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Schur dynamics of the Schur processes

SCHUR DYNAMICS OF THE SCHUR PROCESSES

Alexei Borodin

Abstract. We construct discrete time Markov chains that preserve the class of Schur processes on partitions and signatures.

One application is a simple exact sampling algorithm for q^{volume}-distributed skew plane partitions with an arbitrary back wall. Another application is a construction of Markov chains on infinite Gelfand-Tsetlin schemes that represent deterministic flows on the space of extreme characters of the infinite-dimensional unitary group.

Contents

Introduction

The *Schur processes* were introduced in [OR1] as a class of measures on sequences of partitions in order to study large random plane partitions with weights proportional to q^{volume}, $0 < q < 1$. The concept generalized that of the *Schur measures* introduced earlier in [Ok]. The asymptotic techniques of [OR1] were developed further in [OR2] to study the asymptotics of large skew plane partitions, see also [BMRT].

The range of applications of the Schur measures and Schur processes expanded quickly; apart from random plane partitions they have been applied to harmonic analysis on the infinite symmetric group [Ok], Szegö-type formulas for Toeplitz determinants [BO], relative Gromov-Witten theory of $\mathbb{C}^*$ [OP], random domino tilings of the Aztec diamond [J2], discrete and continuous polynuclear growth processes in 1+1 dimensions [PS], [J1], topological string theory [ORV], and so forth.

The goal of this paper is to define discrete time Markov chains that map Schur processes to themselves, possibly modifying the parameters. We also define Markov chains on the *two-sided Schur processes* introduced below; the principal difference

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of those from the Schur processes is that they live on sequences of *signatures* that, unlike partitions, may have negative parts.

The dynamics we construct is also 'Schur like'; for example, an evolution of a partition or a signature that represents a fixed slice of the (possibly two-sided) Schur process is also a (possibly two-sided) Schur process.

We present two applications of the construction.

First, we give an exact sampling algorithm for measures of type q^{volume} on skew plane partitions. Other sampling algorithms for such measures are known, see [BFP] and references therein. However, it seems that the algorithm we suggest is simpler; for skew plane partitions with support fitting in $A \times B$ box, the algorithm consists in sampling no more that $AB(B+1)/2$ independent one-dimensional geometric distributions. A short 'code' for the algorithm can be found in Section 7. Exact sampling algorithms for *boxed* plane partitions based on similar ideas were constructed in [BG], [BGR].

The second application is a construction of Markov chains on infinite Gelfand-Tsetlin schemes that preserve the class of Fourier transforms of the extreme characters of the infinite-dimensional unitary group, see Section 4 for details. For similar developments on the infinite-dimensional orthogonal group see [BK].

A special case of the Markov dynamics that we construct has been studied in detail in [BF]. One of the goals of this paper is to provide a more general setup (a broad class of initial conditions and a multi-parameter family of update rules) for large time asymptotic analysis of the dynamics.

The construction below is based on a formalism developed in [BF], which in its turn was based on an idea from [DF]. However, our exposition is self-contained.

1. Nonnegative specializations of the Schur functions

In what follows we use the notation of [M].

Let Λ be the algebra of symmetric functions. A *specialization* ρ of Λ is an algebra homomorphism of Λ to $\mathbb{C}$; we denote the application of ρ to $f \in \Lambda$ as $f(\rho)$. The *trivial* specialization $\varnothing$ takes value 1 at the constant function $1 \in \Lambda$ and takes value 0 at any homogeneous $f \in \Lambda$ of degree ≥ 1.

For two specializations ρ_1 and ρ_2 we define their union $\rho = (\rho_1, \rho_2)$ as the specialization defined on Newton power sums via

$$p_n(\rho_1, \rho_2) = p_n(\rho_1) + p_n(\rho_2), \qquad n \geq 1.$$

Definition 1. We say that a specialization ρ of Λ is *nonnegative* if it takes nonnegative values on the Schur functions: $s_\lambda(\rho) \geq 0$ for any partition λ.

The classification of all nonnegative specializations is a classical result proved independently by Aissen, Edrei, Schoenberg, and Whitney [AESW] (see also [E]) and Thoma [T]. It says that a specialization ρ is nonnegative if and only if the generating function of the images of complete homogeneous functions has the form

$$(1) \qquad H(\rho; u) := \sum_{n=0}^{\infty} h_n(\rho) u^n = e^{\gamma u} \prod_{i \geq 1} \frac{1 + \beta_i u}{1 - \alpha_i u}$$

for certain nonnegative $\{\alpha_i\}$, $\{\beta_i\}$, and γ such that $\sum_i(\alpha_i + \beta_i) < \infty$.

It turns out that nonnegativity of $s_\lambda(\rho)$ for all λ is equivalent to nonnegativity of the images of the skew Schur functions $s_{\lambda/\mu}(\rho)$ for all λ and μ. Hence, via the Jacobi-Trudi formula

$$s_{\lambda/\mu} = \det[h_{\lambda_i-i-\mu_j+j}]_{i,j=1}^r, \qquad r \geq \max\{\ell(\lambda), \ell(\mu)\},$$

the classification of nonnegative specializations is equivalent to that of totally nonnegative triangular Toeplitz matrices with diagonal entries equal to 1. An excellent exposition of deep relations of this classification result to representation theory of the infinite symmetric group can be found in Kerov's book [K].

For a single α or a single β specialization, the values of skew Schur functions are easy to compute:

(2)

$$H(\rho; u) = \frac{1}{1 - \alpha u} \quad \text{implies} \quad s_{\lambda/\mu}(\rho) = \begin{cases} \alpha^{|\lambda|-|\mu|}, & \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots, \\ 0, & \text{otherwise}; \end{cases}$$

(3)

$$H(\rho; u) = 1 + \beta u \quad \text{implies} \quad s_{\lambda/\mu}(\rho) = \begin{cases} \beta^{|\lambda|-|\mu|}, & \lambda_j - \mu_j \in \{0,1\} \text{ for all } j \geq 1, \\ 0, & \text{otherwise}. \end{cases}$$

We say that a nonnegative specialization ρ of Λ is *admissible* if the generating function (1) is holomorphic in a disc $D_r = \{u \in \mathbb{C} \mid |u| < r\}$ with $r > 1$. In other words, ρ is admissible iff $\alpha_i < r^{-1} < 1$ for all i.

Since $H(\rho_1, \rho_2; u) = H(\rho_1; u)H(\rho_2; u)$, the union of admissible specializations is admissible (unions of nonnegative specializations are also nonnegative).

For a nonnegative specialization ρ, denote by $\mathbb{Y}(\rho)$ the set of partitions (or Young diagrams) λ such that $s_\lambda(\rho) > 0$. We also call $\mathbb{Y}(\rho)$ the *support* of ρ. The set of all partitions will be denoted as $\mathbb{Y}$.

Using the combinatorial formula for the Schur functions [M, Sect. I.5 (5.12)] and the involution ω [M, Sect. I.2], it is not hard to show that if, for a nonnegative specialization ρ, in (1) $\gamma = 0$ and there are $p < \infty$ nonzero α_j's and $q < \infty$ nonzero β_j's, then $\mathbb{Y}(\rho)$ consists of the Young diagrams that fit into the Γ-shaped figure with p rows and q columns. Otherwise it is easy to see that $\mathbb{Y}(\rho) = \mathbb{Y}$.

In particular, if in (1) all β_j's and γ vanish, and there are p nonzero α_j's, then $\mathbb{Y}(\rho)$ consists of Young diagrams with no more than p rows. Such a specialization consists in assigning values α_j to p of the symmetric variables used to define Λ, and 0's to all the other symmetric variables.

We will also need minors of arbitrary (not necessarily triangular) doubly-infinite totally nonnegative Toeplitz matrices. The classification of such matrices was obtained by Edrei in [E], who proved an earlier conjecture of Schoenberg. The result is as follows.

A matrix $M = [M_{i-j}]_{i,j=-\infty}^{+\infty}$ is totally nonnegative if and only if, after a transformation of the form $M_n \mapsto cR^n M_n$ with $c > 0$, $R \geq 0$, the generating function of its entries has the form

$$(4) \quad H(M; u) := \sum_{n=-\infty}^{+\infty} M_n u^n$$

$$= e^{\gamma^+(u-1)+\gamma^-(u^{-1}-1)} \prod_{i=1}^\infty \left(\frac{1 + \beta_i^+(u-1)}{1 - \alpha_i^+(u-1)} \frac{1 + \beta_i^-(u^{-1}-1)}{1 - \alpha_i^-(u^{-1}-1)} \right)$$

for certain nonnegative $\{\alpha_j^\pm\}$, $\{\beta_j^\pm\}$, and $\gamma^\pm$ such that $\sum(\alpha_i^+ + \alpha_i^- + \beta_i^+ + \beta_i^-) < \infty$ and $\beta_j \leq 1$ for all j. The parametrization of M by $(\{\alpha_j^\pm\}, \{\beta_j^\pm\}, \gamma^\pm)$ becomes unique if one adds the condition $\max_j\{\beta_j^+\} + \max_j\{\beta_j^-\} \leq 1$.

The generating function on the left is understood as the Laurent series of the holomorphic function in a neighborhood of the unit circle $|u| = 1$ that stands on the right. We call the largest annulus of the form $\{u \in \mathbb{C} \mid 0 \leq r_1 < |u| < r_2\}$ where $H(M; u)$ is holomorphic (the unit circle must be inside the annulus) the *analyticity annulus* of $H(M; u)$.

Definition 2. We say that a totally nonnegative Toeplitz matrix M is *admissible* if the generating function of its entries is given by (4) (i.e., no multiplication by cR^n is involved).

Note that since multiplying Toeplitz matrices corresponds to multiplying the generating functions (4), the product of two admissible matrices is admissible.

It will be convenient for us to use a similar notation for the minors of general Toeplitz matrices as in the triangular case (Jacobi-Trudi formula).

Define *signatures* of length n as n-tuples $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ of nonincreasing integers. We will also write $\ell(\lambda) = n$ and $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. By convention, there is a unique signature $\varnothing$ of length 0 with $|\varnothing| = 0$.

For any two signatures λ and μ of length n and an admissible M we set

$$s_{\lambda/\mu}(M) = \det\left[M_{\lambda_i - i - \mu_j + j}\right]_{i,j=1}^n.$$

For totally nonnegative M with only one $\alpha^\pm$ or $\beta^\pm$ parameter nonzero (and all other parameters being zero), one obtains formulas analogous to (2), (3):

$$(5) \quad H(\rho; u) = \frac{1}{1 - \alpha(u^{\pm 1} - 1)} \quad \text{implies} \quad s_{\lambda/\mu}(\rho) = \frac{1}{(1+\alpha)^n}\left(\frac{\alpha}{1+\alpha}\right)^{\pm|\lambda|\mp|\mu|}$$

if $\pm\lambda_j \mp \mu_j \geq 0$ for all $1 \leq j \leq n$, and 0 otherwise;

$$(6) \quad H(\rho; u) = 1 + \beta(u^{\pm 1} - 1) \quad \text{implies} \quad s_{\lambda/\mu}(\rho) = (1-\beta)^n\left(\frac{\beta}{1-\beta}\right)^{\pm|\lambda|\mp|\mu|}$$

if $\pm\lambda_j \mp \mu_j \in \{0, 1\}$ for all $1 \leq j \leq n$, and 0 otherwise.

Also, mimicking the property of the Schur functions, for a constant $c \in \mathbb{C}$, a signature ν of length $n + 1$, and a signature λ of length n, we set

$$(7) \qquad s_{\lambda/\mu}(c) := \begin{cases} c^{|\lambda| - |\mu|}, & \lambda_{n+1} \leq \mu_n \leq \lambda_n \leq \cdots \leq \lambda_2 \leq \mu_1 \leq \lambda_1, \\ 0, & \text{otherwise,} \end{cases}$$

with the convention that $0^0 = 1$.

2. The Schur processes

Pick a natural number N and admissible specializations $\rho_0^+, \ldots, \rho_{N-1}^+, \rho_1^-, \ldots, \rho_N^-$ of Λ. For any sequences $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(N)})$ and $\mu = (\mu^{(1)}, \ldots, \mu^{(N-1)})$ of partitions satisfying

$$(8) \qquad \varnothing \subset \lambda^{(1)} \supset \mu^{(1)} \subset \lambda^{(2)} \supset \mu^{(2)} \subset \cdots \supset \mu^{(N-1)} \subset \lambda^{(N)} \supset \varnothing$$

define their weight as

$$
\mathcal{W}(\lambda,\mu) := s_{\lambda^{(1)}}(\rho_0^+)\, s_{\lambda^{(1)}/\mu^{(1)}}(\rho_1^-) s_{\lambda^{(2)}/\mu^{(1)}}(\rho_1^+) \cdots s_{\lambda^{(N)}/\mu^{(N-1)}}(\rho_{N-1}^+)\, s_{\lambda^{(N)}}(\rho_N^-). \tag{9}
$$

There is one Schur function factor for any two neighboring partitions in (8).

The fact that all the specializations are nonnegative implies that all the weights are nonnegative. The admissibility of ρ's implies that

$$
Z(\rho_0^+,\ldots,\rho_{N-1}^+;\rho_1^-,\ldots,\rho_N^-) := \sum_{\lambda,\mu}\mathcal{W}(\lambda,\mu) = \prod_{0\le i<j\le N} H(\rho_i^+;\rho_j^-) < \infty, \tag{10}
$$

where $H(\rho_1;\rho_2) = \sum_{\lambda\in\mathbb{Y}} s_\lambda(\rho_1)s_\lambda(\rho_2) = \exp\left(\sum_{n\ge 1} p_n(\rho_1)p_n(\rho_2)/n\right)$, and p_n's are the Newton power sums. Indeed, this follows from the repeated use of identities, cf. [M, I(5.9) and Ex. I.5.26(1)],

$$
\sum_{\kappa\in\mathbb{Y}} s_{\kappa/\nu}(\rho_1)s_{\kappa/\hat{\nu}}(\rho_2) = H(\rho_1;\rho_2)\sum_{\tau\in\mathbb{Y}} s_{\nu/\tau}(\rho_2)s_{\hat{\nu}/\tau}(\rho_1), \tag{11}
$$

$$
\sum_{\nu\in\mathbb{Y}} s_{\kappa/\nu}(\rho_1)s_{\nu/\tau}(\rho_2) = s_{\kappa/\tau}(\rho_1,\rho_2), \tag{12}
$$

and from the fact that for an admissible specialization ρ with $H(\rho;u)$ holomorphic in a disc of radius r, we have $p_n(\rho) = O(r^{-n})$.

The same argument shows that the partition function (10) is finite under the weaker assumption of finiteness of all $H(\rho_i^+;\rho_j^-)$ for $0\le i<j\le N$.

Definition 3. The *Schur process* $\mathbf{S}(\rho_0^+,\ldots,\rho_{N-1}^+;\rho_1^-,\ldots,\rho_N^-)$ is the probability distribution on sequences (λ,μ) as in (8) with

$$
\mathbf{S}(\rho_0^+,\ldots,\rho_{N-1}^+;\rho_1^-,\ldots,\rho_N^-)(\lambda,\mu) = \frac{\mathcal{W}(\lambda,\mu)}{Z(\rho_0^+,\ldots,\rho_{N-1}^+;\rho_1^-,\ldots,\rho_N^-)}.
$$

The Schur process with $N=1$ is called the *Schur measure*.

Using (11)-(12) it is not difficult to show that a projection of the Schur process to any subsequences of (λ,μ) is a also a Schur process. In particular, the projection of $\mathbf{S}(\rho_0^+,\ldots,\rho_{N-1}^+;\rho_1^-,\ldots,\rho_N^-)$ to $\lambda^{(j)}$ is the Schur measure $\mathbf{S}(\rho_{[0,j-1]}^+;\rho_{[j,N]}^-)$, and its projection to $\mu^{(k)}$ is a slightly different Schur measure $\mathbf{S}(\rho_{[0,k-1]}^+;\rho_{[k+1,N]}^-)$. Here we used the notation $\rho_{[a,b]}^\pm$ to denote the union of specializations $\rho_m^\pm$, $m=a,\ldots,b$.

We now aim at defining a Schur like process for signatures.

Pick a natural number N, real numbers $a_1,\ldots,a_N > 0$, nonnegative integers $c(1),\ldots,c(N)$, and $c(1)+\cdots+c(N)$ admissible Toeplitz matrices

$$
\mathbf{M} = \{M^{(k,l)} \mid 1\le k\le N,\ 1\le l\le c(k)\}.
$$

If all $c(k)$ are zero then $\mathbf{M}$ is empty.

We will also need a totally nonnegative matrix of size $\mathbb{Z}\times N$, denote it as

$$
\Psi = [\Psi_{ij}]_{i\in\mathbb{Z},\,-1\ge j\ge -N}.
$$

For any sequences

$$(13) \quad \vec{\lambda}^{(1)} = (\lambda^{(1,0)}, \lambda^{(1,1)}, \ldots, \lambda^{(1,c(1))}), \ldots, \vec{\lambda}^{(N)} = (\lambda^{(N,0)}, \lambda^{(N,1)}, \ldots, \lambda^{(N,c(N))})$$

of signatures of lengths $\ell(\lambda^{(k,*)}) = k$, define their (nonnegative) weight as

$$(14) \quad \mathcal{W}(\vec{\lambda}^{(1)}, \ldots, \vec{\lambda}^{(N)}) := \det\left[\Psi_{\lambda_i^{(N,c(N))} - i, -j}\right]_{i,j=1}^N$$

$$\times \prod_{k=1}^N \left(s_{\lambda^{(k,0)}/\lambda^{(k-1,c(k-1))}}(a_k) \prod_{l=1}^{c(k)} s_{\lambda^{(k,l)}/\lambda^{(k,l-1)}}\left(M^{(k,l)}\right) \right)$$

with $\lambda^{(0,c(0))} = \varnothing$.

We assume that the generating functions

$$(15) \qquad\qquad \Psi_j(u) := \sum_{n=-\infty}^{+\infty} \Psi_{n,-j}\, u^{n+j}$$

are holomorphic in an open set containing the unit circle. As we will see in Section 10, if for any $j \leq N$, a_j lies in the common analyticity annulus for $\{H(M^{(k,l)}; u^{-1})\}_{k \geq j}$, $\{\Psi_i(u)\}_{i=1}^N$, then the partition function of weights (14) is finite and it has the form

$$(16) \quad Z(a_1, \ldots, a_N; \mathbf{M}; \Psi) := \sum_{\vec{\lambda}^{(1)}, \ldots, \vec{\lambda}^{(N)}} \mathcal{W}(\vec{\lambda}^{(1)}, \ldots, \vec{\lambda}^{(N)})$$

$$= \frac{\det\left[a_i^{-j} \Psi_j(a_i)\right]_{i,j=1}^N}{\det\left[a_i^{-j}\right]_{i,j=1}^N} \prod_{1 \leq j \leq k \leq N} \prod_{l=1}^{c(k)} H\left(M^{(k,l)}; a_j^{-1}\right).$$

In the important special case when the matrix Ψ is actually Toeplitz, $\Psi_{i,-j} = \psi_{i+j}$, (16) simplifies:

$$(17) \qquad Z(a_1, \ldots, a_N; \mathbf{M}; \Psi) = \prod_{i=1}^N \psi(a_i) \prod_{1 \leq j \leq k \leq N} \prod_{l=1}^{c(k)} H\left(M^{(k,l)}; a_j^{-1}\right),$$

where $\psi(u) = \sum_{n \in \mathbb{Z}} \psi_n u^n$.

Definition 4. The *two-sided Schur process* $\mathbf{T}(a_1, \ldots, a_N; \mathbf{M}; \Psi)$ is the probability distribution on sequences $(\lambda^{(1)}, \ldots, \lambda^{(N)})$ as in (13) with

$$\mathbf{T}(a_1, \ldots, a_N; \mathbf{M}; \Psi)(\vec{\lambda}^{(1)}, \ldots, \vec{\lambda}^{(N)}) = \frac{\mathcal{W}(\vec{\lambda}^{(1)}, \ldots, \vec{\lambda}^{(N)})}{Z(a_1, \ldots, a_N; \mathbf{M}; \Psi)}.$$

Remark 5. If in the Schur process of Definition 3 each of the specializations ρ_j^+ is a one-variable specialization with $H(\rho_j^+; u) = (1 - a_{j+1}u)^{-1}$, $j = 0, \ldots, N-1$, then the Schur process can be viewed as a special case of the two-sided Schur process with $c(1) = \cdots = c(N-1) = 1$, $c(N) = 0$, and identification

$$\lambda^{(j)} = \lambda^{(j,0)}, \quad j = 1, \ldots, N, \qquad \mu^{(j)} = \lambda^{(j,1)}, \quad j = 1, \ldots, N-1,$$

$$H(\rho_k^-; u) = H(M^{(k,1)}; u^{-1}), \quad k = 1, \ldots, N-1; \qquad H(\rho_N^-; u) = \psi(u).$$

The corresponding two-sided Schur process lives on signatures with nonnegative parts that can also be viewed as partitions.

Observe that under this identification the formulas (10) and (17) coincide.

3. Example 1. Measures q^{volume} on skew plane partitions

Fix two natural numbers A and B. For a Young diagram $\pi \subset B^A$, set $\bar{\pi} = B^A/\pi$.

A (skew) plane partition Π with support $\bar{\pi}$ is a filling of all boxes of $\bar{\pi}$ by nonnegative integers $\Pi_{i,j}$ (we assume that $\Pi_{i,j}$ is located in the ith row and jth column of B^A) such that $\Pi_{i,j} \geq \Pi_{i,j+1}$ and $\Pi_{i,j} \geq \Pi_{i+1,j}$ for all values of i, j.

The volume of the plane partition Π is defined as

$$vol(\Pi) = \sum_{i,j} \Pi_{i,j}.$$

The goal of the section is to explain that the measure on plane partitions with given support $\bar{\pi}$ and weights proportional to $q^{vol(\cdot)}$, $0 < q < 1$, is a Schur process. This fact has been observed and used in [OR1], [OR2], [BMRT].

The Schur process will be such that for any two neighboring specializations ρ_k^-, ρ_k^+ at least one is trivial. This implies that each $\mu^{(j)}$ coincides either with $\lambda^{(j)}$ or with $\lambda^{(j+1)}$. Thus, we can restrict our attention to $\lambda^{(j)}$'s only.

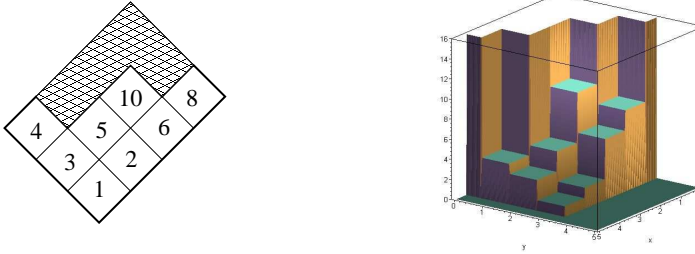For a plane partition Π, we set ($1 \leq k \leq A + B + 1$)

$$\lambda^{(k)}(\Pi) = \left\{\Pi_{i,i+k-A-1} \mid (i, i+k-A-1) \in \bar{\pi}\right\}.$$

Note that $\lambda^{(1)} = \lambda^{(A+B+1)} = \varnothing$.

We need one more piece of notation. Define

$$\mathcal{L}(\pi) = \{A + \pi_i - i + 1 \mid i = 1, \dots, A\}.$$

This is an A-point subset in $\{1, 2, \dots, A + B\}$, and all such subsets are in bijection with the partitions π contained in the box B^A. The elements of $\mathcal{L}(\pi)$ mark the "up-steps" in the boundary of π (=back wall of Π).

The figure above shows a plane partition Π and its plot with

$$A = 4, \;\; B = 3, \;\; \pi = (2, 1, 1, 0),$$

$$\lambda^{(2)} = (4), \;\; \lambda^{(3)} = 3, \;\; \lambda^{(4)} = (5, 1), \;\; \lambda^{(5)} = (10, 2), \;\; \lambda^{(6)} = (6), \;\; \lambda^{(7)} = (8),$$

$$vol(\Pi) = \sum_{i=2}^{A+B} |\lambda^{(i)}| = 39, \quad \mathcal{L}(\pi) = \{1, 3, 4, 6\}.$$

Proposition 6. *Let π be a partition contained in the box B^A. The measure on the plane partitions Π with support $\bar\pi$ and weights proportional to $q^{vol(\Pi)}$, is the Schur process with $N = A + B + 1$ and nonnegative specializations $\{\rho_i^+\}$, $\{\rho_j^-\}$ defined by*

$$H(\rho_0^+; u) = H(\rho_N^-; u) = 1,$$

$$H(\rho_j^+; u) = \begin{cases} \dfrac{1}{1 - q^{-j}u}, & j \in \mathcal{L}(\pi), \\ 1, & j \notin \mathcal{L}(\pi); \end{cases} \qquad H(\rho_j^-; u) = \begin{cases} 1, & j \in \mathcal{L}(\pi), \\ \dfrac{1}{1 - q^j u}, & j \notin \mathcal{L}(\pi). \end{cases}$$

Note that not all specializations are admissible, but the weaker assumption of finiteness of $H(\rho_i^+; \rho_j^-)$ for $0 \le i < j \le N$ guarantees that the partition function is finite.

Proof. Observe that the set of all plane partitions supported by $\bar\pi$, as well as the support of the Schur process from the statement of the proposition, consists of sequences $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N)})$ with

$$\lambda^{(1)} = \lambda^{(N)} = \varnothing,$$

$$\lambda^{(j)} \prec \lambda^{(j+1)} \text{ if } j \in \mathcal{L}(\lambda), \qquad \lambda^{(j)} \succ \lambda^{(j+1)} \text{ if } j \notin \mathcal{L}(\lambda),$$

where we write $\mu \prec \nu$ or $\nu \succ \mu$ iff $\nu_1 \ge \mu_1 \ge \nu_2 \ge \mu_2 \ge \ldots$.

On the other hand, (2) implies that the weight of $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N)})$ with respect to the Schur process from the hypothesis is equal to q raised to the power

$$\sum_{j=2}^{A+B} |\lambda^{(j)}| \Big(-(j-1)\mathbf{1}_{j-1 \in \mathcal{L}(\pi)} - (j-1)\mathbf{1}_{j-1 \notin \mathcal{L}(\pi)} + j\mathbf{1}_{j \in \mathcal{L}(\pi)} + j\mathbf{1}_{j \notin \mathcal{L}(\pi)} \Big),$$

where the four terms are the contributions of $\rho_{j-1}^+, \rho_{j-1}^-, \rho_j^+, \rho_j^-$, respectively.

Clearly, the sum is equal to $\sum_{j=2}^{A+B} |\lambda^{(j)}| = vol(\Pi)$. $\square$

Remark 7. A similar statement holds for any measure on plane partitions with weights proportional to $\prod q_j^{|\lambda_j|}$ with possibly different positive parameters q_j, as long as the partition function is finite. The proof is very similar.

4. Example 2. Path measures for extreme characters of $U(\infty)$

Let $U(N)$ denote the group of $N \times N$ unitary matrices. It is a classical result that the irreducible representations of $U(N)$ can be paramterized by signatures $\lambda = (\lambda_1 \ge \ldots \ge \lambda_N)$ of length N also called *highest weights*. Thus, there is a natural bijection $\lambda \leftrightarrow \chi^\lambda$ between signatures of length N and the conventional irreducible characters (=traces of irreducible representations) of $U(N)$.

For each N, embed $U(N)$ in $U(N+1)$ as the subgroup fixing the $(N+1)$st basis vector. Equivalently, each $U \in U(N)$ can be thought of as an $(N+1) \times (N+1)$ matrix by setting $U_{i,N+1} = U_{N+1,j} = 0$ for $1 \le i, j \le N$ and $U_{N+1,N+1} = 1$. The union $\bigcup_{N=1}^{\infty} U(N)$ is denoted $U(\infty)$ and called the *infinite-dimensional unitary group*.

A *character* of $U(\infty)$ is a positive definite function $\chi : U(\infty) \to \mathbb{C}$ which is constant on conjugacy classes and normalized by $\chi(\mathbf{1}) = 1$. We further assume

that χ is continuous on each $U(N) \subset U(\infty)$. The set of all characters of $U(\infty)$ is convex, and the extreme points of this set are called *extreme* characters.

Remarkably, the extreme characters of $U(\infty)$ are in one-to-one correspondence with admissible Toeplitz matrices M from Definition 2, see [Vo], [VK], [OO]. The values of the character χ^M corresponding to M are given by

$$\chi^M(U) = \prod_{u \in \mathrm{Spectrum}(U)} H(M; u),$$

where $H(M; u)$ is given in (4).

Let $\mathbb{GT}_N$ be the set of all signatures of length N; set $\mathbb{GT} = \bigcup_N \mathbb{GT}_N$. Turn $\mathbb{GT}$ into a graph by drawing an edge between signatures $\lambda \in \mathbb{GT}_N$ and $\mu \in \mathbb{GT}_{N+1}$ if λ and μ satisfy the branching relation $\lambda \prec \mu$, where $\lambda \prec \mu$ means that $\mu_1 \leq \lambda_1 \leq \mu_2 \leq \lambda_2 \leq \ldots \leq \lambda_N \leq \mu_{N+1}$. $\mathbb{GT}$ is known as the *Gelfand-Tsetlin graph*.

A *path* in $\mathbb{GT}$, or an *infinite Gelfand-Tsetlin scheme*, is an infinite sequence $t = (t_1, t_2, \ldots)$ such that $t_i \in \mathbb{GT}_i$ and $t_i \prec t_{i+1}$. Let $\mathcal{T}$ be the set of all such paths.

One can also look at finite paths, or finite Gelfand-Tsetlin schemes, which are sequences $\tau = (\tau_1, \tau_2, \ldots, \tau_N)$ such that $\tau_i \in \mathbb{GT}_i$ and $\tau_1 \prec \tau_2 \prec \ldots \prec \tau_N$. Denote the set of all paths of length N by $\mathcal{T}_N$.
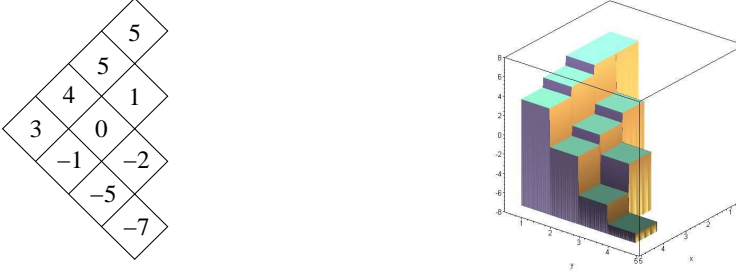
The figure above depicts a Gelfand-Tsetlin scheme $\tau \in \mathcal{T}_4$ and its plot with

$$\tau_1 = (3), \quad \tau_2 = (4, -1), \quad \tau_3 = (5, 0, -5), \quad \tau_4 = (5, 1, -2, -7).$$

Each character χ of $U(\infty)$ defines a probability measure P_N^χ on $\mathbb{GT}_N$: Restricting the character to $U(N)$, we have

$$\chi\Big|_{U(N)} = \sum_{\lambda \in \mathbb{GT}_N} P_N^\chi(\lambda) \frac{\chi^\lambda}{\chi^\lambda(\mathbf{1}_N)}.$$

For each finite path $\tau \in \mathcal{T}_N$, let $C_\tau \subset \mathcal{T}$ be the set

$$C_\tau = \{t \in \mathcal{T} : (t_1, t_2, \ldots, t_N) = \tau\}.$$

A character χ of $U(\infty)$ also defines a probability measure P^χ on $\mathcal{T}$ (with a suitably defined Borel structure), which can be uniquely specified by setting

$$P^\chi(C_\tau) = \frac{P_N^\chi(\lambda)}{\chi^\lambda(\mathbf{1}_N)},$$

where τ is an arbitrary finite path ending at λ, see [Ol, Section 10] for details. Note that we assign the same weight to all finite paths with the same end.

We use the same formula to define a probability measure $P_{[1,N]}^\chi$ on $\mathcal{T}_N$, which is just the projection of P^χ from $\mathcal{T}$ to $\mathcal{T}_N$.

Proposition 8. *For any admissible Toeplitz matrix M as in Definition 2, the measure $P_{[1,N]}^{\chi^M}$ on $\mathcal{T}_N$ coincides with the two-sided Schur process of Definition 4 with*

$$a_1 = \cdots = a_N = 1, \quad c(1) = \cdots = c(N) = 0, \quad \Psi = M,$$

and with sequences $(\lambda^{(1,0)}, \ldots, \lambda^{(N,0)})$ viewed as elements of $\mathcal{T}_N$.

Proof. Directly follows from (7) and Lemma 6.5 of [Ol]. $\square$

5. Markov chains on the Schur processes

Let us introduce some notation.

For two nonnegative specializations ρ_1, ρ_2 of Λ such that $H(\rho_1; \rho_2) < \infty$, and $\lambda, \mu \in \mathbb{Y}$, set

$$P_{\rho_1,\rho_2}(\lambda, \mu \uparrow\uparrow \nu) = const \cdot s_{\nu/\lambda}(\rho_1) s_{\nu/\mu}(\rho_2), \qquad \nu \in \mathbb{Y},$$

where we assume that

(18) $$\{\nu \in \mathbb{Y} \mid s_{\nu/\lambda}(\rho_1) s_{\nu/\mu}(\rho_2) > 0\} \neq \varnothing,$$

and the constant prefactor is chosen so that we obtain a probability measure in ν:

$$\sum_{\nu \in \mathbb{Y}} P_{\rho_1,\rho_2}(\lambda, \mu \uparrow\uparrow \nu) = 1.$$

Given (18), the existence of such constant follows from (11).

Similarly, dropping the assumption $H(\rho_1; \rho_2) < \infty$, we define

$$P_{\rho_1,\rho_2}(\lambda, \mu \downarrow\uparrow \nu) = const \cdot s_{\lambda/\nu}(\rho_1) s_{\nu/\mu}(\rho_2),$$
$$P_{\rho_1,\rho_2}(\lambda, \mu \uparrow\downarrow \nu) = const \cdot s_{\nu/\lambda}(\rho_1) s_{\mu/\nu}(\rho_2),$$
$$P_{\rho_1,\rho_2}(\lambda, \mu \downarrow\downarrow \nu) = const \cdot s_{\lambda/\nu}(\rho_1) s_{\mu/\nu}(\rho_2),$$

where in all three cases we assume that the set of ν giving nonzero values on the right-hand side is nonempty (it is finite in all three cases), and we choose constants so that we obtain probability distributions in $\nu \in \mathbb{Y}$.

If both ρ_1 and ρ_2 are single-α or single-β specializations, relations (2), (3) show that all four distributions P_{ρ_1,ρ_2} are products of geometric distributions conditioned to stay in segments and Bernoulli measures.

Example 8. Assume that $H(\rho_1; u) = (1 - au)^{-1}$, $H(\rho_2; u) = (1 - bu)^{-1}$. Denote by $G_{m,n}^\xi$, $m \leq n$, the probability distribution on the set $\{m, m+1, \ldots, n\}$ given by

$$G_{m,n}^\xi(\{k\}) = \frac{\xi^k}{\sum_{j=m}^n \xi^j} = \frac{1 - \xi^{n-m+1}}{\xi^m(1-\xi)} \cdot \xi^k, \qquad m \leq k \leq n.$$

Then

$$P_{\rho_1,\rho_2}(\lambda, \mu \uparrow\uparrow \nu) = G_{\max\{\lambda_1,\mu_1\},+\infty}^{ab}(\nu_1) \prod_{j \geq 2} G_{\max\{\lambda_j,\mu_j\},\min\{\lambda_{j-1},\mu_{j-1}\}}^{ab}(\nu_j),$$

$$P_{\rho_1,\rho_2}(\lambda, \mu \downarrow\uparrow \nu) = G_{\max\{\lambda_2,\mu_1\},\lambda_1}^{b/a}(\nu_1) \prod_{j \geq 2} G_{\max\{\lambda_{j+1},\mu_j\},\min\{\lambda_j,\mu_{j-1}\}}^{b/a}(\nu_j),$$

where in the first case we need to additionally assume that $ab < 1$ (equivalently, $H(\rho_1; \rho_2) < \infty$).

Further, assume that $H(\rho_3; u) = (1 + cu)$. Denote by $B^p_{m,n}$, $n \in \{m, m+1\}$, the probability distribution on $\{m, m+1\}$ given by

$$B^p_{m,m}(\{k\}) = \begin{cases} 1, & k = m, \\ 0, & k = m+1, \end{cases} \qquad B^p_{m,m+1}(\{k\}) = \begin{cases} \dfrac{1}{1+c}, & k = m, \\ \dfrac{c}{1+c}, & k = m+1. \end{cases}$$

Then

$$P_{\rho_3,\rho_2}(\lambda, \mu \uparrow\uparrow \nu) = B^{bc}_{\max\{\lambda_1,\mu_1\},\lambda_1+1}(\nu_1) \prod_{j \geq 2} B^{bc}_{\max\{\lambda_j,\mu_j\},\min\{\lambda_{j-1},\lambda_j+1,\mu_{j-1}\}}(\nu_j),$$

$$P_{\rho_3,\rho_2}(\lambda, \mu \downarrow\uparrow \nu) = B^{b/c}_{\max\{\lambda_1-1,\lambda_2,\mu_1\},\lambda_1}(\nu_1) \prod_{j \geq 2} B^{b/c}_{\max\{\lambda_j-1,\lambda_{j+1},\mu_j\},\min\{\lambda_j,\mu_{j-1}\}}(\nu_j).$$

Let $(\rho_0^+, \ldots, \rho_{N-1}^+; \rho_1^-, \ldots, \rho_N^-)$ be nonnegative specializations of Λ defining a Schur process as in Definition 3. Let π be another nonnegative specialization of Λ such that $H(\pi, \rho_j^+) < \infty$ for all $0 \leq j < N$.

Let $\mathcal{X}$ be the set of pairs of sequences (λ, μ) as in (8) with

$$s_{\lambda^{(1)}}(\rho_0^+) \, s_{\lambda^{(1)}/\mu^{(1)}}(\rho_1^-) s_{\lambda^{(2)}/\mu^{(1)}}(\rho_1^+) \cdots s_{\lambda^{(N)}/\mu^{(N-1)}}(\rho_{N-1}^+) > 0$$

The product above is the same as in (9) without the last factor. Thus, the support of $\mathbf{S}(\rho_0^+, \ldots, \rho_{N-1}^+; \rho_1^-, \ldots, \rho_N^-)$ is contained in $\mathcal{X}$.

Define a matrix $\mathfrak{P}_\pi^\uparrow$ with rows and columns parameterized by elements of $\mathcal{X}$ via

$$(19) \quad \mathfrak{P}_\pi^\uparrow((\lambda, \mu), (\tilde{\lambda}, \tilde{\mu})) = P_{\rho_0^+, \pi}(\varnothing, \lambda^{(1)} \uparrow\uparrow \tilde{\lambda}^{(1)})$$

$$\times \prod_{j=1}^{N-1} P_{\rho_j^-, \pi}(\tilde{\lambda}^{(j)}, \mu^{(j)} \downarrow\uparrow \tilde{\mu}^{(j)}) P_{\rho_j^+, \pi}(\tilde{\mu}^{(j)}, \lambda^{(j+1)} \uparrow\uparrow \tilde{\lambda}^{(j+1)}).$$

In other words, starting from (λ, μ), one first finds $\tilde{\lambda}^{(1)}$ using $\lambda^{(1)}$, then $\tilde{\mu}^{(1)}$ using $\tilde{\lambda}^{(1)}$ and $\mu^{(1)}$, then $\tilde{\lambda}^{(2)}$ using $\tilde{\mu}^{(1)}$ and $\lambda^{(2)}$, and so on. One could say that we perform *sequential update*.

Note that some of the entries of $\mathfrak{P}_\pi^\uparrow$ might remain undefined if one of the conditions of type (18) is not satisfied. Part of the theorem below is that this never happens.

Theorem 10. *In the above assumptions, the matrix $\mathfrak{P}_\pi^\uparrow$ is well-defined and it is stochastic. Moreover,*

$$\mathbf{S}(\rho_0^+, \ldots, \rho_{N-1}^+; \rho_1^-, \ldots, \rho_N^-) \, \mathfrak{P}_\pi^\uparrow = \mathbf{S}(\rho_0^+, \ldots, \rho_{N-1}^+; \rho_1^-, \ldots, {}^\sharp\rho_N^-),$$

where ${}^\sharp\rho_N^- = (\rho_N^-, \pi)$. In other words, $\mathfrak{P}_\pi^\uparrow$ changes the last specialization of the Schur process by adding π to it.

The proof of Theorem 10 will be given in Section 9.

Matrices $\mathfrak{P}^\uparrow_\pi$ describe a certain growth process. In a similar fashion, one obtains a process of decay. Let us describe it.

Let σ be a nonnegative specialization of Λ that 'divides' ρ_0^+, that is, there exists a nonnegative specialization $^\flat\rho_0^+$ such that $\rho_0^+ = (^\flat\rho_0^+, \sigma)$. For example, σ may coincide with ρ_0^+; in that case $^\flat\rho_0^+$ is trivial.

Let $\mathcal{Y}$ be the set of pairs of sequences (λ, μ) as in (8) with

$$s_{\lambda^{(1)}}(^\flat\rho_0^+)\, s_{\lambda^{(1)}/\mu^{(1)}}(\rho_1^-) s_{\lambda^{(2)}/\mu^{(1)}}(\rho_1^+) \cdots s_{\lambda^{(N)}/\mu^{(N-1)}}(\rho_{N-1}^+) > 0.$$

Note that if $\sigma = \rho_0^+$ then $\lambda^{(1)}$ and $\mu^{(1)}$ must be empty in order for (λ, μ) to lie in $\mathcal{Y}$.

Define a matrix $\mathfrak{P}^\downarrow_\sigma$ with rows parameterized by $\mathcal{X}$ and columns parameterized by $\mathcal{Y}$ via

$$\mathfrak{P}^\downarrow_\sigma((\lambda, \mu), (\tilde{\lambda}, \tilde{\mu})) = P_{\rho_0^+, \sigma}\big(\varnothing, \lambda^{(1)} \uparrow\downarrow \tilde{\lambda}^{(1)}\big)$$
$$\times \prod_{j=1}^{N-1} P_{\rho_j^-, \sigma}\big(\tilde{\lambda}^{(j)}, \mu^{(j)} \downarrow\downarrow \tilde{\mu}^{(j)}\big) P_{\rho_j^+, \sigma}\big(\tilde{\mu}^{(j)}, \lambda^{(j+1)} \uparrow\downarrow \tilde{\lambda}^{(j+1)}\big).$$

Notice that the only difference of this definition and that of $\mathfrak{P}^\uparrow_\pi$ above, is switching π and σ and changing the second arrows from $\uparrow$ to $\downarrow$.

Theorem 11. *In the above assumptions, the matrix $\mathfrak{P}^\downarrow_\sigma$ is well-defined and it is stochastic. Moreover,*

$$\mathbf{S}(\rho_0^+, \ldots, \rho_{N-1}^+; \rho_1^-, \ldots, \rho_N^-)\, \mathfrak{P}^\downarrow_\sigma = \mathbf{S}(^\flat\rho_0^+, \ldots, \rho_{N-1}^+; \rho_1^-, \ldots, \rho_N^-),$$

where $(^\flat\rho_0^+, \sigma) = \rho_0^+$. In other words, $\mathfrak{P}^\downarrow_\sigma$ changes the first specialization of the Schur process by removing σ from it.

The proof of Theorem 11 will also be given in Section 9.

Remark 12. Both Theorems 10 and 11 can be generalized as follows. Assume we have an arbitrary sequence of Markov steps of types $\mathfrak{P}^\uparrow$ and $\mathfrak{P}^\downarrow$ applied to an initial Schur process, and let us denote by $(\lambda(t), \mu(t))$ the result of the application of t first members of the sequence. One can show that any finite sequence of random partitions of the form

$$(\lambda^{(1)}(t_{1,1}), \lambda^{(1)}(t_{1,2}), \ldots, \mu^{(1)}(t'_{1,1}), \mu^{(1)}(t'_{1,2}), \ldots, \ldots,$$
$$\ldots, \mu^{(N-1)}(t'_{N-1,1}), \mu^{(N-1)}(t'_{N-1,2}), \ldots, \lambda^{(N)}(t_{N,1}), \lambda^{(N)}(t_{N,2}), \ldots)$$

forms a Schur process with an explicitly known specializations as long as

$$t_{1,1} \geq t_{1,2} \geq \cdots \geq t'_{1,1} \geq t'_{1,2} \geq \cdots \geq t'_{N-1,1} \geq t'_{N-1,2} \geq \cdots \geq t_{N,1} \geq t_{N,2} \geq \ldots,$$

cf. the last sentence of Section 8.

6. Markov chains on the two-sided Schur processes

We now aim at formulating (and later proving) a statement for the two-sided Schur processes that is analogous to Theorem 10.

For two admissible matrices M_1 and M_2 ('admissible' is explained in Definition 2), and two signatures λ and μ of length $n \geq 1$, we define a probability distribution on $\mathbb{GT}_n$ (=the set of all signatures of length n) via

$$P_{M_1,M_2}(\lambda, \mu \,\|\, \nu) = const \cdot s_{\nu/\lambda}(M_1)s_{\nu/\mu}(M_2), \qquad \nu \in \mathbb{GT}_n.$$

For an admissible matrix M and a positive number a in the annulus of analyticity of $H(M; u)$, and for two signatures $\lambda \in \mathbb{GT}_{n-1}$ and $\mu \in \mathbb{GT}_n$, we define a probability distribution on $\mathbb{GT}_n$ via

$$P_{a,M}(\lambda, \mu \,\|\, \nu) = const \cdot s_{\nu/\lambda}(a)s_{\nu/\mu}(M), \qquad \nu \in \mathbb{GT}_n.$$

In both definitions, we suppose that the set of ν's giving nonzero contributions to the right-hand sides is nonempty. Then our assumptions imply the existence of the normalizing constants.

Similarly to the one-sided Schur process, if M_1 and M_2 are both single-$\alpha^{\pm}$ or single-$\beta^{\pm}$ matrices, then P_{M_1,M_2} splits into a product of geometric/Bernoulli random variables, cf. (5)-(6) and Example 8. For $P_{a,M}$ the same holds if M is a single-$\alpha^{\pm}$ or single-$\beta^{\pm}$ matrix.

Consider the two-sided Schur process of Definition 4, and let

$$\mathcal{X} = \left\{ (\vec{\lambda}^{(1)}, \dots, \vec{\lambda}^{(N)}) \in (\mathbb{GT}_1)^{c(1)+1} \times \cdots \times (\mathbb{GT}_N)^{c(N)+1} \,\middle|\, \right.$$

$$\left. \prod_{k=1}^{N} \left(s_{\lambda^{(k,0)}/\lambda^{(k-1,c(k-1))}}(a_k) \prod_{l=1}^{c(k)} s_{\lambda^{(k,l)}/\lambda^{(k,l-1)}}\left(M^{(k,l)}\right) > 0 \right) \right\},$$

where $\lambda^{(0,c(0))} = \varnothing$, cf. (14). Clearly, $\operatorname{supp} \mathbf{T}(a_1, \dots, a_N; \mathbf{M}; \Psi) \subset \mathcal{X}$.

Let Q be an additional admissible matrix such that all the parameters a_j lie in the analyticity annulus of $H(Q; u)$. Define a matrix $\mathfrak{P}_Q$ with rows and columns parameterized by $\mathcal{X}$ via

$$\mathfrak{P}_Q((\vec{\lambda}^{(1)}, \dots, \vec{\lambda}^{(N)}), (\vec{\mu}^{(1)}, \dots, \vec{\mu}^{(N)})) =$$

$$\prod_{k=1}^{N} \left(P_{a_k,Q}\left(\lambda^{(k-1,c(k-1))}, \lambda^{(k,0)} \,\|\, \mu^{(k,0)}\right) \prod_{l=1}^{c(k)} P_{M^{(k,l)},Q}\left(\lambda^{(k,l-1)}, \lambda^{(k,l)} \,\|\, \mu^{(k,l)}\right) \right)$$

The structure of $\mathfrak{P}_Q$ is such that to compute its row indexed by $(\vec{\lambda}^{(1)}, \dots, \vec{\lambda}^{(N)})$, one first finds $\mu^{(1,0)}$ using $\lambda^{(1,0)}$, then $\mu^{(1,1)}$ using $\lambda^{(1,1)}$ and $\mu^{(1,0)}$, then $\mu^{(1,2)}$ using $\lambda^{(1,2)}$ and $\mu^{(1,1)}$, and so on.

Theorem 13. *In the above assumptions, the matrix $\mathfrak{P}_Q$ is well-defined and it is stochastic. Moreover,*

$$\mathbf{T}(a_1, \dots, a_N; \mathbf{M}; \Psi)\, \mathfrak{P}_Q = \mathbf{T}(a_1, \dots, a_N; \mathbf{M}; Q\Psi).$$

The proof of Theorem 13 will be given in Section 10.

Remark 14. Similarly to Remark 12, a more general statement can be proved. Assume we have an arbitrary sequence of matrices $\mathfrak{P}_Q$ applied to a two-sided Schur process $\mathbf{T}(a_1, \ldots, a_N; \mathbf{M}; \Psi)$. Denote by $(\vec{\lambda}^{(1)}(t), \ldots \vec{\lambda}^{(N)}(t))$ the random sequence obtained after the application of t first matrices. Then any sequence $\{\lambda^{(k,l)}(t_{k,l})\}$ forms (a marginal of) an explicitly describable two-sided Schur process as long as $(k_1, l_1) \leq (k_2, l_2)$ lexicographically implies $t_{k_1, l_1} \geq t_{k_2, l_2}$.

Remark 15. The matrices $\mathfrak{P}_Q$ are similar to the growth process defined by $\mathfrak{P}_\pi^\uparrow$ of the previous section. One could also define a 'decay process' for the two-sided Schur processes that would be similar to $\mathfrak{P}_\sigma^\downarrow$; the application of the corresponding matrix to $\mathbf{T}(a_1, \ldots, a_N; \mathbf{M}; \Psi)$ would reduce N by 1 and remove a_1 and $\{M^{(1,l)}\}_{l=1}^{c(1)}$ from the set of parameters.

Remark 16. In the setting of Remark 5, one easily shows that $\mathfrak{P}_\pi^\uparrow$ and $\mathfrak{P}_Q$ coincide if $H(\pi; u) = H(Q; u)$.

7. Exact sampling algorithms

Let us start with (one-sided) Schur processes. Theorem 10 yields an exact sampling algorithm that is inductive in N.

As the base one can take the empty sequence and $N = 0$. Let us explain the induction step. Assume we already know how to sample from the Schur process $\mathcal{P}_{n-1} = \mathbf{S}(\rho_0^+, \ldots, \rho_{N-2}^+; \rho_1^-, \ldots, \rho_{N-1}^-)$.

Consider the process $\tilde{\mathcal{P}}_n = \mathbf{S}(\rho_0^+, \ldots, \rho_{N-2}^+, \rho_{N-1}^+; \rho_1^-, \ldots, \rho_{N-1}^-, \varnothing)$, where $\varnothing$ is the trivial specialization. The definition of the Schur process implies that for this process $\lambda^{(N)} = \mu^{(N-1)} = \varnothing$ with probability 1, and the distribution of the remaining partitions $(\lambda^{(1)}, \mu^{(1)}, \ldots, \mu^{(N-2)}, \lambda^{(N-1)})$ is the same as for $\mathcal{P}_{n-1}$ that we already know how to sample from by the induction hypothesis.

In order to obtain a sample of $\mathcal{P}_n = \mathbf{S}(\rho_0^+, \ldots, \rho_{N-2}^+, \rho_{N-1}^+; \rho_1^-, \ldots, \rho_{N-1}^-, \rho_N^-)$ we apply the stochastic matrix $\mathfrak{P}_\pi^\uparrow$ with $\pi = \rho_N^-$ to $\tilde{\mathcal{P}}_n$, cf. Theorem 10. The application of this matrix requires sequential update from $\lambda^{(1)}$ up, cf. (19).

We thus see that if each of $(\rho_0^+, \ldots, \rho_{N-2}^+, \rho_{N-1}^+; \rho_1^-, \ldots, \rho_{N-1}^-, \rho_N^-)$ is a single-α or a single-β specializations (or trivial), then exact sampling is reduced to sampling a finite number of independent geometric/Bernoulli random variables. Noting that in the algorithm for the Nth step one does not have to use a single $\mathfrak{P}_\pi^\uparrow$ with $\pi = \rho_N^-$, but can instead use a sequence of $\mathfrak{P}_{\pi_k}^\uparrow$ with $\rho_N^- = (\pi_1, \pi_2, \ldots)$, we see that the a similar reduction holds for the Schur processes with all specializations having finitely many nonzero α's and β's (and $\gamma = 0$).

For the measures q^{volume} on skew plane partitions considered in Section 3, the algorithm can be implemented as follows (we use Section 3 and Example 8 below).

> Initiate by assigning $\lambda^{(1)} = \cdots = \lambda^{(A+B)} = \varnothing$.
> For k running from 2 to $(A+B)$
>   If $k \notin \mathcal{L}(\pi)$ then
>   For l running from 1 to $(k-1)$
>   If $l \in \mathcal{L}(\pi)$ then $\lambda^{(l+1)} := \nu$ with ν distributed as
> $$G^{q^{k-l}}_{\max\{\lambda_1^{(l)}, \lambda_1^{(l+1)}\}, +\infty}(\nu_1) \prod_{j \geq 2} G^{q^{k-l}}_{\max\{\lambda_j^{(l)}, \lambda_j^{(l+1)}\}, \min\{\lambda_{j-1}^{(l)}, \lambda_{j-1}^{(l+1)}\}}(\nu_j)$$
>   If $l \notin \mathcal{L}(\pi)$ then $\lambda^{(l+1)} := \nu$ with ν distributed as

$$G^{q^{k-l}}_{\max\{\lambda^{(l)}_2,\lambda^{(l+1)}_1\},\lambda^{(l+1)}_1}(\nu_1) \prod_{j\geq 2} G^{q^{k-l}}_{\max\{\lambda^{(l)}_{j+1},\lambda^{(l+1)}_j\},\min\{\lambda^{(l)}_j,\lambda^{(l+1)}_{j-1}\}}(\nu_j)$$

End of l-cycle

End of k-cycle

At the end of each k-step we see an exact sample of the measure q^{volume} on plane partitions with a smaller support. The number of nontrivial one-dimensional samples needed to go through the k-step with $k \notin \mathcal{L}(\pi)$ is the number of boxes in this support. It is not difficult to see that this number is at most A for the smallest $k \notin \mathcal{L}(\pi)$, it is at most $2A$ for the next one and so on, so that the total number of one-dimensional samples needed is at most $AB(B+1)/2$. The maximum is achieved at $\mathcal{L}(\pi) = \{1,\ldots,A\}$, i.e. when the plane partitions are supported by the full $A \times B$ box.
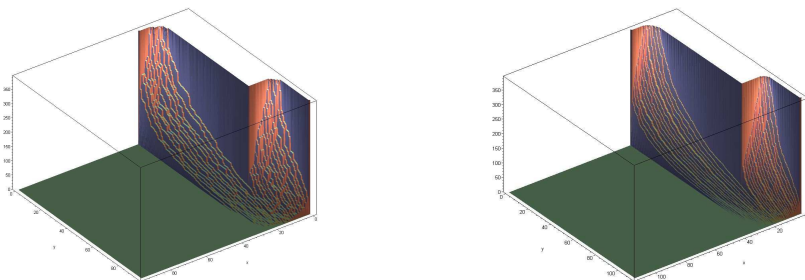

The above figures show a sample for a specific back wall profile, and an average over ten samples with the same back wall. A limit shape and its cusp are clearly visible, cf. [OR2].

Finally, note that a very similar algorithm would sample skew plane partitions with weights of the form $\prod q_j^{|\lambda^{(j)}|}$.

Let us now discuss the two-sided Schur process. First, let us restrict ourselves to the case when Ψ is Toeplitz. Then if all $H(M^{(k,l)}; u^{-1})$ and $\psi(u)$ are analytic in a disc of radius > 1 (not just in an annulus containing the unit circle), then the two-sided Schur process lives on signatures with nonnegative coordinates and it constitutes a special case of the (one-sided) Schur process, cf. Remark 5. Consequently, if all $M^{(k,l)}$ and Ψ are admissible matrices with $M^{(k,l)}$ having finitely many α^- and β^- nonzero parameters (all others are zero), and Ψ having finitely many α^+ and β^+ nonzero parameters, the inductive algorithm for the Schur process described above reduces sampling to a finite number of independent samples of geometric/Bernoulli random variables.

On the other hand, Theorem 13 allows us to add finitely many $\alpha^\pm$ and $\beta^\pm$ parameters to Ψ by sampling from independent geometric/Bernoulli distributions. Hence, we can relax the assumption on Ψ in the previous paragraph by requiring that it has finitely many $\alpha^\pm$ and $\beta^\pm$ parameters.
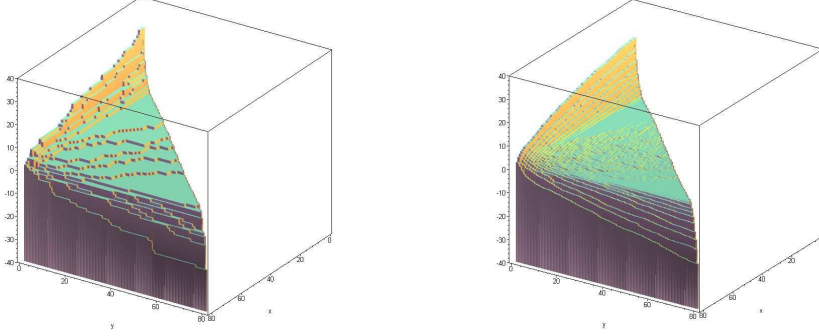
The figure above shows a sample of the path measure and the average over ten samples for the extreme character of $U(\infty)$ with

$$\alpha_1^+ = \ldots \alpha_{10}^+ = \frac{1}{10}, \quad \beta_1^+ = \ldots \beta_5^+ = \frac{1}{2}, \quad \alpha_1^- = \cdots = \alpha_{10}^- = \frac{1}{10},$$

and all other parameters being zero, cf. Section 4. The first order asymptotic behavior of such measures as the path length goes to infinity and parameters remain fixed is known, see [OO].

8. A general construction of multivariate Markov chains

The general construction of this section will be used in Sections 9 and 10 to prove Theorems 10, 11, and 13.

Let $(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ and $(\tilde{\mathcal{S}}_1, \ldots, \tilde{\mathcal{S}}_n)$ be two n-tuples of discrete countable sets, $P_1, \ldots, P_n$ be stochastic matrices defining Markov chains $\mathcal{S}_j \to \tilde{\mathcal{S}}_j$. Also let $\Lambda_1^2, \ldots,$ Λ_{n-1}^n and $\tilde{\Lambda}_1^2, \ldots, \tilde{\Lambda}_{n-1}^n$ be stochastic links between these sets:

$$P_k : \mathcal{S}_k \times \tilde{\mathcal{S}}_k \to [0,1], \quad \sum_{y \in \tilde{\mathcal{S}}_k} P_k(x,y) = 1, \quad x \in \mathcal{S}_k, \quad k = 1, \ldots, n;$$

$$\Lambda_{k-1}^k : \mathcal{S}_k \times \mathcal{S}_{k-1} \to [0,1], \quad \sum_{y \in \mathcal{S}_{k-1}} \Lambda_{k-1}^k(x,y) = 1, \quad x \in \mathcal{S}_k, \quad k = 2, \ldots, n;$$

$$\tilde{\Lambda}_{k-1}^k : \tilde{\mathcal{S}}_k \times \tilde{\mathcal{S}}_{k-1} \to [0,1], \quad \sum_{y \in \tilde{\mathcal{S}}_{k-1}} \tilde{\Lambda}_{k-1}^k(x,y) = 1, \quad x \in \tilde{\mathcal{S}}_k, \quad k = 2, \ldots, n.$$

Assume that these matrices satisfy the commutation relations

(20) $$\Delta_{k-1}^k := \Lambda_{k-1}^k P_{k-1} = P_k \tilde{\Lambda}_{k-1}^k, \qquad k = 2, \ldots, n.$$

We will define a *multivariate* Markov chain $P^{(n)}$ between the state spaces

$$\mathcal{S}^{(n)} = \left\{ (x_1, \ldots, x_n) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \mid \prod_{k=2}^n \Lambda_{k-1}^k(x_k, x_{k-1}) \neq 0 \right\}$$

and

$$\tilde{\mathcal{S}}^{(n)} = \Big\{ (x_1, \ldots, x_n) \in \tilde{\mathcal{S}}_1 \times \cdots \times \tilde{\mathcal{S}}_n \mid \prod_{k=2}^{n} \tilde{\Lambda}_{k-1}^{k}(x_k, x_{k-1}) \neq 0 \Big\}.$$

The transition probabilities for the Markov chain $P^{(n)}$ are defined as (we use the notation $X_n = (x_1, \ldots, x_n)$, $Y_n = (y_1, \ldots, y_n)$)

$$(21) \qquad P^{(n)}(X_n, Y_n) = P_1(x_1, y_1) \prod_{k=2}^{n} \frac{P_k(x_k, y_k) \tilde{\Lambda}_{k-1}^{k}(y_k, y_{k-1})}{\Delta_{k-1}^{k}(x_k, y_{k-1})}$$

if $\prod_{k=2}^{n} \Delta_{k-1}^{k}(x_k, y_{k-1}) > 0$, and 0 otherwise.

One way to think of $P^{(n)}$ is as follows.

Starting from $X = (x_1, \ldots, x_n)$, we first choose y_1 according to the transition matrix $P_1(x_1, y_1)$, then choose y_2 using $\frac{P_2(x_2, y_2) \tilde{\Lambda}_1^2(y_2, y_1)}{\Delta_1^2(x_2, y_1)}$, which is the conditional distribution of the middle point in the successive application of P_2 and $\tilde{\Lambda}_1^2$ provided that we start at x_2 and finish at y_1, after that we choose y_3 using the conditional distribution of the middle point in the successive application of P_3 and $\tilde{\Lambda}_2^3$ provided that we start at x_3 and finish at y_2, and so on. Thus, one could say that Y is obtained from X by the sequential update.

Proposition 17. *Let m_n be a probability measure on $\mathcal{S}_n$. Let $m^{(n)}$ be a probability measure on $\mathcal{S}^{(n)}$ defined by*

$$m^{(n)}(X_n) = m_n(x_n) \Lambda_{n-1}^{n}(x_n, x_{n-1}) \cdots \Lambda_1^2(x_2, x_1), \qquad X_n = (x_1, \ldots, x_n) \in \mathcal{S}^{(n)}.$$

Set $\tilde{m}_n = m_n P_n$ and

$$\tilde{m}^{(n)}(X_n) = \tilde{m}_n(x_n) \tilde{\Lambda}_{n-1}^{n}(x_n, x_{n-1}) \cdots \tilde{\Lambda}_1^2(x_2, x_1), \qquad X_n = (x_1, \ldots, x_n) \in \tilde{\mathcal{S}}^{(n)}.$$

Then $m^{(n)} P^{(n)} = \tilde{m}^{(n)}$.

Proof. The argument is straightforward. Indeed,

$$m^{(n)} P^{(n)}(Y_n) = \sum_{X_n \in \mathcal{S}^{(n)}} m_n(x_n) \Lambda_{n-1}^{n}(x_n, x_{n-1}) \cdots \Lambda_1^2(x_2, x_1)$$

$$\times P_1(x_1, y_1) \prod_{k=2}^{n} \frac{P_k(x_k, y_k) \tilde{\Lambda}_{k-1}^{k}(y_k, y_{k-1})}{\Delta_{k-1}^{k}(x_k, y_{k-1})}.$$

Extending the sum to $x_1 \in \mathcal{S}_1$ adds 0 to the right-hand side. Then we can use relation (20) to compute the sum over x_1, removing $\Lambda_1^2(x_2, x_1)$, $P_1(x_1, y_1)$ and $\Delta_1^2(x_2, y_1)$ from the expression. Similarly, we sum consecutively over $x_2, \ldots, x_n$, and this gives the needed result. $\square$

Proposition 17 will be used to prove Theorems 10, 11, and 13. A more general [BF, Proposition 2.7] is needed to prove the statements mentioned in Remarks 12 and 14.

9. Application to the Schur processes

In this section we prove Theorems 10 and 11.

Let us start by putting the Schur process $\mathbf{S}(\rho_0^+, \ldots, \rho_{N-1}^+; \rho_1^-, \ldots, \rho_N^-)$ of Definition 3 into the framework of the previous section.

We need some general definitions.

Let y, z, t be nonnegative specializations of Λ. Set

$$p_{\lambda\mu}^\uparrow(y; z) = \frac{1}{H(y; z)} \frac{s_\mu(y)}{s_\lambda(y)} s_{\mu/\lambda}(z), \qquad \lambda, \mu \in \mathbb{Y}(y),$$

$$p_{\lambda\nu}^\downarrow(y; t) = \frac{s_\nu(y)}{s_\lambda(y, t)} s_{\lambda/\nu}(t), \qquad \lambda \in \mathbb{Y}(y, t), \ \nu \in \mathbb{Y}(y),$$

where $\mathbb{Y}(\rho) = \{\kappa \in \mathbb{Y} \mid s_\kappa(\rho) > 0\}$, and for the first definition we assume that $H(y; z) = \sum_{\kappa \in \mathbb{Y}} s_\kappa(y) s_\kappa(z) < \infty$.

Relations (11) and (12) imply that the matrices

$$p^\uparrow(y; z) = \left[p_{\lambda\mu}^\uparrow(y; z)\right]_{\lambda, \mu \in \mathbb{Y}(y)} \quad \text{and} \quad p^\downarrow(y; t) = \left[p_{\lambda\nu}^\downarrow(y; t)\right]_{\lambda \in \mathbb{Y}(y, t), \nu \in \mathbb{Y}(y)}$$

are stochastic:

$$\sum_{\mu \in \mathbb{Y}(y)} p_{\lambda\mu}^\uparrow(y; z) = \sum_{\nu \in \mathbb{Y}(y)} p_{\lambda\nu}^\downarrow(y; t) = 1.$$

It is immediate to see that $p^\uparrow$ and $p^\downarrow$ act well on the Schur measures:

(22) $$\mathbf{S}(x; y) p^\uparrow(y; z) = \mathbf{S}(x, z; y), \qquad \mathbf{S}(x; y, t) p^\downarrow(y; t) = \mathbf{S}(x; y).$$

Observe that $\mathbf{S}(\rho_1; \rho_2) = \mathbf{S}(\rho_2; \rho_1)$, so the parameters of the Schur measures in these relations can also be permuted.

Proposition 18. *Let y, z, z_1, z_2, t_1, t_2 be nonnegative specializations of Λ. Then we have the commutativity relations*

$$p^\uparrow(y; z_1) p^\uparrow(y; z_2) = p^\uparrow(y; z_2) p^\uparrow(y; z_1),$$

$$p^\downarrow(y, t_2; t_1) p^\downarrow(y; t_2) = p^\downarrow(y, t_1; t_2) p^\downarrow(y; t_1),$$

$$p^\uparrow(y, t; z) p^\downarrow(y; t) = p^\downarrow(y; t) p^\uparrow(y; z),$$

where for the first relation we assume $H(y; z_1, z_2) < \infty$, and for the third relation we assume $H(y, t; z) < \infty$.

Proof. The arguments for all three identities are similar; we only give the proof of the third one which is in a way the hardest. We have

$$\sum_\mu p_{\lambda\mu}^\uparrow(y, t; z) p_{\mu\nu}^\downarrow(y; t) = \frac{1}{H(y, t; z)} \sum_{\mu \in \mathbb{Y}(y, t)} \frac{s_\mu(y, t)}{s_\lambda(y, t)} s_{\mu/\lambda}(z) \frac{s_\nu(y)}{s_\mu(y, t)} s_{\mu/\nu}(t)$$

$$= \frac{1}{H(y, t; z)} \frac{s_\nu(y)}{s_\lambda(y, t)} \sum_{\mu \in \mathbb{Y}} s_{\mu/\lambda}(z) s_{\mu/\nu}(t) = \frac{H(t; z)}{H(y, t; z)} \frac{s_\nu(y)}{s_\lambda(y, t)} \sum_{\kappa \in \mathbb{Y}} s_{\lambda/\kappa}(t) s_{\nu/\kappa}(z)$$

$$= \frac{1}{H(y; z)} \sum_{\kappa \in \mathbb{Y}(y)} \frac{s_\kappa(y)}{s_\lambda(y, t)} s_{\lambda/\kappa}(t) \frac{s_\nu(y)}{s_\kappa(y)} s_{\nu/\kappa}(z) = \sum_{\kappa \in \mathbb{Y}(y)} p_{\lambda\kappa}^\downarrow(y; t) p_{\kappa\nu}^\uparrow(y; z),$$

where along the way we extended the summation in μ from $\mathbb{Y}(y,t)$ to $\mathbb{Y}$ because $s_\nu(y)s_{\mu/\nu}(t) > 0$ implies $s_\mu(y,t) > 0$ by (12); we used (11) to switch from μ to κ, and finally we restricted the summation in κ from $\mathbb{Y}$ to $\mathbb{Y}(y)$ because $s_\nu(y)s_{\nu/\kappa}(z) > 0$ implies $\kappa \subset \nu$ and $s_\kappa(y) > 0$. $\square$

We are now ready to return to the Schur process $\mathbf{S}(\rho_0^+, \ldots, \rho_{N-1}^+; \rho_1^-, \ldots, \rho_N^-)$. Set $n = 2N - 1$ and

$$\mathcal{S}_{2j-1} = \mathbb{Y}\big(\rho_{[0,j-1]}^+\big), \qquad j = 1, \ldots, N;$$
$$\mathcal{S}_{2k} = \mathbb{Y}\big(\rho_{[0,k-1]}^+\big), \qquad k = 1, \ldots, N-1.$$

Since $\lambda^{(j)}$ and $\mu^{(k)}$ are distributed according to the Schur measures $\mathbf{S}(\rho_{[0,j-1]}^+; \rho_{[j,N]}^-)$ and $\mathbf{S}(\rho_{[0,k-1]}^+; \rho_{[k+1,N]}^-)$ respectively, the projections of the support of the Schur process to these coordinates lie inside $\mathcal{S}_{2j-1}$ and $\mathcal{S}_{2k}$, respectively.

Define the stochastic links by

$$\Lambda_{2j}^{2j+1} = p^\downarrow(\rho_{[0,j-1]}^+; \rho_j^+), \qquad j = 1, \ldots, N-1;$$
$$\Lambda_{2j-1}^{2j} = p^\uparrow(\rho_{[0,j-1]}^+; \rho_j^-), \qquad j = 1, \ldots, N-1.$$

One immediately verifies the formula

(23) $\quad \mathbf{S}(\rho_0^+, \ldots, \rho_{N-1}^+; \rho_1^-, \ldots, \rho_N^-)(\lambda, \mu)$

$$= \mathbf{S}\big(\rho_{[0,N-1]}^+; \rho_N^-\big)(\lambda^{(N)}) \prod_{k=1}^{N-1} \left(\Lambda_{2k}^{2k+1}\big(\lambda^{(k+1)}, \mu^{(k)}\big) \, \Lambda_{2k-1}^{2k}\big(\mu^{(k)}, \lambda^{(k)}\big) \right),$$

cf. the definition of $m^{(n)}$ in Proposition 17.

Proof of Theorem 10. We apply Proposition 17. Set $\tilde{\mathcal{S}}_j = \mathcal{S}_j$ for $j = 1, \ldots, n$, $\tilde{\Lambda}_{j-1}^j = \Lambda_{j-1}^j$ for $j = 2, \ldots, n$, and also

$$m_n = \mathbf{S}(\rho_{[0,N-1]}^+; \rho_N^-),$$
$$P_{2j-1} = p^\uparrow(\rho_{[0,j-1]}^+; \pi), \quad j = 1, \ldots, N;$$
$$P_{2j} = p^\uparrow(\rho_{[0,j-1]}^+; \pi), \quad j = 1, \ldots, N-1.$$

The commutation relations (20) follow from Proposition 18, and the matrix of transition probabilities $P^{(n)}$ from (21) is easily seen to coincide with $\mathfrak{P}_\pi^\uparrow$. The claim now follows from (23), Proposition 17, and the relation (cf. (22))

$$\mathbf{S}\big(\rho_{[0,N-1]}^+; \rho_N^-\big)P_n = \mathbf{S}(\rho_{[0,N-1]}^+; \rho_N^-, \pi). \quad \square$$

Proof of Theorem 11. We also apply Proposition 17. This time we need to modify the state spaces:

$$\tilde{\mathcal{S}}_{2j-1} = \mathbb{Y}\big(^\flat\rho_0^+, \rho_{[1,j-1]}^+\big), \qquad j = 1, \ldots, N;$$
$$\tilde{\mathcal{S}}_{2k} = \mathbb{Y}\big(^\flat\rho_0^+, \rho_{[1,k-1]}^+\big), \qquad k = 1, \ldots, N-1.$$

Also set

$$\tilde{\Lambda}_{2j}^{2j+1} = p^{\downarrow}({}^{\flat}\rho_0^+, \rho_{[1,j-1]}^+; \rho_j^+), \qquad j = 1, \ldots, N-1;$$
$$\tilde{\Lambda}_{2j-1}^{2j} = p^{\uparrow}({}^{\flat}\rho_0^+, \rho_{[1,j-1]}^+; \rho_j^-), \qquad j = 1, \ldots, N-1;$$

and

$$m_n = \mathbf{S}(\rho_{[0,N-1]}^+; \rho_N^-),$$
$$P_{2j-1} = p^{\downarrow}({}^{\flat}\rho_0^+, \rho_{[1,j-1]}^+; \sigma), \qquad j = 1, \ldots, N;$$
$$P_{2j} = p^{\downarrow}({}^{\flat}\rho_0^+, \rho_{[1,j-1]}^+; \sigma), \qquad j = 1, \ldots, N-1,$$

Again, the commutation relations (20) follow from Proposition 18, and the matrix of transition probabilities $P^{(n)}$ from (21) coincides with $\mathfrak{P}_{\sigma}^{\downarrow}$. The claim follows from (23), Proposition 17, and the relation (cf. (22))

$$\mathbf{S}\left(\rho_{[0,N-1]}^+; \rho_N^-\right) P_n = \mathbf{S}({}^{\flat}\rho_0^+, \rho_{[1,N-1]}^+; \rho_N^-). \quad \square$$

10. Application to the two-sided Schur processes

Let us put the two-sided Schur process of Definition 4 into the general framework.

We need some notation. For $n \geq 1$, an admissible matrix M, cf. Definition 2, and $a_1, \ldots, a_n > 0$ in the analyticity annulus of $H(M; u^{-1})$, define

$$T_{\lambda\mu}(a_1, \ldots, a_n; M) = \frac{1}{\prod\limits_{j=1}^{n} H(M; a_j^{-1})} \frac{\det\left[a_i^{\mu_j - j}\right]_{i,j=1}^{n}}{\det\left[a_i^{\lambda_j - j}\right]_{i,j=1}^{n}} s_{\lambda/\mu}(M), \qquad \lambda, \mu \in \mathbb{GT}_n.$$

For arbitrary $a_1, \ldots, a_n > 0$ also set ($\lambda \in \mathbb{GT}_n$, $\mu \in \mathbb{GT}_{n-1}$)

$$T_{\lambda\mu}(a_1, \ldots, a_n) = \frac{1}{a_n} \prod_{j=1}^{n-1} \left(\frac{1}{a_n} - \frac{1}{a_j}\right) \frac{\det\left[a_i^{\mu_j - j}\right]_{i,j=1}^{n-1}}{\det\left[a_i^{\lambda_j - j}\right]_{i,j=1}^{n}} s_{\lambda/\mu}(a_n).$$

Thus, we have matrices $T(a_1, \ldots, a_n; M)$ with rows and column parameterized by $\mathbb{GT}_n$, and matrices $T(a_1, \ldots, a_n)$ with rows parameterized by $\mathbb{GT}_n$ and columns parameterized by $\mathbb{GT}_{n-1}$.

Proposition 19. *In the above assumptions, the matrices $T(a_1, \ldots, a_n; M)$ and $T(a_1, \ldots, a_n)$ are stochastic, and the following commutation relation holds:*

$$T(a_1, \ldots, a_n; M)\, T(a_1, \ldots, a_n) = T(a_1, \ldots, a_n)\, T(a_1, \ldots, a_{n-1}; M).$$

For admissible matrices M_1, M_2 and $a_1, \ldots, a_n > 0$ in the analyticity annuli of $H(M_i; u^{-1})$, $i = 1, 2$, we also have the commutation relation

$$T(a_1, \ldots, a_n; M_1)\, T(a_1, \ldots, a_n; M_2) = T(a_1, \ldots, a_n; M_2)\, T(a_1, \ldots, a_n; M_1).$$

Proof. Follows from Propositions 2.8-2.10 and Lemma 2.13(ii) of [BF]. $\square$

Consider now the two-sided Schur process $\mathbf{T}(a_1,\ldots,a_N;\mathbf{M};\Psi)$ of Definition 4. Set $n = c(1) + \cdots + c(N) + N$, and $(c(0) := 0)$

$$\mathcal{S}_j = \mathbb{GT}_k, \qquad c(k-1) + k \leq j \leq c(k) + k, \quad k = 1,\ldots,N.$$

Define the stochastic links by

$$\Lambda^{c(k-1)+k}_{c(k-1)+k-1} = T(a_1,\ldots,a_k), \qquad\qquad k = 2,\ldots,N;$$
$$\Lambda^{c(k-1)+k+l}_{c(k-1)+k+l-1} = T(a_1,\ldots,a_k;M^{(k,l)}), \qquad k = 1,\ldots,N, \quad l = 1,\ldots,c(k).$$

Also define a probability distribution m_n^Ψ on $\mathcal{S}_n = \mathbb{GT}_N$ via

$$m_n^\Psi(\lambda) = \frac{\det\left[a_i^{\lambda_j-j}\right]_{i,j=1}^N \det\left[\Psi_{\lambda_i-i,-j}\right]_{i,j=1}^N}{\det\left[a_i^{-j}\Psi_j(a_i)\right]_{i,j=1}^N}, \qquad \lambda \in \mathbb{GT}_N,$$

where we used the notation (15).

These definitions imply that

$$\mathbf{T}(a_1,\ldots,a_N;\mathbf{M};\Psi)(\vec{\lambda}^{(1)},\ldots,\vec{\lambda}^{(N)})$$
$$= m_n^\Psi\left(\lambda^{(N,c(N))}\right)\Lambda_{n-1}^n\left(\lambda^{(N,c(N))},\lambda^{(N,c(N-1))}\right)\ldots\Lambda_1^2\left(\lambda^{(1,1)},\lambda^{(1,0)}\right).$$

Note that this proves formula (16) for the partition function since

$$\det\left[a_i^{-j}\right]_{i,j=1}^N = \prod_{k=1}^n \frac{1}{a_k}\prod_{j=1}^{k-1}\left(\frac{1}{a_k} - \frac{1}{a_j}\right).$$

Proof of Theorem 13. Once again we apply Proposition 17. We set $\tilde{\mathcal{S}}_j = \mathcal{S}_j$ for $j = 1,\ldots,n$; $\tilde{\Lambda}_{j-1}^j = \Lambda_{j-1}^j$ for $j = 2,\ldots,n$; and $m_n = m_n^\Psi$,

$$P_j = T(a_1,\ldots,a_k;Q^t), \qquad c(k-1) + k \leq j \leq c(k) + k, \quad k = 1,\ldots,N.$$

Note that $H(Q^t;u) = H(Q;u^{-1})$ and $s_{\lambda/\mu}(Q^t) = s_{\mu/\lambda}(Q)$ for signatures λ and μ of the same length.

The claim now follows from Proposition 17 as the needed commutativity relations are given in Proposition 19, and by the Cauchy-Binet identity

$$(m_n^\Psi P_n)(\mu) = \frac{1}{\det\left[a_i^{-j}\Psi_j(a_i)\right]_{i,j=1}^N}\frac{1}{\prod_{j=1}^n H(Q;a_j)}$$
$$\times \sum_{\lambda\in\mathbb{GT}_N}\det\left[a_i^{\lambda_j-j}\right]_{i,j=1}^N \det\left[\Psi_{\lambda_i-i,-j}\right]_{i,j=1}^N \frac{\det\left[a_i^{\mu_j-j}\right]_{i,j=1}^n}{\det\left[a_i^{\lambda_j-j}\right]_{i,j=1}^n} s_{\mu/\lambda}(Q) = m_n^{Q\Psi}(\mu).$$

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