Consumer Heterogeneity, Uncertainty, and Product Policies
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Abstract

This dissertation consists of three essays on the implications of consumer heterogeneity and uncertainty for firms’ strategies.

The first essay analyzes how firms should develop add-on policies when consumers have heterogeneous tastes and firms are vertically differentiated. The theory provides an explanation for the seemingly counter-intuitive phenomenon that higher-end hotels are more likely than lower-end hotels to charge for Internet service, and predicts that selling an add-on as optional intensifies competition, in sharp contrast to standard conclusions found in the literature.

The second essay examines how firms should develop product and pricing policies when customer reviews provide informative feedback about improving product or service quality. The analysis provides an alternative view of customer reviews such that they not only can help consumers learn about product quality, but also can help firms learn about problems with their products or services.

The third essay studies the implications of cognitive simplicity for consumer learning problems. We explore one viable decision heuristic - index strategies, and demonstrate that they are intuitive, tractable, and plausible. Index strategies are much simpler for consumers to use but provide close-to-optimal utility. They also avoid exponential growth in computational complexity, enabling researchers to study learning models in more-complex situations.

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Essay 1

Add-on Policies under Vertical Differentiation: Why Do Luxury Hotels Charge for Internet Whereas Economy Hotels Do Not?
Add-on Policies under Vertical Differentiation: Why Do Luxury Hotels Charge for Internet Whereas Economy Hotels Do Not? *

Abstract

That higher-end hotels are more likely than lower-end hotels to charge for Internet service is a seemingly counter-intuitive phenomenon. Why does it persist? Solving this puzzle sheds light on product policy decisions for firms selling an add-on to a base good: Should they sell the add-on separately from the base as optional, or bundle it with the base as standard, or not sell it at all?

I propose that vertical differentiation plays a role, and develop a theory to explain why and when a divergence in product policy arises as an equilibrium outcome. The theory uncovers the differential role of an add-on for vertically differentiated firms. A firm with higher base quality sells an add-on as optional so that higher-taste consumers self-select to buy it. Although a lower-quality firm also wants to price discriminate, it is incentivized to lower the add-on price to lure consumers who may buy the higher-quality base without the add-on. This trade-off renders its policy sensitive to the cost of providing the add-on. When the cost is small, the lower-quality firm sells the add-on as standard, whereas the higher-quality firm sells it as optional. Examining a sample of hotels that are likely to be in a monopoly or vertical duopoly market, I find suggestive evidence for this theoretical prediction. Surprisingly, the theory predicts that selling an add-on as optional intensifies competition, in sharp contrast to standard conclusions in the literature. If both firms sell an optional add-on, they price aggressively to compete for consumers who trade off the higher-quality base versus the lower-quality base including the add-on. Although selling the add-on as optional is unilaterally optimal, both firms lose profit in equilibrium - a Prisoner's Dilemma outcome.

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1 Introduction

Although Internet service is a necessity and is available in most hotels, only 54% of luxury hotels provide it for free. In contrast, the percentage grows to 72%, 81%, 93%, and 91% for upscale, mid-priced, economy, and budget hotels, respectively. This phenomenon appears to be counter-intuitive, attracting considerable public attention and media coverage. Why does the phenomenon persist? The answer to the question may have important managerial implications. In many other industries, it is common for firms to sell a base good or service as their primary business, and then sell a complementary item or upgrade (hereafter, “add-on”). Examples include airlines selling drinks and snacks on a flight, car manufacturers selling upgrades such as GPS and leather seats on top of a base model, and mobile applications offering in-app purchases or premium upgrades. Solving the puzzle elucidates what product policies firms should adopt: Should they sell an add-on separately from the base good as optional, sell it as standard (i.e., free), or not sell it at all?

Existing pricing theories do not offer adequate explanations regarding why this stylized fact exists. Conventional wisdom from monopoly pricing suggests that selling an add-on as optional allows a firm with market power to price discriminate to enhance profit. This argument, however, contradicts the practice of lower-end hotels. A simple explanation would be that consumers staying at higher-end hotels are less price-
However, this argument would suggest that the higher-end hotels charge a higher total price rather than separate Internet prices from room rates. Shugan and Kumar (2014) compare the hotel industry to the airline industry and argue that it is optimal for a monopolist with a product line of base services to bundle add-ons with the lower-end base to decrease the base differentiation when the differentiation is large (e.g., the hotel industry), and to unbundle add-ons with the lower-end base to increase the base differentiation when the differentiation is small (e.g., the airline industry). However, this theory does not explain why the phenomenon persists when higher-end and lower-end hotels are not owned by the same company, and why policies are different for different types of add-ons within an industry (e.g., mini-bar, laundry, or airport shuttle services in the hotel industry). I propose a different but complementary explanation that vertical differentiation between competing firms plays a role. Using a duopoly theory, I explain why and when a divergence in product policy arises as an equilibrium outcome. The theory leads to three interesting insights.

First, the theory identifies that the role of an add-on can be quite different for vertically differentiated firms. A firm with higher base quality behaves just like a monopolist. Selling an add-on as optional at a high price serves as a screening or segmentation device so consumers with higher tastes for quality self-select to buy the expensive add-on. This incentive to screen consumers also applies to a firm with lower base quality. However, the lower-quality firm is incentivized to lower the add-on price to lure those consumers who may buy the higher-quality base without paying for the add-on to switch to the lower-quality base with the add-on. This vertical differentiation role of the add-on is absent from extant literature on add-on pricing, which focuses on unobserved add-on prices with horizontal or no differentiation (Lal

There are several related explanations along the same line. For example, one may argue that consumers at higher-end hotels have corporate accounts covering their expenses whereas consumers at lower-end hotels travel with their own accounts. Another related argument is that higher-end hotels customers are typically business travelers whereas lower-end hotels have more leisure travelers.
and Matutes 1994, Verboven 1999, Ellison 2005, Gabaix and Laibson 2006).\textsuperscript{5} Due to the trade-off between screening and differentiation, the lower-quality firm’s policy is more sensitive to the efficiency of supplying the add-on. The firm does not sell the add-on when it is too costly to provide it to improve quality because it has to charge a high add-on price that discourages consumers from buying (e.g., mini-bar). In equilibrium, the lower-quality firm sells the add-on only when it is not too costly. When the cost becomes sufficiently small (e.g., Internet service), the add-on price is so low that all consumers who buy the base from the lower-quality firm also pay for the add-on. Consequently, in equilibrium, the higher-quality firm sells the add-on as optional, whereas the lower-quality firm sells it as standard. This equilibrium outcome provides an explanation of the stylized fact.

Examining a sample of monopoly and duopoly markets with vertical differentiation in the American hotel industry, I find support for theoretical predictions when the cost of an add-on is very small (i.e., Internet service). On the one hand, a hotel at the higher end of a vertical duopoly market is as likely as a monopoly hotel to charge for Internet service, consistent with the theory that higher-quality firms focus on screening consumers, behaving like a monopolist. On the other hand, a hotel is more likely to offer free Internet service if it is at the lower end of a vertical duopoly market. This finding supports the theory that vertical differentiation introduces a trade-off for lower-quality firms, which find it optimal to sell an add-on as standard rather than sell it as optional. Conclusions are robust even after controlling for a number of potentially confounding factors such as hotel segment, location, size, age, operation, etc.

Second, the theory leads to a surprising prediction that selling an add-on as optional intensifies competition. Since a higher-quality firm sells an add-on to its higher-

\textsuperscript{5}The term “add-on pricing” has been used to refer to a specific situation with unobserved add-on prices (Ellison 2005). In this essay the term refers to broader problems that involve pricing of a base good and an add-on, regardless of the observability of the add-on price.
taste consumers, leaving some lower-taste consumers who do not buy the add-on, it creates an opportunity for a lower-quality firm to lower its add-on price to induce switching. The firms then price aggressively to compete for these marginal consumers who trade off the higher-quality base good versus the lower-quality base good plus the add-on. Although the optional-add-on policy is unilaterally optimal, a Prisoner’s Dilemma emerges in which both firms lose profits in equilibrium. The result is striking. Extant literature predicts that selling an add-on as optional either has no impact on firm profits under competition (Lal and Matutes 1994), or softens price competition (Ellison 2005). The profit-irrelevant result is essentially a “Chicago School” argument that any profit earned from selling a high-priced add-on is competed away on the base price. The competition-softening result hinges on the idea that with the add-on prices unobserved naturally firms create an adverse selection problem that makes price-cutting unappealing, thereby raising equilibrium profits. In contrast, I identify a mechanism by which selling an add-on hurts firm profits. The mechanism does not rely on the unobservability of add-on prices. It is the interaction between the self-selection effect and the differentiation effect that reverses standard conclusions on the profitability of selling an add-on. This competition-intensifying effect incentivizes the higher-quality firm to commit to a standard-add-on policy, and the lower-quality firm to commit to a no-add-on policy, if such commitments are possible. Luxury cars, for example, are more likely than economy cars to offer some advanced features such as standard leather seats, GPS navigation, side airbags, etc.

Third, when consumers do not observe add-on prices, hold-up problems arise naturally. However, unlike other settings in the literature, vertical differentiation moderates the effects of hold-up on firm profits in this context. The higher-quality firm’s policy is unaffected by the unobservability of the add-on price, because its consumers already expect the add-on to be expansive due to the firm’s screening incentive. The hold-up effect coincides with the self-selection effect for the higher-
quality firm. Anticipating being held up by the lower-quality firm, some consumers refrain from buying from it and switch to the higher-quality base good without paying for the add-on. Consequently, the higher-quality firm demands a higher base price, but the lower-quality firm is forced to lower its base price while keeping the add-on price high. In equilibrium the higher-quality firm is better off, whereas the lower-quality firm is worse off, suggesting that the higher-quality firm has no incentive to advertise the add-on price, whereas the lower-quality firm has a strict incentive to advertise. This prediction contrasts sharply with extant results that a hold-up problem has no impact on firm profits under competition because profits earned by holding up consumers ex post are competed away by lowering base prices (i.e., “loss leader”; Lal and Matutes 1994).

Related Literature

This study relates to broader literature on price discrimination and multi-product pricing. Not surprisingly, add-on pricing can be a form of second-degree price discrimination. A base good plus an add-on versus the base good alone can be viewed as two quality levels. If firms sell an add-on as standard, they essentially sell the same quality to all consumers with no price discrimination. If firms instead adopt an optional-add-on policy, they sell the bundle (i.e., the higher-quality level) to consumers with higher tastes for quality while selling only the base (i.e., the lower-quality level) to lower-taste consumers. In monopoly markets, it is generally optimal for firms to price discriminate. The problem is much harder to analyze under imperfect competition. Extant studies of competitive second-degree price discrimination (Stole 1995, Armstrong and Vickers 2001, Rochet and Stole 2002, Ellison 2005, Schmidt-Mohr and Villas-Boas 2008) examine settings in which firms are symmetric, finding that firms engage in price discrimination if sufficient consumer heterogeneity exists. They do not consider the possibility of vertical pressure competing firms face, a scenario that arises often in the real world. One exception is Champsaur and Rochet (1989), who
study differentiated firms in a duopoly market competing by offering ranges of quality to heterogeneous consumers. After firms’ investments in quality ranges, they determine the optimal nonlinear pricing policies. The price-setting game, given differential quality ranges, is similar to the pricing game this essay examines. However, unlike those authors, I study the context of add-on pricing in which quality is discrete, and derive implications of competitive price discrimination on profitability.

The result that selling an add-on intensifies competition also relates to several findings in the literature. Although it is generally true that monopolists find it optimal to price discriminate among consumers, this conclusion is no longer robust under competitive settings. Price discrimination can intensify competition, thereby hurting profits. Extant literature has documented two mechanisms that lead to this surprising result. The first one arises in the context of competitive third-degree price discrimination (Thisse and Vives 1988, Shaffer and Zhang 1995, Corts 1998). Corts (1998) points out that the critical factor is best response asymmetry. He shows that when firms have a divergent view on the ranking of consumer segments (i.e., a strong market for one firm is weak for the other), it is possible that price discrimination leads to lower prices in all market segments, reducing profits. The second mechanism arises in competitive mixed bundling in the two-stop shopping framework (Matutes and Regibeau 1992, Anderson and Leruth 1993, Armstrong and Vickers 2010). In this case, competing firms offer multiple products, and they can either sell the component products alone (i.e., component pricing), or sell the products in a bundle (i.e., pure bundling), or both (i.e., mixed bundling). Practicing mixed bundling might trigger fierce price competition, lowering prices for component products. Consequently, profits reduce. This essay identifies an alternative mechanism that differs substantially in terms of modeling framework and business context. The add-on pricing problem studied here differs from third-degree price discrimination because firms do not know

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6 Many cases exist in which this condition does not hold, and thus competing firms are better off in equilibrium (Chen et al. 2001, Shaffer and Zhang 2002).
consumers’ preferences, and thus rely on incentive compatibilities to practice price discrimination. It is also different from multi-product bundling in that an add-on is only available and valuable conditional on the purchase of a base good. Nevertheless, the recurring theme underlying the competition-intensifying result appears to be that competing firms price aggressively to acquire consumers who trade off between buying a higher quality level at one firm versus buying a lower quality level at another.

Finally, this essay relates to growing academic and industrial interest in drip pricing. The Federal Trade Commission defines drip pricing as “a pricing technique in which firms advertise only part of a product’s price and reveal other charges later as the customer goes through the buying process. The additional charges can be mandatory charges, such as hotel resort fees, or fees for optional upgrades and add-ons.” The leading theory that explains why firms adopt drip pricing is that it is profitable to exploit myopic consumers who do not anticipate the hidden cost. Gabaix and Laibson (2006) show that firms may choose not to advertise add-on prices (i.e., shrouding) even under perfect competition. Shulman and Geng (2012) extend the mechanism to the situation in which firms are ex ante different in both horizontal and vertical dimensions. Dahremöller (2013) introduces a commitment decision of shrouding or unshrouding, which can alter underlying incentives to unshroud. Unlike these authors, I examine long-run market outcomes in which consumers know or correctly anticipate add-on prices in equilibrium. This assumption is not unreasonable, given that consumers might learn about prices through repeat purchases and/or word-of-mouth, and in many cases, firms advertise add-on policies because they care about reputation or regulations require it. It is unlikely that the stylized fact addressed in

7Consumers cannot buy the add-on without buying the base good, but they can buy the base good alone without buying the add-on. For example, a consumer cannot access Internet service in a hotel if she does not stay at the hotel, but she can stay in the hotel room without using the Internet service.

8See http://www.ftc.gov/be/workshops/drippricing/.

9For example, having stayed at Marriott and learned that Internet service costs $13, a consumer may keep this in mind the next time she books a room at the same hotel or at another location of the same chain.
this essay is driven by consumer myopia, given that repeat purchases are common in the hotel industry and Internet service is a highly expected and frequently used feature.\footnote{If consumers are boundedly rational, then Internet fees should be shrouded and higher than marginal costs at both the luxury and economy hotels.} Results from this study suggest that vertical differentiation not only has profound impacts on add-on policies even under complete price information, but also interacts with firms’ incentives to advertise add-on prices when they are unobservable.

The rest of the essay is structured as follows. Section 2 introduces the main theory. Section 3 presents the empirical evidence that supports the theoretical predictions. Section 4 discusses the implications of the theory on firm profits. Section 5 extends the mechanism to the case in which add-on prices are unobserved. Section 6 concludes the essay.

2 A Theory of Add-on Policy under Vertical Differentiation

In this section, I present a theory that explains why and when a divergence in product policy arises as an equilibrium outcome. I first write down the simplest model that illustrates the key underlying mechanism, and then relax some of the simplifying assumptions to demonstrate that they do not alter the main message of the mechanism.

2.1 Model Setup

There are two firms $j \in \{l, h\}$, that differ in terms of the quality of the base good $V$ such that $V_h > V_l > 0$. The quality difference, or \textit{quality premium}, is $\Delta V$. The marginal cost of the base good is normalized to zero for both firms. In an extension discussed later, I allow the marginal cost to be different for the two firms. In addition to the base good, an add-on technology is available. The add-on has value $w$ and costs $c$, the same for both firms. Again, the symmetric assumption is made only to simplify analysis. Qualitative conclusions remain largely unaffected in an extension with asymmetric add-on discussed later. The efficiency of supplying the add-on is
measured by the cost-to-value ratio, $\alpha = \frac{\varphi}{\omega}$, which plays an important role in the equilibrium analysis. It is assumed that the quality premium is greater than the value of the add-on, $\Delta V > \omega$, allowing interesting equilibria to arise. Each firm can set base price $P_j$ and add-on price $p_j$.

A continuum of consumers differ in their marginal valuation, or taste, for quality. The taste parameter, $\theta$, is distributed uniformly with $\theta \in [\underline{\theta}, \overline{\theta}]$ and $\underline{\theta} > 0$. Two assumptions are made throughout the analysis. First, $\overline{\theta} > 2\theta$, so there is a sufficient amount of consumer heterogeneity in the market. Second, $\overline{\theta} > \alpha$, so there are positive sales of the add-on in equilibrium. The utility of buying from firm $j$ for type-$\theta$ consumer is

$$U_{\theta j} = \begin{cases} 
\theta V_j - P_j & \text{if only the base good is purchased;} \\
\theta (V_j + \omega) - P_j - p_j & \text{if both the base good and the add-on are purchased.}
\end{cases}$$

(1)

The base and add-on values $V$ and $\omega$ are both common knowledge to all parties. This is not an unreasonable assumption for industries such as the hotel industry in which consumers possess sufficient knowledge or information due to say repeat purchases. However, consumers know their own tastes $\theta$, but the firms do not. The firms only know the distribution of tastes, and thus rely on incentive compatibilities to screen consumers or price discriminate.

It is worth noting that the assumption that the unobserved consumer preferences are summarized entirely in one dimension, $\theta$, may appear strong, but it keeps the model tractable. An alternative interpretation is that tastes are the inverse of price

\[11\text{The assumption that the lower bound is positive, combined with a sufficiently large base quality } V_j, \text{ ensures that the market is fully covered. It simplifies analysis by focusing on the interaction between the two firms, assuming away the outside option of not buying from any firm.}

\[12\text{There are, of course, situations in which consumers experience uncertainty about } V \text{ and/or } \omega. \text{ If firms have superior knowledge on these values, then the add-on can signal the base quality. For example, Bertini et al. (2009) show that add-on features can influence consumers' evaluations of a base good about which they are uncertain. How firms design product policies under this situation is an interesting direction to explore, but it is beyond the scope of this essay. If firms are also uncertain about these values, insights from this simpler specification of consumer utility may apply.}
sensitivities. The implicit restriction behind this setup is that willingness-to-pay for the base good and for the add-on ($\theta V_j$ and $\theta w$) correlate perfectly. Nevertheless, the mechanism does not rely on this assumption, as shown in an extension with imperfectly correlated tastes in the Appendix. What is necessary is that there is unobserved heterogeneity in both the base good and add-on, enabling consumers to trade off between the add-on and the base quality. For the self-selection effect to arise, the single-crossing property is necessary such that higher taste for the value of an add-on and for a base quality implies a higher willingness-to-pay for the add-on and for the higher-quality base.

The strategic interaction is modeled as a full-information simultaneous-move game. Both firms announce prices simultaneously, $(P_h, P_h)$ and $(P_l, P_l)$. Consumers observe all prices and decide which firm to visit and whether to pay for an add-on from the chosen firm. The formulation naturally builds on two stylized models. If there is no add-on, the model reduces to a duopoly model of vertical differentiation (Shaked and Sutton 1983). If there is no competition, the model reduces to a monopoly model of nonlinear pricing (Mussa and Rosen 1978) with continuous-type consumers and discrete qualities. Each reduced model is straightforward to solve. However, combining the two features complicates equilibrium analysis dramatically due to the many possibilities of equilibrium outcome. Specifically, each firm can price a base and an add-on to implement any of the following three outcomes:

**No Add-on:** The price of the add-on is sufficiently large such that no consumer buys it. All consumers buy just the base good.

**Standard Add-on:** The price of the add-on is sufficiently small such that all consumers buy it. The add-on is always bundled with the base good, and thus

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13 One can specify an equivalent preference model such that the utility of buying both a base and an add-on is given by $U_{o_j} = V_j + w - \frac{1}{\theta}(P_j + p_j)$.

14 This feature distinguishes the paper from Dogan et al. (2010). In their model of competitive second-degree price discrimination with vertical differentiation in the context of rebates, there is no unobserved heterogeneity in tastes for the base quality.
only the total or bundle price matters.

Optional Add-on: The price of the add-on is moderate such that some but not all consumers buy it. The higher-type consumers buy the add-on, and the lower-type consumers buy the base only.

There are nine possible market configurations, depending on the implementations by both firms. Each configuration constitutes a possible equilibrium profile. To prove the existence of an equilibrium for each profile, one has to examine, for each firm, all non-local deviations that lead to any form of the remaining eight possible market configurations. However, the solution to the game can be simplified by observing that the higher-quality firm always serves the highest-type consumers and thus may have a strong incentive to sell the add-on as optional in equilibrium. The next sub-section formalizes this intuition.

2.2 The Higher-quality Firm Focuses on Screening

If the higher-quality firm implements the optional-add-on policy, it essentially divides its pool of consumers into two segments. The higher-type consumers, $\theta \in [\hat{\theta}_h, \bar{\theta}]$, buy both the base and the add-on. Type $\hat{\theta}_h$ is the intra-marginal consumer who is indifferent between buying the bundle and buying only the base, and it is equal to $p_h/w$. The remaining lower-type consumers, $\theta \in [\hat{\theta}_{hl}, \hat{\theta}_h]$, buy the base only. This segmentation is a result of the single-crossing property so that higher-type consumers self-select to buy the add-on. Type $\hat{\theta}_{hl}$ is the marginal consumer who is indifferent between buying from the higher-quality or lower-quality firm. This marginal consumer is only affected by base price $P_h$, and the add-on price is irrelevant. The firm’s profit decomposes into two additive profit components

$$\Pi_h = (\bar{\theta} - \hat{\theta}_{hl})P_h + (\bar{\theta} - \hat{\theta}_h)(p_h - c) \cdot \pi_h(p_h).$$ (2)
The first is the profit from selling the base good, independent from the add-on price since it is irrelevant to the marginal consumer \( \hat{\theta}_{hl} \). The second is the additional profit from selling the add-on, independent from the base price. Without considering incentive constraint \( \hat{\theta}_h \geq \hat{\theta}_{hl} \), to maximize profit, the higher-quality firm simply chooses the appropriate base and add-on prices to maximize the two components separately. If, however, the incentive constraint is binding, all consumers buy the bundle. In this case, the firm is selling the add-on as standard and a single bundle price \( P_h^+ \) suffices. Since the firm serves consumers with the highest tastes in the market, it relaxes the pressure for a binding constraint. Just as in a monopoly market, the higher-quality firm can easily find the optional-add-on policy strictly better than the standard-add-on policy. The following lemma formalizes this observation.

**Lemma 1.** For the higher-quality firm, selling the add-on as standard is strictly dominated by selling the add-on as optional, if:

1. \( p_l > -\alpha \Delta V \) when the lower-quality firm does not sell the add-on, or
2. \( p_l^+ > -\alpha (\Delta V - w) \) when the lower-quality firm sells the add-on.

**Proof.** See Appendix B. ■

The intuition behind it is simple. When the higher-quality firm sells the add-on as standard, the add-on price is set sufficiently low so that all of its consumers buy it. Consider a local deviation whereby the firm increases add-on price \( p_h \) by a small amount \( \epsilon \) and decreases the base price by the same amount. Then some consumers refrain from paying for the add-on (i.e., the firm implements the optional-add-on policy). Increasing the add-on price does not lose many consumers who originally bought the add-on, but it generates additional revenue from those higher-type consumers who continue to pay for it. On the other hand, decreasing the base price expands the market. Acquired consumers are of the lower types so the loss due to the lower base price is quite limited. The total profit is then increased. Note that the
argument requires that there is sufficient number of different types (more than two) so that a local deviation by separating the prices is profitable.\textsuperscript{15} This intuition is the same as that in a monopoly market. Despite facing competitive pressures from the lower-quality firm, the higher-quality firm behaves like a monopolist. The first main proposition follows.

**Proposition 1.** *In any equilibrium, if it exists, the higher-quality firm sells the add-on as optional.*

*Proof.* See Appendix C. ■

The mechanism in the higher-quality firm accords with what many managers have in mind. One reason luxury hotels charge for Internet service is “because they can”. Selling optional Internet service is so lucrative that these hotels do not want to give up this source of revenue. One driving force is the firm’s incentive to screen consumers, an effective way to boost short-term profits. This incentive should also apply to the lower-quality firm, given that it also serves a pool of consumers with heterogeneous tastes. Why does the lower-quality firm behave differently? The next sub-section provides the answer.

### 2.3 The Lower-quality Firm Trades off Screening and Differentiation

Having shown that the only possible implementation of the higher-quality firm is the optional-add-on policy, there are only three possible equilibrium profiles to be considered, depending on the lower-quality firm’s implementation. In what follows I develop the equilibrium in which the lower-quality firm sells the add-on as optional. The other two possible equilibria become straightforward given this development.

The lower-quality firm now targets consumers of lower types, $\theta \in [\hat{\theta}, \hat{\theta}_l]$. Similar to the higher-quality firm, it segments consumers into two groups. The lower-type consumers, $\theta \in [\hat{\theta}, \hat{\theta}_l]$, buy only the base good. The intra-marginal consumer, who

\textsuperscript{15}With just two types of consumers, it is easy to conclude that selling the add-on as standard is optimal, as in Ellison (2005) and Shugan and Kumar (2014).
is indifferent between buying the bundle and buying only the base, is given as $\hat{\theta}_l = p_l/w$. This consumer does not react to the base price. Consumers of higher types, $\theta \in [\hat{\theta}_l, \hat{\theta}_hl]$, buy both the base and add-on. The marginal consumer, who is indifferent between the two firms, $\hat{\theta}_hl$, depends on the lower-quality firm’s total price of the base and add-on, $P^+_l = P_l + p_l$. The profit is

$$\Pi_l = (\hat{\theta}_hl - \theta)P_l + (\hat{\theta}_hl - \hat{\theta}_l)(p_l - c).$$

Like its rival, the lower-quality firm is incentivized to set a high add-on price to allow the consumers self-select. Those who have a higher taste for quality consider the high-priced add-on, and the lower-type consumers consider only the base. Unlike its rival, however, the lower-quality firm also uses the add-on to attract consumers who consider only the base from the competitor. The firm attempts to keep its add-on price reasonably low to attract these potential switchers. To see the optimal pricing strategy that resolves this trade-off, it is instructive to rewrite the firm’s profit as

$$\Pi_l = (\hat{\theta}_hl - \theta)(P^+_l - c) - (\hat{\theta}_l - \hat{\theta}_l)(p_l - c).$$

(3)

In this decomposition, the first component depends only on bundle price $P^+_l$, and the second depends only on add-on price $p_l$. The unconstrained problem is solved by maximizing each component separately. The intuition can be understood as follows. The lower-quality firm advertises an attractive bundle package (e.g., a 3-star hotel offers “free-Internet” or “stay-connected” packages) to its potential consumers, $\theta \in [\theta, \hat{\theta}_hl]$, and convinces all to visit. Once consumers accept the offer, the firm excludes the lowest-type consumers, $\theta \in [\theta, \hat{\theta}_l]$, from consuming the add-on by subsidizing them to not consume it. In this way, the firm attracts its most valuable consumers, $\theta \in [\hat{\theta}_l, \hat{\theta}_hl]$, while avoiding unnecessary costs of supplying the add-on to the lowest-
type consumers, who do not value it much.

Strategic interactions between the two firms reduce to competition between the lower-quality firm’s bundle and the higher-quality firm’s base good. They compete for the marginal consumer who is indifferent between the two, given by \( \hat{\theta}_{hl} = (P_h - P^+_l)/(\Delta V - w) \). The competition is the same as a duopoly model of vertical differentiation,\(^{16}\) except that the quality premium is \( \Delta V - w \) instead of \( \Delta V \), and that the lower-quality firm’s price is \( P^+_l \) instead of \( P_l \). In equilibrium, the prices are

\[
P^*_h = \frac{1}{3}(\Delta V - w)(2\bar{\theta} - \bar{\vartheta}) + \frac{1}{3}c, \quad \text{and} \quad P^{++}_l = \frac{1}{3}(\Delta V - w)(\bar{\theta} - 2\vartheta) + \frac{2}{3}c, \tag{4}
\]

and the marginal consumer is

\[
\hat{\theta}^*_hl = \frac{1}{3}(\bar{\theta} + \vartheta) - \frac{c}{3(\Delta V - w)}, \tag{5}
\]

which determines the equilibrium market share of each firm. Note that equilibrium marginal consumer, \( \hat{\theta}_{hl}^* \), decreases with add-on value \( w \). As the value grows, two opposite effects occur. On the one hand, there is a direct effect of increasing demand for the lower-quality firm, and decreasing demand for the higher-quality firm, because consumers obtain higher utility from the lower-quality firm due to the added value of the add-on. For fixed prices, the marginal consumer moves upward as \( w \) increases. On the other hand, there is an indirect effect stemming from strategic price responses to changes in product quality. The higher-quality firm lowers its base price as value \( w \) increases. Contrarily, the lower-quality firm raises its bundle price to exploit acquired consumers who have higher tastes. The price gap is decreased, moving the marginal consumer downward. In equilibrium, which effect dominates depends on the cost of the add-on relative to the cost of the quality premium. In the current setup, the strategic effect dominates given that the cost of the quality premium is small (i.e.,

\(^{16}\)See Tirole (1988) for a stylized model.
assumed to be zero), and hence the marginal consumer becomes lower as w increases.\footnote{In the extension in which the marginal cost of the base good is asymmetric, this conclusion holds as long as the cost of the add-on is greater than the cost of the quality premium.}

Independent of strategic interactions that determine equilibrium market shares, the firms set an optimal add-on price. Add-on prices $p_h$ and $p_l$ are chosen to maximize $\pi_h(p_h)$ and $\pi_l(p_l)$ in Equations (2) and (3) respectively, which lead to

$$p_h^* = \frac{1}{2} (\overline{\theta} + \alpha) w, \quad \text{and} \quad p_l^* = \frac{1}{2} (\underline{\theta} + \alpha) w,$$

with resulting intra-marginal consumers: $\hat{\theta}_h^* = \frac{1}{2} (\overline{\theta} + \alpha)$ and $\hat{\theta}_l^* = \frac{1}{2} (\underline{\theta} + \alpha)$. The optimal add-on prices reflect underlying consumer tastes. Although the add-on is the same for both firms, the price is higher at the higher-quality firm. This accords with the casual observation that Internet fees at higher-end hotels are higher than those at lower-end hotels (if they charge for it). Furthermore, the add-on price at the higher-quality firm is higher than the marginal cost, suggesting that it is a profitable business. However, the price is considerably lower at the lower-quality firm, even lower than the marginal cost. In fact, the lower-quality firm prices the add-on significantly lower than what it would have charged if there were no competition. To see that, imagine that the lower-quality firm is the monopolist for a market of consumers with types $\theta \in [\underline{\theta}, \bar{\theta}]$. The maximization problem would lead to an add-on price of $\bar{p}_l^* = \frac{1}{2} (\bar{\theta}_l + \alpha) w$, which is greater than $p_l^*$ in the equilibrium under vertical differentiation because $\bar{\theta}_l > \underline{\theta}$. This is true even though $\bar{\theta}_l$ can be much smaller than $\bar{\theta}$. The fact that the lower-quality firm serves the consumers with lower tastes is not the main driver of the low add-on price; vertical differentiation forces the firm to price it considerably low, even below the marginal cost.

\subsection*{2.4 Incentive Compatibility and Equilibrium Outcomes}

The preceding analysis uncovers the equilibrium pricing under the scenario in which both firms sell the add-on as optional. This equilibrium exists as long as the following
incentive constraints hold:

1. \( \hat{q}_h^* > \hat{q}_l^* \),
2. \( \hat{q}_h^* > \hat{q}_l^* \),
3. \( \hat{q}_l^* > q \).

The first incentive constraint holds as long as \( \Delta V > w \). Intuitively, this means that as long as the quality premium is greater than the add-on value the lower-quality firm provides, some consumers prefer the higher-quality base good to the lower-quality bundle.

The last two constraints depend crucially on the unit cost of supplying the add-on, and determine whether the lower-quality firm sells the add-on, or sell it as standard or as optional. On the one hand, the marginal consumer who is indifferent between the two firms, \( \hat{q}_l^* \), decreases in marginal cost of add-on \( c \) for a fixed value of the add-on, according to Equation (5). This implies that \( \hat{q}_l^* \) decreases as \( \alpha \) increases. Intuitively, as the cost of the add-on increases, so does the bundle price of the lower-quality firm. Some consumers would rather buy only the base from the higher-quality firm. On the other hand, the intra-marginal consumer for the lower-quality firm, \( \hat{q}_l^* \), increases with \( \alpha \). As the add-on becomes more costly to provide, it is optimal for the lower-quality firm to exclude more consumers who do not value it much. Consequently, the segment of consumers that buys the bundle from the lower-quality firm shrinks. As cost-to-value ratio \( \alpha \) becomes sufficiently large, the firm excludes all consumers from buying the add-on, thereby not selling it.\(^{18}\) Therefore, incentive constraint \( \hat{q}_l^* > \hat{q}_l^* \) ensures that the firm sells the add-on in equilibrium, leading to

\[
\alpha < \frac{1}{3}(2\theta - \theta) \quad \text{and} \quad \Delta V > \frac{2\theta - \theta - \alpha}{2\theta - \theta - 3\alpha} \cdot w \equiv \Delta_1. \quad (6)
\]

If however cost-to-value ratio \( \alpha \) becomes smaller, \( \hat{q}_l^* \) increases whereas \( \hat{q}_l^* \) de-

\(^{18}\)More generally, not selling the add-on is strictly dominated by selling the add-on provided that \( \alpha \) is sufficiently small. This result is summarized in a lemma, analogous to Lemma 1, in the proof of the next proposition in the Appendix.
creases. The segment of consumers that buys the bundle from the lower-quality firm expands, deriving from two sources. One source of switchers comes from those who originally considered only the base from the higher-quality firm, and are now drawn to the lower-quality firm due to its lower bundle price. The other switchers originally considered only the base from the lower-quality firm, and are now drawn to the add-on because it becomes affordable. As $\alpha$ becomes sufficiently small, all consumers who decided to buy from the lower-quality firm are willing to buy the add-on. The add-on is then essentially standard, or free, because all consumers pay just the bundle price.

In this case, $\hat{\theta}_i^* \leq \bar{\theta}$, which is equivalent to $\alpha \leq \bar{\theta}$. Constraints (2) and (3) now reduce to $\hat{\theta}_h^* > \bar{\theta}$, leading to

$$\Delta V > \frac{\bar{\theta} - 2\bar{\theta} + \alpha}{\bar{\theta} - 2\bar{\theta}} \cdot w \equiv \Delta_2.$$  

The following proposition summarizes pure-strategy equilibrium outcomes of the game.

**Proposition 2.** There exist pure-strategy Nash equilibria, in which firms may adopt different add-on policies.

1. If $\alpha$ is large such that $\alpha \geq \frac{1}{3}(2\bar{\theta} - \bar{\theta})$, there exists an equilibrium in which the higher-quality firm sells the add-on as optional whereas the lower-quality firm does not sell it;

2. If $\alpha$ is moderate such that $\bar{\theta} < \alpha < \frac{1}{3}(2\bar{\theta} - \bar{\theta})$, there exists an equilibrium when $\Delta V > \Delta_1$, in which both firms sells the add-on as optional;

3. If $\alpha$ is small such that $\alpha \leq \bar{\theta}$, there exists an equilibrium when $\Delta V > \Delta_2$, in which the higher-quality firm sells the add-on as optional whereas the lower-quality firm sells it as standard;

4. If $\alpha < \frac{1}{3}(2\bar{\theta} - \bar{\theta})$ and $\Delta V \leq \max\{\Delta_1, \Delta_2\}$, there is no pure-strategy equilibrium.

**Proof.** See Appendix D.
2.5 Remarks

A few remarks are in order. The theory explains the relative difference in add-on policies between higher-quality and lower-quality firms. While a higher-quality firm concentrates on using an add-on to screen consumers, a lower-quality firm's policy is far more sensitive to the unit cost of providing the add-on because of its trade-off between screening and differentiation. The trade-off depends on whether the firm improves quality efficiently by providing the add-on. Internet service has arguably a very low marginal cost and/or a large value, making unit cost a very small. Consumers who visit lower-end hotels always pay for Internet service. In this sense, Internet service is essentially bundled with room rates, and thus hotels are selling it as standard or for free, and quoting only the total price. For an add-ons with a larger unit cost, the add-on price has to be higher to recover the cost, discouraging consumers from buying. It is then optimal for a lower-quality firm to offer it as optional. This case can explain why lower-end hotels are equally likely as higher-end hotels to offer add-ons such as laundry or airport shuttle services as optional. It can also explain why lower-end airlines or cruise lines are equally likely as their higher-end competitors to charge for Internet service, because unit costs of providing Internet access remain large based on today's technology.\footnote{Unlike hotels, airlines use ground stations or satellites to provide Internet access on a flight and most cruise lines use satellite technology to provide Internet access. Currently in-flight Internet fees range from $10 to $20 per hour, and Internet charges on a cruise ship can be as high as 90 cents per minute.} For an add-on with a very large unit cost, the add-on price may be so high that very few consumers choose to pay for it. This case can explain why lower-end hotels do not offer amenities such as mini-bar or room services whereas higher-end hotels sell them as optional at high prices.

Clearly the competition between higher-quality and lower-quality firms for marginal consumers who trade off a higher-quality base and a lower-quality bundle is driving
the result. This competition is the reason why a higher-quality firm is able to concentrate on price discrimination whereas its lower-quality rival has to think more in terms of using an add-on to differentiate. If both firms are owned by the same managing company, then it is optimal not to allow the two brands to price aggressively for the marginal consumer. In Appendix E, I present a monopoly model with a product line and show that, under the same set of assumptions, selling the add-on for the lower-quality brand will cannibalize profit of the higher-quality brand. Therefore, absent other forces, the monopolist will sell the add-on as optional for the higher-quality brand but does not sell it for the lower-quality brand.

The theory assumes no horizontal differentiation and focuses on the mechanism of vertical differentiation. If two firms are differentiated horizontally, they will be incentivized to lower add-on prices. Without asymmetry generated by quality difference, however, the competing firms will tend to adopt the same policy. Therefore, horizontal differentiation alone is unlikely to explain why two competing firms diverge in their add-on policies. An alternative explanation is that horizontal differentiations both at a higher-end market and at a lower-end market can lead to different outcomes due to different characteristics of the two markets. Applying theories of competitive second-degree price discrimination with horizontal differentiation, under the assumption that consumers' brand preferences are independent of their preferences for quality, may predict that an add-on is sold at the marginal cost (Verboven 1999). However, marginal costs for Internet service arguably are small and similar at both the higher-end and lower-end markets, suggesting that hotels at both ends will offer Internet service as standard, contradicting the stylized fact. If the independence assumption is violated, as Ellison (2005) shows, firms will bundle an add-on when consumers are less heterogeneous and unbundle when they are more heterogeneous. The theory can then explain the stylized fact if consumers are more heterogeneous at the higher-end markets than at the low-end markets. However, this argument relies
on the modeling assumption that only two types of consumers exist (in terms of their tastes for a higher quality), and thus firms often find it optimal to sell the add-on to both types. The analysis of the higher-quality firm in Section 2.2 implies that the incentive to unbundle is generally strong provided a sufficient number of types exists.

I focus on the simplest setting in which there is only one add-on with one quality level. In reality, firms can supply multiple add-ons (e.g., breakfast, local calls, or airport shuttle), or various qualities of an add-on (e.g., high-speed Internet access). These applications share the common feature that consumers who value a higher quality more self-select to buy more add-ons or the higher-quality level of the add-on. The primary intuition of the mechanism applies. For example, “all-inclusive” hotels are typically not the most luxurious; less-than-luxurious hotels are more likely to offer all-inclusive services. It is increasingly common that higher-end hotels use tiered pricing to charge for Internet service. They offer complimentary Internet access for basic use such as emailing but charge for higher-speed Internet or heavy use such as video conferencing and streaming movies. This practice is rare at lower-end hotels.

An add-on may evolve due to, for example, technology improvement or changing consumer preferences. This may reduce the cost of supplying the add-on and/or enhance the value of the add-on. For example, the cost of Internet service has decreased, and the value has increased over time. Proposition 2 provides a straightforward prediction about the dynamics of add-on policies.

**Corollary 1.** As the cost of an add-on decreases over time, the lower-quality firm tends to sell it as standard. As the value of the add-on increases over time, the lower-quality firm tends to sell it as standard.

Recall the three simplifying assumptions. First, the add-on is homogeneous even though the firms are differentiated vertically with respect to the base good. This may be a reasonable assumption for some applications like Internet service. More realistically, the add-on may be asymmetric across firms in terms of cost and/or
value. For example, a 5-star hotel may provide higher-speed Internet access while the Internet speed may be slower at a 3-star hotel. Breakfast may be of higher quality but it costs more at a 5-star hotel. Second, the marginal cost of the base good is assumed equal for both firms even though the base quality differs. This assumption may be reasonable if the quality premium originates from the fixed costs of investing in the product design of the base good. A 5-star hotel, for example, can invest in better locations and views, high ceilings, swimming pools, fitness centers and other facilities that provide better quality. The investment costs may be substantially higher than those at a 3-star hotel, but less so for the marginal costs. Nevertheless, it is more realistic to allow asymmetric marginal cost. Third, consumers have the same marginal utility or taste for the base good and for the add-on, making the preferences for the two perfectly correlated. It is more realistic to assume that consumers have separate tastes, one for the base good and the other for the add-on. However, it is not unreasonable to allow for some degree of positive correlation between the two tastes, perhaps through price sensitivity. For example, a less price-sensitive consumer may be willing to pay more for a more comfortable room and for Internet service. I relax each of these three convenient assumptions and demonstrate that they do not fundamentally alter the main conclusions. Appendices F, G, and H summarize the details of the results.

3 Empirical Evidence

The theory is constructed to explain the motivating stylized fact that higher-end hotels are more likely to charge for Internet service. It leads to the insight that the role of an add-on is different for vertically differentiated firms. There are of course other plausible explanations for the stylized fact. It is beyond the scope of the essay to test each of them. In what follows I provide some suggestive evidence that is consistent with the theory. Notice that the theory leads to the following prediction
that can be examined empirically.

Prediction. If a monopoly market is compared to a duopoly market with vertical differentiation in terms of add-on policy when the unit cost of the add-on is small, then the theory suggests: (a) there is no difference between the monopolist and the higher-quality firm in the vertical duopoly, and (b) the lower-quality firm in the duopoly is more likely than the monopolist to sell the add-on as standard.

3.1 Data

The primary data source came from the American Hotel and Lodging Association (AH&LA). Started in 1988, AH&LA has been conducting the “Lodging Survey” study every two years. The surveys present respondents (i.e., hotel managers) with a list of amenities, and ask whether their properties provide them. The list is comprehensive, ranging from in-room and bathroom amenities such as high-definition TV, coffee makers, and Internet services, to general services such as swimming pools, airport shuttles, and guest parking. For a few amenities (e.g., Internet service, breakfast, local calls, etc.), respondents report whether they are provided free. The surveys also obtain relevant hotel information such as open dates, and numbers of guest rooms and floors. The survey population includes all properties in the United States with 15 or more rooms (more than 52,000). The overall response rate is 23% with more than 12,000 participants, believed representative of the U.S. lodging industry.

Since 2004, AH&LA outsourced the Lodging Survey to a private research company, Smith Travel Research (STR). I obtained the individual-level data for the most recent four surveys (2006, 2008, 2010, and 2012) from this company. The total number of respondents for the four surveys was 25,179. Among them, 65% completed one survey, 24% two, 9% three, and 2% all four. The company groups hotels in a general market into five price segments according to actual or estimated average room rates: luxury, upscale, mid-priced, economy, and budget.20 In addition, STR provides numbers of

20The company defines a general market as “a geographic area composed of a Metropolitan Sta-
census properties and rooms within each zip code. There are 163 general markets and 11,154 zip codes in the data. Additional data include the parent chains of surveyed hotels.

To isolate the hotels most likely to be monopolists or vertically differentiated duopolists, I focused on small markets with one or two hotels within a zip code. I obtained a dataset from the company that contained the number of hotels for each price segment within each zip code. This allowed me to identify monopoly and duopoly markets, and infer the vertical relationship within a duopoly market. There were 4,229 observations in these markets, 1,441 of which had no information concerning in-room Internet service. According to the company, not all properties completed all questions for a variety of reasons. One major reason could be that the surveys were too long, fatiguing respondents. There is no strong evidence that would suggest the pattern of missing information correlates with the hotels’ incentives to offer free Internet service. 21 Analysis was performed on the remaining 2,269 observations identified to provide Internet service, either free or otherwise. In Appendix A, I collected additional data from a major online travel agency, Expedia, and verified that the qualitative conclusions remained valid, even though a large number of observations were excluded from analysis. There were 1,285 hotels in a monopoly, 705 higher-end hotels in a vertical duopoly, and 279 lower-end hotels in a vertical duopoly. The 519 hotels in a horizontal duopoly in which both hotels fall into the same price segment were excluded from analysis. 22

21 However, it is worth noting that lower-end hotels were less likely than higher-end hotels to answer questions on the surveys.

22 The relatively high ratio of vertical duopoly to horizontal duopoly is also an evidence that hotels are often vertically differentiated and possibly acting strategically.
3.2 Empirical Findings

Figure 1 visualizes the likelihood of offering free Internet service under various market conditions. 82% of the hotels in the monopoly markets provided free Internet service. Higher-end hotels in the duopoly markets were equally likely (81%) to offer Internet free. In contrast, the likelihood was much higher for lower-end hotels in the duopoly markets: 87% provided free Internet service. These descriptive statistics are consistent with the theoretical predictions. Notice that the absolute percentages are quite high across all three market conditions. The primary reason is that these markets (i.e., zip codes) with one or two hotels are most likely in suburban or small towns rather than in big cities. These hotels are less likely to be luxury or upscale, and thus more likely to face upward competition from other markets. Furthermore, it is likely that many higher-end hotels offer basic Internet access for free but charge for heavy uses. These hotels would report free Internet policy even though they were practicing price discrimination. Therefore, the observed difference in Internet policy between lower-end and higher-end hotels might be smaller than the actual difference, suggesting that the analysis is conservative.

A number of factors might have influenced Internet policies. For example, many hotels have VIP or club floors that target consumers with higher willingness-to-pay. The VIP floors typically charge customers for higher room rates, but provide additional benefits that likely include Internet access. This represents a classic example of second-degree price discrimination. These hotels are likely not to offer free Internet service to all consumers. Another example is that hotels vary in terms of number of rooms. A larger size likely increases setup, maintenance, or labor costs of supplying Internet service, or the cost of implementing price discrimination. Other potential confounding factors include the age, location (i.e., airport, interstate, resort, small town, suburban, or urban), and type of operation (i.e., chain operated, franchise, or
Table 1 reports the summary statistics for all of these variables.

I used regression analysis to control for these possible confounds. The dependent variable was a binary indicator of whether a hotel offered free Internet service. Independent variables were dummies for three market conditions: monopoly, high-end in a vertical duopoly, and low-end in a vertical duopoly. Monopoly markets were treated as a benchmark group. The first column of Table 2 suggests that (a) the likelihood of offering free Internet service at a high-end hotel in a duopoly market was not significantly different from that at a monopoly hotel, and (b) the likelihood was significantly higher at a lower-end hotel in a duopoly market in comparison to a monopoly hotel. The second column of the table reports regression results after controlling for potential confounding factors. Conclusions remained robust even when these confounds were controlled. Effects of the confounds varied. As expected, hotels with a VIP floor were less likely to offer free Internet service. Larger hotels were also less likely to offer free Internet service. However, no trend was apparent over the eight years.

A further test is to compare the higher-end and lower-end hotels in vertical duopoly markets, a direct test of the stylized fact. Notice that in the preceding analysis, the objective was to compare duopoly markets to monopoly markets. It did not require both hotels in the same market reported their Internet policies. However, to make within-market comparison, I restricted attention to duopoly markets in which both hotels reported their Internet policies. There were 86 such duopoly markets. 74% of the higher-end hotels provided free Internet service, whereas 88% of the higher-end hotels provided it for free. A paired t-test suggests that this difference was statistically significant ($p = 0.022, t = 2.325$). This provides an evidence that in a market with vertical differentiation, the lower-quality firm is significantly more

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23 There were some hotels with missing information on VIP floor or age. These hotels were flagged in the regressions.
likely than the higher-quality firm to sell an add-on as standard when it has a very small unit cost.

3.3 Restricting Analysis to Upscale Hotels

Next I focused on the segment of upscale hotels and showed that the patterns found are not attributed to the differences in price segments. A hotel in the restricted sample could be in the higher-end condition if there is a mid-priced or below hotel in the same zip code, or in the lower-end condition if there is a luxury hotel nearby. A hotel could also be the monopolist in a zip code. There were 646 observations, of which 355 were monopolists, 250 higher-end, and 41 lower-end.

Figure 2 summarizes the descriptive statistics. Table 3 summarizes the coefficients of the logistic regressions. The results are qualitatively similar to the preceding analysis. Notice that the higher-end hotels appeared to be more likely to offer free Internet service than monopoly hotels did. This is most likely because hotels in the duopoly condition tend to be in a zip code with more competition from neighboring zip codes. These upscale hotels are more likely in urban or suburban areas. The significance of the difference, however, was weakened when control variables such as location types were included.

Restricting the analysis to upscale hotels allows us to rule out several alternative explanations for the observed patterns. One possible explanation is that higher-end hotels face more heterogeneous consumers than lower-end hotels do. Similarly, the upscale hotels in the sample are likely to have very similar levels of consumer heterogeneity across different market conditions. The difference in heterogeneity is unlikely to explain the patterns observed here. It is also plausible that higher-end hotels tend to attract consumers who do not anticipate Internet charges, and thus charge for Internet service at a high price trying to exploit these myopic consumers. Again, the restriction to upscale hotels implies that the patterns are not attributed to the difference in consumer biases. Finally, an alternative argument would be that
implementing price discrimination costs more at lower-end markets than at higher-end markets due to efficiency of management. Consequently, lower-end hotels tend to offer free Internet service (no price discrimination). However, presumably the costs do not differ substantially across the upscale hotels in the sample, and thus the cost explanation does not support the observed patterns.\textsuperscript{24}

4 Profitability of Add-on Policies

The focus of analysis thus far has been on why and when a divergence of product policy arises as an equilibrium outcome. The role of vertical differentiation has been identified. A higher-quality firm behaves like a monopolist, who finds it optimal to sell an add-on as optional to screen consumers. A lower-quality firm faces a trade-off between screening and differentiation. It sells an add-on as optional in equilibrium only when it is not too costly to supply. That selling an optional add-on is unilaterally optimal to both firms does not necessarily imply that equilibrium profits improve over situations in which neither firm sells it as optional. This is because strategic interaction may render the optional-add-on policy unprofitable.

To investigate the firms' \textit{ex ante} incentives to implement add-on policies, I introduce, before firms set price levels, a commitment stage in which they can simultaneously choose whether to commit to a no-add-on policy or standard-add-on policy. By committing to a no-add-on policy, firms do not introduce an add-on, and charge only for the base good. By committing to a standard-add-on policy, firms always bundle the base and add-on, and charge only for the bundle. Once they commit to either of two policies, they are unable to price the add-on and the base separately in the second stage of price setting. By not committing to these policies, however, firms retain the flexibility of selling an add-on as optional to screen the consumers. Timing of the game is such that both firms first simultaneously decide their commitment

\textsuperscript{24}The cost explanation is, to some extent, controlled by the variable hotel size (number of rooms in a hotel) in the regressions.
choices during the first stage, and compete in prices during the second stage given, their chosen policies. This formulation allows us to compare equilibrium profits when both firms sell an add-on as optional to those when neither firm does.

Although the two-stage game with strategic commitment is not intended to fit the business environment in which the stylized fact arises, it may be applicable to other environments in which selling an add-on involves a large fixed cost of investment. One example could be the automobile industry in which many features such as side airbags, GPS navigation, and leather seats require large investments in production. It may be harder for manufacturers to sell these features as options once investments have been made.

Nine commitment outcomes are possible in the first stage, and each can lead to a pricing equilibrium in the second-stage pricing game. To allow equilibrium in which both firms sell the add-on as optional, I focus on the case with moderate add-on cost: $\theta < \alpha < \frac{1}{3}(2\bar{\theta} - \theta)$. The equilibrium of the full game is found by first solving second-stage pricing equilibrium for each of the 9 commitment outcomes, and then finding the equilibrium commitment choices taking into account the second-stage equilibrium outcomes. The following proposition summarizes the equilibrium of the full game.

**Proposition 3.** Suppose that the cost of an add-on is moderate, $\theta < \alpha < \frac{1}{3}(2\bar{\theta} - \theta)$. There exists threshold $A^*$ such that when $AV > A^*$, the higher-quality firm commits to the standard-add-on policy, whereas the lower-quality firm commits to the no-add-on policy in equilibrium.

**Proof.** See Appendix I. ■

Equilibrium prices are

$$P_{h}^{++} = \frac{1}{3}(\Delta V + w)(2\bar{\theta} - \theta) + \frac{2}{3}c, \text{ and } P_{l}^{*} = \frac{1}{3}(\Delta V + w)(\bar{\theta} - 2\theta) + \frac{1}{3}c.$$  

The result that in equilibrium neither firm has an incentive to sell an optional add-on
is striking. This contrasts sharply with the insight from monopoly settings in which the optional-add-on policy is always profit enhancing (weakly). As shown in Section 2, fixing its rival’s pricing, each firm finds it optimal to sell an add-on as optional. However, taking into account rivals’ reactions, firms find themselves trapped in a Prisoner’s Dilemma; they can both be better off to commit not to sell an add-on as optional.

To understand why this result arises, it is helpful to examine what might have happened if, under the equilibrium of Proposition 3, firms fail to commit. When commitment fails, Lemma 1 implies that it is optimal for the higher-quality firm to separate the base and add-on prices so some lower-type consumers do not buy the add-on. Its best-response base price is

\[ P_h^{(1)} = \frac{1}{2} (P_1^* + \Delta V \bar{\theta}). \]

In response, the lower-quality firm sells the add-on to those who buy only the base from its rival, inducing them to switch and get the extra benefit of the add-on. The firm chooses optimal bundle price \( P_1^{+(1)} \) that responds to the higher-quality firm’s base price \( P_h^{(1)} \), and optimal add-on price \( p_1^{(1)} \) that minimizes the cost of over selling the add-on:

\[ P_1^{+(1)} = \frac{1}{2} (P_h^{(1)} + c - (\Delta V - w)\bar{\theta}), \quad \text{and} \quad p_1^{(1)} = \frac{1}{2} (w\bar{\theta} + c). \]

Note that resulting base price \( P_1^{(1)} = P_1^{+(1)} - p_1^{(1)} \) is lower than the equilibrium base price:

\[ P_1^{(1)} - P_1^* = \frac{w}{4} (\bar{\theta} - 2\bar{\theta} + \alpha) < 0. \]

Further, in response to its rival’s bundle price, the higher-quality firm sets optimal
base price

\[ P_h^{(2)} = \frac{1}{2} (P_l^{(1)} + (\Delta V - w)\overline{\theta}), \]

which is lower than its previous price \( P_h^{(1)} \) given that \( \alpha < \frac{1}{3}(2\overline{\theta} - \overline{\theta}) \):

\[ P_h^{(2)} - P_h^{(1)} = -\frac{w}{8} (5\overline{\theta} - 4\overline{\theta} - \alpha) < 0. \]

The lower base price further motivates the lower-quality firm to lower its bundle price. This dynamic iterates and converges to an equilibrium in which both firms may end up being worse off, even though they both sell the add-on as optional eventually.

The fact that the two firms end up being maximally differentiated under the two-stage game may bear some similarity to the same result that arises from a stylized duopoly model of vertical differentiation with \( ex \ ante \) product-design decisions. However, the results of this analysis suggest that even with the presence of potential benefits from screening consumers, the maximal-differentiation principle remains dominating. In fact, what the screening does is the opposite of what the firms expect; it opens the opportunity for fiercer price competition, thereby hurting profits. The negative result of the optional-add-on policy on profitability presents a challenge for firms selling an add-on. In the short run, firms may be better off selling an add-on as optional. In the long run, however, profits may be damaged if firms are vertically differentiated. Hence, it is valuable for firms to have commitment powers. For many add-ons in the hotel industry (Internet, phone calls, breakfast, etc.), firms lack such commitment powers because it is quite flexible for them to add these items. For many features in the automobile industry (side airbags, GPS navigation, leather seats, etc.), however, firms cannot flexibly add these features once a base model is built. These features tend to be standard in luxury cars but not in economy cars, a phenomenon that can be explained by Proposition 3.
5 Further Analysis: Unobserved Add-on Prices

Thus far, I have assumed add-on prices are observable by consumers. This is not unreasonable given that consumers may learn about prices through repeat purchases or word-of-mouth, and that in many cases, firms advertise add-on policies because either they care about reputation or regulations require it. Much of the focus in the literature, however, has been on situations in which add-on prices are unobserved by consumers. For example, banks often hide information on fees such as ATM and minimum balance fees, and many consumers are unaware of the costs of these extra fees. In this section, I explore how the assumption of unobserved add-on prices influences equilibrium outcomes and firm profits. The game is modeled as:

- at $t = 0$, both firms set prices for the base and add-on;

- at $t = 1$, consumers observe only base prices $P_h$ and $P_l$, and decide which firm to buy from, and pay the base price;

- at $t = 2$, consumers visit the firms they have chosen; the add-on price is revealed and they decide whether to pay.

Consumers have rational expectations about add-on prices $p_h^* \text{ and } p_l^*$. In this setup, consumers cannot learn about add-on prices by searching. This assumption is somewhat strict, but it is sufficient to illustrate the phenomenon. In an alternative specification, consumers can incur positive search costs to discover add-on prices. The standard argument by Diamond (1971) suggests that even though search costs may be very small, in equilibrium, firms still enjoy monopoly power ex post after consumers patronize, and thus the add-on is charged at a monopoly price. The equilibrium outcome is the same as the one presented here. I examine each firm’s problem in turn and derive the \textit{sequential equilibrium} in which both firms sell an add-on.
5.1 The Higher-quality Firm

The higher-quality firm’s problem is analyzed backwards. Suppose a fraction of consumers \([\tilde{\theta}_{hl}, \overline{\theta}]\) decide to visit the higher-quality firm at time \(t = 1\). The marginal consumer \(\hat{\theta}_{hl}\) is a function of base price \(P_h\) and the total price of buying from the lower-quality firm. Among these consumers, the higher types, \(\theta \in [\hat{\theta}_{hl}, \overline{\theta}]\), buy the add-on if add-on price \(p_h^*\) is not too high. The intra-marginal consumer is given by \(\hat{\theta}_{hl} = p_h^* / \omega\). The firm charges optimal price \(p_h^*\) that maximizes add-on profit, \(\pi_h = (\overline{\theta} - \hat{\theta}_{hl})(p_h^* - c)\). This leads to the monopoly price, \(p_h^* = \frac{1}{2}(\overline{\theta}w + c)\), and the equilibrium intra-marginal consumer becomes \(\tilde{\theta}_{hl}^* = \frac{1}{2}(\overline{\theta} + \alpha)\). Note that this price is independent of the lowest-type consumer, \(\hat{\theta}_{hl}\), as long as the solution is interior. Therefore, at the beginning period \(t = 0\), maximizing total profit is equivalent to maximizing base profit, \((\overline{\theta} - \hat{\theta}_{hl})P_h\). The key observation is that the higher-quality firm’s problem is the same as the one with observable add-on price. The add-on price is chosen to maximize \textit{ex-post} profit given the firm’s local monopoly power, and thus is expected by consumers even though it is unobservable.

5.2 The Lower-quality Firm

The lower-quality firm’s problem is also analyzed backwards. At time \(t = 1\), remaining consumers \([\hat{\theta}, \hat{\theta}_{hl}]\) decide to pay the base price to visit the lower-quality firm. At time \(t = 2\), the consumers observe add-on price \(p_l\) and are faced with the decision of whether to buy it. Given these consumers, the firm maximizes \textit{ex-post} profit, \(\pi_l = (\hat{\theta}_{hl} - \hat{\theta}_l)(p_l^* - c)\) with \(\hat{\theta}_l = p_l^* / \omega\). It is useful to change the control variable from add-on price \(p_l^*\) to intra-marginal consumer \(\hat{\theta}_l\) who is indifferent regarding buying the add-on. The optimal intra-marginal consumer is chosen as a function of the marginal consumer

\[\hat{\theta}_l^*(\hat{\theta}_{hl}) = \frac{1}{2}(\hat{\theta}_{hl} + \alpha)\].

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Taking into account the second-period problem, the firm’s problem at time \( t = 0 \) is then to find the optimal marginal consumer \( \hat{\theta}_{hl} \) that maximizes total profit

\[
\max_{\hat{\theta}_{hl}} (\hat{\theta}_{hl} - \vartheta)(P_h - \hat{\theta}_{hl}(\Delta V - w) - c) - (\hat{\theta}_{hl}^* - \vartheta)(w \cdot \hat{\theta}_{hl}^* - c).
\]

Given the higher-quality firm’s base price \( P_h \), the best-response for the lower-quality firm is to choose a marginal consumer of

\[
\hat{\theta}_{hl}^{BR} = \frac{2P_h - 2c + (2\Delta V - w)\vartheta}{4\Delta V - 3w}.
\]

This leads to a total price of \( P_l + P_l^c = P_h - \hat{\theta}_{hl}^{BR}(\Delta V - w) \). Combined with the best response of the higher-quality firm, the resulting equilibrium profile is

\[
\hat{\theta}_{hl}^* = \frac{(\vartheta + \vartheta)\Delta V - (2\vartheta + \vartheta + 2\alpha)w}{6\Delta V - 5w}; \\
P_h^* = \frac{2(2\vartheta - \vartheta)\Delta V - (3\vartheta - \vartheta - 2\alpha)w}{6\Delta V - 5w} \cdot (\Delta V - w).
\]

Equilibrium outcomes are summarized in the following proposition.

**Proposition 4.** Consider an add-on that is not too costly to provide, \( \alpha < \frac{1}{3}(\vartheta + \vartheta) \). The game has equilibria in which the higher-quality firm always sells the add-on as optional, whereas the lower-quality firm’s policy depends on cost-to-value ratio \( \alpha \) such that

1. if \( \alpha \) is moderate such that \( \alpha \in (\vartheta, \frac{1}{3}(\vartheta + \vartheta)) \), there exists an equilibrium when \( \Delta V > \Delta_1^{(u)} \), in which the lower-quality firm sells the add-on as optional;

2. if \( \alpha \) is small such that \( \alpha \leq \vartheta \), then there exists an equilibrium when \( \Delta V \in [\Delta_0^{(u)}, \Delta_2^{(u)}] \) in which the lower-quality firm sells the add-on as standard, and there exists an equilibrium when \( \Delta V > \Delta_2^{(u}) \) in which the firm sells it as optional.
This proposition reveals that the interaction between screening and vertical differentiation remains critical even when add-on prices are unobserved. The trade-off between the two forces at the lower-quality firm renders its add-on policy sensitive to the cost of providing the add-on. The unobserved add-on price at the lower-quality firm leads to the hold-up problem that keeps the add-on price high. This excludes the lowest types from buying the expensive add-on, and thus the firm is less likely to sell it as standard even when it is not costly to provide. The next sub-section further investigates the impacts of this hold-up effect on equilibrium outcomes and profits.

5.3 Unobserved-price Equilibrium versus Observed-price Equilibrium

How does the equilibrium under unobservable prices differ from the one under observable prices? To facilitate the comparison, I restrict analysis to an add-on with moderate cost so both firms sell it as optional under either scenario.

**Proposition 5.** Consider a moderately costly add-on such that \( \alpha \in (\bar{\theta}, \frac{1}{3}(\bar{\theta} + \theta)) \) and a sufficiently large quality premium such that \( \Delta V > \Delta_1^{(u)} \). Compared to the observed-price equilibrium, under the unobserved-price equilibrium,

1. the lower-quality firm has smaller market share and fewer consumers buying the add-on;

2. the higher-quality firm charges a higher base price but the same add-on price, whereas the lower-quality firm charges a lower base price and higher add-on price; the total prices for both firms are higher;

3. the higher-quality firm’s profit increases, whereas the lower-quality firm’s profit decreases.

**Proof.** See Appendix K. ■
The driving force of the difference in equilibrium outcomes and profits is the varying effects of the hold-up problem for vertical differentiated firms. At the higher-quality firm, the hold-up problem has no effect because consumers anticipate an \textit{ex-post} high price of the add-on. However, the problem arises at the lower-quality firm. The higher-type consumers anticipate being held up at a high price, and thus would rather switch to the higher-quality firm and buy the base only, without paying for the add-on. Consequently, the market share of the lower-quality firm falls. This also relaxes the higher-quality firm's competitive pressure, which thereby increases its base price. In response, the lower-quality firm also increases its total price. Clearly, the higher-quality firm benefits from the add-on price being unobserved at the lower-quality firm. Although the lower-quality firm's total price increases, its segment of consumers who pay for the expensive add-on shrinks. Eventually the firm's profit drops.

This result suggests that vertically differentiated firms experience disparate incentives regarding whether to advertise add-on prices. The higher-quality firm has no incentive to do so because advertising does not reveal a better deal that induces its rival's consumers to switch. Its lower-type consumers are uninterested in the add-on, so whether the price is advertised is irrelevant to them. Contrarily, the lower-quality firm is incentivized to advertise to its rival's consumers that it has a better deal by lowering the bundle price. Despite the fact that the total price drops, the gains from acquiring new consumers who originally bought the higher-quality base good and from persuading more lower-type consumers to buy the add-on, make advertising profitable.

6 Concluding Remarks

This essay seeks to explain the seemingly counter-intuitive phenomenon that higher-end hotels are more likely than lower-end hotels to charge for Internet service. I
propose that vertical differentiation plays a role, and develop a theory to explain why and when a divergence in product policy can arise as an equilibrium outcome. The theory uncovers the differential role of an add-on for vertically differentiated firms. Because the firms primarily competing for marginal consumers who trade off a higher-quality base good alone versus a lower-quality base good plus an add-on, the firm with the higher base quality focuses on screening consumers, whereas the lower-quality firm has to trade off screening and differentiation. Surprisingly, the theory suggests that selling an add-on as optional intensifies competition. This is driven by the mechanism that the firms price aggressively to attract these marginal consumers.

These insights are particularly relevant to product policy decisions that managers face in many industries. When a firm sells an add-on in addition to a primary base product, managers must decide whether they should sell the add-on separately from the base as optional, or sell it as standard (i.e., free), or not sell it at all. This essay sheds some light on what product policies a firm should adopt in the presence of vertical differentiation. Managers should evaluate their add-on policies based on positioning of the base good. They should consider whether the base good is differentiated vertically from competitors before designing a product policy. They should be cautious regarding whether an optional-add-on policy hurts profit under competition in the long run.

A number of questions remain unaddressed, and may be worth future investigation. First, the comparative statics implied by Proposition 2 with respect to the unit cost of an add-on provide a direction for future empirical work. Detailed data with exogenous changes in the cost or value of an add-on are needed for such a study. Second, the proposition that selling an add-on as optional can hurt profits of vertically differentiated firms is an interesting hypothesis to test. Unfortunately, data used in this essay were unsuitable to test it. Future empirical research into this topic is worth exploring. Third, the theory focuses on pure vertical differentiation, assuming away...
horizontal differentiation. In reality, firms are often differentiated both vertically and horizontally. Further research allowing for both aspects is interesting. Fourth, the theory assumes a simultaneous-move strategic interaction. It would be interesting to explore whether insights apply to situations in which firms enter a market sequentially.
References


Figure 1: Vertical Duopoly vs. Monopoly in Small Markets

- Monopoly: 81.7%
- Duopoly - Higher-end: 81.4%
- Duopoly - Lower-end: 86.7%

Figure 2: Vertical Duopoly vs. Monopoly in Small Markets (Upscale Hotels)

- Monopoly: 77.7%
- Duopoly - Higher-end: 84.4%
- Duopoly - Lower-end: 92.7%
Table 1: Summary Statistics

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<th>Variable</th>
<th>Mean</th>
<th>Std. Dev.</th>
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<th>Max</th>
<th>Numb. Obs.</th>
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Notes: This table reports the summary statistics for the variables included in the regression analysis for monopoly and vertical duopoly markets.
Table 2: Regression Results

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Notes: This table reports the coefficients of the logit model. The dependent variable is whether a hotel offers free Internet service (1 - yes, 0 - no). The benchmark group is the monopoly markets. Robust standard errors are reported. Significantly different from zero: * for $p < .1$, ** for $p < .05$, and *** for $p < .01$. 

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Table 3: Regression Results (Upscale Hotels)

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<td>Location - Small Town</td>
<td>1.198*</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.706)</td>
<td></td>
</tr>
<tr>
<td>Location - Suburban</td>
<td>0.944</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.675)</td>
<td></td>
</tr>
<tr>
<td>Location - Urban</td>
<td>1.813**</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.755)</td>
<td></td>
</tr>
<tr>
<td>Operation - Franchised</td>
<td>0.229</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.499)</td>
<td></td>
</tr>
<tr>
<td>Operation - Independent</td>
<td>0.106</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.559)</td>
<td></td>
</tr>
<tr>
<td>Size - 75 to 149 rooms</td>
<td>0.660*</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.392)</td>
<td></td>
</tr>
<tr>
<td>Size - 150 to 299 rooms</td>
<td>-1.556***</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.448)</td>
<td></td>
</tr>
<tr>
<td>Size - 300 to 500 rooms</td>
<td>-2.744***</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.622)</td>
<td></td>
</tr>
<tr>
<td>Size &gt;500 rooms</td>
<td>-3.753**</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.605)</td>
<td></td>
</tr>
<tr>
<td>Age</td>
<td>-0.003</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.006)</td>
<td></td>
</tr>
</tbody>
</table>

| Observations | 646 | 646 |

Notes: This table reports the coefficients of the logit model. The dependent variable is whether a hotel offers free Internet service (1 - yes, 0 - no). The benchmark group is the monopoly markets. Robust standard errors are reported. Significantly different from zero: * for $p < .1$, ** for $p < .05$, and *** for $p < .01$. 

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A Additional Evidence from Expedia Data

The data based on the Lodging Survey documents some empirical evidence that supports the theoretical predictions. However, the survey-based data has its limitations. Not all surveyed hotels provided information regarding in-room Internet service. To evaluate whether missing information might have biased the conclusions, I collected an additional dataset from a major online travel agency, Expedia. Although this data has its own limitations that present new challenges, it demonstrates that the qualitative conclusions made using the survey data are robust.

Expedia Affiliate Network, part of Expedia Inc, provides a database that includes full text descriptions, amenity lists, hotel IDs, star ratings, etc, for every property that Expedia offers.\(^\text{25}\) I obtained all the relevant datasets in December 2013. Internet information was embedded in the texts in multiple datasets on hotel description, amenity, policy, and room type.\(^\text{26}\) I conducted text mining to create a variable indicating whether a hotel provides free Internet service and whether it charges it for a fee. Unfortunately 26% of the hotels in the database had no Internet information that can be identified. Ignoring these observations presents an even more challenging statistical problem than in the survey data. This is because the Internet information appears in the search engine of Expedia, being revealed to all potential customers. It is not unreasonable to suspect that hotels that charge for Internet service may have an incentive to hide the information, in line with the literature in drip pricing with myopic consumers (Gabaix and Laibson 2006).

To address the issue I restricted the analysis to the markets in the six states of New England.\(^\text{27}\) and filled out the Internet information whenever it is missing. Similar to the analysis on the survey data, I focused on the hotels in small markets such that

\(^{25}\)http://developer.ean.com/database-catalogs/

\(^{26}\)The most common Internet information in the data is Wi-Fi, so I restrict attention to Wi-Fi instead of other types of Internet service (e.g., wired Internet). Therefore I use Wi-Fi and Internet service interchangeably in this section.

\(^{27}\)The states are Maine, Massachusetts, New Hampshire, Vermont, Rhode Island, and Connecticut.
there were only one or two hotels identified within the same zip code. Among this sample of 432 hotels, 33% of them did not have identified Internet information. For these hotels I first searched the hotel websites to find out whether they provided free Internet service. If there was no available information, I called up the hotels directly to request for the information. Out of the 141 with missing information, 111 were reached and 108 of them (more than 97%) were identified to offer free Internet service. This surprising large number suggests that the missing information is very unlikely to correlate with hotels’ incentives to disclose the information. For the remaining 30 hotels with no relevant information, they were either closed during the winter season, or not reachable after three attempts of phone contact. Either ignoring them or treating all of them as providing free Internet service does not affect the qualitative conclusions. I reported the results using the latter approach.

Vertical differentiation is defined using the variable High Rate provided in the data. If hotels A and B were in the same market (zip code), then the market was vertically differentiated if and only if |A’s High Rate - B’s High Rate| > D, where D = $0. If otherwise, the market was treated as horizontally differentiated. I used High Rate, instead of star ratings and Low Rate, to define the vertical structure of a market because this information is more complete. A small threshold D = $0 could maximize the sample size for the duopoly markets. Robustness checks suggested that using alternative measures such as star ratings or Low Rate or the average of High Rate and Low Rate, and using different thresholds D = $20, $40 do not alter the qualitative conclusions.

There were 232 monopoly markets in the sample and 198 duopoly markets with vertical differentiation. Table A-1 provides a summary of the statistics for the 430 hotels included in the analysis.²⁸ Figure A-1 summarizes the descriptive statistics graphically. Among the monopoly markets, 93% of hotels offered free Internet ser-

²⁸Two hotels were dropped because they have identical High Rate, and thus they were unlikely to be vertically differentiated.
Among the higher-end hotels in the duopoly markets, 91% of them offered free Internet service, very close to the monopoly markets. However, 98% of the lower-end hotels in the duopoly markets offered free Internet service, much higher than the hotels in the other two market conditions. These patterns are qualitatively similar to the ones found in the survey data. Notice that the overall percentages are higher than those in the survey data because the survey data excluded very small hotels with less than 15 rooms, which were included in the Expedia data. These hotels are typically very low-end, and thus, on average, they are very likely to offer free Internet.

I further validated the patterns using regression analysis. Table A-2 reports the estimates of the logit regressions. The dependent variable was again whether or not a hotel provided free Internet service. The independent variables were the three market conditions: (1) a monopoly market, (2) the higher-end hotel in a duopoly, and (3) the lower-end hotel in a duopoly. The monopoly markets were treated as the benchmark group. The first column reports the baseline model that includes the main variables of interest. The non-significance of the coefficient for the higher-end hotels suggests that they were equally likely to offer free Internet service as the monopoly hotels. The significant positive coefficient for the lower-end hotels suggests that they were much more likely than the monopoly hotels to offer free Internet service. Finally, the coefficient for the lower-end hotels is significantly higher than that for the higher-end hotels, suggesting that the lower-end hotels were more likely to offer free Internet service than the higher-end hotels in duopoly markets. In the second column of the table, I further controlled for the state fixed effects. The conclusions are qualitatively the same.
Figure A-1: Vertical Duopoly vs. Monopoly in Small Markets (Expedia Data)

- Monopoly: 93.1%
- Duopoly - Higher-end: 90.9%
- Duopoly - Lower-end: 98.0%

% Hotels Offering Free Internet
Table A-1: Summary Statistics (Expedia Data)

<table>
<thead>
<tr>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Min</th>
<th>Max</th>
<th>Numb. Obs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monopoly</td>
<td>0.540</td>
<td>0.499</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Higher End</td>
<td>0.230</td>
<td>0.421</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Lower End</td>
<td>0.230</td>
<td>0.421</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Notes: This table reports the summary statistics for the variables included in the regression analysis.

Table A-2: Regression Results (Expedia Data)

<table>
<thead>
<tr>
<th></th>
<th>No Controls</th>
<th>With Controls</th>
</tr>
</thead>
<tbody>
<tr>
<td>High-end in Duopoly</td>
<td>-0.300</td>
<td>-0.447</td>
</tr>
<tr>
<td></td>
<td>(0.436)</td>
<td>(0.547)</td>
</tr>
<tr>
<td>Low-end in Duopoly</td>
<td>1.279*</td>
<td>1.159**</td>
</tr>
<tr>
<td></td>
<td>(0.761)</td>
<td>(0.472)</td>
</tr>
<tr>
<td>State Fixed Effects</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Observations</td>
<td>430</td>
<td>371</td>
</tr>
</tbody>
</table>

Notes: This table reports the coefficients of the logit model. The dependent variable is whether a hotel offers free Internet service (1 - yes, 0 - no). The benchmark group is the monopoly markets. There were less observations in the second column because all of the 59 hotels in Maine offered free Internet service and thus were dropped. Robust clustered (at state level) standard errors are reported. Significantly different from zero: * for \( p < .1 \), ** for \( p < .05 \), and *** for \( p < .01 \).
B Proof of Lemma 1

When the Lower-quality Firm Does not Sell the Add-on

Suppose the higher-quality firm sells standard add-on at price $P^+_h$. Then the marginal consumer who is indifferent between the two firms is

$$\hat{\theta}_{hl} = \frac{P^+_h - P_l}{\Delta V + w}.$$ 

The higher-quality firm obtains a profit of $\Pi_h = (\bar{\theta} - \hat{\theta}_{hl})(P^+_h - c)$. This strategy is exactly equivalent to separating the bundle price $P^+_h$ into a base price $P_h$ and an add-on price $p_h$ such that $P^+_h = P_h + p_h$, and that

$$\frac{p_h}{w} = \frac{P^+_h - P_l}{\Delta V + w} = \frac{P_h - P_l}{\Delta V}. \tag{B-1}$$

Let’s now consider a small increase of the add-on price to $p'_h = p_h + \epsilon$ with $\epsilon > 0$, and a small decrease of the base price to $P'_h = P_h - \epsilon$ by the same amount. The bundle price $P^+_h$ is then unchanged. Therefore we have

$$\frac{p'_h}{w} > \frac{P^+_h - P_l}{\Delta V + w} > \frac{P'_h - P_l}{\Delta V}.$$ 

The profit function now becomes

$$\Pi_h(P'_h, p'_h) = (\bar{\theta} - \frac{P'_h - P_l}{\Delta V})p'_h + (\bar{\theta} - \frac{p'_h}{w})(p'_h - c).$$
Evaluating this function around the original prices \((P_h, p_h)\) by Taylor series yields:

\[
\Pi_h(P_h', p_h') = \Pi_h(P_h, p_h) - \epsilon \cdot \Pi_{Ph}(P_h, p_h) + \epsilon \cdot \Pi_{pph}(P_h, p_h) \\
+ \frac{1}{2} \left[ \epsilon^2 \cdot \Pi_{pph}(P_h, p_h) + 2\epsilon^2 \cdot \Pi_{ppph}(P_h, p_h) + \epsilon^2 \cdot \Pi_{pppph}(P_h, p_h) \right] + R_2 \\
= \Pi_h(P_h, p_h) - \epsilon \cdot \hat{\theta} - \frac{P_h - P_l}{\Delta V} - \frac{P_h}{\Delta V} + \epsilon \cdot \frac{p_h}{w} - \frac{p_h - c}{w} + \frac{\epsilon^2}{2} \left[ -\frac{2}{\Delta V} - \frac{2}{w} \right] \\
= \Pi_h(P_h, p_h) + \frac{\epsilon}{\Delta V} (P_l + \alpha \Delta V) - \frac{\epsilon^2}{w \Delta V} (\Delta V + w)
\]

where the second equality follows from the remainder term \(R_2 = 0\) because the higher order derivatives are all zero, and the last equality follows from Equation (B-1). Therefore, \(M > 0\) for small positive \(\epsilon\) when \(P_l + \alpha \Delta V > 0\).

**When the Lower-quality Firm Sells the Add-on**

Suppose the higher-quality firm sells standard add-on at price \(P_h^+\). Then the marginal consumer who is indifferent between the two firms is

\[
\hat{\theta}_{hl} = \frac{P_h^+ - P_l^+}{\Delta V}.
\]

The higher-quality firm obtains a profit of \(\Pi_h = (\hat{\theta} - \hat{\theta}_{hl})(P_h^+ - c)\). This strategy is exactly equivalent to separating the bundle price \(P_h^+\) into a base price \(P_h\) and an add-on price \(p_h\) such that \(P_h^+ = P_h + p_h\), and that

\[
\frac{p_h}{w} = \frac{P_h^+ - P_l^+}{\Delta V} = \frac{P_h - P_l^+}{\Delta V - w}.
\]  

(B-2)

Let's now consider a small increase of the add-on price to \(p_h' = p_h + \epsilon\) with \(\epsilon > 0\), and a small decrease of the base price to \(P_h' = P_h - \epsilon\) by the same amount. The bundle price \(P_h^+\) is then unchanged. Therefore we have

\[
\frac{p_h'}{w} > \frac{P_h^+ - P_l^+}{\Delta V} > \frac{P_h - P_l^+}{\Delta V - w}.
\]
The profit function now becomes

$$\Pi_h(P'_h, p'_h) = (\bar{\theta} - \frac{P'_h - P^+_l}{\Delta V - w}) P'_h + (\bar{\theta} - \frac{p'_h}{w})(p'_h - c)$$

Similar to the above case, evaluating this function around the original prices \((P_h, p_h)\) by Taylor series yields:

$$\Pi_h(P'_h, p'_h) = \Pi_h(P_h, p_h) - \epsilon \left[ \bar{\theta} - \frac{P_h - P^+_l}{\Delta V - w} - \frac{P_h}{\Delta V - w} \right] + \epsilon \left[ \bar{\theta} - \frac{p_h}{w} - \frac{p_h - c}{w} \right]$$

$$+ \frac{\epsilon^2}{2} \left[ -\frac{2}{\Delta V - w} - \frac{2}{w} \right]$$

$$= \Pi_h(P_h, p_h) + \frac{\epsilon}{\Delta V - w} (P^+_l + \alpha(\Delta V - w)) - \frac{\epsilon^2 \Delta V}{w(\Delta V - w)}$$

where the last equality follows from Equation (B-2). Therefore, \(M > 0\) for small positive \(\epsilon\) when \(P^+_l + \alpha(\Delta V - w) > 0\).

C Proof of Proposition 1

Suppose the statement is not true, then in equilibrium the higher-quality firm can either (1) sell standard add-on, or (2) does not sell the add-on at all. In each case there are two possible equilibria, depending on whether the low quality firm sells the add-on. In what follows, I show that in each case the higher-quality firm always finds it profitable to deviate to selling optional add-on.

When the Higher-quality Firm Sells Standard Add-on

Case (a): the lower-quality firm sells the add-on. The marginal consumer who is indifferent between the two firms is

$$\hat{\theta}_{hl} = \frac{P^+_h - P^+_l}{\Delta V}.$$
Equilibrium prices are

\[ P_{h}^{**} = \frac{1}{3} \Delta V (2\bar{\theta} - \bar{\theta}) + \frac{2}{3} c, \quad p_{h}^{*} < w\hat{\theta}_{hl}^{*}; \]  
\[ P_{l}^{**} = \frac{1}{3} \Delta V (\bar{\theta} - 2\bar{\theta}) + \frac{1}{3} c, \quad p_{l}^{*} < w\hat{\theta}_{hl}^{*}. \]  
\[ \text{(C-1)} \]
\[ \text{(C-2)} \]

with equilibrium threshold \( \hat{\theta}_{hl}^{*} = (\bar{\theta} + \bar{\theta})/3 > \bar{\theta} \). Notice that whether or not the lower-quality firm sells only the base or the bundle to its lowest-type consumers (i.e., whether \( p_{l}^{*} > \bar{\theta} \)) does not affect the argument below. Because \( P_{l}^{**} > 0 \), by Lemma 1 the higher-quality firm can profitably deviate to selling optional add-on by increasing the add-on price while lowering the base price.

**Case (b): the lower-quality firm does not sell the add-on.** The marginal consumer who is indifferent between the two firms is

\[ \hat{\theta}_{hl} = \frac{P_{h}^{+} - P_{l}}{\Delta V + w}. \]

Equilibrium prices are

\[ P_{h}^{**} = \frac{1}{3} (\Delta V + w)(2\bar{\theta} - \bar{\theta}) + \frac{2}{3} c, \quad p_{h}^{*} < w\hat{\theta}_{hl}^{*}; \]  
\[ P_{l}^{*} = \frac{1}{3} (\Delta V + w)(\bar{\theta} - 2\bar{\theta}) + \frac{1}{3} c, \quad p_{l}^{*} > w\hat{\theta}_{hl}^{*}, \]  
\[ \text{(C-3)} \]
\[ \text{(C-4)} \]

where the equilibrium threshold \( \hat{\theta}^{*} = (\bar{\theta} + \bar{\theta})/3 + c/(3\Delta V + 3w) > \bar{\theta} \). Because \( P_{l}^{*} > 0 \), by Lemma 1 the higher-quality firm can profitably deviate to selling optional add-on by increasing the add-on price while lowering the base price.

**When the Higher-quality Firm Does not Sell the Add-on**

In this case, the higher-quality firm’s profit is then \( \Pi_{h} = (\bar{\theta} - \hat{\theta}_{hl}) P_{h} \). The threshold \( \hat{\theta}_{hl} \) is either \( (P_{h} - P_{l}^{+})/(\Delta V - w) \) or \( (P_{h} - P_{l})/\Delta V \), depending on whether the lower-quality firm sells the add-on to its highest-type consumers. The best-response prices lead to
an equilibrium threshold $\hat{\theta}_{hl} < \bar{\theta}$. The higher-quality firm would need to set a high add-on price $p_h^* > \bar{\theta}w$ so that no one buys the add-on. However, it is always profitable for the higher-quality firm to lower its add-on price $p_h < \bar{\theta}w$ without affecting the profit from selling the base as long as $p_h > w\hat{\theta}_{hl}$. Therefore, no equilibrium exists where the higher-quality firm sells only the base.

**D Proof of Proposition 2**

I first establish a lemma for the lower-quality firm and then proceed to prove the two cases of the proposition (1) when the add-on is costly $\alpha > \frac{1}{3}(2\bar{\theta} - \bar{\theta})$ and (2) when the add-on is not too costly $\alpha < \frac{1}{3}(2\bar{\theta} - \bar{\theta})$.

**D-1 A Lemma for the Lower-quality Firm**

**Lemma 2.** For the lower-quality firm, not selling the add-on while leaving some consumers to buy from the higher-quality firm is strictly dominated by selling the add-on, if:

1. $P_h > \alpha \Delta V$ when the higher-quality firm sells optional add-on, or
2. $P_h^+ > \alpha(\Delta V + w)$ when the higher-quality firm sells standard add-on.

**When the Higher-quality Firm Sells Optional Add-on**

Suppose the lower-quality firm sells only the base to consumers at price $P_l$. Then the marginal consumer who is indifferent between the two firms is

$$\hat{\theta}_{hl} = \frac{P_h - P_l}{\Delta V}.$$ 

The lower-quality firm obtains a profit of $\Pi_l = (\hat{\theta}_{hl} - \bar{\theta})P_l$. This strategy is exactly equivalent to charging a bundle price $P_l^+$ and an add-on price $p_l$ such that $P_l^+ = P_l + p_l$, and that

$$\frac{P_h - P_l^+}{\Delta V - w} = \frac{P_h - P_l}{\Delta V} = \frac{p_l}{w}. \quad \text{(D-1)}$$
Let's now consider a small decrease of the bundle price to 
\( P_l^{+'} = P_l^+ - \epsilon \) with \( \epsilon > 0 \),
and a small decrease of the add-on price to 
\( p_l' = p_l - \epsilon \) by the same amount. The base price \( P_l \) is then unchanged. Therefore we have
\[
\frac{P_h - P_l^{+'}}{\Delta V - w} > \frac{P_h - P_l}{\Delta V} > \frac{p_l'}{w}.
\]
The profit function now becomes
\[
\Pi_l(P_l^{+'}, p_l') = \left( \frac{P_h - P_l^{+'}}{\Delta V - w} - \theta \right) (P_l^{+'} - c) - \left( \frac{p_l'}{w} - \theta \right) (p_l' - c)
\]
Evaluating this function around the original prices \((P_l^+, p_l)\) by Taylor series yields:
\[
\Pi_l(P_l^{+'}, p_l') = \Pi_l(P_l^+, p_l) - \epsilon \cdot \Pi_{P_l^+} (P_l^+, p_l) - \epsilon \cdot \Pi_{p_l} (P_l^+, p_l)
+ \frac{1}{2} \left[ \epsilon^2 \cdot \Pi_{P_l^+ P_l^+} (P_l^+, p_l) + 2 \epsilon^2 \cdot \Pi_{P_l^+ p_l} (P_l^+, p_l) + \epsilon^2 \cdot \Pi_{p_l p_l} (P_l^+, p_l) \right] + R_2
= \Pi_l(P_l^+, p_l) - \epsilon \cdot \left[ \frac{P_h - P_l^+}{\Delta V - w} - \theta + \frac{P_l^+ - c}{\Delta V - w} \right] + \epsilon \cdot \left[ \frac{p_l}{w} - \theta + \frac{p_l - c}{w} \right]
+ \frac{\epsilon^2}{2} \left[ - \frac{2}{\Delta V - w} - \frac{2}{w} \right]
= \Pi_l(P_l^+, p_l) + \frac{\epsilon}{\Delta V - w} \left( P_h - \alpha \Delta V \right) - \frac{\epsilon^2}{w(\Delta V - w) \Delta V}
\]
where the second equality follows from the remainder term \( R_2 = 0 \) because the higher order derivatives are all zero, and the last equality follows from Equation (D-1). Therefore, \( M > 0 \) for small positive \( \epsilon \) when \( P_h > \alpha \Delta V \).

**When the Higher-quality Firm Sells Standard Add-on**

Suppose the lower-quality firm sells only the base to consumers at price \( P_l \). Then the marginal consumer who is indifferent between the two firms is
\[
\hat{\theta}_h = \frac{P_h^+ - P_l}{\Delta V + w}.
\]
The lower-quality firm obtains a profit of $\Pi_l = (\hat{\theta}_{hl} - \theta)P_l$. This strategy is exactly equivalent to charging a bundle price $P^+_l$ and an add-on price $p_i$ such that $P^+_l = P_i + p_i$, and that
\[
\frac{P^+_h - P^+_l}{\Delta V} = \frac{P^+_h - P_i}{\Delta V + w} = \frac{p_i}{w}. \tag{D-2}
\]

Let's now consider a small decrease of the bundle price to $P^+_l = P^+_l - \epsilon$ with $\epsilon > 0$, and a small decrease of the add-on price to $p'_i = p_i - \epsilon$ by the same amount. The base price $P_i$ is then unchanged. Therefore we have
\[
\frac{P^+_h - P^+_l}{\Delta V} > \frac{P^+_h - P_i}{\Delta V + w} > \frac{p'_i}{w}.
\]

The profit function now becomes
\[
\Pi_l(P^+_l, p'_i) = \left(\frac{P^+_h - P^+_l}{\Delta V} - \theta\right)(P^+_l - c) - \left(\frac{p'_i}{w} - \theta\right)(p'_i - c)
\]

Evaluating this function around the original prices $(P^+_l, p_i)$ by Taylor series yields:
\[
\Pi_l(P^+_l, p'_i) = \Pi_l(P^+_l, p_i) - \epsilon \cdot \left[\frac{P^+_h - P^+_l}{\Delta V} - \theta - \frac{P^+_h - c}{\Delta V}\right] + \epsilon \cdot \left[\frac{p_i}{w} - \theta + \frac{p_i - c}{w}\right]
\]
\[
+ \frac{\epsilon^2}{2} \left[\frac{\Delta V}{w} - \frac{2}{w}\right]
\]
\[
= \Pi_l(P^+_l, p_i) + \frac{\epsilon}{\Delta V} (P^+_h - \alpha(\Delta V + w)) - \frac{\epsilon^2}{w\Delta V} (\Delta V + w)
\]

where the second equality follows from the remainder term $R_2 = 0$ because the higher order derivatives are all zero, and the last equality follows from Equation (D-2). Therefore, $M > 0$ for small positive $\epsilon$ when $P^+_h > \alpha(\Delta V + w)$. 

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D-2 Equilibrium When $\alpha \geq \frac{1}{3} (2\bar{\theta} - \bar{\theta})$

Recall that the equilibrium profile consists of the following pricing strategies:

$$P_h^* = \frac{1}{3} (2\bar{\theta} - \bar{\theta}) \Delta V, \quad p_h^* = w\hat{\theta}_h^*, \quad (D-3)$$

$$P_l^* = \frac{1}{3} (\bar{\theta} - 2\bar{\theta}) \Delta V, \quad p_l^* > w\hat{\theta}_{hl}^*, \quad (D-4)$$

where the thresholds are given by

$$\hat{\theta}_{hl}^* = \frac{1}{3} (\bar{\theta} + \bar{\theta}), \quad \hat{\theta}_h^* = \frac{1}{2} (\bar{\theta} + \alpha).$$

In what follows I establish that this there is no profitable deviation for either firm when $\alpha > (2\bar{\theta} - \bar{\theta})/3$.

No Profitable Deviation for the Lower-quality Firm

This is established by considering three possible non-local deviations by the lower-quality firm: (a) it does not sell the add-on and leaves no consumers buy $H$, (b) it sells the add-on and leaves some consumers buy $H$, and (c) it sells the add-on but leaves no consumers buy $H$. In the first case, the firm has to lower the base price substantially in order to compete for the higher-type consumers of the higher-quality firm, leading to a loss in profit. The key idea behinds the last two cases is that if the lower-quality firm were to sell the add-on, the relatively high cost of the add-on drives the lower-quality firm's add-on price up, leaving fewer consumers buy the high priced add-on.

Case (a): the lower-quality firm does not sell the add-on and leaves no consumers buy $H$. The marginal consumer who is indifferent between the two firms is

$$\hat{\theta}_{hl} = \frac{P_h^{**} - P_l}{\Delta V + w}.$$
For this situation to arise, we need that $\hat{\theta}_{hl} \geq \frac{p_h^*}{w}$ so that no consumers buy $H$. By contrast the marginal consumer in the equilibrium profile satisfies:

$$\hat{\theta}_{hl}^* < \hat{\theta}_{hl} \iff \frac{P_h^* - P_l^*}{\Delta V} < \frac{p_h^*}{w} \iff \frac{P_h^{++} - P_l^*}{\Delta V + w} < \frac{p_h^*}{w}.$$  \hfill (D-5)

The lower-quality firm’s profit is given by $\Pi_l = (\hat{\theta}_{hl} - \theta)P_l$. It turns out that the optimal profit under this case is achieved by the corner solution $\hat{\theta}_{hl} = \frac{p_h^*}{w}$. To see this, note that the first order condition of the Lagrangian is given by

$$P_l = \frac{1}{2}(P_h^{++} - (\Delta V + w)\theta - \lambda)$$

The constraint becomes:

$$\hat{\theta}_{hl} - \frac{p_h^*}{w} = \frac{P_h^{++} - P_l}{\Delta V + w} - \frac{p_h^*}{w} \leq \frac{P_l^* - P_l}{\Delta V + w} = \frac{1}{2(\Delta V + w)}(-p_h^* + w\theta + \lambda)$$

where the inequality follows from Equation (D-5) and the last equality is obtained by substituting the best responses of the lower-quality firm. Because $p_h^* > w\bar{\theta}$, to guarantee the constraint is nonnegative, $\lambda > 0$. This implies that the constraint has to be binding by complementary slackness. This corner solution coincides with the corner solution for the problem under equilibrium because:

$$\frac{P_h^{++} - P_l}{\Delta V + w} = \frac{p_h^*}{w} \iff \frac{P_l^* - P_l}{\Delta V} = \frac{p_h^*}{w}.$$

Therefore the deviation does not improve the profit.

Case (b): the lower-quality firm sells the add-on and leaves some consumers buy $H$. The market is divided into four consumer segments: the consumers $\theta \in [\hat{\theta}_{hl}, \bar{\theta}]$ buy $H^+$, the consumers $\theta \in [\hat{\theta}_{hl}, \hat{\theta}_{hl}]$ buy $H$, the consumers with $\theta \in [\hat{\theta}_l, \hat{\theta}_{hl}]$ buy $L^+$,
and the consumers with \( \theta \in [\hat{\theta}, \tilde{\theta}] \) buy \( L \). The thresholds are given by

\[
\hat{\theta}_h = \frac{p_h^*}{w}, \quad \hat{\theta}_l = \frac{P_h^* - P_l - p_l}{\Delta V - w}, \quad \text{and} \quad \hat{\theta}_l = \frac{p_l}{w}.
\]

It will later become evident that when the lower-quality firm sells the add-on to all of its consumers (i.e., \( \hat{\theta}_l < \tilde{\theta} \)) the profit is not improved. Given the higher-quality firm’s equilibrium pricing the lower-quality firm maximizes profit

\( \Pi_l = (\hat{\theta}_h - \tilde{\theta})(P_l^+ - c) - (\hat{\theta}_l - \tilde{\theta})(p_l - c) \)

subject to the constraints

\[
\begin{align*}
(1) \quad & \hat{\theta}_l \geq \hat{\theta}_i; \\
(2) \quad & \hat{\theta}_i \geq \tilde{\theta}; \\
(3) \quad & \hat{\theta}_h \geq \hat{\theta}_h.
\end{align*}
\]

The objective function is concave and thus the necessary and sufficient condition for the optimization program is Karush-Kuhn-Tucker conditions. Let \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) be the Lagrangian multipliers. The first-order conditions are:

\[
P_l^+ = \frac{1}{2}(P_h^* - (\Delta V - w)\tilde{\theta} + c - \lambda_1 + \lambda_3), \quad p_l = \frac{1}{2}(w\tilde{\theta} + c - \lambda_1 + \lambda_2). \quad (D-6)
\]

Therefore, Constraint (1) is given by

\[
\hat{\theta}_l - \hat{\theta}_i = \frac{P_h^* - P_l^+}{\Delta V - w} - \frac{p_l}{w}
= \frac{1}{2w(\Delta V - w)} \left[ w\Delta V \left( \frac{1}{3}(2\tilde{\theta} - \theta) - \alpha \right) - \lambda_2(\Delta V - w) - \lambda_3w + \lambda_1\Delta V \right]
\]

Because \( \alpha > (2\tilde{\theta} - \theta)/3 \) and both \( \lambda_2 \) and \( \lambda_3 \) are nonnegative, the term \( M \) in the bracket is strictly negative \( (M < 0) \). To guarantee the constraint is nonnegative, \( \lambda_1 > 0 \). This implies that Constraint (1) has to be binding by complementary slackness. Therefore the profit is no greater than that one obtained from not selling the add-on (i.e., the equilibrium strategy).

It remains to verify that the lower-quality firm would not deviate to selling the
add-on to all of its consumers. In such case the profit is \( \Pi_l = (\hat{\theta}_{hl} - \bar{\theta})(P_l^+ - c) \). This profit can be strictly improved if the firm sets the add-on price \( p_t \) to be anywhere between \([\theta w, c]\) so that some consumers decide not to buy the add-on. The incremental benefit is equal to \(- (\hat{\theta}_l - \bar{\theta})(p_t - c)\) which is strictly positive.

**Case (c): the lower-quality firm sells the add-on but leaves no consumers buy H.**

The marginal consumer who is indifferent between the two firm is

\[
\hat{\theta}_{hl} = \frac{P_h^{**} - P_l^+}{\Delta V}.
\]

To ensure that no consumers buy \( H \), it is necessary that \( \hat{\theta}_{hl} \geq p_h^*/w \). Consider that the lower-quality firm maximizes profit \( \Pi_l = (\hat{\theta}_{hl} - \bar{\theta})(P_l^+ - c) - (\hat{\theta}_l - \bar{\theta})(p_l - c) \).

The profit differs from Case (b) only in terms of the profit of selling the bundle, \((\hat{\theta}_{hl} - \bar{\theta})(P_l^+ - c)\). Maximizing this bundling profit under the constraint yields:

\[
P_l^+ = \frac{1}{2}(p_h^{**} - \Delta V \bar{\theta} + c - \lambda)
\]

Substituting this best response back to the constraint

\[
\hat{\theta}_{hl} - \frac{p_h^*}{w} = \frac{P_h^{**} - P_l^+}{\Delta V} - \frac{p_h^*}{w} = \frac{1}{2\Delta V} \left[ -\frac{1}{3}(\bar{\theta} - 2\bar{\theta} + 3\alpha)\Delta V + \frac{1}{2}(\bar{\theta} - \alpha)w + \lambda \right]
\]

Because \( M < 0 \) when \( \alpha > (2\bar{\theta} - \theta)/3 \), for the constraint to be nonnegative, it is necessary that \( \lambda > 0 \). This implies that the constraint has to be binding by complementary slackness. Therefore the profit is no greater than that one obtained from Case (b), which is not a profitable deviation.

**No Profitable Deviation for the Higher-quality Firm**

There are four non-local deviations by the higher-quality firm: (a) it sells optional add-on and leaves some consumers buy \( L^+ \), (b) it sells standard add-on and leaves
some consumers buy $L^+$, (c) it sells standard add-on but leaves no consumers buy $L^+$, (d) it does not sell the add-on at all.

Case (a): the higher-quality firm sells optional add-on and leaves some consumers buy $L^+$. The marginal consumer who is indifferent between the two firm becomes

$$\hat{\theta}_{hl} = \frac{P_h - p^*_l - p^*_l}{\Delta V - w}.$$ 

Consider the higher-quality firm’s maximization problem $\Pi_h = (\bar{\theta} - \hat{\theta}_{hl})P_h + (\bar{\theta} - \hat{\theta}_{hl})(p_h - c)$ subject to the constraint $\hat{\theta}_{hl} \geq p^*_l / w$. Notice that the profit function is the same as the equilibrium one except the threshold $\hat{\theta}_{hl}$. It then suffices to compare the profit from selling the base only. The first-order condition of the constrained problem is given by:

$$p_h = \frac{1}{2}(p^*_l + p^*_l + (\Delta V - w)\bar{\theta} + \lambda).$$

Therefore,

$$\hat{\theta}_{hl} - \frac{p^*_l}{w} = \frac{P_h - p^*_l - p^*_l}{\Delta V - w} - \frac{p^*_l}{w}$$

$$= \frac{1}{2(\Delta V - w)}\left[\frac{2}{3}(\bar{\theta} + \bar{\theta})\Delta V - \bar{\theta}w - \frac{p^*_l}{w}(2\Delta V - w) + \lambda\right]$$

$$\leq \frac{1}{2(\Delta V - w)}\left[\frac{2}{3}(\bar{\theta} + \bar{\theta})\Delta V - \bar{\theta}w - \frac{1}{3}(\bar{\theta} + \bar{\theta})(2\Delta V - w) + \lambda\right]$$

$$= \frac{1}{2(\Delta V - w)}\left(-\frac{1}{3}(2\bar{\theta} - \bar{\theta})w + \lambda\right)$$

where the inequality is implied by the equilibrium add-on price of the lower-quality firm $p^*_l > (\bar{\theta} + \bar{\theta})w/3$. Because $(2\bar{\theta} - \bar{\theta})$ is positive, to ensure the constraint is nonnegative it is necessary that $\lambda > 0$. Therefore, by complementary slackness the constraint must be binding. That is, the higher-quality firm leaves no demand for the $L^+$. Therefore, the optimal profit obtained in this case is no greater than the equilibrium profit.
Case (b): the higher-quality firm sells standard add-on and leaves some consumers buy $L^+$. Because $P_{l}^{++} > 0$, by Lemma 1 this strategy is strictly dominated by a local deviation where the higher-quality firm sells optional add-on by increasing the add-on price while lowering the base price. This deviation is exactly in the form of Case (a), which yields no greater profit than the equilibrium one.

Case (c): the higher-quality firm sells standard add-on but leaves no consumers buy $L^+$. Because $P_1^* > 0$, by Lemma 1 this strategy is strictly dominated by a local deviation where the higher-quality firm sells optional add-on by increasing the add-on price while lowering the base price. This local deviation is exactly in the form of the maximization problem in the equilibrium. Therefore, there is no profitable deviation.

Case (d): the higher-quality firm does not sell the add-on at all. This deviation can take two forms. First, the higher-quality firm can leave no consumer buy the add-on from the lower-quality firm. The optimal base price is then the same as the equilibrium strategy $P_h^*$. However, since the deviation forgoes the profit from selling the add-on, the deviation profit is smaller. Second, the higher-quality firm can leave some consumers buy the add-on from the lower-quality firm. The profit is smaller than that in case (a) because a small positive sales of the add-on can improve profit without affecting profit from the base good. Therefore, in either case, the deviation is not profitable.

**D-3  Equilibrium When $\alpha < \frac{1}{3}(2\theta - \bar{\theta})$**

Recall that the equilibrium profile consists of the following pricing strategies:

\[
\begin{align*}
P_h^* &= \frac{1}{3}(2\bar{\theta} - \bar{\theta})(\Delta V - w) + \frac{1}{3}c, \quad p_h^* = w\bar{\theta}_h^*, \\
P_{l}^{++} &= \frac{1}{3}(\bar{\theta} - 2\bar{\theta})(\Delta V - w) + \frac{2}{3}c, \quad p_{l}^{++} \begin{cases} 
= w\hat{\theta}^*_l & \text{if } \alpha > \bar{\theta} \\
\leq w\bar{\theta} & \text{if } \alpha \leq \bar{\theta}
\end{cases}
\end{align*}
\]
where the thresholds are given by

\[
\hat{\theta}_A^* = \frac{1}{3}(\bar{\theta} + \theta) - \frac{c}{3(\Delta V - w)}, \quad \hat{\theta}^* = \frac{1}{2}(\bar{\theta} + \alpha), \quad \hat{\theta}_B^* = \frac{1}{2}(\bar{\theta} + \alpha)
\]

The next two subsections establish the second and the third statements of the proposition by showing that neither firm will deviate when \( \alpha < \frac{1}{3}(2\bar{\theta} - \theta) \) and \( \Delta V > \max\{\Delta_1, \Delta_2\} \). The last subsection shows that no pure strategy equilibrium exists when \( \alpha < \frac{1}{3}(2\bar{\theta} - \theta) \) and \( \Delta V \leq \max\{\Delta_1, \Delta_2\} \).

**No Profitable Deviation for the Lower-quality Firm**

There are three possible non-local deviations of the lower-quality firm to consider: (a) the firm does not sell the add-on while leaving some consumers to buy \( H \), (b) the firm does not sell the add-on and leaves no consumers to buy \( H \), (c) the firm sells the add-on but leaves no consumers to buy \( H \).

**Case (a): the lower-quality firm does not sell the add-on while leaving some consumers to buy \( H \).** By Lemma 2 this is not a profitable deviation as long as \( P_h^* > \alpha \Delta V \).

Notice that

\[
P_h^* - \alpha \Delta V = \frac{1}{3}(2\bar{\theta} - \theta)(\Delta V - w) + \frac{c}{3} - \alpha \Delta V
\]

\[
= \frac{1}{3}
\left[
(2\bar{\theta} - \theta - 3\alpha)\Delta V - (2\bar{\theta} - \theta - \alpha)w
\right]
\]

which is positive exactly when \( \alpha < \frac{1}{3}(2\bar{\theta} - \theta) \) and \( \Delta V > \Delta_1 \). Therefore, the deviation is not profitable.

**Case (b): the lower-quality firm does not sell the add-on and leaves no consumers to buy \( H \).** By Lemma 2 this is not a profitable deviation as long as \( P_h^{**} > \alpha(\Delta V + w) \).
Notice that
\[
P_{h}^{++} - \alpha(\Delta V + w) = \frac{1}{3}(2\bar{\theta} - \theta)(\Delta V - w) + \frac{c}{3} + \frac{1}{2}(\bar{\theta}w + c) - \alpha(\Delta V + w)
\]
\[
= \frac{1}{3} \left[ (2\bar{\theta} - \theta - 3\alpha)\Delta V - \frac{1}{2}(\bar{\theta} - \theta + \alpha)w \right]
\]
\[
> \frac{1}{3} \left[ (2\bar{\theta} - \theta - 3\alpha)\Delta V - (2\bar{\theta} - \theta - \alpha)w \right]
\]
where the inequality follows from \( \alpha < \bar{\theta} \). The last equation is again positive when \( \alpha < \frac{1}{3}(2\bar{\theta} - \theta) \) and \( \Delta V > \Delta_1 \).

Case (c): the lower-quality firm sells the add-on but leaves no consumers to buy \( H \). The marginal consumer who is indifferent between the two firms now becomes
\[
\hat{\theta}_{hl} = \frac{P_{h}^{++} - P_{i}^{+}}{\Delta V}
\]
To ensure that no consumer buy \( H \), the lower-quality firm has to set the bundle price \( P_{i}^{+} \) low enough such that \( \hat{\theta}_{hl} \geq p_{h}^{*}/w \). By contrast the marginal consumer in the equilibrium profile satisfies:
\[
\hat{\theta}_{hl}^* < \hat{\theta}_{h}^* \iff \frac{P_{h}^{*} - P_{i}^{++}}{\Delta V - w} < \frac{P_{h}^{*} - P_{i}^{++}}{\Delta V} < \frac{p_{h}^{*}}{w}. \quad (D-9)
\]
The lower-quality firm's profit is given by \( \Pi_l = (\hat{\theta}_{hl} - \theta)(P_{i}^{+} - c) - (\hat{\theta}_{l} - \theta)(p_{l} - c) \). This profit is different from the equilibrium one only in terms of the bundling component \((\hat{\theta}_{hl} - \theta)(P_{i}^{+} - c)\). The first order condition of the Lagrangian for the constrained maximization of the bundling profit is given by
\[
P_{i}^{+} = \frac{1}{2}(P_{h}^{++} - \Delta V \theta + c - \lambda)
\]
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The constraint becomes:

$$\hat{\theta}_{hl} - \frac{p_h^*}{w} = \frac{P_{h^*} - P_l^*}{\Delta V} - \frac{p_h^*}{w} < \frac{P_{h^*} - P_l^*}{\Delta V} = \frac{1}{2\Delta V} (-p_h^* + w\theta + \lambda)$$

where the inequality follows from Equation (D-9) and the last equality is obtained by substituting the best responses of the lower-quality firm. Because $p_h^* > w\theta$, to guarantee the constraint is nonnegative, $\lambda > 0$. This implies that the constraint has to be binding by complementary slackness. This corner solution coincides with the one for the problem under the equilibrium because:

$$\frac{P_{h^*} - P_l^*}{\Delta V} = \frac{p_h^*}{w} \implies \frac{P_{h^*} - P_l^*}{\Delta V - w} = \frac{p_h^*}{w}.$$ 

Therefore the deviation profit is no greater than that obtained from the equilibrium strategy.

**No Profitable Deviation for the Higher-quality Firm**

There are four possible non-local deviations of the higher-quality firm: (a) the firm sells optional add-on but leaves no demand for $L^+$, (b) the firm sells standard add-on and leaves some demand for $L^+$, (c) it sells standard add-on and leaves no demand for $L^+$, and (d) it sells no add-on at all.

**Case (a): the higher-quality firm sells optional add-on but leaves no demand for $L^+$.** The the marginal consumer who is indifferent between the two firms is

$$\hat{\theta}_{hl} = \frac{P_h - P_l^*}{\Delta V}.$$ 

The higher-quality firm maximizes profit $\Pi_h = (\bar{\theta} - \hat{\theta}_{hl})P_h + (\bar{\theta} - \hat{\theta}_h)(p_h - c)$ subject to the constraint $\hat{\theta}_{hl} \leq \frac{p_l^*}{w}$ so that there is no demand for consumers who buy $L^+$. Note that the profit is also separable into a component for the base good and a component for the add-on price. It suffices to examine the profit component of the
base good which is different from the equilibrium one. The first-order condition for the constrained optimization problem of the base profit is

\[ P_h = \frac{1}{2}(P_i^* + \Delta V \bar{\theta} - \lambda). \]

There are two cases to consider. First, if \( \alpha > \bar{\theta} \), then the equilibrium add-on price is \( p_i^* = (\bar{\theta} w + c)/2 \) and satisfies

\[ \hat{\theta}_i^* > \hat{\theta}_i^* \Leftrightarrow \frac{P_h - P_i^*}{\Delta V} > \frac{p_i^*}{w}. \]

The constraint becomes

\[ \frac{p_i^*}{w} - \hat{\theta}_i = \frac{p_i^*}{w} - \frac{P_h - P_i^*}{\Delta V} < \frac{P_h - P_i^*}{\Delta V} = \frac{1}{2\Delta V}(p_i^* - \bar{\theta} w + \lambda) \]

where the inequality follows from the Equation (D-3) and the last equality is obtained by substituting the best responses of the higher-quality firm. Because \( p_i^* < \bar{\theta} w \), for the constraint to be nonnegative we have \( \lambda > 0 \). This implies that the constraint must be binding. The corner solution is the same as the one under the equilibrium problem because:

\[ \frac{P_h - P_i^*}{\Delta V} = \frac{p_i}{w} \iff \Delta V \frac{P_h - P_i^{++}}{\Delta V - w} = \frac{p_i}{w} \]

Therefore the deviation profit is no greater than the equilibrium profit.

Second, if \( \alpha \leq \bar{\theta} \), then \( p_i^* \leq \bar{\theta} w \). In this deviation case, there is no demand for the lower-quality firm at all. The equilibrium strategy is optimal.

**Case (b):** the higher-quality firm sells standard add-on and leaves some demand for \( L^+ \). Because \( P_i^{++} > 0 \), by Lemma 1 this strategy is strictly dominated by selling optional add-on by increasing the add-on price and lowering the base price. This is exactly the equilibrium strategy. Therefore, deviation in this case is not profitable.

**Case (c):** the higher-quality firm sells standard add-on to and leaves no demand
for $L^+$. Because $P^*_l > 0$, by Lemma 1 this strategy is strictly dominated by selling optional add-on by increasing the add-on price and lowering the base price. This is exactly the strategy in Case (a), the profit of which is no greater than the equilibrium profit.

**Case (d): the higher-quality firm does not sell the add-on at all.** This deviation can take two forms. First, the higher-quality firm can leave no consumer buy the add-on from the lower-quality firm. The profit is smaller than that in case (a) because a small positive sales of the add-on can improve profit without affecting profit from the base good. Second, the higher-quality firm can leave some consumers buy the add-on from the lower-quality firm. The optimal base price is then the same as the equilibrium strategy $P^*_h$. However, since the deviation forgoes the profit from selling the add-on, the deviation profit is smaller than the equilibrium profit. Therefore, in either case, the deviation is not profitable.

**No Pure Strategy Equilibrium Exists When** $\Delta V \leq \max\{\Delta_1, \Delta_2\}$

Because in equilibrium the higher-quality firm will always sell the add-on as optional by Proposition 1, there are two possible equilibrium outcomes depending on the lower-quality firm’s implementation: (a) the lower-quality firm sells the add-on, and (b) the firm does not sell the add-on at all. Neither case is sustainable when $\Delta V \leq \max\{\Delta_1, \Delta_2\}$.

**Case (a): the lower-quality firm sells the add-on.** In this case, the equilibrium profiles are the same as those in the second and the third statements of the proposition. When $\alpha > \bar{\theta}$, both firms sells optional add-on. The fraction of consumers who buy the bundle from the lower-quality firm is given by

$$\hat{\theta}_h^* - \hat{\theta}_l^* = \frac{1}{6(\Delta V - w)} \left[ (2\bar{\theta} - \theta - 3\alpha)\Delta V - (2\bar{\theta} - \theta - \alpha)w \right].$$
which is positive when $\Delta V > \Delta_1$. Therefore when $\Delta V \leq \Delta_1$ the equilibrium does not sustain.

When $\alpha \leq \theta$, the lower-quality firm will the add-on as standard, and the fraction of consumers who buy the bundle from the lower-quality firm becomes

$$\hat{\theta}_l^* - \theta = \frac{1}{3(\Delta V - w)} \left[ (\hat{\theta} - 2\theta) \Delta V - (\hat{\theta} - 2\theta + \alpha)w \right].$$

which is positive when $\Delta V > \Delta_2$. Therefore when $\Delta V \leq \Delta_2$ the equilibrium does not sustain.

Case (b): the lower-quality firm does not sell the add-on at all. In this case, the equilibrium profile is the same as that in the first statement of the proposition. The higher-quality firm’s base price is then $P_h^* = (2\hat{\theta} - \theta)\Delta V/3$. Since $\alpha < (2\hat{\theta} - \theta)/3$, we have $P_h^* > \alpha \Delta V$. Therefore, by Lemma 2 the lower-quality firm can profitably deviate by selling the add-on.

**E Monopoly with a Product Line**

Imagine that both the higher-quality and lower-quality firms belong to the same managing company. That is, the monopolist manages a product line of a higher-quality brand and a lower-quality brand. In this modification I explore the optimal product line decision. The timing of the decision problem is the same as the simultaneous-move game except that the monopolist controls the base and add-on prices for both brands: $(P_h, p_h)$ and $(P_l, p_l)$. All other assumptions remain the same. To enable comparison to the duopoly model, I focus on full market coverage. The main result from of this model is that the monopolist chooses to sell the add-on as optional for the higher-quality brand, but not to sell the add-on for the lower-quality brand.

**Proposition 6.** It is always optimal for the higher-quality brand to sell the add-on as optional, and for the lower-quality brand not to sell the add-on.

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29Note that $\Delta_1 > \Delta_2$ when $\alpha > \theta$, and $\Delta_2 > \Delta_1$ when $\alpha < \theta$.  

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The proof is given as follows.

**E-1 When the Lower-quality Brand Sells the Add-on**

Consider the first case when the higher-quality brand sells the add-on as optional. The two brands are competing for the marginal consumer \( \hat{\theta}_{hl} = (P_h - P_t^+)/ (\Delta V - w) \). The monopolist maximizes the joint profit of the two brands, \( \Pi_h \) as in Equation (2) and \( \Pi_t \) as in Equation (3). The maximization problem can decompose into two parts. The first part involves maximizing \( \tilde{\Pi} \) by choosing the optimal price pair \((P_h, P_t^+)\):

\[
\max_{P_h, P_t^+} \tilde{\Pi} = (\tilde{\theta} - P_h - P_t^+) P_h + (P_h - P_t^+) (\Delta V - w) \tilde{\theta} (P_t^+ - c).
\]

Since the market is fully covered, the monopolist can choose the maximum \( P_t^+ \) for which the type-\( \hat{\theta} \) consumer is willing to pay and then choose the optimal \( P_h \) to maximize profit. Therefore

\[
P_t^{**} = \tilde{\theta} (V_t + w) \quad \text{and} \quad P_h^* = P_t^{**} + \frac{1}{2} \tilde{\theta} (\Delta V - w) - \frac{1}{2} c.
\]

The second part involves maximizing \( \pi_h(p_h) \) and \( \pi_t(p_t) \) within each brand segment by choosing the appropriate add-on prices \((p_h, p_t)\). They are exactly the same as the ones in the duopoly model. The optimal total profit is then

\[
\Pi^{(opt, opt)} = (\tilde{\theta} - \theta) \tilde{\theta} (V_t + w) + \frac{1}{4 (\Delta V - w)} \left[ \theta (\Delta V - w) - c \right]^2 + \theta c + \frac{1}{4w} (\theta w - c)^2 + \frac{1}{4w} (\theta w - c)^2.
\]

For this solution to hold, it is necessary that \( \hat{\theta}_{hl} > \theta_t^* \) which leads to

\[
\alpha < \tilde{\theta} - \theta \quad \text{and} \quad \Delta V > \frac{\tilde{\theta} - \theta}{\tilde{\theta} - \theta - \alpha} w. \tag{E-1}
\]

Consider now the second case when the higher-quality brand sells the add-on as standard. The two brands now compete for the marginal consumer \( \hat{\theta}_{hl} = (P_h^+ - \)
\( P_I^+ / \Delta V \). The maximization problem can decompose into two parts. The first part involves maximizing \( \Pi \) by choosing the optimal price pair \((P_h^+, P_I^+)\):

\[
\max_{P_h^+, P_I^+} \Pi = (\bar{\theta} - \frac{P_h^+ - P_I^+}{\Delta V})(P_h^+ - c) + \left( \frac{P_h^+ - P_I^+}{\Delta V} - \bar{\theta} \right)(P_I^+ - c).
\]

The optimal pricing is

\[
P_I^{++} = \bar{\theta}(V + w) \quad \text{and} \quad P_h^* = P_I^{++} + \frac{1}{2} \bar{\theta} \Delta V.
\]

The second part involves maximizing \( \pi_i(p_l) \) by choosing the add-on price \( p_l \). The optimal profit is

\[
\Pi^{(\text{std, opt})} = (\bar{\theta} - \bar{\theta})(\bar{\theta}V + \bar{\theta}w - c) + \frac{1}{4} \bar{\theta}^2 \Delta V + \frac{1}{4w}(\bar{\theta}w - c)^2.
\]

It is straightforward to show that \( \Pi^{(\text{opt, opt})} \geq \Pi^{(\text{std, opt})} \). Notice that when it is optimal to sell the add-on to all the consumers at the lower-quality brand (i.e., \( \alpha < \bar{\theta} \)), the same relationship holds: \( \Pi^{(\text{opt, std})} \geq \Pi^{(\text{std, std})} \).

**E-2 When the Lower-quality Brand Does Not Sell the Add-on**

Consider the first case when the higher-quality brand sells the add-on as optional. The two brands are competing for the marginal consumer \( \hat{\theta}_h = (P_h - P_l) / \Delta V \). The monopolist’s maximization problem can decompose into two parts. The first part involves maximizing \( \Pi \) by choosing the optimal price pair \((P_h, P_I)\):

\[
\max_{P_h, P_I} \Pi = (\bar{\theta} - \frac{P_h - P_I}{\Delta V})P_h + \left( \frac{P_h - P_I}{\Delta V} - \bar{\theta} \right)P_I.
\]

The optimal pricing is

\[
P_I^* = \bar{\theta}V \quad \text{and} \quad P_h^* = P_I^* + \frac{1}{2} \bar{\theta} \Delta V.
\]
The second part involves maximizing $\pi_h(p_h)$ by choosing the add-on price $p_h$. The optimal profit becomes

$$\Pi^{(opt, no)} = (\bar{\theta} - \theta)\theta V_i + \frac{1}{4(\Delta V + w)} [\bar{\theta}(\Delta V + w) - c]^2 + \frac{1}{4w}(\bar{\theta}w - c)^2.$$ 

Consider now the second case when the higher-quality brand sells the add-on as standard. The two brands now compete for the marginal consumer $\hat{\theta}_{hl} = (P_h^+ - P_i)/(\Delta V + w)$. The monopolist only needs to maximize $\Pi$ by choosing the optimal price pair $(P_h^+, P_i)$:

$$\max_{P_h^+, P_i} \Pi = (\bar{\theta} - \theta)\theta V_i + \frac{1}{2}\bar{\theta}(\Delta V + w) + \frac{1}{2}c.$$ 

The optimal pricing is

$$P_i^* = \bar{\theta}V_i \quad \text{and} \quad P_h^* = \frac{1}{2}(\Delta V + w) + \frac{1}{2}c.$$ 

with optimal profit

$$\Pi^{(std, no)} = (\bar{\theta} - \theta)\theta V_i + \frac{1}{4(\Delta V + w)} [\bar{\theta}(\Delta V + w) - c]^2.$$ 

It follows that $\Pi^{(opt, no)} \geq \Pi^{(std, no)}$. Therefore, it is always optimal for the monopolist to sell the add-on as optional for the higher-quality brand.

It remains to compare the profit obtained by the strategy of selling the add-on to the lower-quality brand to the one of not selling it. Note that

$$\Pi^{(opt, no)} - \Pi^{(opt, opt)} = \frac{w}{4(\Delta V - w)} [(M - c^2)\Delta V - M \cdot w],$$

where $M = (\bar{\theta} - \theta)(\bar{\theta} - 3\bar{\theta} + 2\alpha)$. Given the condition in Equation (E-1), we have $\Pi^{(opt, no)} \geq \Pi^{(opt, opt)}$. Therefore, it is optimal for the monopolist to sell the add-on as
optional for the higher-quality brand while not selling the add-on to the lower-quality brand.

F Extension 1: Asymmetric Add-on

Consider the first extension that the cost and the value of the add-on are different for the two firms: \((c_h, w_h)\) for the higher-quality firm and \((c_l, w_l)\) for the lower-quality firm. The cost-to-value ratio also varies across the firms: \(\alpha_h\) and \(\alpha_l\). No restriction is placed on whether the cost and/or value should be higher for the higher-quality firm or not. The cost-to-value ratio can be either higher or lower for the higher-quality firm. It is, however, assumed that the quality premium \(\Delta V\) remains higher than the maximum value of the add-ons, \(\Delta V > \max\{w_h, w_l\}\). In addition, the assumption that the cost of the add-on is not unreasonably large, \(\alpha_h < \bar{\theta}\), still applies so that the higher-quality firm always sells the add-on in equilibrium. Other assumptions follow from the main analysis. With this specification, the conclusions in the preceding analysis are qualitatively unchanged. Let

\[
\Delta_{1,t} = \frac{2\bar{\theta} - \theta - \alpha_l}{2\bar{\theta} - \theta - 3\alpha_l} \cdot w_l, \quad \text{and} \quad \Delta_{2,t} = \frac{\bar{\theta} - 2\theta + \alpha_l}{\theta - 2\bar{\theta}} \cdot w_l.
\]

**Proposition 7.** Under asymmetric add-on, Propositions 1 and 2 still hold by replacing \(\alpha, w, \Delta_1, \text{and } \Delta_2\) with \(\alpha_l, w_l, \Delta_{1,t}, \text{and } \Delta_{2,t}\).

The proof of the proposition largely follows the proof of Propositions 1 and 2. I first establish two lemmas and then proceed to prove the main proposition.

**Lemma 3.** For the higher-quality firm, selling standard add-on while leaving some consumers to buy from the lower-quality firm is strictly dominated by selling optional add-on, if:

1. \(P_t > -\alpha_h\Delta V\) when the lower-quality firm does not sell the add-on, or

2. \(P_t^+ > -\alpha_h(\Delta V - w_l)\) when the lower-quality firm sells the add-on.
Lemma 4. For the lower-quality firm, not selling the add-on while leaving some consumers to buy from the higher-quality firm is strictly dominated by selling the add-on, if:

1. $P_h > \alpha_l \Delta V$ when the higher-quality firm sells optional add-on, or

2. $P_h^+ > \alpha_l (\Delta V + w_h)$ when the higher-quality firm sells standard add-on.

F-1 Proof of Lemma 3

First, when the lower-quality firm does not sell the add-on, the argument in the first section of the proof of Lemma 1 follows except that $\alpha$ and $w$ are replaced by $\alpha_h$ and $w_h$. Second, when the lower-quality firm sells the add-on, the argument is also similar. Suppose the higher-quality firm sells standard add-on at price $P_h^+$. Then the marginal consumer who is indifferent between the two firms is

$$\hat{\theta}_{hl} = \frac{P_h^+ - P_l^+}{\Delta V + w_h - w_l}.$$  

The higher-quality firm obtains a profit of $\Pi_h = (\bar{\theta} - \hat{\theta}_{hl})(P_h^+ \ - c)$. This strategy is exactly equivalent to separating the bundle price $P_h^+$ into a base price $P_h$ and an add-on price $p_h$ such that $P_h^+ = P_h + p_h$, and that

$$\frac{p_h}{w_h} = \frac{P_h^+ - P_l^+}{\Delta V + w_h - w_l} = \frac{P_h - P_l^+}{\Delta V - w_l}. \quad (F-1)$$

Let’s now consider a small increase of the add-on price to $p'_h = p_h + \epsilon$ with $\epsilon > 0$, and a small decrease of the base price to $P_h' = P_h - \epsilon$ by the same amount. The bundle price $P_h^+$ is then unchanged. Therefore we have

$$\frac{p'_h}{w_h} > \frac{P_h^+ - P_l^+}{\Delta V + w_h - w_l} > \frac{P_h' - P_l^+}{\Delta V - w_l}.$$
The profit function now becomes

$$\Pi_h(P'_h, p'_h) = (\bar{\theta} - \frac{P'_h - P^+_l}{\Delta V - w_l})P'_h + (\bar{\theta} - \frac{p'_h}{w_h})(p'_h - c_h)$$

Similar to the above case, evaluating this function around the original prices \((P_h, p_h)\) by Taylor series yields:

$$\Pi_h(P'_h, p'_h) = \Pi_h(P_h, p_h) - \epsilon \cdot [\bar{\theta} - \frac{P_h - P^+_l}{\Delta V - w_l} - \frac{P_h}{\Delta V - w_l}] + \epsilon \cdot [\bar{\theta} - \frac{p_h}{w_h} - \frac{p_h - c_h}{w}]
+ \frac{\epsilon^2}{2} \left[ \frac{2}{\Delta V - w_l} - \frac{2}{w_h} \right]
= \Pi_h(P_h, p_h) + \frac{\epsilon}{\Delta V - w_l} \left( P^+_l + \alpha_h(\Delta V - w_l) \right) - \frac{\epsilon^2(\Delta V + w_h - w_l)}{w_h(\Delta V - w_l)}$$

where the last equality follows from Equation (F-1). Therefore, \(M > 0\) for small positive \(\epsilon\) when \(P^+_l + \alpha_h(\Delta V - w_l) > 0\).

**F-2 Proof of Lemma 4**

First, when the higher-quality firm sells optional add-on, the argument in the first section of the proof of Lemma 2 follows except that \(\alpha\) and \(w\) are replaced by \(\alpha_l\) and \(w_l\). Second, when the higher-quality firm sells standard add-on, the argument is also similar. Suppose the lower-quality firm sells only the base to consumers at price \(P_l\). Then the marginal consumer who is indifferent between the two firms is

$$\hat{\theta}_{hl} = \frac{P^+_h - P_l}{\Delta V + w_h}.$$  

The lower-quality firm obtains a profit of \(\Pi_l = (\hat{\theta}_{hl} - \theta)P_l\). This strategy is exactly equivalent to charging a bundle price \(P^+_l\) and an add-on price \(p_l\) such that \(P^+_l = P_l + p_l\), and that

$$\frac{P^+_h - P^+_l}{\Delta V + w_h - w_l} = \frac{P^+_h - P_l}{\Delta V + w_h} = \frac{p_l}{w_l}. \tag{F-2}$$
Let's now consider a small decrease of the bundle price to \( P_i^{+'} = P_i^+ - \epsilon \) with \( \epsilon > 0 \), and a small decrease of the add-on price to \( p_i' = p_i - \epsilon \) by the same amount. The base price \( P_i \) is then unchanged. Therefore we have

\[
\frac{P_h^+ - P_i^{+'}}{\Delta V + w_h - w_i} > \frac{P_h^+ - P_i}{\Delta V + w_h} > \frac{p_i'}{w_i}.
\]

The profit function now becomes

\[
\Pi_i(P_i^{+'}, p_i') = \left( \frac{P_h^+ - P_i^{+'}}{\Delta V + w_h - w_i} - \theta \right)(P_i^{+'} - c_i) - \left( \frac{p_i'}{w_i} - \theta \right)(p_i' - c_i).
\]

Evaluating this function around the original prices \((P_i^+, p_i)\) by Taylor series yields:

\[
\Pi_i(P_i^{+'}, p_i') = \Pi_i(P_i^+, p_i) - \epsilon \cdot \left[ \frac{P_h^+ - P_i^+}{\Delta V + w_h - w_i} - \theta - \frac{P_i^+ - c_i}{\Delta V + w_h} \right] + \epsilon \cdot \left[ \frac{p_i}{w_i} - \theta + \frac{p_i - c_i}{w_i} \right] + \frac{\epsilon^2}{2} \left[ -\frac{2}{\Delta V + w_h - w_i} - \frac{2}{w_i} \right]
\]

\[
= \Pi_i(P_i^+, p_i) + \frac{\epsilon}{\Delta V}(P_h^+ - \alpha_i(\Delta V + w_h)) - \frac{\epsilon^2}{w_i(\Delta V + w_h)}(\Delta V + w_h)
\]

where the second equality follows from the remainder term \( R_2 = 0 \) because the higher order derivatives are all zero, and the last equality follows from Equation (F-2). Therefore, \( M > 0 \) for small positive \( \epsilon \) when \( P_h^+ > \alpha_i(\Delta V + w_h) \).

### F-3 Proof of Proposition 7

**The Higher-quality Firm Always Sells Optional Add-on in Equilibrium**

There are two alternative equilibria: (1) the higher-quality firm sells standard add-on, and (2) the firm does not sell the add-on. In the first case, the argument from Section C applies by replacing Lemma 1 with Lemma 3. In the second case, the argument from Section D-1 applies.
Equilibrium When the Add-on is Too Costly

The equilibrium profile in the case of large $\alpha_l$ ($\alpha_l > (2\bar{\theta} - \theta)/3$) now becomes:

$$P_h^* = \frac{1}{3}(2\bar{\theta} - \theta)\Delta V, \quad p_h^* = w_h\hat{\theta}_h^*,$$

$$P_l^* = \frac{1}{3}(\bar{\theta} - 2\theta)\Delta V, \quad p_l^* > w_l\hat{\theta}_{hl}^*,$$

where the thresholds are given by

$$\hat{\theta}_{hl}^* = \frac{1}{3}(\bar{\theta} + \theta), \quad \hat{\theta}_h^* = \frac{1}{2}(\bar{\theta} + \alpha_h).$$

First consider whether the lower-quality firm would deviate. There are three possible non-local deviations. Case (a): the firm does not sell the add-on. Replacing $w$ with $w_h$, the proof of Case (a) in Section D-2 applies here. Case (b): the firm sells the add-on and leaves some consumers buy $H$. Replacing $w$ and $c$ with $w_l$ and $c_l$ in the first order conditions and the incentive constraint $\hat{\theta}_{hl} \geq \hat{\theta}_l$, the proof of Case (b) in Section D-2 follows given the condition $\alpha_l > (2\bar{\theta} - \theta)/3$. Case (c): the firm sells the add-on but leaves no consumers to buy $H$. The argument is very similar to the one in Section D-3 with slight change of notations. The first order condition of the Lagrangian for the constrained maximization of the bundling profit now becomes

$$P_l^+ = \frac{1}{2}(P_h^{**} - (\Delta V + w_h - w_l)\bar{\theta} + c_l - \lambda)$$

The constraint becomes:

$$\hat{\theta}_{hl} - \frac{p_h^*}{w_h} = \frac{P_h^{**} - P_l^+}{\Delta V + w_h - w_l} - \frac{p_h^*}{w_h}$$

$$= \frac{1}{2(\Delta V + w_h - w_l)} \left[ -\frac{1}{3}(\bar{\theta} - 2\theta + 3\alpha_h)(\Delta V - w_l) ight.$$  

$$- \frac{w_h}{2}(\bar{\theta} - 2\theta + \alpha_h) - (\alpha_l - \frac{1}{3}(\bar{\theta} - 2\theta))w_l + \lambda \right]$$
Given that \( \alpha_l > (2\bar{\theta} - \theta)/3 \), to guarantee the constraint is nonnegative, it is necessary that \( \lambda > 0 \). This implies that the constraint has to be binding by complementary slackness. Therefore the deviation profit is no greater than that obtained from Case (b), which is not a profitable either.

Next consider whether the higher-quality firm would deviate. There are four possible non-local deviations. Case (a): the firm sells optional add-on and leaves some consumers buy \( L^+ \). Replacing \( w \) with \( w_l \), the proof of Case (a) in Section D-2 applies here. Case (b): it sells standard add-on and leaves some consumers buy \( L^+ \). The proof of Case (b) in Section D-2 applies by replacing Lemma 1 with Lemma 3. Case (c): it sells standard add-on but leaves no consumers buy \( L^+ \). Again, the proof of Case (c) in Section D-2 applies by replacing Lemma 1 with Lemma 3. Case (d): it does not sell the add-on at all. The proof of Case (d) in Section D-2 applies here.

Therefore neither the lower-quality firm nor the higher-quality would deviate.

**Equilibrium When the Add-on is Not Too Costly**

The equilibrium profile now becomes:

\[
P_h^* = \frac{1}{3}(2\bar{\theta} - \theta)(\Delta V - w_l) + \frac{1}{3}c_l,
\]

\[
p_h^* = w_h\hat{\theta}_h^*,
\]

\[
P_l^{**} = \frac{1}{3}(\bar{\theta} - 2\theta)(\Delta V - w_l) + \frac{2}{3}c_l,
\]

\[
p_l^* = \begin{cases} w_l\hat{\theta}_l^* & \text{if } \alpha > \theta \\ \leq w_l\theta & \text{if } \alpha \leq \theta \end{cases}
\]

where the thresholds are given by

\[
\hat{\theta}_h^* = \frac{1}{3}(\bar{\theta} + \theta) - \frac{c_l}{3(\Delta V - w_l)}, \quad \hat{\theta}_l^* = \frac{1}{2}(\bar{\theta} + \alpha_l), \quad \hat{\theta}_l^* = \frac{1}{2}(\theta + \alpha_l)
\]

First consider whether the lower-quality firm would deviate. There are three possible non-local deviations. Case (a): the firm does not sell the add-on while leaving some consumers to buy \( H \). Replacing \( w, c, \alpha, \Delta_1 \) and Lemma 2 with \( w_l, c_l, \).
\( \alpha_l, \Delta_{1,l} \) and Lemma 4, the proof of Case (a) in Section D-3 applies here.

Case (b): the firm does not sell the add-on and leaves no consumers to buy \( H \). By Lemma 4 this is not a profitable deviation as long as \( P_{h}^{++} > \alpha_l(\Delta V + w_h) \). Notice that

\[
P_{h}^{++} - \alpha_l(\Delta V + w_h) = \frac{1}{3}(2\bar{\theta} - \theta)(\Delta V - w_l) + \frac{\alpha_l}{3} + \frac{1}{2}(\bar{\theta}w_h + c_h) - \alpha_l(\Delta V + w_h)
\]

\[
= \left(\frac{1}{3}(2\bar{\theta} - \theta) - \alpha_l\right)\Delta V - \frac{1}{3}(2\bar{\theta} - \theta)w_l + \left(\frac{1}{2}(\bar{\theta} + \alpha_h) - \alpha_l\right)w_h
\]

\[
> \frac{1}{3}(2\bar{\theta} - \theta)\Delta V - (2\bar{\theta} - \theta)w_l
\]

where the inequality follows from \( \alpha_l < \frac{1}{3}(2\bar{\theta} - \theta) \). The last equation is positive when \( \Delta V > \Delta_{1,l} \).

Case (c): the firm sells the add-on but leaves no consumers to buy \( H \). The argument is very similar to the one in Section D-3. The marginal consumer who is indifferent between the two firms now becomes

\[
\hat{\theta}_{hl} = \frac{P_{i}^{++} - P_{l}^{+}}{\Delta V + w_h - w_l}
\]

To ensure that no consumer buy \( H \), the lower-quality firm has to set the bundle price \( P_{l}^{+} \) low enough such that \( \hat{\theta}_{hl} \geq p^*_h/w_h \). By contrast the marginal consumer in the equilibrium profile satisfies:

\[
\hat{\theta}_{hl}^{*} < \hat{\theta}^{*}_h \Leftrightarrow \frac{P_{i}^{*} - P_{l}^{++}}{\Delta V - w_l} < \frac{p^*_h}{w_h} \Rightarrow \frac{P_{h}^{++} - P_{l}^{++}}{\Delta V + w_h - w_l} < \frac{p^*_h}{w_h}.
\]

The lower-quality firm’s profit is given by \( \Pi_l = (\hat{\theta}_{hl} - \theta)(P_{i}^{+} - c_l) - (\hat{\theta}_l - \theta)(p_l - c_l) \). This profit is different from the equilibrium one only in terms of the bundling component \( (\hat{\theta}_{hl} - \theta)(P_{l}^{+} - c_l) \). The first order condition of the Lagrangian for the constrained
maximization of the bundling profit is given by

\[ P^*_l = \frac{1}{2}(P^{**}_h - \Delta V \theta + c_l - \lambda) \]

The constraint becomes:

\[
\hat{\theta}_{hl} - \frac{p^*_h}{w_h} = \frac{P^{**}_h - P^+_l}{\Delta V + w_h - w_l} - \frac{p^*_h}{w_h} < \frac{P^{**}_h - P^+_l}{\Delta V + w_h - w_l} = \frac{1}{2(\Delta V + w_h - w_l)}(-p^*_h + w_l \theta + \lambda)
\]

where the inequality follows from Equation (F-7) and the last equality is obtained by substituting the best responses of the lower-quality firm. Because \( p^*_h > w_l \theta \), to guarantee the constraint is nonnegative, \( \lambda > 0 \). This implies that the constraint has to be binding by complementary slackness. This corner solution coincides with the one for the problem under the equilibrium because:

\[
\frac{P^{**}_h - P^+_l}{\Delta V + w_h - w_l} = \frac{p^*_h}{w_h} \Rightarrow \frac{P^*_h - P^+_l}{\Delta V - w_l} = \frac{p^*_h}{w_h}.
\]

Therefore the deviation profit is no greater than that obtained from the equilibrium strategy.

Next consider whether the higher-quality firm would deviate. There are four possible non-local deviations. Case (a): the firm sells optional add-on but leaves no demand for \( L^+ \). Replacing \( w \) and \( \alpha \) with \( w_l \) and \( \alpha_l \), the proof of Case (a) in Section D-3 applies here. Case (b): it sells standard add-on and leaves some consumers buy \( L^+ \). The proof of Case (b) in Section D-2 applies by replacing Lemma 1 with Lemma 3. Case (c): it sells standard add-on but leaves no consumers buy \( L^+ \). Again, the proof of Case (c) in Section D-3 applies by replacing Lemma 1 with Lemma 3. Case (d): it does not sell the add-on at all. The proof of Case (d) in Section D-3 applies here.

Finally, there exists no pure-strategy equilibrium when \( \Delta V \leq \max\{\Delta_{1,l}, \Delta_{2,l}\} \).
The argument is exactly the same as the one in Section D-3, by replacing \( w, \alpha, \Delta_1, \Delta_2 \) and Lemma 2 with \( w_t, \alpha_t, \Delta_{1,t}, \Delta_{2,t} \) and Lemma 4.

**G Extension 2: Asymmetric Marginal Cost of the Base Good**

Consider next the extension that the marginal cost for the higher-quality firm is larger. The marginal cost of the lower-quality firm remains zero by normalization, but the cost for the higher-quality firm now becomes \( \Delta C \geq 0 \). Define the cost-to-value ratio for the base good as \( A = \Delta C / \Delta V \), which measures the unit cost of the quality premium. Let

\[
A = \frac{(\bar{\theta} - 2\theta + 3\alpha)\Delta V - (\bar{\theta} - 2\theta + \alpha)w}{2\Delta V}.
\]

This is the bound above which it is costly for the higher-quality firm to serve additional consumers who buy only the base by lowering the base price. Therefore we shall focus on the case \( A < \bar{A} \).\(^{30}\) Other assumptions follow from the main analysis. Let

\[
\Delta'_1 = \max\{w, \frac{2\bar{\theta} - \bar{\theta} - \alpha}{2\bar{\theta} + 2\bar{\theta} - \bar{\theta} - 3\alpha} \cdot w\}, \quad \text{and} \quad \Delta'_2 = \max\{w, \frac{\bar{\theta} - 2\bar{\theta} + \alpha}{A + \bar{\theta} - 2\bar{\theta}} \cdot w\}.
\]

**Proposition 8.** Under asymmetric marginal cost, Propositions 1 and 2 still hold by replacing \( \frac{1}{3}(2\bar{\theta} - \bar{\theta}), \Delta_1 \) and \( \Delta_2 \) with \( \frac{1}{3}(2\bar{\theta} - \bar{\theta}) + \frac{2}{3}A, \Delta'_1, \) and \( \Delta'_2 \).

The proof largely follows the proof of Propositions 1 and 2. Again I start with the lemmas before proving the main proposition.

**Lemma 5.** For the higher-quality firm, selling standard add-on while leaving some consumers to buy from the lower-quality firm is strictly dominated by selling optional add-on, if:

1. \( P_1 > \Delta C - \alpha \Delta V \) when the lower-quality firm does not sell the add-on, or

\(^{30}\)When \( A \geq \bar{A} \) there exists equilibrium where the higher-quality firm sells the add-on as standard.
2. \( P_l^+ > \Delta C - \alpha(\Delta V - w) \) when the lower-quality firm sells the add-on.

**Lemma 6.** For the lower-quality firm, not selling the add-on while leaving some consumers to buy from the higher-quality firm is strictly dominated by selling the add-on, if:

1. \( P_h > \alpha \Delta V \) when the higher-quality firm sells optional add-on, or
2. \( P_h^+ > \alpha(\Delta V + w) \) when the higher-quality firm sells standard add-on.

**G-1 Proof of Lemma 5**

The argument is very similar to that in the proof of Lemma 1, except that the marginal cost of the base good \( \Delta C \) is subtracted from the base price in the profit function. When the lower-quality firm does not sell the add-on, selling optional add-on is a strictly dominating strategy when \( P_l + \alpha \Delta V > \Delta C \). When the lower-quality firm sells the add-on, selling optional add-on is a strictly dominating strategy when \( P_l^+ + \alpha(\Delta V - w) > \Delta C \).

**G-2 Proof of Lemma 6**

Since the marginal cost of the base good for the lower-quality firm is normalized to zero, Lemma 2 immediately applies.

**G-3 Proof of Proposition 8**

The Higher-quality Firm Always Sells Optional Add-on in Equilibrium

First, consider the possible equilibrium where the higher-quality firm sells standard add-on. There are two further cases. Case (a): the lower-quality firm sells the add-on. The marginal consumer who is indifferent between the two firms is

\[
\hat{\theta}_{hl} = \frac{P_h^+ - P_l^+}{\Delta V}.
\]
Equilibrium prices are

\[ P_h^{**} = \frac{1}{3} \Delta V(2\bar{\theta} - \hat{\theta}) + c + \frac{2}{3} \Delta C, \quad p_h^* < w\hat{\theta}_h^*; \]  
\[ P_l^{**} = \frac{1}{3} \Delta V(\bar{\theta} - 2\hat{\theta}) + c + \frac{1}{3} \Delta C, \quad p_l^* < w\hat{\theta}_l^*. \]  

(G-1)  

(G-2)

Notice that whether or not the lower-quality firm sells only the base or the bundle to its lowest-type consumers (i.e., whether \( p_l^* > \bar{\theta} \)) does not affect the argument below.

To apply Lemma 5, it is necessary that \( P_l^{**} + \alpha(\Delta V - w) > \Delta C \). That is

\[ P_l^{**} + \alpha(\Delta V - w) - \Delta C = \left( \frac{1}{3} (\bar{\theta} - 2\hat{\theta}) + \alpha \right) \Delta V - \frac{2}{3} \Delta C > 0 \Rightarrow A < \frac{1}{2} (\bar{\theta} - 2\hat{\theta} + 3\alpha) \]

Since \( A < \bar{A} \), the above inequality is satisfied.

Case (b): the lower-quality firm does not sell the add-on. The marginal consumer who is indifferent between the two firms is

\[ \hat{\theta}_{hl} = \frac{P_h^* - P_l^*}{\Delta V + w}. \]

Equilibrium prices are

\[ P_h^{**} = \frac{1}{3} (\Delta V + w)(2\bar{\theta} - \hat{\theta}) + \frac{2}{3} c + \frac{2}{3} \Delta C, \quad p_h^* < w\hat{\theta}_h^*; \]  
\[ P_l^{**} = \frac{1}{3} (\Delta V + w)(\bar{\theta} - 2\hat{\theta}) + \frac{1}{3} c + \frac{1}{3} \Delta C, \quad p_l^* > w\hat{\theta}_l^*. \]  

(G-3)  

(G-4)

To apply Lemma 5, it is necessary that \( P_l^* + \alpha \Delta V > \Delta C \). That is

\[ P_l^* + \alpha \Delta V - \Delta C = \left( \frac{1}{3} (\bar{\theta} - 2\hat{\theta}) + \alpha \right) \Delta V - \frac{2}{3} \Delta C + \frac{1}{3} (\bar{\theta} - 2\hat{\theta} + \alpha) w > 0 \]

which holds whenever the condition in Case (a) is satisfied. Therefore, there is always a profitable deviation for the higher-quality firm.
Finally, in a possible equilibrium where the higher-quality firm does not sell the add-on, the higher-quality firm can always profitably deviate to selling optional add-on. The argument is exactly the same as the one in Section C.

**Equilibrium When the Add-on is Too Costly**

The equilibrium profile now becomes:

\[
P_h^* = \frac{1}{3}(2\bar{\theta} - \underline{\theta})\Delta V + \frac{2}{3}\Delta C, \quad p_h^* = \theta^*_h, \quad (G-5)
\]

\[
P_l^* = \frac{1}{3}(\bar{\theta} - 2\underline{\theta})\Delta V + \frac{1}{3}\Delta C, \quad p_l^* > \theta^*_h, \quad (G-6)
\]

where the thresholds are given by

\[
\theta^*_h = \frac{1}{3}(\bar{\theta} + \theta + A), \quad \theta^*_h = \frac{1}{2}(\bar{\theta} + \alpha).
\]

In what follows I establish that there is no profitable deviation for either firm when \( \alpha > (2\bar{\theta} - \underline{\theta})/3 + 2A/3 \).

First, consider the three possible deviations by the lower-quality firm. Case (a): the firm does not sell the add-on and leaves no demand for \( H \). The argument in Section D-2 immediately follows, and thus there is no profitable deviation in this case. Case (b): it sells the add-on and leaves some demand for \( H \). The argument is the same as that in Case (b) of Section D-2, except that the constraint now becomes

\[
\hat{\theta}_h - \hat{\theta}_l = \frac{P_h^* - P_l^*}{\Delta V - w} - \frac{p_l}{w}
\]

\[
= \frac{1}{2w(\Delta V - w)} \left[ w\Delta V(\frac{1}{3}(2\bar{\theta} - \underline{\theta}) - \alpha) + \frac{2}{3}w\Delta C - \lambda_2(\Delta V - w) - \lambda_3w + \lambda_1\Delta V \right]
\]

Because \( \alpha > (2\bar{\theta} - \underline{\theta})/3 + 2A/3 \) we have \( M < 0 \). To guarantee the constraint is non-negative, \( \lambda_1 > 0 \), implying that Constraint (1) has to be binding by complementary slackness. Therefore the profit is no greater than that one obtained from not selling
the add-on (i.e., the equilibrium strategy).

Case (c): the firm sells the add-on but leaves no demand for $H$. The argument is the same as that in Case (c) of Section D-2, except that the constraint now becomes

$$\frac{\hat{p}_h}{w} - \frac{p_h^*}{w} = \frac{P_{h}^{++} - P_{l}^{+} - p_h^*}{\Delta V} = \frac{1}{2\Delta V} \left[ -\frac{1}{3} (\bar{\theta} - 2\overline{\theta} + 3\alpha) \Delta V + \frac{2}{3} \Delta C + \frac{1}{2} (\bar{\theta} - \alpha) + \lambda \right]$$

where $M < 0$ when $\alpha > (2\bar{\theta} - \overline{\theta})/3 + 2A/3$. Again, for the constraint to be nonnegative, it is necessary that $\lambda > 0$. This implies that the constraint has to be binding by complementary slackness. Therefore the profit is no greater than that one obtained from Case (b), which is not a profitable deviation.

Second, consider the four possible deviations by the higher-quality firm. Case (a): the firm sells optional add-on and leaves some demand for $L^+$. The argument is the same as that in Case (a) of Section D-2, The first-order condition of the constrained problem now becomes:

$$P_h = \frac{1}{2} (P_i^* + p_i^* + (\Delta V - w)\bar{\theta} + \Delta C + \lambda).$$

Therefore,

$$\hat{\theta}_h - \frac{p_i^*}{w} = \frac{P_h - P_i^* - p_i^*}{\Delta V - w} - \frac{p_i^*}{w}$$

$$= \frac{1}{2(\Delta V - w)} \left[ \frac{2}{3} (\bar{\theta} + \overline{\theta}) \Delta V + \frac{2}{3} \Delta C - \bar{\theta} w - \frac{p_i^*}{w} (2\Delta V - w) + \lambda \right]$$

$$\leq \frac{1}{2(\Delta V - w)} \left[ \frac{2}{3} (\bar{\theta} + \overline{\theta}) \Delta V + \frac{2}{3} \Delta C - \bar{\theta} w - \frac{1}{3} (\bar{\theta} + \overline{\theta} + A)(2\Delta V - w) + \lambda \right]$$

$$= \frac{1}{2(\Delta V - w)} \left( -\frac{1}{3} (2\overline{\theta} - \bar{\theta} - A) w + \lambda \right)$$

where the inequality is implied by the equilibrium add-on price of the lower-quality firm $p_i^* > \hat{\theta}_h w = (\theta + \overline{\theta} + A) w/3$. Because $2\overline{\theta} - \bar{\theta} > A$ given that $A < \overline{A}$ and $\alpha < \bar{\theta}$, to ensure the constraint is nonnegative it is necessary that $\lambda > 0$. Therefore, by
complementary slackness the constraint must be binding. That is, the higher-quality firm leaves no demand for the $L^+$. Therefore, the optimal profit obtained in this case is no greater than the equilibrium profit.

Case (b): the firm sells standard add-on and leaves some demand for $L^+$. Note that:

$$\begin{align*}
P_i^{++} + \alpha(\Delta V - w) - \Delta C &= \left(\frac{1}{3}(\bar{\theta} - 2\overline{\theta}) + \alpha\right)\Delta V - \frac{2}{3}\Delta C + p_i^* - c \\
> (\frac{1}{3}(\bar{\theta} - 2\overline{\theta}) + \alpha)\Delta V - \frac{2}{3}\Delta C + \frac{1}{3}(\bar{\theta} + \overline{\theta} + A)w - c \\
= (\frac{1}{3}(\bar{\theta} - 2\overline{\theta}) + \alpha)\Delta V - \frac{1}{3}(2\Delta V - w)A + (\frac{1}{3}(\bar{\theta} + \overline{\theta}) - \alpha)w \\
> (\frac{1}{3}(\bar{\theta} - 2\overline{\theta}) + \alpha)\Delta V - \frac{1}{2}(\frac{1}{3}(\bar{\theta} - 2\overline{\theta}) + \alpha)(2\Delta V - w)A + (\frac{1}{3}(\bar{\theta} + \overline{\theta}) - \alpha)w \\
= \frac{1}{2}(\bar{\theta} - \alpha)w > 0
\end{align*}$$

where the first inequality comes from the equilibrium constraint $p_i^* > \hat{\theta}_{hi}^* w = (\bar{\theta} + \overline{\theta} + A)w/3$, while the second inequality follows from the condition $A < \overline{A}$. Therefore, the rest of the argument is the same as Case (b) of Section D-2 by replacing Lemma 1 with Lemma 5.

Case (c): the firm sells standard add-on but leaves no demand for $L^+$. Note that:

$$\begin{align*}
P_i^* + \alpha\Delta V - \Delta C &= \left(\frac{1}{3}(\bar{\theta} - 2\overline{\theta}) + \alpha\right)\Delta V - \frac{2}{3}\Delta C = \frac{1}{3}(\bar{\theta} - 2\overline{\theta} + \alpha - 2A)\Delta V > 0
\end{align*}$$

where the inequality is true given $A < \overline{A}$. Therefore, the rest of the argument is the same as Case (c) of Section D-2 by replacing Lemma 1 with Lemma 5.

Case (d): the firm does not sell the add-on at all. The argument is exactly the same as Case (d) of Section D-2. To conclude, there is no profitable deviation by the higher-quality firm.
Equilibrium When the Add-on is Not Too Costly

The equilibrium profile now becomes:

\[ P_h^* = \frac{1}{3}(2\bar{\theta} - \theta)(\Delta V - w) + \frac{1}{3}c + \frac{2}{3}\Delta C, \quad p_h^* = w\hat{\theta}_h^* \tag{G-7} \]

\[ P_i^{**} = \frac{1}{3}(\bar{\theta} - 2\theta)(\Delta V - w) + \frac{2}{3}c + \frac{1}{3}\Delta C, \quad p_i^* \begin{cases} = w\hat{\theta}_i^* & \text{if } \alpha > \theta \\ \leq w\theta & \text{if } \alpha \leq \theta \end{cases} \tag{G-8} \]

where the thresholds are given by

\[ \hat{\theta}_h^* = \frac{1}{3}(\bar{\theta} + \theta) - \frac{c - \Delta C}{3(\Delta V - w)}, \quad \hat{\theta}_h^* = \frac{1}{2}(\bar{\theta} + \alpha), \quad \hat{\theta}_i^* = \frac{1}{2}(\theta + \alpha) \]

First, consider the three possible deviations by the lower-quality firm. Case (a): the lower-quality firm does not sell the add-on while leaving some consumers to buy \( H \). By Lemma 6 this is not a profitable deviation as long as \( P_h^* > a\Delta V \). Notice that

\[ P_h^* - a\Delta V = \frac{1}{3}[(2\bar{\theta} - \theta - 3\alpha + 2A)\Delta V - (2\bar{\theta} - \theta - \alpha)w] \]

which is positive exactly when \( \alpha < \frac{1}{3}(2\bar{\theta} - \theta) + \frac{2}{3}A \) and \( \Delta V > \Delta_i' \). Therefore, the deviation is not profitable.

Case (b): the lower-quality firm does not sell the add-on and leaves no consumers to buy \( H \). By Lemma 6 this is not a profitable deviation as long as \( P_h^{**} > a(\Delta V + w) \). Notice that

\[ P_h^{**} - a(\Delta V + w) = \frac{1}{3}[(2\bar{\theta} - \theta - 3\alpha + 2A)\Delta V - (\frac{1}{2}\bar{\theta} - \theta + \frac{1}{2}\alpha)w] \]

\[ > \frac{1}{3}[(2\bar{\theta} - \theta - 3\alpha + 2A)\Delta V - (2\bar{\theta} - \theta - \alpha)w] \]

where the inequality follows from \( \alpha < \bar{\theta} \). The last equation is again positive when \( \alpha < \frac{1}{3}(2\bar{\theta} - \theta) + \frac{2}{3}A \) and \( \Delta V > \Delta_i' \).
Case (c): the firm sells the add-on but leaves no demand for $H$. Since the marginal cost of the base good for the lower-quality firm is normalized to zero, the argument of Case (c) in Section D-3 immediately applies.

Second, consider the four possible deviations by the higher-quality firm. Case (a): the firm sells optional add-on but leaves no demand for $L^+$. The argument follows Case (a) in Section D-3 except that the marginal cost $ΔC$ enters the profit function and the best response functions without affecting the argument.

Case (b): the firm sells standard add-on and leaves some demand for $L^+$. Note that:

\[
P_{i}^{++} + α(ΔV - w) - ΔC = \left(\frac{1}{3}(\bar{θ} - 2\bar{θ}) + α\right)(ΔV - w) + \frac{2}{3}c + \frac{1}{3}ΔC + α(ΔV - w) - ΔC
\]
\[
= \left(\frac{1}{3}(\bar{θ} - 2\bar{θ}) + α - \frac{2}{3}A\right)ΔV - \frac{1}{3}(\bar{θ} - 2\bar{θ} + α)w
\]

where the first inequality is true given $A < \bar{A}$. Therefore, the rest of the argument is the same as Case (b) of Section D-3 by replacing Lemma 1 with Lemma 5.

Case (c): the firm sells standard add-on but leaves no demand for $L^+$. Note that:

\[
P_{i}^{*} + αΔV - ΔC = \left(\frac{1}{3}(\bar{θ} - 2\bar{θ}) + α\right)(ΔV - w) + \frac{2}{3}c + \frac{1}{3}ΔC - p_{i}^{*} + αΔV - ΔC
\]
\[
= \left(\frac{1}{3}(\bar{θ} - 2\bar{θ}) + α - \frac{2}{3}A\right)ΔV - \frac{1}{3}(\bar{θ} - 2\bar{θ} + α)w + c - p_{i}^{*}
\]

There are two possibility. If $α > \bar{θ}$, then $p_{i}^{*} = \frac{1}{2}(θw + c) < c$. In this case the above equation is positive given $A < \bar{A}$. By Lemma 5 this deviation is strictly dominated by selling optional add-on which is exactly the equilibrium strategy. If $α \leq \bar{θ}$, then $p_{i}^{*} ≤ \bar{θ}w$. In this deviation case, there is no demand for the lower-quality firm at all, which is not profitable.

Case (d): the firm does not sell the add-on at all. The argument is exactly the same as Case (d) of Section D-3. To conclude, there is no profitable deviation by the
Finally, it remains to verify that there is no pure-strategy equilibrium when \( \Delta V \leq \max\{\Delta'_1, \Delta'_2\} \). There are two cases. Case (a): the lower-quality firm sells the add-on. The argument is exactly the same as the one in Section D-3, by replacing \( \Delta_1, \Delta_2 \) and Lemma 2 with \( \Delta'_1, \Delta'_2 \) and Lemma 6. Case (b): the lower-quality does not sell the add-on at all. In this case, the equilibrium is the same as that in Section G-3. The higher-quality firm’s base price is then \( P_h^* = (2\bar{\theta} - \bar{\theta} + 2A)\Delta V/3 \). Since \( \alpha < (2\bar{\theta} - \bar{\theta} + 2A)/3 \), we have \( P_h^* > \alpha \Delta V \). Therefore, by Lemma 6 the lower-quality firm can profitably deviate by selling the add-on.

**H Extension 3: Imperfectly Correlated Tastes**

In this extension, I relax the assumption that the tastes for the base and the add-on are perfectly correlated. For tractability, I focus on the case where there is at least some correlation between the two tastes. Suppose the taste for the base good remains \( \theta \), but the taste for the add-on is given by

\[
\lambda = \beta \theta + e
\]

where the constant \( \beta > 0 \) and \( e \sim U[\underline{e}, \bar{e}] \). Let \( \Delta e = \bar{e} - \underline{e} \). Therefore, each consumer is summarized by a pair of taste parameters \((\theta, e)\). The utility of buying from firm \( j \) for type-\((\theta, e)\) consumer is then

\[
U_j = \begin{cases} 
\theta V_j - P_j & \text{if only the base good is purchased;} \\
\theta V_j + (\beta w + e)w - P_j - p_j & \text{if both the base good and the add-on are purchased.}
\end{cases}
\]

With this specification, it is possible to find a closed-form solution to the equilibrium in which both firms sell the add-on as optional. Because consumers who have a higher taste for the base good also have a higher taste for the add-on, it is pos-
sible to have a situation where the high-quality firm and the low-quality firm only competes for the marginal consumers who trade off between the higher-quality base alone versus the lower-quality base plus the add-on. These consumers are given by \( \theta(\Delta V - \beta w) - ew = P_h - P_j - p_j \). To determine the market shares, it is sufficient to know the upper and lower bounds of the marginal consumers

\[
\hat{\theta}_{hl,u} = \frac{\bar{w} + P_h - P_j - p_j}{\Delta V - \beta w}, \quad \text{and} \quad \hat{\theta}_{hl,l} = \frac{\bar{w} + P_h - P_j - p_j}{\Delta V - \beta w}.
\]

Therefore, the high-quality firm’s profit is

\[
\Pi_h = \Delta e \cdot \left[ \bar{\theta} - \frac{1}{2}(\hat{\theta}_{hl,u} + \hat{\theta}_{hl,l}) \right] P_h + \frac{1}{2} \left( \bar{\theta} - \hat{\theta}_h \right) (\bar{\varepsilon} - \hat{\varepsilon}_h)(p_h - c),
\]

where intra-marginal consumers are given by

\[
\hat{\theta}_h = \frac{1}{\beta} \left( \frac{p_h}{w} - \bar{\varepsilon} \right), \quad \text{and} \quad \hat{\varepsilon}_h = \frac{p_h}{w} - \beta \bar{\theta}.
\]

Similar to the case of perfect correlation, the profit maximization problem reduces to maximizing two profit components separately using the base price \( P_h \) and the add-on price \( p_h \).

The low-quality firm’s profit is

\[
\Pi_l = \Delta e \cdot \left[ \frac{1}{2}(\hat{\theta}_{hl,u} + \hat{\theta}_{hl,l}) - \bar{\theta} \right] (P_l^+ - c) - \frac{1}{2} \left( \bar{\theta}_l - \hat{\theta}_l \right) (\bar{\varepsilon}_l - \hat{\varepsilon}_l)(p_l - c),
\]

where intra-marginal consumers are given by

\[
\hat{\theta}_l = \frac{1}{\beta} \left( \frac{p_l}{w} - \varepsilon \right), \quad \text{and} \quad \hat{\varepsilon}_l = \frac{p_l}{w} - \beta \bar{\theta}.
\]

\(^{31}\)The implicit assumption is that type-(\( \bar{\theta}, \varepsilon \)) consumer does not buy the add-on in equilibrium. However, this assumption is not important. The qualitative conclusions still hold when this consumer buys the add-on in equilibrium.
Similar to the case of perfect correlation, maximizing profit reduces to maximizing the two components of the profit separately using the bundle price $P_i^+$ and the add-on price $p_i$.

The price competition between the two firms pins down the equilibrium prices $P_h$ and $P_i^+$:

$$P_h^* = \frac{1}{3}(\Delta V - \beta w)(2\bar{\theta} - \theta) + \frac{1}{3}c + \frac{1}{2}(\bar{e} + \varepsilon)w,$$

and

$$P_i^{**} = \frac{1}{3}(\Delta V - \beta w)(\bar{\theta} - 2\theta) + \frac{2}{3}c + \frac{1}{2}(\bar{e} + \varepsilon)w,$$

Independent from the strategic interaction that determines equilibrium market shares, the firms set the optimal add-on prices. Add-on prices $p_h$ and $p_i$ are chosen to maximize $\pi_h(p_h)$ and $\pi_i(p_i)$ in Equations (H-2) and (H-3) respectively, which lead to

$$p_h^* = \frac{1}{3}(\bar{\theta} + \bar{e} + 2\alpha)w,$$

and

$$p_i^* = \frac{1}{3}(\beta \theta + \varepsilon + 2\alpha)w.$$

This equilibrium exists as long as the following incentive constraints hold:

$$\begin{align*}
(1) \quad \hat{\theta}_h^* > \hat{\theta}_{h,l,u}, & \quad (2) \quad \hat{\theta}_{h,l} > \hat{\theta}_i^*, & \quad (3) \quad \hat{\theta}_i^* > \hat{\theta}.
\end{align*}$$

These incentive constraints provide some intuition of equilibrium outcomes. As $\alpha$ drops, the add-on price becomes lower so that more lower-taste consumers are willing to buy the add-on. When it is sufficiently small, constraint (3) becomes violated, driving the low-quality firm to sell the add-on as standard. As $\alpha$ increases, the add-on price has to be higher to recover the cost of supplying the add-on, and thus more lower-taste consumers choose to buy only the base good. When it is sufficiently large, constraint (2) becomes violated, driving the low-quality firm to give up selling the add-on. However, constraint (1) stays satisfied within a reasonable range of $\alpha$. The equilibrium outcomes are qualitatively the same as the ones derived from the main model.
Proposition 9. The game has pure-strategy Nash equilibria, and firms can adopt different add-on policies in equilibrium.

1. If $\alpha$ is large such that $\alpha > \alpha_1$, there exists an equilibrium in which the high-quality firm sells the add-on as optional whereas the low-quality firm does not sell it;

2. If $\alpha$ is moderate such that $\alpha_3 < \alpha < \alpha_2 \leq \alpha_1$, there exists an equilibrium when $\Delta V > \Delta_1$, in which both firms sells the add-on as optional;

3. If $\alpha$ is small such that $\alpha < \alpha_4 \leq \alpha_3$, there exists an equilibrium when $\Delta V > \Delta_2$, in which the high-quality firm sells the add-on as optional whereas the low-quality firm sells it as standard.

The proof is logically the same as that for the default model, and hence the details are omitted.

I Proof of Proposition 3

I-1 The Higher-quality Firm Never Commits to No-Add-on Policy

By committing to no-add-on policy, the higher-quality firm always uses the base good to compete with the lower-quality firm in the second stage. Deviating to not making the commitment allows it to sell the add-on to a fraction of consumers who have the highest taste for quality. This does not affect the profit from selling the base good but can gain some additional benefit from selling the add-on. Profit is then strictly improved. Therefore, committing to no-add-on policy is always strictly dominated by not committing.

In what follows, I derive the equilibrium outcomes for the remaining six cases depending on whether the higher-quality firm commits to standard add-on or not.
I-2 When the Higher-quality Firm Does Not Commit to Standard Add-on

In this case, the equilibrium outcomes are the same as those summarized in Proposition 2. The difference is that here the lower-quality firm can not deviate to selling optional add-on in the second-stage pricing game once it has made the commitment to either standard-add-on or no-add-on. The equilibrium outcome depends on each of the three commitment choices of the lower-quality firm.

When the Lower-quality Firm Commits to No Add-on

Following the equilibrium analysis in the first case of Proposition 2, the equilibrium pricing in the second stage is

\[ P^*_h = \frac{1}{3}(2\bar{\theta} - \theta)\Delta V, \quad p^*_h = w\hat{\theta}_h^*, \]
\[ P^*_l = \frac{1}{3}(\bar{\theta} - 2\theta)\Delta V, \]

where the marginal consumers are given by

\[ \hat{\theta}_h^* = \frac{1}{3}(\bar{\theta} + \theta), \quad \hat{\theta}_h^* = \frac{1}{2}(\bar{\theta} + \alpha). \]

The incentive constraints \( \hat{\theta}_h^* > \hat{\theta}_h^* > \theta \) hold given that \( \bar{\theta} > 2\theta \). The resulting profits are given by

\[ \Pi_h^{(opt, no)} = \frac{1}{9}(2\bar{\theta} - \theta)^2\Delta V + \frac{1}{4w}(\bar{\theta}w - c)^2; \quad (I-1) \]
\[ \Pi_l^{(opt, no)} = \frac{1}{9}(\bar{\theta} - 2\theta)^2\Delta V. \quad (I-2) \]
When the Lower-quality Firm Commits to Standard Add-on

Following the equilibrium analysis in the third case of Proposition 2, the equilibrium pricing in the second stage is

\[ P_h^* = \frac{1}{3}(2\tilde{\theta} - \theta)(\Delta V - w) + \frac{1}{3}c, \quad p_h^* = w\hat{\theta}_h^*, \]
\[ P_i^{**} = \frac{1}{3}(\bar{\theta} - 2\tilde{\theta})(\Delta V - w) + \frac{2}{3}c, \]

where the marginal consumers are given by

\[ \hat{\theta}_h^* = \frac{1}{3}(\bar{\theta} + \theta) - \frac{c}{3(\Delta V - w)}, \quad \hat{\theta}_h^* = \frac{1}{2}(\bar{\theta} + \alpha). \]

The incentive constraints that \( \hat{\theta}_h^* > \hat{\theta}_{hl}^* > \theta \) holds given that \( \Delta V > \Delta_2 \). The resulting profits are given by

\[ \Pi_h^{(opt, std)} = \frac{1}{9(\Delta V - w)} \left[ (2\bar{\theta} - \theta)(\Delta V - w) + c \right]^2 + \frac{1}{4w}(\bar{\theta}w - c)^2; \quad (I-3) \]
\[ \Pi_i^{(opt, std)} = \frac{1}{9(\Delta V - w)} \left[ (\bar{\theta} - 2\tilde{\theta})(\Delta V - w) - c \right]^2. \quad (I-4) \]

When the Lower-quality Firm Does Not Commit

Following the equilibrium analysis in the second case of Proposition 2, the equilibrium pricing in the second stage is

\[ P_h^* = \frac{1}{3}(2\tilde{\theta} - \theta)(\Delta V - w) + \frac{1}{3}c, \quad p_h^* = w\hat{\theta}_h^*, \]
\[ P_i^{**} = \frac{1}{3}(\bar{\theta} - 2\tilde{\theta})(\Delta V - w) + \frac{2}{3}c, \quad p_i^* = w\hat{\theta}_i^*, \]

where the marginal consumers are given by

\[ \hat{\theta}_h^* = \frac{1}{3}(\bar{\theta} + \theta) - \frac{c}{3(\Delta V - w)}, \quad \hat{\theta}_h^* = \frac{1}{2}(\bar{\theta} + \alpha), \quad \hat{\theta}_i^* = \frac{1}{2}(\bar{\theta} + \alpha). \]

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The incentive constraints that \( \hat{\theta}_b^* > \hat{\theta}_h^* > \hat{\theta}_l^* > \theta \) holds given that \( \Delta V > \Delta_1 \). The resulting profits are given by

\[
\Pi_h^{(opt, opt)} = \frac{1}{9(\Delta V - w)} \left[ (2\bar{\theta} - \theta)(\Delta V - w) + c \right]^2 + \frac{1}{4w}(\bar{\theta}w - c)^2; \quad (I-5)
\]

\[
\Pi_l^{(opt, opt)} = \frac{1}{9(\Delta V - w)} \left[ (\bar{\theta} - 2\theta)(\Delta V - w) - c \right]^2 + \frac{1}{4w}(\theta w - c)^2. \quad (I-6)
\]

### I-3 When the Higher-quality Firm Commits to Standard Add-on

In this case the higher-quality firm commits to a uniform price \( P_h^+ \) that bundles the base good and the add-on. In the second-stage pricing game, the firm then cannot sell the add-on separately to screen consumers. The equilibrium outcome depends on each of the three commitment choices of the lower-quality firm.

#### When the Lower-quality Firm Commits to No Add-on

The equilibrium pricing in the second stage now becomes

\[
P_h^{++} = \frac{1}{3}(2\bar{\theta} - \theta)(\Delta V + w) + \frac{2}{3}c,
\]

\[
P_l^* = \frac{1}{3}(\bar{\theta} - 2\theta)(\Delta V + w) + \frac{1}{3}c,
\]

and the marginal consumer is given by

\[
\hat{\theta}_h^* = \frac{1}{3}(\bar{\theta} + \theta) + \frac{c}{3(\Delta V + w)}.
\]

The incentive constraints \( \bar{\theta} > \hat{\theta}_h^* > \theta \) hold given that \( \bar{\theta} > 2\theta \). The resulting profits are given by

\[
\Pi_h^{(std, no)} = \frac{1}{9(\Delta V + w)} \left[ (2\bar{\theta} - \theta)(\Delta V + w) - c \right]^2; \quad (I-7)
\]

\[
\Pi_l^{(std, no)} = \frac{1}{9(\Delta V + w)} \left[ (\bar{\theta} - 2\theta)(\Delta V + w) + c \right]^2. \quad (I-8)
\]
When the Lower-quality Firm Commits to Standard Add-on

The equilibrium pricing in the second stage now becomes

\[ P_{h}^{**} = \frac{1}{3}(2\bar{\theta} - \vartheta)\Delta V + c, \]
\[ P_{l}^{**} = \frac{1}{3}(\bar{\theta} - 2\theta)\Delta V + c, \]

and the marginal consumer is given by

\[ \hat{\theta}_{hl}^{*} = \frac{1}{3}(\bar{\theta} + \vartheta). \]

The incentive constraints \( \bar{\theta} > \hat{\theta}_{hl}^{*} > \vartheta \) hold given that \( \bar{\theta} > 2\theta \). The resulting profits are given by

\[ \Pi_{h}^{(std,std)} = \frac{1}{9}(2\bar{\theta} - \vartheta)^2 \Delta V; \]  \hspace{1cm} (I-9)
\[ \Pi_{l}^{(std,std)} = \frac{1}{9}(\bar{\theta} - 2\theta)^2 \Delta V. \]  \hspace{1cm} (I-10)

When the Lower-quality Firm Does Not Commit

The equilibrium pricing in the second stage now becomes

\[ P_{h}^{**} = \frac{1}{3}(2\bar{\theta} - \theta)\Delta V + c, \]
\[ P_{l}^{**} = \frac{1}{3}(\bar{\theta} - 2\theta)\Delta V + c, \]
\[ p_{l}^{*} = w\hat{\theta}_{l}^{*}, \]

where the marginal consumers are given by

\[ \hat{\theta}_{hl}^{*} = \frac{1}{3}(\bar{\theta} + \vartheta), \hspace{1cm} \hat{\theta}_{l}^{*} = \frac{1}{2}(\theta + \alpha). \]
The incentive constraints $\bar{\theta} > \hat{\theta}^*_h > \hat{\theta}^*_i > \bar{\theta}$ hold given that $\bar{\theta} > 2\theta$ and $\bar{\theta} < \alpha < \frac{1}{3}(2\bar{\theta} - \theta)$. The resulting profits are given by

$$\Pi_h^{(std,opt)} = \frac{1}{9}(2\bar{\theta} - \theta)^2 \Delta V;$$  \hspace{1cm} (I-11)

$$\Pi_i^{(std,opt)} = \frac{1}{9}(\bar{\theta} - 2\theta)^2 \Delta V + \frac{1}{4w}(\theta w - c)^2. \hspace{1cm} (I-12)$$

**I-4 First-Stage Commitment Choices**

**The Higher-quality Firm Commits to Standard Add-on**

First, when the lower-quality firm commits to not selling the add-on, the higher-quality firm is better off to commit to standard add-on because the improvement in profit is positive

$$\Pi_h^{(std,no)} - \Pi_h^{(opt,no)} = \frac{w}{9} \left[ (2\bar{\theta} - \theta)^2 - 2\alpha(2\bar{\theta} - \theta) - \frac{9}{4}(\bar{\theta} - \alpha)^2 + \frac{\alpha^2 w}{\Delta V + w} \right] > 0.$$  \hspace{1cm} (I-13)

Second, when the lower-quality firm sells the add-on, the higher-quality firm’s profit is unaffected by whether its rival commits to standard add-on or not. This is because in either case the higher-quality firm is competing with its rival’s bundle good. The firm, however, benefits from committing to standard add-on as long as the quality premium is large enough. The improvement in profit becomes

$$\Pi_h^{(std,opt)} - \Pi_h^{(opt,opt)} = \frac{w}{9} \left[ (2\bar{\theta} - \theta)^2 - 2\alpha(2\bar{\theta} - \theta) - \frac{9}{4}(\bar{\theta} - \alpha)^2 - \frac{\alpha^2 w}{\Delta V - w} \right],$$  \hspace{1cm} (I-13)

which is increasing in the quality premium $\Delta V$ and is positive if

$$\Delta V = (1 + \frac{\alpha^2}{X})w \equiv \Delta_3$$
The Lower-quality Firm Commits to No Add-on

Given that the higher-quality firm commits to standard add-on, it remains to study the three possibilities depending on the lower-quality firm’s commitment choices. First, it is straightforward to see that committing to standard add-on is dominated by not committing, \( \Pi_{l}^{(std, opt)} > \Pi_{l}^{(std, std)} \). This is because the lower-quality firm is selling the add-on to its higher-type consumers anyway, the commitment choice does not matter for the competition with its rival. This is very different from the higher-quality firm’s problem.

Second, selling optional add-on is dominated by committing to not selling any add-on. To see that, the value from commitment is given by

\[
\Pi_{l}^{(std, no)} - \Pi_{l}^{(std, opt)} = \frac{w}{9} \left[ (\theta - 2\theta)^2 + 2\alpha(\theta - 2\theta) - \frac{9}{4}(\theta - \alpha)^2 + \frac{\alpha^2w}{\Delta V - w} \right].
\]

which is also increasing in the quality premium \( \Delta V \). It can be shown that \( Y > 0 \) when \( \theta < \alpha < \frac{1}{3}(2\theta - \theta) \). Therefore, \( \Pi_{l}^{(std, no)} > \Pi_{l}^{(std, opt)} \).

Therefore, there exists an equilibrium of the overall game such that the higher-quality firm commits to standard add-on, while the lower-quality firm commits to no add-on, provided the quality premium is large enough such that \( \Delta V > \max\{\Delta_1, \Delta_3\} \).

J Proof of Proposition 4

For this equilibrium to exist, the incentive constraints have to be satisfied:

(a) \( \tilde{\theta}_{hl}^* > \tilde{\theta}_{l}^* \); (b) \( \tilde{\theta}_{l}^* > \theta \Rightarrow \tilde{\theta}_{hl}^* > 2\theta - \alpha \); (c) \( \tilde{\theta}_{hl}^* > \theta \)
where (a) and (b) imply (c) and (c) is needed only when the lower-quality firm sells the add-on to all consumers. Define the following quantities:

\[
\Delta_0^{(u)} = \frac{\bar{\theta} - 2\theta + \alpha}{\bar{\theta} - 2\theta} \cdot w, \quad \Delta_1^{(u)} = \frac{2\bar{\theta} + \bar{\theta} - 3\alpha}{2(\theta + \theta - 3\alpha)} \cdot w, \quad \text{and} \quad \Delta_2^{(u)} = \frac{2\bar{\theta} - 9\theta + 7\alpha}{2(\bar{\theta} - 5\theta + 3\alpha)} \cdot w
\]

Constraint (a) is equivalent to \( \hat{\theta}^*_h > \alpha \) since \( \hat{\theta}^*_b = \frac{1}{2}(\hat{\theta}^*_h + c) \), and that

\[
\hat{\theta}^*_h - \alpha = \frac{2(\bar{\theta} + \theta - 3\alpha)\Delta V - (2\bar{\theta} + \theta - 3\alpha)w}{6\Delta V - 5w}
\]

This quantity is positive as long as \( \Delta V > \Delta_1^{(u)} \).

Constraint (b) is equivalent to \( \hat{\theta}^*_h > 2\theta - \alpha \) since \( \hat{\theta}^*_b = \frac{1}{2}(\hat{\theta}^*_h + c) \), and that

\[
\hat{\theta}^*_h - (2\theta - \alpha) = \frac{2(\bar{\theta} - 5\theta + 3\alpha)\Delta V - (2\bar{\theta} - 9\theta + 7\alpha)w}{6\Delta V - 5w}
\]

This quantity is positive as long as \( \Delta V > \Delta_2^{(u)} \).

Finally, to see when Constraint (c) holds, note that

\[
\hat{\theta}^{*(u)}_{hl} - \theta = \frac{2(\bar{\theta} - 2\theta)\Delta V - (\bar{\theta} - 2\theta + \alpha)w}{6\Delta V - 5w}
\]

which is positive as long as \( \Delta V > \Delta_0^{(u)} \).

It remains to verify that there is no profitable deviation from either firm.

**J-1 No Profitable Deviation for the Lower-quality Firm**

The only non-local deviation is such that the lower-quality firm does not leave demand for \( H \). Fixing the higher-quality firm’s equilibrium prices, \((P^*_h, p^*_h)\), and given the lower-quality firm’s total price, \( P^{+}_l \), the consumers with \( \theta \in [\bar{\theta}, \hat{\theta}^*_h] \) choose to pay the base price to visit the lower-quality firm at time \( t = 1 \). The marginal consumer is given by

\[
\hat{\theta}^*_{hl} = \frac{p^{+*}_h - P_l^+}{V}
\]
At time $t = 2$, these consumers observe the ad-on price $p_l$ and are faced with the decision to buy it or not. The firm maximizes the ex post profit, $\pi_t = (\hat{\theta}_{hl} - \hat{\theta}_l)(p_l^* - c)$ with $\hat{\theta}_l = p_l^*/w$. Again the optimal strategy by the firm is to set $\hat{\theta}_l$ as a function of the marginal consumer $\hat{\theta}_{hl}$

\[ \hat{\theta}_l(\hat{\theta}_{hl}) = \frac{1}{2}(\hat{\theta}_{hl} + \alpha). \]

The firm's problem at time $t = 0$ is then finding the optimal $\hat{\theta}_{hl}$ that maximizes the total profit taking into account the second-period problem

\[ \max_{\hat{\theta}_{hl}} (\hat{\theta}_{hl} - \bar{\theta})\left(\frac{P_h^{**} - \hat{\theta}_{hl} \Delta V}{h_l} - c\right) - \left(\hat{\theta}_l(\hat{\theta}_{hl}) - \bar{\theta}\right)(w \cdot \hat{\theta}_l(\hat{\theta}_{hl}) - c). \]

The (unconstrained) optimal choice of the marginal consumer is

\[ \hat{\theta}_{hl}^d = \frac{2P_h^{**} - 2c + (2\Delta V + w)\bar{\theta}}{4\Delta V + w}. \]

This deviation requires that no consumer would buy only the base good from the higher-quality firm, that is, $\hat{\theta}_h^d \geq \hat{\theta}_h^*$. However, notice that

\[ \hat{\theta}_h^d - \hat{\theta}_h^* = \frac{2P_h^{**} - 2c + (2\Delta V + w)\bar{\theta}}{4\Delta V + w} - \frac{p_h^*}{w} \]

\[ = -\frac{4(\bar{\theta} - 2\bar{\theta} + 3\alpha)\Delta V^2 - (\bar{\theta} - 2\bar{\theta} - 5\alpha)w\Delta V + \frac{7}{2}(\bar{\theta} - 2\bar{\theta} + \alpha)w^2}{(4\Delta V + w)(6\Delta V - 5w)}, \]

\[ < 0 \]

where the inequality comes from $\Delta V > w$. Therefore, the optimal solution is binding such that $\hat{\theta}_{hl} = \hat{\theta}_h^*$. The corner solution clearly yields a profit that is no greater than the equilibrium profit. The lower-quality firm has no incentive to deviate.
The only non-local deviation that could be profitable is such that the higher-quality firm implements standard add-on policy. Since it is always optimal for the higher-quality firm to set the add-on price, \( p_h^* = \frac{1}{2}(\bar{\theta}w + c) \), that maximizes ex post profit, the deviation is achieved by increasing the base price so that no consumer will buy the base good. In the first case the firm deviates by charging the total price \( P_h^+ = P_h + p_h^* \) at \( t = 0 \) while leaving some demand for \( L^+ \):

\[
\max_{p_h^*} (\bar{\theta} - \frac{P_h^+ - P_i^{**}}{\Delta V})(P_h^+ - c) \tag{J-1}
\]

The optimal deviation is \( P_h^{+d} = \frac{1}{2}(P_i^{++} + \Delta V\bar{\theta} + c) \). The profit from this deviation is \( \Pi_h^d \) is, however, no greater than the equilibrium profit, because

\[
\Pi_h^* - \Pi_h^d = \left[ \frac{P_i^{++} + \bar{\theta}(\Delta V - w)}{4(\Delta V - w)} \right]^2 + \frac{(\bar{\theta}w - c)^2}{4w} - \frac{(P_i^{++} + \bar{\theta}\Delta V - c)^2}{4\Delta V}
\]

\[
= \frac{\left[ w[P_i^{++} + \bar{\theta}(\Delta V - w)] - (\Delta V - w)(\bar{\theta}w - c) \right]^2}{4(\Delta V - w)w\Delta V} \\
\geq 0.
\]

In the second case the firm charges \( P_i^{++} \) such that there is no demand for \( L^+ \)

\[
\max_{P_i^*} (\bar{\theta} - \frac{P_h^+ - P_i^*}{\Delta V + w})(P_h^+ - c) \tag{J-2}
\]
The optimal deviation is \( P^{+d}_h = \frac{1}{2}(P^*_l + \Delta V(\bar{\theta} + w) + c) \). The profit from this deviation is \( \Pi^{d'}_h \) is, however, no greater than the deviation profit in the first case, because

\[
\Pi^{d'}_h - \Pi^d_h = \frac{1}{4\Delta V} \left( \frac{P^{+d} + \bar{\theta} \Delta V - c}{A + B} \right)^2 - \frac{1}{4(\Delta V + w)} \left( \frac{P^*_l + \bar{\theta}(\Delta V + w) - c}{A} \right)^2
\]

\[
= \frac{B^2(\Delta V + w) + 2AB(\Delta V + w) + A^2w}{4(\Delta V + w)\Delta V}
\]

\[
\geq 0
\]

where the inequality holds given that \( A + B > 0 \) and \( B < 0 \). Therefore, there is no profitable deviation for the higher-quality firm.

**K Proof of Proposition 5**

For notational convenience the observed-price case is denoted as “\( o \)” while the unobserved-price case is denoted as “\( u \)”.

**K-1 Market Shares**

The difference in the marginal consumer who is indifferent between the two firms:

\[
\hat{\theta}_{hl}^{(u)} - \hat{\theta}_{hl}^{(o)} = -\left[ (\bar{\theta} - 2\theta)\Delta V - (\bar{\theta} - 2\theta + \alpha)w \right] w \frac{3(\Delta - w)(6\Delta V - 5w)}{3(\Delta - w)(6\Delta V - 5w)} < 0
\]

The inequality follows from \( \Delta V > \Delta_0^{(u)} \). The difference in the marginal consumer who is indifference between buying the add-on and not buying:

\[
\hat{\theta}_{h}^{(u)} - \hat{\theta}_{h}^{(o)} = \frac{1}{2}(\hat{\theta}_{hl}^{(u)} - \theta) = \frac{(\bar{\theta} - 2\theta)\Delta V - (\bar{\theta} - 2\theta + \alpha)w}{6\Delta V - 5w} > 0
\]

The inequality holds if and only if \( \Delta V > \Delta_0^{(u)} \). Therefore the lower-quality firm is also losing consumers who buy the add-on. The fact that the lower-quality firm is losing consumers to higher-quality firm as well as attracting fewer consumers to buy the add-on is precisely because those higher-type consumers of the lower-quality firm
fear being held up by the high add-on price and thus switch to the higher-quality firm without buying the add-on.

**K-2 Prices**

The difference in the base price of the higher-quality firm is given by:

\[ P^{*}_{h} - P^{*}_{o} = \frac{w(\Delta V - w)(\bar{\theta} - 2\theta + 6\alpha)}{3(6\Delta V - 5w)} > 0 \]

Therefore the higher-quality firm charges higher base price when the add-on prices are not observed.

The difference in the add-on price of the lower-quality firm is given by:

\[ p^{*}_{l} - p^{*}_{o} = \frac{1}{2}(\hat{\theta}^{*}_{hl} - \bar{\theta})w = \frac{\left((\bar{\theta} - 2\theta)\Delta V - (\bar{\theta} - 2\theta + \alpha)w\right)w}{6\Delta V - 5w} > 0 \]

The inequality follows from \( \Delta V > \Delta^{(u)} \). Therefore the lower-quality firm charges a higher add-on price under the unobserved-price equilibrium.

Note that under either unobserved-price or observed-price equilibrium, the indifference condition and the higher-quality firm's best response satisfy:

\[
\begin{aligned}
h^{+*} &= P^{*}_{h} - \hat{\theta}^{*}_{hi}(\Delta V - w) \\
&\Rightarrow P^{+*}_{l} = (\Delta V - w)(\bar{\theta} - 2\hat{\theta}^{*}_{hi}) \quad \text{(K-1)}
\end{aligned}
\]

Therefore, the difference in the total price is given by

\[ p^{+*}_{l(u)} - p^{+*}_{l(o)} = 2(\Delta V - w)(\hat{\theta}^{*}_{hi} - \hat{\theta}^{*}_{hi(u)}) > 0 \]
The difference in the base price is given by

\[ P_i^{*}(u) - P_i^{*}(o) = P_i^{++}(u) - P_i^{++}(o) - (p_i^{*}(u) - p_i^{*}(o)) = 2(\Delta V - w)(\hat{\theta}_{hi}^{*}(u) - \hat{\theta}_{hi}^{*}(o)) - \frac{1}{2}(\hat{\theta}_{hi}^{*}(u) - \theta)w \]

\[ = \frac{w[(\bar{\theta} - 2\theta)\Delta V - (\bar{\theta} - 2\theta + \alpha)w]}{3(6\Delta V - 5w)} < 0 \]

where the inequality follows from \( \Delta V > \Delta_0^{(u)} \).

**K-3 Profits**

The higher-quality firm’s profit is clearly increased because it sells the base to more consumers at a higher price while keeping the add-on profit unchanged. For the lower-quality firm, the profit difference is given by

\[ \Delta \Pi_i = \Pi_i^{(u)} - \Pi_i^{(o)} = (\hat{\theta}_{hi}^{*}(u) - \bar{\theta})(P_i^{++}(u) - c) - (\hat{\theta}_{hi}^{*}(u) - \bar{\theta})(p_i^{*}(u) - c) - (\hat{\theta}_{hi}^{*}(o) - \bar{\theta})(P_i^{++}(o) - c) + (\hat{\theta}_{hi}^{*}(o) - \bar{\theta})(p_i^{*}(o) - c) \]

Let

\[ B = \frac{[(\bar{\theta} - 2\theta)\Delta V - (\bar{\theta} - 2\theta + \alpha)w]w}{3(\Delta - w)(6\Delta V - 5w)} \]

Then from the above results imply \( \hat{\theta}_{hi}^{*}(u) = \hat{\theta}_{hi}^{*}(o) - B \), \( \hat{\theta}_{hi}^{*}(u) = \hat{\theta}_{hi}^{*}(o) + 3B(\Delta - w)/w \),

\( P_i^{++}(u) = P_i^{++}(o) + 2(\Delta - w)B \), and \( p_i^{*}(u) = p_i^{*}(o) + 3(\Delta - w)B \). Substituting these quantities into the profit difference yields

\[ \Delta \Pi_i = -\frac{(3\Delta V - 2w)(\Delta V - w)B^2}{w} < 0 \]

the equality follows from \( 3\Delta V > 2w \).
Essay 2

Customer Reviews, Two-sided Learning, and

Endogenous Product Quality
Customer Reviews, Two-sided Learning, and Endogenous Product Quality*

Abstract

This essay examines how firms should develop product and pricing policies when customer reviews provide informative feedback about improving product or service quality. Departing from the existing literature that assumes product quality stays unaffected by abundant customer reviews, I posit that firms can enhance quality based on these reviews, particularly when they appear negative. The fact that customer reviews can provide informative but imperfect quality signals gives rise to a fundamental tension between information – how likely a firm can obtain useful feedback – and efficiency – how much the firm can invest in quality enhancement by changing product design. This trade-off shapes the firm’s product and pricing policies depending on the informativeness of quality signals embedded in customer reviews and the degree to which the firm and consumers are uncertain about the quality. If the prior quality belief is less (more) accurate, the firm will invest less (more) in quality enhancement and price higher (lower) in an early period, as opposed to later. Interestingly, the firm’s profit can be non-monotone in the uncertainty about the product quality. As the prior belief becomes more accurate, the firm finds it less costly to investigate problems behind negative reviews and to develop solutions to them, leading to a greater quality enhancement and thus higher profit. However, as the prior belief becomes very accurate, it becomes much less likely for the firm to obtain useful feedback, and hence its profit may be reduced. Implications for customer-review platforms and firms’ product and pricing policies are discussed.

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1 Introduction

*Just as bitter medicine cures sickness, so unpalatable advice benefits conduct.*

— Chinese proverb

The last decade has witnessed the growing popularity of public customer reviews. Online or mobile platforms such as Yelp, TripAdvisor, eBay, Amazon, or the Apple App Store that aggregate customer reviews on products or services have become an important source for consumers to learn product information. Firms have realized that positive reviews can boost sales, whereas negative reviews can hurt their business quite significantly. A number of research studies have documented evidence for this demand-shifting effect and investigated different underlying mechanisms (e.g., Chevalier and Mayzlin 2006, Li and Hitt 2008, Moe and Trusov 2011, Godes and Silva 2012, Moe and Schweidel 2012, Sun 2012, Anderson and Simester 2014). Managing reputation and word-of-mouth has become an important agenda for online platforms and marketing managers. One strategic tool firms can employ is promotional chats or reviews, which have been found effective and received considerable academic attention (e.g., Dellarocas 2006, Mayzlin 2006, Godes and Mayzlin 2009, Mayzlin et al. 2014).

However, much of the attention, both academic and managerial, has been paid to how customer reviews affect consumer decision making and hence how firms can manipulate reviews to stimulate sales. The basic premise is that firms do not change product or service quality in response to customers’ reviews. Little attention is paid to the fact that firms can benefit from negative reviews that provide useful feedback about how to improve product quality. Anecdotal evidence suggests that firms often take proactive actions on negative reviews by analyzing and resolving potential drawbacks of their products or services to improve quality. For example, restaurants often investigate reviews on Yelp to identify problems with their dining services, such as
foods being too salty or spicy, or waiting taking too long, etc. Many hotel managers actively monitor customer reviews on TripAdvisor and try to improve their services if they receive feedback about potential problems, such as lighting being bad in guest rooms, or the Internet being too slow or too expensive. Many developers of mobile applications constantly inspect user reviews, which can provide information about possible bugs or additional features that customers would like the apps to possess. Such feedback often leads to upgraded versions of their mobile applications.

While whether or not firms improve quality based on customer reviews remains an open empirical question, I present a theoretical analysis to address how firms should develop product and pricing policies when online customer reviews provide informative feedback about quality improvement. Departing from the standard assumption in the literature that product quality stays unaltered by abundant customer reviews, I posit that firms instead can enhance product quality based on these reviews, particularly when they appear negative.

Specifically, this essay builds on the fundamental feature of customer review systems that reviews provide imperfect but potentially informative quality signals about a product. This feature has two implications. First, firms do not always find reviews trustworthy or informative. As suggested in the aforementioned examples, informative feedback that is helpful for firms to improve quality often arises when reviews appear negative relative to consumers' expectation. This phenomenon, however, is far more general than in the setting of customer reviews. Casual observations suggest that honest advice is often unpalatable to hear. Examples include parents' advice to children when they have done things wrong, and reviewers' criticisms of research papers when they do not meet academic standards. Furthermore, informative feedback becomes more likely to arise when consumers are more uncertain about product quality, because they may have heterogeneous tastes or simply disagree on the perception of quality. Second, it is costly for the firms to investigate the problems giving rise to
a negative review, and to develop and implement solutions to them. The labor, mate-
rial, or opportunity costs involved in the process of changing product design increase
when the firms are more uncertain about the product quality. As a result of this
fundamental tension between information and efficiency, optimal product and pricing
policies depend crucially on the informativeness of the quality signals embedded in
customer reviews and the degree to which the firms and consumers are uncertain
about the quality.

If the prior quality belief is less accurate, a firm will invest less in quality en-
hancement and price higher in an early period, as opposed to later, waiting for more
customer feedback. If the prior belief is more accurate, then the firm will invest
more in the early period so that it can profit from the higher quality. There is also
a reduction in price so as to generate more sales which result in a higher chance of
getting trustworthy and informative review. Interestingly, the firm’s profit can be
non-monotone in the firm and consumers’ uncertainty about quality. As the prior
belief becomes more accurate, the firm finds it less costly to investigate problems
behind negative reviews and to develop solutions to them, leading to a greater qual-
ity enhancement and thus higher profit. However, as the prior belief becomes very
accurate, it becomes much less likely for the firm to obtain useful feedback, and hence
profit may fall.

Understanding the informational role of customer reviews has important impli-
cations both for online platforms’ design of review systems and for firms’ product
and pricing policies. It is well documented that customer reviews provide poten-
tially useful information for consumers who are new to a product or market, and
hence instrumental in their decision making. A positive review is likely to raise new
consumers’ expectation about the quality of a product; they will then become more
likely to try out the product. But reviews are not always trustworthy. They are
potentially biased in a number of ways, due to heterogeneous tastes of consumers (Li
and Hitt 2008), costs of writing reviews (Godes and Silva 2012), reciprocity (Bolton et al. 2013), retaliation (Horton and Golden 2015), unusual consumers (Anderson and Simester 2014), firms’ manipulation (Mayzlin et al. 2014), etc. Many consumers are not entirely aware of these biases and hence make sub-optimal decisions. This further incentivizes firms to produce fake positive reviews\(^1\) or threaten consumers who give negative reviews.\(^2\) These strategic manipulations of online reputation, though effective in the short run, could lead to market inefficiency in the long run (Nosko and Tadelis 2015).

This essay advocates an alternative, complementary view of customer reviews such that they not only can help consumers learn about product quality, but also can help firms learn about problems with their products or services, enabling them to improve quality. In contrast to the widespread view that negative reviews are always harmful to firms, I show that some amounts of unpalatable reviews may actually benefit the firms. It is in the firms’ interest to collect “wisdom of the crowd” from these negative reviews when the product quality is relatively uncertain in the market. Firms may become better off even when these reviews are noisy signals of quality. When consumers develop more confidence in the product, it is likely that firms will become worse off because of the lack of useful feedback.

Seller reputation dynamics has long been an important topic in information economics. Seminal papers by Kreps and Wilson (1982), Kreps et al. (1982), and Milgrom and Roberts (1982) suggest that, even when there is a small probability that a seller is a good type who offers a high quality in every period (and thus is irrational), a bad-type seller might find it optimal to imitate the good type over a long horizon, trying to convince buyers that he or she is the good type. This essay, however, differs

\(^1\)Some firms hire online reputation-management agencies to publish reviews filled with positive news (See a WSJ report: http://on.wsj.com/1w3trCb). Many firms often reward customers who give positive reviews by offering discounts for their products or other benefits.

\(^2\)For example, a hotel in New York fine wedding parties 500 for any negative online reviews, according to a CNN report (http://www.cnn.com/2014/08/04/travel/bad-hotel-review-fine-backlash/).
from that literature in two important ways. First, instead of assuming that firms can choose any quality type costlessly in every period and that higher quality is suboptimal for the firms, I focus on the scenario where higher quality is optimal for the firms but is costly to maintain. Once a product design has been changed that leads to a higher quality, it carries over for a long horizon, providing a strong incentive for the firms to improve quality. Second, instead of assuming that firms are better informed about quality and hence can change product design to signal their type to consumers, I focus on the case where both the firms and consumers possess the same level of information concerning some product component with uncertain quality.\(^3\) The quality of this product component, or how much consumers may like it, is not completely known by either the firm or the consumers when it is introduced to the market. The consumers may be able to resolve the uncertainty through their own experience of using the product. The advent of online platforms allows consumers to reveal such information to the public (both firms and consumers), thereby providing useful feedback to firms. In this sense, the essay adds to the growing literature that studies the implications of two-sided learning (e.g., Bergemann and Välimäki 1997, Bonatti 2011).

The rest of the essay is structured as follows. Section 2 introduces a model that endogenizes a firm’s product quality based on customer reviews. Section 3 characterizes the equilibrium product and pricing policies. Section 4 studies how the firm’s equilibrium profit changes with respect to the uncertainty about product quality and the noisiness of quality signals embedded in customer reviews. Section 5 concludes the essay.

\(^3\)This assumption needs not suggest that the firms do not have better knowledge about the product than the consumers do. In reality, firms often do have better information. The purpose of this analysis is to abstract from the possibility that firms may signal quality using prices because of being better informed, and focus on the residual uncertainty that both the firms and consumers may have.
2 Model

Consider a monopolist selling an experience good to consumers over multiple periods. Initially the product has an unknown quality of $q_0$ with a common prior $\mathcal{N}(\mu_0, \sigma_0)$, distributed normally and shared by both the firm and the consumers. For notational convenience, I will use the inverse of the standard deviation, $\tau_t = 1/\sigma_t$, to denote the precision or accuracy of the quality belief in each period $t$. The marginal cost is normalized to zero. Each consumer is characterized by $\theta \in [0, 1]$ from a uniform distribution. The consumers live for only one period, but there are new consumers of the same size entering the market in every period. In each period, the consumers decide whether to buy it or not given the price $p_t$, and, if they buy the product, they can get a utility of $U_t = \theta q_t - p_t$.

2.1 Review Generating Process

Given the price set by the firm, a number of $d_t = 1 - p_t/\mu_t$ consumers will choose to buy the product, where $\mu_t$ is the expected quality according to the prior or posterior belief. Upon purchasing (and consuming) the product, some consumers may write a review. However, not all reviews are trustworthy. There is abundant empirical evidence that reviews may be fake, either by firms (Mayzlin et al. 2014), or by consumers who are angry or emotional, or by consumers who have not even purchased the products (Anderson and Simester 2014). In light of this phenomenon and without loss of generality, I assume that with probability $h(d_t, \mu_t) \in [0, 1)$, one consumer will write a truthful review about the product. It is not unreasonable to assume $h(d_t, \mu_t)$ increases with $d_t$. That is, the more consumers buy the product, the more likely a truthful review is generated. The following functional-form assumption captures this idea and will prove useful for tractability

$$h(d_t, \mu_t) = \lambda \mu_t [1 - (1 - d_t)^2].$$

(1)
Notice that \( h(0, \mu_t) = 0 \) and \( h(1, \mu_t) = \lambda \mu_t \), where \( \lambda \) is a scaling constant. It is assumed that \( \lambda \) is positive and sufficiently small to ensure that \( h(1, \mu_t) < 1 \).\(^4\) The idea is that, even when all consumers purchase the product, it is not guaranteed that there is any truthful review. Moreover, the function \( h(d_t, \mu_t) \) increases with the expectation \( \mu_t \) for every \( d_t \). The higher \( \mu_t \), the more likely a truthful review will be available. While this assumption is not important in driving the results, it keeps the analysis tractable and is consistent with recent empirical findings that consumers with more positive experience are more likely to review products whereas unsatisfied consumers are less likely to leave reviews (Dellarocas and Wood 2008, Nosko and Tadelis 2015). The review provides a signal, \( x_t \), about the quality such that

\[
x_t = q_t + \epsilon_t, \quad \text{where } \epsilon_t \sim \mathcal{N}(0, 1/\tau_t).
\]  

(2)

The noise \( \epsilon_t \) reflects either idiosyncratic tastes in the population, or consumers’ disagreement on the underlying quality. All parties observe the signal and update their belief about product quality \( q_t \). The posterior belief is passed to the new consumers, so that all parties possess the same level of information. This is not an unreasonable assumption given that, on most online marketplaces or platforms, reviews and ratings are publicly available to all consumers.

A review can be either positive or negative, relative to the market’s expectation. However, only a negative one contains information about quality issues that may motivate the firm to investigate and improve. To capture this idea I assume that when \( x_t \leq \mu_t - m \), where \( m > 0 \) is a constant, the review provides constructive feedback. If \( m \) is larger, then the review is less diagnostic and hence it is less likely that a helpful feedback is available. However, when \( x_t > \mu_t - m \), the review does not contain any information for the firm to improve quality, although the review

\(^4\)For the results derived in this essay, it is sufficient, though not necessary, that \( \lambda < 34mc \), where \( m \) and \( c \) are to be defined later. I will make this assumption throughout the essay.
might provide a positive quality signal (when \( x_t > \mu_t \)) leading to a higher posterior expectation.

### 2.2 Quality Enhancement

Once the firm has investigated the problem revealed by a negative review, it then decides how much to enhance the quality additively by an amount of \( w_t \) such that

\[
q_{t+1} = q_t + w_t. \tag{3}
\]

The underlying change in product design for a quality enhancement is enduring. For example, hotels may change the lighting of all guest rooms because this is what many guests have recommended, or a mobile game application may fix a bug after many users have complained about it. These changes, once they are integrated, will become an indivisible part of the product. The quality enhancement, however, is costly. Investigating the problem as well as improving the quality costs the firm an amount of

\[
C(w_t, \tau_t) = \frac{c}{2} \cdot \frac{1}{\tau_t} \cdot w_t^2. \tag{4}
\]

The cost function is assumed to be convex in \( w_t \) and increasing with the market-level uncertainty, \( 1/\tau_t \). If there is more uncertainty about the quality belief, then the firm needs to exert more efforts to investigate the underlying problems, and to develop and implement solutions to them. The costs could be any opportunity costs, material costs, or labor costs involved in this process. This assumption will play an important role in driving the results. The immediate implication is that, when \( \tau_t \) is smaller, the firm finds it more costly, or less efficient, to improve quality. In addition, I assume that the quality enhancement requires only a lump-sum investment and has no effect on the marginal cost of the production.
2.3 Learning Process

At the beginning of each period $t + 1$, the posterior belief is assumed to be updated in a Bayesian fashion after the signal $x_t$ is obtained such that

$$
\mu_{t+1} = \rho_t \mu_t + (1 - \rho_t)x_t + w_t, \quad \text{and} \quad \tau_{t+1}^2 = \tau_t^2 + \tau^2_c,
$$

where $\rho_t = \tau_t^2/(\tau_c^2 + \tau_t^2)$. Notice that when the firm has enhanced the quality by an amount of $w_t$, the posterior belief changes accordingly. Because the quality enhancement involves only additive changes, the updated posterior mean increases by an amount of $w_t$, whereas the posterior uncertainty remains unchanged. If the firm has not enhanced the quality, then $w_t = 0$ and the posterior mean reduces to the standard belief updating.

The posterior belief is also common knowledge to all parties. This assumption reflects the nature of many online mechanisms which attempt to aggregate public opinions. Both the firm and new consumers can learn about the information from the reviews or ratings of earlier consumers.

2.4 Timing

The game has three time periods. In each of the first two periods, both the firm and the new set of consumers have the same prior belief or updated the posterior belief ($\mu_t, 1/\tau_t$). The firm first sets the price $p_t$. The consumers then decide whether to buy from the firm or not. After the purchase decision, a trustful review is generated with probability $h_t$ and provides a signal $x_t$ about the product quality. When the signal is negative such that $x_t \leq \mu_t - m$, it triggers the firm to invest in improving the quality leading to an enhancement of $w_t$. In the last period, the firm only needs to set the price based on the market-level posterior belief. Figure 1 visualizes the timing of the game.
3 Equilibrium Policies

3.1 At $t = 2$

The game is solved backward. At $t = 2$, the firm determines the price $p_2$ to maximize profit given the updated belief $\mu_2 = \rho_1 \mu_1 + (1 - \rho_1)x_1 + w_1$

$$\max_{p_2} (1 - \frac{p_2}{\mu_2})p_2.$$ 

The profit-maximizing price, $p_2^*(\mu_2) = \frac{1}{2}\mu_2$, is a function of the posterior belief, $\mu_2$. The resulting optimal profit is $\pi^*(\mu_2) = \frac{1}{4}\mu_2$. Therefore, higher quality is more desirable, motivating the firm to improve its product quality.
3.2 At $t = 1$

At the end of period $t = 1$, anticipating the future price and profit at $t = 2$, the firm decides how much to enhance the quality

$$\max_{w_1} \pi^*(\rho_1 \mu_1 + (1 - \rho_1)x_1 + w_1) - C(w_1, \tau_1).$$

The optimal level of investment is

$$w_1^* = \frac{\tau_1}{4c}.$$  \hspace{1cm} (6)

At the beginning of this period, given the updated belief $\mu_1 = \rho_0 \mu_0 + (1 - \rho_0)x_0 + w_0$, the firm determines the price to maximize profit:

$$\max_{p_1} (1 - \frac{p_1}{\mu_1})p_1 + \mathbb{E}\left[ (1 - h(d_1(p_1, \mu_1))) \cdot \pi^*(\mu_1) + h(d_1(p_1, \mu_1)) \right. \cdot \left. (\mathbb{I}\{x_1 > \mu_1 - m\} \pi^*(\mu_2|w_1 = 0) + \mathbb{I}\{x_1 \leq \mu_1 - m\} \{\pi^*(\mu_2|w_1 = w_1^*) - C(w_1^*, \tau_1)\} \right]_{\mu_1, \tau_1}.$$  

Using the fact that $\mathbb{E}[\pi^*(\mu_2|w_1 = 0)|\mu_1] = \pi^*(\mu_1)$ because $\pi^*$ is linear in $\mu$, and rearranging terms, we can simplify the above equation as

$$\max_{p_1} (1 - \frac{p_1}{\mu_1})p_1 + \pi^*(\mu_1) + h(d_1(p_1, \mu_1))$$

$$\cdot \int_{x_1 \leq \mu_1 - m} \left( \pi^*(\mu_2|w_1 = w_1^*) - \pi^*(\mu_2|w_1 = 0) - C(w_1^*, \tau_1) \right) dN(x_1|\mu_1, 1/\tau_1),$$  \hspace{1cm} (7)

where $R_1(\mu_1, \tau_1)$ measures the expected future value of obtaining useful feedback relative to the value of no useful feedback. Substituting the optimal enhancement $w_1^*$ obtained from Equation 6 into the above expression yields

$$R_1(\mu_1, \tau_1) = \Phi(-m \tau_1) \cdot \frac{\tau_1}{32c},$$  \hspace{1cm} (8)

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.
(with mean of zero and standard deviation of one). It is worth noting that the function \( R \) is independent from \( \mu \), because the probability of obtaining critical and useful feedback \textit{conditional} on the posterior belief, \( P(x_1 \leq \mu_1 - m|\mu_1, 1/\tau_1) = \Phi(-m\tau_1) \), is not a function of the expectation. This is due to the distributional assumption that \( x_1 \) is normally distributed. This simplifies the solution and allows us to focus on the effect of posterior precision \( \tau_1 \). In particular, the following result holds.

**Lemma 1.** There exists a threshold \( \tau^* = \gamma/m \) such that, (a) \( R_1 \) decreases with \( \tau_1 \) if \( m > m^* = \gamma/\tau \); (b) \( R_1 \) first increases in \( \tau_1 \in (0, \tau^*) \), but then decreases in \( \tau_1 \in [\tau^*, +\infty) \) if \( m < m^* \).

The relative value of informative customer feedback, \( R_1 \), is shaped by two opposing forces. To see that, differentiating \( R_1 \) with respect to \( \tau_1 \) yields

\[
\frac{\partial R_1}{\partial \tau_1} = \frac{1}{32c} \left( \Phi(-m\tau_1) - m\tau_1\phi(-m\tau_1) \right). \tag{9}
\]

An increase in \( \tau_1 \) leads to a higher future profit by enhancing the quality. However, a higher \( \tau_1 \) results in a lower chance of getting informative feedback. Which effect dominates depends on the size of \( m\tau_1 \). Using the facts that \( \Phi(-m\tau_1) = 1 - \Phi(m\tau_1) \) and \( \phi(-m\tau_1) = \phi(m\tau_1) \) by symmetry, where \( \phi(\cdot) \) denotes the probability density function for the standard normal distribution, we have the following relationship

\[
\frac{\partial R_1}{\partial \tau_1} > 0 \iff m\tau_1 - \frac{1 - \Phi(m\tau_1)}{\phi(m\tau_1)} < 0. \tag{10}
\]

The expression on the right hand side resembles the well-known “virtual value” in the mechanism design literature. Under regular distributional assumptions (satisfied for the normal distribution), this expression is increasing in \( m\tau_1 \) and equals to zero at \( m\tau_1 = \gamma \), where \( \gamma \approx 0.75 \) is a positive constant less than one standard deviation. It can be shown that for any \( m > m^* = \gamma/\tau \), \( \partial R_1/\partial \tau_1 < 0 \) always holds. Intuitively,
when reviews are not diagnostic (\(m\) very large), it is very unlikely for the firm to obtain helpful feedback. Therefore, an increase in precision has a more profound impact on lowering the likelihood of useful information. But when reviews are more diagnostic, \(m < m^*\), whether the efficiency or informational effect dominates depends on the precision of the belief. For less accurate belief, \(\tau_1 \in (\tau_e, \gamma/m)\), the efficiency effect dominates and hence \(\partial R_1/\partial \tau_1 > 0\). In contrast, for more accurate belief, \(\tau_1 \geq \gamma/m\), the informational effect dominates and hence \(\partial R_1/\partial \tau_1 < 0\). The relative value \(R_1\) reaches the maximum when \(m \tau_1\) is approximately three quarters of one standard deviation.

Given the relative future value, the firm sets the optimal price

\[
p_1^* = \frac{1}{2\lambda R_1 + 2\mu_1}.
\]

Clearly, the optimal price is lower than what the firm would have charged if there were no feedback (i.e., \(\frac{1}{2\mu_1}\)) since \(R_1 > 0\). Furthermore, the price drops as \(R_1\) increases. This is due to the fact that a lower price can generate more demand which leads to a higher chance of obtaining a truthful review and thus quality improvement. The following proposition summarizes the behavior of optimal policies in this period.

**Proposition 1.** At time \(t = 1\), the optimal policies exhibit the following behavior

1. Quality enhancement \(w_1^*\) increases with posterior precision \(\tau_1\);

2. Price \(p_1^*\) (a) increases with posterior expectation \(\mu_1\); (b) increases with posterior precision \(\tau_1\) if \(m \geq m^*\); (c) first decreases but then increases with \(\tau_1\) if \(m < m^*\).

The intuitive fact that the product quality improves more with a higher posterior precision follows immediately from the expression for the optimal enhancement in Equation 6. Because there is no further quality enhancement based on customer reviews in the future, only the present value of quality enhancement matters. When
the market belief about quality becomes less uncertain, it is more efficient for the firm to identify and resolve the problems given the available review. The second result follows immediately from the fact that the optimal price, \( p_1 \), decreases monotonically with \( R_1(\tau_1) \), and Lemma 1.

The optimal product and pricing policies lead to a total profit, starting from \( t = 1 \), of

\[
V_1(\mu_1) = \left( \lambda R_1 + \frac{1}{4\lambda R_1 + 4} + \frac{1}{4} \right) \mu_1. \tag{12}
\]

3.3 At \( t = 0 \)

At the end of period \( t = 0 \), anticipating the future price and profit at \( t = 1 \), the firm decides how much to invest in enhancing the quality

\[
\max_{w_0} V_1(\rho_0 \mu_0 + (1 - \rho_0)x_0 + w_0) - C(w_0, \tau_0).
\]

The optimal enhancement becomes

\[
w_0^* = \left( \lambda R_1 + \frac{1}{4\lambda R_1 + 4} + \frac{1}{4} \right) \mu_0. \tag{13}
\]

Notice that, unlike the second-period quality enhancement, the initial-period enhancement takes into account of the effect of customer feedback on future quality improvement, captured by \( R_1 \). A higher \( R_1 \) induces the firm to invest more in the initial period. Given this product quality choice, the firm determines the initial price for the product at the beginning of this period

\[
\max_{p_0} (1 - \frac{p_0}{\mu_0})p_0 + \mathbb{E} \left[ (1 - h(d_0(p_0, \mu_0))) \cdot V_1(\mu_0) + h(d_0(p_0, \mu_0)) \cdot (1 \{x_0 > \mu_0 - m\} V_1(\mu_1|w_0 = 0) + 1 \{x_0 \leq \mu_0 - m\}(V_1(\mu_1|w_0 = w_0^*) - C(w_0^*, \tau_0)) \right] | \mu_0, \tau_0.
\]
Again, we can simplify the above equation as

\[
\max_{p_0} \left(1 - \frac{p_0}{\mu_0}\right)p_0 + V_1(\mu_0) + h(d_0(p_0, \mu_0)) \cdot \int_{x_0 \leq \mu_0 - m} \left(V_1(\mu_1|w_0 = w_0^*) - V_1(\mu_1|w_0 = 0) - C(w_0^*, \tau_0)\right) dN(x_0|\mu_0, 1/\tau_0),
\]  

where \( R_0(\mu_0, \tau_0) \) is the expected future value of obtaining useful feedback relative to the value of no useful feedback. Substituting the optimal enhancement, \( w_0^* \), from Equation 13 into the above expression gives

\[
R_0(\mu_0, \tau_0) = \Phi(-m\tau_0) \cdot \left(\lambda R_1 + \frac{1}{4\lambda R_1 + 4} + \frac{1}{4}\right)^2 \tau_0 \frac{s}{2c}. \tag{15}
\]

This relative value is different from that in the second period. In the initial period, the firm has to internalize the effect of customer feedback on future policies and profit. A higher \( R_1 \) motivates the firm to invest more in quality enhancement, thereby increasing the value of getting customer feedback in the initial period. However, similar to \( R_1 \), the value of \( R_0 \) is also driven by the two competing forces, one arising from the information effect and the other from the efficiency effect.

**Lemma 2.** There exists a threshold \( \tau^{**} < \gamma/m \) such that, \( R_0 \) first increases in \( \tau_0 \in (0, \tau^{**}) \), but then decreases in \( \tau_0 \in [\tau^{**}, +\infty) \).

The pattern of \( R_0 \) is very similar to that of \( R_1 \), except that it is always non-monotone in \( \tau_0 \). This is due to the fact that the prior precision, \( \tau_0 \), can start out being very small, resulting in a very small relative future value \( R_0 \). Therefore, increasing \( \tau_0 \) leads to a larger efficiency effect for small \( \tau_0 \)'s, implying that \( R_0 \) becomes greater. However, the later-period precision, \( \tau_1 \), is bounded below by \( \tau \), after a noisy signal is incorporated. It is possible that \( \tau_1 \) is large enough that the informational effect dominates, and therefore increasing \( \tau_1 \) leads to a lower \( R_1 \).
Taking into account the relative future value, the firm sets the price as

\[ p_0^* = \frac{1}{2\lambda R_0 + 2\mu_0}. \]  

The following proposition summarizes the properties of the optimal product and pricing policies in the initial period.

**Proposition 2.** At time \( t = 0 \), the optimal policies exhibit the following behavior

1. Quality enhancement \( w_0^* \) increases with prior precision \( \tau_0 \);

2. Price \( p_0^* \) (a) increases with prior expectation \( \mu_0 \); (b) first decreases but then increases with prior precision \( \tau_0 \).

First, the fact that the initial-period enhancement \( w_0^* \) increases with the precision of the prior belief is quite intuitive but not at all obvious \( a \ priori \). There is a dynamic effect that increasing \( \tau_0 \) might lead to a lower probability of getting informative feedback in the future and hence \( R_1 \) in the next period might become smaller. As a result, the firm might want to lower today's quality enhancement. However, because the future is uncertain, efficiency concern dominates when the firm is deciding how much to invest in enhancing the quality today. Second, the pricing behavior is similar to that in the second period, except that it is always non-monotone in \( \tau_0 \) due to the behavior of \( R_0 \) in Lemma 2.

Comparing the first- versus second-period policies, we have the following result.

**Proposition 3.**

1. There exists a threshold \( \tau^w \) such that, (a) \( w_0^* < w_1^* \) when \( \tau_0 < \tau^w \); (b) \( w_0^* > w_1^* \) when \( \tau_0 > \tau^w \).

2. There exists a threshold \( \tau^p \) such that, (a) \( p_0^* > p_1^* \) when \( \tau_0 < \tau^p \); (b) \( p_0^* < p_1^* \) when \( \tau_0 > \tau^p \).

Intuitively, the initial-period enhancement \( w_0^* \) can start out being very small because of the lack of efficiency when the prior belief is very uncertain (i.e., \( \tau_0 \) very
small). It is optimal for the firm to wait for more customer feedback. This leads to a very low relative future value $R_0$ and thus a relatively high initial price $p^*$. However, if the prior belief is more accurate, it is more efficient for the firm to invest more in the initial period. Since customer review in the later period provides an additional quality signal, the posterior quality belief will become even more accurate. It becomes less likely to obtain useful feedback in the later period, reducing the benefit of having additional customer feedback. Therefore, the firm is better off to invest more in improving product quality as well as to price lower in the early period.

The total profit given the optimal product and pricing policies now becomes

$$V_0(\mu_0) = \left( \lambda R_0 + \frac{1}{4\lambda R_0 + 4} \right) \mu_0 + V_1(\mu_0),$$

where $V_1(\mu_0)$ is obtained by replacing $\mu_1$ with $\mu_0$ in Equation 12.

4 Profitability

In this section, I investigate the profitability of the firm given the equilibrium policies. Evidently, the firm’s profit always increases with the prior expectation $\mu_0$ according to Equation 17. This is consistent with many empirical findings in the existing literature. The main contribution of this essay, however, is to uncover how firms’ profit can be improved in the presence of negative customer reviews. The primary interests are how equilibrium profit changes with respect to the precision of prior belief, $\tau_0$, and to the precision of the review mechanism, $\tau_e$.

Proposition 4. There exists a threshold $\tau^V$ such that, as the precision of the prior belief $\tau_0$ increases, $V_0$ first increases in $(0, \tau^V)$ but then decreases in $[\tau^V, +\infty)$.

Recall that there are two competing forces when the precision of prior belief becomes larger. The first effect stems from the decreased likelihood of obtaining negative but useful information. That is, $Pr(x_t \leq \mu_t - m|\mu_t, 1/\tau_t) = \Phi(-m \tau_t)$ becomes
smaller. However, when the precision of prior belief increases, it is more efficient for the firm to improve quality. Therefore, the firm invests more efforts in improving the quality. As a result of the trade-off between these two effects, if \( m \) is not too large, then Lemmas 1 and 2 suggest that the relative future values in both periods, \( R_0 \) and \( R_1 \), first increase but then decrease as \( \tau_0 \) becomes larger. Consequently, the prices in both periods, \( p_0^* \) and \( p_1^* \), first decrease but then increase. Hence, the demand in each period first increases but then decreases, and thus the probability of obtaining a truthful review first increases but then decreases, further strengthening the informational effect. Therefore, the expected profit first grows but then drops, despite the fact that prices fall initially. If \( m \) is too large, then according to Lemma 1, \( R_1 \) decreases with \( \tau_1 \) and thus decreases with \( \tau_0 \). This might reduce \( V_0 \) through \( V_1 \). But this effect is quite limited because when \( \tau_0 \) is very small, an increase in \( \tau_0 \) results in only a very small increase in \( \tau_1 \). Figure 2 provides an example of the equilibrium profit as a function of the prior precision.  

\[ V_0 \]

Figure 2: Profit as a Function of Prior Precision: an Example

\[ \begin{align*}
V_0 & \quad \tau_0 \\
0.86 & \quad 0.82 \\
0.82 & \quad 0.78 \\
0.78 & \quad 0.74 \\
0 & \quad 5 \\
5 & \quad 10 \\
10 & \quad 15 \\
15 & \quad 20 \\
20 & \quad 25 \\
25 & \quad 0
\end{align*} \]

\[ \text{Prior precision } \tau_0 \]

**Proposition 5.** There exist a threshold \( m^c = \gamma/\tau_0 \) such that,

\[ m^c = \frac{\gamma}{\tau_0} \]

5The parameters of the example are set as follows: \( m = 0.1, c = 1, \lambda = 0.5, \tau_c = 1, \mu_0 = 1. \]
1. if $m \geq m^c$, $V_0$ decreases with $\tau_\epsilon$;

2. if $m < m^c$, $V_0$ first increases with $\tau_\epsilon \in (0, \gamma/m)$ but then decreases with $\tau_\epsilon \in [\gamma/m, +\infty)$.

The intuition of this result is very similar to the case where $\tau_0$ is varied while $\tau_\epsilon$ is fixed. The effect of $\tau_\epsilon$ on the firm's profit is entirely attributed to its effect on the precision of posterior belief, $\tau_1$. When the quality signals from customer reviews are more accurate, the posterior belief will also become more accurate. Therefore, the firm has to face the fundamental tradeoff between efficiency and information in the second period. Following Lemma 1, if reviews are not diagnostic, $m > m^c$, then the relative value of quality enhancement, $R_1$, always decreases with $\tau_1$ and thus with $\tau_\epsilon$. This is because an increase in precision has a more profound impact on lowering the likelihood of obtaining critical and useful feedback. However, if reviews are diagnostic, $m < m^c$, then for less accurate belief, the efficiency effect dominates and hence $R_1$ increases with $\tau_1$ and thus with $\tau_\epsilon$, whereas for more accurate belief, the informational effect dominates and hence $R_1$ decreases with $\tau_1$ and thus with $\tau_\epsilon$. Furthermore, a higher $R_1$ will also motivate the firm to invest more in the initial period (i.e., $w^*_0$ is greater), without affecting the efficiency concern (because changing $\tau_\epsilon$ does not change $\tau_0$), and thus $R_0$ increases as well.

The result that the firm's profit may drop with a higher precision of the quality signals embedded in customer reviews may appear counter-intuitive at first glance, contradicting what conventional wisdom would have predicted. However, it highlights the benefits of negative reviews. If a review provides a noisier signal about product quality, it is more likely that a negative but useful review will arise. This feedback is particularly beneficial to the firm in the long run because of possible quality enhancement. Propositions 4 and 5 together provide an alternative view of the effects of review mechanisms on profitability. While the existing literature and managerial practices predominantly focus on how negative reviews can hurt sales and
profit through lowering quality expectation, the above results suggest a more positive role of negative reviews and a more prominent role of uncertainty about quality belief. Under certain market conditions, firms may be better off if they distill the constructive feedback from these negative reviews.

5 Concluding Remarks

Departing from the existing literature that assumes product quality stays unaffected by abundant customer reviews, in this essay I investigate how firms can enhance quality based on customers’ reviews, particularly when reviews provide informative feedback. There appear to be a fundamental tension between information – how likely a firm can obtain useful feedback – and efficiency – how much the firm can invest in quality enhancement by changing product designs. This trade-off impacts on firms’ product and pricing policies depending on the informativeness of quality signals embedded in customer reviews and the degree to which firms and consumers are uncertain about product quality. It is shown that a firm’s profit can be non-monotone in the uncertainty about product quality.

This essay advocates an alternative view of customer reviews that they not only can help consumers learn about product quality, but also can help firms learn about problems with their products or services, enabling them to improve quality, an outcome desired by both firms and consumers. I explore the implications of this perspective for firms’ product and pricing policies, focusing on the impact of firms and consumers’ uncertainty about product quality. There are limitations about this work and there is room for future work to address these issues. First, there is little empirical work on how informative reviews may affect firms’ product and pricing policies, although casual empiricism suggests there might be interesting relationships and the theoretical analysis in this essay provides one possible mechanism. Second, there are a number of ways that consumer decision making is more complex than the one I
model here. Future work can investigate the situations when consumers make repeat purchases or when some consumers are myopic. Last but not least, I have restricted the analysis to the monopoly setting as a natural first step. Understanding the role of informational feedback in competitive markets seems to be a potentially fruitful area.
References


Appendix

Proof of Lemma 1

*Proof.* Notice that the “virtual value”

\[ v(m\tau_1) = m\tau_1 - \frac{1 - \Phi(m\tau_1)}{\phi(m\tau_1)}, \]

is increasing in \( \tau_1 \) and \( m \), and \( v(0) < 0 \). The unique solution to \( v(m\tau_1) = 0 \) is \( \gamma \). Therefore, as long as \( m\tau_1 < \gamma \), we have \( v(m\tau_1) < 0 \) and hence \( \frac{\partial R_1}{\partial \tau_1} > 0 \). If, however, \( m\tau_1 > \gamma \), we have \( v(m\tau_1) > 0 \) and hence \( \frac{\partial R_1}{\partial \tau_1} < 0 \).

Since \( \tau_1 = \sqrt{\tau_0^2 + \tau_\epsilon^2} > \tau_\epsilon \), if \( m > \gamma/\tau_\epsilon \), then \( m\tau_1 > m\tau_\epsilon > \gamma \). It follows that \( v(m\tau_1) > 0 \) and hence \( \frac{\partial R_1}{\partial \tau_1} < 0 \) always hold. If \( m < \gamma/\tau_\epsilon \), then \( \tau_1 \) can be greater or smaller than the cutoff \( \gamma/m \). If \( \tau_1 < \gamma/m \), then \( \frac{\partial R_1}{\partial \tau_1} > 0 \). If \( \tau_1 > \gamma/m \), then \( \frac{\partial R_1}{\partial \tau_1} < 0 \).

\[ \]  

Proof of Proposition 1

*Proof.* The results in 1.1 and 1.2(a) are evidence from the expressions of \( w_1^* \) and \( p_1^* \).

The statements in 1.2(b) and 1.2(c) follow immediately from Lemma 1 and the fact that \( p_1^* \) is monotonically decreasing in \( R_1 \).

\[ \]  

Proof of Lemma 2

*Proof.* The following lemma is useful for the proof. Let \( N = \tau_0 \Phi(-m\tau_0)/2c \), then

**Lemma 3.** (a) For all \( \tau_0 \in (0, \gamma/m) \), \( \frac{\partial N}{\partial \tau_0} > 0 \); for all \( \tau_0 > \gamma/m \), \( \frac{\partial N}{\partial \tau_0} < 0 \). (b) For all \( \tau_0 \in (0, \gamma/m) \), \( \frac{\partial^2 N}{\partial \tau_0^2} < 0 \).

The first result is straightforward following the same logic of the proof of Proposition 1, with the fact that the virtual value \( v(m\tau_0) = m\tau_0 - \frac{1 - \Phi(m\tau_0)}{\phi(m\tau_0)} \) approaches
the lower bound when $\tau_0 \downarrow 0$ and the minimum is negative. To obtain the second statement, notice that the second derivative is given by:

$$
\frac{\partial^2 N}{\partial \tau_0^2} = \frac{1}{2c} \frac{\partial}{\partial \tau_0} \left( \Phi(-m\tau_0) - m\tau_0\phi(-m\tau_0) \right)
= -m\phi(-m\tau_0) - m\left( \phi(-m\tau_0) + \tau_0 \cdot m\tau_0 \cdot (-m) \cdot \phi(-m\tau_0) \right)
= m\phi(-m\tau_0)(m^2\tau_0^2 - 2),
$$

where the second equality uses the fact that $\phi'(x) = -x\phi(x)$. The above expression is negative if and only if $m\tau_0 < \sqrt{2}$. Since $\gamma = \frac{1-\Phi(\gamma)}{\Phi(\gamma)} < \frac{1-\Phi(0)}{\Phi(0)} = \frac{\sqrt{2}}{2} < \sqrt{2}$, it follows that $\frac{\partial^2 N}{\partial \tau_0^2} < 0$ for all $\tau_0 \in (0, \gamma/m)$.

Let

$$
M_1 = \frac{1}{2} A_1 + \frac{1}{2A_1} - \frac{3}{4},
$$

where $A_1 = 2(\lambda R_1 + 1)$. The remainder of the proof is to show that $R_0(\tau_0) = M_1(\tau_0)^2 \cdot N(\tau_0)$ has the same pattern as the function $N(\tau_0)$ such that it first increases and then decreases as $\tau_0$ grows. Because the optimal price is inversely related to $R_0$, it immediately follows that the price first decreases and then increases as $\tau_0$ becomes larger. Two possible cases are considered depending on the size of $m$.

**Case 1: Large $m$**

First consider the case when $m > m^* = \gamma/\tau_e$. From Proposition 1, we know that $\frac{\partial R_1}{\partial \tau_1} < 0$ always holds. Therefore, $\frac{\partial R_1}{\partial \tau_0} < 0$ for all $\tau_0$. Notice that

$$
\frac{\partial M_1}{\partial \tau_1} = \frac{1}{2} \frac{\partial}{\partial \tau_1} \left( A_1 + \frac{1}{A_1} \right) \cdot \frac{\partial R_1}{\partial \tau_1}
= \frac{1}{2} \left( \frac{\partial A_1}{\partial \tau_1} - \frac{1}{A_1^2} \frac{\partial A_1}{\partial \tau_1} \right) \cdot \frac{\partial R_1}{\partial \tau_1}
= \lambda(1 - \frac{1}{A_1^2}) \frac{\partial R_1}{\partial \tau_1}.
$$
Because $M_1$ monotonically increases with $A_1$ and hence monotonically increases with $R_1$, we have $\frac{\partial M_1}{\partial \tau_0} < 0$ for all $\tau_0$. Differentiating $R_0$ with respect to $\tau_0$ yields

$$\frac{\partial R_0}{\partial \tau_0} = 2M_1 \frac{\partial M_1}{\partial \tau_0} N + M_1 \frac{\partial N}{\partial \tau_0} = M_1 \left( 2N \frac{\partial M_1}{\partial \tau_0} + M_1 \frac{\partial N}{\partial \tau_0} \right).$$

Since $M_1 > 0$, the sign of $\frac{\partial R_0}{\partial \tau_0}$ is the same as the sign of $Z(\tau_0)$. To prove that $R_0$ first increases and then eventually decreases after $\tau_0 > \gamma/m$, it suffices to show the following results:

1. $\lim_{\tau_0 \downarrow 0} Z(\tau_0) > 0$;
2. $Z(\tau_0) < 0$ for all $\tau_0 \geq \gamma/m$;
3. $\frac{\partial Z(\tau_0)}{\partial \tau_0} < 0$ for all $\tau_0 < \gamma/m$.

Claim (1) can be validated by noticing that $\lim_{\tau_0 \downarrow 0} N(\tau_0) > 0$ and $\lim_{\tau_0 \downarrow 0} \frac{\partial N(\tau_0)}{\partial \tau_0} > 0$.

Claim (2) follows from Lemma 3 such that, for all $\tau_0 > \gamma/m$, $\frac{\partial N}{\partial \tau_0} < 0$. To prove Claim (3), first differentiate $Z(\tau_0)$ with respect to $\tau_0$

$$\frac{\partial Z}{\partial \tau_0} = 2 \frac{\partial N}{\partial \tau_0} \frac{\partial M_1}{\partial \tau_0} + 2N \frac{\partial^2 M_1}{\partial \tau_0^2} + \frac{\partial M_1}{\partial \tau_0} \frac{\partial N}{\partial \tau_0} + M_1 \frac{\partial^2 N}{\partial \tau_0^2}$$

$$= 3 \frac{\partial N}{\partial \tau_0} \frac{\partial M_1}{\partial \tau_0} + 2N \left( \frac{\partial^2 M_1}{\partial \tau_1^2} \left( \frac{\partial \tau_1}{\partial \tau_0} \right)^2 + \frac{\partial M_1}{\partial \tau_1} \frac{\partial^2 \tau_1}{\partial \tau_0^2} \right) + M_1 \frac{\partial^2 N}{\partial \tau_0^2}$$

$$< 2N \frac{\partial^2 M_1}{\partial \tau_1^2} \left( \frac{\partial \tau_1}{\partial \tau_0} \right)^2 + M_1 \frac{\partial^2 N}{\partial \tau_0^2}$$

where the first inequality follows from the fact that the terms $3 \frac{\partial N}{\partial \tau_0} \frac{\partial M_1}{\partial \tau_0}$ and $\frac{\partial M_1}{\partial \tau_1} \frac{\partial^2 \tau_1}{\partial \tau_0^2}$ are both negative, and the second inequality follows from the fact that

$$\frac{\partial^2 M_1}{\partial \tau_1^2} = \frac{\partial}{\partial \tau_1} \left( \lambda(1 - \frac{1}{A_1^2}) \frac{\partial R_1}{\partial \tau_1} \right) = \frac{\partial}{\partial \tau_1} \left( \lambda(1 - \frac{1}{A_1^2}) \right) \frac{\partial R_1}{\partial \tau_1} + \lambda(1 - \frac{1}{A_1^2}) \frac{\partial^2 R_1}{\partial \tau_1^2} < \lambda(1 - \frac{1}{A_1^2}) \frac{\partial^2 R_1}{\partial \tau_1^2}.$$
If $\frac{\partial^2 R_1}{\partial \tau_1^2} \leq 0$, then clearly $\frac{\partial Z}{\partial \tau_0} < 0$ is negative because $\frac{\partial^2 N}{\partial \tau_0^2} < 0$ by Lemma 3, concluding that Claim (3) holds. If, however, $\frac{\partial^2 R_1}{\partial \tau_1^2} > 0$, then it suffices to show that

$$2N\lambda \frac{\partial^2 R_1}{\partial \tau_1^2} \left( \frac{\partial \tau_1}{\partial \tau_0} \right)^2 + M_1 \frac{\partial^2 N}{\partial \tau_0^2} < 0.$$  

Note further that

$$M_1 = \lambda R_1 + \frac{1}{4\lambda R_1 + 4} + \frac{1}{4} > \lambda R_1 + \frac{1}{4},$$
and

$$\left( \frac{\partial \tau_1}{\partial \tau_0} \right)^2 = \frac{\tau_0^2}{\tau_1} < \frac{\tau_0}{\tau_1},$$

we have

$$2N\lambda \frac{\partial^2 R_1}{\partial \tau_1^2} \left( \frac{\partial \tau_1}{\partial \tau_0} \right)^2 + M_1 \frac{\partial^2 N}{\partial \tau_0^2} < 2N\lambda \frac{\partial^2 R_1}{\partial \tau_1^2} \frac{\tau_0}{\tau_1} + (\lambda R_1 + \frac{1}{4}) \frac{\partial^2 N}{\partial \tau_0^2}$$

$$= 2\frac{1}{2c} \tau_0 \Phi(-m\tau_0) \lambda \frac{1}{32c} m\phi(-m\tau_1)(m^2\tau_1^2 - 2) \frac{\tau_0}{\tau_1} + (\lambda R_1 + \frac{1}{4}) \frac{1}{2c} m\phi(-m\tau_0)(m^2\tau_0^2 - 2)$$

$$= \frac{\lambda}{64c^2} \left[ 2\tau_0 \Phi(-m\tau_0) \frac{\tau_0}{\tau_1} Q_1 + (\tau_1 \Phi(-m\tau_1) + \frac{8c}{\lambda}) Q_0 \right].$$

The above expression is negative if and only if

$$\frac{8c}{\lambda} > -\tau_1 \Phi(-m\tau_1) + 2\tau_0 \Phi(-m\tau_0) \frac{\tau_0}{\tau_1} \frac{Q_1}{Q_0}.$$  

Notice that $Q(\tau) = m\phi(-m\tau)(m^2\tau^2 - 2)$ is negative when $\tau$ is very small, and increases with $\tau \in (0, 2/m]$, and then decreases towards zero with $\tau > 2/m$. The function equals zero when $\tau = \sqrt{2}/m$ and reaches maximum at $\tau = 2/m$. Since $\tau_0 < \gamma/m < \sqrt{2}/m$ and for $\frac{\partial^2 R_1}{\partial \tau_1^2} > 0$ it requires that $\tau_1 > \sqrt{2}/m$. Therefore, $Q_0$ has an upper bound of $Q(\gamma/m) = (\gamma^2 - 2)\phi(-\gamma)m$, and $Q_1$ has an upper bound of
\( Q(2/m) = 2\phi(-2)m \). It follows that

\[
\frac{Q_1}{-Q_0} < \frac{2\phi(-2)m}{-(\gamma^2 - 2)\phi(-\gamma)m} = \frac{2\phi(2)}{(2 - \gamma^2)\phi(\gamma)},
\]

which approximately equals to 0.25. Then

\[
-\tau_1 \Phi(-m\tau_1) + 2\tau_0 \Phi(-m\tau_0) \frac{\tau_0}{\tau_1} \cdot \frac{Q_1}{-Q_0} < -\tau_1 \Phi(-m\tau_1) + \frac{1}{2} \tau_0 \Phi(-m\tau_0) \frac{\tau_0}{\tau_1}
\]

\[
< \frac{1}{\tau_0} \left( \frac{1}{2} \tau_0^2 \Phi(-m\tau_0) - \tau_1^2 \Phi(-m\tau_1) \right)
\]

\[
< \frac{1}{2} \tau_0 \Phi(-m\tau_0)
\]

\[
< \frac{1}{2} \frac{\gamma}{m} \Phi(-\gamma).
\]

It suffices that

\[
\frac{8c}{\lambda} > \frac{\gamma \Phi(-\gamma)}{2m}, \quad \text{or equivalently} \quad \frac{\lambda}{c} < \frac{16m}{\gamma \Phi(-\gamma)}.
\]

This upper bound \((\approx 94m)\) is greater than 34\(m\), and hence is satisfied.

**Case 2: Small \(m\)**

Next consider the case when \(m < m^* = \gamma/\tau\). From Proposition 1, we know that for all \(\tau_1 \in (\tau, \gamma/m)\), \(\frac{\partial R_1}{\partial \tau_1} > 0\), and that for all \(\tau_1 > \gamma/m\), \(\frac{\partial R_1}{\partial \tau_1} < 0\). Since \(\tau_1 = \sqrt{\tau_0^2 + \tau_\epsilon^2}\), \(\tau_1\) increases monotonically with \(\tau_0\) and reaches \(\gamma/m\) when \(\tau_0 = \sqrt{\gamma^2/m^2 - \tau_\epsilon^2} \equiv \tau' < \gamma/m\). Therefore, for all \(\tau_0 \in (0, \tau')\), \(\frac{\partial R_1}{\partial \tau_0} > 0\) and thus \(\frac{\partial M_1}{\partial \tau_0} > 0\), and that for all \(\tau_0 > \gamma/m\), \(\frac{\partial R_1}{\partial \tau_0} < 0\) and thus \(\frac{\partial M_1}{\partial \tau_0} < 0\).

When \(\tau_0 \in (0, \tau')\), both \(\frac{\partial M_1}{\partial \tau_0}\) and \(\frac{\partial N}{\partial \tau_0}\) are positive, and hence \(Z(\tau_0) > 0\). When \(\tau_0 > \tau'\), the pattern of \(Z(\tau_0)\) is exactly the same as the one in Case 1. Therefore, \(Z(\tau_0)\) is initially positive, and monotonically decreasing in \(\tau_0 \in (0, \gamma/m)\). This implies that, there exists a threshold \(\tau''\) such that \(R_0\) first increases with \(\tau_0 \in (0, \tau'')\) and then eventually decreases with \(\tau_0 \geq \tau''\).
Proof of Proposition 2

Proposition 2.1

Proof. Let $\Delta_w = w_0^* - w_1^*$. From Proposition 3.1 we learn that $\frac{\partial \Delta_w}{\partial \tau_0} > 0$. From Proposition 1.1 we have $\frac{\partial w_1^*}{\partial \tau_1} > 0$. Then

$$\frac{\partial w_0^*}{\partial \tau_0} = \frac{\partial \Delta_w}{\partial \tau_0} + \frac{\partial w_1^*}{\partial \tau_1} \frac{\partial \tau_1}{\partial \tau_0} > 0,$$

which establishes the proof.

Proposition 2.2

Proof. Again, result (a) is evident from the expression of the optimal price $p_0^*$. Result (b) is obtained from Lemma 2 and the fact $p_0^*$ decreases monotonically with $R_0$.

Proof of Proposition 3

Proposition 3.1

Proof. Let $\Delta_w = w_0^* - w_1^*$. Clearly, $\lim_{\tau_0 \to 0} \Delta_w(\tau_0) = -\frac{r_0}{4c} < 0$. To prove this proposition, it suffices to show that $\frac{\partial \Delta_w}{\partial \tau_0} > 0$.

$$\frac{\partial \Delta_w}{\partial \tau_0} = \frac{1}{c} \left( \frac{\partial M_1}{\partial \tau_0} \frac{\partial \tau_0}{\tau_0} + M_1 - \frac{1}{4} \frac{\partial \tau_1}{\partial \tau_0} \right)$$

$$= \frac{1}{c} \left( \frac{\partial M_1}{\partial \tau_1} \frac{\partial \tau_1}{\partial \tau_0} \frac{\partial \tau_0}{\tau_0} + M_1 - \frac{1}{4} \frac{\partial \tau_1}{\partial \tau_0} \right)$$

$$= \frac{1}{c} \left[ \lambda(1 - \frac{1}{A_1^2}) \frac{\partial R_1}{\partial \tau_1} \frac{\partial \tau_1}{\partial \tau_0} + \lambda R_1 + \frac{1}{4(\lambda R_1 + 1)} + \frac{1}{4} \left( 1 - \frac{\partial \tau_1}{\partial \tau_0} \right) \right].$$
Notice that \( R_1 > 0, A_1 > 2, \) and

\[
\frac{\partial \tau_1}{\partial \tau_0} = \frac{\tau_0}{\sqrt{\tau_0^2 + \tau_1^2}} = \frac{\tau_0}{\tau_1} \in (0, 1). \tag{1}
\]

If \( \frac{\partial R_1}{\partial \tau_1} > 0, \) then clearly \( \frac{\partial \Delta_w}{\partial \tau_0} > 0. \) If \( \frac{\partial R_1}{\partial \tau_1} < 0, \) then

\[
\frac{\partial \Delta_w}{\partial \tau_0} > \frac{1}{c} \left[ \lambda \frac{\partial R_1}{\partial \tau_1} \tau_0 \frac{\tau_0}{\tau_1} + \frac{1}{4(\lambda R_1 + 1)} \right]
\]

Next I shall show that \( Y \) has a minimum that is negative, and that \( 1/4(\lambda R_1 + 1) \) has a minimum that is positive. Differentiating \( Y \) with respect to \( \tau_1 \) yields

\[
\frac{\partial Y}{\partial \tau_1} = \Phi(-m\tau_1) - m\tau_1\phi(-m\tau_1) + m\tau_1\phi(-m\tau_1)(m^2\tau_1^2 - 2)
\]

Therefore, \( \frac{\partial Y}{\partial \tau_1} > 0 \) is equivalent to

\[
\iff \frac{1 - \Phi(m\tau_1)}{\phi(m\tau_1)} > m\tau_1(3 - m^2\tau_1^2).
\]

The left-hand side is monotonically decreasing towards zero, and equals to \( (1 - \Phi(0))/\phi(0) > 0 \) when \( m\tau_1 = 0. \) The right-hand side first increases and then decreases with the maximum achieved at \( m\tau_1 = 1. \) It equals to 0 when \( m\tau_1 = 0 \) and approaches negative infinity as \( m\tau_1 \to \infty. \) Therefore, as \( \tau_1 \) increases, \( \frac{\partial Y}{\partial \tau_1} \) starts out being positive, and then becomes negative, and eventually becomes positive. The minimum of \( Y, \) achieved at \( m\tau_1 = \eta \approx 1.65, \) where the two functions above intersect at the second time, is equal to \( Y_{\text{min}} = (\Phi(-\eta) - \eta\phi(-\eta))\eta/m. \)
The fact that \(1/4(\lambda R_1 + 1)\) has a minimum follows from Lemma 1 such that \(R_1\) reaches maximum when \(m\tau_1 = \gamma \approx 0.75\) and equals to \(\gamma \Phi(-\gamma)/32mc\). Therefore, it suffices to show that

\[
\frac{\lambda}{32c} \left( \Phi(-\eta) - \eta \phi(-\eta) \right) + \frac{1}{4\lambda \frac{1}{2c}} \Phi(-\eta) + 4 > 0
\]

\[
\Leftrightarrow a \left( \frac{\lambda}{16mc} \right)^2 + b \frac{\lambda}{16mc} - 1 < 0
\]

\[
\Leftrightarrow \frac{\lambda}{16mc} < \frac{-b + \sqrt{b^2 + 4a}}{2a}
\]

\[
\Leftrightarrow \frac{\lambda}{c} < 34m,
\]

where \(a = - \left( \Phi(-\eta) - \eta \phi(-\eta) \right) \gamma \Phi(-\gamma)\) and \(b = -2 \left( \Phi(-\eta) - \eta \phi(-\eta) \right) \eta\). ■

**Proposition 3.2**

*Proof.* Since \(p_t^*\) decreases monotonically with \(R_t\), it suffices to compare \(R_0\) and \(R_1\).

Let \(\Delta_R = R_0 - R_1\). Two cases are considered depending on the size of \(m\).

**Case 1: Large \(m\)**

First consider the case when \(m > m^* = \gamma/\tau_1\). From Lemma 1, we know that \(\frac{\partial R_1}{\partial \tau_1} < 0\) always holds. From Lemma 2, we have that \(\frac{\partial R_0}{\partial \tau} > 0\) when \(\tau_0 < \tau^{**}\), and \(\frac{\partial R_0}{\partial \tau_0} < 0\) when \(\tau_0 > \tau^{**}\).

When \(\tau_0 < \tau^{**}\), clearly \(\frac{\partial \Delta_R}{\partial \tau_0} > 0\). When \(\tau_0 > \tau^{**}\), we have

\[
\Delta_R = M_1 \frac{1}{\tau_0} \Phi(-m\tau_0) - \frac{1}{32c} \Phi(-m\tau_1)
\]

\[
> \tilde{M}_1 \frac{1}{\tau_0} \Phi(-m\tau_1) - \frac{1}{32c} \Phi(-m\tau_1)
\]

\[
> \frac{1}{16} \frac{1}{\tau_1} \Phi(-m\tau_1) - \frac{1}{32c} \Phi(-m\tau_1)
\]

\[
= 0,
\]

where the first inequality follows from the fact that \(\frac{\partial R_0}{\partial \tau_0} < 0\), and \(\tilde{M}_1\) is obtained by
replacing \( \tau_0 \) with \( \tau_1 \) in the function \( M_1(\tau_0) \), and the second inequality follows from \( M_1 > 1/4 \).

**Case 2: Small \( m \)**

Next consider the case when \( m < m^* = \gamma/\tau_c \). From Lemma 1, we know that \( \frac{\partial R_1}{\partial \tau_1} < 0 \) when \( \tau_1 > \tau^* \), or equivalently, \( \tau_0 > \sqrt{\tau^* - \tau_c^2} \). The analysis follows exactly the same logic as Case 1. It remains to show that when \( \tau_0 < \sqrt{\tau^* - \tau_c^2} \), we have \( \frac{\partial \Delta_R}{\partial \tau_0} > 0 \).

\[
\frac{\partial \Delta_R}{\partial \tau_0} = M_1 \left( 2N \frac{\partial M_1}{\partial \tau_0} + M_1 \frac{\partial N}{\partial \tau_0} \right) - \frac{\partial R_1}{\partial \tau_1} \frac{\partial \tau_1}{\partial \tau_0} - M_1 \frac{\partial N}{\partial \tau_0} - \frac{\partial R_1}{\partial \tau_1} \frac{\tau_0}{\tau_1}
\]

\[
> M_1 \frac{\partial N}{\partial \tau_0} - \frac{\partial R_1}{\partial \tau_1} \frac{\tau_0}{\tau_1}
\]

\[
> \frac{1}{16} \frac{\partial}{\partial \tau_0} (\tau_0 \Phi(-m\tau_0)) - \frac{1}{32c} \frac{\partial}{\partial \tau_1} (\tau_1 \Phi(-m\tau_1)) \frac{\tau_0}{\tau_1}
\]

\[
> \frac{1}{32c} \left[ \frac{\partial}{\partial \tau_0} (\tau_0 \Phi(-m\tau_0)) - \frac{\partial}{\partial \tau_1} (\tau_1 \Phi(-m\tau_1)) \right]
\]

\[
> 0,
\]

where the first inequality follows from \( \frac{\partial M_1}{\partial \tau_0} > 0 \), the second inequality follows from \( M_1 > \frac{1}{4} \), the forth inequality follows from the fact that \( \frac{\partial^2}{\partial \tau_1^2} (\tau \Phi(-m\tau)) < 0 \).

■

**Proof of Proposition 4**

*Proof.* Let

\[ M_0 = \frac{1}{2} A_0 + \frac{1}{2A_0} - 1, \]

where \( A_0 = 2(\lambda R_0 + 1) \). Differentiating the profit function \( V_0 \) with respect to \( \tau_0 \) yields

\[
\frac{\partial V_0}{\partial \tau_0} = \left( \frac{\partial M_1}{\partial \tau_0} + \frac{\partial M_0}{\partial \tau_0} \right) \mu_0 = \left( \frac{\partial M_1}{\partial \tau_1} \frac{\partial \tau_1}{\partial \tau_0} + \frac{\partial M_0}{\partial \tau_0} \right) \mu_0.
\]

Again, we consider two possible cases depending on the size of \( m \).
Case 1: Large \( m \)

First consider the case when \( m > m^* = \gamma / \tau_e \). Following Lemma 1 and the fact that \( M_1 \) monotonically increases with \( A_1 \) and hence monotonically increases with \( R_1 \), we have \( \frac{\partial M_1}{\partial \tau_0} < 0 \) for all \( \tau_0 \). Notice that \( \frac{\partial A_1}{\partial \tau_0} = 0 \) when \( \tau_0 \downarrow 0 \) and is increasing and positive for all \( \tau_0 > 0 \). By Lemma 2, \( \frac{\partial R_0}{\partial \tau_0} > 0 \) and thus \( \frac{\partial M_0}{\partial \tau_0} > 0 \) when \( \tau_0 = 0 \) and \( \frac{\partial M_0}{\partial \tau_0} < 0 \) eventually when \( \tau_0 > \tau^{**} \). Therefore \( \frac{\partial V_0}{\partial \tau_0} \) start out being positive and then eventually becomes negative. It remains to show that \( \frac{\partial V_0}{\partial \tau_0} \) drops monotonically as \( \tau_0 \) increases within \((0, \tau^{**})\). This is true given the facts that (a) \( \frac{\partial M_1}{\partial \tau_1} \) is negative and decreases with \( \tau_0 \), (b) \( \frac{\partial M_1}{\partial \tau_0} \) is positive and increases with \( \tau_0 \), and (c) \( \frac{\partial M_0}{\partial \tau_0} \) is positive and decreases with \( \tau_0 \).

Case 2: Small \( m \)

Next consider the case when \( m < m^* = \gamma / \tau_e \). Following Lemma 1 and the fact that \( M_1 \) monotonically increases with \( A_1 \) and hence monotonically increases with \( R_1 \), we have \( \frac{\partial M_1}{\partial \tau_1} > 0 \) for all \( \tau_1 \in (\tau_e, \gamma / m) \), and \( \frac{\partial M_1}{\partial \tau_1} < 0 \) for all \( \tau_1 > \gamma / m \). Applying the same logic as in the proof of Case 2 in Lemma 2, for all \( \tau_0 \in (0, \tau') \), \( \frac{\partial M_1}{\partial \tau_0} > 0 \), and for all \( \tau_0 > \tau' \), \( \frac{\partial M_1}{\partial \tau_0} < 0 \).

When \( \tau_0 \in (0, \tau') \), both \( \frac{\partial M_1}{\partial \tau_0} \) and \( \frac{\partial M_0}{\partial \tau_0} \) are positive, and hence \( \frac{\partial V_0}{\partial \tau_0} > 0 \). When \( \tau_0 > \tau' \), the pattern of \( \frac{\partial V_0}{\partial \tau_0} \) is exactly the same as the one in Case 1. Therefore, \( \frac{\partial V_0}{\partial \tau_0} \) is initially positive, and monotonically decreasing in \( \tau_0 \in (0, \gamma / m) \). This implies that there exists a threshold \( \tau V' < \gamma / m \) such that \( V_0 \) first increases when \( \tau_0 < \tau V \) and then eventually decreases when \( \tau_0 > \tau V \).
Proof of Proposition 5

Proof. Differentiating $R_1$ with respect to $\tau_\epsilon$ leads to

$$\frac{\partial R_1}{\partial \tau_\epsilon} = \frac{\partial R_1}{\partial \tau_1} \cdot \frac{\partial \tau_1}{\partial \tau_\epsilon} = \frac{1}{32c} \left( \Phi(-m\tau_1) - m\tau_1 \phi(-m\tau_1) \right) \cdot \frac{\tau_\epsilon}{\sqrt{\tau_0^2 + \tau_\epsilon^2}}.$$

Therefore, $\text{sign}\left(\frac{\partial R_1}{\partial \tau_\epsilon}\right) = \text{sign}\left(\frac{\partial R_1}{\partial \tau_1}\right)$. It then follows that the sign depends on the size of $m$. When $m > m^e$ where $m^e = \gamma/\tau_0$, $\frac{\partial R_1}{\partial \tau_\epsilon} < 0$ always holds. When $m < m^e$, for more noisy reviews, $\tau_\epsilon \in (0, \sqrt{\gamma^2/m^2 - \tau_0^2})$, we have $\frac{\partial R_1}{\partial \tau_\epsilon} > 0$, whereas for more accurate reviews, $\tau_\epsilon > \sqrt{\gamma^2/m^2 - \tau_0^2}$ we have $\frac{\partial R_1}{\partial \tau_\epsilon} < 0$.

Differentiating $R_0$ with respect to $\tau_\epsilon$ leads to

$$\frac{\partial R_0}{\partial \tau_\epsilon} = 2M_1 \frac{\partial M_1}{\partial \tau_\epsilon} N = 2M_1 N \frac{\partial M_1}{\partial \tau_1} \frac{\partial \tau_1}{\partial \tau_\epsilon}.$$

Clearly, $\text{sign}\left(\frac{\partial R_0}{\partial \tau_\epsilon}\right) = \text{sign}\left(\frac{\partial M_1}{\partial \tau_1}\right)$. Because $M_1$ increases monotonically with $A_1$ and thus $R_1$, $\frac{\partial R_0}{\partial \tau_\epsilon}$ has the same sign as $\frac{\partial R_1}{\partial \tau_\epsilon}$.

Since $V_0 = (M_1 + M_0)\mu_0$, it follows that when $m > m^e$, $V_0$ always decreases. When $m < m^e$, for more noisy reviews, $\tau_\epsilon \in (0, \sqrt{\gamma^2/m^2 - \tau_0^2})$, the total profit $V_0$ increases with $\tau_\epsilon$, whereas for more accurate reviews, $\tau_\epsilon > \sqrt{\gamma^2/m^2 - \tau_0^2}$, the total profit $V_0$ decreases with $\tau_\epsilon$.

\[\blacksquare\]
Essay 3

Learning from Experience, Simply
Learning from Experience, Simply*

Abstract

There is substantial academic interest in modeling consumer experiential learning. However, (approximately) optimal solutions to forward-looking experiential learning problems are complex, limiting their behavioral plausibility and empirical feasibility. We propose that consumers use cognitively simple heuristic strategies. We explore one viable heuristic – index strategies, and demonstrate that they are intuitive, tractable, and plausible. Index strategies are much simpler for consumers to use but provide close-to-optimal utility. They also avoid exponential growth in computational complexity, enabling researchers to study learning models in more-complex situations.

Well-defined index strategies depend upon a structural property called indexability. We prove the indexability of a canonical forward-looking experiential learning model in which consumers learn brand quality while facing random utility shocks. Following an index strategy, consumers develop an index for each brand separately and choose the brand with the highest index. Using synthetic data, we demonstrate that an index strategy achieves nearly optimal utility at substantially lower computational costs. Using IRI data for diapers, we find that an index strategy performs as well as an approximately optimal solution and better than myopic learning. We extend the analysis to incorporate risk aversion, other cognitively simple heuristics, heterogeneous foresight, and an alternative specification of brands.

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1. Introduction and Motivation

Considerable effort in marketing is devoted to studying the dynamics by which consumers learn from their consumption experience (e.g., Roberts and Urban 1988; Erdem and Keane 1996; Ching, Erdem and Keane 2013a). As an example, imagine new parents who have to shop for diapers, perhaps with little pre-existing knowledge about this category. As these parents find out more about diaper brands through usage experience, they face a strategic choice. They can exploit their knowledge to date and select the most appealing brand. They can also explore further, which may entail sampling a currently-less-than-ideal brand, so that they can make a more-informed decision in the future.

Researchers have developed theory-rich models of optimizing forward-looking consumers who balance exploitation with exploration. Pillars of these models include an explicitly specified description of consumer utility and an explicitly specified process by which consumers learn. Most models assume consumers choose brands by solving a dynamic program which maximizes expected total utility taking learning into account. Researchers argue that theory-based models are more likely to uncover insight and be invariant for new-domain policy simulations (Chintagunta et al. 2006, p. 604). However, these advantages often come at the expense of difficult problems and time-consuming solution methods.

The dynamic programs for forward-looking experiential learning models are, themselves, extremely difficult to solve optimally. We cite evidence below that the problems are PSPACE-hard – they are at least as hard to solve as any problem that requires
PSPACE computational memory. This intractability presents both practical and theoretical challenges. Practically, researchers have had to rely on approximate solutions. Without explicit comparisons to the optimal solution, we do not know the impact of the approximations on estimation results. Moreover, the well-known "curse of dimensionality" prevents researchers from investigating problems with moderate or large numbers of brands or marketing variables, whereby even approximate solutions may not be feasible.

Theoretically, it is reasonable to posit that a consumer cannot solve optimally in his or her head a dynamic problem that requires vast amounts of memory and computation. In fact, well-developed theories in marketing, psychology, and economics suggest that observed consumer decision rules are often cognitively simple (e.g., Payne et al. 1988, 1993; Gigerenzer and Goldstein 1996).

We propose that consumers use cognitively simple heuristics to solve learning problems. As an example of the class of cognitively simple heuristics, we investigate an attractive candidate heuristic, index strategies, whereby a consumer develops a numerical score, or an index, for each brand separately and then chooses the brand with the largest index. Index strategies are a solution concept that decomposes an intractable problem into a set of tractable sub-problems. We retain basic pillars of structural modeling such as an explicit description of the decision process and an assumption that consumers seek to optimize. We posit in addition a cost to solving complex problems (e.g., Shugan 1980; Johnson and Payne 1985). We assume the consumer chooses a strategy that optimizes expected discounted utility minus this cognitive cost. While the cost of cognitive com-

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1 PSPACE is the set of problems which use polynomial-sized memory—memory proportional to $|\Omega|^n$, where $|\Omega|$ is a measure of the size of the problem and $n$ can be extremely large. PSPACE-hard problems are at least as hard as NP-hard problems, which are themselves suspected of being unsolvable in polynomial time.
plexity might be observable in the laboratory, say through response latency, it is unobservable in vivo. Instead, we identify domains where index strategies are nearly optimal in the sense of maximizing expected discounted utility. If, in such domains, index strategies are substantially simpler for the consumer to implement, then it is likely that savings in cognitive costs exceed the slight deviation from optimality and, hence, provide the consumer with greater utility net of cognitive costs. In the special cases where index strategies provide optimal expected utility, we argue that index strategies are superior as a description of forward-looking learning. Following the same logic, we establish conditions where myopic learning strategies (i.e., exploiting posterior beliefs without exploration) suffice to model consumer behavior.

To motivate the viability of index strategies as a descriptive model of consumers we (1) establish whether well-defined index strategies exist, (2) explain why they are intuitive and hence might be used by consumers, (3) investigate when index strategies are (near) optimal solutions to the reduced problem of utility maximization, and whether they are computationally simpler than the approximately optimal solution, and (4) test whether index strategies explain observed consumer behavior at least as well as alternative models.

We address #1 analytically by proving the “indexability” property of canonical forward-looking experiential learning models. (Indexability is hard to establish in general.) We address #2 by examining the form and properties of index strategies and arguing they are behaviorally intuitive relative to the approximately optimal solution assumed in most forward-looking learning models. We address #3 using synthetic data. We ad-

\footnote{We use computational simplicity as a surrogate for cognitive simplicity in this essay.}
dress #4 by estimating alternative models using IRI data on the purchase of diapers, a product category where we expect to see forward-looking experiential learning.

Our basic hypothesis is that consumers can use a cognitively simple index strategy to solve forward-looking experiential learning problems. Figure 1 is a conceptual summary of our hypothesis. We demonstrate viability by showing that there exists a well-defined index that satisfies the four criteria. We do not argue that consumers actually use this well-defined index. Rather we argue that the well-defined index is a better “as if” description than the (approximately) optimal solution strategy.³

We first describe a canonical learning problem. We next review briefly literatures that address learning dynamics, cognitive simplicity, and related optimization problems. We then examine index strategies from the perspectives of theory, synthetic data, and empirical estimation. We close with extensions.

[Insert Figure 1 about here.]

2. Canonical Forward-Looking Experiential Learning Problem

We consider the following canonical forward-looking experiential learning problem. A consumer sequentially chooses from a set $A$ containing $J$ brands. Let $j$ index brands and $t$ index purchase occasions. The consumer’s utility, $u_{jt}$, from choosing $j$ at $t$ has three components. The first component is quality, $q_{jt}$, which can be defined to include enjoyment, fit with needs, weighted sum of brand features, etc. Quality is drawn independently from a distribution $F_j(q_{jt}; \theta_j)$ with parameters $\theta_j$. The $F_j$ distributions are independent.

³ An empirical search among heuristics would risk exploiting random variation in the data. Instead we demonstrate that an index strategy, and at least one other cognitively simple heuristic, perform well on the data.
across \( j \). This independence assumption rules out learning about a brand by choosing another. The consumer, however, does not know the value of the parameters \( \theta_j \) and observes quality draws to infer the value of \( \theta_j \). A quality draw of a brand is only realized after the consumer has chosen that brand.

The second component of utility is a set of observable shocks, \( \bar{x}_{jt} \), such as advertising, price, promotion, and other control variables that are observable to the researcher and consumer.\(^4\) For simplicity, we assume that observable shocks affect utility directly, although the model is extendable to indirect effects through the quality component as in Erdem and Keane (1996), Ackerberg (2003), and Narayanan et al. (2005). The third component of utility is an unobservable shock, \( \epsilon_{jt} \), which represents random fluctuations in realized utility that are observed by the consumer, but not by the researcher.

Consumer decision-making depends on quality and the weighted sum of observable and unobservable shocks, \( \bar{\beta}' \bar{x}_{jt} + \epsilon_{jt} \), where \( \bar{\beta} \) is a vector of weight parameters. We refer to this weighted sum as utility shocks. We let utility shocks be drawn from a joint distribution, \( H_j(\bar{x}_{jt}, \epsilon_{jt}; \phi) \), independently over purchase occasions with parameters, \( \phi \).\(^5\) The \( H_j \)'s are independent across \( j \). The consumer knows the distribution \( H_j \) and the value of \( \phi \), observes the current utility shocks prior to making a purchase decision, but does not know future realizations of the shocks. Notice that, unlike the quality draws, the utility shocks of a brand are realized regardless of whether the consumer has chosen that

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\(^4\) Our consumer learning model treats observable utility shocks as exogenous. However, the same insight applies to endogenous observable utility shocks as long as (1) each “atomic” consumer’s learning does not affect these shocks (e.g., a brand’s advertising expenditure), and (2) these shocks do not directly convey quality information.

\(^5\) Observable shocks can be independently distributed over purchase occasions for a number of reasons. For example, firms may intentionally randomize price promotions in response to competition. Such “mixed strategies” can generate observed prices that appear to be independently drawn at each purchase occasion from a known distribution (Narasimhan 1988).
brand. We make the conservative assumption that utility shocks are independent of \( q_{jt} \) and thus do not help the consumer learn quality directly. However, utility shocks do shape learning indirectly by varying the consumer’s utility from exploitation, which in turn affects incentive for exploration.

In summary, we write the consumer’s utility from choosing brand \( j \) at purchase occasion \( t \) as follows:

\[
u_{jt} = q_{jt} + \beta^j z_{jt} + \epsilon_{jt}.
\]

For ease of exposition, in the main analysis, we assume that the consumer is risk neutral. We extend the model to incorporate risk aversion in §8.1.

We model each consumer as if the consumer uses Bayes Theorem to update beliefs about the quality parameter \( \theta_j \) after each consumption experience (assumed to occur after choice but before the next choice). Let \( s_{jt} \) be the information set that summarizes the consumer’s beliefs about \( \theta_j \) at purchase occasion \( t \). At \( t = 0 \), beliefs about \( \theta_j \) are summarized by a prior distribution, \( B_{j0}(\theta_j; s_{j0}) \), where \( s_{j0} \) is based on all relevant prior experience. After the \( t^{th} \) consumption experience the consumer’s posterior beliefs are summarized by \( B_{jt}(\theta_j; s_{jt}) \). When both \( F_j \) and prior beliefs are normal, Bayesian updating is naturally conjugate. We obtain \( s_{jt} = (\mu_{jt}, \sigma_{jt}) \) using standard updating formulae. The parameters of posterior beliefs, \( s_{jt} \in \Omega \) and the realized utility shocks, \( z_{jt} \in X \) and \( \epsilon_{jt} \in E \), summarize the state of information about brand \( j \). The collection of brand-specific states, \( (s_{jt}, x_{jt}, \epsilon_{jt}) = (s_{1jt}, s_{2jt}, ..., s_{jt}, x_{1jt}, x_{2jt}, ..., x_{jt}, \epsilon_{1jt}, \epsilon_{2jt}, ..., \epsilon_{jt}) \) represents the set of states relevant to the decision problem at \( t \).

We seek to model a decision strategy, \( \Pi: (\Omega \times X \times E)^t \rightarrow A \), that maps the state
space to the choice set. Without further assumptions, the consumer must choose a decision strategy to maximize expected discounted utility:

\[
V(\bar{s}_{t}, \bar{x}_{t}, \bar{\epsilon}_{t}) = \max_{\bar{\pi}} \mathbb{E}_{\pi} \left[ \sum_{t=1}^{\infty} \delta^{t-t} \left( q_{jt} + \tilde{\beta}^{t} \bar{x}_{jt} + \epsilon_{jt} \right) |(\bar{s}_{t}, \bar{x}_{t}, \bar{\epsilon}_{t}) \right],
\]

where \( \delta \) is the discount factor. The expectation \( \mathbb{E} \) is taken over the stochastic process generated by the decision strategy (in particular, the transition between states that may depend on the consumer’s brand choice). The infinite horizon can be justified either by consumption over a long horizon or by the consumer’s subjective belief that the decision problem will end randomly.

The optimal solution to the consumer’s decision problem can be characterized as the solution to the Bellman equation

\[
V(\bar{s}_{t}, \bar{x}_{t}, \bar{\epsilon}_{t}) = \max_{j \in \mathcal{A}} \left\{ \tilde{\beta}^{t} \bar{x}_{jt} + \epsilon_{jt} + \mathbb{E}[q_{jt} + \delta V(\bar{s}_{t+1}, \bar{x}_{t+1}, \bar{\epsilon}_{t+1}) |(\bar{s}_{t}, \bar{x}_{t}, \bar{\epsilon}_{t})] \right\}.
\]

While the Bellman equation is conceptually simple, the full solution is computationally difficult because, even after integrating out the utility shocks \( \bar{x}_{t} \) and \( \bar{\epsilon}_{t} \), it evolves on a state space of size \( |\Omega|^{J} \), where \( |\Omega| \) is the number of elements in \( \Omega \). Not only is \( |\Omega|^{J} \) exponential in the number of brands \( J \), it becomes extremely large if \( \Omega \) contains many elements, even when the optimal solution is approximated by choosing discrete points to represent \( \Omega \), as is common in the literature. We provide an illustrative example in §4.

3. Related Literatures

Before we introduce index strategies, it is helpful to review concepts from literatures on learning dynamics, cognitive simplicity, and related optimization problems.

3.1. Learning Dynamics
Many influential papers study consumer learning dynamics and apply learning models to explain or forecast consumer choices in problems related to the canonical learning problem. For example, using data from automotive consumers, Roberts and Urban (1988) estimate a model in which consumers use Bayesian learning to integrate information from a variety of sources to resolve uncertainty about brand quality. Erdem and Keane (1996) build upon the concept of Bayesian learning and include forward-looking consumers who tradeoff exploitation with exploration. For frequently-purchased goods, their model fits data better than a no-learning model (reduced form of Guadagni and Little 1983) and the myopic learning model of Roberts and Urban.

These papers stimulated a line of research that estimates the dynamics of consumer learning – for a comprehensive review see Ching et al. (2013a). Some models focus on myopic consumers with Bayesian learning (e.g., Narayanan et al. 2005; Mehta et al. 2008; Chintagunta et al. 2009; Narayanan and Manchanda 2009; Ching and Ishihara 2010, 2012), while others explicitly model forward-looking consumers (e.g., Ackerberg 2003; Crawford and Shum 2005; Erdem et al. 2005, 2008). The computational complexity of forward-looking learning has been one of the reasons that some applications assume myopic learning. However, if a theory is accurately descriptive, more-complex forward-looking models should improve policy simulations.

Because forward-looking choice problems that involve continuous state space generally cannot be solved optimally, significant effort has been spent on developing approximate solutions. For example, Keane and Wolpin (1994) use Monte Carlo integration and interpolation, Rust (1997a) introduces a randomization approach, and Imai et al. (2009) develop an estimator that combines dynamic programming solutions with a
Bayesian Markov chain Monte Carlo algorithm. While these solution methods vary in speed, all attempt to approximate the Bellman equation to the overall problems and thus may suffer from the curse of dimensionality (\(|\Omega|\)).

At the same time of technical developments, there is a growing recognition of the need for richer theories of consumer behavior. For example, Chintagunta et al. (2006, p. 614) suggest that “the future development of structural models in marketing will focus on the interface between economics and psychology.”

3.2. Cognitive Simplicity

Parallel literatures in marketing, psychology, and economics provide evidence that consumers use decision rules that are cognitively simple. In marketing, Payne et al. (1988, 1993) and Bettman et al. (1998) present evidence that consumers use simple heuristic decision rules to evaluate products. For example, under time pressure, consumers often use conjunctive rules (require a few “must have” features) rather than more complicated compensatory rules. Using simulated thinking costs with “elementary information processes,” Johnson and Payne (1985) illustrate how heuristic decision rules can be rational when balancing utility and thinking costs. Methods to estimate the parameters of cognitively simple decision rules vary, but such rules often predict difficult consumer decisions as well as or better than compensatory rules (e.g., Bröder 2000; Gilbride and Allenby 2004; Kohli and Jedidi 2007; Yee et al. 2007; Hauser et al. 2010).

Building on Simon’s (1995, 1956) theory of bounded rationality, researchers in psychology argue that human beings use cognitively simple rules that are “fast and fru-

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6 There is a related literature on neuro-dynamic programming, which uses neural networks and other approximation architectures to overcome the curse of dimensionality (Bertsekas and Tsitsiklis 1996).
gal” (e.g., Gigerenzer and Goldstein 1996; Martignon and Hoffrage 2002). Fast and frugal rules evolve when consumers learn decision rules from experience. Consumers continue to use the decision rules because they lead to good outcomes in familiar environments (Goldstein and Gigerenzer 2002). For example, when judging the size of cities, “take the best” often leads to sound judgments. In 2010-2011, two issues of Judgment and Decision Making were devoted to the recognition heuristic alone (e.g., Marewski et al. 2010).

The costly nature of cognition has also received attention in economics (see Camerer 2003 for a review). A line of research looks to extend or revise standard dynamic decision-making models with the explicit recognition that cognition is costly. For example, Gabaix and Laibson (2000) empirically test a behavioral solution to decision-tree problems, whereby decision-makers actively eliminate low-probability branches to simplify the task. Gabaix et al. (2006) develop a “directed cognition model,” in which a decision-maker acts as if there is only one more opportunity to search. In the laboratory, the directed cognition model explains subjects’ behavior better than a standard search model with costless cognition. Houser, Keane, and McCabe (2004) provide further evidence that consumers might use heuristic rules to solve dynamic programs.

Cognitive process mechanisms are debated in the marketing, psychology, and economics literatures. Our hypothesis, that consumers use heuristics such as index strategies, need only the observation that consumers favor decision rules that are cognitively simple and that such rules often lead to good outcomes. The simplicity hypothesis assumes that consumers trade off utility gains versus cognitive costs, but does not require

7 The take-the-best rule is, simply, if you recognize one city and not the other it is likely larger; if you recognize both use the most diagnostic feature to make the choice.
explicit measurement of cognitive costs.

3.3. Cognitively Simple Solutions to Complex Optimization Problems

If a ball player wants to catch a ball that is already high in the air and traveling directly toward the player, then all the player need do is gaze upon the ball, start running, and adjust his or her speed to maintain a constant gaze angle with the ball (Hutchinson and Gigerenzer 2005, p. 102). The gaze heuristic is an example where a cognitively simple rule accomplishes a task that might otherwise involve solving difficult differential equations. But the principle is more general: simple solutions often perform well in complex optimization problems.

There are many examples in marketing and economics where descriptive decision rules solve more-complex problems. In domains such as consumer-budget allocation, the choice of which information source to search, and the evaluation of products via agendas, heuristic solutions appear to describe consumer behavior well (Hauser 1986; Hauser and Urban 1986; Hauser, et al. 1993). Rust (1997b) argues that it is likely consumers solve problems requiring an “infeasibly large number of calculations” by using heuristic solutions such as decomposition into sub-problems. He states that “[t]he challenge is to recognize whether or not a problem is nearly decomposable, and if so, to identify its approximately independent sub-problems, [and] determine whether they can be solved separately (p. 18).” This view is closely related to our index approach to complex forward-looking learning problems.

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8 Professional athletes use more-complicated heuristics that give them greater range, for example, in baseball, pre-positioning based on prior tendencies and the expected pitch, and the sound as the bat hits the ball.

9 Of course, the empirical performance of descriptive solutions is not guaranteed. Gilovich et al. (2002) provide a comprehensive survey of human decision heuristics and their possible biases.
3.4. Related Optimization Problems: Bandit Problems and Index Solutions

The model we formulate in §2 is closely related to the *multi-armed bandit problem*, a prototypical problem that illustrates the fundamental tradeoff between exploration and exploitation in sequential decision making under uncertainty. In a bandit problem, the decision-maker faces a finite number of choices, each of which yields an uncertain payoff. The decision-maker must make choices, observe outcomes, and update beliefs with a sequential decision rule. The decision-maker seeks to maximize expected discounted values.

The bandit problems was first formulated by the British in World War II, and, for over thirty years, no simple solution was known. Then Gittins and Jones (1974) demonstrated a simple index solution – develop an index for each “arm” (i.e., each choice alternative) by solving a sub-problem that involves only that arm, then choose the arm with the largest index. This index solution reduces an exponentially complex problem to a set of one-dimensional problems. Gittins and Jones (1974) proved the surprising result that the index solution is the optimal solution to the classic bandit problem.\(^{10}\)

However, the Gittins-Jones’ striking result comes at the cost of a strict assumption that the states of the non-chosen choice alternatives do not evolve. When this assumption is violated, say due to random shocks, Gittins’ index is no longer guaranteed to be optimal. Such problems are known as *restless bandits* (Whittle 1988) and, in general, are computationally intractable (Papadimitriou and Tsitsiklis 1999). In his seminal paper,

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\(^{10}\) Hauser et al. (2009) apply Gittins’ index to derive optimal “website morphing” strategies that match website design with customers’ cognitive styles. Urban et al. (2014) field-test morphing for AT&T’s banner advertising on CNET and General Motors’ banner advertising on a variety of websites. Other well-known applications of index strategies include job-match-learning (Jovanovic 1979; Miller 1984) and pharmaceutical-product learning (Dickstein 2012). See Ching et al. (2013a) for a survey.
Whittle (1988) proposed a tractable heuristic solution. The solution generalizes Gittins’ index such that the problems can be solved optimally or near optimally by associating an index, referred to as Whittle’s index, separately with each alternative and choosing the alternative with the largest index.

The existence of well-defined index solutions relies on a structural property called indexability, which is not guaranteed for all restless bandit problems. Whittle (1988, p. 292) wrote that “One would very much like to have simple sufficient conditions for indexability; at the moment, none are known” (see also Niño-Mora 2001). Gittins et al. (2011, p. 154) also lament that “the question of indexability is subtle, and a complete understanding is yet to be achieved.” In an important class of marketing models, choice models, consumer utility tends to be restless over purchase occasions. For example, in most random-utility choice models there is an idiosyncratic “error term” as well as other changes in the choice environment (e.g., McFadden 1986). Without further study, we do not know whether an index strategy is a good solution to such restless problems.

We recognize that the canonical forward-looking experiential learning problem belongs to the general class of restless bandits because of the presence of utility shocks. In §5 we prove that the problem is indexable and, thus, a well-defined index solution ex-

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11 The indexability of restless bandits is problem-specific. For example, Niño-Mora (2001) takes the achievable region approach (Bertsimas and Niño-Mora 2000) and establishes the indexability of a class of restless bandit problems with linear performance measures (e.g., queue input control). Glazebrook, et al. (2006) show that a special class of restless bandit problems — stochastic scheduling — is indexable. To our knowledge, no general result analogous to Gittins’ Index Theorem exists as of today.

12 The error term has been modeled as an unobserved (to the researcher) state variable in structural applications (Rust 1994, Chapter 51, Sections 3.1 to 3.2). This modeling approach “provides a natural way to ‘rationalize’ discrepancies between observed behavior and the predictions of the discrete decision process model” (Rust 1994, p. 3101). This is different from the “optimal choice plus noise/measurement error” approach.
ists in the sense of Whittle (1988). Moreover, we explore the key properties of such an index, which shed light on how consumers may behave in solving the learning problem.

4. An Index Strategy in the Absence of Utility Shocks

The learning problem we examine includes utility shocks, but it is easier to illustrate the intuition of index strategies using a problem without utility shocks. Temporarily assume $\bar{x}_{jt} = \epsilon_{jt} = 0$ for all $j$ and $t$, although the same result holds when there is no inter-temporal variation in $\bar{x}_{jt}$ and $\epsilon_{jt}$. In this special case, the consumer’s decision problem is a classic multi-arm bandit.

Gittins’ insight is as follows. To evaluate a brand $j$, the consumer thinks as if he or she is choosing between this brand and a reward $\lambda_j$ that is fixed for all future purchase occasions. The consumer thus solves a sub-problem at each purchase occasion – the consumer can either sample this brand to gain more information about it, or exploit the fixed reward $\lambda_j$. In the latter case, the consumer’s belief about brand $j$ ceases to evolve, such that $s_{j,t+1} = s_{jt}$. The optimal solution to this sub-problem is determined by a greatly simplified version of the Bellman equation:

$$V(s_{jt}, \lambda_j) = \max \{ \lambda_j + \delta V(s_{jt}, \lambda_j), \mathbb{E}[q_j + \delta V(s_{j,t+1}, \lambda_j)|s_{jt}] \}.$$  

Notice that each sub-problem only depends on the state evolution of a single brand, $j$. The sub-problem is much simpler than the full problem specified in Equation 3.

Gittins’ index, $G(s_{jt})$, is defined as the smallest value of $\lambda_j$ such that the consumer at purchase occasion $t$ is just indifferent between experiencing brand $j$ and receiving the fixed reward. That is, we obtain $G(s_{jt})$ by equating the two terms inside the maximization operator of Equation 4. Gittins proposed that $G(s_{jt})$ could be used as a measuring
device for the value of exploring brand $j$ – if there is more uncertainty about a brand left to explore, the consumer will demand a higher fixed reward to be willing to stop exploration. Naturally, Gittins’ index is updated when new information arrives.

Gittins’ surprising result is the Index Theorem. The optimal solution is to choose the brand with the highest index at each purchase occasion. A computationally difficult problem has thus been decomposed into $J$ simpler sub-problems.

**Index Theorem** (Gittins and Jones 1974). The optimal decision strategy when there are no utility shocks is

\[ \Pi_G(\bar{s}_t) = \arg\max_{j \in A} G(s_{jt}). \]

Figure 2 illustrates intuitive properties of Gittins’ index. We consider one brand. The solid line plots one realization of Gittins’ index as it evolves when the brand is chosen repeatedly. The dashed line plots the consumer’s posterior mean quality belief. It is updated by brand experience and converges toward the true brand quality. Myopic consumers would exploit experience and choose the brand that yields the highest posterior mean quality. Forward-looking consumers may want to explore further. The dotted curve, which is simply the difference between Gittins’ index and the posterior mean quality, measures the value of exploration. This curve declines smoothly with experience because the value of exploration decreases as the consumer learns more about brand quality.

When we plot Gittins’ index as a function of the consumer’s posterior quality uncertainty $\bar{\sigma}_{jt}$ (not shown), it is also intuitive – the index increases with $\bar{\sigma}_{jt}$ because the value of exploration increases with the remaining amount of quality uncertainty. Figure 2 and the simple relationship between Gittins’ index and posterior quality beliefs suggest that a consumer might intuit something close to the dotted curve if there were no utility shocks.

[Insert Figure 2 about here.]
5. An Index Strategy in the Presence of Utility Shocks

We now allow utility shocks. Observable shocks \( x_{jt} \) include effects that researchers observe and model, such as changes in advertising, promotion, or price. Unobservable shocks \( e_{jt} \) include effects that researchers do not observe and which do not provide a signal about quality. The presence of unobservable shocks is central to many empirical consumer choice models. Because shocks enter the utility function regardless of the consumer’s decisions, the consumer may, in any purchase occasion, switch among brands.\(^{13}\)

When the model includes utility shocks, the Gittins-Jones Index Theorem no longer applies because the states of non-chosen brands do not remain constant. With shocks, the consumer’s problem belongs to the class of restless-bandit problems as introduced by Whittle (1988). In general, such optimization problems are PSPACE-hard (Papadimitriou and Tsitsiklis 1999, Theorem 4) making the problem extremely difficult, if not infeasible, to solve and making it implausible that the consumer would use a solution strategy based on Equation 3. Among other difficulties, PSPACE-hard problems require extremely large memory – a particularly scarce resource for consumers (e.g., Bettman 1979, p. 140; Lindsay and Norman 1977, p. 306). We develop a theoretical solution to this problem in this section. We will show that the canonical forward-looking experiential learning problem is indexable and index strategies have intuitive properties.

5.1. The Canonical Forward-Looking Experiential Learning Problem is Indexable

Whittle (1988) proposed a solution that generalizes Gittins’ index. At each purchase oc-

\(^{13}\) Even when there is no learning, a typical empirical model of consumer choices may include a shock, or an idiosyncratic error, \( e_{jt} \), that is treated as unobservable by researchers. Without this shock, the model would predict that the consumer makes the same choice over purchase occasions if all other observable factors remain constant. In the context of learning, incorporating this shock allows for switching among brands even when the consumer has learned much about brand quality.
casion, to evaluate a brand \( j \), the consumer thinks as if he or she must choose between brand \( j \) and a reward \( \lambda_j \) that is fixed for all future purchase occasions. The Bellman equation for the \( j^{th} \) sub-problem, which now includes utility shocks, becomes

\[
V(s_{jt}, \tilde{x}_{jt}, \epsilon_{jt}, \lambda_j) = \max \{ \lambda_j + \delta \mathbb{E}[V(s_{jt}, \tilde{x}_{jt+1}, \epsilon_{jt+1}, \lambda_{jt})], \beta^t \tilde{x}_{jt} + \epsilon_{jt} + \mathbb{E}[q_{jt} + \\
\delta V(s_{jt+1}, \tilde{x}_{jt+1}, \epsilon_{jt+1}, \lambda_j) | s_{jt}] \}.
\]

The index is defined as the smallest value of \( \lambda_j \) such that the consumer at purchase occasion \( t \) is just indifferent between choosing brand \( j \) and receiving the fixed reward. For such an index to be well-defined and meaningful, the indexability condition need to be satisfied (Whittle 1988). Let \( S_t(\lambda_j) \subseteq \Omega \times X \times E \) be the set of states for which choosing \( \lambda_j \) at purchase occasion \( t \) is optimal:

\[
S_t(\lambda_j) = \{(s_{jt}, \tilde{x}_{jt}, \epsilon_{jt}) \in \Omega \times X \times E : \lambda_j + \delta \mathbb{E}[V(s_{jt}, \tilde{x}_{jt+1}, \epsilon_{jt+1}, \lambda_{jt})] \geq \beta^t \tilde{x}_{jt} + \epsilon_{jt} + \mathbb{E}[q_{jt} + \delta V(s_{jt+1}, \tilde{x}_{jt+1}, \epsilon_{jt+1}, \lambda_j) | s_{jt}] \}.
\]

Indexability is defined as follows:

**Definition:** A brand \( j \) is indexable if, for any \( t \), \( S_t(\lambda_j) \subseteq S_t(\lambda'_j) \) for any \( \lambda_j < \lambda'_j \).

Indexability requires that, as the fixed reward increases, the collection of states for which the fixed reward is optimal does not decrease. In other words, if in some state it is optimal to choose the fixed reward, it must also be optimal to choose a higher fixed reward. Indexability implies a consistent ordering of brands for any state, so an index strategy is meaningful. However, indexability need not always hold in general and can not be
taken for granted (Whittle 1988).\footnote{For example, Whittle (1988, p. 297) provides a simple example where indexability fails.} Thus, before we can posit an index strategy as a consumer heuristic, we must establish indexability for a model that includes utility shocks. In Online Appendix A we prove the following proposition.

**Proposition 1.** The canonical forward-looking experiential learning problem defined in §2 is indexable.

Once the indexability condition is established, then a well-defined strategy is to choose at each purchase occasion the brand with the largest index. The index strategy breaks the curse of dimensionality by decomposing a problem with exponential complexity into $J$ much-simpler sub-problems, each on a state space of $|\Omega|$ after integrating out the utility shocks $\tilde{x}_{jt}$ and $\epsilon_{jt}$. With this simplification, it is more plausible that the consumer might use the index strategy. As a bonus, estimation is much faster. The difference in the size of the state space can be dramatic. For example, suppose we were interested in the mean and variance of quality and discretized them with $M$ and $N$ grid points, respectively. With $J$ brands, the state space for index strategies is $M \times N$ for each brand, rather than $(M \times N)^J$ for the original optimization problem given in Equation 3. For $M = N = 10$ and $J = 6$, this is the difference between a state space of 100 (for each of six brands) and 1,000,000,000,000.

**5.2. The Index Strategy is Invariant to Scale and Behaves Intuitively**

Index strategies dramatically simplify the solution, but can the consumer intuit (perhaps approximately) an index strategy? We expect future laboratory experiments to address this issue empirically. In this essay, we argue that index strategies have intuitive proper-
ties and that it is not unreasonable for the consumer to intuit those properties.

An index strategy would be difficult for the consumer to use if the strategy were not invariant to permissible scale transformations. If it is invariant the consumer can intuit (or learn) the basic shape of the index function and use that intuited shape in many situations. Invariance facilitates ecological rationality.\textsuperscript{15} The following results hold for fairly general distributions of quality, $F_j(q_{jt}; \theta_j)$, and joint distributions of utility shocks $\tilde{x}_{jt}$ and $\varepsilon_{jt}$, as long as they have scale and location parameters and the quality belief $B_{jt}(\theta_j; s_{jt})$ is conjugate. To ease interpretation, we assume that $F_j$ and $B_{jt}$ are normal distributions with parameters defined earlier: $\mu_j$ and $\sigma_j$ for true quality; $\tilde{\mu}_{jt}$ and $\tilde{\sigma}_{jt}$ for posterior beliefs about quality; and $\mu_{j,x,e}$ and $\sigma_{j,x,e}$ for utility shocks. In Online Appendix B we prove the following proposition.

Proposition 2. Let $\tilde{W}$ be Whittle’s index for the canonical forward-looking experiential learning problem computed when the posterior mean quality ($\tilde{\mu}_{jt}$) is zero, the mean utility shock ($\mu_{j,x,e}$) is zero, and the inherent variation of quality ($\sigma_j$) is 1. Whittle’s index for any values of these parameters is the following simple function of $\tilde{W}$.

$$W_j(\tilde{\mu}_{jt}, \tilde{\sigma}_{jt}, \tilde{\beta}^{x,j} \tilde{x}_{jt} + \varepsilon_{jt}, \sigma_j, \mu_{j,x,e}, \sigma_{j,x,e}, \delta) = \tilde{\mu}_{jt} + \mu_{j,x,e} + \sigma_j \tilde{W}_j(0, \frac{\tilde{\beta}^{x,j} \tilde{x}_{jt} + \varepsilon_{jt} - \mu_{j,x,e}}{\sigma_j}, 0, 0, \frac{\sigma_{j,x,e}}{\sigma_j}, \delta).$$

Proposition 2 implies that the consumer can simplify his or her mental evaluations by decomposing the index for each brand into (1) the mean utility gained from myopic learning, $\tilde{\mu}_{jt} + \mu_{j,x,e}$, which reflects the exploitation of posterior beliefs, and (2) the incremental benefit of looking forward, $\sigma_j \tilde{W}$, which captures quality information gained.

\textsuperscript{15} Gittins’ index exhibits invariance properties (Gittins 1989).
Through exploration. To assess the value of exploration, the consumer need only intu\textsuperscript{it} the shape of \( \bar{W} \) for a limited range of parameter values and scale it by \( \sigma_f \). Proposition 2 also helps researchers understand which parameters can be identified in the index-strategy model.

To provide further intuition, we prove the following proposition in Online Appendix C. The proposition shows that Whittle’s index behaves as expected when the parameters of the problem vary. The consumer likes increases in quality and utility shocks, dislikes inherent uncertainty in quality and utility shocks, but values the ability to learn and, hence, resolve the uncertainty in posterior beliefs about quality.

**Proposition 3.** Whittle’s Index for the canonical forward-looking experiential learning problem (1) increases with the posterior mean of quality (\( \bar{\mu}_f \)), the observable utility shocks (\( \tilde{\gamma}_f \bar{x}_{ft} \)), and the unobservable utility shock (\( \epsilon_{ft} \)), (2) weakly decreases with the inherent uncertainty in quality (\( \sigma_f \)) and the magnitude of uncertainty in the utility shocks (\( \sigma^x_{\varepsilon} \)), and (3) increases with the consumer’s posterior uncertainty about quality (\( \bar{\alpha}_f \)).

Figure 3 illustrates Whittle’s index where we set the posterior mean quality to zero, so that the curve represents the value of exploration. (More generally, Whittle’s index fluctuates with the posterior mean quality in a way similar to Figure 2.) As was the case for Gittins’ index, the value-of-exploration component of Whittle’s index is a smooth decreasing function of experience because experience reduces posterior quality uncertainty. With sufficient experience, the value of exploration converges toward zero implying that, asymptotically, the value of a brand is based on the posterior mean of quality (Proposition 2). Unlike Gittins’ index, Whittle’s index is a function of the magnitude of utility.
shocks \((\sigma_j^{x,e})\). As the magnitude of utility shocks becomes larger, it is less important for the consumer to explore, and the value of exploration decreases as shown in Figure 3. These properties and the shape of the curve itself, are intuitive.

[Insert Figure 3 about here.]

Figure 3 and Proposition 3 suggest that, other things being equal, when the magnitude of the uncertainty in utility shocks is larger, the realized utility shocks are more likely to be the deciding factor in consumers’ brand choices. For example, as the depth of price promotions increases, consumers are more likely to base their purchase decisions on price. When \(\sigma_j^{x,e} = 5\) (compared with inherent quality uncertainty normalized as \(\sigma_j = 1\)), Whittle’s index is almost flat implying an almost myopic strategy. To formalize this insight, we state the following Corollary to Proposition 3:

**Corollary.** (1) As the consumer’s posterior uncertainty in quality increases relative to the magnitude of the utility shocks, the value to the consumer from looking forward increases. (2) As the magnitude of the utility shocks increases relative to the consumer’s posterior uncertainty in quality, the value from looking forward decreases. In this latter case, a myopic leaning strategy (i.e., exploiting posterior beliefs) may suffice, and could be the optimal strategy if it is cognitively simpler than a forward-looking learning strategy.

These results highlight the intricate relationship between the consumer’s uncertainty in quality and uncertainty caused by utility shocks. The two types of uncertainty complement each other in driving the consumer’s value of exploitation, but may compete with each other in shaping the consumer’s value of exploration. The index solution offers an intuitive description of this relationship. In §6 and §7, we examine the empirical per-
formance of the index strategy.

6. Examination of the Near Optimality of an Index Strategy

We now examine whether an index strategy implies a reasonable tradeoff between optimality and simplicity. Indexability guarantees existence of a well-defined index strategy but does not guarantee its optimality. For the canonical forward-looking experiential learning problem, the performance of the index strategy is an empirical question. Cognitive costs remain unobservable, but §4 and §5 suggest that an index strategy could be substantially simpler than the direct solution of the Bellman equation to the overall problem. To examine whether the loss in utility is small, we switch from analytic derivations to synthetic data because the loss in utility is an issue of magnitude rather than direction. Synthetic data establish existence (rather than universality) of situations where index strategies are close to optimal.

For concreteness we examine the special case when $F_j$ and $B_{jt}$ are normal distributions. From the perspective of consumer decision-making, what matters is the joint distribution of observable shocks ($\bar{x}_{jt}$) and unobservable shocks ($\epsilon_{jt}$). Therefore, for the synthetic-data analysis we set observable shocks to zero without loss of generality. Practically, even if there are no observable shocks (e.g., no price promotions), unobservable shocks (e.g., idiosyncratic taste fluctuations) are still likely to prevail in most choice models. We allow for both observable and unobservable shocks in the field-data analysis.

We compare four decision strategies that the consumer might use.

1. **No Learning.** In this strategy the consumer chooses the brand based only on the

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16 Many performance bounds have been developed under different contexts and conditions. See Gittins et al. (2011) for a review of recent developments.
consumer’s prior beliefs of quality and the current utility shocks. This strategy provides a baseline to evaluate the incremental value of learning.

2. **Myopic Learning.** In this strategy the consumer chooses the brand based only on the consumer’s posterior quality beliefs and the current utility shocks. This strategy exploits the consumer’s posterior knowledge about brand quality. The Corollary predicts that this strategy will suffice when the magnitude of utility shocks is relatively high compared with posterior quality uncertainty.

3. **Index Strategy.** This strategy assumes the consumer can intuit the shape of Whittle’s index. As per Proposition 2, this strategy improves on the myopic learning strategy to take into account the exploration value of learning. Brand choices reflect the consumer’s tradeoff between exploitation and exploration.

4. **Approximately Optimal.** The PSPACE-hard forward-looking experiential learning problem cannot be solved optimally, hence researchers resort to approximate solutions (e.g., Keane and Wolpin 1994; Erdem and Keane 1996; Rust 1997a; Ackerberg 2003; Crawford and Shum 2006; Imai et al. 2009; Ching 2010; Ching et al. 2013b). Although approximation methods vary (see Online Appendix G for a review), discrete optimization is a representative method and should converge to the optimal solution with a larger number of grids (Chow and Tsitsiklis 1991; Rust 1996).

We choose parameters that illustrate the phenomena and are empirically plausible. The simulation requires a finite horizon; we select $T = 50$ purchase occasions. If the discount factor is $\delta = 0.90$, truncation to a finite horizon is negligible. We discretize the state space, $s_{jt} = (\bar{\mu}_{jt}, \tilde{\sigma}_{jt})$ into a set of $M \times N$ grid points for each of $J$ brands. We
choose $M \times N = 200 \times 50 = 10^4$, which should be close to optimal in the continuous problem. To simplify integration we draw the utility shocks from a Gumbel distribution with parameters $(\mu_f^j, \sigma_f^j)$ and normalize the location parameter such that the utility shocks have zero unconditional means (Rust 1987, 1994). Inherent uncertainties in quality for both brands, $\sigma_f$, are equal and normalized to 1.

The index strategy evolves on a state-space of size $(M \times N)$ for each of the $J$ brands, whereas the approximately optimal solution evolves on a state-space of size $(M \times N)^J$. We choose $J = 2$ for a conservative test of the relative simplicity of the index strategy.

We vary the parameter values to capture three possibilities: (1) the means and uncertainty both favor one brand, (2) the means are the same but uncertainty favors one brand, and (3) the means and uncertainty favor different brands. Because quality beliefs are relative, we fix the prior mean quality belief of Brand 1 as $\bar{\mu}_{10} = 0$ and vary the prior mean quality belief for Brand 2 as $\bar{\mu}_{20} \in \{-0.3, 0, 0.3\}$. We normalize the standard deviation of Brand 2’s prior quality belief as $\bar{\sigma}_{20} = 1$ and the standard deviation of Brand 1’s prior quality belief as $\bar{\sigma}_{10} = 0.5$. Finally, to test the Corollary we allow the uncertainty in shocks to vary from relatively small to relatively large: $\sigma_f^c = \sigma_f^c \in \{0.1, 1\}$.

We compute the indices and the consumer’s expected total utilities for 50 purchase occasions under the four decision strategies. Details are provided in Online Appendices D and E. Table 1 summarizes the results.

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17 We choose $M = 200$ grid points for the posterior mean quality. Meanwhile, we fix each brand’s prior quality variance. Posterior quality variance evolves deterministically following Bayesian updating formulae. Because there are $T = 50$ purchase occasions, a brand’s posterior quality variance has $N = T = 50$ possible values, depending on how many times this brand has been chosen. Therefore, the size of the state space for the index strategy is $M \times N = 200 \times 50 = 10^4$. 

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We first examine computation times as surrogates for cognitive complexity. As expected, the no-learning and myopic learning strategies impose negligible computation time, the index strategy requires moderate computation time, and the approximately optimal solution is substantially slower—600 times as time-consuming as the index strategy even for this basic problem. Faster approximation algorithms would reduce the computational time for the approximately optimal solution (Keane and Wolpin 1994; Rust 1997a; Imai, et al. 2009), but they would also expedite the index strategy because we use the same algorithm for solving the Bellman equations in both models (see Online Appendix G for implementation details). Moreover, faster approximation algorithms do not address the curse of dimensionality. The ratio of computational times in Table 1 could be made arbitrarily large with finer grid points or with a larger number of brands.

We next examine the consumer’s expected utilities. In all cases, the no-learning strategy leads to the lowest utility, which suggests that learning is valuable. Furthermore, the index strategy is statistically indistinguishable from the approximately optimal strategy. As long as cognitive simplicity matters even a little, the index strategy will be better on utility minus complexity.

Finally, the results are consistent with the Corollary. When there is relatively low uncertainty in utility shocks (upper panel of Table 1), the index strategy and the approximately optimal strategy generate higher utility than myopic learning, and, in two of the three cases, significantly higher utility. When there is relatively high uncertainty in utility shocks (lower panel of Table 1), the myopic learning model performs virtually the same as either the index strategy or the approximately optimal strategy. The differences are not
significant. In this case, the consumer might achieve the best utility minus complexity with a myopic strategy, among the models tested.

Analysis of synthetic data never covers all cases. Table 1 is best interpreted as providing evidence that (1) there exist reasonable situations where an index solution is better than the approximately optimal solution on utility minus complexity and (2) there exist domains where myopic learning is best on utility minus complexity. We now examine field data.

7. Field Estimation of an Index Strategy

We examine how an index solution fits and predicts behaviors compared with an approximately optimal solution and myopic learning. As a first test, we seek a product category and sample where consumers are likely to be forward-looking. Even if an index solution does no better than an approximately optimal solution, we consider the result promising because an index solution is cognitively simpler. As a test of face validity, we expect learning strategies to outperform no-learning strategies and, because we focus on a situation that favors forward-looking behavior, we expect forward-looking strategies to outperform myopic learning.

7.1. IRI Data on Diaper Purchases

We select the diaper category from the IRI Marketing Dataset that is maintained by the SymphonyIRI Group and available to academic researchers (Bronnenberg, et al. 2008).18 Diaper consumers are likely to be learning and forward-looking. Parents typically begin

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18 In comparison, durable goods may induce different learning dynamics. Because of the low purchase frequency, consumers may not have the opportunity to learn by sampling. Also, because the stakes are often high, consumers may have the motivation to acquire other types of information (e.g., Consumer Reports reviews) prior to purchase. The ISMS durables goods dataset (Ni et al. 2012) provides a good resource to study these learning dynamics.
purchasing diapers based on a discrete birth event, and their entry to the category is arguably exogenous (Ching et al. 2010, 2012). Even if the birth is a second or subsequent child, diaper quality may have changed. Informal qualitative interviews suggest that parents learn about whether diaper brands match their needs through experience (with often more than one purchase), that diapers are sufficiently important that parents take learning seriously, and that parents often try multiple brands before settling on a favorite brand. In fact, Ching et al. (2012) find that diaper consumers conduct strategic trials of various brands.\textsuperscript{19} There are observable shocks due to price promotions and shocks due to unobservable events. For example, a baby might go through a stage where a different brand is best suited to the parent/child's needs. Finally, diapers have the advantage of being regular purchases, where the no-choice option is less of a concern, and consumers tend to be in the market for many purchase occasions.

To isolate a situation favoring forward-looking learning, we apply the following sample screening criteria. First, to focus on consumers whose purchases are likely triggered by a birth event, we select households whose first purchase occurs 30 weeks after the start of data collection (73% of the entire sample). Second, we focus on frequent buyers. Compared with occasional buyers who might be shopping for a baby shower, frequent buyers are more likely to have both the motivation and the opportunity to explore different diaper brands. Therefore, we select households who have made at least 5 pur-

\textsuperscript{19} Ching et al. (2012) use a quasi-structural approach, where they model the consumer's expected future payoffs as a function of state variables. Their model detects strategic trial if the coefficients of expected future payoffs are significant and if model fit improves significantly over the myopic model.
chases during the observation window (39% of the entire sample). Third, to focus further, we eliminate any consumers who have purchased private labels and restrict attention to consumers who buy exclusively branded products (64% of the entire sample). To the extent that private label buyers are more price sensitive (Hansen, Singh, and Chintagunta 2006), they may be less interested in learning about product quality. (In §8.4, we reanalyze the data by including private labels.) After applying these screening criteria, the data contain 262 households who made 3,379 purchases (13 purchases per household on average). We randomly select 131 households for estimation and 131 households for validation.

The market is dominated by three major brands, Pampers, Huggies, and Luvs. We aggregate all other branded purchases as “Other Brands” and do not model the no-purchase option. As a first-order view, Table 2a compares market-shared-weighted switching behavior during the first 13 purchases with that after the first 13 purchases. There is a noticeable change in switching patterns. For example, the relative brand loyalty of Huggies increases after 13 purchases. This suggests that consumers may care about brand quality over purchases. Although the category is chosen as a likely test-bed for

20 Analyses based on a random selection of buyers rather than frequent buyers are available from the authors. Likely because infrequent buyers have less incentive or opportunity to learn, the myopic learning model does better on this random selection of buyers than on frequent buyers.

21 The data only record the week, as opposed to the exact time, of purchase. Therefore, if a consumer makes multiple purchases during the same week, we do not observe the sequence of brands purchased. Rather than make potentially erroneous assumptions about the data, we remove consumers who make multiple-brand purchases in any week of the observation window (11% of the entire sample). An alternative analysis strategy might have been to randomize purchase orders. However, there is no reason to expect that removing consumers who make multiple purchases a week will affect the comparison between the index strategy and the approximately optimal solution. We also do not model purchase quantity decisions. Instead we assume that consumers update their quality beliefs after each purchase (and consumption) occasion.

22 We define market share at the purchase level across the observation window, so that market shares before and after the first 13 purchases add up to 100%. For readers who wish to normalize Table 2 in other ways, the raw counts are obtained by multiplying the percentages in Table 2 by 1,407, the total number of purchases in the estimation sample except the last purchase of each household.
consumer learning, high brand loyalty, even during the initial 13 purchases, suggests that there is no guarantee a forward-looking strategy will fit the data.

[Insert Table 2 about here.]

7.2. Empirical Specification

We denote households by $i$ and denote by $T_i$ household $i$’s purchase-occasion horizon. We assume that the quality and quality-belief distributions, $F_j$ and $B_{ijt}$, are normal and that unobservable shock distributions are Gumbel. For this initial test of an index solution, we limit $\tilde{x}_{jt}$ to the weekly average prices. The decision strategies are specified below ($\beta$ and $x_{jt}$ are now scalars).

**No Learning:**

$$\Pi_N = \text{argmax}_j \{ \tilde{\mu}_{ij0} + \beta x_{jt} + \epsilon_{ijt} \} ,$$

**Myopic Learning:**

$$\Pi_M = \text{argmax}_j \{ \tilde{\mu}_{ijt} + \beta x_{jt} + \epsilon_{ijt} \} ,$$

**Index Strategy:**

$$\Pi_W = \text{argmax}_j \left\{ \tilde{\mu}_{ijt} + \mu_j^{x,e} + \sigma_j \tilde{W}_j \left( 0, \frac{\tilde{\sigma}_{ijt}}{\sigma_j}, \frac{\beta \tilde{\sigma}_{ijt} + \epsilon_{ijt} - \mu_j^{x,e}}{\sigma_j}, 1, 0, \frac{\tilde{\sigma}_{ijt}}{\sigma_j}, \delta \right) \right\} ,$$

**Approximately Optimal:**

$$\Pi_A = \text{argmax}_j \{ \tilde{\mu}_{ijt} + \beta x_{jt} + \epsilon_{ijt} + \delta \tilde{E}[V(\tilde{s}_{i,t+1}, \tilde{x}_{t+1, i}, \tilde{\epsilon}_{i,t+1}, \tilde{s}_{it}, j)] \} .$$

7.3. Issues of Identification

Although we would like to identify all parameters of the various models, we cannot do so from choice data alone because utility is only specified to an affine transformation, and because the parameters that matter are relative parameters. For the no-learning model we can identify only the relative means of prior beliefs, as well as the price sensitivity parameter $\beta$. For the myopic learning model we can identify only the relative means of prior beliefs, the relative uncertainties of prior beliefs, the true means of quality, and price sensitivity. For the no-learning and myopic learning models time discounting does not matter.
For the index strategy and approximately optimal strategy we set the mean prior belief of one brand ($\tilde{\mu}_{i10}$) to zero and normalize its variance of quality ($\sigma_1$) to one to set the scale of quality. (Only $\bar{\sigma}_{i10}/\sigma_j$ matters.) We cannot simultaneously identify a brand-specific mean of quality and a brand-specific mean of the unobservable shock, so we set the latter to zero ($\mu_j^e = 0$). The standard deviation of $\chi_j$ is observed in the data. We can then compute $\sigma_j^e$ from $\sigma_j^{x,e}$ because the observable and unobservable shocks are independent. As in most dynamic discrete choice processes (Rust 1994), the discount factor $\delta$ is difficult to estimate; we set it to 0.90.23

Finally, as in Erdem and Keane (1996), we suppress “parameter heterogeneity” among households. We continue to allow each household’s quality beliefs to evolve idiosyncratically, but we do not attempt to estimate heterogeneity in prior beliefs, true mean quality, or the magnitude of utility shocks. We abstract away from parameter heterogeneity for the following reasons. First, there are, on average, only 13 purchases per household. We would overly strain the model by attempting to estimate heterogeneity in all of the parameters.24 Second, we wish to focus on behavioral heterogeneity that arises endogenously from forward-looking learning. Even if households start with exogenously homogeneous prior beliefs, different quality realizations and utility shocks lead to different posterior beliefs, different exploitation-versus-exploration tradeoffs, and different learning paths (e.g., Ching, et al. 2013a). We seek to evaluate heterogeneous learning dy-

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23 Sensitivity analyses with other discount rates (e.g., 0.95 and 0.99) yield almost identical log-likelihood statistics and similar parameter estimates for the index strategy model. Anticipating the results of §7.4, we expect a similar lack of sensitivity for the approximately optimal strategy. The ease with which such sensitivity checks can be run is a benefit of the computational tractability of the index strategy model.

24 Doing so is technically feasible, but would likely over-parameterize the model and exploit noise in the data. More importantly, our goal is to demonstrate that an index solution is a viable representation of cognitive simplicity and that cognitive simplicity is a phenomenon worth studying in structural models. We leave explicit modeling of parameter heterogeneity to future research. §8.3 explores foresight heterogeneity.
amics based on the data, rather than using heterogeneous parameters to fit the data. For an initial test of an index strategy, this simplification is conservative because it biases against a good model fit.

We estimate each model’s parameters with maximum simulated likelihood estimation. Estimation details are provided in Online Appendices F and G.

7.4. Estimation Results

Table 3 summarizes the fit statistics for the 1,538 diaper purchases in the in-sample estimation and the 1,841 purchases in the out-of-sample validation. $U^2$ is an information-theoretic measure that calculates the percent of uncertainty explained by the model (Hauser 1978); AIC and BIC attempt to correct the likelihood function based on the number of parameters in the in-sample estimation, BIC more so than AIC. (There are no free parameters in the out-of-sample validation.)

For comparability, we estimated the index strategy in two ways. The first estimation discretizes the state space in the same manner as the approximately optimal model ($M = N = 5$). This enables an “apples-to-apples” comparison. Then, because the index model does not suffer from the curse of dimensionality, we re-estimate the model with a finer grid ($M = 200, N = 75$). There were only trivial differences. For example, $U^2 = 88.18\%$ for both estimations, and parameter values are not significantly different (nor different from the approximately optimal model). We will report the results associated with the finer grid for the rest of the essay.

[Insert Table 3 about here.]

First, on all measures there are sizable gains to learning – all learning models explain and predict brand choices substantially better than the no-learning strategy. Second,
the index strategy improves in-sample fit and out-of-sample predictions relative to myopic learning. The likelihood is significantly better (Vuong test significance is \( p = 0.0002 \) in-sample and 0.0429 out-of-sample).\(^{25}\) This result is consistent with our expectation that frequent buyers of branded diapers are forward-looking. Third, the index strategy performs as well as the approximately optimal solution in terms of both in-sample fit and out-of-sample predictions. This result is consistent with the synthetic-data analysis – when two strategies yield almost the same expected utilities and hence predict almost the same brand choices, they are observationally equivalent and statistically indistinguishable.

As a further visualization of model fit, Table 2b reports the predicted market-share-weighted switching patterns. The predicted switching patterns are qualitatively similar to actual switching patterns in Table 2a. For example, the index strategy model picks up the fact that consumers are more loyal to Huggies than to the other brands because, as we will discuss below, the true mean quality is higher for Huggies and it is likely more rewarding to learn about Huggies (\( \sigma_f x^e \) being relatively small). Although predictions are not perfect and could be improved if other \( x \)-variables were observed, the overall MAE is within 0.8% of actual switching. Moreover, the predicted switching patterns from the index strategy model are virtually identical to those from the approximately optimal solution model (reported in Online Appendix H). The market-share-weighted mean absolute error (MAE) is approximately \( 3/100^{ths} \) of 1%.

Table 4 summarizes the estimated parameter values. As expected, the price sensitivity coefficient is negative in every model. Across all learning models, all four brands

\(^{25}\) We use the Vuong test to compare non-nested models (Vuong 1989).
increase in mean quality relative to prior beliefs, which implies that diaper buyers learn to purchase these brands more through experience. These results are consistent with the switching patterns in Table 2a.

Forward-looking models identify the magnitude of utility shocks relative to inherent quality uncertainty (last panel of Table 4). Because the relative shock uncertainty varies across brands, the index curve implies different behavior than myopic learning for those brands. This explains why forward-looking models fit and predict better than myopic learning. For example, Huggies has lower relative shock uncertainty than other brands, which may provide greater incentives for consumers to explore Huggies. Because the myopic learning model ignores this difference, it compensate by overestimating the mean prior belief of Huggies. Managerially, Huggies has a higher true mean quality than Pampers and Luvs, but also higher inherent relative uncertainty in quality across consumption. (The table reports the ratio of shock uncertainty to quality uncertainty — a smaller number means higher relative quality uncertainty.)

Both the index strategy and the approximately optimal strategy lead to similar parameter estimates. Parameter estimates of either model are usually within confidence regions of the alternative model. This result is consistent with the synthetic-data analysis, which suggests that both strategies lead to near optimal utility. The index strategy will be a more plausible description of consumer behavior if it is cognitively simpler. We explore this last point below.

Computation time in the embedded optimization problem is one surrogate for cognitive complexity. The last row of Table 3 reports the time necessary to compute one
likelihood function in each model. For the index strategy model we report the computation time for both the original grid \((M = N = 5)\) and the finer grid \((M = 200, N = 75)\) — the latter is in parentheses. Consistent with the synthetic-data analysis, the index strategy is substantially faster than the approximately optimal strategy (74-to-1 ratio based on the same grid density of \(M = N = 5\)).

The size of the state space is another surrogate for cognitive complexity (e.g., a consumer’s memory). The state space for the approximately optimal strategy is 15,625 times as large as the state space for the index strategy given the same grid density of \(M = N = 5\). Computational-time ratios are not equal to state-space ratios due to computational overhead. Nonetheless, if we were to attempt to use the finer grid of \(M = 200, N = 75\) for the approximately optimal strategy, we would increase the state space of the approximately optimal solution by a factor of 130 billion. It is unlikely that approximately optimal computations would be feasible for the finer grid. Detailed calculations are presented in Online Appendix G.

In summary, using IRI data on diaper purchases we find that (1) learning models fit and predict substantially better than the no-learning model; (2) forward-looking learning models fit and predict significantly better than the myopic learning model; (3) the index strategy and the approximately optimal solution achieve similar in-sample fit and out-of-sample forecasts, as well as reasonably close parameter estimates; and (4) computational (and cognitive) simplicity favors the index strategy model relative to the approximately optimal model.

8. Further Explorations

We have shown that the canonical forward-looking experiential learning model is index-
able and that an index strategy performs well. We now extend the analysis to explore consumer risk aversion, other cognitively simple heuristics, heterogeneous consumer foresight, and private-labels.

8.1. Risk Aversion

For ease of exposition, in previous sections we assumed that consumers are risk neutral. However, risk aversion can be an important issue for decision-making under uncertainty (see Ching et al. 2013a for a review). We generalize our model to incorporate risk aversion following the standard discounted-utility approach (e.g., Samuelson 1937; Erdem and Keane 1996). At each purchase occasion $t$, the consumer maximizes

$$
\sum_{t=1}^{\infty} \delta^{t-t} u(w_t),
$$

where $w_t$ is the net payoff the consumer receives at purchase occasion $t$, and $u(\cdot)$ is the consumer's utility function. Utility increases with net payoff (i.e., $u' > 0$). In addition, the curvature of the utility function captures general risk preferences: the consumer is risk neutral if $u'' = 0$, risk averse if $u'' < 0$, and risk seeking if $u'' > 0$. The Bellman equation for the sub-problem of the $j^{th}$ brand (Equation 5) is generalized as:

$$
V(s_{jt}, \bar{x}_{jt}, \epsilon_{jt}, \lambda_j) = \max \left\{ u(\lambda_j) + \delta \mathbb{E}[V(s_{jt}, \bar{x}_{j,t+1}, \epsilon_{j,t+1}, \lambda_j)], \mathbb{E}[u(\bar{\beta}\bar{x}_{jt} + \epsilon_{jt} + q_j)|s_{jt}] + \delta \mathbb{E}[V(s_{j,t+1}, \bar{x}_{j,t+1}, \epsilon_{j,t+1}, \lambda_j)|s_{jt}] \right\}
$$

We prove in Online Appendix A.2 that the generalized canonical forward-looking experiential learning model is indexable. This is true for all consumer utility functions satisfying $u' > 0$.

The indexability result allows us to test for risk aversion at low computational costs. We do so using the diaper data. To parameterize the test, we assume that consumers exhibit constant absolute risk aversion: $u(w) = 1 - e^{-rw}$, where $r > 0$ measures the
degree of risk aversion (e.g., Roberts and Urban 1988). Based on this utility function, we re-estimate the index strategy model. In addition, we re-estimate the myopic learning model to see whether general risk preferences as opposed to the exploration incentive suffice to explain consumer choices.

Table 3 reports the fit statistics. Allowing for risk aversion brings little improvement to the likelihood and $U^2$, and worsens the AIC and BIC because of the extra risk-aversion parameter. Table 4 reports the parameter estimates. The risk-aversion parameter is insignificant for the myopic learning model; it is marginally significant for the index strategy model but the magnitude is small. Diaper buyers in our sample do not seem to be strongly risk averse. Because the approximately optimal model provides parameter estimates that are close to the index model for the risk neutral case, we expect similar results if we were to estimate the approximately optimal model for the risk averse case.

8.2. Other Cognitively Simple Heuristics

The canonical forward-looking experiential learning model assumes that consumers have perfect foresight. But the degree of foresight is an empirical question. Ho and Chong (2003) find that a parsimonious myopic model accurately describes and predicts SKU demand. Models in which the decision-maker looks one period ahead sometimes explain choices well (Hauser et al. 1993; Gabaix et al. 2006; Che et al. 2007). In the bandit literature, Ny and Feron (2006) explore one-period look-ahead heuristics as approximate solutions to restless bandits with switching costs. A one-period look-ahead model is ar-

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26 For constant risk aversion, $u(w) \rightarrow w$ as $r \rightarrow 0$. Erdem and Keane (1996) express risk aversion with a quadratic utility function.

27 The risk aversion parameter cannot be separately identified from the mean prior beliefs in the no-learning model.

28 The model was used by Procter & Gamble to predict SKU purchases.
guably simpler than the full dynamic optimization problem, and is a heuristic consumers might use. For a one-period look-ahead model, the Bellman equation (Equation 3) is modified as:

\[
V(\tilde{s}_t, \tilde{x}_t, \tilde{e}_t) = \max_{j \in A} \left\{ \beta' \tilde{x}_{jt} + \epsilon_{jt} + \mathbb{E} \left[ q_{jt} + \delta \max_{k \in A} \{ q_{k,t+1} + \beta' \tilde{x}_{k,t+1} + \epsilon_{k,t+1} \} \right] \right\};
\]

(8)

Tables 3 and 4 report the empirical results. The one-period look-ahead model has worse in-sample fit than the index strategy model (Vuong test \( p = 0.0244 \)) and approximately the same out-of-sample prediction (Vuong test \( p = 0.2529 \)). The one-period look-ahead model fits better than the myopic learning model both in-sample (Vuong test \( p = 0.0033 \)) and out-of-sample (Vuong test \( p = 0.0009 \)). These results suggest that diaper consumers are not myopic, although they may not be perfectly forward-looking.

We could easily estimate a variety of cognitively simple heuristics including \( T_l \)-period look-ahead models for \( T_l < T \), Gittins’-index models modified to allow for utility shocks,\(^29\) and various heuristics such as those proposed by Bertsimas and Niño-Mora (2000). For example, a modified-Gittins’-index model \( (U^2 = 88.09\% \text{ in-sample}; \ U^2 = 88.84\% \text{ out-of-sample}) \) does better than myopic learning, but not as well as the index strategy model (which is based on Whittle’s index). We strongly caution against choosing a best-predicting model based on a single dataset. Unrestricted search among models would likely exploit random variation. However, from the good fit and predictive ability of the three tested heuristics (Whittle’s index, one-period look-ahead, and modified Gittins’ index), we are comfortable in our hypotheses that (1) cognitively simple heuristics

\(^{29}\) Specifically, the modified Gittins’ index assumes that the consumer \( i \) at purchase occasion \( t \) chooses the brand with the highest value of \( G(s_{jt}) + \beta' \tilde{x}_{jt} + \epsilon_{jt} \), where \( G(s_{jt}) \) is Gittins’ index derived from the optimization problem in the absence of utility shocks (see §4). The modified Gittins’ index is an ad hoc solution compared with the Whittle’s-index model.
are plausible alternatives to modeling forward-looking behavior and (2) an index strategy is one viable model.

8.3. Heterogeneous Foresight

In an alternative approach we allow for heterogeneous consumer foresight. We assume there are two latent consumer segments that represent the two “extremes” of the foresight spectrum. One segment engages in myopic learning and the other segment has perfect foresight. Because the index strategy and the approximately optimal solution are observationally indistinguishable, we assume that the perfect-foresight segment follows the computationally favorable index strategy. We use the latent class method (Kamakura and Russell 1989) to estimate the fraction of consumers belonging to each segment, as well as the set of parameters associated with each segment.

Not surprisingly, as Table 3 shows, the latent class model generates higher likelihood and $U^2$ than both the myopic learning model ($p < 0.0001$ in-sample; $p < 0.0001$ out-of-sample) and the index strategy model ($p = 0.0003$ in-sample; $p = 0.018$ out-of-sample). The flexibility of the latent class model comes at the cost of extra parameters. It produces a slightly better AIC but a worse BIC than the index strategy model.

The last two columns of Table 4 report the parameter estimates of the latent class model. The parameter estimates associated with the respective segments are comparable to values in the homogeneous models. Meanwhile, 77% of diaper buyers are forward-looking. This finding echoes the result from the one-period look-ahead model that the average diaper buyers is neither myopic nor perfectly forward-looking.

8.4. Private Labels

Our primary analyses eliminated any consumer who purchased a private label during the
observation window. This restriction allowed us to focus on a case where we expected forward-looking learning. The decision was also driven by the curse of dimensionality inherent in the approximately optimal solution – adding another brand increases the size of the state space by $M \times N = 25$ times. As a robustness check, we repeat our estimations replacing “other brands” with private labels. Table 5 presents the fit statistics. The relative fit and predictive accuracies are the same as in Table 3. Furthermore, the (unreported) parameter estimates for Pampers, Huggies, and Luvs are not significantly different when comparing the index strategy and approximately optimal models in Tables 3 and 5.

9. Summary, Conclusions, and Future Research

Models of forward-looking experiential learning are important to marketing. These theory-driven models examine how consumers make tradeoffs between exploiting and exploring brand information. Managerially, these models enable researchers to investigate effects due to quality uncertainty, learning, and the variation in utility shocks. However, the consumer problem in these models is computationally intractable (PSPACE-hard). Existing solutions via the Bellman equation require vast computational resources (time and memory) that may contradict cognitive simplicity theories of consumers.

In this essay we propose that consumers use cognitively simple heuristics to solve forward-looking experiential learning problems. We explore one viable heuristic – index strategies. Index strategies represent a solution concept that decomposes a complex problem into a set of much simpler sub-problems. We prove analytically that an index strategy exists for canonical forward-looking experiential learning models and that the index function has simple properties that consumers might intuit. Using synthetic data, we
demonstrate that a well-defined index solution achieves near optimal expected utility and is fast to compute. Using IRI data on diaper purchases, we show that at least one index solution fits the data and predicts out-of-sample significantly better than either a no-learning model or a myopic learning model. Compared with an approximately optimal solution, the index strategy fits equally well, produces similar estimation results (and hence managerial implications), requires significantly lower computational costs and, we believe, is more likely to describe consumer behavior.

We address many issues, but many issues remain. We do not model advertising as a quality signal (the IRI dataset for the diaper category does not track advertising). The consequence of incorporating advertising signals depends on how consumers learn. We abstract away from inventory problems. Inventory effects are found to be insignificant in previous research (Ching et al. 2012), but nevertheless add a dimension to consumers’ dynamic planning. We study standard settings where consumers do not learn from non-chosen alternatives. It would be interesting to model correlated learning or extend index strategies to incorporate hypothetical reinforcement of non-chosen options (Camcer and Ho 1999). Technically, it would also be interesting to examine the indexability of learning models when there are switching costs.30

Diaper buyers are likely forward-looking, but consumers in other product categories may not be. Our theory suggests that consumers are most likely to be forward-looking when shock uncertainty is small compared to quality uncertainty; we expect myopic learning models to do well when shock uncertainty is large. This prediction is testa-

30 Bank and Sundaram (1994) prove that there is no consistent way to define an optimal index in the presence of switching costs among choice alternatives. However, a bandit problem with switching cost can be re-formulated as a restless bandit problem, which could be indexable (Glazebrook et al., 2006, Niño-Mora 2008).
ble using cross-category analysis. For instance, shock uncertainty may be large in hedonic goods categories where consumption value swings with idiosyncratic mood. Shock uncertainty may also be dominant in markets characterized by volatile marketing mix variables. The recent rise of flash sales introduced remarkable price volatility to categories such as food, gadgets, and apparel. It will be interesting to study whether this change serves to promote myopic purchase behaviors.

Finally, an index solution appears to be a reasonable tradeoff for diaper consumers, but our basic hypothesis is that consumers use cognitively simple heuristic strategies. Other cognitively simple heuristics might explain consumer behavior even better than index strategies. §8.2 suggests testable alternatives. Future research can explore these and other heuristics using either field data or laboratory experiments.
References


Rust, J. 1997b. Dealing with the complexity of economic calculations, working paper, Yale University.


Table 1. Comparing Decision Strategies on Utility and Simplicity (Synthetic Data)

<table>
<thead>
<tr>
<th>Size of state space</th>
<th>Expected Discounted Utility (standard errors in parentheses)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No Learning</td>
</tr>
<tr>
<td>Computation time (surrogate for cognitive complexity) $^\dagger$</td>
<td>n/a</td>
</tr>
<tr>
<td>negligible</td>
<td>negligible</td>
</tr>
</tbody>
</table>

Relatively low uncertainty in utility shocks ($\sigma_1^2 = \sigma_2^2 = 0.1$)

Mean of prior quality beliefs (Brand 1, Brand 2)

- $(\bar{\mu}_{10}, \bar{\mu}_{20}) = (0.0, -0.3)$
  - Mean: 0.041, 1.801, 1.992, 1.996
  - Standard Errors: (0.003), (0.043), (0.045), (0.045)

- $(\bar{\mu}_{10}, \bar{\mu}_{20}) = (0.0, 0.0)$
  - Mean: 0.618, 3.352, 3.544, 3.547
  - Standard Errors: (0.003), (0.049), (0.052), (0.052)

- $(\bar{\mu}_{10}, \bar{\mu}_{20}) = (0.0, +0.3)$
  - Mean: 3.036, 5.298, 5.323, 5.327
  - Standard Errors: (0.003), (0.056), (0.056), (0.056)

Relatively high uncertainty in utility shocks ($\sigma_1^2 = \sigma_2^2 = 1$)

Mean of prior quality beliefs (Brand 1, Brand 2)

- $(\bar{\mu}_{10}, \bar{\mu}_{20}) = (0.0, -0.3)$
  - Mean: 4.919, 5.762, 5.767, 5.768
  - Standard Errors: (0.026), (0.047), (0.047), (0.047)

- $(\bar{\mu}_{10}, \bar{\mu}_{20}) = (0.0, 0.0)$
  - Mean: 6.182, 7.150, 7.190, 7.190
  - Standard Errors: (0.027), (0.050), (0.052), (0.052)

- $(\bar{\mu}_{10}, \bar{\mu}_{20}) = (0.0, +0.3)$
  - Mean: 7.946, 8.912, 8.911, 8.912
  - Standard Errors: (0.026), (0.054), (0.054), (0.054)

$^\dagger$ This is the time required to compute one utility function using a university computing system based on Sun Grid Engine (SGE) and Red Hat Enterprise Linux.
Table 2. Switching among Diaper Brands

(a) Actual Switching Matrix

Percent of Times that Row Brand is Purchased at Occasion \( t \) and Column Brand is Purchased at Occasion \( t + 1 \)

<table>
<thead>
<tr>
<th></th>
<th>Pampers</th>
<th>Huggies</th>
<th>Luvs</th>
<th>Other Brands</th>
</tr>
</thead>
<tbody>
<tr>
<td>Within the first 13 purchases</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td>20.3%</td>
<td>3.9%</td>
<td>2.9%</td>
<td>0.5%</td>
</tr>
<tr>
<td>Huggies</td>
<td>3.8%</td>
<td>21.5%</td>
<td>1.6%</td>
<td>0.2%</td>
</tr>
<tr>
<td>Luvs</td>
<td>2.5%</td>
<td>2.4%</td>
<td>12.6%</td>
<td>0.3%</td>
</tr>
<tr>
<td>Other Brands</td>
<td>0.6%</td>
<td>0.4%</td>
<td>0.3%</td>
<td>1.0%</td>
</tr>
</tbody>
</table>

After the first 13 purchases

<table>
<thead>
<tr>
<th></th>
<th>Pampers</th>
<th>Huggies</th>
<th>Luvs</th>
<th>Other Brands</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pampers</td>
<td>6.3%</td>
<td>0.9%</td>
<td>0.7%</td>
<td>0.1%</td>
</tr>
<tr>
<td>Huggies</td>
<td>0.7%</td>
<td>11.2%</td>
<td>0.2%</td>
<td>0.1%</td>
</tr>
<tr>
<td>Luvs</td>
<td>0.9%</td>
<td>0.0%</td>
<td>4.0%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Other Brands</td>
<td>0.1%</td>
<td>0.1%</td>
<td>0.0%</td>
<td>0.0%</td>
</tr>
</tbody>
</table>

(b) Predicted Switching Matrix – Index Strategy Model

Percent of Times that Row Brand is Purchased at Occasion \( t \) and Column Brand is Purchased at Occasion \( t + 1 \)

<table>
<thead>
<tr>
<th></th>
<th>Pampers</th>
<th>Huggies</th>
<th>Luvs</th>
<th>Other Brands</th>
</tr>
</thead>
<tbody>
<tr>
<td>Within the first 13 purchases</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td>20.3%</td>
<td>3.0%</td>
<td>1.9%</td>
<td>0.2%</td>
</tr>
<tr>
<td>Huggies</td>
<td>2.0%</td>
<td>26.1%</td>
<td>1.3%</td>
<td>0.1%</td>
</tr>
<tr>
<td>Luvs</td>
<td>1.6%</td>
<td>1.6%</td>
<td>15.2%</td>
<td>0.1%</td>
</tr>
<tr>
<td>Other Brands</td>
<td>0.3%</td>
<td>0.1%</td>
<td>0.1%</td>
<td>0.9%</td>
</tr>
</tbody>
</table>

After the first 13 purchases

<table>
<thead>
<tr>
<th></th>
<th>Pampers</th>
<th>Huggies</th>
<th>Luvs</th>
<th>Other Brands</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pampers</td>
<td>4.6%</td>
<td>0.5%</td>
<td>0.4%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Huggies</td>
<td>0.3%</td>
<td>15.0%</td>
<td>0.1%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Luvs</td>
<td>0.5%</td>
<td>0.2%</td>
<td>3.4%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Other Brands</td>
<td>0.0%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>0.1%</td>
</tr>
</tbody>
</table>

\(^*\) Switching percentages are weighted by market share so that the percentages in the same table add up to 100%.
Table 3. In-Sample and Out-of-Sample Fit Statistics for Diaper Data

<table>
<thead>
<tr>
<th>No Learning</th>
<th>Myopic Learning</th>
<th>Index Strategy</th>
<th>Approximately Optimal</th>
<th>Myopic Learning with Risk Aversion</th>
<th>Index Strategy with Risk Aversion</th>
<th>One-Period Look-ahead</th>
<th>Heterogeneous Foresight</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Calibration sample</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Log likelihood</td>
<td>-1760.26</td>
<td>-1051.39</td>
<td>-1008.19</td>
<td>-1008.95</td>
<td>-1051.39</td>
<td>-1008.01</td>
<td>-1023.30</td>
</tr>
<tr>
<td>U²</td>
<td>79.36%</td>
<td>87.67%</td>
<td>88.18%</td>
<td>88.17%</td>
<td>87.67%</td>
<td>88.18%</td>
<td>88.00%</td>
</tr>
<tr>
<td>AIC</td>
<td>3528.53</td>
<td>2126.77</td>
<td>2048.39</td>
<td>2049.89</td>
<td>2128.77</td>
<td>2050.02</td>
<td>2072.61</td>
</tr>
<tr>
<td>BIC</td>
<td>3549.88</td>
<td>2190.83</td>
<td>2133.80</td>
<td>2135.31</td>
<td>2198.17</td>
<td>2140.77</td>
<td>2142.00</td>
</tr>
<tr>
<td># parameters</td>
<td>4</td>
<td>12</td>
<td>16</td>
<td>16</td>
<td>13</td>
<td>17</td>
<td>13</td>
</tr>
<tr>
<td># observations</td>
<td>1,538</td>
<td>1,538</td>
<td>1,538</td>
<td>1,538</td>
<td>1,538</td>
<td>1,538</td>
<td>1,538</td>
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<tr>
<td>Hold-out sample</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Log likelihood</td>
<td>-1998.25</td>
<td>-1165.47</td>
<td>-1126.87</td>
<td>-1127.35</td>
<td>-1165.47</td>
<td>-1124.31</td>
<td>-1115.04</td>
</tr>
<tr>
<td>U²</td>
<td>80.43%</td>
<td>88.58%</td>
<td>88.96%</td>
<td>88.96%</td>
<td>88.58%</td>
<td>88.99%</td>
<td>89.08%</td>
</tr>
<tr>
<td># observations</td>
<td>1,841</td>
<td>1,841</td>
<td>1,841</td>
<td>1,841</td>
<td>1,841</td>
<td>1,841</td>
<td>1,841</td>
</tr>
<tr>
<td>Computation time in seconds †</td>
<td>negligible</td>
<td>2</td>
<td>1.4 (22)</td>
<td>104</td>
<td>2</td>
<td>23</td>
<td>11</td>
</tr>
</tbody>
</table>

† This is the time required to compute one likelihood function using a university computing system based on Sun Grid Engine (SGE) and Red Hat Enterprise Linux. For the index strategy model, the computation time is 1.4 seconds for the original grid (M = N = 5) and 22 seconds for the finer grid (M = 200, N = 75).
Table 4. Parameter Estimates for Diaper Data

<table>
<thead>
<tr>
<th></th>
<th>No Learning</th>
<th>Myopic Learning</th>
<th>Index Strategy</th>
<th>Approximately Optimal</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Relative mean of prior beliefs</strong> $\bar{\mu}_j$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Huggies</td>
<td>0.095</td>
<td>0.079</td>
<td>-0.798</td>
<td>-0.198</td>
</tr>
<tr>
<td>(0.059)</td>
<td>(0.156)</td>
<td>(0.717)</td>
<td>(0.194)</td>
<td></td>
</tr>
<tr>
<td>Luvs</td>
<td>-0.716</td>
<td>-0.641</td>
<td>-2.351</td>
<td>-1.381</td>
</tr>
<tr>
<td>(0.081)</td>
<td>(0.177)</td>
<td>(1.626)</td>
<td>(0.534)</td>
<td></td>
</tr>
<tr>
<td>Other Brands</td>
<td>-3.223</td>
<td>-2.761</td>
<td>-3.143</td>
<td>-2.978</td>
</tr>
<tr>
<td>(0.190)</td>
<td>(0.321)</td>
<td>(2.107)</td>
<td>(0.853)</td>
<td></td>
</tr>
<tr>
<td><strong>Uncertainty of prior beliefs</strong> $\sigma_j$ $\bar{\sigma}_j$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td></td>
<td>0.734</td>
<td>0.694</td>
<td>0.724</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.099)</td>
<td>(0.118)</td>
<td>(0.119)</td>
</tr>
<tr>
<td>Huggies</td>
<td></td>
<td>0.476</td>
<td>0.455</td>
<td>0.494</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.066)</td>
<td>(0.294)</td>
<td>(0.153)</td>
</tr>
<tr>
<td>Luvs</td>
<td></td>
<td>0.773</td>
<td>1.126</td>
<td>1.394</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.139)</td>
<td>(0.730)</td>
<td>(0.600)</td>
</tr>
<tr>
<td>Other Brands</td>
<td></td>
<td>1.332</td>
<td>1.428</td>
<td>1.394</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.491)</td>
<td>(1.750)</td>
<td>(0.818)</td>
</tr>
<tr>
<td><strong>True mean quality</strong> $\mu_j$ $\bar{y}_j$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td></td>
<td>3.902</td>
<td>3.686</td>
<td>3.551</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.329)</td>
<td>(2.092)</td>
<td>(0.844)</td>
</tr>
<tr>
<td>Huggies</td>
<td></td>
<td>5.852</td>
<td>8.438</td>
<td>7.850</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.688)</td>
<td>(5.088)</td>
<td>(2.192)</td>
</tr>
<tr>
<td>Luvs</td>
<td></td>
<td>3.666</td>
<td>2.724</td>
<td>2.544</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.502)</td>
<td>(1.544)</td>
<td>(0.704)</td>
</tr>
<tr>
<td>Other Brands</td>
<td></td>
<td>1.630</td>
<td>1.343</td>
<td>1.364</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.820)</td>
<td>(1.080)</td>
<td>(0.856)</td>
</tr>
<tr>
<td><strong>Magnitude of utility shocks</strong> $\bar{k}_j$ $\sigma_j^x$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td></td>
<td></td>
<td>0.837</td>
<td>0.836</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.508)</td>
<td>(0.221)</td>
</tr>
<tr>
<td>Huggies</td>
<td></td>
<td></td>
<td>0.138</td>
<td>0.151</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.084)</td>
<td>(0.040)</td>
</tr>
<tr>
<td>Luvs</td>
<td></td>
<td></td>
<td>0.316</td>
<td>0.347</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.192)</td>
<td>(0.092)</td>
</tr>
<tr>
<td>Other Brands</td>
<td></td>
<td></td>
<td>1.272</td>
<td>1.347</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.772)</td>
<td>(0.357)</td>
</tr>
<tr>
<td><strong>Price Sensitivity</strong> $(\beta)$</td>
<td>-0.126</td>
<td>-0.128</td>
<td>-0.152</td>
<td>-0.153</td>
</tr>
<tr>
<td></td>
<td>(0.020)</td>
<td>(0.026)</td>
<td>(0.094)</td>
<td>(0.048)</td>
</tr>
</tbody>
</table>

For identification: 1. $\bar{\mu}_{10} = 0. 2. \mu_j^* = 0. 3. \sigma_j^x$ observed in the data, $\sigma_j^x$ computed from $\sigma_j^x$ via independence.
Table 4. Parameter Estimates for Diaper Data (Continued)

<table>
<thead>
<tr>
<th></th>
<th>Myopic Learning with Risk Aversion</th>
<th>Index Strategy with Risk Aversion</th>
<th>One-Period Look-ahead</th>
<th>Heterogeneous Foresight (Myopic Learning)</th>
<th>Heterogeneous Foresight (Index Strategy)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Relative mean of prior beliefs ($\bar{\mu}_{10}$)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Huggies</td>
<td>0.007</td>
<td>-0.346</td>
<td>-0.014</td>
<td>-0.479</td>
<td>-1.169</td>
</tr>
<tr>
<td></td>
<td>(0.151)</td>
<td>(0.036)</td>
<td>(0.092)</td>
<td>(0.774)</td>
<td>(1.776)</td>
</tr>
<tr>
<td>Luvs</td>
<td>-0.627</td>
<td>-1.941</td>
<td>-0.819</td>
<td>-1.686</td>
<td>-3.926</td>
</tr>
<tr>
<td></td>
<td>(0.163)</td>
<td>(1.079)</td>
<td>(0.224)</td>
<td>(1.412)</td>
<td>(5.717)</td>
</tr>
<tr>
<td></td>
<td>(0.488)</td>
<td>(2.566)</td>
<td>(3.195)</td>
<td></td>
<td>(2.419)</td>
</tr>
<tr>
<td><strong>Uncertainty of prior beliefs ($\bar{\sigma}<em>{10}$) relative to inherent quality uncertainty ($\sigma</em>{1}$)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td>0.734</td>
<td>0.699</td>
<td>0.564</td>
<td>0.142</td>
<td>0.887</td>
</tr>
<tr>
<td></td>
<td>(0.099)</td>
<td>(0.116)</td>
<td>(0.038)</td>
<td>(0.018)</td>
<td>(0.356)</td>
</tr>
<tr>
<td>Huggies</td>
<td>0.476</td>
<td>0.463</td>
<td>0.424</td>
<td>3.404</td>
<td>0.545</td>
</tr>
<tr>
<td></td>
<td>(0.066)</td>
<td>(0.074)</td>
<td>(2.898)</td>
<td></td>
<td>(0.237)</td>
</tr>
<tr>
<td>Luvs</td>
<td>0.773</td>
<td>1.215</td>
<td>0.740</td>
<td>0.623</td>
<td>1.397</td>
</tr>
<tr>
<td></td>
<td>(0.138)</td>
<td>(0.064)</td>
<td>(0.428)</td>
<td></td>
<td>(1.806)</td>
</tr>
<tr>
<td>Other Brands</td>
<td>1.333</td>
<td>1.423</td>
<td>0.779</td>
<td>0.666</td>
<td>1.391</td>
</tr>
<tr>
<td></td>
<td>(0.487)</td>
<td>(1.931)</td>
<td>(1.146)</td>
<td></td>
<td>(5.697)</td>
</tr>
<tr>
<td><strong>True mean quality ($\mu_{1}$)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.361)</td>
<td>(2.571)</td>
<td>(0.182)</td>
<td>(1.905)</td>
<td>(0.841)</td>
</tr>
<tr>
<td>Huggies</td>
<td>5.728</td>
<td>8.189</td>
<td>6.374</td>
<td>0.226</td>
<td>9.083</td>
</tr>
<tr>
<td></td>
<td>(0.703)</td>
<td>(5.914)</td>
<td>(1.404)</td>
<td>(3.164)</td>
<td>(3.164)</td>
</tr>
<tr>
<td>Luvs</td>
<td>3.542</td>
<td>2.569</td>
<td>2.956</td>
<td>2.774</td>
<td>2.908</td>
</tr>
<tr>
<td></td>
<td>(0.499)</td>
<td>(1.857)</td>
<td>(0.276)</td>
<td>(2.460)</td>
<td>(0.974)</td>
</tr>
<tr>
<td>Other Brands</td>
<td>1.504</td>
<td>1.293</td>
<td>1.985</td>
<td>2.247</td>
<td>2.058</td>
</tr>
<tr>
<td></td>
<td>(0.824)</td>
<td>(1.239)</td>
<td>(0.961)</td>
<td>(9.340)</td>
<td>(0.921)</td>
</tr>
<tr>
<td><strong>Magnitude of utility shocks ($\sigma_{1}\varepsilon$) relative to inherent quality uncertainty</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td>-</td>
<td>1.081</td>
<td>0.273</td>
<td>-</td>
<td>0.885</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.813)</td>
<td>(0.021)</td>
<td>-</td>
<td>(0.219)</td>
</tr>
<tr>
<td>Huggies</td>
<td>-</td>
<td>0.141</td>
<td>0.271</td>
<td>-</td>
<td>0.128</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.106)</td>
<td>(0.020)</td>
<td>-</td>
<td>(0.032)</td>
</tr>
<tr>
<td>Luvs</td>
<td>-</td>
<td>0.317</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>(0.238)</td>
<td>(0.022)</td>
<td>-</td>
<td>(0.057)</td>
</tr>
<tr>
<td>Other Brands</td>
<td>-</td>
<td>1.090</td>
<td>0.275</td>
<td>-</td>
<td>1.156</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.820)</td>
<td>(0.022)</td>
<td>-</td>
<td>(0.289)</td>
</tr>
<tr>
<td><strong>Price Sensitivity ($\beta$)</strong></td>
<td>-0.128</td>
<td>-0.152</td>
<td>-0.160</td>
<td>-0.274</td>
<td>-0.081</td>
</tr>
<tr>
<td></td>
<td>(0.026)</td>
<td>(0.115)</td>
<td>(0.028)</td>
<td>(0.077)</td>
<td>(0.045)</td>
</tr>
<tr>
<td><strong>Risk Aversion ($r$)</strong></td>
<td>0.463</td>
<td>0.084</td>
<td>-</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.305)</td>
<td>(0.039)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>% Forward-Looking</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.773</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.091)</td>
</tr>
</tbody>
</table>

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Table 5. In-Sample and Out-of-Sample Fit Statistics for Diaper Data (Private Labels Included)

<table>
<thead>
<tr>
<th></th>
<th>No Learning</th>
<th>Myopic Learning</th>
<th>Index Strategy</th>
<th>Approximately Optimal</th>
<th>Myopic Learning with Risk Aversion</th>
<th>Index Strategy with Risk Aversion</th>
<th>One-Period Look-ahead</th>
<th>Heterogeneous Foresight</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Log likelihood</strong></td>
<td>-3301.57</td>
<td>-1955.16</td>
<td>-1853.22</td>
<td>-1857.04</td>
<td>-1955.16</td>
<td>-1852.43</td>
<td>-1862.60</td>
<td>-1814.66</td>
</tr>
<tr>
<td><strong>U^2</strong></td>
<td>76.58%</td>
<td>86.13%</td>
<td>86.85%</td>
<td>86.83%</td>
<td>86.13%</td>
<td>86.86%</td>
<td>86.79%</td>
<td>87.13%</td>
</tr>
<tr>
<td><strong>AIC</strong></td>
<td>6611.14</td>
<td>3934.33</td>
<td>3738.45</td>
<td>3746.08</td>
<td>3936.33</td>
<td>3738.86</td>
<td>3751.20</td>
<td>3687.31</td>
</tr>
<tr>
<td><strong>BIC</strong></td>
<td>6634.50</td>
<td>4004.41</td>
<td>3831.90</td>
<td>3839.53</td>
<td>4012.26</td>
<td>3838.15</td>
<td>3827.12</td>
<td>3856.69</td>
</tr>
<tr>
<td><strong># parameters</strong></td>
<td>4</td>
<td>12</td>
<td>16</td>
<td>16</td>
<td>13</td>
<td>17</td>
<td>13</td>
<td>29</td>
</tr>
<tr>
<td><strong># observations</strong></td>
<td>2,542</td>
<td>2,542</td>
<td>2,542</td>
<td>2,542</td>
<td>2,542</td>
<td>2,542</td>
<td>2,542</td>
<td>2,542</td>
</tr>
<tr>
<td><strong>Hold-out sample</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Log likelihood</strong></td>
<td>-3173.91</td>
<td>-1935.58</td>
<td>-1855.17</td>
<td>-1859.30</td>
<td>-1935.59</td>
<td>-1852.32</td>
<td>-1861.23</td>
<td>-1870.81</td>
</tr>
<tr>
<td><strong>U^2</strong></td>
<td>76.87%</td>
<td>85.90%</td>
<td>86.48%</td>
<td>86.45%</td>
<td>85.90%</td>
<td>86.50%</td>
<td>86.44%</td>
<td>86.37%</td>
</tr>
<tr>
<td><strong># observations</strong></td>
<td>2,475</td>
<td>2,475</td>
<td>2,475</td>
<td>2,475</td>
<td>2,475</td>
<td>2,475</td>
<td>2,475</td>
<td>2,475</td>
</tr>
<tr>
<td><strong>Computation time in seconds</strong></td>
<td>negligible</td>
<td>1</td>
<td>1.5 (23)</td>
<td>146</td>
<td>1</td>
<td>24</td>
<td>6</td>
<td>24</td>
</tr>
</tbody>
</table>

† This is the time required to compute one likelihood function using a university computing system based on Sun Grid Engine (SGE) and Red Hat Enterprise Linux. For the index strategy model, the computation time is 1.5 seconds for the original grid (M = N = 5) and 23 seconds for the finer grid (M = 200, N = 75).
Figure 1. Index Strategies Balance Utility and Simplicity

(Conceptual Diagram)
Figure 2. Gittins’ Index, Posterior Mean Quality, and the Value of Exploration

Figure 3. Whittle’s Index as Experience and Utility Shock Magnitude Vary

Notes. Posterior mean quality is set to zero in this figure, so that Whittle’s index captures the value of exploration. Inherent quality uncertainty \( \sigma_j \) is normalized as 1.
Appendix

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A. Proof of Proposition 1 (Indexability)
   A.1 When Consumers are Risk Neutral
   A.2 When Consumers Exhibit General Risk Preferences
B. Proof of Proposition 2 (Invariance)
C. Proof of Proposition 3 (Comparative Statics)
D. Computation of the Index Function
E. Computation of the Value Function
F. Maximum Simulated Likelihood Estimation
G. User’s Guide to Implementation
H. Predicted Switching Matrices
A Proof of Proposition 1 (Indexability)

A.1 Proof of Proposition 1 When Consumers Are Risk Neutral

Without loss of generality, we set observable shocks $x_t$ to zero and use $\epsilon_t$ to represent all utility shocks. The focus is on the sub-problem where the consumer chooses between an uncertain brand $j$ and a certain reward $\lambda$. To simplify notation, we drop the brand identifier $j$. The Bellman equation for this problem is

$$(A1) \ V(s_t, \epsilon_t, \lambda) = \max\{\lambda + \delta E[V(s_t, \epsilon_{t+1}, \lambda)|\epsilon_t], \ \epsilon_t + E[q_t|s_t] + \delta E[V(s_{t+1}, \epsilon_{t+1}, \lambda)|s_t, \epsilon_t]\},$$

where $s_t$ summarizes the consumer’s belief about brand quality at purchase occasion $t$. The definition of indexability is that, for any state $(s_t, \epsilon_t)$, if it is optimal to choose the fixed reward $\lambda$, then it must be also optimal to choose the fixed reward $\lambda'$ for any $\lambda' > \lambda$. This is equivalent to the following condition:

$$(A2) \ \frac{\partial}{\partial \lambda} (\lambda + \delta E[V(s_t, \epsilon_{t+1}, \lambda)|\epsilon_t] - \epsilon_t - E[q_t|s_t] - \delta E[V(s_{t+1}, \epsilon_{t+1}, \lambda)|s_t, \epsilon_t]) \geq 0,$$

$$\Leftrightarrow 1 + \delta \frac{\partial}{\partial \lambda} E[V(s_t, \epsilon_{t+1}, \lambda)|\epsilon_t] - \delta \frac{\partial}{\partial \lambda} E[V(s_{t+1}, \epsilon_{t+1}, \lambda)|s_t, \epsilon_t] \geq 0.$$ 

Intuitively, condition (A2) requires that, as $\lambda$ increases, the expected future value of choosing the uncertain brand should not grow too fast compared to that of choosing the fixed reward $\lambda$. It turns out that the assumptions of the canonical forward-looking experiential learning problem as specified in §2 are sufficient for condition (A2) to hold. We prove this result below.

We first define the expected value function $EV(s_t, \lambda)$ by integrating out $\epsilon_t$:

$$(A3) \ EV(s_t, \lambda) = E_{\epsilon_t}[V(s_t, \epsilon_t, \lambda)].$$

Given the assumptions that $\epsilon_{t+1}$ and $\epsilon_t$ are i.i.d. and that $\epsilon_t$ is independent of $s_t$, we have

$$E[V(s_t, \epsilon_{t+1}, \lambda)|\epsilon_t] = E[V(s_t, \epsilon_{t+1}, \lambda)] = E[V(s_t, \epsilon_t, \lambda)] = EV(s_t, \lambda),$$

and
Therefore, Equation (A1) implies the following fixed point:

\[ EV(st, \lambda) = \int \max(\lambda + \delta EV(st, \lambda), \epsilon_t + \mathbb{E}[q_t|s_t] + \delta \mathbb{E}[EV(st+1, \lambda)|s_t]) dH(\epsilon_t) \]

Denote 0 as the option of the certain reward $\lambda$, and 1 as the uncertain brand. We define the following quantities:

\[ v_0(st, \lambda) = \lambda + \delta EV(st, \lambda) \quad \text{and} \quad v_1(st, \lambda) = \mathbb{E}[q_t|s_t] + \delta \mathbb{E}[EV(st+1, \lambda)|s_t]. \]

Observe that the conditional probability of choosing 1 is given by

\[ P(1|s_t, \lambda) = \int \mathbb{1}[v_0(st, \lambda) \leq \epsilon_t + v_1(st, \lambda)] dH(\epsilon_t) \]

\[ = \int \frac{\partial}{\partial v_1(st, \lambda)} \max(v_0(st, \lambda), \epsilon_t + v_1(st, \lambda)) dH(\epsilon_t) \]

\[ = \frac{\partial EV(st, \lambda)}{\partial v_1(st, \lambda)}. \]

The last equality is obtained by interchanging integration and differentiation and evoking Equations (A4) and (A5). Similarly we have

\[ P(0|s_t, \lambda) = \frac{\partial EV(st, \lambda)}{\partial v_0(st, \lambda)}. \]

Differentiating both sides of Equation (A4) with respect to $\lambda$ and using the Chain Rule, we obtain

\[ EV_\lambda(st, \lambda) = \frac{\partial EV(st, \lambda)}{\partial \lambda} = \frac{\partial EV(st, \lambda)}{\partial v_0(st, \lambda)} \frac{\partial v_0(st, \lambda)}{\partial \lambda} + \frac{\partial EV(st, \lambda)}{\partial v_1(st, \lambda)} \frac{\partial v_1(st, \lambda)}{\partial \lambda} \]

\[ = P(0|s_t, \lambda)(1 + \delta EV_\lambda(st, \lambda)) + P(1|s_t, \lambda)(\delta \mathbb{E}[EV_\lambda(st+1, \lambda)|s_t]), \]

where the last equality is obtained from Equations (A5), (A6), and (A7). Next, we prove the following lemma.

**Lemma 1.** For all $s, \lambda$, we have
\[
0 \leq EV_\lambda(s, \lambda) \leq \frac{1}{1-\delta}.
\]

**Proof.** Fix any \(s, \epsilon, \) and \(\lambda.\) Let \(\pi^*\) denote the optimal policy that achieves \(V(s, \epsilon, \lambda).\) First, if a positive constant \(c\) is added only to the fixed reward \(\lambda\) in every period but the uncertain brand remains unchanged, then following \(\pi^*\) yields an expected total utility at least as large as \(V(s, \epsilon, \lambda).\) Therefore, \(V(s, \epsilon, \lambda + c) \geq V(s, \epsilon, \lambda).\) Second, if a positive constant \(c\) is added to both the fixed reward and the uncertain brand in every period, then \(\pi^*\) is still optimal and yields an expected total utility of \(V(s, \epsilon, \lambda) + c/(1 - \delta).\) By construction, adding a positive constant to both options yields expected utility at least as high as adding the constant only to the fixed reward: \(V(s, \epsilon, \lambda) + c/(1 - \delta) \geq V(s, \epsilon, \lambda + c).\) Integrating out \(\epsilon\) we have

\[
EV(s, \lambda) \leq EV(s, \lambda + c) \leq EV(s, \lambda) + \frac{c}{1-\delta}.
\]

It follows that

\[
0 \leq \frac{EV(s, \lambda + c) - EV(s, \lambda)}{c} \leq \frac{1}{1-\delta}.
\]

Taking the limit on both sides as \(c \to 0\) establishes the lemma.

Lemma 1 implies that \(0 \leq EV_\lambda(s_t, \lambda) \leq 1 + \delta EV_\lambda(s_t, \lambda).\) This result, together with Equation (A8) and the fact that \(P(0|s_t, \lambda) + P(1|s_t, \lambda) = 1,\) in turn implies that:

\[
1 + \delta EV_\lambda(s_t, \lambda) \geq \delta \mathbb{E}[EV_\lambda(s_{t+1}, \lambda)|s_t],
\]

which establishes the indexability condition (A2).

**A.2. Proof of Proposition 1 When Consumers Exhibit General Risk Preferences**

In this section, we extend the proof of §A.1 to show that the canonical forward-looking experiential learning problem is indexable when consumers exhibit general risk preferences.

The Bellman equation in the case of general risk preferences is
For the ease of comparison, we denote the above equation as \((A1')\), meaning that it corresponds to Equation (A1) of §A.1. The same notational rule applies throughout §A.2. The indexability condition becomes

\[
\begin{align*}
\frac{\partial}{\partial \lambda}(u(\lambda) + \delta \mathbb{E}[V(s_{t+1}, \epsilon_{t+1}, \lambda)]\epsilon_t) - \mathbb{E}[u(\epsilon_t + q_t)|s_t] - \delta \mathbb{E}[V(s_{t+1}, \epsilon_{t+1}, \lambda)|s_t, \epsilon_t] &\geq 0 \\
\iff u'(\lambda) + \delta \frac{\partial}{\partial \lambda} \mathbb{E}[V(s_t, \epsilon_{t+1}, \lambda)]\epsilon_t - \delta \frac{\partial}{\partial \lambda} \mathbb{E}[V(s_{t+1}, \epsilon_{t+1}, \lambda)|s_t, \epsilon_t] &\geq 0.
\end{align*}
\]

We again define the expected value function \(EV(s_t, \lambda)\) by integrating out \(\epsilon_t\):

\[
(A3') \quad EV(s_t, \lambda) \triangleq \mathbb{E}_{\epsilon_t}[V(s_t, \epsilon_t, \lambda)].
\]

Given the assumptions that \(\epsilon_{t+1}\) and \(\epsilon_t\) are i.i.d. and that \(\epsilon_t\) is independent of \(s_t\), we have:

\[
\mathbb{E}[V(s_t, \epsilon_{t+1}, \lambda)|\epsilon_t] = \mathbb{E}[V(s_t, \epsilon_{t+1}, \lambda)] = \mathbb{E}[V(s_t, \epsilon_t, \lambda)] = EV(s_t, \lambda),
\]

and

\[
\mathbb{E}[V(s_{t+1}, \epsilon_{t+1}, \lambda)|s_t, \epsilon_t] = \mathbb{E}[V(s_{t+1}, \epsilon_{t+1}, \lambda)|s_t] = \mathbb{E}[V(s_{t+1}, \epsilon_t, \lambda)|s_t] = \mathbb{E}[EV(s_{t+1}, \lambda)|s_t].
\]

Therefore, the Bellman equation of the sub-problem implies the following fixed point:

\[
(A4') \quad EV(s_t, \lambda) = \int \max\{u(\lambda) + \delta EV(s_t, \lambda),
\]

\[
\mathbb{E}[u(\epsilon_t + q_t)|s_t] + \delta \mathbb{E}[EV(s_{t+1}, \lambda)|s_t]\} dH(\epsilon_t).
\]

Denote 0 as the option of the certain reward \(\lambda\), and 1 as the uncertain brand. We define the following quantities:

\[
(A5') \quad v_0(s_t, \lambda) \triangleq u(\lambda) + \delta EV(s_t, \lambda) \quad \text{and} \quad v_1(s_t, \lambda) \triangleq \delta \mathbb{E}[EV(s_{t+1}, \lambda)|s_t].
\]

The conditional probability of choosing 1 is given by
The last equality is obtained by interchanging integration and differentiation and evoking Equations (A4') and (A5'). Similarly we have

\[(A7')\]  
P(0|s_t, \lambda) = \frac{\partial EV(s_t, \lambda)}{\partial v_0(s_t, \lambda)}.

Differentiating both sides of Equation (A4') with respect to \(\lambda\) and using the Chain Rule, we obtain

\[(A8')\]  
EV_\lambda(s_t, \lambda) = \frac{\partial EV(s_t, \lambda)}{\partial \lambda} = \frac{\partial EV(s_t, \lambda)}{\partial v_0(s_t, \lambda)} \frac{\partial v_0(s_t, \lambda)}{\partial \lambda} + \frac{\partial EV(s_t, \lambda)}{\partial v_1(s_t, \lambda)} \frac{\partial v_1(s_t, \lambda)}{\partial \lambda}

\[= P(0|s_t, \lambda)(u'(\lambda) + \delta EV_\lambda(s_t, \lambda)) + P(1|s_t, \lambda)(\delta EV(s_{t+1}, \lambda)|s_t),\]

where the last equality is obtained from Equations (A5'), (A6'), and (A7'). Next, we prove the following lemma.

**Lemma 1'**. For all \(s, \lambda\), we have

\[(A9')\]  
0 \leq EV_\lambda(s, \lambda) \leq \frac{u'(\lambda)}{1 - \delta}.

**Proof.** Fix any \(s, \epsilon, \) and \(\lambda\). Let \(\pi^*\) denote the optimal policy that solves \(V(s, \epsilon, \lambda)\). Suppose a positive constant \(c\) is added only to the fixed reward \(\lambda\) in every period but the uncertain brand remains unchanged. First, the consumer is weakly better off after this change. Even if the consumer maintains \(\pi^*\) – and the consumer can do weakly better – the consumer’s expected utility re-
mains unchanged in periods when the uncertain brand is chosen but increases in periods when
the fixed reward is chosen. Therefore, $V(s, \varepsilon, \lambda + c) \geq V(s, \varepsilon, \lambda)$.

Second, let $\pi'$ denote the optimal policy that solves $V(s, \varepsilon, \lambda + c)$. Suppose the consumer
adopts $\pi'$ in state $(s, \varepsilon, \lambda)$. After the increase in the fixed reward, the consumer’s expected utility
from adopting $\pi'$ remains unchanged in periods when $\pi'$ indicates choosing the uncertain brand
but increases in periods when $\pi'$ indicates choosing the fixed reward. The improvement in the
consumer’s expected discounted utility is thus weakly less than if $\pi'$ indicated choosing the fixed
reward in each period, in which case the improvement in the consumer’s expected discounted
utility would equal $\frac{u(\lambda + c) - u(\lambda)}{1 - \delta}$. Recall that $\pi^*$ is the optimal policy that solves $V(s, \varepsilon, \lambda)$. By def-
inition, the consumer is weakly better off choosing $\pi^*$ than $\pi'$ in state $(s, \varepsilon, \lambda)$. Therefore:

$$V(s, \varepsilon, \lambda + c) - V(s, \varepsilon, \lambda) \leq \frac{u(\lambda + c) - u(\lambda)}{1 - \delta}.$$  

Integrating out $\varepsilon$ we have:

$$EV(s, \lambda) \leq EV(s, \lambda + c) \leq EV(s, \lambda) + \frac{u(\lambda + c) - u(\lambda)}{1 - \delta}.$$  

It follows that:

$$0 \leq \frac{EV(s, \lambda + c) - EV(s, \lambda)}{c} \leq \frac{u(\lambda + c) - u(\lambda)}{c(1 - \delta)}.$$  

Taking the limit on both sides as $c \to 0$ establishes the lemma.

Lemma 1’ implies that $0 \leq EV_\lambda(s_t, \lambda) \leq u'(\lambda) + \delta EV_\lambda(s_t, \lambda)$. This result, together with
Equation (A8’) and the fact that $P(0|s_t, \lambda) + P(1|s_t, \lambda) = 1$, in turn implies that:

(A10’)

$$u'(\lambda) + \delta EV_\lambda(s_t, \lambda) \geq \delta \mathbb{E}[EV_\lambda(s_{t+1}, \lambda)|s_t],$$

which establishes the indexability condition (A2’).
B Proof of Proposition 2 (Invariance)

We first prove two useful lemmas. The focus is again on the sub-problem of a single brand and thus we drop the brand identifier $j$.

Lemma 2. Fix a prior quality belief $s_0 = (\bar{\mu}_0, \bar{\sigma}_0)$, and a sequence of quality draws \{\(q_t: t \geq 0\}\). Consider a modified version of the original sub-problem where the utility shocks become \(\epsilon_t^m = \epsilon_t + c\) for all \(t\), and the fixed reward becomes \(\lambda^m = \lambda + c\). Denote \(E^m\) and \(W^m\) as the expected value and index value for the modified problem. Then for any belief state \(s\), we have:

\[
\begin{align*}
(B1) & \quad E^m(s, \lambda + c) = E(s, \lambda) + \frac{c}{1-\delta}, \\
(B2) & \quad W^m(s, \epsilon + c; \mu^c + c, \sigma^c) = W(s, \epsilon; \mu^c, \sigma^c) + c.
\end{align*}
\]

Proof. To prove the first part of the lemma, it suffices to show that the proposed identity in Equation (B1) satisfies the fixed-point relationship implied by the modified problem:

\[
(B3) \quad E^m(s_t, \lambda^m) = \max(\lambda^m + \delta E^m(s_t, \lambda^m), \epsilon_t^m + \mathbb{E}[q_t|s_t])
\]

\[
+ \delta \mathbb{E}[E^m(s_{t+1}, \lambda^m)|s_t] dH^m(\epsilon_t^m).
\]

Suppose Equation (B1) holds. Following the definitions given in Equation (A5) we have

\[
(B4) \quad v^m_0(s_t, \lambda^m) \triangleq \lambda + c + \delta E^m(s_t, \lambda + c)
\]

\[
= \lambda + \delta E(s_t, \lambda) + \frac{c}{1-\delta} = v_0(s_t, \lambda) + \frac{c}{1-\delta},
\]

and

\[
(B5) \quad v^m_1(s_t, \lambda^m) \triangleq \mathbb{E}[q_t|s_t] + \delta \mathbb{E}[E^m(s_{t+1}, \lambda + c)|s_t]
\]

\[
= \mathbb{E}[q_t|s_t] + \delta \mathbb{E}[E(s_{t+1}, \lambda)|s_t] + \frac{c\delta}{1-\delta} = v_1(s_t, \lambda) + \frac{c\delta}{1-\delta}.
\]
Define $\Delta(s_t, \lambda) \triangleq v_0(s_t, \lambda) - v_1(s_t, \lambda)$. Then $\Delta^m(s_t, \lambda + c) = \Delta(s_t, \lambda) + c$ for the modified problem. The assumption that the distribution of $\varepsilon$ has scale and location parameters implies:

\begin{equation}
H^m(\Delta^m) = \Pr(\varepsilon^m \leq \Delta^m) = \Pr(\varepsilon \leq \Delta) = H(\Delta), \text{ and}
\end{equation}

\[
\int_{\varepsilon^m \geq \Delta^m} \varepsilon^m d H^m(\varepsilon^m) = \int_{\varepsilon \geq \Delta} (\varepsilon + c) d H(\varepsilon).
\]

The right-hand side of Equation (B3) becomes:

\[
\int_{\varepsilon^m \geq \Delta^m} \varepsilon^m d H^m(\varepsilon^m) + v_1^m(s_t, \lambda^m)(1 - H^m(\Delta^m)) + v_0^m(s_t, \lambda^m)H^m(\Delta^m)
\]

\[
= \int_{\varepsilon \geq \Delta} (\varepsilon + c) d H(\varepsilon) + \left( v_1(s_t, \lambda) + \frac{c \delta}{1 - \delta} \right)(1 - H(\Delta)) + \left( v_0(s_t, \lambda) + \frac{c}{1 - \delta} \right) H(\Delta)
\]

\[
= \int_{\varepsilon \geq \Delta} \varepsilon d H(\varepsilon) + v_1(s_t, \lambda)(1 - H(\Delta)) + v_0(s_t, \lambda) H(\Delta) + \frac{c}{1 - \delta}
\]

\[
= EV(s, \lambda) + \frac{c}{1 - \delta}
\]

\[
= EV^m(s, \lambda + c),
\]

which is the left-hand side of Equation (B3). The first equality follows from Equations (B4) to (B6). The third equality uses the fact that $EV(s, \lambda)$ is the fixed point of Equation (A4). Therefore, $EV^m(s, \lambda^m)$ also satisfies the fixed-point relationship implied by the modified problem.

For the second part of the lemma, we use the definition of Whittle’s index, which is the value of $\lambda^m$ such that

\begin{equation}
v_0^m(s_t, \lambda^m) = \varepsilon^m + v_1^m(s_t, \lambda^m).
\end{equation}

It suffices to show that the proposed identity in Equation (B2) solves the above equality. Note that the right-hand side of Equation (B7) is:

\[
\varepsilon^m + v_1^m(s_t, \lambda^m) = \varepsilon + c + v_1(s_t, \lambda) + \frac{c \delta}{1 - \delta}
\]
which equals the left-hand side of Equation (B7) following Equation (B4). The first equality follows from Equation (B5). The third equality follows from the definition of Whittle’s index for the original problem (setting $W(s, \epsilon) = \lambda$). Then by the definition of Whittle’s index, we have $W^m(s, \epsilon + c; \mu^\epsilon + c, \sigma^\epsilon) = W(s, \epsilon; \mu^\epsilon, \sigma^\epsilon) + c$.

**Lemma 3.** Fix the original sub-problem. Consider a modified problem where the quality sample becomes $q^m_t = b q_t + c: t \geq 0$, the utility shocks becomes $\epsilon^m_t = b \epsilon_t$ for all $t$, the prior belief becomes $s^m_0 = (\mu^m_0, \sigma^m_0) = (b \mu_0 + c, b \sigma_0)$, and the fixed reward becomes $\lambda^m = b \lambda + c$. Then for all $t$, $s^m_t = (\mu^m_t, \sigma^m_t) = (b \mu_t + c, b \sigma_t)$. Denote $EV^m$ and $W^m$ as the expected value and index value for the modified problem. Then for any belief state $s$, we have:

\[(B8)\quad EV^m(s^m, b \lambda + c) = b EV(s, \lambda) + \frac{c}{1 - \delta},\]
\[(B9)\quad W^m(s^m, b \epsilon; b \mu^\epsilon, b \sigma^\epsilon) = b W(s, \epsilon; \mu^\epsilon, \sigma^\epsilon) + c.\]

**Proof.** The strategy of the proof is similar to that of Lemma 2. Note that the Bayesian updating implies that for all $t$ the precision $w^m_t$ of the modified problem remains the same as that of the original problem:

$$w^m_t = \frac{\sigma^m_t}{\sigma^m_t^2 + \sigma^2} = \frac{b^2 \sigma_t^2}{b^2 \sigma_t^2 + b^2 \sigma^2} = w_t.$$ 

It follows that the updated posterior mean and variance in the next period become

$$\bar{\mu}^m_{t+1} = w^m_t q^m_t + (1 - w^m_t) \bar{\mu}_t = w_t(b q_t + c) + (1 - w_t)(b \mu_t + c) = b \mu_{t+1} + c,$$

$$\bar{\sigma}^m_{t+1} = \bar{\sigma}^m_t \sqrt{1 - w^m_t} = b \sigma_t \sqrt{1 - w_t} = b \sigma_{t+1}.$$ 

Therefore, the belief state in the next period preserves the relationship:
\[ s_{t+1}^m = \left( \bar{\mu}_{t+1}^m, \bar{\sigma}_{t+1}^m \right) = \left( b\bar{\mu}_{t+1} + c, b\bar{\sigma}_{t+1} \right). \]

For the first part of the lemma we show that the identity in Equation (B8) satisfies the fixed-point relationship implied by the modified problem

\[
EV^m(s_t^m, \lambda^m) = \int \max\{\lambda^m + \delta EV^m(s_t^m, \lambda^m), \epsilon_t + \mathbb{E}[q_t^m | s_t^m] + \delta \mathbb{E}[EV^m(s_{t+1}^m, \lambda^m) | s_t^m] \} dH^m(\epsilon_t^m).
\]

Suppose Equation (B8) holds, then

\[
v_0^m(s_t^m, \lambda^m) = b\lambda + c + \delta EV^m(s_t^m, b\lambda + c) = b\lambda + \delta bEV(s_t, \lambda) + \frac{c}{1 - \delta} = bv_0(s_t, \lambda) + \frac{c}{1 - \delta}.
\]

Similarly,

\[
v_1^m(s_t^m, \lambda^m) = \mathbb{E}[q_t^m | s_t^m] + \mathbb{E}[EV^m(s_{t+1}^m, b\lambda + c) | s_t^m] = b \int \int (q_{t+1} + \delta EV(s_{t+1}, \lambda)) dF(q_{t+1} | \mu, \sigma) dB_t(\mu | s_t) + \frac{c}{1 - \delta} = bv_1(s_t, \lambda) + \frac{c}{1 - \delta}.
\]

The first equality uses the fact that \( EV^m(s_{t+1}^m, b\lambda + c) = bEV(s_{t+1}, \lambda) + c/(1 - \delta) \). The second equality follows from normality and conjugate prior assumptions for the distribution of qualities \( F \) and beliefs \( B_t \). Then \( \Delta^m(s_t^m, b\lambda + c) = v_0^m(s_t^m, b\lambda + c) - v_1^m(s_t^m, b\lambda + c) = b\Delta(s_t, \lambda) \) for the modified problem. The assumption that the distribution of \( \epsilon \) has scale and location parameters implies
(B13) \[ H^m(\Delta^m) = H(\Delta), \quad \text{and} \quad \int_{\epsilon \in \Delta} \epsilon^m d H^m(\epsilon^m) = b \int_{\epsilon \in \Delta} (\epsilon + c) d H(\epsilon). \]

The right-hand side of Equation (B10) becomes

\[
\int_{\epsilon \in \Delta} \epsilon^m d H^m(\epsilon^m) + v_1^m(s_t^m, b\lambda + c)(1 - H^m(\Delta^m)) + v_0^m(s_t^m, b\lambda + c)H^m(\Delta^m)
\]

\[= b \int_{\epsilon \in \Delta} \epsilon d H(\epsilon) + b v_1(s_t, \lambda) \left(1 - H(\Delta) \right) + b v_0(s_t, \lambda) \left(1 - \frac{c}{1 - \delta} \right) H(\Delta)
\]

\[= b \int_{\epsilon \in \Delta} \epsilon d H(\epsilon) + b v_1(s_t, \lambda) \left(1 - H(\Delta) \right) + b v_0(s_t, \lambda) H(\Delta) + \frac{c}{1 - \delta}
\]

\[= b EV(s_t, \lambda) + \frac{c}{1 - \delta}
\]

\[= EV^m(s_t^m, b\lambda + c), \]

which is the left-hand side of Equation (B10). The first equality follows from Equations (B11) to (B13). The third equality uses the fact that \(EV(s, \lambda)\) is the fixed point of Equation (A4). Therefore, \(EV^m(s_t^m, \lambda^m)\) also satisfies the fixed-point relationship implied by the modified problem.

For the second part of the lemma, we again use the definition of Whittle’s index, which is the value of \(\lambda^m\) such that

(B14) \[ v_0^m(s_t^m, \lambda^m) = \epsilon^m + v_1^m(s_t^m, \lambda^m). \]

It suffices to show the proposed relation in Equation (B9) solves the above equality. Note that the right-hand side of Equation (B14) is

\[
\epsilon^m + v_1^m(s_t^m, \lambda^m) = b \epsilon + b v_1(s_t, \lambda) + \frac{c}{1 - \delta} = b (\epsilon + v_1(s_t, \lambda)) + \frac{c}{1 - \delta} = b v_0(s_t, \lambda) + \frac{c}{1 - \delta},
\]

which equals the left-hand side of Equation (B14) following Equation (B11). The first equality follows from Equation (B12). The third equality follows from the definition of Whittle’s index for the original problem (setting \(W(s, \epsilon) = \lambda\)). Then by the definition of Whittle’s index, we have \(W^m(s^m, b\epsilon; b\mu^\epsilon, b\sigma^\epsilon) = bW(s, \epsilon; \mu^\epsilon, \sigma^\epsilon) + c.\)
To complete the proof of the proposition, note that by Lemma 3 we have:

\[ W(b\bar{ \mu} + c, b\bar{ \sigma}, b\epsilon; b\sigma, b\mu^e, b\sigma^e) = bW(\bar{ \mu}, \bar{ \sigma}, \epsilon; \sigma, \mu^e, \sigma^e) + c. \]

Setting \( b = 1/\sigma \) and \( c = -\bar{ \mu}/\sigma \) and evoking Lemma 2 yields

\[
W(\bar{ \mu}, \bar{ \sigma}, \epsilon; \sigma, \mu^e, \sigma^e) = \tilde{\mu} + \sigma W\left(0, \frac{\bar{ \sigma}}{\sigma}, \frac{\epsilon}{\sigma}; 1, \frac{\mu^e}{\sigma}, \frac{\sigma^e}{\sigma}\right)
\]

\[
= \tilde{\mu} + \frac{\epsilon}{\sigma} \left[ W\left(0, \frac{\bar{ \sigma}}{\sigma}, \frac{\epsilon - \mu^e}{\sigma}; 1, 0, \frac{\sigma^e}{\sigma}\right) + \frac{\mu^e}{\sigma} \right]
\]

\[
= \tilde{\mu} + \frac{\epsilon}{\sigma} + \sigma W\left(0, \frac{\bar{ \sigma}}{\sigma}, \frac{\epsilon - \mu^e}{\sigma}; 1, 0, \frac{\sigma^e}{\sigma}\right),
\]

which completes the proof of the proposition.
C. Proof of Proposition 3 (Comparative Statics)

We again focus on the sub-problem of a single brand and thus drop the brand identifier $j$.

Proof of Proposition 3(1)

The first part that Whittle’s index increases with posterior mean $\bar{\mu}$ is evident from Proposition 2. For the second part, fix some belief state $s$ and consider any $\epsilon' > \epsilon$. Let $W'$ and $W$ be the corresponding Whittle’s indices. Recall that $\Delta(s, \lambda) \triangleq \nu_0(s, \lambda) - \nu_1(s, \lambda)$. Then by the definition of an index, $\Delta(s, W') = \epsilon' > \epsilon = \Delta(s, W)$. Note that:

\[ \Delta_\lambda(s, \lambda) = \frac{\partial \Delta(s, \lambda)}{\partial \lambda} = \frac{\partial \nu_0(s, \lambda)}{\partial \lambda} - \frac{\partial \nu_1(s, \lambda)}{\partial \lambda} \geq 0, \]

where the inequality is implied by (A10). It then follows that $W' > W$.

Proof of Proposition 3(2)

We will prove the first part. The second part holds following a similar argument. Fix some $\sigma_2^2 > \sigma_1^2$. Let $\{Y_t: t \geq 0\}$ be a sequence of random variables conditionally i.i.d. from the distribution $N(\mu, \sigma_2^2)$. Let $\{\omega_t: t \geq 0\}$ be a sequence of random variables conditionally i.i.d. from the distribution $N(\mu, \sigma_2^2 - \sigma_1^2)$. The two sequences are independent. Construct a sequence of random variables such that $Z_t = Y_t + \omega_t$ for all $t$. Then $\{Z_t: t \geq 0\}$ are conditionally i.i.d. from the distribution $N(\mu, \sigma_2^2)$. Fix some policy $\pi$ that solves the problem under $\{Z_t: t \geq 0\}$. Denote $\pi(s_t, \epsilon_t) = 0$ if the fixed reward $\lambda$ is chosen, and $\pi(s_t, \epsilon_t) = 1$ if the uncertain brand is chosen. Then the value function in state $(s, \epsilon)$ when $\pi$ is applied becomes
\[ V_\pi(s, \epsilon, \lambda; \sigma_2) \]
\[ = \mathbb{E} \left[ \sum_{t=0}^{\infty} \delta^t \{1[\pi(s_t, \epsilon_t) = 1](Z_t + \epsilon_t) + 1[\pi(s_t, \epsilon_t) = 0]\lambda\} \mid (s_0, \epsilon_0) = (s, \epsilon) \right] \]
\[ = \mathbb{E} \left[ \sum_{t=0}^{\infty} \delta^t \{1[\pi(s_t, \epsilon_t) = 1](Y_t + \omega_t + \epsilon_t) + 1[\pi(s_t, \epsilon_t) = 0]\lambda\} \mid (s_0, \epsilon_0) = (s, \epsilon) \right] \]
\[ = \mathbb{E} \left[ \sum_{t=0}^{\infty} \delta^t \{1[\pi(s_t, \epsilon_t) = 1](Y_t + \epsilon_t) + 1[\pi(s_t, \epsilon_t) = 0]\lambda\} \mid (s_0, \epsilon_0) = (s, \epsilon) \right] \]
\[ + \mathbb{E} \left[ \sum_{t=0}^{\infty} \delta^t 1[\pi(s_t, \epsilon_t) = 1]\omega_t \mid (s_0, \epsilon_0) = (s, \epsilon) \right] \]
\[ = V_\pi(s, \epsilon, \lambda; \sigma_1), \]

where the last equality uses the fact that the second term is equal to zero. Note that
\[ V_\pi(s, \epsilon, \lambda; \sigma_1) \leq V(s, \epsilon, \lambda; \sigma_1) \] because the latter is the optimal value function. Therefore
\[ V_\pi(s, \epsilon, \lambda; \sigma_2) \leq V(s, \epsilon, \lambda; \sigma_1) \] for all \( \pi \). Taking maximum on the left-hand side gives
\[ V(s, \epsilon, \lambda; \sigma_2) \leq V(s, \epsilon, \lambda; \sigma_1). \] Integrating out \( \epsilon \) further yields \( EV(s, \lambda; \sigma_2) \leq EV(s, \lambda; \sigma_1). \)

Then we have \( EV_{\sigma}(s, \lambda; \sigma) \equiv \partial EV(s, \lambda; \sigma)/\partial \sigma \leq 0 \) for all \( s, \lambda, \sigma \). Differentiating both sides of Equation (A4) with respect to \( \sigma \) and using Chain Rule gives
\[ EV_{\sigma}(s_t, \lambda; \sigma) = \frac{\partial EV(s_t, \lambda; \sigma)}{\partial v_0(s_t, \lambda; \sigma)} \frac{\partial v_0(s_t, \lambda; \sigma)}{\partial \sigma} + \frac{\partial EV(s_t, \lambda; \sigma)}{\partial v_1(s_t, \lambda; \sigma)} \frac{\partial v_1(s_t, \lambda; \sigma)}{\partial \sigma} \]
\[ = P(0|s_t, \lambda; \sigma)(\delta EV_{\sigma}(s_t, \lambda; \sigma)) + P(1|s_t, \lambda; \sigma)(\delta E[EV_{\sigma}(s_{t+1}, \lambda; \sigma)|s_t]) \]

The last equality implies that
\[ EV_{\sigma}(s_t, \lambda; \sigma) = \frac{\delta P(1|s_t, \lambda; \sigma)}{1 - \delta P(0|s_t, \lambda; \sigma)} E[EV_{\sigma}(s_{t+1}, \lambda; \sigma)|s_t]. \]

It then follows that
\[
\frac{\partial \Delta(s_t, \lambda; \sigma)}{\partial \sigma} = \delta(\mathbb{E}_\sigma(s_t, \lambda; \sigma) - \mathbb{E}[\mathbb{E}_\sigma(s_{t+1}, \lambda; \sigma)|s_t])
\]

\[
= \delta \left( \frac{\delta P(1|s_t, \lambda; \sigma)}{1 - \delta P(0|s_t, \lambda; \sigma)} - 1 \right) \mathbb{E}[\mathbb{E}_\sigma(s_{t+1}, \lambda; \sigma)|s_t]
\]

\[
= \delta \left( \frac{\delta - 1}{1 - \delta P(0|s_t, \lambda; \sigma)} \right) \mathbb{E}[\mathbb{E}_\sigma(s_{t+1}, \lambda; \sigma)|s_t] \geq 0,
\]

where the last inequality uses the fact that \( \mathbb{E}[\mathbb{E}_\sigma(s_{t+1}, \lambda; \sigma)|s_t] \leq 0 \). Let \( W_2 \) and \( W_1 \) be the Whittle's indices corresponding to \( \sigma_2 \) and \( \sigma_1 \). This inequality implies \( \Delta(s, W_1; \sigma_2) \geq \Delta(s, W_1; \sigma_1) \). Since by the definition of an index, \( \Delta(s, W_2; \sigma_2) = \Delta(s, W_1; \sigma_1) = \epsilon \), we have \( \Delta(s, W_1; \sigma_2) \geq \Delta(s, W_2; \sigma_2) \). It then follows that \( W_2 \leq W_1 \) by Equation (C1).

**Proof of Proposition 3(3)**

First, we prove a lemma, analogous to the result we have in the proof of Proposition 3(2).

**Lemma 4.** Consider any \( \bar{\sigma}^\prime_t > \bar{\sigma}_t \). If \( V(\bar{\mu}_t, \bar{\sigma}_t, \epsilon_{t+1}, \lambda) \leq V(\bar{\mu}_t, \bar{\sigma}^\prime_t, \epsilon_{t+1}, \lambda) \), then

\[
W(\bar{\mu}_t, \bar{\sigma}_t, \epsilon_{t+1}) \leq W(\bar{\mu}_t, \bar{\sigma}^\prime_t, \epsilon_{t+1})
\]

**Proof.** Since \( V(\bar{\mu}_t, \bar{\sigma}_t, \epsilon_{t+1}, \lambda) \leq V(\bar{\mu}_t, \bar{\sigma}^\prime_t, \epsilon_{t+1}, \lambda) \) for every \( \epsilon_{t+1} \), we have \( \mathbb{E}V(\bar{\mu}_t, \bar{\sigma}_t, \lambda) \leq \mathbb{E}V(\bar{\mu}_t, \bar{\sigma}^\prime_t, \lambda) \) by integrating out \( \epsilon_{t+1} \). This implies that \( \mathbb{E}V(\bar{\mu}_t, \bar{\sigma}_t, \lambda) = \frac{\partial \mathbb{E}V(\bar{\mu}_t, \bar{\sigma}_t, \lambda)}{\partial \bar{\sigma}_t} \geq 0 \) for all \( \bar{\mu}_t, \bar{\sigma}_t, \lambda \). Differentiating both sides of Equation (A4) with respect to \( \bar{\sigma}_t \) and using Chain Rule gives

\[
\mathbb{E}V_\bar{\sigma}_t(\bar{\mu}_t, \bar{\sigma}_t, \lambda) = \frac{\partial \mathbb{E}V(\bar{\mu}_t, \bar{\sigma}_t, \lambda)}{\partial \bar{\nu}_0(\bar{\mu}_t, \bar{\sigma}_t, \lambda)} \frac{\partial \bar{\nu}_0(\bar{\mu}_t, \bar{\sigma}_t, \lambda)}{\partial \bar{\sigma}_t} + \frac{\partial \mathbb{E}V(\bar{\mu}_t, \bar{\sigma}_t, \lambda)}{\partial \bar{\nu}_1(\bar{\mu}_t, \bar{\sigma}_t, \lambda)} \frac{\partial \bar{\nu}_1(\bar{\mu}_t, \bar{\sigma}_t, \lambda)}{\partial \bar{\sigma}_t}
\]

\[
= P(0|\bar{\mu}_t, \bar{\sigma}_t, \lambda) \left( \frac{\partial \mathbb{E}V_\bar{\sigma}_t(\bar{\mu}_t, \bar{\sigma}_t, \lambda)}{\partial \bar{\sigma}_t} \right) + P(1|\bar{\mu}_t, \bar{\sigma}_t, \lambda) \left( \frac{\partial \mathbb{E}V(\bar{\mu}_t, \bar{\sigma}_t, \lambda)}{\partial \bar{\sigma}_t} \right) \mathbb{E}V(\bar{\mu}_t+1, \bar{\sigma}_t+1, \lambda)|\bar{\mu}_t, \bar{\sigma}_t, \lambda)
\]

The last equality implies that

\[
\mathbb{E}V_\bar{\sigma}_t(\bar{\mu}_t, \bar{\sigma}_t, \lambda) = \frac{\delta P(1|\bar{\mu}_t, \bar{\sigma}_t, \lambda)}{1 - \delta P(0|\bar{\mu}_t, \bar{\sigma}_t, \lambda)} \frac{\partial \mathbb{E}V(\bar{\mu}_t+1, \bar{\sigma}_t+1, \lambda)|\bar{\mu}_t, \bar{\sigma}_t, \lambda)}{\partial \bar{\sigma}_t}
\]

It then follows that
\[
\frac{\partial \Delta(\bar{\mu}_t, \bar{\sigma}_t, \lambda)}{\partial \bar{\sigma}_t} = \delta \left( EV(\bar{\mu}_t, \bar{\sigma}_t, \lambda) - \frac{\partial}{\partial \bar{\sigma}_t} EV(\bar{\mu}_{t+1}, \bar{\sigma}_{t+1}, \lambda) | \bar{\mu}_t, \bar{\sigma}_t \right)
\]

\[
= \delta \left( \frac{\delta P(1|\bar{\mu}_t, \bar{\sigma}_t, \lambda)}{1 - \delta P(0|\bar{\mu}_t, \bar{\sigma}_t, \lambda)} - 1 \right) \frac{\partial}{\partial \bar{\sigma}_t} EV(\bar{\mu}_{t+1}, \bar{\sigma}_{t+1}, \lambda) | \bar{\mu}_t, \bar{\sigma}_t
\]

\[
= \delta \left( \frac{\delta - 1}{1 - \delta P(0|\bar{\mu}_t, \bar{\sigma}_t, \lambda)} \right) \frac{\partial}{\partial \bar{\sigma}_t} EV(\bar{\mu}_{t+1}, \bar{\sigma}_{t+1}, \lambda) | \bar{\mu}_t, \bar{\sigma}_t \right) \leq 0,
\]

where the last inequality uses the fact that \( \frac{\partial}{\partial \bar{\sigma}_t} EV(\bar{\mu}_{t+1}, \bar{\sigma}_{t+1}, \lambda) | \bar{\mu}_t, \bar{\sigma}_t \right) \geq 0 \) from above. Let \( W \) and \( W' \) be the Whittle’s indices corresponding to \((\bar{\mu}_t, \bar{\sigma}_t)\) and to \((\bar{\mu}_t, \bar{\sigma}'_t)\). This inequality implies \( \Delta(\bar{\mu}_t, \bar{\sigma}_t, W') \geq \Delta(\bar{\mu}_t, \bar{\sigma}'_t, W') \). Since by the definition of an index, we set equal the two arguments in the right hand side of (A1) and get \( \Delta(\bar{\mu}_t, \bar{\sigma}'_t, W') = \Delta(\bar{\mu}_t, \bar{\sigma}_t, W) = \epsilon_t \), we have \( \Delta(\bar{\mu}_t, \bar{\sigma}_t, W') \geq \Delta(\bar{\mu}_t, \bar{\sigma}_t, W) \). It then follows that \( W \leq W' \) because \( \Delta(\bar{\mu}_t, \bar{\sigma}_t, \lambda) \geq 0 \) by Equation (C1).

We now prove Proposition 3(3) using this lemma. The challenge of the proof comes from finding the right way to pass the ordering of priors to the ordering of posteriors. Let \( \{q_t: t \geq 0\} \) be the sampling process that starts from the prior belief \((\bar{\mu}_t, \bar{\sigma}_t)\). Following Gittins and Wang (1992) and Magnac and Robin (1999), we can construct another sampling process \( \{q'_t: t \geq 0\} \) that starts from the prior belief \((\bar{\mu}_t, \bar{\sigma}'_t)\) such that

\[
F(q'_t|\bar{\mu}_t, \bar{\sigma}'_t) = F(q_t|\bar{\mu}_t, \bar{\sigma}_t),
\]

where \( F(\cdot |\bar{\mu}_t, \bar{\sigma}_t) \) is the distribution of another observation conditional on the prior belief. Therefore, \( \{q'_t: t \geq 0\} \) is another realization of the same sampling process, this time starting from the prior belief \((\bar{\mu}_t, \bar{\sigma}'_t)\). If \( \bar{\sigma}'_t > \bar{\sigma}_t \), then the process \( \{q'_t: t \geq 0\} \) stochastically dominates the process \( \{q_t: t \geq 0\} \) in the dynamic sense.
Let $\pi^*(\bar{\mu}_t, \bar{\sigma}_t)$ be the optimal policy that solves the problem under the original process $\{q_t: t \geq 0\}$. Denote $\pi^*(\bar{\mu}_t, \bar{\sigma}_t) = 0$ if the fixed reward $\lambda$ is chosen, and $\pi^*(\bar{\mu}_t, \bar{\sigma}_t) = 1$ if the uncertain brand is chosen. Then the value function in state $(\bar{\mu}_t, \bar{\sigma}_t)$ is

$$V(\bar{\mu}_t, \bar{\sigma}_t, \epsilon_t, \lambda)$$

$$= \mathbb{E} \left[ \sum_{t=0}^{\infty} \delta^t \{ \mathbb{I}[\pi^*(\bar{\mu}_t, \bar{\sigma}_t) = 1](q_t + \epsilon_t) + \mathbb{I}[\pi^*(\bar{\mu}_t, \bar{\sigma}_t) = 0]\;\lambda \} \mid (\mu_0, \sigma_0, \epsilon_0) = (\bar{\mu}_t, \bar{\sigma}_t, \epsilon_t) \right].$$

Applying this decision sequence to the process $\{q'_t: t \geq 0\}$ starting from $(\bar{\mu}_t, \bar{\sigma}_t')$ gives the expected value of

$$V'_\pi(\bar{\mu}_t, \bar{\sigma}_t', \epsilon_t, \lambda)$$

$$= \mathbb{E} \left[ \sum_{t=0}^{\infty} \delta^t \{ \mathbb{I}[\pi^*(\bar{\mu}_t, \bar{\sigma}_t) = 1](q'_t + \epsilon_t) + \mathbb{I}[\pi^*(\bar{\mu}_t, \bar{\sigma}_t) = 0]\;\lambda \} \mid (\mu_0, \sigma_0, \epsilon_0) = (\bar{\mu}_t, \bar{\sigma}_t', \epsilon_t) \right].$$

Given the construction that $\{q'_t: t \geq 0\}$ dominates $\{q_t: t \geq 0\}$, we have $V(\bar{\mu}_t, \bar{\sigma}_t, \epsilon_t, \lambda) \leq V'_\pi(\bar{\mu}_t, \bar{\sigma}_t', \epsilon_t, \lambda)$. Since the optimal policy under the process $\{q'_t: t \geq 0\}$ is $V(\bar{\mu}_t, \bar{\sigma}_t', \epsilon_t, \lambda)$, which is not less than $V'_\pi(\bar{\mu}_t, \bar{\sigma}_t', \epsilon_t, \lambda)$. It follows that $V(\bar{\mu}_t, \bar{\sigma}_t, \epsilon_t, \lambda) \leq V(\bar{\mu}_t, \bar{\sigma}_t', \epsilon_t, \lambda)$. Applying Lemma 4 completes the proof.

**Additional Reference**


D Computation of the Index Function

We can use the invariance property to simplify the computation of Whittle's index. The computation is based on the fixed point problem in Equation (A4) and the definition of Whittle's index. Product identifier \( j \) is dropped to simplify notation. Note that the \( EV \) function evaluated at \( \bar{\mu}_t = 0 \) is

\[
EV(0, \bar{\sigma}_t, \lambda) = \int \max \{ \lambda + \sigma EV(0, \sigma_t, \lambda), \epsilon_t + \delta \mathbb{E}[EV(\bar{\mu}_{t+1}, \sigma_{t+1}, \lambda)|0, \bar{\sigma}_t] \} dH(\epsilon_t)
\]

\[
= \int \max \{ \lambda + \sigma EV(0, \sigma_t, \lambda), \epsilon_t + \delta \mathbb{E}_{q_t}[EV(\mu_{t+1}q_t + (1 - w_t)\bar{\mu}_t, \sigma_{t+1}, \lambda)|0, \bar{\sigma}_t] \} dH(\epsilon_t)
\]

\[
= \int \max \{ \lambda + \sigma EV(0, \sigma_t, \lambda), \epsilon_t + \delta \mathbb{E}_{q_t}[EV(0, \sigma_{t+1}, \lambda - w_tq_t) + \frac{w_tq_t}{1 - \delta}|0, \bar{\sigma}_t] \} dH(\epsilon_t)
\]

where \( w_t = \bar{\sigma}_t^2 / (\bar{\sigma}_t^2 + \sigma^2) \) is the precision. The first equality is implied by Bayesian updating formulae for the normal distribution. The second equality uses Equation (B8) from Lemma 3 and the fact that the expectation of \( q_t \) and \( \bar{\mu}_t \) conditional on \( \bar{\mu}_t = 0 \) are both zero. The last equality again uses the zero expectation of \( q_t \) conditional on \( \bar{\mu}_t = 0 \).

We now treat \( \lambda \) as a state variable. Let \( \lambda_{t+1} = \lambda_t - w_tq_t \). Note that the distribution of \( q_t \) conditional on the belief \( (\mu_t, \sigma_t) \) is normal with mean \( \bar{\mu}_t \) and standard deviation \( \sqrt{\sigma^2 + \sigma_t^2} \).

Therefore \( \lambda_{t+1}|\lambda_t \sim N(\lambda_t, w_t\sqrt{\sigma^2 + \sigma_t^2}) \). The fixed point problem now only involves the \( EV \) function fixed at \( \bar{\mu}_t = 0 \), and evolves on the state space \( (\sigma_t, \lambda_t) \):

\[
EV(0, \sigma_t, \lambda_t) = \int \max \{ \lambda_t + \sigma EV(0, \sigma_t, \lambda_t), \epsilon_t + \delta \mathbb{E}_{\lambda_{t+1}}[EV(0, \sigma_{t+1}, \lambda_{t+1})|0, \sigma_t, \lambda_t] \} dH(\epsilon_t).
\]

Standard dynamic programming algorithms can be used to solve the above fixed point.
Given the solution of $EV(0, \bar{\sigma}_t, \lambda_t)$, we can find the value of $\lambda$ such that the two terms inside the max operator in the above equation are equal for various values of random shocks $\epsilon$ under $\bar{\mu} = 0$. This value is then the corresponding Whittle's index: $W(0, \bar{\sigma}, \epsilon; \sigma, \mu^\epsilon, \sigma^\epsilon)$. The index evaluated at any value of posterior mean $\bar{\mu}$ is then computed by linear summation as implied by the invariance property. We present further implementation details in Online Appendix G.2.
E Computation of the Value Function

Given a decision rule $\Pi$, we compute the value function (expected total utilities) by forward simulating utilities for a sufficiently long horizon. Starting at a given state $(\tilde{s}_0, \tilde{e}_0) = (\tilde{s}, \tilde{e})$, we sample a large number $D$ of Markov chains for each brand $\{ (s_{jk}^{(d)}, e_{jk}^{(d)}): k = 1, 2, \ldots K \}_{d=1}^{D}$, where $K$ is greater than the truncated horizon $T$. Bayesian updating of the normal distribution leads to the following state transition probabilities:

$$
\mu_{j,k+1}^{(d)} \sim N\left( \mu_{jk}^{(d)}, w_{jk}^{(d)} \sqrt{\sigma_j^2 + \bar{\sigma}_{jk}^2 (d)} \right), \quad \bar{\sigma}_{j,k+1}^{(d)} = \bar{\sigma}_{jk}^{(d)} \sqrt{1 - w_{jk}^{(d)}}, \text{ and } \\
\epsilon_{jk}^{(d)} \sim H(\epsilon; \mu^\epsilon, \sigma^\epsilon), \text{ where } w_{jk}^{(d)} = \bar{\sigma}_{jk}^2 (d) / (\sigma_j^2 + \bar{\sigma}_{jk}^2 (d)).
$$

These sequences of belief states are then fixed in advance and reused for each decision rule. Under a decision rule $\Pi$, the empirical estimate of its expected total utility for a truncated horizon $T$ is given by

$$
V_{\Pi}(\tilde{s}, \tilde{e}) = \frac{1}{D} \sum_{d=1}^{D} \left\{ \sum_{t=0}^{T-1} \delta^t \sum_{j=1}^{J} \mathbb{1} \left( (\tilde{z}_t^{(d)}, \tilde{e}_t^{(d)}) = j \right) \left( \bar{\mu}_{j,n_{jt}}^{(d)} + \epsilon_{jt}^{(d)} \right) \right\} (\tilde{s}_0, \tilde{e}_0) = (\tilde{s}, \tilde{e}),
$$

where $n_{jt}$ is the cumulative number of trials for brand $j$ up to period $t$. Note that the realized state values are chosen from the pre-drawn sample paths, with $n_{jt}$ indicating which state in the sample path is chosen.
F Maximum Simulated Likelihood Estimation

We estimate each model’s parameters with maximum simulated likelihood estimation. To simplify notation let \( \tilde{\alpha} \) denote the vector of parameters to be estimated. Let \( d_{it} \in A \) denote household \( i \)'s decision at period \( t \) and let \( \tilde{d}_{it}^T = \{d_{itk}\}_{k=1}^T \) denote \( i \)'s decision sequence up to period \( t \). The likelihood of observing the choice sequences as a function of \( \tilde{\alpha} \) is:

\[
L(\tilde{\alpha}) = \prod_{i=1}^I L_i(\tilde{\alpha}) = \prod_{i=1}^I \Pr(\tilde{d}_{it}^T; \tilde{\alpha}).
\]

Learning strategies depend upon the evolution of the unobserved belief states, which complicates the inference process. If we were to write the likelihood function as a function of each consumer’s unobserved belief states and shocks over periods, we would need to sample from an extremely complicated joint density of belief states and shocks. Instead, we augment the data and sample directly from the more-fundamental unobservables—the quality experiences \( q_{ijt} \) that are drawn conditionally i.i.d. from normal distributions with mean \( \mu_j \) and standard deviation \( \sigma_j \). Given a set of quality experiences and a set of prior beliefs, we obtain the unobserved belief states, \( s_{ijt} = (\tilde{\mu}_{ijt}, \tilde{\sigma}_{ijt}) \), by conjugate updating formulae:

\[
\tilde{\mu}_{ijt} = \frac{\tilde{\sigma}_0^2}{\tilde{\sigma}_0^2 + \sigma_j^2 / n_{ijt}} \bar{q}_{ijt} + \frac{\sigma_j^2 / n_{ijt}}{\tilde{\sigma}_0^2 + \sigma_j^2 / n_{ijt}} \tilde{\mu}_{j0}, \quad \text{and} \quad \tilde{\sigma}_{ijt}^2 = \frac{\tilde{\sigma}_0^2 \sigma_j^2 / n_{ijt}}{\tilde{\sigma}_0^2 + \sigma_j^2 / n_{ijt}},
\]

where \( n_{ijt} \) is the cumulative number of purchases of brand \( j \) by consumer \( i \) through period \( t \). We use \( \bar{q}_{ijt} = \frac{1}{n_{ijt}} \sum_{k=1}^{n_{ijt}} q_{ijk} \) to denote the average quality experience observed by the consumer through period \( t \). Note that the posterior variance \( \tilde{\sigma}_{ijt}^2 \) decreases over time towards zero as more information is incorporated and the speed of convergence depends on the prior and the true variance. When the prior \( \tilde{\sigma}_0^2 \) is much larger than the true variance \( \sigma_j^2 \), then after just one update, the
posterior quickly converges to $\sigma^2_j$.  

We introduce vector notation to simplify exposition. Let $\bar{\mu}$ be the vector (over $j$) of mean qualities, let $\bar{\sigma}$ be the vector (over $j$) of the standard deviations of quality draws, and let $\hat{\sigma}^\varepsilon$ be the vector (over $j$) of the standard deviations of unobservable shocks. Let the sequence of quality draws through period $t$ be $\bar{q}_t^j = \{\bar{q}_{ik}^j\}_{k=1}^T$. Let $\vec{x}_t$ and $\vec{e}_it$ be the vectors (over $j$) of prices and unobservable utility shocks. Finally, let $f_q(\cdot)$ and $p_\varepsilon(\cdot)$ be the probability density functions for the quality draws and the unobservable shocks. The likelihood for household $i$ is given by

\[
(F1) \quad L_i(\bar{\alpha}) = \int \int \prod_{t=1}^{T_i} \mathbb{I}[s_{it}(\bar{d}_i^{T_i} - 1, \bar{q}_i^{T_i} - 1), \vec{x}_t, \vec{e}_it] f_q(\bar{q}_t^{T_i}; \bar{\mu}, \bar{\sigma}) p_\varepsilon(\vec{e}_it)d\bar{q}_t^{T_i} d\vec{e}_it.
\]

To compute the likelihood we integrate over quality draws and unobservable shocks. To integrate numerically we sample $R$ sequences of quality draws (each sequence has $T_i$ draws for consumer $i$) from a multivariate normal distribution with parameters $\bar{\mu}$ and $\bar{\sigma}$. We assume that the unobservable shocks follow zero-mean Gumbel distribution with homogenous variance $\sigma^\varepsilon$ for all brands. This assumption allows us to use the well-known logit formula to substantially simplify the computation of choice probabilities for all models. Based on Proposition 2, we specify the index function as a linear function of the unobserved shocks $\varepsilon_{ijt}$, while preserving monotonicity:

$$\tilde{f}_W = \text{argmax}_j \left\{ \frac{\bar{\sigma}_{ijt}}{\sigma_{ij}} \left( 0, \frac{\bar{\sigma}_{ij}^\varepsilon}{\sigma_{ij}} \right), 1, 0, \frac{\bar{\sigma}_{ij}^\varepsilon}{\sigma_{ij}}, \delta \right\}. $$

We provide further implementation details of the estimation procedures in Online Appendix G.

---

1 This introduces difficulty in estimating the variance of prior quality beliefs. For example, we encounter difficulty when estimating the mixture model. The likelihood function is flat over the regions where the values of $\sigma^2_j$ are large. As a solution, we terminate the iterations of estimation when the likelihood improves by less than 0.02%. We expect the same difficulty to apply to the approximately optimal solution.
G User’s Guide to Implementation

This user’s guide documents the implementation details of the optimization solutions and the estimation procedures. §§G.1 and G.2 provide the details of solving the single-agent problem via the approximately optimal solution and the index solution using discrete approximation. §§G.3 and G.4 summarize the details of estimating the approximately optimal solution and the index solution.

G.1 Computing the Approximately Optimal Solution

G.1.1 Overview

Recall that the goal is to solve the following Bellman equation of the overall problem:

\[ V(\tilde{s}_t, x_t, \tilde{e}_t) = \max_{j \in A} \{ \beta' \tilde{x}_{jt} + \epsilon_{jt} + \mathbb{E}[q_{jt} + \delta V(\tilde{s}_{t+1}, \tilde{x}_{t+1}, \tilde{e}_{t+1})|\tilde{s}_t, j] \} \].

Under the assumption that unobservable shocks \( \epsilon_{jt} \) are i.i.d., we can integrate out this component and transform the problem to (Rust 1994):

\[ EV(\tilde{s}_t, \tilde{x}_t) = \int_{\tilde{e}_t} \max_{j \in A} \{ \beta' \tilde{x}_{jt} + \epsilon_{jt} + \mathbb{E}[q_{jt} + \delta EV(\tilde{s}_{t+1}, \tilde{x}_{t+1})|\tilde{s}_t, j] \} dH(\tilde{e}_t). \]

Further assumption on the distribution of \( \tilde{e}_t \) can simplify the above integration. If \( \epsilon_{jt} \) follows i.i.d. Gumbel distribution, then we have a closed-form expression (Rust 1994):

\[ EV(\tilde{s}_t, \tilde{x}_t) = \mu^e + \sigma^e \gamma + \sigma^e \log \left( \sum_{j=1}^{J} \exp \left( \frac{\beta' \tilde{x}_{jt} + \mathbb{E}[q_{jt} + \delta EV(\tilde{s}_{t+1}, \tilde{x}_{t+1})|\tilde{s}_t, j]}{\sigma^e} \right) \right), \]

where \( \gamma \) is the Euler constant. Notice that for this simplification to hold, we need to assume the distribution of the unobservable shocks is the same for all brands: \( (\mu_j^e, \sigma_j^e) = (\mu^e, \sigma^e) \) for all \( j \).

The observable shocks \( \tilde{x}_t \) remain in the state space. One can also integrate out \( \tilde{x}_t \) given the independence assumption, so that Equation (G1) becomes

\[ EV(\tilde{s}_t) = \int_{(\tilde{x}_t, \tilde{e}_t)} \max_{j \in A} \{ \beta' \tilde{x}_{jt} + \epsilon_{jt} + \mathbb{E}[q_{jt} + \delta EV(\tilde{s}_{t+1})|\tilde{s}_t, j] \} dH(\tilde{x}_t, \tilde{e}_t). \]
We solve Equation (G3) with simulation by integrating over the joint distribution of \((\hat{\beta}' \vec{z}_j + \epsilon_j)\).\(^2\) The modified Bellman equation still cannot be solved exactly because the state space of each brand \(s_j = (\mu_j, \sigma_j)\) is continuous. There are many algorithms to approximate the solution (see a survey by Rust 1996 for methods to solve continuous-state Markov decision processes).

We will use a "discrete approximation" approach that first discretizes the state space, then solves the discrete-state dynamic programming problem, and finally finds the value function of the continuous-state problem by aggregating the discrete-state solution using interpolation. While discrete approximation may be slower than "smooth approximation" (e.g., the Keane-Wolpin 1994 algorithm adopted by Erdem and Keane 1996; see Ching et al. 2013b for details of its implementation), it can fully preserve the contraction property of the Bellman equation, and is guaranteed to converge to the true solution as the discretization becomes finer (e.g., Chow and Tsitsiklis 1991). The absolute computation time depends on various factors such as the algorithm, computer memory, software package, coding, etc. In this study we are interested in comparing the relative computation time of the two solution concepts, the approximately optimal solution and the index solution, using the same discrete approximation approach to solve the fixed-point problem in both solutions.

**G.1.2 State Space and Transition Probabilities**

We discretize the state space into a finite collection of state points \(\mathbb{D}(\mu_j; M) = \{\vec{\mu}_j^{(m)}\}_{m=1}^M\) and \(\mathbb{D}(\sigma_j; N) = \{\vec{\sigma}_j^{(n)}\}_{n=1}^N\) for each brand. Given the discretized state space, we convert the transition

\(^2\) Alternatively, one can assume that the distribution of the sum \((\hat{\beta}' \vec{z}_j + \epsilon_j)\) is Gumbel and is homogenous, which leads to a value function similar to Equation (G3). However, this assumption is not appealing in this setting because (1) we need \(\epsilon_j\) alone to be Gumbel to obtain a simple logit expression of choice probabilities, and (2) the uncertainty in utility shocks may vary across brands.
probabilities from the continuous problem to the discrete problem. Recall that the transitions in the continuous problem are given as follows. Suppose the consumer’s current belief about brand \( j \) at period \( t \) is \((\bar{\mu}_{jt}, \bar{\sigma}_{jt})\). If the consumer chooses brand \( j \), then the consumer’s beliefs about brand \( j \) in the next period become

\[
\bar{\mu}_{j,t+1} \sim f \left( \mu_{jt}, \sigma_{jt} \right) \quad \text{and} \quad \bar{\sigma}_{j,t+1} = \sigma_{jt} \sqrt{1 - w_{jt}},
\]

where \( f(\cdot) \) is the normal density, and \( w_{jt} = \sigma_{jt}^2 / (\sigma_{jt}^2 + \bar{\sigma}_{jt}^2) \). If the consumer does not choose brand \( j \), then the consumer’s beliefs about \( j \) remain the same in the next period \((\bar{\mu}_{j,t+1}, \bar{\sigma}_{j,t+1}) = (\bar{\mu}_{jt}, \bar{\sigma}_{jt})\).

The transition of the variance of belief \( \bar{\sigma}_{jt} \) is deterministic. Given the variance of prior belief \( \sigma_{j0} \), we know the exact values that the future \( \bar{\sigma}_{jt} \) would fall in. This pins down the discretization of \( \bar{\sigma}_{jt} \). We can set \( \bar{\sigma}_{j1} = \sigma_{j0} \) and \( \bar{\sigma}_{j1} = \bar{\sigma}_{jt} \), and so on: \( \bar{\sigma}_{j}^{(N)} = \bar{\sigma}_{j,N-1} \). Note that for large values of \( N \), \( \bar{\sigma}_{j}^{(N)} \) will be close to zero.

The transition of the posterior mean of belief \( \bar{\mu}_{jt} \) is probabilistic on the entire unbounded continuous space. We choose a bound \([-B, B]\) and discretize it uniformly into \( M \) points. The size \( B \) is chosen to be large enough so that the states outside the bounds are rarely visited. We set \( B \) to be 5 deviation from the mean of the conditional distribution. We then define the transition probabilities on the discretized state space from one state point \((\hat{\mu}_{j}, \hat{\sigma}_{j})\) to another

\[
(\hat{\mu}_{j}^{(m)}, \hat{\sigma}_{j}^{(n)}) \text{ as } p(\hat{\mu}_{j}^{(m)}, \hat{\sigma}_{j}^{(n)} \mid \hat{\mu}_{j}^{(m)}, \hat{\sigma}_{j}^{(n)}). \]

If the consumer does not choose brand \( j \), then

\[
(\hat{\mu}_{j}^{(m)}, \hat{\sigma}_{j}^{(n)}) \text{ transits to } (\hat{\mu}_{j}^{(m)}, \hat{\sigma}_{j}^{(n)} \) with probability one. If the consumer chooses brand \( j \), then

the transition is as follows. First, the variance of posterior belief \( \bar{\sigma}_{j}^{(n)} \) transits to \( \bar{\sigma}_{j}^{(n')} \) determinis-
tically with \( n' = n + 1 \) for \( n < N \). Note that the difference between \( \hat{\sigma}_j^{(n)} \) and \( \hat{\sigma}_j^{(n+1)} \) converges to zero as \( n \) becomes larger. Therefore, we will set \( N' = N \) such that \( \hat{\sigma}_j^{(N)} \) transits to itself. Second, the mean of posterior belief \( \hat{\mu}_j^{(m)} \in \mathcal{D}({\bar{\mu}_j; M}) \) transits to any state point \( \hat{\mu}_j^{(m')} \in \mathcal{D}({\bar{\mu}_j; M}) \) with normalized probability:
\[
(p(\hat{\mu}_j^{(m')} | \hat{\mu}_j^{(m)}, \hat{\sigma}_j^{(n)})) = \frac{f(\hat{\mu}_j^{(m')} | \hat{\mu}_j^{(m)}, \hat{\sigma}_j^{(n)})}{\sum_{m'=1}^M f(\hat{\mu}_j^{(m')} | \hat{\mu}_j^{(m)}, \hat{\sigma}_j^{(n)})},
\]
where \( f(\hat{\mu}_j^{(m')} | \hat{\mu}_j^{(m)}, \hat{\sigma}_j^{(n)}) \) is the density conditional on \( (\hat{\mu}_j^{(m)}, \hat{\sigma}_j^{(n)}) \) defined in the continuous-state problem (see Equation G4).

G.1.3 Algorithm

After obtaining the transition probabilities for the discrete problem we can then solve the discrete-state dynamic programming problem using any standard dynamic programming algorithm. Here we use value iteration, which is easy to implement albeit not particularly fast. One can adopt the multi-grid approach (Chow and Tsitsiklis 1991) or the random-grid approach (Rust 1996) to speed up the algorithm. The value iteration algorithm proceeds as follows:

Step 1: Initialize the \((M \times N)^j\) matrix \( E V_0(\hat{\mu}, \hat{\sigma}) \).

Step 2: Iterate the modified Bellman equation until \( \| E V_{k+1} - E V_k \| < \text{Tolerance} \):
\[
E V_{k+1}(\hat{\mu}, \hat{\sigma}) = \max_{(\hat{x}, \hat{e})} \left\{ \beta' \hat{x} + \epsilon_j + \hat{\mu}_j^{(m)} \right\} + \delta \sum_{m'=1}^M E V_k \left( \hat{\mu}_j^{(m')}, \hat{\sigma}_j^{(n')} \right) p(\hat{\mu}_j^{(m')} | \hat{\mu}_j^{(m)}, \hat{\sigma}_j^{(n)}) \right\} dH(\hat{x}, \hat{e}),
\]

In Step 2, numerical integration is needed. We use direct simulation to integrate the function over the joint distribution of utility shocks. Once we have found the solution \( E V \) to the discrete...
problem, we can aggregate it to produce the solution $EV$ defined on the entire continuous state space using interpolation. In this study, we use linear interpolation.

**G.2 Computing the Index Solution**

**G.2.1 Overview**

Recall that in Appendix D we have shown that the index function reduces to solving the following modified Bellman equation (after integrating out both $x_j$ and $e_j$):

$$
EV(0, \bar{\sigma}_j, \lambda_j) = \int_{(\bar{x}_j, \epsilon_j)} \max\left\{ \lambda_j + \delta EV(0, \bar{\sigma}_j, \lambda_j), \beta^T \bar{x}_j + \epsilon_j \right\} dH(\bar{x}_j, \epsilon_j)
$$

where the $EV$ function is fixed at $\mu_j = 0$, and $\lambda_j$ is treated as a state variable with the following transition probabilities:

$$
\lambda_{j+1}|\lambda_j \sim \text{Normal}\left(\lambda_j, w_j \sqrt{\sigma^2_j + \bar{\sigma}^2_j}\right), \quad \text{and} \quad \bar{\sigma}_{j+1}|\bar{\sigma}_j = \bar{\sigma}_j \sqrt{1 - w_j}.
$$

We will remove $0$ from the $EV$ function for notational convenience, with a slight abuse of notation of the $EV$ function:

$$
EV(\bar{\sigma}_j, \lambda_j) = \int_{(\bar{x}_j, \epsilon_j)} \max\left\{ \lambda_j + \delta EV(\bar{\sigma}_j, \lambda_j), \beta^T \bar{x}_j + \epsilon_j \right\} dH(\bar{x}_j, \epsilon_j).
$$

The integration over the distribution of utility shocks ($\beta^T \bar{x}_j + \epsilon_j$) in general has no closed-form expression. We can, however, compute its value given any distributional assumptions using simulation or Gaussian quadrature. For example, we can assume that $\bar{x}_j$ is normal and $\epsilon_j$ is Gumbel, or the sum $\beta^T \bar{x}_j + \epsilon_j$ is normal or Gumbel. The computation is relatively easy given this is a one-dimensional problem (i.e., involving one brand). For the synthetic-data analysis, we as-
sume that \( \tilde{x}_{jt} \) is 0 and \( \epsilon_{jt} \) is Gumbel. For the field-data analysis on diaper purchases, we assume that \( \tilde{x}_{jt} \) is normal and \( \epsilon_{jt} \) is Gumbel. These assumptions are made simply to ease the computation of the approximately optimal solution, to which the index solution is compared, rendering ours a conservative test of the relative simplicity of the index strategy.

The procedure of computing the index function involves two stages. The first stage is to find the solution to the \( EV \) function from Equation (G7). We will use discrete approximation, the same method used for the approximately optimal solution. The second stage is to find the value of \( \lambda \) such that the two terms inside the max operator in Equation (G7) are equal. This \( \lambda \) value then equals Whittle’s index evaluated at \( \tilde{\mu}_{jt} = 0 \) and for a particular variance \( \tilde{\sigma}_{jt} \) and utility shocks \( \tilde{\beta}' \tilde{x}_{jt} + \epsilon_{jt} \), that is, \( W(0, \tilde{\alpha}_{jt}, \tilde{\beta}' \tilde{x}_{jt} + \epsilon_{jt}; \sigma_{j}, \mu_{k}^{X}, \sigma_{k}^{X}) \).

G.2.1 Algorithm

The discretization of the state space is the same as that for the approximately optimal solution described in §G.1.2, except that we now treat \( \lambda \) as the state variable rather than \( \tilde{\mu}_{jt} \).

Step 1: Initialize \( EV_{0} \left( \tilde{\sigma}_{j}^{(n)}, \tilde{\lambda}_{j}^{(m)} \right) \), \( \forall m, n \).

Step 2: Iterate the modified Bellman equation until \( ||EV_{k+1} - EV_{k}|| < \text{Tolerance} \):

\[
EV_{k+1} \left( \tilde{\sigma}_{j}^{(n)}, \tilde{\lambda}_{j}^{(m)} \right) = \int_{(\tilde{x}_{j}, \epsilon_{j})} \max \left\{ \tilde{\lambda}_{j}^{(m)} + \delta EV_{k} \left( \tilde{\sigma}_{j}^{(n)}, \tilde{\lambda}_{j}^{(m)} \right), \tilde{\beta}' \tilde{x}_{j} + \epsilon_{j} \right\} \delta \sum_{m=1}^{M} EV_{k} \left( \tilde{\sigma}_{j}^{(n)}; \tilde{\lambda}_{j}^{(m)} \right) p(\tilde{\lambda}_{j}^{(m)} | \tilde{\sigma}_{j}^{(n)}, \tilde{\lambda}_{j}^{(m)}) dH(\tilde{x}_{j}, \epsilon_{j}).
\]

Step 3: Obtain \( EV^{*} \left( \tilde{\sigma}_{j}^{(n)}, \tilde{\lambda}_{j}^{(m)} \right) \) after Step 2. For every \( n \) and value of \( \tilde{\beta}' \tilde{x}_{j} + \epsilon_{j} \), find the root \( \lambda^{*} \) such that the difference \( \Delta(\lambda) = 0 \), where
\[ \Delta(\lambda) = \hat{\lambda}_j^{(m)} + \delta E \hat{V}^n \left( \hat{\sigma}_j^{(n)}, \hat{\lambda}_j^{(m)} \right) - \beta \hat{\lambda}_j + \epsilon_j \]
\[ - \delta \sum_{m=1}^{M} E \hat{V}_k \left( \hat{\sigma}_j^{(n)}, \hat{\lambda}_j^{(m)} \right) p(\hat{\lambda}_j^{(m)} | \hat{\sigma}_j^{(n)}, \hat{\lambda}_j^{(m)}) . \]

Note that the difference \( \Delta(\lambda) \) increases with \( \lambda \). One easy way is to locate the \((\hat{\lambda}^{(a)}, \hat{\lambda}^{(b)}) \in \{ \hat{\lambda}_j^{(m)} \}_{m=1}^{M} \) such that \( \Delta(\hat{\lambda}^{(a)}) > 0 \) and \( \Delta(\hat{\lambda}^{(b)}) < 0 \) and then use linear interpolation to find \( \lambda_j^* \) such that \( \Delta(\lambda_j^*) = 0 \). \( \lambda_j^* \) is then Whittle’s index for brand \( j \), \( \hat{\lambda}_j^* \), evaluated at \( \hat{\sigma}_j \) and \( \beta \hat{\lambda}_j + \epsilon_j \).

Once we have the \( E \hat{V} \) function and Whittle’s index \( \hat{\lambda}_j \) defined on the discrete state space, we can use interpolation to find values of \( E \hat{V} \) and \( \hat{\lambda}_j \) over the entire continuous state space.

**G.3 Estimating the Approximately Optimal Solution**

We estimate the model using maximum simulated likelihood. For the approximately optimal solution, the likelihood function in Equation (F1) can be expressed as:
\[ L_t(\tilde{\alpha}) = \int \prod_{t=1}^{T_t} \frac{f_q(\tilde{q}^T_t; \tilde{\mu}, \tilde{\sigma}) d\tilde{q}^T_t}{\sum_{j=1}^{J} \exp(\beta x_{it,t+} + \mu_{i,t,t+} (d_{it,t+}^{t-1}, \tilde{q}_{it,t+}^{t-1}) + \delta E[EV(\mu_{i,t,t+}, \tilde{\mu}_{i,t,t+}, \tilde{\sigma}_{i,t,t+}, \tilde{\sigma}_{i,t,t+})])} \]

where the last equation uses the assumption that \( \epsilon_{ijt} \) follows the i.i.d. Gumbel distribution. The outer integration is computed by simulating \( R \) sequences of quality signals \( \{\tilde{q}^T_t\}_{r=1}^{R} \) from a multivariate normal distribution of true quality \( f_q(\cdot) \) parameterized by \( (\tilde{\mu}, \tilde{\sigma}) \).

To find the maximum simulated likelihood estimates \( \tilde{\alpha}^* = \arg \max \prod_{t=1}^{T_t} L_t(\tilde{\alpha}) \), we adopt the nested fixed point algorithm (Rust 1994):

- In the inner loop, for each guess of parameters \( \tilde{\alpha} \), solve for the \( E \hat{V} \) function based on
Equation (G3) using the procedure in §G.1 and then evaluate the likelihood function.

- In the outer loop, find the parameters $\hat{\alpha}^*$ that maximize the likelihood value.

The inner loop is computationally intense. For each parameter guess, we obtain an $EV$ function, which is then used to initialize the $EV$ function for the next parameter guess. This allows for faster convergence.

Another important issue is the choice of the size of discretization (i.e., the values of $M$ and $N$). The larger these numbers, the greater accuracy we obtain. But computation memory and time will increase exponentially as well. If we choose $M = N = 10$ then the size of the state space is $(M \times N)^4 = 10^8$. If $M = N = 100$ the size is $10^{16}$. In the estimation, we set $M = N = 5$ which yields a state space of size $5^8 = 390,625$.

**G.4 Estimating the Index Strategy**

The estimation procedure is similar to the one for the approximately optimal solution. Recall that the index rule is defined as

$$
\Pi_w = \arg\max_j \left\{ \bar{\mu}_{ijt} + \mu_j x_{ijt} + \sigma_j \bar{W}_j \left( 0, \frac{\bar{\sigma}_{ijt}}{\sigma_j}, \frac{\beta^{\xi_{ijt}} + \epsilon_{ijt} - \mu_j^{\xi_{ijt}}}{\sigma_j}, 1, 0, \frac{\sigma_j^{\xi_{ijt}}}{\sigma_j}, \delta \right) \right\}.
$$

We rewrite the index as a linear function of $\epsilon_{ijt}$:

$$
\hat{\Pi}_w = \arg\max_j \left\{ \bar{\mu}_{ijt} + \beta x_{ijt} + \epsilon_{ijt} + \sigma_j \bar{W}_j \left( 0, \frac{\bar{\sigma}_{ijt}}{\sigma_j}, 0, 1, 0, \frac{\sigma_j^{\xi_{ijt}}}{\sigma_j}, \delta \right) \right\}.
$$

This transformed function preserves all the properties of Whittle’s index and simplifies the computation of choice probabilities.

Assuming $\epsilon_{ijt}$ follows i.i.d. Gumbel, the likelihood in Equation (F1) for household $i$ is
We again use the nested fixed point algorithm to find the maximum simulated likelihood estimates:

- In the inner loop, for each guess of parameters $\tilde{\alpha}$, first compute Whittle’s index $\tilde{W}_j$ using procedures described in §G.2 with utility shocks normalized to zero. Then evaluate the likelihood given the index function.

- In the outer loop, find the parameters $\tilde{\alpha}^*$ that maximize the likelihood value.

We need to determine the number of grid points for the posterior mean and the posterior variance (i.e., the values of $M$ and $N$). For an “apples-to-apples” comparison between the approximately optimal solution and the index strategy, we set $M = 5$ and $N = 5$ for both models.

The state space is $5^4 = 625$ for the approximately optimal solution and $5 \cdot 5 = 25$ for the index strategy, a ratio of 15,625. (We solve for $J = 4$ indices.)

Because the size of the state space does not grow exponentially with the number of brands under the index strategy, it is feasible to choose finer grids for the index strategy than for the approximately optimal solution. Thus, for greater accuracy we set $M = 200$ and set $N$ to the maximum number of repeat purchases of households in the sample, which is 75. The size of the state space for the index strategy is $200 \cdot 75 = 15,000$. If we were to attempt this finer grid for the approximately optimal solution, the state space would be $(200 \cdot 75)^4 = 50,625,000,000,000,000$. This is about 130 billion times the approximately optimal solution’s
state space under the original grid of $M = N = 5$. It is unlikely that computations would be feasible with such a large state space, even with a more efficient search of the grid.
H Predicted Switching Matrices

Table H presents the predicted switching matrices among diaper brands, using the parameter estimates from the (a) no-learning model, (b) myopic learning model, and (c) approximately optimal solution model, respectively. The predicted switching matrix based on the parameter estimates from the index strategy model is presented in Table 2b.

Table H. Switching among Diaper Brands
(a) Predicted Switching Matrix – No-Learning Model

<table>
<thead>
<tr>
<th></th>
<th>Pampers</th>
<th>Huggies</th>
<th>Luvs</th>
<th>Other Brands</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Within the first 13 purchases</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td>17.5%</td>
<td>4.4%</td>
<td>4.7%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Huggies</td>
<td>8.1%</td>
<td>19.8%</td>
<td>3.6%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Luvs</td>
<td>4.4%</td>
<td>2.9%</td>
<td>9.2%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Other Brands</td>
<td>n/a*</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td><strong>After the first 13 purchases</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td>5.3%</td>
<td>1.7%</td>
<td>1.6%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Huggies</td>
<td>2.5%</td>
<td>7.5%</td>
<td>1.0%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Luvs</td>
<td>1.2%</td>
<td>0.7%</td>
<td>3.7%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Other Brands</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
</tr>
</tbody>
</table>

Switching percentages are weighted by market share so that the percentages in the same table add up to 100%.

Switching probability not applicable because the model predicts no purchase of Other Brands.
Table H. Switching among Diaper Brands (continued)

(b) Predicted Switching Matrix – Myopic Learning Model

<table>
<thead>
<tr>
<th></th>
<th>Pampers</th>
<th>Huggies</th>
<th>Luvs</th>
<th>Other Brands</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Within the first 13 purchases</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td>23.3%</td>
<td>2.6%</td>
<td>1.6%</td>
<td>0.1%</td>
</tr>
<tr>
<td>Huggies</td>
<td>1.8%</td>
<td>23.9%</td>
<td>1.1%</td>
<td>0.1%</td>
</tr>
<tr>
<td>Luvs</td>
<td>1.7%</td>
<td>1.2%</td>
<td>15.7%</td>
<td>0.1%</td>
</tr>
<tr>
<td>Other Brands</td>
<td>0.3%</td>
<td>0.1%</td>
<td>0.1%</td>
<td>1.0%</td>
</tr>
<tr>
<td><strong>After the first 13 purchases</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td>6.2%</td>
<td>0.6%</td>
<td>0.5%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Huggies</td>
<td>0.3%</td>
<td>12.4%</td>
<td>0.1%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Luvs</td>
<td>0.5%</td>
<td>0.2%</td>
<td>4.3%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Other Brands</td>
<td>0.0%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>0.1%</td>
</tr>
</tbody>
</table>

(c) Predicted Switching Matrix – Approximately Optimal Solution Model

<table>
<thead>
<tr>
<th></th>
<th>Pampers</th>
<th>Huggies</th>
<th>Luvs</th>
<th>Other Brands</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Within the first 13 purchases</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td>20.2%</td>
<td>3.0%</td>
<td>2.0%</td>
<td>0.2%</td>
</tr>
<tr>
<td>Huggies</td>
<td>2.0%</td>
<td>25.9%</td>
<td>1.4%</td>
<td>0.1%</td>
</tr>
<tr>
<td>Luvs</td>
<td>1.5%</td>
<td>1.6%</td>
<td>15.3%</td>
<td>0.1%</td>
</tr>
<tr>
<td>Other Brands</td>
<td>0.3%</td>
<td>0.1%</td>
<td>0.1%</td>
<td>1.0%</td>
</tr>
<tr>
<td><strong>After the first 13 purchases</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pampers</td>
<td>4.6%</td>
<td>0.4%</td>
<td>0.4%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Huggies</td>
<td>0.3%</td>
<td>15.0%</td>
<td>0.1%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Luvs</td>
<td>0.5%</td>
<td>0.2%</td>
<td>3.4%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Other Brands</td>
<td>0.0%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>0.1%</td>
</tr>
</tbody>
</table>