DERIVED FUNCTORS

by

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ABSTRACT

The purpose of this thesis is to give a very general treatment of the basic machinery of relative homological algebra. This is done by combining the ideas of Eilenberg and Moore (closed classes of sequences) with the technique of Verdier (derived categories).

Part I is an exposition of the needed results of Eilenberg and Moore. Part II is an exposition of Verdier's results.

In part III these ideas are combined to define relative derived categories and relative derived functors. Two theorems giving conditions for the existence of relative derived functors are proved. The relation of this approach to that of Buchsbaum, MacLane and Yoneda is discussed and their results are recovered in a new fashion.

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INTRODUCTION

The notion of a derived functor was introduced by Cartan and Eilenberg in 1956 [5]. In the succeeding decade it has become a vital notion in a variety of contexts. During the same period it has been generalized in a multitude of ways.

These generalizations have taken three basic forms:
(1.) The notion has been extended to relative homological algebra. The original impetus in this direction was due to Hochschild [15] who studied extension classes of short exact sequences of modules over a ring R which were split when considered over a subring R^*.

Relative homological algebra has since been codified in a variety of ways by Heller [14], Buchsbaum [2], Butler and Horrocks [4], MacLane [16] and most recently by Eilenberg and Moore [8].

(2.) Another generalization has been in the direction of defining derived functors - and proving their existence - without the use of projectives or injectives. The first step was Yoneda's definition of Ext in terms of equivalence classes of exact sequences [19, 20]. Although variations of Yoneda's technique have been used to define derived functors other than Ext, there is not a general theory.

A different direction was suggested by Godement's [12] use of flasque resolutions in studying the cohomology of sheaves. Also suggestive were his remarks about the possible role of "standard constructions" in homological algebra.

His ideas show up quite clearly in the work of Eilenberg and Moore [9] and, less visibly, in the work of Verdier and others mentioned below.

The most extensive work in the direction of eliminating the use of projectives and injectives is Verdier's [18] theory of derived categories of abelian categories.

(3.) The third direction in which generalization has gone is in eliminating the additivity requirement on the functor or category. This was begun by Dold and Puppe [7] who defined derived functors for non-additive functors between categories of modules.

Very recently a number of mathematicians have begun to study derived functors of very general functors from small categories to the traditional nice categories. The basic idea here is to use semi-simplicial complexes rather than chain complexes. These ideas are still unpublished.

The primary purpose of this work is to obtain a resolution free treatment of relative derived functors. This is done by combining both the Eilenberg-Moore relative theory and Verdier's theory of derived categories.

There are three parts. The first part is a sketch of the needed definitions and results from Eilenberg and Moore [8].

The second part is a summary of the required definitions and results from Verdier [18].

The third part presents a study of relative derived functors via relative derived categories. There are various existence results. In addition this theory is applied to recover many of the old results in relative homological algebra in a new way.

I. CLOSED AND INJECTIVE CLASSES

Everything in part I can be found in Eilenberg and Moore [8], though usually in dual form.

Let $\mathcal{A}$ be a category. If A and B are objects of $\mathcal{A}$, $\underline{\mathcal{A}(A,B)}$ will denote the set of morphisms from A to B.

All categories considered will have a zero object which will be denoted by 0. All zero maps will also be denoted by 0. All functors will be assumed to carry 0 to 0. Thus in particular $\mathcal{A}(\cdot , \cdot)$ is a bifunctor with values in the category of pointed sets.

A category $\mathcal{A}$ is called a pre-additive category if for every pair of objects A and B in $\mathcal{A}$, $\mathcal{A}(A,B)$ is an abelian group in a functorial manner. If in addition $\mathcal{A}$ has finite products and finite coproducts, then $\mathcal{A}$ is an additive category. (Note that in a pre-additive category finite products and coproducts of the same family are canonically isomorphic. We write $A_n \xrightarrow{i_n} \bigoplus_n A_n \xrightarrow{p_n} A_n$ for the biproduct of the finite family A_n .)

FROM NOW ON ALL CATEGORIES ARE ADDITIVE!

$A \xrightarrow{a} B \xrightarrow{b} C$, a and b in $\mathcal{A}$, is called a sequence iff $ba = 0$. The notion of a sequence extends in the usual way to longer diagrams. In particular a sequence which is infinite in both directions is called a complex.

Let I be an object of $\mathcal{A}$ and $E^{\cdot}$ a sequence of $\mathcal{A}$. Then $\mathcal{A}(E^{\cdot},I)$ is a sequence of abelian groups.

Given a class $\mathcal{E}$ of sequences of $\mathcal{A}$, define $\underline{\mathcal{J}(\mathcal{E})}$ to be the class of all objects I of $\mathcal{A}$ such that $\mathcal{A}(E^{\bullet}, I)$ is an exact sequence for every $E^{\bullet}$ in $\mathcal{E}$.

Similiarly if $\mathcal{I}$ is any class of objects of $\mathcal{A}$, let <u>$\mathcal{E}(\mathcal{I})$</u> be the class of all sequences $E^{\bullet}$ in $\mathcal{A}$ such that $\mathcal{A}(E^{\bullet}, I)$ is an exact sequence for all I in $\mathcal{I}$.

Now write $\bar{\mathcal{E}}$ for $\mathcal{E}(\mathcal{I}(\mathcal{E}))$ and $\bar{\mathcal{I}}$ for $\mathcal{I}(\mathcal{E}(\mathcal{I}))$. Clearly $\mathcal{E} \leq \bar{\mathcal{E}} = \bar{\bar{\mathcal{E}}}$ and $\mathcal{I} \leq \bar{\mathcal{I}} = \bar{\bar{\mathcal{I}}}$. A class $\mathcal{I}$ of objects is (<u>injectively</u>) <u>closed</u> iff $\mathcal{I} = \bar{\mathcal{I}}$. Similiarly a class $\mathcal{E}$ of sequences is (<u>injectively</u>) <u>closed</u> iff $\mathcal{E} = \bar{\mathcal{E}}$. In particular if $\mathcal{E} = \mathcal{E}(\mathcal{I})$, $\mathcal{I}$ any class of objects in $\mathcal{A}$, then $\mathcal{E}$ is closed. Also if $\mathcal{I} = \mathcal{I}(\mathcal{E})$, $\mathcal{E}$ any class of sequences in $\mathcal{A}$, then $\mathcal{I}$ is a closed class.

The objects in $\mathcal{I} = \mathcal{I}(\mathcal{E})$ are called $\mathcal{E}$-injectives, while the sequences in $\mathcal{E} = \mathcal{E}(\mathcal{I})$ are called $\mathcal{I}$-exact sequences. The motivation for this notation comes from the situation where $\mathcal{A}$ is an abelian category, $\mathcal{E}$ is the class of all exact sequences and $\mathcal{I}$ is the class of all injective objects.

Let $\mathcal{E}$ be a closed class and let $\mathcal{I} = \mathcal{I}(\mathcal{E})$. $\mathcal{E}$ is called an <u>injective class</u> iff the following condition holds ("$\mathcal{A}$ has enough $\mathcal{E}$-injectives"): For every morphism $A \longrightarrow B$ in $\mathcal{A}$ there exists a morphism $B \longrightarrow I$ with I in $\mathcal{I}$ and $A \longrightarrow B \longrightarrow I$ in $\mathcal{E}$.

If $\mathcal{E}$ is an injective class, right (or co-) derived functors with respect to $\mathcal{E}$ can be defined by mimicing the original definition of derived functors.

Let A be an object of $\mathcal{A}$. A <u>right complex under A</u>, $A \longrightarrow A^{\bullet}$, is a complex $A^{\bullet}$: $\ldots \longrightarrow 0 \longrightarrow A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \ldots$

together with an augmentation $\varepsilon: A \to A^{\cdot}$, i. e. a morphism $\varepsilon: A \to A^0$ such that $d^0\varepsilon = 0$. The right complex $A \to A^{\cdot}$ under A is called <u>acyclic</u> iff $0 \to A \xrightarrow{\varepsilon} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \ldots$ is in $\mathcal{E}$. A complex $I^{\cdot}$ is called <u>$\mathcal{E}$-injective</u> iff each I^n is $\mathcal{E}$-injective. If $I^{\cdot}$ is $\mathcal{E}$-injective and $A \xrightarrow{\varepsilon} I^{\cdot}$ is acyclic, then $A \xrightarrow{\varepsilon} I^{\cdot}$ is called an <u>$\mathcal{E}$-injective resolution of A</u>.

The two basic results needed to define right derived functors are:

<u>Proposition 1</u>: If $\mathcal{E}$ is an injective class in $\mathcal{A}$, then every object of $\mathcal{A}$ has an $\mathcal{E}$-injective resolution.

<u>Proposition 2</u>: Let $A \xrightarrow{\varepsilon} A^{\cdot}$ be an $\mathcal{E}$-injective right complex under A and let $B \xrightarrow{\eta} B^{\cdot}$ be an acyclic right complex under B. Then for any morphism $f: A \to B$ there is a morphism of complexes $F: A^{\cdot} \to B^{\cdot}$ such that $\eta f = F\varepsilon$. Furthermore any two such morphisms F are chain homotopic.

Now let $T: \mathcal{A} \to \mathcal{B}$ be an additive functor with range an abelian category. Let $\mathcal{E}$ be an injective class in $\mathcal{A}$. For A an object of $\mathcal{A}$, let $A \to I^{\cdot}$ be an $\mathcal{E}$-injective resolution. Then $T(I^{\cdot})$ is a complex of $\mathcal{B}$, and its cohomology depends only on A. The <u>n^{th} right derived functor of T relative to $\mathcal{E}$</u> is defined by $R^n_{\mathcal{E}}T(A) = H^n(T(I^{\cdot}))$. This is indeed a functor.

The usual properties of resolutions of short exact sequences (i. e. sequences $0 \to A' \to A \to A'' \to 0$ in $\mathcal{E}$) hold, so the connecting morphism $\delta: R^nT(A'') \to R^{n+1}T(A')$ exists and has the usual properties.

By dualization we get the notion of the class of

$\mathcal{E}'$-projectives associated to a class $\mathcal{E}'$ of sequences. Also we have the notion of the class of $\mathcal{P}$-exact sequences associated to a class $\mathcal{P}$ of potential projectives. Following the dual of the above procedure we get (projectively) closed classes of objects and of sequences, projective classes and eventually left derived functors with respect to a projective class $\mathcal{E}'$. These notions should of course relate in the study of Ext. For this we define an injectively closed class $\mathcal{E}$ and a projectively closed class $\mathcal{E}'$ to be complementary if the complexes in the two are the same. As usual we define $\mathrm{Ext}^n_{\mathcal{E}}(A, \ .\ )$ to be $R^n_{\mathcal{E}}\mathcal{H}(A, \ .\ )$ and $\mathrm{Ext}^n_{\mathcal{E}'}(\ .\ ,B)$ to be $R^n_{\mathcal{E}'}\mathcal{H}(\ .\ ,B)$. Considering both of these as bifunctors, they are naturally isomorphic when $\mathcal{E}$ and $\mathcal{E}'$ are complementary.

II. DERIVED CATEGORIES

0. Introduction

Everything in part II can be found, in slightly different form, in Verdier [18]. Much of it can also be found in Hartshorne [13].

Since the constructions in this part are long and obscure, some historical background seems worthwhile.

The story begins with Cartier's [6] account of Yoneda's [19] construction of Ext^n in an abelian category. In outline the account goes as follows: Let $\mathcal{A}$ be an abelian category and let $C(\mathcal{A})$ be the category of complexes of $\mathcal{A}$ with homotopy classes of chain maps as morphisms. Let $S:\mathcal{A} \longrightarrow \mathcal{A}$ be the suspension functor (see sect. 4). Define $C^n(\mathcal{A})(A^\cdot,B^\cdot) = C(\mathcal{A})(A^\cdot,S^n(B^\cdot))$. The main proposition says that if $A^\cdot$ and $B^\cdot$ are injective resolutions of A and B respectively, then $C^n(\mathcal{A})(A^\cdot,B^\cdot) = \mathrm{Ext}^n(A,B)$.

Cartier extended this viewpoint to a theory of "derived categories" which enabled him to give a similiar description of other derived functors. A "morsel" of his theory is described in Gabriel's thesis [10]. It goes roughly as follows: Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $T:\mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Let $\mathcal{J}$ be the full subcategory of injectives in $\mathcal{A}$. Consider the morphism category of $\mathcal{A}$. If $d:A \longrightarrow B$, $d':A' \longrightarrow B'$ and $(a,b):d \longrightarrow d'$, say $(a,b) \simeq 0$ iff there exists $h:B \longrightarrow A'$ (in $\mathcal{A}$) such that $a = hd$. This gives a two-sided ideal in the morphism category; denote the quotient category by $K(\mathcal{A})$. Similiarly define $K(\mathcal{B})$ and

$K(\mathcal{J})$. Kernels are homotopy invariants, so we have $\mathrm{Ker}: K(\mathcal{J}) \longrightarrow \mathcal{A}$. If $\mathcal{A}$ has enough injectives, Ker has a left adjoint $I: \mathcal{A} \rightarrow K(\mathcal{J})$ (the beginning of an injective resolution), and Ker, I give an equivalence of categories: $\mathcal{A} \cong K(\mathcal{J})$. Further the 0^{th} right derived functor of T, R^0T, is just

$\mathcal{A} \xrightarrow{I} K(\mathcal{J}) \xrightarrow{K(T)} K(\mathcal{B}) \xrightarrow{\mathrm{Ker}} \mathcal{B}$.

By working with complexes rather than just morphisms this extends rather naturally to define the higher right derived functors. Unfortunately it does not eliminate the need for injectives. This was done by Verdier.

In order to eliminate injectives Verdier introduced an additional element of structure into the category of complexes, namely the triangles arising from the mapping cone (see sect. 4). Using this and quotient categories the desired results were obtained in a very elegant form.

1. Graded Categories

Let $\mathcal{A}$ be an additive category and $S: \mathcal{A} \longrightarrow \mathcal{A}$ an additive automorphism. For any two objects A and B in $\mathcal{A}$, define $\underline{\mathcal{A}^n(A,B)} = \mathcal{A}(A,S^n(B))$ (n an integer). If C is a third object of $\mathcal{A}$, $a \in \mathcal{A}^n(A,B)$ and $b \in \mathcal{A}^m(B,C)$, define $\underline{b \cdot a}$ in $\mathcal{A}^{n+m}(A,C)$ to be $S^n(b) \cdot a \in \mathcal{A}(A,S^{n+m}(B))$.

Clearly we can define a new category $\underline{\mathcal{A}^{\cdot}}$ with the same objects as $\mathcal{A}$ and where $\mathcal{A}^{\cdot}(A,B)$ is the $\mathbb{Z}$-graded abelian group $\mathcal{A}^n(A,B) : n \in \mathbb{Z}$. $\mathcal{A}^{\cdot}$ is called the <u>graded category over $\mathcal{A}$ with suspension functor S</u>.

Note that $\mathcal{A}$ is the subcategory of $\mathcal{A}^{\cdot}$ consisting of all morphisms of degree zero. A morphism of $\mathcal{A}^\bullet$ will always be assumed of degree zero unless otherwise specified. Also note that S extends naturally to an automorphism of $\mathcal{A}^\bullet$.

2. Triangulated Categories

Let $\mathcal{A}^\bullet$ be a graded category. A <u>triangle</u> in $\mathcal{A}^\bullet$ is a diagram of the form:

$$\begin{tikzcd} A \arrow[dr, "a"'] & & C \arrow[ll, "c"'] \\ & B \arrow[ur, "b"'] & \end{tikzcd}$$

where $\deg(c) = 1$.

A <u>morphism of triangles</u> is a diagram of the form:

$$\begin{tikzcd} A \arrow[rrr, "f"] \arrow[dr, "a"] & & & A' \arrow[dl, "a'"'] \\ & B \arrow[r, "g"] \arrow[dl, "b"] & B' \arrow[dr, "b'"'] & \\ C \arrow[uu, "c"] \arrow[rrr, "h"] & & & C' \arrow[uu, "c'"'] \end{tikzcd}$$

$\deg(c) = 1$

$\deg(c') = 1$

such that the three squares commute.

A <u>triangulated category</u> is a graded category $\mathcal{A}^\bullet$ together with a family of <u>distinguished triangles</u> in $\mathcal{A}^\bullet$ satisfying:

(T.1) Every triangle isomorphic to a distinguished triangle is distinguished. For every A in $\mathcal{A}^\bullet$,

$$\begin{tikzcd} A \arrow[dr, "1_A"'] & & 0 \arrow[ll, "0"'] \\ & A \arrow[ur, "0"'] & \end{tikzcd}$$

is distinguished. For every $A \xrightarrow{a} B$ there exists a distinguished triangle

$$\begin{tikzcd} A \arrow[dr, "a"'] & & C \arrow[ll, "c"'] \\ & B \arrow[ur, "b"'] & \end{tikzcd}$$

$\deg(c) = 1$.

(T.2)

$$\begin{tikzcd} A \arrow[dr, "a"'] & & C \arrow[ll, "c"'] \\ & B \arrow[ur, "b"'] & \end{tikzcd}$$

$\deg(c) = 1$

is distinguished if and only if

$$\begin{tikzcd}[column sep=small]
B & & S(A) \arrow[ll, "-S(a)"'] \\
& C \arrow[ur, "c"'] \arrow[from=1-1, "b"']
\end{tikzcd}$$

$\deg(-S(a)) = 1$

is distinguished.

(T.3) Every diagram

$$\begin{tikzcd}
A \arrow[rrr, "f"] \arrow[dr, "a"] & & & A' \arrow[dl, "a'"'] \\
& B \arrow[r, "g"] \arrow[dl, "b"] & B' \arrow[dr, "b'"'] & \\
C \arrow[uu, "c"] \arrow[rrr, dashed, "h"] & & & C' \arrow[uu, "c'"']
\end{tikzcd}$$

$\deg(c) = 1$

$\deg(c') = 1$

where the two ends are distinguished triangles and the square commutes can be completed by h to a morphism of triangles.

(T.4) Let

$\deg(a') = 1$

$$\begin{tikzcd}[column sep=small]
A & & C' \arrow[ll, "(\deg = 1)"'] \\
& B \arrow[ur, "b'"'] \arrow[from=1-1, "a"']
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
B & & A' \arrow[ll, "a'"'] \\
& C \arrow[ur] \arrow[from=1-1, "b"']
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
A & & B' \arrow[ll, "(\deg = 1)"'] \\
& C \arrow[ur] \arrow[from=1-1, "c"']
\end{tikzcd}$$

be three distinguished triangles such c = ba. Then there exist morphisms $f:C' \longrightarrow B'$ and $g:B' \longrightarrow A'$ such that:

(i)

$$\begin{tikzcd}
A \arrow[rrr, "1_A"] \arrow[dr, "a"] & & & A \arrow[dl, "c"'] \\
& B \arrow[r, "b"] \arrow[dl, "b'"] & C \arrow[dr] & \\
C' \arrow[uu, "(\deg = 1)"] \arrow[rrr, "f"] & & & B' \arrow[uu, "(\deg = 1)"']
\end{tikzcd}$$

is a morphism of triangles.

(ii)

$$\begin{tikzcd}
A \arrow[rrr, "a"] \arrow[dr, "c"] & & & B \arrow[dl, dash, "b"'] \\
& C \arrow[r, "1_C"] \arrow[dl] & C \arrow[dr] & \\
B' \arrow[uu, "(\deg = 1)"] \arrow[rrr, "g"] & & & A' \arrow[uu, "(\deg = 1)"']
\end{tikzcd}$$

is a morphism of triangles.

(iii)

$$\begin{tikzcd}[column sep=small]
C' & & A' \arrow[ll, "S(b')a'"'] \\
& B' \arrow[ur, "g"'] \arrow[from=1-1, "f"']
\end{tikzcd}$$

$\deg(S(b')a') = 1$

is a distinguished triangle.

3. Exact and Cohomological Functors

A <u>graded functor</u> F between two graded categories $\mathcal{A}^{\bullet}$ and $\mathcal{B}^{\bullet}$ is an additive functor $F:\mathcal{A}^{\bullet}\longrightarrow\mathcal{B}^{\bullet}$ such that $F:\mathcal{A}^{n}(A,B)\longrightarrow\mathcal{B}^{n}(F(A),F(B))$ for all A, B in $\mathcal{A}^{\bullet}$ and all $n\in\mathbb{Z}$.

An <u>exact functor</u> between two triangulated categories is a graded functor which carries distinguished triangles into distinguished triangles.

A <u>cohomological</u> functor H from a triangulated category to an abelian category is an additive functor such that if

$$\begin{tikzcd} A \arrow[dr, "a"'] & & C \arrow[ll, "c"'] \\ & B \arrow[ur, "b"'] & \end{tikzcd} \qquad \deg(c) = 1$$

is distinguished, then $H(A)\xrightarrow{H(a)}H(B)\xrightarrow{H(b)}H(C)$ is exact. Denoting by <u>H^n</u> the functor $H\cdot S^n$, (T.2) gives the long exact sequence:

$$\cdots\longrightarrow H^n(A)\longrightarrow H^n(B)\longrightarrow H^n(C)\longrightarrow H^{n+1}(A)\longrightarrow\cdots$$

4. Basic Examples

Let $\mathcal{A}$ be an additive category and let <u>$C(\mathcal{A})$</u> be the category of complexes in $\mathcal{A}$, morphisms being homotopy classes of chain maps. Define the <u>suspension functor</u> <u>$S:C(\mathcal{A})\longrightarrow C(\mathcal{A})$</u> by: $S(A^{\bullet})^n = A^{n+1}$, $d^n_{S(A^{\bullet})} = -d^{n+1}_{A^{\bullet}}$ and $S(f^{\bullet})^n = f^{n+1}$. S is an additive automorphism. We write <u>$\mathcal{A}^{\bullet}$</u>, rather than $C(\mathcal{A})^{\bullet}$, for the graded category over $C(\mathcal{A})$ with suspension functor S.

Note that $\mathcal{A}$ is a full subcategory of $C(\mathcal{A})$ under $A\longrightarrow A^{\bullet}$ where $A^n = \begin{cases} A & n = 0 \\ 0 & n \neq 0 \end{cases}$. Thus $\mathcal{A}$ is also a subcategory of $\mathcal{A}^{\bullet}$ and $\mathcal{A}(A,B) = \mathcal{A}^{0}(A,B)$ for all A, B in $\mathcal{A}$.

Define a <u>special triangle</u> in $\mathcal{A}^{\bullet}$ to be one of the form

$$\begin{tikzcd} A^{\bullet} \arrow[dr, "a^{\bullet}"'] & & C(a^{\bullet}) \arrow[ll, "c^{\bullet}"'] \\ & B^{\bullet} \arrow[ur, "b^{\bullet}"'] & \end{tikzcd} \qquad \deg(c^{\bullet}) = 1$$

where $C(a^{\bullet})$ is the mapping cone of $a^{\bullet}$, $b^{\bullet}$ is the canonical inclusion and $c^{\bullet}:C(a^{\bullet})\longrightarrow S(A^{\bullet})$ is the canonical quotient map. A distinguished triangle in $\mathcal{A}^{\bullet}$ is any triangle isomorphic to a special triangle.

The category $\mathcal{A}^{\bullet}$ together with this family of distinguished triangles is a triangulated category.

If $\mathcal{A}^{\bullet}$ is a triangulated category and **A** is an object of $\mathcal{A}^{\bullet}$, then $\mathcal{A}^{n}(A,\ .\ ):\mathcal{A}^{\bullet}\longrightarrow\mathcal{Ab}$ is a cohomological functor. Also $\mathcal{A}^{n}(\ .\ ,A):\mathcal{A}^{\bullet op}\longrightarrow\mathcal{Ab}$ is a cohomological functor.

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be three additive categories and $F:\mathcal{A}\times\mathcal{B}\longrightarrow\mathcal{C}$ a bilinear functor. F induces a functor <u>$F^{\bullet}:\mathcal{A}^{\bullet}\times\mathcal{B}^{\bullet}\longrightarrow\mathcal{C}^{\bullet}$</u> by: If $A^{\bullet}$ is in $\mathcal{A}^{\bullet}$ and $B^{\bullet}$ is in $\mathcal{B}^{\bullet}$, then $F(A^{\bullet},B^{\bullet})$ is a bicomplex of $\mathcal{C}$. Define $F^{\bullet}(A^{\bullet},B^{\bullet})$ to be the simple complex associated to the bicomplex $F(A^{\bullet},B^{\bullet})$. $F^{\bullet}$ is an exact bifunctor. In particular consider the case $\mathcal{A}(\ .\ ,\ .\ ):\mathcal{A}^{op}\times\mathcal{A}\longrightarrow\mathcal{Ab}$. This induces an exact bifunctor <u>$\mathcal{A}^{\bullet}(\ .\ ,\ .\ )$</u>$:\mathcal{A}^{\bullet op}\times\mathcal{A}^{\bullet}\longrightarrow\mathcal{Ab}^{\bullet}$. (This $\mathcal{A}^{\bullet}$ is not the same as $\mathcal{A}^{\bullet}(\ .\ ,\ .\ )$ in sect. 1, but no confusion should result.)

Now suppose $\mathcal{A}$ is an abelian category and let $H^{0}:\mathcal{A}^{\bullet}\longrightarrow\mathcal{A}$ be the 0-th cohomology functor. H^{0} is a cohomological functor. Also note that $H^{n} = H^{0}\circ S^{n}$ is the n-th cohomology functor.

Combining these last two paragraphs note that
$H^n \circ \mathcal{A}^\bullet(\, . \, , \, . \,) = \mathcal{A}^n(\, . \, , \, . \,)$.

5. Thick Subcategories and Multiplicative Systems

Let $\mathcal{A}^\bullet$ be a triangulated category. A subcategory $\mathcal{E}^\bullet$ of $\mathcal{A}^\bullet$ is a <u>triangulated subcategory</u> iff it is a triangulated category and the inclusion functor is exact. Another way of saying this is that if a distinguished triangle in $\mathcal{A}^\bullet$ has one of its morphisms in $\mathcal{E}^\bullet$, then it is isomorphic to a distinguished triangle of $\mathcal{E}^\bullet$.

A subcategory $\underline{\mathcal{E}^\bullet}$ of $\mathcal{A}^\bullet$ is called <u>thick</u> iff $\mathcal{E}^\bullet$ is a full triangulated subcategory of $\mathcal{A}^\bullet$ satisfying:

If $a: A \longrightarrow B$ factors through an object of $\mathcal{E}^\bullet$ and

$$\begin{tikzcd} A \arrow[dr, "a"'] & & C \arrow[ll, "(\deg = 1)"'] \\ & B \arrow[ur, "b"'] & \end{tikzcd}$$

is a distinguished triangle with C in $\mathcal{E}^\bullet$, then A and B are in $\mathcal{E}^\bullet$.

The class of thick subcategories of $\mathcal{A}^\bullet$ is ordered by inclusion and closed under arbitrary intersections.

A family $\underline{M}$ of morphisms of $\mathcal{A}^\bullet$ is called a <u>multiplicative system</u> iff it has the following properties:

(MS.1) Every identity map is in M. M is closed under composition.

(MS.2) Every diagram

$$\begin{tikzcd} & B_1 \arrow[d, "n \in M"] \\ A_2 \arrow[r] & B_2 \end{tikzcd}$$

can be completed to a commutative square

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "m"'] & B_1 \arrow[d, "n"] \\ A_2 \arrow[r] & B_2 \end{tikzcd}$$

with m also in M.

The dual statement also holds.

(MS.3) Let a and b be morphisms in $\mathcal{A}^\bullet$. There exists m in M such that ma = mb iff there exists n in M such that an = bn.

(MS.4) S(m) is in M for all m in M. (S is the suspension functor of $\mathcal{A}^\bullet$.)

(MS.5) If

$$\begin{tikzcd} A_1 \arrow[dr, "a_1"'] & & A_3 \arrow[ll, "a_3"'] \\ & A_2 \arrow[ur, "a_2"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B_1 \arrow[dr, "b_1"'] & & B_3 \arrow[ll, "b_3"'] \\ & B_2 \arrow[ur, "b_2"'] & \end{tikzcd}$$

$$\deg(a_3) = 1 \qquad\qquad \deg(b_3) = 1$$

are distinguished triangles and $(m_1, m_2): a_1 \longrightarrow b_1$ with m_1 and m_2 in M, then there exists $m_3: A_3 \longrightarrow B_3$, also in M, so that (m_1, m_2, m_3) is a morphism of triangles.

A multiplicative system is <u>saturated</u> iff m is in M when and only when there are two morphisms a and b so that am and mb are both in M.

The class of saturated multiplicative systems of $\mathcal{A}^\bullet$ is ordered by inclusion and closed under arbitrary intersections.

If $\mathcal{E}^\bullet$ is a thick subcategory of $\mathcal{A}^\bullet$, put <u>$\mathcal{M}(\mathcal{E}^\bullet)$</u> equal to the family of all morphisms $a: A \longrightarrow B$ such that there is a distinguished triangle

$$\begin{tikzcd} A \arrow[dr, "a"'] & & C \arrow[ll, "(\deg = 1)"'] \\ & B \arrow[ur] & \end{tikzcd}$$

with C in $\mathcal{E}^\bullet$.

$\mathcal{M}(\mathcal{E}^\bullet)$ is a saturated multiplicative system.

Similiarly let M be a saturated multiplicative system and put <u>$\mathcal{E}(M)$</u> equal to the full subcategory of $\mathcal{A}^\bullet$ generated by all objects C which are contained in a distinguished triangle

$$\begin{tikzcd} A \arrow[dr, "a"'] & & C \arrow[ll, "(\deg = 1)"'] \\ & B \arrow[ur] & \end{tikzcd}$$

where a is in M.

$\epsilon(M)$ is a thick subcategory.

$\mathcal{M}$ is an order isomorphism of the class of thick subcategories of $\mathcal{A}^{\bullet}$ onto the class of saturated multiplicative systems of $\mathcal{A}^{\bullet}$. $\mathcal{M}$ preserves intersections. The inverse of $\mathcal{M}$ is ϵ.

Let $\mathcal{A}^{\bullet}$ and $\mathcal{B}^{\bullet}$ be two triangulated categories and $F:\mathcal{A}^{\bullet}\longrightarrow\mathcal{B}^{\bullet}$ an exact functor. Let $\underline{M(F)}$ be the family of all morphisms of $\mathcal{A}^{\bullet}$ which are transformed into isomorphisms by F. Let $\underline{\mathcal{E}^{\bullet}(F)}$ be the full subcategory of $\mathcal{A}^{\bullet}$ generated by the objects annihilated by F. M(F) is a saturated multiplicative systen, $\mathcal{E}^{\bullet}(F)$ is a thick subcategory and $\mathcal{M}(\mathcal{E}^{\bullet}(F)) = M(F)$.

Now let $\mathcal{b}$ be an abelian category and $H:\mathcal{A}^{\bullet}\longrightarrow\mathcal{b}$ a cohomological functor. Let $\underline{\mathcal{E}^{\bullet}(H)}$ be the full subcategory of $\mathcal{A}^{\bullet}$ generated by all A in $\mathcal{A}^{\bullet}$ such that $H^n(A) = 0$ for all n. Also let $\underline{M(H)}$ be the family of all morphisms a such that $H^n(a)$ is an isomorphism for all n. Then $\mathcal{E}^{\bullet}(H)$ is a thick subcategory, M(H) is a saturated multiplicative system and $M(H) = \mathcal{M}(\mathcal{E}^{\bullet}(H))$.

6. Quotient Categories

Let $\mathcal{A}^{\bullet}$ be a triangulated category, $\mathcal{E}^{\bullet}$ a thick subcategory and $M = \mathcal{M}(\mathcal{E}^{\bullet})$ the corresponding saturated multiplicative system. The following universal problems are equivalent.

<u>Problem I</u>: Find a triangulated category $\mathcal{A}^{\bullet}/\mathcal{E}^{\bullet}$ and an exact functor $Q:\mathcal{A}^{\bullet}\longrightarrow\mathcal{A}^{\bullet}/\mathcal{E}^{\bullet}$ such that if $F:\mathcal{A}^{\bullet}\longrightarrow\mathcal{B}^{\bullet}$ is an exact functor which annihilates $\mathcal{E}^{\bullet}$, there is a unique exact functor $G:\mathcal{A}^{\bullet}/\mathcal{E}^{\bullet}\longrightarrow\mathcal{B}^{\bullet}$ so that

$$\begin{tikzcd} \mathcal{A}^\cdot \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{B}^\cdot \\ \mathcal{A}^\cdot/\mathcal{E}^\cdot \arrow[ur, "G"'] & \end{tikzcd}$$ is commutative.

<u>Problem II</u>: Find a triangulated category $\mathcal{A}^\cdot_M$ and an exact functor $Q:\mathcal{A}^\cdot \longrightarrow \mathcal{A}^\cdot_M$ such that for every exact functor $F:\mathcal{A}^\cdot \longrightarrow \mathcal{B}^\cdot$ which carries every element of M into an isomorphism there is a unique exact functor $G:\mathcal{A}^\cdot_M \longrightarrow \mathcal{B}^\cdot$ so that

$$\begin{tikzcd} \mathcal{A}^\cdot \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{B}^\cdot \\ \mathcal{A}^\cdot_M \arrow[ur, "G"'] & \end{tikzcd}$$ is commutative.

Next we show that **Problem II** (and hence Problem I) always has a solution. The solution is obtained as follows: For each object A in $\mathcal{A}^\cdot$ define the category <u>M_A</u> of morphisms in M with range A, i. e. an object of M_A is $m:B \longrightarrow A$ where m is in M. Now for each B in $\mathcal{A}^\cdot$ define a functor <u>H_B</u> $:M_A{}^{op} \longrightarrow \mathcal{Ab}$ by $H_B(m) = \mathcal{A}^0(\text{domain}(m),B)$. Define

$$\mathcal{A}^0[M^{-1}](A,B) = \varinjlim_{M_A^{op}} H_B$$

(Note: Since M is a multiplicative system, M_A^{op} is a pseudo-filtering category and so $\varinjlim:\text{Funct}(M_A^{op},\mathcal{Ab}) \longrightarrow \mathcal{Ab}$ has good properties which are used below. For details see Artin [1; Chapter 1, sect. 1].)

If A, B and C are three objects of $\mathcal{A}^\cdot$, a is in $\mathcal{A}^0[M^{-1}]$ (A,B) and b is in $\mathcal{A}^0[M^{-1}]$ (B,C), define ba in $\mathcal{A}^0[M^{-1}]$ (A,C) as follows. Let m be an object of M_A and <u>a</u> an element of $\mathcal{A}^0(\text{domain}(m),B)$ whose image in $\mathcal{A}^0[M^{-1}]$ (A,B) is a. Let n be an object of M_B and $\underline{b}$ an element of $\mathcal{A}^0(\text{domain}(n),C)$ whose image in $\mathcal{A}^0[M^{-1}](B,C)$ is b. Schematically

$$\begin{tikzcd}
\bullet \arrow[d, "m"'] \arrow[drr, "\underline{a}"] & & \bullet \arrow[d, "n"'] \arrow[dr, "\underline{b}"] & \\
A & & B & C
\end{tikzcd}$$

Now by (MS.2) we can complete this to:

$$\begin{tikzcd}
\bullet \arrow[d, "n'"'] \arrow[drr, "\underline{a}'"] & & & \\
\bullet \arrow[d, "m"'] \arrow[drr, "\underline{a}"] & & \bullet \arrow[d, "n"'] \arrow[dr, "\underline{b}"] & \\
A & & B & C
\end{tikzcd}$$

with n' in M. Denote by ba the image of $\underline{ba}'$ in $\mathcal{A}^0[M^{-1}](A,C)$. Using (MS.1), (MS.2) and (MS.3) it is easy to check that ba is well-defined and gives us a new category with the same objects as $\mathcal{A}^\bullet$. We denote by $\underline{Q}:\mathcal{A}^0 \longrightarrow \mathcal{A}^0[M^{-1}]$ the obvious quotient functor.

Since $\mathcal{A}^\bullet$ is an additive category, so is $\mathcal{A}^0[M^{-1}]$ and Q is an additive functor.

(MS.4) allows us to define, in a unique manner, a suspension functor S on $\mathcal{A}^0[M^{-1}]$ satisfying SQ = QS. Using S we define the graded category over $\mathcal{A}^0[M^{-1}]$ and denote it by $\mathcal{A}^\bullet[M^{-1}]$. Q extends uniquely to $Q:\mathcal{A}^\bullet \longrightarrow \mathcal{A}^\bullet[M^{-1}]$, a graded functor.

Finally, using (MS.5), we find that there is a unique triangulation of $\mathcal{A}^\bullet[M^{-1}]$ so that Q is exact. In fact the distinguished triangles are precisely those triangles isomorphic to the image under Q of some distinguished triangle of $\mathcal{A}^{\cdot}$. We write $\underline{\mathcal{A}^{\cdot}_{M}}$ for the category $\mathcal{A}^{\cdot}[M^{-1}]$ with this triangulation. Then $(\mathcal{A}^{\cdot}_{M}, Q)$ is the desired solution to Problem II.

We will write $(\mathcal{A}^{\cdot}/\mathcal{E}^{\cdot}, Q)$ for the solution of Problem I (which we now know exists) and call $\mathcal{A}^{\cdot}/\mathcal{E}^{\cdot}$ the <u>quotient of $\mathcal{A}^{\cdot}$ by $\mathcal{E}^{\cdot}$</u>.

7. Derived Categories

Let $\mathcal{A}$ be an abelian category. As remarked earlier (sect. 4) $H^0: \mathcal{A}^{\cdot} \longrightarrow \mathcal{A}$, the 0-th cohomology functor, is a cohomological functor. Thus we have $\mathcal{E}^{\cdot}(H^0)$, a thick subcategory of $\mathcal{A}^{\cdot}$ (see sect. 5). $\mathcal{E}^{\cdot}(H^0)$ is the full subcategory generated by the acyclic complexes in $\mathcal{A}^{\cdot}$.

The quotient category $\mathcal{A}^{\cdot}/\mathcal{E}^{\cdot}(H^0)$ is called the <u>derived category of $\mathcal{A}$</u> and is denoted by $\underline{D^{\cdot}(\mathcal{A})}$. The canonical functor $\mathcal{A}^{\cdot} \longrightarrow D^{\cdot}(\mathcal{A})$ will be denoted by $\underline{D}$.

D restricted to the subcategory $\mathcal{A}$ in $\mathcal{A}^{\cdot}$ will also be denoted by D. For all A, B in $\mathcal{A}$, $\mathcal{A}(A,B) \xrightarrow[D]{\cong} D^0(\mathcal{A})(D(A),D(B))$. Thus $\mathcal{A}$ is a subcategory of $D^{\cdot}(\mathcal{A})$.

There are several important subcategories of $D^{\cdot}(\mathcal{A})$ coming from various subcategories of $\mathcal{A}^{\cdot}$. Let $\underline{\mathcal{A}^{\cdot}_{+}}$ be the full triangulated subcategory of $\mathcal{A}^{\cdot}$ generated by all complexes $A^{\cdot}$ such that $A^n = 0$ for $n \ll 0$. Similiarly $\underline{\mathcal{A}^{\cdot}_{-}}$ is generated by all $A^{\cdot}$ such that $A^n = 0$ for $n \gg 0$. Finally let

$\underline{\mathcal{A}^{\cdot}_{0}} = \mathcal{A}^{\cdot}_{+} \cap \mathcal{A}^{\cdot}_{-}$.

NOTATIONAL CONVENTION: If $\mathcal{B}^\bullet$ is a full triangulated subcategory of $\mathcal{A}^\bullet$, we write $\underline{\mathcal{B}^\bullet_+}$ (resp. $\underline{\mathcal{B}^\bullet_-}$, $\underline{\mathcal{B}^\bullet_0}$) for $\mathcal{B}^\bullet \cap \mathcal{A}^\bullet_+$ (resp. $\mathcal{B}^\bullet \cap \mathcal{A}^\bullet_-$, $\mathcal{B}^\bullet \cap \mathcal{A}^\bullet_0$). $\mathcal{B}^\bullet_+$ is a full triangulated subcategory of $\mathcal{A}^\bullet_+$. Similiarly for the other two cases. Finally we will commonly write $\underline{\mathcal{B}^\bullet_*}$ and will mean that * can be either +, -, 0 or nothing in the given context.

$\mathcal{E}^\bullet_*(H^0)$ is a thick subcategory of $\mathcal{A}^\bullet_*$, so we define $D^\bullet_*(\mathcal{A}) = \mathcal{A}^\bullet_*/\mathcal{E}^\bullet_*(H^0)$. The functors in the following diagram:

$$\begin{tikzcd} & D^\bullet(\mathcal{A}) & \\ D^\bullet_+(\mathcal{A}) \arrow[ur] & & D^\bullet_-(\mathcal{A}) \arrow[ul] \\ & D^\bullet_0(\mathcal{A}) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

are all fully faithful and injective on objects, thereby exhibiting $D^\bullet_0(\mathcal{A})$, $D^\bullet_+(\mathcal{A})$ and $D^\bullet_-(\mathcal{A})$ as full subcategories of $D^\bullet(\mathcal{A})$.

Denote by $\underline{I^\bullet(\mathcal{A})}$ the full triangulated subcategory of $\mathcal{A}^\bullet$ generated by the injective complexes.

<u>Theorem</u>: If $\mathcal{A}$ has enough injectives, then $D_+ = D\big|_{\mathcal{A}^\bullet_+} : \mathcal{A}^\bullet_+ \longrightarrow D^\bullet_+(\mathcal{A})$ has a right adjoint $\underline{I}$. Furthermore $I : D^\bullet_+(\mathcal{A}) \longrightarrow I^\bullet_+(\mathcal{A})$ and D_+ , I define an equivalence of categories.

Finally if $\mathcal{A}$ has finite homological dimension (i. e. there is an integer $N \geq 0$ such that for all $n > N$ and all A, B in $\mathcal{A}$, $D^n(\mathcal{A})(A,B) = 0$), then the subscript + may be surpressed.

Note: The dual of this theorem is also true. This is obtained by replacing "injective" by "projective", "I" by "P" and

"+" by "-".

8. Derived Functors

Let $\mathcal{A}^\cdot$ and $\mathcal{B}^\cdot$ be two graded categories (with the suspension functor of each denoted by S.) Suppose $F,\ G:\mathcal{A}^\cdot \longrightarrow \mathcal{B}^\cdot$ are graded functors. A *morphism of graded functors* is a morphism of functors $f: F \longrightarrow G$ such that for every A in $\mathcal{A}^\cdot$

$$\begin{tikzcd} FS(A) \arrow[r] \arrow[d, phantom, "\wr"] & GS(A) \arrow[d, phantom, "\wr"] \\ SF(A) \arrow[r] & SG(A) \end{tikzcd}$$

commutes.

A *morphism of exact functors* is simply a morphism of graded functors.

Now let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $F^\cdot : \mathcal{A}^\cdot_* \longrightarrow \mathcal{B}^\cdot_*$ an exact functor. The *total right derived functor of $F^\cdot$* is an exact functor $\underline{\underline{R}}F^\cdot : D^\cdot_*(\mathcal{A}) \longrightarrow D^\cdot_*(\mathcal{B})$ together with a morphism of exact functors $D_* \circ F^\cdot \longrightarrow \underline{\underline{R}}F^\cdot \circ D_*$ (see Helpful Diagram below) which is universal in the sense that if $G^\cdot : D^\cdot_*(\mathcal{A}) \longrightarrow D^\cdot_*(\mathcal{B})$ is another exact functor and $D_* \circ F^\cdot \longrightarrow G^\cdot \circ D_*$ a morphism of exact functors, then there is a unique morphism of exact functors $\underline{\underline{R}}F^\cdot \dashrightarrow G^\cdot$ so that

$$\begin{tikzcd} D_* \circ F^\cdot \arrow[r] \arrow[d] & \underline{\underline{R}}F^\cdot \circ D_* \arrow[dl, dashed] \\ G^\cdot \circ D_* & \end{tikzcd}$$

commutes.

Helpful Diagram:

$$\begin{tikzcd} \mathcal{A}^\cdot_* \arrow[r, "F^\cdot"] \arrow[d, "D_*"'] & \mathcal{B}^\cdot_* \arrow[d, "D_*"] \\ D^\cdot_*(\mathcal{A}) \arrow[r, "\underline{\underline{R}}F^\cdot"] & D^\cdot_*(\mathcal{B}) \end{tikzcd}$$

If $F^{\bullet}:\mathcal{A}^{\bullet}_{*}\longrightarrow\mathcal{B}^{\bullet}_{*}$ is induced by an additive functor $F:\mathcal{A}\longrightarrow\mathcal{B}$, then $\underline{\underline{R}}F^{\bullet}$ will be denoted by $\underline{\underline{R}_{*}F}$.

Dually we can define the total left derived functor $\underline{\underline{L}}F^{\bullet}$ of $F^{\bullet}$. The convention analogous to the last paragraph will be observed.

Similiarly let $H:\mathcal{A}^{\bullet}_{*}\longrightarrow\mathcal{B}$ be a cohomological functor and define the right derived functor of H as a cohomological functor $\underline{RH}:D^{\bullet}_{*}(\mathcal{A})\longrightarrow\mathcal{B}$ together with a morphism of functors $H\longrightarrow RH\circ D_{*}$ (see diagram below) such that if $G:D^{\bullet}_{*}(\mathcal{A})\longrightarrow\mathcal{B}$ is another cohomological functor and $H\longrightarrow G\circ D_{*}$ a morphism of functors, then there is a unique morphism of functors $RH\dashrightarrow G$ so that

$$\begin{tikzcd} H \arrow[r] \arrow[d] & RH\circ D_{*} \arrow[dl, dashed] \\ G\circ D_{*} & \end{tikzcd}$$

commutes.

Diagram:

$$\begin{tikzcd} \mathcal{A}^{\bullet}_{*} \arrow[rr, "H"] \arrow[d, "D_{*}"'] & & \mathcal{B} \\ D^{\bullet}_{*}(\mathcal{A}) \arrow[urr, "RH"'] & & \end{tikzcd}$$

Now if $F:\mathcal{A}\longrightarrow\mathcal{B}$ is an additive functor, there is an induced functor $F^{\bullet}:\mathcal{A}^{\bullet}_{*}\longrightarrow\mathcal{B}^{\bullet}_{*}$ and when $\mathcal{B}$ is abelian this gives a cohomological functor $H^{0}\circ F^{\bullet}:\mathcal{A}^{\bullet}_{*}\longrightarrow\mathcal{B}$. The right derived functor of $H^{0}\circ F^{\bullet}$ will be denoted by $\underline{R_{*}F}$. Whenever no confusion is possible both $R_{+}F\circ S^{n}$ and $RF\circ S^{n}$ will be denoted by $\underline{R^{n}F}$.

By dualization we define the left derived functor $\underline{LH}$ of the cohomological functor H. We also define $\underline{L_{*}F}$ and $\underline{L_{n}F} = L_{-}F\circ S^{-n}$ or $LF\circ S^{-n}$ when F is an additive functor.

9. Existence of Derived Functors

The elements of $M = \mathcal{M}(\mathcal{E}^{\cdot}(H^0))$ (see sects. 5 & 7) are called <u>quasi-isomorphisms</u>. Recall that m is in M iff it has the following equivalent properties:

(a.) $D(m)$ is an isomorphism in $D^{\cdot}(\mathcal{A})$.

(b.) $H^n(m)$ is an isomorphism for all n.

(c.) $C^{\cdot}(m)$, the mapping cone of m, is in $\mathcal{E}^{\cdot}(H^0)$.

<u>Theorem 1</u>: Let $F^{\cdot}: \mathcal{A}^{\cdot}_{*} \longrightarrow \mathcal{B}^{\cdot}_{*}$ be an exact functor. Suppose there is a triangulated subcategory $\mathcal{A}c^{\cdot}$ of $\mathcal{A}^{\cdot}_{*}$ such that:

(1.) Every object $A^{\cdot}$ of $\mathcal{A}^{\cdot}_{*}$ admits a quasi-isomorphism $A^{\cdot} \longrightarrow I^{\cdot}$ with $I^{\cdot}$ in $\mathcal{A}c^{\cdot}$, and

(2.) If $A^{\cdot}$ is in $\mathcal{A}c^{\cdot}$ and also in $\mathcal{E}^{\cdot}_{*}(H^0)$, then $F^{\cdot}(A^{\cdot})$ is acyclic, i. e. in $\mathcal{E}^{\cdot}_{*}(H^0) \subseteq \mathcal{B}^{\cdot}_{*}$,

then

(a.) $\underline{\underline{R}}F^{\cdot}: D^{\cdot}_{*}(\mathcal{A}) \longrightarrow D^{\cdot}_{*}(\mathcal{B})$ exists, and

(b.) if $A^{\cdot} \longrightarrow I^{\cdot}$ is a quasi-isomorphism with $I^{\cdot}$ in $\mathcal{A}c^{\cdot}$, then $\underline{\underline{R}}F^{\cdot} \circ D_{*}(A^{\cdot}) \xrightarrow{\cong} D_{*} \circ F^{\cdot}(I^{\cdot})$. Also

(c.) $RH^0 \circ F^{\cdot}$ exists and in fact equals $H^0 \underline{\underline{R}}F^{\cdot}$.

<u>Corollary 1</u>: Let $F^{\cdot}: \mathcal{A}^{\cdot}_{+} \longrightarrow \mathcal{B}^{\cdot}_{+}$ be as above. Let $\mathcal{A}c^{\cdot} = I^{\cdot}_{+}(\mathcal{A})$ (see page 20). Then $\underline{\underline{R}}F^{\cdot}$ and $RH^0 \circ F^{\cdot}$ both exist. In particular if $F^{\cdot}$ comes from an additive functor $F: \mathcal{A} \longrightarrow \mathcal{B}$, then R^nF exists and $R^nF(A) = H^nF(I^{\cdot}(A))$ for all A in $\mathcal{A}$. (Here $I^{\cdot}(A)$ is the injective resolution of A.)

<u>Corollary 2</u>: Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor. Let $\mathcal{A}c$ be a full subcategory of $\mathcal{A}$ satisfying:

(1.) Every object of $\mathcal{A}$ is a subobject of some object in $\mathcal{A}c$.

(2.) If $A^{\bullet}$ is in $\mathcal{A}c^{\bullet}_{+}$ and is acyclic, then $F(A^{\bullet})$ is acyclic. Then $\underline{\underline{R}}_{+}F$ and $\mathbf{R}_{+}F$ both exist. In this situation $\mathcal{A}c$ is a subcategory of <u>F-acyclic objects</u>, i. e. if A is in $\mathcal{A}c$, then $R^{n}F(A) = 0$ for all $n \geq 1$.

Remarks: (1.) Corollary 2 is summarized by saying that "the derived functors of F exist whenever $\mathcal{A}$ has enough F-acyclic objects."

(2.) Theorem 1 and its corollaries have obvious duals which also have dual proofs.

<u>Theorem 2</u>: Let $\mathcal{B}$ be an abelian category in which direct limits from pseudo-filtering categories exist and are exact. Let $F:\mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor. Then $RF:D^{\bullet}(\mathcal{A}) \longrightarrow \mathcal{B}$ exists.

Remarks: (1.) The proof proceeds by exhibiting $\mathbf{RF}(A^{\bullet})$ as the direct limit of a canonically associated pseudo-filtering system.

(2.) This is the immediate analogue of a result of Buchsbaum [3] on satellites.

(3.) The dual of Theorem 2 holds and has a dual proof. However Theorem 2 does introduce a certain asymmetry in applications. This is because $\mathcal{A}b$ satisfies the conditions on $\mathcal{B}$, but <u>not</u> the conditions in the dual theorem. Thus for $F:\mathcal{A} \longrightarrow \mathcal{A}b$ RF will exist, while LF generally does not. This can be repaired at the expense of considering $LF:D^{\bullet}(\mathcal{A}) \longrightarrow \text{Pro-}\mathcal{A}b$, the category of pro-abelian groups.

(4.) Foundational difficulties make the full applicability of this theorem unclear. For example does $RF:D^{\bullet}(\mathcal{A}) \longrightarrow \mathcal{A}b$

exist for all $F:\mathcal{A} \longrightarrow \mathcal{Ab}$?

10. Technicalities on Quotient Categories

The following rather technical results will be needed in Part III. Indeed they are actually needed for the proof of the results stated in sections 7-9.

Throughout this section $\mathcal{A}^{\bullet}$ will be a triangulated category, $\mathcal{E}^{\bullet}$ a thick subcategory and $M = \mathcal{M}(\mathcal{E}^{\bullet})$ the corresponding saturated multiplicative system.

<u>Proposition 1</u>: Let $\mathcal{B}^{\bullet}$ be a triangulated subcategory of $\mathcal{A}^{\bullet}$. Then

(1.) $\mathcal{E}^{\bullet} \cap \mathcal{B}^{\bullet}$ is a thick subcategory of $\mathcal{A}^{\bullet}$ and the corresponding saturated multiplicative system is $M \cap \mathcal{B}^{\bullet}$.

(2.) The following are equivalent:

(a.) For each object B in $\mathcal{B}^{\bullet}$ and every morphism $m:A \longrightarrow B$ in M, there is a morphism $m':A' \longrightarrow A$ so that A' is in $\mathcal{B}^{\bullet}$ and mm' is in $M \cap \mathcal{B}^{\bullet}$.

(b.) Every morphism of an object of $\mathcal{B}^{\bullet}$ into an object of $\mathcal{E}^{\bullet}$ factors through an object in $\mathcal{E}^{\bullet} \cap \mathcal{B}^{\bullet}$.

(3.) The dual of (2.) also holds. The statements dual to (a.) and (b.) will be denoted by (a.') and (b.').

(4.) If (a.) and (b.) hold, or if (a.') and (b.') hold, then the canonical functor $\mathcal{B}^{\bullet}/\mathcal{E}^{\bullet} \cap \mathcal{B}^{\bullet} \longrightarrow \mathcal{A}/\mathcal{E}^{\bullet}$ is faithful. Since this functor is injective on objects it realizes $\mathcal{B}^{\bullet}/\mathcal{E}^{\bullet} \cap \mathcal{B}^{\bullet}$ as a subcategory of $\mathcal{A}^{\bullet}/\mathcal{E}^{\bullet}$. Further if $\mathcal{B}^{\bullet}$ is a full subcategory of $\mathcal{A}^{\bullet}$, then $\mathcal{B}^{\bullet}/\mathcal{E}^{\bullet} \cap \mathcal{B}^{\bullet}$ is a full subcategory of $\mathcal{A}^{\bullet}/\mathcal{E}^{\bullet}$.

Proposition 2: Let A be an object of $\mathcal{A}^{\cdot}$. The following are equivalent:

(1.) For every object B of $\mathcal{A}^{\cdot}$, $\mathcal{A}^{\cdot}(A,B) \xrightarrow{Q} \mathcal{A}^{\cdot}/\mathcal{E}^{\cdot}(Q(A),Q(B))$ is an isomorphism.

(2.) Every morphism of A into an object of $\mathcal{E}^{\cdot}$ is zero.

Dually the following are equivalent:

(1!) For every object B of $\mathcal{A}^{\cdot}$, $\mathcal{A}^{\cdot}(B,A) \xrightarrow{Q} \mathcal{A}^{\cdot}/\mathcal{E}^{\cdot}(Q(B),Q(A))$ is an isomorphism.

(2!) Every morphism of an object of $\mathcal{E}^{\cdot}$ into A is zero.

Now we associate to $\mathcal{E}^{\cdot}$ two other subcategories of $\mathcal{A}^{\cdot}$. Namely define $\underline{\mathcal{E}^{\cdot\perp}}$ (resp. $\underline{{}^{\perp}\mathcal{E}^{\cdot}}$) to be the full subcategory of $\mathcal{A}^{\cdot}$ generated by the objects A such that $\mathcal{A}^{\cdot}(B,A) = 0$ (resp. $\mathcal{A}^{\cdot}(A,B) = 0$) for all B in $\mathcal{E}^{\cdot}$. These are both thick subcategories of $\mathcal{A}^{\cdot}$. They are called the <u>right</u> and <u>left</u> <u>orthogonal complement</u> of $\mathcal{E}^{\cdot}$ respectively. Write $\underline{M^{\perp}} = \mathcal{M}(\mathcal{E}^{\cdot\perp})$ and $\underline{{}^{\perp}M} = \mathcal{M}({}^{\perp}\mathcal{E}^{\cdot})$.

Proposition 3: For an object A of $\mathcal{A}^{\cdot}$ consider the statements:

(1.) A is the range of a morphism in M with domain an object of ${}^{\perp}\mathcal{E}^{\cdot}$.

(2.) A is the domain of a morphism in ${}^{\perp}M$ with range an object of $\mathcal{E}^{\cdot}$.

(3.) The category ${}^{\perp}M_A$ (cf. page 17) of morphisms of ${}^{\perp}M$ with domain A has a final object.

(4.) The category M^A of morphisms in M with range A has an initial object.

(5.) The category ${}^{\perp}\mathcal{E}^{\cdot}/A$ of objects of ${}^{\perp}\mathcal{E}^{\cdot}$ over A has an initial object.

(6.) The category $\mathcal{E}^{\cdot}\backslash A$ of objects of $\mathcal{E}^{\cdot}$ under A has a final object.

Then we have: (1.) $\Leftrightarrow$ (2.) $\Leftrightarrow$ (3.) $\Leftrightarrow$ (6.) $\begin{matrix} \nearrow (4.) \\ \searrow (5.) \end{matrix}$

If also $({}^{\perp}\mathcal{E}^{\cdot})^{\perp} = \mathcal{E}^{\cdot}$, then all of the statements are equivalent.

<u>Proposition 4</u>: Let $i: \mathcal{E}^{\cdot} \longrightarrow \mathcal{A}^{\cdot}$ and ${}^{\perp}i: {}^{\perp}\mathcal{E}^{\cdot} \longrightarrow \mathcal{A}^{\cdot}$ be the inclusion functors. The following are equivalent:

(1.) The functor i has a left adjoint.

(2.) The functor Q has a left adjoint.

(3.) The functor ${}^{\perp}i$ has a right adjoint and $({}^{\perp}\mathcal{E}^{\cdot})^{\perp} = \mathcal{E}^{\cdot}$.

(This proposition also holds with right and left interchanged.)

III. RELATIVE DERIVED FUNCTORS

1. Relative Derived Categories

Let $\mathcal{A}$ be an additive category. $\mathcal{A}^{\cdot}$ is the triangulated category associated to $\mathcal{A}$ in II.4. Let $\mathcal{E}$ be an (injectively) closed class of sequences in $\mathcal{A}$. Denote by $\underline{\mathcal{E}^{\cdot}}$ the full subcategory of $\mathcal{A}^{\cdot}$ generated by the complexes in $\mathcal{E}$.

<u>Proposition 1</u>: $\mathcal{E}^{\cdot}$ is a thick subcategory of $\mathcal{A}^{\cdot}$.

<u>Proof</u>: Let $\mathcal{J} = \mathcal{d}(\mathcal{E})$. For each $I^{\cdot}$ in $\mathcal{J}$, $\mathcal{A}^{0}(\ .\ ,I^{\cdot}):\mathcal{A}^{\cdot}\longrightarrow\mathcal{Ab}$ is a cohomological functor (II.4). But then (II.5) $\mathcal{E}^{\cdot}(\mathcal{A}^{0}(\ .\ ,I^{\cdot})) \stackrel{\text{def}}{=} \mathcal{E}^{\cdot}(I^{\cdot})$ is a thick subcategory of $\mathcal{A}^{\cdot}$. Thus since $\mathcal{E}^{\cdot} = \bigcap_{I^{\cdot}\in\mathcal{J}} \mathcal{E}^{\cdot}(I^{\cdot})$, it is a thick subcategory of $\mathcal{A}^{\cdot}$. ▮

<u>Definition</u>: $\underline{D_{\mathcal{E}}^{\cdot}(\mathcal{A})} = \mathcal{A}^{\cdot}/\mathcal{E}^{\cdot}$ is called the <u>derived category of $\mathcal{A}$ relative to $\mathcal{E}$</u>.

We denote the quotient functor by $\underline{D}:\mathcal{A}^{\cdot}\longrightarrow D_{\mathcal{E}}^{\cdot}(\mathcal{A})$. Recall that $D:\mathcal{A}(A,B)\xrightarrow{\cong} D_{\mathcal{E}}^{0}(\mathcal{A})(A,B)$ for all A and B in $\mathcal{A}$.

Denote by $\underline{\mathcal{J}^{\cdot}}$ the full subcategory of $\mathcal{A}^{\cdot}$ generated by the complexes $I^{\cdot}$ with I^{n} in $\mathcal{J} = \mathcal{d}(\mathcal{E})$ for all n. I. e. $\mathcal{J}^{\cdot}$ is the full subcategory of $\mathcal{E}$-injective complexes. Note that $\mathcal{J}^{\cdot}_{+} = \mathcal{E}^{\cdot\perp}_{+}$ (II.10).

A morphism $a^{\cdot}:A^{\cdot}\longrightarrow B^{\cdot}$ in $\mathcal{A}^{\cdot}$ is called a <u>quasi-isomorphism</u> iff $D(a^{\cdot}):D(A^{\cdot})\longrightarrow D(B^{\cdot})$ is an isomorphism in $D^{\cdot}(\mathcal{A})$. Recall (II.6) that this is equivalent to the requirement that the mapping cone of $a^{\cdot}$ be in $\mathcal{E}^{\cdot}$.

Lemma 1: Let $\mathcal{B}$ be a full additive subcategory of $\mathcal{A}$ satisfying: For every $A_1\longrightarrow A_2$ in $\mathcal{A}$, there exists $A_2\longrightarrow B$ in $\mathcal{A}$ with

B in $\mathcal{B}$ and $A_1 \longrightarrow A_2 \longrightarrow B$ in $\mathcal{E}$. Then for every $A^\cdot$ in $\mathcal{A}^\cdot_+$ there is a quasi-isomorphism $A^\cdot \longrightarrow B^\cdot$ in $\mathcal{A}^\cdot_+$ with $B^\cdot$ in $\mathcal{B}^\cdot_+$.

<u>Proof</u>: Let $A^\cdot$ be in $\mathcal{A}^\cdot_+$. We may assume that $A^n = 0$ for $n < 0$. We define $a^\cdot : A^\cdot \longrightarrow B^\cdot$ inductively. First choose $a^0 : A^0 \longrightarrow B^0$ so that B^0 is in $\mathcal{B}$ and $0 \longrightarrow A^0 \longrightarrow B^0$ is in $\mathcal{E}$. Now assuming that $a^n : A^n \longrightarrow B^n$ and $\partial^{n-1} : B^{n-1} \longrightarrow B^n$ have been properly defined, choose B^{n+1} in $\mathcal{B}$ and $B^n \oplus A^{n+1} \longrightarrow B^{n+1}$ so that $A^n \xrightarrow{(a^n, -d^n)} B^n \oplus A^{n+1} \longrightarrow B^{n+1}$ is in $\mathcal{E}$.

$B^n \oplus A^{n+1} \longrightarrow B^{n+1}$ defines a^{n+1} and ∂^n by being $\begin{pmatrix} \partial^n \\ a^{n+1} \end{pmatrix}$.

Now $(B^\cdot, \partial^\cdot)$ is a chain complex and $a^\cdot : A^\cdot \longrightarrow B^\cdot$ is a chain map.

To show that $a^\cdot : A^\cdot \longrightarrow B^\cdot$ is a quasi-isomorphism we show that the mapping cone of $a^\cdot$, $C(a^\cdot)$, is in $\mathcal{E}^\cdot_+$.

$$C(a^\cdot) = \ldots \longrightarrow B^n \oplus A^{n+1} \xrightarrow{\begin{pmatrix} \partial^n & 0 \\ a^{n+1} & -d^{n+1} \end{pmatrix}} B^{n+1} \oplus A^{n+2} \longrightarrow \ldots$$

Let I be in $\mathcal{J}$. Then $\mathcal{A}(C(a^\cdot), I)$ is:

$$\ldots \rightarrow \mathcal{A}(B^{n+1} \oplus A^{n+2}, I) \rightarrow \mathcal{A}(B^n \oplus A^{n+1}, I) \rightarrow \mathcal{A}(B^{n-1} \oplus A^n, I) \rightarrow \ldots$$

and we must show it is exact.

By construction we know that

$\mathcal{A}(B^{n+1}, I) \longrightarrow \mathcal{A}(B^n \oplus A^{n+1}, I) \longrightarrow \mathcal{A}(A^n, I)$ is exact. Now the kernel of $\mathcal{A}(B^n \oplus A^{n+1}, I) \longrightarrow \mathcal{A}(A^n, I)$ is

$\left\{ \begin{pmatrix} f \\ g \end{pmatrix} \,\middle|\, fa^n - gd^n = 0 \right\} \overset{\text{def}}{=} K_2$, and the image of $\mathcal{A}(B^{n+1}, I) \longrightarrow \mathcal{A}(B^n \oplus A^{n+1}, I)$ is

$$\left\{ \begin{pmatrix} \bar{f}\partial^n \\ \bar{f}a^{n+1} \end{pmatrix} \,\middle|\, \bar{f} : B^{n+1} \longrightarrow I \right\} \overset{\text{def}}{=} I_2.$$

By comparison the kernel of

$\mathcal{A}(B^n \oplus A^{n+1}, I) \longrightarrow \mathcal{A}(B^{n-1} \oplus A^n, I)$ is

$\left\{\begin{pmatrix} f \\ g \end{pmatrix} \middle| \begin{pmatrix} f\partial^{n-1} \\ fa^n - gd^n \end{pmatrix} = 0\right\} \stackrel{\text{def}}{=} K_1$, while the image of

$\mathcal{A}(B^{n+1} \oplus A^{n+2}, I) \longrightarrow \mathcal{A}(B^n \oplus A^{n+1}, I)$ is

$\left\{\begin{pmatrix} \bar{f}\partial^n \\ \bar{f}a^{n+1} - \bar{g}d^{n+1} \end{pmatrix} \middle| \begin{pmatrix} \bar{f} \\ \bar{g} \end{pmatrix} : B^{n+1} \oplus A^{n+2} \longrightarrow I\right\} \stackrel{\text{def}}{=} I_1$.

Thus $I_2 \leq I_1 \leq K_1 \leq K_2$. But $I_2 = K_2$! Therefore $I_1 = K_1$ and the sequence is exact. ∎

The first application of Lemma 1 is to the situation where $\mathcal{A}$ has enough $\mathcal{E}$-injectives and $\mathcal{B}$ is the full subcategory of $\mathcal{A}$ generated by $\mathcal{J}$. This subcategory will also be denoted by $\mathcal{J}$. Lemma 1 shows that for every $A^\cdot$ in $\mathcal{A}^\cdot_+$ there is an $I^\cdot$ in $\mathcal{J}^\cdot_+$ and a quasi-isomorphism $A^\cdot \longrightarrow I^\cdot$. Now either by mimicing the standard arguments about injective resolutions or by applying Proposition II.10.3, we see that $A^\cdot \rightsquigarrow I^\cdot$ defines a functor $\underline{I} : \mathcal{A}^\cdot_+ \longrightarrow \mathcal{J}^\cdot_+$. Furthermore the unique quasi-isomorphism $A^\cdot \longrightarrow I(A^\cdot)$ guaranteed by the above argument exhibits I as the left adjoint of the inclusion functor $\mathcal{J}^\cdot_+ \longrightarrow \mathcal{A}^\cdot_+$.

Now note that if $A^\cdot$ is in $\mathcal{E}^\cdot_+$, then $I(A^\cdot) = 0$ ("every acyclic injective complex is null-homotopic"), so I factors through $D^\cdot_+(\mathcal{A})$ to give a functor $\underline{D^\wedge_+} : D^\cdot_+(\mathcal{A}) \longrightarrow \mathcal{J}^\cdot_+$ which is easily seen to be the right adjoint of $D_+ : \mathcal{A}^\cdot_+ \longrightarrow D^\cdot_+(\mathcal{A})$. (cf. Proposition II.10.4) Invoking Proposition II.10.2 we see that D_+ and $D^\wedge_+$ define an equivalence of categories: $D^\cdot_+(\mathcal{A}) \cong \mathcal{J}^\cdot_+$. Thus:

<u>Theorem 1</u>: If $\mathcal{E}$ is an injective class in $\mathcal{A}$, then $D^\cdot_+(\mathcal{A}) = \mathcal{J}^\cdot_+$.

Remark: If $\mathcal{E}'$ is a projective class in $\mathcal{A}$ with $\mathcal{P} = \mathcal{P}(\mathcal{E}')$, then $D^{\cdot}_{\mathcal{E}'-}(\mathcal{A}) \cong \mathcal{P}^\cdot_-$. Further if $\mathcal{E}$ and $\mathcal{E}'$ are complementary classes in $\mathcal{A}$, then $D^{\cdot}_{\mathcal{E}}(\mathcal{A}) = D^{\cdot}_{\mathcal{E}'}(\mathcal{A})$ and in particular $\mathcal{J}^{\cdot}_{0} \cong D^{\cdot}_{0}(\mathcal{A}) \cong \mathcal{P}^{\cdot}_{0}$. ("0" refers to bounded complexes)

2. Relative Derived Functors

The definitions of the various (right, left; total, cohomological) derived functors relative to $\mathcal{E}$ are the obvious modifications of those in II.8 and will not be given. The notation will be the same as in II.8 except that an $\mathcal{E}$ may be added for emphasis.

As in part II there are two main results on the existence of derived functors.

In the following $\mathcal{B}$ will always be an abelian category and $D^{\cdot}(\mathcal{B})$ will mean the derived category of $\mathcal{B}$ as discussed in II.

Theorem 1: Let $F^{\cdot}:\mathcal{A}^{\cdot}_{*}\longrightarrow\mathcal{B}^{\cdot}_{*}$ (where * is either +, -, 0 or nothing) be an exact functor. Suppose there is a triangulated subcategory $\mathcal{B}^{\cdot}_{*}$ of $\mathcal{A}^{\cdot}_{*}$ such that:

(1.) Every $A^{\cdot}$ in $\mathcal{A}^{\cdot}_{*}$ admits a quasi-isomorphism $A^{\cdot}\longrightarrow B^{\cdot}$ with $B^{\cdot}$ in $\mathcal{B}^{\cdot}_{*}$.

(2.) If $B^{\cdot}$ is in $\mathcal{B}^{\cdot}_{*}\cap\mathcal{E}^{\cdot}_{*}$, then $F^{\cdot}(B^{\cdot})$ is acyclic.

Then

(a.) $\underline{\underline{R}}F^{\cdot}:D^{\cdot}_{\mathcal{E}_{*}}(\mathcal{A})\longrightarrow D^{\cdot}_{*}(\mathcal{B})$ exists.

(b.) If $A^{\cdot}\longrightarrow B^{\cdot}$ is a quasi-isomorphism with $B^{\cdot}$ in $\mathcal{B}^{\cdot}_{*}$, then $\underline{\underline{R}}F^{\cdot}\circ D_{*}(A^{\cdot}) \cong D_{*}\circ F^{\cdot}(B^{\cdot})$.

(c.) $H^{0}\circ F^{\cdot}:\mathcal{A}^{\cdot}_{*}\longrightarrow\mathcal{B}$ has a right derived functor $RH^{0}\circ F^{\cdot}$ which is just $H^{0}\circ\underline{\underline{R}}F^{\cdot}$.

Proof: By Proposition II.10.1 $\mathcal{B}^{\cdot}_{*}\cap\mathcal{E}^{\cdot}_{*}$ is a thick subcategory of $\mathcal{B}^{\cdot}_{*}$ and the natural inclusion

$j: \mathcal{B}^{\cdot}_{*}/\mathcal{B}^{\cdot}_{*} \cap \mathcal{E}^{\cdot}_{*} \xrightarrow{j} D^{\cdot}_{\mathcal{E}_*}(\mathcal{A})$ is fully faithful. In fact from (1.) it is an equivalence of categories. Now since $F^{\cdot}(\mathcal{B}^{\cdot}_{*} \cap \mathcal{E}^{\cdot}_{*}) \subseteq \mathcal{E}^{\cdot}_{*}(H^0)$, $F^{\cdot}$ induces a unique functor $\mathcal{B}^{\cdot}_{*}/\mathcal{B}^{\cdot}_{*} \cap \mathcal{E}^{\cdot}_{*} \longrightarrow D^{\cdot}_{*}(\mathcal{B})$. $\underline{\underline{R}}F^{\cdot}$ is defined to be this functor composed with j^{-1}.

Conclusion (b.) is satisfied by the construction of $\underline{\underline{R}}F^{\cdot}$. The universality of $\underline{\underline{R}}F^{\cdot}$ comes as follows: For any $A^{\cdot}$ in $\mathcal{A}^{\cdot}_{*}$, choose a quasi-isomorphism $A^{\cdot} \longrightarrow B^{\cdot}$, $B^{\cdot}$ in $\mathcal{B}^{\cdot}_{*}$. Then the morphisms $D_* \circ F^{\cdot}(A^{\cdot}) \longrightarrow D_* \circ F^{\cdot}(B^{\cdot})$ define a morphism of functors $\xi: D_* \circ F^{\cdot} \longrightarrow \underline{\underline{R}}F^{\cdot} \circ D_*$ (since $\underline{\underline{R}}F^{\cdot} \circ D_*(A^{\cdot}) \cong D_* \circ F^{\cdot}(B^{\cdot})$). Now given an exact functor $G^{\cdot}: D^{\cdot}_{\mathcal{E}_*}(\mathcal{A}) \longrightarrow D^{\cdot}_{*}(\mathcal{B})$, and $\zeta: D_* \circ F^{\cdot} \longrightarrow G^{\cdot} \circ D_*$, define $\eta: \underline{\underline{R}}F^{\cdot} \longrightarrow G^{\cdot}$ by $\eta_{A^{\cdot}}: \underline{\underline{R}}F^{\cdot}(A^{\cdot}) \longrightarrow G^{\cdot}(A^{\cdot})$ is

$$\underline{\underline{R}}F^{\cdot}(A^{\cdot}) \cong D_* \circ F^{\cdot}(B^{\cdot}) \xrightarrow{\zeta_{B^{\cdot}}} G^{\cdot} \circ D_*(B^{\cdot}) \cong G^{\cdot}(A^{\cdot}) .$$

This is clearly the desired morphism of functors.

(c.) follows by the same sort of argument. ∎

<u>Application 1</u>: Suppose $\mathcal{E}$ is an injective class in $\mathcal{A}$. Then Theorem 1 holds for arbitrary $F^{\cdot}$ with $* = +$ and $\mathcal{B}^{\cdot}_{*} = \mathcal{J}^{\cdot}_{+}$.

Note: This also follows from Theorem 1 of sect. 1.

<u>Application 2</u>: Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor. Suppose there is a full additive subcategory $\mathcal{A}c$ of $\mathcal{A}$ such that:

(1.) If $A^{\cdot}$ is in $\mathcal{A}c^{\cdot}_{+} \cap \mathcal{E}^{\cdot}_{+}$, then $F^{\cdot}(A^{\cdot})$ is acyclic; and

(2.) For all $A_1 \longrightarrow A_2$ in $\mathcal{A}$ there exists $A_2 \longrightarrow B$ in $\mathcal{A}$ so that B is in $\mathcal{A}c$ and $A_1 \longrightarrow A_2 \longrightarrow B$ is in $\mathcal{E}$.

Then Theorem 1 applies with $* = +$, $\mathcal{B}^{\bullet}_{*} = \mathcal{A}c^{\bullet}_{+}$ and $F^{\bullet}$ the prolongation of F to $\mathcal{A}^{\bullet}_{+}$.

Also note that if A is in $\mathcal{A}c$, then $R^nF(A) = 0$ for $n > 0$.

Once Lemma 1 of sect. 1 is recalled all that remains to be checked is the last remark. But in that case $R^nF(A) = RF \circ S^n(A) = H^n \circ \underline{\underline{R}}F(D_+(A)) = H^n \circ D_+ \circ F^{\bullet}(A)$ which is clearly 0 for $n > 0$.

<u>Theorem 2</u>: Let $\mathcal{A}$, $\mathcal{L}$ be as above. Let $\mathcal{B}$ be an abelian category for which direct limits from pseudo-filtering categories exist and are exact. Then every cohomological functor $H:\mathcal{A}^{\bullet} \longrightarrow \mathcal{B}$ has a right derived functor $RH:D^{\bullet}(\mathcal{A}) \longrightarrow \mathcal{B}$.

<u>Proof</u>: If $A^{\bullet}$ is an object of $\mathcal{A}^{\bullet}$, let $M^{A^{\bullet}}$ be the category of quasi-isomorphisms with domain $A^{\bullet}$ (cf. pages 17 & 26). I. e. an object in $M^{A^{\bullet}}$ is a quasi-isomorphism $m:A^{\bullet} \longrightarrow B^{\bullet}$. $M^{A^{\bullet}}$ is pseudo-filtering (cf. II.6). Now H defines a functor, also denoted by H, from $M^{A^{\bullet}}$ to $\mathcal{B}$ by $H(m) = H(\mathrm{range}(m))$. Define $RH(D(A^{\bullet})) = \varinjlim_{M^{A^{\bullet}}} H(\ .\ )$. That RH is a functor $RH:D^{\bullet}(\mathcal{A}) \longrightarrow \mathcal{B}$ follows once we notice that

$$D^{\bullet}(\mathcal{A})(D(A^{\bullet}),D(B^{\bullet})) = \varinjlim_{M^{B^{\bullet}}} \mathcal{A}^{\bullet}(A^{\bullet},\ .\ ),$$

and this is an easy consequence of (MS.2) of II.5. It is a cohomological functor because H is a cohomological functor and $\varinjlim$ is exact in the present context.

Now to show that RH is actually the right derived functor of H define $H \longrightarrow RH \circ D$ as the morphism $H(A^{\bullet}) \longrightarrow \varinjlim_{M^{A^{\bullet}}} H(\ .\ )$

resulting from $1_{A^\cdot}$ being in $M^{A^\cdot}$. Now if $H \longrightarrow G \circ D$ is another morphism of cohomological functors, define $RH \longrightarrow G$ by

$$RH(D(A^\cdot)) = \varinjlim_{M^{A^\cdot}} H(\ .\ ) \longrightarrow G(D(A^\cdot)) \text{ has components}$$

$$H(m) = H(B^\cdot) \longrightarrow G(D(B^\cdot)) \xrightarrow[\cong]{G(D(m))^{-1}} G(D(A^\cdot)).$$

(here $m: A^\cdot \longrightarrow B^\cdot \in M^{A^\cdot}$) This is clearly the unique morphism having the desired property. ▮

Remark: The remarks made after Theorem II.9.2 apply here equally well.

3. Applications

The usual approach to relative homological algebra has as data an abelian category $\mathcal{A}$ together with a "proper class" of short exact sequences (MacLane [16]) or equivalently an "h. f. class" of epimorphisms (Buchsbaum [2], Mitchell [17]).

In the present approach we may consider an <u>exact closed class</u>, i. e. a closed class in which every sequence is exact. The relation between such exact closed classes and h. f. classes is discussed by Eilenberg and Moore [8 , sect. 4] and by Mitchell [17, V.7] . It is as follows: If $\mathcal{E}$ is a class of exact sequences, associate to it a class $\mathcal{M}$ of epimorphisms by: $A \longrightarrow B$ is in $\mathcal{M}$ iff $A \longrightarrow B \longrightarrow 0$ is in $\mathcal{E}$. Also associate to $\mathcal{M}$ a class of exact sequences $\bar{\mathcal{E}}$ by: $A \longrightarrow B \longrightarrow C$ is in $\bar{\mathcal{E}}$ iff it factors as

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & B \arrow[r] & C \\ & \bar{B} \arrow[u] & \end{tikzcd}$$

where $\bar{B} \longrightarrow B$ is the kernel of $B \longrightarrow C$ and $A \longrightarrow \bar{B}$ is in $\mathcal{M}$. $\bar{\mathcal{E}}$ is closed, and if $\mathcal{E}$ was already closed, $\mathcal{E} = \bar{\mathcal{E}}$.

Now if $\mathcal{E}$ is a closed class of exact sequences, then $\mathcal{M}$ is an h. f. class of epimorphisms. Conversely if $\mathcal{M}$ is an h. f. class of epimorphisms, then $\mathcal{E}$, the class of sequences associated to $\mathcal{M}$, is a (projectively) closed class. Further $\mathcal{E}$ is the class of proper exact sequences associated to $\mathcal{M}$ in MacLane's approach. Thus our approach applies to this situation and of course gives the same results when $\mathcal{A}$ has enough proper, i. e. $\mathcal{E}$-projectives. Also note that this shows that Theorem 2 of sect. 2 is a strict generalization of Buchsbaum's result [3].

Now both Buchsbaum and MacLane show that Yoneda's construction of Ext^n as equivalence classes of n-fold extensions works also with proper extensions. Below we recover and partially generalize this result.

Anticipating our next result, we define <u>$\mathrm{Ext}^n_{\mathcal{P}}(A,B)$</u> $= D^n_{\mathcal{P}}(\mathcal{A})(D(A),D(B))$ for all A and B in $\mathcal{A}$. More precisely define <u>$\mathrm{Ext}^n_{\mathcal{P}}(\ .,\ .)$</u> $= D^n_{\mathcal{P}}(\mathcal{A})(.\ ,\ .) : D^{\bullet}_{\mathcal{P}}(\mathcal{A})^{op} \times D^{\bullet}_{\mathcal{P}}(\mathcal{A}) \longrightarrow \mathcal{Ab}$.

<u>Proposition 1</u>: $\mathrm{Ext}^n_{\mathcal{P}}(D(A^{\bullet}),\ .) = R^n_{\mathcal{P}}\mathcal{A}^0(A^{\bullet},\ .)$ and $\mathrm{Ext}^n_{\mathcal{P}}(.\ ,D(B^{\bullet})) = R^n_{\mathcal{P}}\mathcal{A}^0(.\ ,\ B^{\bullet})$.

<u>Proof</u>: Clearly we need consider only one of the two cases and that only when n = 0. We show that $\mathrm{Ext}^0_{\mathcal{P}}(D(A^{\bullet}),\ .) = R^0_{\mathcal{P}}\mathcal{A}^0(A^{\bullet},\ .)$. The morphism

$\mathcal{A}^0(A^{\bullet},\ .) \longrightarrow \mathrm{Ext}^0_{\mathcal{P}}(D(A^{\bullet}),\ D(\ .\ )) = D^0_{\mathcal{P}}(\mathcal{A})(D(A^{\bullet}),\ D(\ .\ ))$

is just D. That this is universal follows immediately from Yoneda's lemma on representable functors. ∎

<u>Corollary</u>: If $\mathcal{P}'$ is a projectively closed class in $\mathcal{A}$

complementary to the injectively closed class $\mathcal{E}$, then, considered as bifunctors, $R^n_{\mathcal{E}}\mathcal{A}^0(A^\bullet, \,.\,)$ and $R^n_{\mathcal{E}}\mathcal{A}^0(\,.\,,B^\bullet)$ are isomorphic.

Note that by application 1 of Theorem 2.1 $\mathrm{Ext}^n_{\mathcal{E}}(\,.\,,\,.\,)$ when considered on $\mathcal{A}^{op}\times\mathcal{A}$ is just the "usual" $\mathrm{Ext}^n_{\mathcal{E}}(\,.\,,\,.\,)$ whenever $\mathcal{A}$ has either enough injectives or enough projectives.

Next let $\underline{E^n_{\mathcal{E}}(A,C)}$ be the set of congruence classes of n-fold extensions of C by A. Recall that an <u>n-fold extension of C by A</u> is an exact sequence, i. e. an element of $\mathcal{E}$, of the form $0\to C\to B^1\to \ldots \to B^n\to A\to 0$. If $0\to C\to {}'B^1\to \ldots \to {}'B^n\to A\to 0$ is another extension, the two are <u>congruent</u> iff there is a third extension $0\to C\to {}''B^1\to \ldots \to {}''B^n\to A\to 0$ and maps as below:

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] & B^1 \arrow[r] & \cdots \arrow[r] & B^n \arrow[r] & A \arrow[r] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[u, "1_C"] \arrow[d, "1_C"'] & {}''B^1 \arrow[r] \arrow[u] \arrow[d] & \cdots \arrow[r] & {}''B^n \arrow[r] \arrow[u] \arrow[d] & A \arrow[r] \arrow[u, "1_A"'] \arrow[d, "1_A"'] & 0 \\
0 \arrow[r] & C \arrow[r] & {}'B^1 \arrow[r] & \cdots \arrow[r] & {}'B^n \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

For details, see MacLane [16; III.5].

We define a map $E^n_{\mathcal{E}}(A,C)\longrightarrow \mathrm{Ext}^n_{\mathcal{E}}(A,C)$ as follows: Associate to an extension (as above) a complex in $\mathcal{A}^\bullet$, namely the complex: $\ldots \to 0\to C\to B^1\to \ldots \to B^n\to 0\to \ldots$ Call this complex $B^\bullet$. Note that the morphism $B^n\longrightarrow A$ gives a morphism $B^\bullet\longrightarrow S^{-n}(A)$ which is actually a quasi-isomorphism. Also the identity map of C gives a morphism $B^\bullet\longrightarrow C$. But the datum $S^{-n}(A)\xleftarrow{\approx}B^\bullet\longrightarrow C$ ($\approx$ means quasi-isomorphism) is exactly an element of the pseudo-filtering system with direct limit $D^n_{\mathcal{E}}(\mathcal{A})(D(A),D(C)) = \mathrm{Ext}^n_{\mathcal{E}}(A,C)$.

Now consider the above diagram and the three complexes $B^\bullet$, $''B^\bullet$, and $'B^\bullet$ arising from it. The morphisms of extensions involved in the diagram give rise to morphisms of complexes $B^\bullet \xleftarrow{\approx} ''B^\bullet \xrightarrow{\approx} 'B^\bullet$ which are actually quasi-isomorphisms. Thus we get the diagram

$$\begin{tikzcd} & B^\bullet \arrow[dl, "\approx"'] \arrow[dr] & \\ S^{-n}(A) & ''B^\bullet \arrow[l, "\approx"'] \arrow[u, "\approx"'] \arrow[d, "\approx"] \arrow[r] & C \\ & 'B^\bullet \arrow[ul, "\approx"] \arrow[ur] & \end{tikzcd}$$

which is precisely what is needed to guarantee that $S^{-n}(A) \xleftarrow{\approx} B^\bullet \longrightarrow C$ and $S^{-n}(A) \xleftarrow{\approx} 'B^\bullet \longrightarrow C$ define the same element in $\mathrm{Ext}^n(A,C)$.

Thus we have a well-defined function $E^n_{\mathcal{E}}(A,C) \longrightarrow \mathrm{Ext}^n_{\mathcal{E}}(A,C)$. Indeed the above argument shows that it is actually a monomorphism. Now $E^n_{\mathcal{E}}$, just as $\mathrm{Ext}^n_{\mathcal{E}}$, is actually an additive bifunctor to the category of abelian groups. Simple checking of the definitions shows that $E^n_{\mathcal{E}} \longrightarrow \mathrm{Ext}^n_{\mathcal{E}}$ as defined above is actually a morphism of bifunctors.

Now suppose $\mathcal{A}$ is an abelian category and $\mathcal{E}$ is an exact closed class. Then we can define a map $\mathrm{Ext}^n_{\mathcal{E}}(A,C) \longrightarrow E^n_{\mathcal{E}}(A,C)$ as follows: If s is in $\mathrm{Ext}^n_{\mathcal{E}}(A,C)$ it is represented by $S^{-n}(A) \xleftarrow{\approx} B^\bullet \longrightarrow C$. Since $B^\bullet \xrightarrow{\approx} S^{-n}(A)$ is a quasi-isomorphism, the sequence $0 \to \mathrm{Ker}\, d^0 \to B^1 \to B^2 \to \ . \ . \ . \to B^n \to A \to 0$ is in $\mathcal{E}$. Further $B^\bullet \longrightarrow C$ gives a morphism $\mathrm{Ker}\, d^0 \longrightarrow C$. Now the above sequence gives an element of $E^n_{\mathcal{E}}(\mathrm{Ker}\, d^0, C)$ while $\mathrm{Ker}\, d^0 \longrightarrow C$ is in $E^0_{\mathcal{E}}(\mathrm{Ker}\, d^0, C)$. Composition of the two gives an element of $E^n_{\mathcal{E}}(A,C)$. This is well-defined and gives the desired map. Again checking multitudinous details shows that this gives a morphism of bifunctors $\mathrm{Ext}^n_{\mathcal{E}}(\cdot\,,\,\cdot) \longrightarrow E^n_{\mathcal{E}}(\cdot\,,\,\cdot)$ which is the inverse of the morphism of bifunctors in the last paragraph.

This is summarized in

<u>Theorem 1</u>: Let $\mathcal{A}$, $\mathcal{E}$ be as usual. Let $E^n_{\mathcal{E}}(A,C)$ be the group of congruence classes of n-fold extensions of C by A. There is an inclusion of bifunctors $E^n_{\mathcal{E}}(\cdot\,,\,\cdot) \longrightarrow \mathrm{Ext}^n_{\mathcal{E}}(\cdot\,,\,\cdot)$. Furthermore if $\mathcal{A}$ is abelian and $\mathcal{E}$ is an exact closed class of sequences, then it is actually an isomorphism.

Another topic which has been treated but will be here mentioned only briefly is the Yoneda pairing. If $F:\mathcal{A} \longrightarrow \mathcal{A}b$ is an additive functor and $RF:D^\cdot(\mathcal{A}) \longrightarrow \mathcal{A}b$ exists, then there is a natural bilinear map

$$\mathrm{Ext}^n_{\mathcal{E}}(A^\cdot,B^\cdot) \times R^m_{\mathcal{E}}F(A^\cdot) \longrightarrow R^{n+m}_{\mathcal{E}}(B^\cdot)$$

which is gotten from the identification $\mathrm{Ext}^n_{\mathcal{E}}(A^\cdot,B^\cdot) = D^n_{\mathcal{E}}(\mathcal{A})(D(A^\cdot),D(B^\cdot))$. This is precisely the pairing defined by Yoneda [20], whenever his definition of Ext coincides with ours. In particular this gives the Yoneda product in $\mathrm{Ext}^\cdot_{\mathcal{E}}(A^\cdot,A^\cdot)$ and makes it into a graded ring.

The properties of this pairing are very easily derived from this definition.

4. Problems and Prospects

There are several interesting problems which are now open. First, can the isomorphism of the last theorem be proved in a more general setting? Probably the proof sketched above works whenever $\mathcal{A}$ has kernels and $\mathcal{E}$ is an exact closed class. Too, Yoneda's definition of the Yoneda pairing, particularly $\mathrm{Ext}^0(A,B) \times \mathrm{Ext}^n(B,C) \longrightarrow \mathrm{Ext}^n(A,C)$, clearly doesn't work in the most general case considered here, so the situation is unclear.

Second, can $RF/_{\mathcal{A}}$ be characterized by something analogous to the "classical" characterization of right derived functors?

There is some hope for this since short exact sequences still give rise to long exact sequences. This happens as follows: If $0 \to A \to B \to C \to 0$ is in $\mathcal{E}$, then

$$\begin{tikzcd} A \arrow[dr] & & C(A \to B) \arrow[ll] \\ & B \arrow[ur] & \end{tikzcd}$$

is a distinguished triangle and $C(A \to B)$ is quasi-isomorphic to C. Thus there is a map (in $\mathcal{A}^{\bullet}$) $C \to A$ of degree one so that

$$\begin{tikzcd} A \arrow[dr] & & C \arrow[ll] \\ & B \arrow[ur] & \end{tikzcd}$$

is a distinguished triangle. But as remarked much earlier distinguished triangles give long exact sequences. In the case Verdier considered (the absolute case) all distinguished triangles arise in this way. Is this true in general?

Finally we mention that the stable homotopy category is a triangulated category. A cohomological functor is then exactly a generalized homology theory.

Now Freyd has devised a construction which embeds the stable homotopy category fully and faithfully into a very special type of abelian category. (This is as yet unpublished, but should appear in the Proceedings of the Conference on Categorical Algebra to be published by Springer.) This construction turns out to use precisely the fact that the stable homotopy category is a triangulated category. With this observation the same construction imbeds any triangulated category into an abelian category. In this context the problem of finding derived functors is the problem of "approximating" the given functor by an exact functor on the new abelian category.

Several questions arise here. For what is this construction useful? What relation does this have with Freyd's "relative homological algebra made absolute"? How does it happen that both stable homotopy theory and derived categories are triangulated categories? Are they related? Are there any other naturally occuring triangulated categories?

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BIOGRAPHICAL NOTE

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