less straightforward; hence, the present
section.  The exact questions which are addressed here will be
stated shortly, after some minimal preparation.

Let a collectivity <W, J> be given.  Fix attention to a
specific individual $j \in J$, and let ${}_o\theta_J \times R$ be completely
ordered by a preference relation $\leq j$ summarizing $j$'s preferences
between elements of this set.

The first question addressed now is the following:
Under what conditions does there exist a function

$$\nu_j : {}_o\theta_J \times R \to R$$

such that

$$u_j(p_J, r) = \int_{{}_o\theta_J} \nu_j(\omega, r) dp_J(\omega)$$

exists for all $(p_J, r) \in P_J \times R$, and, such that, identifying each

$z \in {}_0\theta_J$ with the degenerate $p_J^z \in P_J$ assigning 1 to the set

$\{z\} \in O_J$ and 0 to the rest of $\theta_J$, the equivalence

$$u_j(p_J^z, r) \geq u_j(p_J^{z'}, r) \text{ iff } (z,r) \geq_j (z',r')$$

holds for all $(z,r), (z',r') \in {}_0\theta_J \times R$?

This question will be answered by use of a result due to

Herstein and Milnor.

Herstein and Milnor [1953] have demonstrated a necessary

and sufficient triplet of conditions for the existence of a

real-valued, linear function on a set M ordered completely

by a relation $\leq_j$ , such that the function preserves this order.

These conditions are:

(1)  M is a "mixture set";

(2) for all a, b, c, $\in$ M, the following sets are closed:

$$\{\lambda \in [0,1] | \lambda a + (1- \lambda)b \geq_j c\} ;$$

$$\{\lambda \in [0,1] | c \geq_j \lambda a + (1 -\lambda)b\} ;$$

(3) if a, b, c $\in$ M with $a =_j b$ (where, for all d, e $\in$ M,

'$d =_j e$' denotes '$d \leq_j e$ and $e \leq_j d$'), then

$$\frac{1}{2} a + \frac{1}{2}c =_j \frac{1}{2}b + \frac{1}{2}c$$

A "mixture set" here is a generalization of a convex set in a real vector space.

Now this result will be used to obtain some reasonable conditions under which the desired sort of function $v_j$ will exist. First, it is assumed that the set $_o\theta_J$ is convex in a real vector space. It follows that $_o\theta_J \times R$ is convex, hence a "mixture set". Next, assume that the conditions (2) and (3) are met by $M = {}_o\theta_J \times R$. Then there exists a linear real-valued function $v_j: {}_o\theta_J \times R \to R$ preserving $\leq_j$. It will now be seen that this function $v_j$ has all the desired properties, after a few more assumptions are made.

Topologize $_o\theta_J \times R$ with the order topology $\leq_j$. That is, topologize $_o\theta_J \times R$ with the coarsest topology for which

$$\{\{b \; \varepsilon \; _o\theta_J \times R | b \geq_j a\}, \{b \; \varepsilon \; _o\theta_J \times R | a \leq_j b\} | a \; \varepsilon \; _o\theta_J \times R\}$$

is a family of closed sets. (Note: This is not the order topology defined by Eilenberg (1941). The definition of Eilenberg would correspond to the quotient space where elements of an equivalence class according to the order are not distinguished, even though they may be distinct in $_o\theta_J \times R$. Although the quotient space of the space defined here is always $T_1$ (in fact, $T_2$, i.e., Hausdorff), this is not true of the space itself, as can be seen from the fact that if $b \neq a$ but $b \geq_j a$ and $a \geq_j b$, then there is no neighborhood of b(a) of which a (b) is not an element. The course of definition chosen here is motivated by the need,

in the present context, to keep distinctness of points distinct from non-equivalence of points).It is obvious that $\nu_j$ is then continuous. Hence, $\nu_j$ is continuous whenever $_o\theta_J \times R$ has a topology finer than the order topology of $\leq_j$ .

The final assumption to be made concerning $_o\theta_J \times R$ is that $_o \theta_J$ is a quasi-compact $T_1$ (topological) subspace of a linear topological space. The first consequence of this comes from the fact that $_o\theta_J \times R$ is $T_1$, i.e., that the singleton subsets $\{a\}$ $_o\theta_J \times R$ are closed. The mentioned consequence of this is that $\nu_j$ is continuous. This is so, for a $T_1$ topology on $_o\theta_J \times R$ = M is finer than the order topology. To see that a $T_1$ topology is finer than the order topology, take a, c $\epsilon$ M such that c $\notin$ $\{b \epsilon M | b \geq_j a\}$ and note that the complement of $\{b \epsilon M | b \geq_j a\}$ is open in the $T_1$ topology by the fact that c $\geq_j$ a implies c to be distinct from a, whereby there is a nbd of c (in the $T_1$ topology) which does not meet $\{b \epsilon M | b \geq_j a\}$ .

Since $_o\theta_J \times R$ fails to be quasi-compact, however, it does not follow from the continuity of $\nu_j$ that $\nu_j$ is bounded. This is a serious deficiency, as boundedness coupled with continuity of $\nu_j$ would guarantee its integrability with respect to each $p_j \epsilon P_J$, as desired. It is this deficiency to which a remedy will now be sought.

Let $\ddot{\gamma}_j$: $_o\theta_J \rightarrow R$ represent a typical element of the set of linear real-valued functions on $_o\theta_J$ for which $\ddot{\gamma}_j(_o\theta_J) \subset B$, where B is a fixed (bounded) closed interval $[r_1, r_2]$.

(The functions $\ddot\gamma_j$ are analogous to the (interpreted) incentive functions $\ddot g_\alpha : X \to R$ of the previous chapters; the symbol $\ddot g_j$, however, is reserved here for a different but closely related function.) By the fact that $\ddot\gamma_j$'s are restricted to have their ranges contained in the bounded set B, it is being assumed that no infinitely large rewards or punishments are distributed in the social systems about to be considered.

From this assumption an important consequence will now be obtained. First to be noted is that each $\ddot\gamma_j$ is obviously bounded. Actually, this follows from the fact that $_0\theta_J$ is quasi-compact and that, by its linearity, $\ddot\gamma_j$ is continuous, as $\ddot\gamma_j(_0\theta_J)$ is therefore quasi-compact, hence bounded as a subset of R. But, also, $\nu_j$ may now be taken to be a function $\nu_j :_0\theta_J \times B \to R$, since its restriction to $_0\theta_J \times B$ is all that matters in the social system which the present development is obviously heading toward.

The result of this is that the function

$$u_j(p_J, r) = \int_{_0\theta_J} \nu_j(\omega, r)\ dp_J(\omega)$$

is now well-defined, having imposed the constraint $r \in B$, as $\nu_j$ is continuous and bounded on $_0\theta_J \times B$, hence integrable with respect to each $p_J \in P_J$. The property desired for $\nu_j$ with respect to degenerate elements of $P_J$ is satisfied by its linearity. Thus, $u_j$ now serves as a "utility function" for the individual j, the domain of definition being $P_J \times B$, where $P_J$ is the joint behavior space of the collectivity $<W, J>$ .

Before turning to the next section, a natural continuation of the above development to be settled here is deriving incentive functions for the individuals in J.  Again, attention will be fixed to the individual $j \in J$.

Define the function $\ddot{g}_j$: $P_J \to R$ by

$$\ddot{g}_j(P_J) = \int_{o}^{\theta_J} \ddot{\gamma}_j(\omega) \, dp_J(\omega).$$

Since $\ddot{\gamma}_j$ is continuous and bounded, $\ddot{g}_j$ is well defined. Now it is clear that this function, $\ddot{g}_j$, will play the role of $\ddot{g}_\alpha$ (in the previous chapters ) for the behavor j of the coming section.

### 5.1.3   Probabilistic Social Systems

Building on the groundwork provided in the last two

sections, it is now possible to define probabilistic social

systems as a variety of the general social system introduced at

the outset of the study.   This will provide an extension of a formal

entity earlier introduced in (Sertel, 1969 a).   The mentioned

formal entity (although called a "collectivity" therein) was

a finite-personnel version of what is about to be defined here

as a probabilistic social system, the behavior spaces being

closed geometric simplexes in Euclidean space, with incentive

and interpretation  schemes missing from the specification while

an impression scheme was present.   The social system specified

was shown in that and an accompanying study (Sertel, 1969 b)

to have a non-empty compact and convex set of equilibria.

The distinguishing characteristics of such equilibria in

comparison to the equilibrium shown by Nash (1950, 1951) to

exist for games of a similar specification consisted of two

components.   Firstly, impression functions were not explicit

- or they were implicitly assumed to be identity maps -

in Nash's specification.   Secondly, the equilibria of Nash were

collective behaviors, while those of Sertel were joint behaviors,

referring to the terminology and very important distinction

(see 5.1.1.8) introduced in 5.1.1.

The importance of being able to deal with joint behaviors
is most clear when an extension of the present study to an
investigation, e.g., of the core is considered. For
the joint randomization of the actions of members has to
be considered as in the very nature of a coalition when the
notion of behavior is probabilistic. This point must therefore
be emphasized as crucial also to any political analysis, if
any, which is to benefit from the methods suggested by the
present study, since it may be expected to be essential especially
in political analysis to be realistic about the workings of
coalitions. The theory of power suggested later in this
study is anticipated in the last remark.

A motivation for defining and studying probabilistic
social systems should be easily extractible from the components
above. With minor effort, relying on the previous sections,
a definition will soon be formalized. The major portion of this
effort has to be directed toward constructing a number of
functions. This is now taken up.

5.1.3.1   <u>Definition-Notation</u>:   Let $P_J$ be the joint behavior space of
a collectivity $<W, J>$ .

Denote $_o\Theta^J = \prod_{J \{j\}} {}_o\Theta_i$ ,   denote the product sigma-field
$\prod_{J \{j\}} \Theta_i$ by $\Theta^J$, and denote the set of probability measures

$p_J^j: \Theta^J \to [0,1]$ by $P_J^j$.   Define the function $\mu_j : P_J \to P_j$ as follows:

$p_{J_j} = \mu_j(p_J)$ has the property that, for any $\theta_j \in \Theta_j$,

$$p_{J_j}(\theta_j) = p_J(\theta_j \times {}_o\theta^j);$$

$\mu_j(p_J)$ will be called the **j-marginal** of $p_J$. Define the function $\mu^j: P_J \to P_J^j$ as follows:

$p_J^j = \mu^j(p_J)$ has the property that, for any $\theta^j \in \Theta^j$,

$$p_J^j(\theta^j) = p_J({}_o\theta_j \times \theta^j);$$

$\mu^j(p_J)$ will be called the **j-exclusive marginal** of $p_J$.

5.1.3.2  <u>Definition:</u>  A function $u_j: P_J \times R \to R$ will be called a

<u>utility function</u> of j.  A function $h_j: P_J^j \to P_J^j$ will be

called an <u>impression function</u> of j.  A function $g_j: P_J \to R$

will be called an <u>incentive function</u> for j.  A function $i_j$

assigning to each incentive function $g_j$ for j an incentive

function $\breve{g}_j = i_j(g_j)$ for j will be called an <u>interpretation</u>

<u>function</u> for j.  A non-empty collection $D_j \subset [P_j]$ of non-empty

subsets $d_j \subset P_j$ will be called a <u>feasibility space</u> for j,

and each element $d_j \in D_j$ will be called a <u>feasibility</u> for j.

Let $\{d_j | j \in J\}$ be a family of feasibilities $d_j$, one for each

$j \in J$.  The set $d \subset P_J$ of all joint behaviors $p_J$ such that,

for each $j \, \epsilon \, J$, $\mu_j(p_J) \, \epsilon \, d_j$ will be called a <u>collective</u>

<u>feasibility</u> for J.  The set of all collective feasibilities

for J will be called the <u>collective feasibility space</u> of J

and denoted by D.  A mapping $t_j \colon P_J \times D \to D_j$ will be called

a <u>feasibility transformation</u> of j.  The mapping $t \colon P_J \times D \to D$

defined by

$$t(p_J, \ d) = \{q_J \, \epsilon \, P_J \, | \, \mu_j(q_J) \, \epsilon \, t_j(p_J, \ d) \ \text{for all} \ j \, \epsilon \, J\}$$

will be called the <u>collective feasibility transformation</u>

of J.


5.1.3.3   <u>Definition</u>:  Let &lt;W, A&gt; be a collectivity, let

$U = \{u_\alpha \, | \, \alpha \ \epsilon \ A\}$ be a family of utility functions,

$H = \{h_\alpha \, | \, \alpha \ \epsilon \ A\}$   a family of impression functions,

$G = \{g_\alpha \, | \, \alpha \ \epsilon \ A\}$ a family of incentive functions,

$I = \{i_\alpha \, | \, \alpha \ \epsilon \ A\}$ a family of interpretation functions,

$T = \{t_\alpha \, | \, \alpha \ \epsilon \ A\}$ a family of feasibility transformations,

and let A be a family of self-indexed mappings

$$\alpha \colon P_A \times D_\alpha \ \to \ [P_\alpha] \quad \text{defined by}$$

$$\alpha(p_A^\alpha, \ d_\alpha) = \{p_\alpha \, \epsilon \, d_\alpha \, | \, \ddot{w}_\alpha(p_\alpha *p_A^\alpha) \geq \sup_{d_\alpha} \ddot{w}_\alpha(q_\alpha *p_A^\alpha)\} \ ,$$

$(q_\alpha \ \epsilon \ d_\alpha)$ where the operator $*$ and the function $\ddot{w}_\alpha$ are

defined below. Then the ordered seven-tuplet S =

$\langle W, U, H, G, I, T, \mathbf{A} \rangle$ will be called a <u>probabilistic social</u>

<u>system</u>.

5.1.3.4 <u>Definition</u>: Let $P_j$ and $P_J^j$ be as in 5.1.3.1. The binary

operation $*$ is defined by

$$ P_j * P_J^j (\theta_j \times \theta^j) = p_j(\theta_j) \, \dot{p}_J^j(\theta^j), \quad (\theta_j \, \varepsilon \, \Theta_j, \theta^j \, \varepsilon \, \Theta^j). $$

5.1.3.5 <u>Definition</u>: Denote $i_\alpha(g_\alpha) = \ddot{g}_\alpha$ and

$$ u_\alpha(p_\alpha * h_\alpha(p_A^\alpha), \, i_\alpha(g_\alpha) \, (p_\alpha * h_\alpha(p_A^\alpha))) $$

$$ = \ddot{u}_\alpha(p_\alpha * h_\alpha(p_A^\alpha), \, \ddot{g}_\alpha(p_\alpha * h_\alpha(p_A^\alpha))) $$

$$ = \ddot{v}_\alpha(p_\alpha * h_\alpha(p_A^\alpha)) $$

$$ = \ddot{w}_\alpha(p_\alpha * p_A^\alpha). $$

The derived function $\ddot{w}_\alpha$ will be called the <u>effective utility</u>

<u>function</u> of $\alpha$.

5.1.3.6   <u>Research Problems</u>:  The investigation of the non-emptiness

and other properties of equilibrium and Pareto sets and the

core of probabilistic social systems in general is a research

problem not undertaken here. A start toward this is the

already mentioned study (Sertel, 1969 a, b) of a finite and

static case - not reproduced here - but the results of that

study can probably be generalized to quite a degree.  The

relation between the equilibrium set, Pareto set and core may

be especially interesting in the case of probabilistic

social systems.  The reason for making this conjecture is

the point made earlier in this section (preceding 5.1.3.1).

concerning the coordinated randomization of members' actions

for coalitions in the case of probabilistic social systems.

## 5.2    Toward a Framework for the Analysis of Power

With the probabilistic social systems of the previous section
in mind, the present section now turns to a topic which is central
to political analysis, namely, power.  A notion of one event (action)
'inducing' another will be introduced. Being related to the concept
of causality between events, it will be used to examine the formal
constituents of power relations between agents.  The main data
governing such power relations will be derived from the joint
behaviors of the social system supposed.  The non-transiency of
these relations will be seen to depend on whether the social system
is at equilibrium.

The present discussion will get further by, rather than starting
from scratch, agreeing in principle with the intuitive idea of power
that guided Dahl: "$\alpha$ has power over $\beta$ to the extent that he can get
$\beta$ to do something that $\beta$ would otherwise not do" [Dahl, 1957, pp.
202-3]  (Dahl's notation is different than the one used here.)
And, taking this as a point of departure, there is no visible route
which both promises to lead toward a fruitful destination and
succeeds in completely by-passing the subject of causality.  For
that reason,it will enhance the exposition to agree from the outset
on a minimal but workable committment as to when a given event will
be considered to be a cause of another given event.  The question of
what are fruitfully to be considered as necessary and sufficient
conditions for such a causal relation to be said to exist between

a given pair of events is too deep to be addressed here. For
what is ahead, however, it will be important to agree on some
necessary conditions. The choice of such conditions will be guided
by the objective of economizing on commitment subject to the con-
straint of obtaining something that is reasonably workable and non-
vague. In doing so, the reader will be left free to add any
further axioms which appear to be desirable. A whole host of
questions concerning time-precedence, contiguity, etc., will thus
be left to the reader to exercise his personal metaphysics with
regard to. Differing from Suppes [1967], the weakest necessary
condition is chosen as expressed in the following definition.

5.2.1 **Definition**: Let $ be a sigma-field of events and p: $ → [0,1]
a probability measure defined on $. A relation $\kappa \subset \$ \times \$$
will be said to satisfy the first axiom of causality with respect
to p  iff the following condition is satisfied:

$(E,F) \varepsilon \kappa$ only if there exists (an event)

$E' \varepsilon \$$ such that $p[E'] > 0$ and

$p[F|E'] < p[F|E]$ .

5.2.2 **Remark**: What is required by the above axiom is quite minimal.
If $(E,F) \varepsilon \kappa$ is ever to be read as "the event E is a cause of
event F when the probability judgement p is made," it is
being demanded that there be some event E' such that F is less

likely, according to that judgement, when it is known that

E' is the case than when E is known to be the case. If this is

not taken as an axiom, then it would have to be considered as

reasonable to say "E causes F although F is at least as likely to

occur under any circumstances which are at all likely (judged

to have non-zero probability measure) as under the circumstance of

E".

It is obvious that the event E' which is demanded must be

distinct from E, i.e., a different subset of the sure event on

which $ is a sigma-field. This is so, for otherwise $p[F|E']$ would

be equal to $p[F|E]$. What is very important to recognize, however,

is that E' is not required to be the complement of $E^C$ of E. In

fact, E' can be an event in $ which is a (proper) subset of E and

still satisfy the requirement imposed, while neither $E^C$ nor,

indeed, any event in $ which is a subset of $E^C$, need satisfy the

requirement.

It is precisely this which constitutes the fundamental

disagreement of the axiom chosen in 5.2.1 with what Suppes

[1967] takes to be minimal as a necessary condition. (Reference

is to his Definition 1 of "prima facie cause", page 11 of

Suppes' Chapter 5.) By concentrating on the complement of E,

Dahl appears in his framework to have anticipated Suppes' point

of departure, although this appearance may be due to the vagueness

of Dahl's notation in this regard, which in turn may be due to

the fact that he did not work with the clear notion of a sigma-field of events and may have been unaware of the distinction between two events being complements and two events being distinct.

Now an additional and final axiom will be introduced to capture the notion of "directionality" in a "causal relation".

5.2.3    Definition:  Let $ and p be as in 5.2.1.  A relation $\kappa \subseteq \$ \times \$$ will be said to satisfy the <u>second axiom of causality with respect to p</u>  iff the following condition is satisfied:

$$(E,F) \; \epsilon \; \kappa \; \text{only if} \; p[F|E] \geq P[E|F].$$

To summarize 5.2.1 and 5.2.3, the following will be useful.

5.2.4    Definition:  Let p be a probability measure defined on a sigma-field $ of events, and let E, F $\epsilon$ $ be two events. F will be said to <u>p-induce</u> F (denoted as $E \to_p F$) iff there exists a relation $\kappa \subseteq \$ \times \$$ such that $\kappa$ satisfies the first and second axioms of causality and $(E,F) \; \epsilon \; \kappa$.

It may be conjectured, as it was by the present author, that $E \to_p F$   and $F \to_p G$ implies $E \to_p G$ (where E, F, G $\epsilon$ $, for some sigma-field $ and where p: $ $\to$ [0,1] is a probability measure).  The conjecture is false, as the counterexample kindly provided by

P. R. Kleindorfer (personal communication), presented below, demonstrates.

5.2.5  <u>Proposition</u>: (The relation) $\underset{p}{\rightarrow}$ need <u>not</u> be transitive.

<u>Proof</u>: Let \$ be the power set of {a, b, c, d} ,and let p: \$ →[0,1] be a probability measure such that p({a}) = p({b}) = p({c}) = p({d}) = 1/4.  Then

$$1/2 = p(\{a,\ b\}|\{b,c\}) > p(\{a,b\}\ \{b,c,d\}\ ) = 1/3,$$

and

$$p\ (\{a,\ b\}|\{b,c\}) = p(\{b,c\}|\{a,b\}) = 1/2,$$

so that

$$\{b,c\}\quad \underset{p}{\rightarrow}\ \{a,\ b\}\ .$$

Also,

$$1/2 = p(\{b,\ c\}|\{c,\ d\}) > p(\{b,\ c\}|\{a,\ b,\ d\}) = 1/3$$

and

$$p(\{b,c\}|\{c,d\}\ ) = p(\{c,\ d\}|\{b,c\}\ ) = 1/2,$$

so that

$$\{b,\ c\}\ \underset{p}{\rightarrow}\ \{a,\ b\}\ \text{and}\{c,\ d\}\ \underset{p}{\rightarrow}\ \{b,\ c\}\ .$$

However, it is <u>not</u> the case that {c, d} $\underset{p}{\rightarrow}$ {a, b} , since p ( {a, b}|{c, d}) = 0 contradicts that the <u>first</u> axiom of causality holds for any $\kappa \subset \$ \times \$$ such that ( {c, d}, {a, b}) $\varepsilon\ \kappa$.

#

5.2.6.    Remark: It has long been recognized that "the causal arrow"

cannot be regarded as the 'copula' of logical implication,

notably for the reason that the latter contraposes while the

former does not.  Another difference between "the causal arrow"

and "the implication arrow" is revealed by 5.2.5, if $\underset{p}{\rightarrow}$ is accepted

as a "causal arrow".  (This is so, for containment '$\subset$' $\underline{is}$ transitive.)

Note also that a sure event can never be p-induced, for

there cannot exist a non-null subset conditional upon which its

probability is less than unity.  Thus, e.g., with reference

to 5.2.5, for any $E \epsilon \$$, $E \underset{p}{\rightarrow} \{a, b, c, d\}$ implies that $p(E) = 0$,

which is a contradiction:    the first axiom of causality effectively

prevents one from saying, for an event, which happens anyway,

that it is "caused" by some event.


Completely sacrificing the formal development of any interesting

mathematical consequences from the axioms or definitions introduced,

the promised application of the above to the topic of power will now

be pursued.  The context of what follows is a proababilistic social

system with a personnel A of typical behaviors $\alpha \epsilon A$.  For any

coalition $B \subset A$, $\Theta_B = \underset{B}{\Pi} \Theta_\beta$ is the joint action field of B, and

$\Theta_\beta$ is shorthand for $\Theta_{\{\beta\}}$ ; $P_B$ is the behavor space of B, consisting

of all joint behaviors $p_B = \Theta_B \rightarrow [0, 1]$ of B.

The following definition formalizes two key concepts.

5.2.7   <u>Definition</u>:   Let A be the personnel of a probabilistic social

system S, and let B, B' $\subset$ A be coalitions.   For any joint

behavior $p_A$ of A, the power relation $M(B, B'; p_A)$ is the set

$$M(B,B'; p_A) = \{ (\theta_B, \theta_{B'}) \mid \theta_B \in \Theta_B, \ \theta_{B'} \in \Theta_{B'}, \ \theta_B \underset{p_A}{\to} \theta_{B'} \}.$$

The power relation $M(B', B; p_A)$ is defined by replacing B with

B' and B' with B in the last expression.   The <u>power structure</u>

of S subject to $p_A$ is the set $M(S; p_A) = \{ M(B, B'; p_A) \mid B, B' \in [A] \}$

of all power relations $M(B, B'; p_A)$ between coalitions B, B' $\subset$ A.

The power relations defined above particularize to "interpersonal"

relations when only singleton coalitions are considered (Cf.(Frey,

n.d; p. 17).)   They, as well as $M(S; p_A)$, can be "quantified" in

the fashion now to be indicated.


5.2.8   <u>Definition</u>: Let $M(B, B'; p_A)$ be a power relation in a power

structure $M(S; p_A)$.   Define the function m for each ordered pair

$(\theta_B, \theta_{B'})$ with $\theta_B \in \Theta_B$ and $\theta_{B'} \in \Theta_{B'}$ by

$$m(\theta_B, \theta_{B'}) = \begin{cases} p_A[\theta_{B'} \mid \theta_B], & \text{if } \theta_B \underset{p_A}{\to} \theta_{B'} \\ \\ 0, & \text{otherwise.} \end{cases}$$

The function m will be called the <u>numerical representation of M(B,</u>

<u>B'; $p_A$</u>).   The set of numerical representations of all elements

of $M(S; p_A)$ will be called the <u>numerical representation of $M(S; p_A)$</u>.

5.2.9    Remark:  The definanda of 5.2.7-8 are all determined by one datum

alone, namely, the joint behavior $p_A$ of S.


5.2.10   Remark:  A brief comparison of the function m with Dahl's "amount

of power" [1957, p. 205] is in order.  Although the probability

measure he uses is not a joint behavior and his notion of action

is not clear, with some harmless change in notation, Dahl's

definition of the amount of power of B  over B', with respect to

the response $\theta_B$', by means of $\theta_B$ sets this amount equal to

$$\Delta(\theta_B, \theta_{B'}) = \text{Prob } [\theta_{B'} | \theta_B] - \text{Prob } [\theta_{B'} | \theta_B^C]$$

If the probability $p_A$ is used, then $\Delta$ becomes more easily

comparable with m.  In that case, m becomes the counterpart

here of $\Delta$ in Dahl's framework.  The two are very different functions

of course, since the formalization  of the underlying notion of

power here fundamentally differs from that of Dahl.  In the

present framework $p_A[\theta_{B'} | \theta_B^C]$ has no particular significance

in the obtaining of the number m $(\theta_B, \theta_{B'})$, as it has no special

role in determining whether or not $\theta_B \not\xrightarrow{p_A} \theta_{B'}$.  It is important

to note that m will detect some cases where $\theta_B \not\xrightarrow{p_A} \theta_{B'}$ by

assigning m$(\theta_B, \theta_{B'}) > 0$ while $\Delta$ will fail to do so.  In fact

$\Delta$  may assign $\Delta(\theta_B, \theta_{B'}) < 0$ while $\theta_B \not\xrightarrow{p_A} \theta_{B'}$, so that m$(\theta_B, \theta_{B'}) > 0$.

        Hence from the standpoint of the development here, Dahl's

measure has to be classified as misleading. It is remarkable that

two formalizations of the same intuitive notion (recorded at the

outset), should give such disagreeing results. The fact is, however, that the notion formalized here is actually the following: B has power over B' with respect to inducee action $\theta_{B'}$, by means of inducing acting $\theta_B$, and subject to $p_A$, iff $\theta_B \xrightarrow{p_A} \theta_{B'}$: the amount of this power is the probability $p_A[\theta_{B'}|\theta_B]$ of the inducee action conditional upon the inducing action.

In the above definitions and remarks concerning power, an arbitrary joint behavior $p_A$ of the social system was used in computing all the necessary probabilities. The importance of that joint behavior being an equilibrium point is clear, if the power relations and power structure are to be considered as non-transient. For if the very fact that a certain power structure (or $p_A$) holds leads to its being altered, as is the case for any non-equilibrium $p_A$, then the power structure (or $p_A$) in question is transient and not a regularity. It is important, therefore, whether there exists an equilibrium $p_A$, for, if there does, then the associated power structure is an equilibrium power structure. That there does exist an equilibrium $p_A$ for certain probabilistic social systems was shown in [Sertel, 1969 a]. Generalizations of that result and the investigation of the attraction and stability properties of equilibrium sets and cores appear clearly to promise an important bearing on political analysis.

## 5.3    Towards the Analysis of Attraction and Stability

Given that the equilibrium set or Pareto set or core of

a social system is non-empty, two important and related types

of question  arise concerning these sets: attraction and stability.

The tools for investigating these topics are to be found in the

theory of dynamical systems.  Naturally, this is also where the

notions themselves of attraction and stability are developed,

so it is there that one has to turn in order to see precisely

what these are.  A few preliminary definitions of this theory

will be presented here to crystallize the required concepts.  Then

it will be shown how a social system may be looked upon as a

dynamical system, so that the theory of the latter may be applied

to the former.  Finally, some discussion will follow.

Possibly the most prominent author on attraction, as well

as the originator of the notion of weak attraction, is Bhatia (1966).

The definitions to follow, however, are borrowed from another

prominent author, Szegö(1968).  They are slightly modified in

harmless fashion to relate most directly to social systems as an

area of application. Because of this, some of the terms have

been changed, in order to avoid confusion.  Notably, the notion of

a dynamical system has been modified and the term "evolutionary

system" attached to the result.

For the following definitions, let X be a locally compact Hausdorff space, and denote the set of non-negative integers by Z, taking the order topology on Z. A bar across the top will indicate topological closure.

5.3.1   Definition: A _evolutionary system_ is an ordered triplet <X, Z, E> , where

(5.3.1.1)   $E: X \times Z \to X$ is an usc point-to-set mapping;

(5.3.1.2)   $E(x, 0) = x$          $(x \in X)$;

(5.3.1.3)   $E(E(x,m),n) = E(x, m+n)$   $(x \in X; m, n \in Z)$.

5.3.2   Definition: The _future_ of a point $x \in X$ is the set $F(x) = E(\{x\} \times Z)$.

5.3.3   Definition: The _limit set_ of a point $x \in X$ is the set

$$L(x) = \bigcap_{y \in F(x)} \overline{F(y)}.$$

5.3.4   Definition: Let $M \subset X$ be compact.

(5.3.4.1)    $A^-(M) = \{x \mid L(x) \cap M \neq \emptyset \}$

is called the <u>region of weak attraction</u> of M. M is called

a <u>weak attractor</u> iff $A^-(M)$ is a nbd of M.


(5.3.4.2)    $A^+(M) = \{x \mid \emptyset \neq L(x) \subset M \}$

is called the <u>region of attraction</u> of M. M is called an

<u>attractor</u> iff $A^+(M)$ is a nbd of M.


5.3.5   <u>Definition</u>:  Let $M \subset X$ be compact.


(5.3.5.1)    M is <u>stable</u> iff, for every nbd V of M, there exists a nbd

U of M such that $F(U) \subset V$.


(5.3.5.2)    M is <u>asymptotically stable</u> iff it is a weak attractor and

stable.


The above definitions will now be interpreted from the viewpoint

of their application to social systems.  The space X is to be

interpreted as the domain of an <u>evolution</u>, so that, depending

on the social system in mind, X will be either simply the collective

(or joint) behavior space, or it will be the product of the

collective (or joint) behavior space with the collective feasibility

space.  In each of the cases where the contractual set was proved

to be non-empty in this study, X was compact, hence satisfying

the requirement that it be a locally compact Hausdorff space.

Now the mapping E can be related to the evolution of a social

system. Denoting the latter by $E^1$, equate $E^1(x) = E(x,1)$. Then

$E^1$ is the one-time-application of E. The condition 5.3.1.2

states that if E is applied zero times, then nothing changes.

If it is applied two times, then $E(x, 2) = E(E(x,1), 1) = E^1(E^1$

$(x))$, and so on. This is clearly consistent with the idea of

an evolution. As to E being usc, it has to be remarked that

$E^1$ was usc in every case where it was shown to have a fixed point.

From the fact that $E^1$ is usc, it follows that, defining

$$E^0(x) = E(x, 0),$$

$$E^n = E^1(E^{n-1}) \ (n = 1, 2, \ldots),$$

$E^n$ is usc for any $n \in Z$.

Now let V be a nbd of $E(x, m)$. To show that E is usc, one

needs to show the existence of nbd U of $(x,m)$ such that $E(U) \subset V$.

Note that $E(x, m) = E^m(x)$ and that $\{m\}$ is a nbd of m. Since

$E^m$ is usc, there exists a nbd N of x such that $E^m(N) \subset V$. Then

$U = N \times \{m\}$ is a nbd of $(x, m)$ such that $E(N \times \{m\}) = E^m(N) \subset V$.

Hence, E is usc, from the assumption that $E^1$ is so. Thus, as

long as usc evolutions are used, as done in this study, to

establish the non-emptiness of a contractual set, it is harmless

to assume 5.3.1.1. This concludes the justification for

5.3.1 as a whole.

Regarding the set M used in the definitions 5.3.4-5, notice

that it can be interpreted as a contractual set (or core, etc.)

as long as compactness is guaranteed for the latter. In the case

of the sets proved to be non-empty in this study, the requirement

is met.

The idea intended to be communicated by the present section

is that the theory of dynamical systems may offer the tools required

for the attraction and stability analysis of the equilibrium set

and core of a social system. The modifications with which the

definitions above were presented amount to incorporating the case

where E is a point-to-set mapping, as corresponding to the fact

that an evolution $E^1$ is, in general, of this nature.

Without some rather stringent assumptions (in the nature

of strict quasi-concavity for certain restrictions of effective

utility functions) the evolution of a social system will usually

not be a point-to-point mapping. In this case, the usual Liapunov

or simpler methods of stability analysis are inapplicable, so that

some remedy has to be sought. The discussion above is the result

of some groping in that direction.

5.4    Towards the Planning and Control of Organizations

The purpose of this section is to indicate selected extensions

and applications of the above framework and theory which bear

especially on the management of social systems. The section has to

be selective for the same sort of reasons that would force one to be

so if one were listing the uses of addition and multiplication. It

has to be selective also for the reason that, after a point, it

is more fruitful to do than to talk about the doing, to extend

and to apply rather than to endlessly converse on where and how to

extend and to apply.

"Management", at least for the present discussion, is the

guessing of what is an achievable "best" and the seeing to it

that such a best is achieved. So, with no great loss in paraphrasing,

it is planning and control. The guessing of what is achievable and

best, i.e., planning, is a matter of knowing what are reachable

points of the universe, having criteria of goodness for those

points, and, last but not at all least, having a framework and

accompanying methods, tools of analysis, to actually select a

point. And by the framework and method of analysis is not meant

as much an optimizing algorithm as is meant a way of thinking,

modelling, faithfully abstracting essentials and simply

representing, in a fashion that is perhaps communicable to some

optimizing algorithm.

Supposing that a best achievable point is known, the actual

seeing to it that the point is approximated in reality is a matter

of controlling the system whose performance is in question and which

performance is one of the reachable "points" in this abstract

discourse. The ability to control this system in turn depends on

what variables - "knobs, buttons, and levers", as it were - can

be set, and on knowing how the system responds to the various

values at which these can be set.

When the system in question is a social system, as it

always is in any non-trivial management problem, and when it

is a large system, there is really no way to manage but to work

with a highly abstract model of it. This is not to say that

one cannot manage or improve the performance of a given hospital,

school, business organization, football team or economy without

such abstraction. It is to say, however, that - as Polya is

known to have remarked - a trick will work once, and it is a method

that works the next time.

It is the obtaining of such methods of management which is

addressed as an application here. It will be taken as granted that

the system to be managed is a social system, its performance

depending on the behavior of the system and that behavior being

a concise way of expressing the behaviors of the members, or ,

in general, the coalitions.

It will be assumed that there is some criterion or objective

functional which numerically represents the performance of the

system as a function of its behavior. If there is no such guide

for comparing one behavior of the system with another behavior,
then it is not possible to guess what is a "best" behavior; so, to
talk of managing the system, such a guide has to be assumed to
exist. In reality, much of the difficulty of management may be
due to the absence of such a guide in clearcut form. But there
is nothing that can be done about that here. If the objective
is too vague, it will not be possible to discriminate good
management from bad anyway.

The nature of this objective functional can be expressed with
the example of an economy. Think of a gross sort of "production
function" which shows national income as a function of levels of
various activities. Supposing that national income is the better
the larger, i.e., that it is a true measure of performance, and
looking at the mentioned "activities" as behaviors of one sort of
another - or as aggregations of and decomposable into such - what
one has is an objective functional of the kind that will be
supposed.

It is an opportune moment to define an organization
$\Omega = \langle S, q \rangle$ as a social system S together with an objective
functional $q:X \to R$, where X is either, as usual, the collective,
or the joint behavior space of S. For the sake of simplicity,
assume that the personnel $N = \langle 1, \ldots, n \rangle$ of S is finite.

Often, when an organization $\Omega = \langle S, q \rangle$ is specified, q is a
"gross" and not a "net" objective functional, in a sense that
will now be seen. Consider the case of a business organization

and let q represent revenue net of all expenses but wages and
salaries. The incentive functions $g_j$ (j $\varepsilon$ N) express amounts,
in money units for the example being considered, given to the
various members of the personnel. Summing up, one obtains the
payroll function $p = \sum\limits_{N} g_j$, so that

$$p(x) = \sum_{N} g_j(x) \qquad\qquad (x\varepsilon\ X)$$

represents the additional payroll expense which has to be
deducted from q(x) to compute the profit $f(x) = q(x)-p(x)$ as
a function of the behavior x of the organization $\Omega$.

Suppose now that one seeks the "optimal" incentive scheme
$g = (g_1, ...., g_n)$. For a profit-maximizing concern, the function
sought is g* such that, among all incentive schemes, g* maximizes
profit. But this is not such a clear statement yet, for the
behavior x of $\Omega$ depends on the incentive scheme imposed, and there
may be more than one possible way in which the system S behaves for
a given g. Furthermore, the set of these behaviors corresponding
to g maybe a large set. Worse, there may be no equilibrium behavior
when g is imposed, the behavior of S cycling around in that set of
behaviors associated with g. It is not possible in general then
to write x as a function $\psi(g)$ of g and then write $f = f(\psi(g))$ to search
for an optimal, i.e., f-maximizing, incentive scheme g*.

If there exists an equilibrium behavior x ε ψ(g), however, then things are different. Suppose it is known then that there is a fairly wide class Γ of incentive schemes such that for every g ε Γ, the system S has an equilibrium. Such a class Γ is identified by the equilibrium existence theorems of this study for a variety of social systems S. Suppose that a fairly wide subset of Γ were identified to be a set of incentive schemes for which the equilibrium set has further nice properties, such as being an attractor and being stable. We already know that the equilibrium set is compact for each of the cases where it was proved in this study to be non-empty. Suppose that further studies, extensions of the present one, teach us how to find a subset Γ' of Γ for which the equilibrium sets are all very small, and stable attractors. Then ψ(g) for each g ε Γ' can be represented, for all practical purposes, by a single point x ε ψ(g).

Now, returning to the original problem of optimizing the incentive scheme, write

$$\text{(Max)} \quad f(\psi(g))$$

$$\text{s.t} \quad g \in \Gamma'$$

as a well-defined optimization problem. Problems of this sort have been considered by Kriebel and Lave [1969]. A particularly interesting case is that of the "constant-share finite organization", where $g_j = \lambda_j q$, $\lambda = (\lambda_0, \lambda_1, \ldots \lambda_n)$ being a non-negative vector in $E^{n+1}$ with $\sum_{j=0}^{n} \lambda_j = 1$, determining $f = \lambda_0 q$ as the profit

function. Assuming negative definite quadratic effective utility

functions $\ddot{w}_j$ and similar q, preliminary results have been obtained

by P. R. Kleindorfer and this author.

The above constitutes one illustration of how one would

usefully proceed from the present study in obtaining results

central to the management of organizations. The basic ideas behind

what was just illustrated arose during the writing of a paper on

organizational structures (Zannetos and Sertel, 1971).

This should be taken as an indication that they will be followed up.

While on the topic of incentive schemes, it is opportune

to mention how this relates to the topics of regulation and

legislation as specific instances of the general topic of controlling

social systems.

No great imagination is required to see that the idea of

an incentive scheme is an idealization of the notion of a

rule, regulation or law in a social context. The sanctions behind

laws are not always real-valued, but usually real vector-valued.

For example, a sentence of 18 months imprisonment and a \$35,000

fine can be pronounced together. Suspensions of licences, etc.,

are also sanctions. So, in general, incentives are not real-

valued. But this is not a tremendous blow to the model of a social

system presented here, for vector-valued incentives can be

incorporated with no serious trouble, except for some tedium in

some of the proofs. In order not to complicate the model any

further, so that its main features stand out more clearly,

incentives were represented as real numbers. Recognizing all this, however, the business of legislation is easily seen to be truly a managerial activity. The choice of incentive scheme for the profit-maximizing organization considered above and the making of laws are essentially the same sort of thing.

How do the existence and various properties of equilibrium or core points relate to the topic of legislation? Perhaps the easiest way to communicate how is by means of an example. Take the case of a typical unsuccessful "rural reform", in a backward and strongly feudal region, which redistributes land and illegalizes all taxes paid by peasants to landlord. The mere passing of the law makes little differences to reality, for the system tends right back to its original equilibrium, if ever it is disturbed in the first place. To prevent what is intended to be prevented, an incentive scheme has to be found under which the undesired status-quo is left outside the equilibrium set. To make sure it really works, the equilibrium set induced by the legislation has to be an attractor and stable, so that the behavior of the system is attracted toward this set and, once attracted, stays in that vicinity.

Successful legislation requires, therefore, an equilibrium analysis of the social system, whether this is based on strong social intuition or mathematics. Often it is not possible to alter the status-quo without a re-socialization of the personnel.

Consider the case where one is illegalizing head-hunting in the

Phillipines, polygamy in a traditional Islamic community, racial

discrimination in South Africa, or slavery in earlier America.

To the people whose behavior has to be altered, there is nothing

wrong with their present behavior, in fact, what is asked of them

seems ridiculous or even immoral to them. There is little

that can be done by legislation, except for that legislation which

affects the socialization process, giving new values to new vintages

of entrants into the personnel, while the older vintage dies.

Black-marketeering and smuggling are typically behaviors

which cannot be prevented except by readjustment of the relative

price vector, for it is the fact that they belong to the equilibrium

set of the present price vector that accounts for their presence.

Adjusting the price vector, of course, often defeats the purpose

of making certain goods and services unavailable in the first place,

so this constitutes no way out. The idea of illegalizing certain

production and trade activities and imposing sanctions tries to

add new "price tags" of possible imprisonment, etc., to the usual

one of pence and piastres, and seeks thus to make the dealing

in the markets in question unattractive. It succeeds to the

extent that the elongated new "price vector" moves the equilibrium

set away from those collective behaviors in which the undesired

behaviors are components.

Another example of a case where some equilibrium analysis

is needed is the prevention of fraud in an accounting system. The usual

rule is well-advised: make it necessary to form as large coalitions
as possible for fraud to become undetectable. Requiring that at
least two people coalesce for any fraud to succeed is the usual
extent to which this rule is carried. The extreme application would
be to expel all fraud from the core. That, of course, may require
a bit more than intuition to successfully do.

The example of the accounting system, however, is interesting
in a different sense than the earlier examples. For the method
of fraud-prevention here is also information-systemic rather than
purely incentive-sanction based. For it is already illegal
and severely punishable to cheat. The key is to yield the undesirable
behavior detectable or observable whenever it becomes possible,
ie., unblocked.

All the above hints at three main means of social control:
remunerational (via incentive schemes), socializational (via
alterations in utility schemes) and informational. The last mentioned
can be explored a bit further. The impression functions of the
framework used in this study can be decomposed into two functions.
Take a typical impression function $h_\alpha : X^\alpha \rightarrow X^\alpha$. Let $\delta_\alpha : X^\alpha \rightarrow X^\alpha$
be a function called the data function reporting to $\alpha$. Look at
the function $\delta : X \rightarrow \Pi X^\alpha$ defined by $\delta(x) = \Pi \delta_\alpha(x)$. This function
is appropriately considered as an information system. The data
flowing through it, the filters and aggregation imposed, are all part
of the information system design. Now respecify the old impression
function $h_\alpha$ as follows:

$$h_\alpha(x^\alpha) = \tilde{h}_\alpha(x^\alpha, \delta_\alpha(x^\alpha)). \qquad (x^\alpha \ \epsilon \ X^\alpha)$$

In the new form, $\alpha$ sees two things, $x^\alpha$ and $\delta_\alpha(x^\alpha)$. Of these $\delta_\alpha(x^\alpha)$ is what he is told about $x^\alpha$. From all this, $\tilde{h}_\alpha$ obtains an impression $h_\alpha(x^\alpha)$. This impression is the $\alpha$-exclusive behavior which $\alpha$ then bases his choice of behavior on.

Now if $\tilde{h}_\alpha$ is given as a parameter of $\alpha$, $h_\alpha$ can be influenced by altering $\delta_\alpha$. That is to say, $h$, the impression scheme, can be altered by altering $\delta$, the information system. This alters the choice of behavior, the equilibria, and so on.

The above indicates a further direction in which the model used in this study can be extended, and identifies a further form of informational control, that is, control via the information system. It is easy to see how the specification of the interpretation scheme could be modifed also in similar fashion.

Finally, the above discussion can now be used to indicate a further extension of the model, towards multi-level social systems. For consider now the fact that there are many people who already realize that a social system can be controlled in the ways described above, each imposing an incentive scheme, socialization process or information system. It is possible to view the behaviors of these controlling agents as points in suitable function spaces, and the agents themselves as behaviors in a "second-level" social

system, affecting the behavior of the underlying social system.
It is possible, therefore, to carry out an analysis of this second
level as one did of the first, and to investigate how the two
levels relate.  For example, how do the respective equilibria
relate? "Hierarchical" social systems thus become an extension of
the ones considered here.

This concludes the present discourse on how to fruitfully
extend the present model. Now is the time to begin the
investigations indicated.

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