Essays on Financial Institutions
by
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Submitted to the Department of Economics in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY June 2015
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Abstract

In the first chapter, I study how banks lend or borrow liquidity in the interbank market and what I can learn about the macro-economy from the interbank market. From a unique database of interbank loan transactions in Mexico, I observe that interest rates vary across different lender-borrower pairs. I find that this variation is driven by the variation across different banks in their cost from handling an excess or a deficit of liquidity. Using my model, I characterize the shape of the interest rate curve as a function of loan size. Moreover, I find that the increased disadvantage that small banks experienced in the interbank market during the 2008 financial crisis can largely be explained by a shift in the liquidity cost.

In the second chapter, joint with Robert Townsend, we study how banks choose their level of cash holdings, taking into account potential payment demands and the short-term interest rate. We develop the notion of a rationing equilibrium in the money market, where a unique equilibrium exists for any given short-term rate. We characterize how changes in the short-term interest rate translate into changes in the banks’ lending activities, thus affecting the economy. In addition, we discuss how banks with different characteristics may respond differently to such changes.

In the third chapter, I study a recent change in the typical form of housing rental contracts in Korea. Traditionally, houses were mostly rented in exchange for a zero-interest loan from the renter to the owner of the house. However, during recent years, such a traditional form of rental agreement has been losing popularity and partially replaced by contracts based on monthly payments to the owner. Using a model of the interaction between the renter and the borrower, I explain how various financial market trends can potentially cause the observed change in the housing rental market.

Thesis Supervisor: Robert M. Townsend
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Acknowledgments

Graduate school feels like a long meditation. I thought about many things, both economic and non-economic. Physically, I do not sense much change about myself, even after several years have passed since I returned to Cambridge. At the end, I had to wake up, and some of what I can recall from my long meditation, I put down into this writing.

Even though I described my years as a meditation, I was not alone in completing this work. I am deeply in debt to Robert Townsend. He genuinely cared about what I was doing and helped me with warm heart and open mind. I have been truly fortunate to have him as my advisor. Another advisor I have been lucky to have is Alp Simsek. He is passionate and serious about economics, and it was a joy to talk with him about my work. Arnaud Costinot gave me invaluable advices on the first chapter, which was my job market paper. He helped me see my work from different perspectives and gave many practical suggestions.

MIT has been a great place, where everyone is willing to help each other. In addition to the three teachers I owe special thanks to, I will remember other faculty members, students and staff as warm and intelligent souls who are just nice to have around.

For the first chapter, I also thank Fabrizio López Gallo Dey, Calixto López Castañón, Ana Mier y Terán, Serafín Martínez-Jaramillo and Juan Pablo Solórzano Margain for their help with the Mexican database, useful suggestions and hospitality during my visit to the Bank of Mexico.

Outside economics, I also thank all my friends and family. In economics, we know that we live to maximize our utility function. My friends and family define my utility function, and my life would be blank without them. I could always count on my father, mother and brother's support. Last of all, I thank Shinae Kang for being with me, always giving me strength, comfort and feeling of love.
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Chapter 1

A Price-Differentiation Model of the Interbank Market and Its Empirical Application

1.1 Introduction

In this paper, we build a model of bank behavior in the overnight interbank market and develop an empirical framework to relate changes in the interbank market to changes in the liquidity condition of the financial system. We focus on the interaction between a large and central bank and a relatively small bank\(^1\) because it is where most of the interesting variations in interest rates occur in the data. We abstract away from the fact that these banks form a trading network\(^2\) and study the interaction between

---

\(^1\)An important question is how we can classify a bank as large or small, and we will discuss the exact classification in the main section of the paper. The idea that banks show different behavior based on their size is not new, and has been employed in empirical studies of interbank markets, such as Furfine (1999) and Afonso, Kovner and Schoar (2011). These papers used different quantiles from bank asset size distribution to classify a bank as large or small. Our approach is different and based on distinctive behaviors of a few largest banks in the system.

\(^2\)There is a growing literature on trading networks. Acemoglu, Ozdaglar and Tahbaz-Salehi (2014) is a recent example of a theoretical study. On the empirical side, Cocco, Gomes and Martins (2009) and Aram and Christophersen (2010) employ common network characteristics of each bank such as centrality to explain some of the variation in the interest rates in the interbank market. Bech and Atalay (2008) computes various common network metrics for the observed trading network in the Federal Funds Market, an interbank loan market in the United States.
a large bank and a small bank in isolation.

The importance of liquidity in the financial system has been studied extensively in the economics and finance literature. However, there are relatively few studies that develop an empirically motivated framework to understand the behavior of a bank in a marketplace of liquidity. One of the reasons is that an interbank market of short-term loans is typically an over-the-counter market, and the transactions in the market are not usually reported to the public. We take advantage of a unique dataset of interbank market transactions from Bank of Mexico which is constructed from the reports from individual banks.

Existing empirical papers on interbank markets study various aspects of the market. One set of these papers (Furfine (2001), Cocco, Gomes and Martins (2009) and Ashcraft and Duffie (2007a and 2007b)) typically estimate regression models to measure how interest rates are affected by various factors, such as default risk and 'relationship strength,' which is typically defined as the frequency of past interaction for a given lender-borrower pair. Another set of papers (Afonso, Kovner and Schoar (2011) and Allen et al. (2012)) study how the interbank market outcomes reflect the state of the financial system. The unique contribution of our approach is to build a model that can rationalize and explain observed empirical regularities with simple assumptions on the primitives of a bank, more specifically, its cost in dealing with an excess or a deficit of liquidity.

In our model of the interbank market, the central object is the cost of handling

---

3There are models of bank behavior in the interbank market at a more abstract level; Ho and Saunders (1985), Coleman, Gilles and Labadi (1996), Gofman (2013) and Afonso and Lagos (2012) are examples. In addition, there are models that are concerned with over-the-counter markets in general, such as Duffie, Gárleanu and Pederson (2005) and Atkeson, Eisfeldt and Weill (2013).

4Existing empirical studies of interbank loan markets mostly reconstruct the interbank market transactions from the records of large-value payments between banks. Furfine (1999) invented the procedure. However, there are debates on how reliable it is; Armantier and Copeland (2012) argues that the procedure is highly unreliable.


6Afonso, Kovner and Schoar (2011) studies the Federal Funds Market during the 2008 financial crisis and finds that interest rates on loans become more sensitive to borrowers' characteristics. Allen et al. (2012) defines a market-wide measure of bargaining power between lenders and borrowers and shows how it is related to the state of the financial market.
an excess or a deficit of liquidity (‘liquidity cost’). A bank may need more liquidity after a period of activity due to, for example, transfer instructions from customers. The bank faces a marginal cost in securing the necessary liquidity, which is increasing with the size of the shortage. Similarly, a bank that has accumulated unnecessarily large amount of liquidity wants to spend it to generate returns and faces a marginal cost of doing so, which is again increasing with the size of the excess. In the view of the model, the interbank market is an alternative to facing this increasing liquidity cost. This cost structure determines the interest rate on a loan in the model.

In the model, a large bank is a bank that has zero liquidity cost. Then, if there is any trade between two such large banks, the interest rate should show little variation, as the lender would not accept a low interest rate and the borrower would not accept a high interest rate. In the data, we map this observation into the fact that there is a small group of largest banks, between which the variation in interest rates is extremely small. Moreover, it turns out that most of the loans are either between two banks in that group of largest banks or between a bank in the group and a bank outside the group. This observation motivates the focus on the trade between a large bank and a small bank.

The large bank acts as a monopolist and offers a schedule of interest rates as a function of the size of the loan that the small bank lends to or borrows from the large bank. This approach to loan pricing in the context of the interbank market is also a novel contribution of our model. In practice, a small bank trades with multiple banks but it typically trades a majority of loans with a certain large bank. The model implies that a small bank gets a ‘better rate’ for a larger loan under broad assumptions. For example, when a small bank lends to a large bank, it tends to receive higher interest rates for larger loans. We confirm that the data support the conclusion of our model.

As an application of the model, we consider the 2008 financial crisis. At the peak

7Aside from anecdotal evidence, there is an empirical support for the view that the interbank market exists to offset cash excess/deficit that is created by payments to/from other banks. Sokolov et al. (2012) studies the network structure of the interbank market in Australia and shows that overnight interbank loan flows largely offset interbank large-value payment flows.
of the financial crisis, we find that (i) the average value (size) of the loans that small banks lend to large banks increased significantly, and (ii) the interest rates that small banks receive for the loan they lend to large banks, relative to the central bank target rate, fall at the peak of the financial crisis. We may interpret the increase in the value of loans as the small banks’ increased precautionary savings; they maintain a higher level of cash holdings, which they lend to the large banks. The fall in the interest rate may be explained by a combination of two reasons: (i) The increased supply of lending by small banks lowers the interest rate they receive on the loans they lend, or (ii) due to the worsening of the financial conditions, the cost of liquidity increases, so small banks accept lower interest rates on the loans that they lend. We can compare the impact of these two different factors by estimating the parameters in our model. We find that the second factor, the increase in the cost of liquidity, can explain a large part of the fall in the interest rates that small banks receive from the large banks.

Section 2 describes the Mexican interbank market and presents empirical observations that motivate our modeling approach. Section 3 presents the model setup and discusses its implications. Section 4 tests some of the implications of the model, develops a framework to estimate parameters of the model, and discusses the implication of the estimation result on the impact of the 2008 financial crisis on the banking sector. Section 5 concludes.

1.2 Data Description and Empirical Patterns

1.2.1 Data Description

The dataset that we use is a record of all transactions in the interbank call money market in Mexico. A call money operation is an uncollateralized loan that can be recalled by the lender before it is due; if the lender recalls a loan, the lender receives back the principal of the loan immediately but earns zero interest on the loan. As far as we know\(^8\), recalling is rare, so we treat these loans as simple uncollateralized

\( ^8 \)This knowledge is based on informal conversations with staff at the Bank of Mexico. There is no official statistic to back up this claim. The lender at least has an incentive to plan its lending
loans. These call money operations can have a maturity of 1, 2 or 3 banking days, but an absolute majority of them are overnight loans and we use only overnight loans in our dataset.

With the repo market, overnight interbank market is a primary source of overnight loans for banks. In the years 2008 and 2009, the total value of loans traded in the interbank market was about half of the total value of loans traded in the repo market.

The market is an over-the-counter market with no centralized exchange. Therefore, a loan is made in the market through a mutual agreement between two banks.

The time span of our dataset is from 01/21/2008 to 12/31/2009 and the total number of transactions is 21,449. We also combine the transaction database with individual banks’ balance sheet information. The number of banks with at least one transaction during this period is 38 and after removing banks without reliable balance sheet information, for example, new entrants in the market, we are left with 30 banks. In this process, we do not lose many observations because the banks that are removed are typically unimportant in the market, with only a small number of transactions.

The mean principal value of the loan is 536 million Mexican pesos, which is roughly 40 million US dollars based on exchange rates that prevailed in the years 2008 and 2009. The cross-sectional standard deviation of interest rates observed on a single day, averaged over all the banking days from 01/21/2008 to 12/31/2009, is 12 basis points, or 0.12 percentage points, in terms of annualized interest rates (all the interest rates that appear in this paper are in terms of annualized rates).

1.2.2 Variation in Interest Rates and Bank Size

We observe that among the few largest banks in the system, the variation in the interest rates on the loans between them is very small. To see this result, we compute, for each number \( n \), the standard deviation of interest rates on the loans between the \( n \) largest banks in terms of total assets. To control for the change in the interest rate properly so that it does not have to recall too often, as recalled loans earn zero interest rate.

\(^9\) We do not know why there does not exist an exchange for this market. One reason may be that the number of transactions is small, so there is not a strong incentive to establish an exchange. More generally, many financial assets are traded outside exchanges.
Interest rate variation among the large banks

Figure 1-1: Variation in Interest Rates within Different Subgroups of Banks

over time, we actually compute the standard deviation of the difference between the interest rate and the central bank target rate.

Figure 1 shows that for \( n \leq 4 \), the standard deviation is very small and there is little variation in interest rates. As \( n \) grows beyond 4, however, the standard deviation tends to increase with \( n \).

Another way to look at this pattern is to define the four largest banks as ‘large banks’ and the remaining banks as ‘small banks.’ As we will show soon, most of the loans, in terms of total loan values, are either (i) between two large banks or (ii) between a large bank and a small bank. If we compare the mean interest rates (as before, we use the difference from the target rate to control for changes over time) on the loans based on (i) whether the lender of the loan is a large bank or a small bank; and (ii) whether the borrower of the loan is a large bank or a small bank, we find that (i) the mean of the interest rates between two large banks is very close to zero; (ii) the mean of the interest rates that small banks pay to large banks when small banks borrow from large banks is significantly positive; and (iii) the mean of the interest rates that small banks receive from large banks when small banks lend to large banks is significantly negative.

\(^{10}\)The number of such cases is small, however, because small banks mostly lend to large banks, rather than borrow from them
Borrower is:

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<td>(0.118)</td>
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<tr>
<td>Small</td>
<td>-0.150</td>
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<td>(0.124)</td>
<td>(0.137)</td>
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* Numbers in () are standard deviations, not standard errors of the mean.

Table 1.1: Mean of Interest Rates for Different Bank Size

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<td>(0.124)</td>
<td>(0.137)</td>
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This result does not concern only the means of interest rates. Figure 2 plots the distribution of interest rates, separately for different size categories that the lender and the borrower belong to. It shows that in most cases, the interest rate that small banks receive from large banks when small banks lend to large banks is negative.

1.2.3 Counterparty Choice of Small Banks

Generally, in the data, the relatively small banks mostly trade, in total loan value terms, with some of the largest banks in the system rather than with other small banks. Therefore, if we divide the banks into two groups, large and small, as we did above, most of the loans will be either between two large banks or between a large bank and a small bank.

To see how this observation depends on how we define the large banks, we compute the total value of the loans between small banks divided by the total value of all the loans. We compute the ratio for each number \( n \), which represents the number of
largest banks that are classified as ‘large.’ When this ratio is small, most of the loans are between two large banks or between a large bank and a small bank.

In figure 3, as we increase the number of large banks, starting from two, the total value of loans between two small banks decreases rapidly until we reach four, then it slowly decreases afterward. The four largest banks seem to act as counterparties to many other banks in the system, while the others mostly trade with one of these largest banks. With \( n = 4 \), in value terms, 47.9% of total loans are between two large banks and 43.6% of the loans are between a large bank and a small bank. Loans between two small banks only account for 8.5% of the loans.

If we define large banks to be the four largest banks in the system, a typical small bank is observed to trade mostly with one of the large banks. Even though a small bank trades with more than one large banks in practice, it is very easy to identify which large bank it principally trades with. On average, 61% of a small bank’s trade with large banks is concentrated on the most frequent trading counterparty. In contrast, the second most frequent trading counterparty accounts for only 20% of trades.
1.2.4 Linking the Observations to Modeling Assumptions

In this subsection, we discuss how the modeling approach that we will lay out in detail in the next section is consistent with the empirical patterns described and can rationalize some of the patterns in a straightforward manner. First, we model the interaction between a large bank and a small bank, in isolation from all the other banks. Also, we assume that the large bank acts as a ‘monopolist,’ in the sense that it offers its profit-maximizing schedule of interest rates as a function of loan value (size) to the small bank.

Since (i) most of the loans are either between two large banks or between a large bank and a small bank; and (ii) there is little variation in interest rates on the loans between two large banks, most of the interesting variations in interest rates are observed on the loans between a large bank and a small bank. Also, since a small bank typically has a single large bank with which it trades a majority of its loans, the assumption that the large bank acts as a type of monopolist toward the small bank is a reasonable simplification.\(^{11}\)

Then, we assume that faced with an excess of liquidity (cash) or a deficit of liquidity, a large bank can handle it costlessly, while a small bank faces some cost (we denote this cost as the ‘cost of liquidity’) to handle the excess or deficit. For example, if a large bank has some excess cash, it can costlessly find an opportunity to invest the cash and generate returns. However, a small bank with excess cash will face some cost in finding and investing the cash to generate returns. This cost, in practice, may represent differential access to potential trading opportunities.

Under these assumptions, it is very easy to rationalize the observations on interest rates. To begin with, let us assume that all the banks face the same price of liquidity, which is the central bank target rate, apart from the cost of liquidity defined above. Since large banks face zero cost of liquidity, their marginal valuation of liquidity is the central bank target rate, and they will lend and borrow at the target rate irrespective

\(^{11}\)In addition, the interest rate that the most preferred trading counterparty offers is not better than what other banks offer. Therefore, it seems that the choice of counterparty is not mainly motivated by price.
of their positions in the market.

However, small banks face a positive cost of liquidity, so their marginal valuation of liquidity is lower than the central bank target rate when they have an excess of liquidity. A profit-increasing trade can be made when small banks lend some of this excess liquidity to a large bank because the large bank faces no cost in handling the excess liquidity. Then, the interest rate that a small bank receives for the loan will be between the central bank target rate and the small bank’s own valuation, and will generally be below the target rate due to the positive cost of liquidity. The same argument can rationalize the observation that when a small bank is borrowing from a large bank, it pays an interest rate higher than the central bank target rate.

1.2.5 Relation to the Existing Literature

The observation that large banks have a relative advantage in interest rates against small banks is consistent with the results from existing empirical literature, for example, Furfine (2001) and Cocco, Gomes and Martins (2009). Also, the analysis on the network structure of transactions in the interbank market in Bech and Atalay (2008) is consistent with our picture of the interbank market, in which a small group of large banks are involved in a large fraction of transactions in the market. Furfine (1999) also finds the same pattern with data on the Federal Funds Market.

1.3 Model

1.3.1 Setup

There are two banks in the model, a large bank (bank $L$) and a small bank (bank $S$). Let $x$ be the amount of excess liquidity that bank $S$ holds and let $l$ be the amount of loan that the small bank lends to the large bank; $x < 0$ means that bank $S$ has $-x$ amount of liquidity deficit and $l < 0$ means that the small bank borrows $-l$ from the large bank.

First, when $l = 0$, the bank $S$’s profit function $\pi_S$ is
\[ \pi_S = \int_0^x (p - c(y))dy. \]  

(1.1)

\(p - c(y)\) is the marginal value of liquidity, where \(p\) is a constant and \(c(\cdot)\) is a strictly increasing function such that \(c(0) = 0\). In this setup, \(p\) is the value of liquidity under no cost and \(c(\cdot)\) is the marginal cost of liquidity, in the sense that \(\int_0^x (p - c(y))dy < px\); \(c(y)\) always works against the small bank because its sign is the same as that of \(y\).

With \(l \neq 0\), bank \(S\) transfers \(l\) amount of liquidity to bank \(L\), so the profit function of bank \(S\) is

\[ \pi_S = \int_0^{x-l} (p - c(y))dy + rl. \]  

(1.2)

\(r\) is the net interest rate on the loan of value \(l\). Compared to equation (1), bank \(S\)'s liquidity position changes from \(x\) to \(x - l\) because it transfers \(l\) amount of liquidity to bank \(L\). In return, bank \(S\) receives interest payment \(rl\).

Bank \(L\)'s profit function \(\pi_L\) has the same form as \(\pi_S\), except that it has no marginal cost term. Therefore, by assumption, bank \(L\) has zero cost of liquidity:

\[ \pi_L = pl - rl. \]  

(1.3)

Bank \(L\) may have its own excess liquidity, but we do not need to consider it here; since the marginal value of liquidity is a constant for bank \(L\), its own excess liquidity does not affect the problem of determining \(r\) and \(l\).

In this setup, bank \(L\) ‘absorbs’ some of the liquidity excess or deficit of bank \(S\) because it can then deal with that absorbed position at a lower cost. The interest rate \(r\) has to be lower (higher) than \(p\) when bank \(S\) is lending (borrowing) so that bank \(L\) does not make a loss from the trade.
1.3.2 Trading Mechanism

$x$, the liquidity position of bank $S$, is a random variable. Bank $S$ knows its exact realization, but bank $L$ only knows its probability distribution. Bank $L$’s problem is to offer a schedule or menu of interest rates as a function of the loan value to maximize its expected profit, taking into account the fact that bank $S$ will choose the point on the interest rate schedule that maximizes its own profit.

Formally, let $r(l)$ be the interest rate schedule offered by bank $L$. $l$ can be either positive or negative; as mentioned before, a negative $l$ means that bank $S$ is borrowing from bank $L$. The problem of bank $S$ is to maximize its own profit given $x$ and the interest rate schedule $r(l)$:

$$
\max_{l} \int_{0}^{x-l} (p - c(y))dy + r(l)l.
$$

(1.4)

The outcome of this optimization problem will characterize the amount of liquidity (cash) that bank $S$ lends to bank $L$. The outcome, $l$, depends on the interest rate schedule offered, $r(y)$, and on $x$. Therefore, we can write $l$ as $l(x|r(y))$.

With this new notation, we can formally write the expected profit maximization problem of bank $L$:

$$
\max_{r(y)} \int_{-\infty}^{\infty} (p - r(l(x|r(y))))(l(x|r(y)))f(x)dx,
$$

(1.5)

where $f(x)$ is the probability density function of $x$. A full characterization, with necessary computational steps, is presented in the appendix. Below, we will omit computational steps and discuss only essential characteristics of the solution.

1.3.3 Characterization of the Solution

In solving bank $L$’s problem, the case for $x > 0$ can be solved separately from $x < 0$. The reason is that bank $L$ will always offer $r \leq p$ for $l > 0$ and offer $r \geq p$ for $l < 0$ to avoid making a loss. Given this fact and the increasing cost function $c(\cdot)$, bank $S$ has no incentive to choose $l < 0$ when $x > 0$ or to choose $l > 0$ when $x < 0$. Therefore,
from this point on, we assume $x > 0$. This case is more relevant because small banks mostly lend to large banks rather than borrow from them, and once we solve the bank $L$’s maximization for this problem, we can solve the problem for $x < 0$ in the same way.

Another result is that given some $r(y)$, $l(x|r(y)) \geq l(x'|r(y))$ if $x > x'$. Since the marginal cost function $c(x)$ is increasing in $x$, bank $S$ benefits more from increasing its lending when its liquidity position $x$ is larger. This result lets us write $l(x) = l(x|r(y))$ as a weakly increasing function of $x$.

Furthermore, for $r(l)$ and $l(x)$ to satisfy the incentive compatibility constraint for bank $S$, bank $S$ should be indifferent to choosing between $(r(l(x)), l(x))$ and $(r(l(x + dx)), l(x + dx))$ when its liquidity position is $x$, which produces the following condition:

$$l'(x)[r(l(x)) + r'(l(x))l(x) - p + c(x - l(x))] = 0. \quad \text{(IC)} \quad (1.6)$$

Finally, to characterize the solution, we use a the first-order condition from differentiating the objective function with respect to $l(x)$. Roughly speaking, the first-order condition is a balance between (i) the profit from increasing $l(x)$ and $r(l(x))$ for some $x$ in such a way to leave bank $S$’s profit the same but increase bank $L$’s profit, and (ii) the cost of increasing $r(l)$ for all $l > l(x)$ to conserve the incentive compatibility of bank $S$. The resulting expression is

$$f(x)c(x - l(x)) - (1 - F(x))c'(x - l(x)) + \lambda(x) = 0, \quad \text{(FOC)} \quad (1.7)$$

where $F(x)$ is the cumulative distribution function of $x$ and $\lambda(x)$ is a shadow cost of the constraints (i) $l(x) \geq 0$ for all $x$, and (ii) $l(x)$ is a weakly increasing function of $x$.

**Proposition 1:** The solution to bank L’s maximization problem can be characterized

---

12This condition makes sense only if $x$ is a continuous random variable. In this section, we are not concerned with presenting exact technical conditions. They are discussed in the appendix.
by the following two equations.

\[
\begin{align*}
l'(x)[r(l(x)) + r'(l(x))l(x) - p + c(x - l(x)) = 0. \quad \text{(IC)} \\
f(x)c(x - l(x)) - (1 - F(x)c'(x - l(x)) + \lambda(x) = 0, \quad \text{(FOC)}
\end{align*}
\]

The proof is in the appendix. ■

The specific functional form of the solutions \( r(l) \) and \( l(x) \) depend on the functional form of \( c \) and the distribution of \( x \). However, there is a general tendency for the interest rate to increase as \( l \) increases, at least for large values of \( l \), as long as the distribution of \( x \) does not have a heavy tail in the sense that either the support of \( x \) is bounded or the inverse of hazard function \( \left( \frac{1-F(x)}{f(x)} \right) \) becomes small for a large \( x \). Then, the first-order condition \( f(x)c(x - l(x)) - (1 - F(x)c'(x - l(x)) = 0 \) implies that \( x - l(x) \) should be close to 0. Intuitively, when the distribution of \( x \) is bounded or does not have a heavy tail, the large bank wants to lend as much as possible for large values of \( x \). The reason is that the cost to conserve the incentive compatibility of bank \( S \), \( (1 - F(x))c'(x - l(x)) \) becomes small relative to the profit from lending more when \( x \) is large.

Then, when \( c(x - l(x)) \) gets small, \( r'(l(x)) = \frac{p - c(x - l(x)) - r(l(x))}{l(x)} \) tends to be positive, given the equation (IC).

1.3.4 Additional Assumptions

We assume that the marginal cost of liquidity, \( c(y) \), takes the form of a power function for \( y \geq 0 \), \( c(y) = \alpha y^\theta \), for positive constants \( \alpha \) and \( \theta \).\footnote{Given that we are now considering the case of bank \( S \) lending, we do not care about the exact form of \( c(y) \) when \( y < 0 \). We only need that \( c(y) \) is an increasing function of \( y \).} If the hazard rate of \( x \), \( \frac{f(x)}{1-F(x)} \) is monotonically weakly increasing in \( x \),\footnote{A normal distribution, a uniform distribution and an exponential distribution are examples of such a distribution.} the solution to the optimization problem of bank \( L \) has a simple solution:

\textbf{Proposition 2:} Under the assumptions on \( c(y) \) and the hazard rate of \( x \) stated
above, \( l(x) \), the amount of loan that bank \( S \) lends to bank \( L \) is
\[
l(x) = \left[ x - \theta \frac{1 - F(x)}{f(x)} \right]^+ \tag{1.10}
\]
where the notation \([ \cdot ]^+\) denotes the maximum of the expression inside the brackets and 0. The condition (IC) can be rewritten as:
\[
r(l(x)) + r'(l(x))l(x) - p + \alpha(\theta(x - l(x)))^\theta = 0, \tag{1.11}
\]
or more conveniently,
\[
r(l) = p - \frac{1}{l} \int_0^l \alpha(\theta \frac{1 - F(l^{-1}(l))}{f(l^{-1}(l))})^\theta dl, \tag{1.12}
\]
where \( l \) denotes both the loan size as a variable in the argument of \( r(l) \) and the loan size \( l(x) \) as a function of \( x \) at the same time, in a slight abuse of notation. Also, \( x_0 = \sup\{x|l(x) = 0\} \).

The proof is in the appendix. ■

Since \( \frac{1 - F(x)}{f(x)} \) is monotonically decreasing in \( x \), \( r(l) \) is a monotonically increasing function of \( l \). Figure 4 shows the solutions for some chosen values of \( \theta \) and some chosen distributions of \( x \).

\subsection{Discussion}

A unique approach of our model is to let bank \( L \) determine the optimal interest rate schedule as a function of loan value. This approach is well known in the industrial organization literature as a screening problem, but it had not been previously applied in the context of interbank loan market. With some additional assumptions, we showed that the interest rate that bank \( S \) receives is an increasing function of the value of the loan.

This result contrasts with that from a profit-sharing approach, which has been
For the marginal cost function $c$, we use $p = 5$ and $\alpha = 0.3$.
Each line corresponds to a different value of $\theta$.
BLACK: $\theta = 0.5$. GREY (DASHED): $\theta = 1$. GREY (SOLID): $\theta = 2$.

Figure 1-4: Interest Rate Schedule $r(l)$ as a Function of Loan Value

used in many studies of interbank market\textsuperscript{15} and over-the-counter markets in general.
Under the same assumptions on the cost function as we made above, if interest rate
is set to share the profit generated from the transfer of liquidity between bank $S$ and
bank $L$ at the ratio of $\beta$ and $1 - \beta$, the interest rate $r(l)$ would be

$$
  r(l(x)) = p - (1 - \beta) \int_{x-l(x)}^{x} c(x - y) dy.
$$

In contrast with the model developed in this section, $r(l(x))$ is a decreasing function
of $l$ as long as $l(x)$ is an increasing function of $x$. Since the marginal cost of liquidity
$c$ is increasing in its argument, larger loans generally correspond to more average cost
per unit, which, in turn, corresponds to low interest rates according to the profit-
sharing rule. As we will show in the next section, we indeed observe that the interest
rate tends to be higher for a larger loan.

\textsuperscript{15}For example, in Allen et al. (2012), the profit is shared according to a fixed ratio, which
represents relative bargaining powers of the lender and the borrower. Please note that, however, the
paper does not study the relationship between interest rate and loan size in particular; it mainly
studies the relative bargaining power between borrowers and lenders.
In developing a model of interest rate, we have not considered the risk level of the borrower. For a risk-neutral lender, the premium (additive) on the interest rate due to a default risk will be approximately the default rate itself. When a large bank is borrowing from a small bank, the default risk of the borrower is typically very small; a large bank, in our model, corresponds to a few largest banks in the system.\textsuperscript{16} The fact that there is very little variance in interest rates also indirectly confirms this assessment because if there is any significant variation in default risk across the large banks, it should be reflected in the interest rate.

However, default risk premium may not be ignored when we consider a small bank borrowing from a large bank. In that case, the observation that small banks pay an interest rate higher than the central bank target rate to large banks may be explained by concerns about default. Even then, the small banks still pay higher interest rates when they borrow from large banks rather than from other small banks, so default risk alone cannot explain the data.

In the empirical analysis in the next chapter, we exclusively use the loans that small banks lend to large banks because small banks mostly lend to large banks rather than borrow from them. Therefore, we do not have to worry much about the role that default risk may play\textsuperscript{17}.

\section*{1.4 Empirical Application of the Model}

\subsection*{1.4.1 Relationship between the Interest Rate and Loan Size}

Here, we briefly confirm that interest and loan size, controlling for the identify of the lender and the borrower, have a positive correlation. We estimate a linear regression

\textsuperscript{16}These largest banks typically are assessed to have a strong ability to repay short-term debt obligations. For example, for the four largest banks in the Mexican system, their most recent ratings by Moody's are P-2; the historical default probability within three months for corporations rated in that category is 0.00 percent. Therefore, annualized, the contribution to the interest rate from default risk should be less than 1 basis point.

\textsuperscript{17}There still can be other variables that matter. For example, there are papers that study the effect on the interest rate of the time of the day when the loan is made; Hamilton (1996) is an example. Unfortunately, we do not have any information on this factor, so we need to assume that the time effect is not large enough to invalidate our analysis.
of the form

\[ r_i - p_t = \alpha + \beta l_t + \epsilon_i \]  

(1.14)

on the loans between a small bank and its most frequent trading partner (a large bank), separately for each small bank and for each 60-business-day time window to control for any significant changes in the relationship over time. \( i \) is simply an index for observations, \( r_i \) is the interest rate, \( p_t \) is the average interest rate between the large banks on day \( t \) (which is practically identical to the central bank target rate), \( \epsilon_i \) is the error term, and \( \alpha \) and \( \beta \) are linear regression coefficients. The coefficient \( \beta \) turns out to be mostly positive; it is positive in about 75% of the regressions.

### 1.4.2 Application to 2008 Financial Crisis

Casual observations indicate that near the peak of the 2008 financial crisis (measured by the peak in the implied stock market volatility index\(^{18}\) in Mexico) both (i) the total value of loans in the interbank market increased\(^{19}\) and (ii) the interest rate ‘discount’ (the absolute value of the difference between individual interest rates and the central bank target rate) on the loans that small banks lend to large banks increased. In this section, we only consider the loans that small banks lend to large banks, not the loans that small banks borrow from large banks.

Since both the total value of loans traded and the interest rate discount seem to have been significantly affected by the financial crisis, at least initially, it makes sense to consider a change in underlying parameters characterizing banks’ liquidity position and liquidity cost during the crisis.

Given the power function parametrization of the cost function, \( c(y) = \alpha y^\beta \), an

---

\(^{18}\)The precise definition of implied stock market volatility is discussed, for example, in Bollerslev, Tauchen and Zhou (2009). The crisis period is defined in our paper as the continuous block of dates around the peak of the implied stock market volatility index over which the daily closing level of the index stayed higher than 50% of the peak value. This definition results in 87 business days of crisis.

\(^{19}\)This observation may surprise some readers who think that an uncollateralized market such as the overnight interbank market would not function well when the financial market is in distress. In fact, overnight interbank market has been functioning well through different crises: Furfine (2002) documents that the Federal Funds Market worked well during the Russian debt crisis, and Afonso, Kovner and Schoar (2011) documents that there was at most a small drop in the total value of loans traded in the Federal Funds Market after the default of Lehman Brothers in 2008.
increase in the interest rate discount can be explained by two changes: (i) an increase in $\alpha$ ('cost change') and (ii) a change in the distribution of $x$, the liquidity position of the small bank. (Usually a shift in the distribution farther away from $x = 0$ causes the interest rate schedule $r(l)$ to shift down, increasing the average interest rate discount, given the increasing cost function.)

Since both the total value of loans and the mean value of loans have increased at the peak of the 2008 financial crisis, part of the observed increase in the interest rate discount should be explained by a shift in the distribution of $x$ to the right. (In the direction of increasing $x$.) The amount by which such a shift affects the interest rate discount is determined by the shape parameter of the cost function, $\theta$.

For example, if we change $\alpha$ to $(1 + \Delta)\alpha$, the interest rate schedule offered by the large bank will shift down from $r(l)$ to $p + (1 + \Delta)(r(l) - p)$. Alternatively, if we shift the distribution of $x$ by using $x' = (1 + \Delta)x$ in place of $x$, the interest rate schedule will shift down to $p + (1 + \Delta)^\theta(r((1 + \Delta)^{-1}l) - p)$\textsuperscript{20}. Intuitively, a larger $\theta$ implies that the marginal cost of liquidity grows faster when the liquidity position of the small bank is large, so a shift in the distribution of $x$ must have a larger effect with a larger

\textsuperscript{20}The mathematics involved in computing these results are simple and presented in the appendix.
Therefore, by measuring $\theta$ from the data, we can estimate whether the data are consistent with an increase in $\alpha$ at the peak of the crisis. In addition, we quantitatively compare the effects of the two sources of the increased interest rate discount.

In summary, by examining the shape of the interest rate schedule $r(l)$, we can estimate how a shift in the distribution in $x$ should affect the interest rate discount; as the formulas derived in the previous section and figure 4 suggest, a larger $\theta$ is associated with a steeper $r(l)$. This information, in turn, lets us estimate how much of the change in interest rate discount is not explained by a shift in the distribution of $x$. This ‘residual’ maps into a change in $\alpha$.

1.4.3 Parameter Estimation Procedure

In this subsection, we describe the estimation procedure briefly. For each small bank, we assume that $x$ follows a distribution with monotonically weakly increasing hazard rate. In particular, we use a linear failure rate distribution,\(^2\) which is characterized by two shape parameters $a$ and $b$ and whose probability density function is $(a + bx)exp(-ax - \frac{b^2x^2}{2})$ for $x > 0$. In principle, we can use any other distribution with reasonable shapes, but we have chosen this particular distribution for computational convenience; its inverse hazard rate is simply given by $a + bx$.

At the peak of the crisis (which we call simply ‘crisis period’), we assume that the distribution of $x$ is shifted to the right so that its distribution becomes that of $Cx$. It means that the parameters of the distribution $a$ and $b$ should be changed to $a' = \frac{a}{C}$ and $b' = \frac{b}{C^2}$ as a function of $C$.

We assume that $\theta$ is a fundamental parameter that does not change over time. Instead, the cost function shifts due to a shift in $\alpha$: We use two parameters, $\alpha$ and $\alpha'$, to parametrize the cost function outside the crisis period and within the crisis period, respectively.

In summary, we estimate the six parameters $(a, b, C, \theta, \alpha, \alpha')$ for each bank. We use the following moment conditions to estimate their values. First, we index individual

\(^2\)For a description of this distribution, see Sarhan and Kundu (2009)
observations by the day \( t^{22} \); \( l_t \) denotes the value of the loan and \( r_t \) denotes the interest rate. \( D_t \) is the dummy variable that takes the value of 1 if and only if \( t \) is inside the crisis period, and \( p_t \) is the average interest rate on the loans between large banks, which is practically identical to the central bank target rate.

The three moment conditions that relate to the shape of the distribution of \( x \) are:

\[
E[(l_t - E(a,b,\theta)(l))(1 - D_t)] = 0, \quad (1.15)
\]
\[
E[(l_t^2 - E(a,b,\theta)(l^2))(1 - D_t)] = 0, \quad (1.16)
\]
\[
E[(l_t^2 - E(a,b,\theta)(l^2))(1 - D_t) + (l_t^2 - E(a,b,\theta)(l^2))D_t] = 0, \quad (1.17)
\]

where \( E(a,b,\theta) \) denotes the theoretically expected value of functions of \( l \) given the parameters \( a, b \) and \( \theta \). Equations (13) and (14) are conditions on the mean of \( l \) and equation (15) is a condition on the second moment of \( l \).

We obtain four additional moment conditions from the (IC) condition derived in the last section, \( r + \frac{dr}{dl}l - p + \alpha \theta (l^{-1}(l) - l)^{\theta} = 0 \). Redefining \( \alpha = \alpha \theta^{\theta} \), we have

\[
E[(r_t + \frac{dr}{dl}l_t - p_t + \alpha(l^{-1}(a,b,\theta)(l_t) - l_t)^{\theta})(1 - D_t)] = 0, \quad (1.18)
\]
\[
E[(r_t + \frac{dr}{dl}l_t - p_t + \alpha'(l^{-1}(a',b',\theta)(l_t) - l_t)^{\theta})D_t] = 0, \quad (1.19)
\]
\[
E[(r_t + \frac{dr}{dl}l_t - p_t + \alpha(l^{-1}(a,b,\theta)(l_t) - l_t)^{\theta})(1 - D_t)l_t] = 0, \quad (1.20)
\]
\[
E[(r_t + \frac{dr}{dl}l_t - p_t + \alpha'(l^{-1}(a',b',\theta)(l_t) - l_t)^{\theta})D_tl_t] = 0, \quad (1.21)
\]

where \( \frac{dr}{dl} \) is the empirically estimated derivative and \( l^{-1}(a,b,\theta) \) is the mapping from \( l \) to liquidity position \( x \) which depends on the parameters \( a, b \) and \( \theta \). Roughly, \( \theta \) corresponds to how steep the curve \( r(l) \) is.

These four moments are from the (IC) condition derived in the last section, interacted with the dummy variable \( D_t \) and the loan size \( l_t \).

\( ^{22} \)In most cases, the number of loans between a small bank and its most frequent trading partner (a large bank) during a day is one. Even when there are more than one loans, the interest rate on the multiple loans made on the same day tend to be the same. Therefore, there is very little ambiguity in this definition of variables.
For all small banks
(n = 24) & For small banks with estimated $\theta > 0$
\hline
Mean & 0.90 & 1.44 \\
Median & 0.43 & 0.90 \\
\hline

*There are a few banks for which the estimated $\theta$ is unusually large, at around 5. These few observations drive up the mean relative to the median.

Table 1.2: Summary of Estimated $\theta$

1.4.4 Estimation Results

Table 2 shows the summary of estimated $\theta$ for individual banks. For several banks, $\theta$ is very close to zero because when there is a negative relationship between the interest rate and the loan size in the data, $\theta = 0$ tends to produce the best fit\textsuperscript{23}. The mean and median of estimated $\theta$ across the small banks is 0.90 and 0.43, respectively, which seem to be reasonable (compared to values such as $\theta = 5$, which implies that the marginal cost of liquidity $c(x)$ grows extremely fast with $x$).

Using the estimated parameters, we can ask how much of the increased discount on the interest rate that small banks receive can be attributed to an increased cost of liquidity (change in $\alpha$) rather than to an increased need to lend by small banks. The estimation result suggests that a large part (about 87%) of the increased discount can be attributed to the increased cost of liquidity rather than simply to the increased need to lend by small banks.

1.4.5 Discussion

Figure 6 is the plot of model-generated (with fitted parameters) change in the interest rate discount versus the observed interest rate discount. The figure shows that the estimation process can reasonably fit the model to the data.

The behavior of the interbank market at the peak of the financial crisis looks much like that of precautionary saving. The small banks lend more to large banks because

\textsuperscript{23}$\theta = 0$ implies a flat $r(l)$.
<table>
<thead>
<tr>
<th>(Average across banks)</th>
<th>Data</th>
<th>Generated by the fitted model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage rise in the loan value during the crisis (×100)</td>
<td>0.072</td>
<td>0.118</td>
</tr>
<tr>
<td>Average interest rate discount outside the crisis (pp)</td>
<td>-0.152</td>
<td>-0.132</td>
</tr>
<tr>
<td>Average interest rate discount during the crisis (pp)</td>
<td>-0.201</td>
<td>-0.197</td>
</tr>
<tr>
<td>Change in the discount, (Row 3) - (Row 2)</td>
<td>-0.0485</td>
<td>-0.0655</td>
</tr>
</tbody>
</table>

Contributions to the model-generated change in discount:

<table>
<thead>
<tr>
<th>Total</th>
<th>Increased cost (change in $\alpha$)</th>
<th>Increased demand to lend</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.0655</td>
<td>-0.0571</td>
<td>-0.0084</td>
</tr>
<tr>
<td>(100%)</td>
<td>(87%)</td>
<td>(13%)</td>
</tr>
</tbody>
</table>

Table 1.3: Different Contributions to the Increased Discount during the Crisis
Figure 1-6: Model Fit of Interest Rate Discount
they maintain a higher level of cash, even though the benefit of doing so in terms of interest rate is smaller in the crisis period.

A regulator monitoring this market may want to know whether the observed increase in the discount on the interest rate that small banks receive is either (i) an indication of an increased supply of lending by small banks, or (ii) an indication of a higher cost of liquidity for small banks. Case (ii) may indicate a worsening of the financial health of the system, while case (i) may simply be regarded as a shift in supply/demand with no strong implication on the financial health of the system.

1.5 Conclusion

In this paper, we have developed a model of bank behavior in the overnight interbank market. The primitive characteristic responsible for determining the interest rate is the cost of liquidity in the model. The model describes how a large and central bank in the system can absorb the liquidity excess and deficit of smaller banks. Our contribution is to build a model that can rationalize important features of the data in an intuitive manner.

In addition, our model, under some broad assumptions, shows that the interest rate disadvantage that a small bank experiences against a large bank decreases with the size of the loan. For example, when a small bank is lending to a large bank, the interest rate tends to be higher for larger loans. This result is also a unique contribution of our model in the context of the interbank market; for example, under the same assumptions, a model in which the price is determined according to a profit-sharing rule in fixed ratio would predict that the interest rate decreased with the loan value. The empirical results are consistent with the prediction of our model.

Finally, we estimate the parameters of our model under the context of 2008 financial crisis. In particular, we ask whether the drop near the peak of the financial crisis in the interest rates on the loans that small banks lend to large banks can be explained by an increased need to lend by small banks, supposedly due to the precautionary increase in cash holdings by small banks. Our estimation suggests that it
is only a small part of the story, and a large part of the drop has been caused by a general increase in the cost of liquidity.

Our paper leads to different avenues for future research. First, we simplified the setup and focused on the one-to-one interaction between a large bank and a small bank. However, in practice, trading occurs on a network of interconnected banks. It will be interesting to expand our setup so that we can also characterize and empirically observe the effect of the shape of the trading network on the observed variation in the interest rates. Another possible extension is to apply our framework to other markets which have a small group of large players that participate in the majority of transactions. A repo market, for example, can be a natural candidate, even though the market analysis will be more complicated with variation in the type of collateral.

\footnote{There are papers that study interbank exposure networks. However, as far as we know, there is no study that builds an analytical framework to study the interest rate variation in an interbank market of liquidity. Existing empirical literature that studies the effect of common network metrics such as centrality typically does not have a comprehensive model of how they should affect the interest rates. Instead, the studies tend to rely on general arguments on market power, outside opportunities, and so on.}
1.6 Bibliography


1.7 Appendix

1.7.1 Proof of Proposition 1

The price-differentiation model that we set up in section 3 is a standard price-differentiation problem. To prove proposition 1, I closely follow the steps outlined in Tadelis and Segal (2005).

We already explained that the maximization problem can be solved separately for the two regions, $x > 0$ and $x < 0$. Therefore, in this subsection, we solve the maximization problem for $x > 0$.

Bank L’s maximization problem is

$$\max_{R(\cdot), l(\cdot)} \int_{0}^{\infty} (p - r(l(x|r(y))))(l(x|r(y)))f(x)dx,$$

subject to the condition that the function $l$ is the optimal response to the following problem, given the function $r(y)$:

$$\max_{l} \int_{0}^{x-l} (p - c(y))dy + r(l)l.$$

First, we rewrite the problem so that the bank L’s problem becomes the one of choosing the shape of both $R(x) \equiv r(l(x))l(x)$ and $l(x)$:

$$\max_{R(\cdot), l(\cdot)} \int_{0}^{\infty} (pl(x) - R(x))f(x)dx.$$

We explained in section 3. 3 that $l(x)$ is an increasing function of $x$: $l'(x) \geq 0$. Also, to impose the choice $l(x)$ on bank S, $l(x)$ must satisfy the incentive compatibility constraint of bank S. In other words, we need to make sure that the change to bank S’s payoff from pretending to be of type $x + dx$ should not be positive.

Therefore, the derivative of the maximand of bank S’s maximization problem
multiplied by \( l'(x) \) should equal zero:

\[
l'(x) \frac{\partial}{\partial l} \left[ \int_0^{x-l} (p - c(y)) dy + r(l)l \right] = 0. \tag{1.25}
\]

By solving this equation, we obtain the following incentive compatibility (IC) constraint:

\[
- (p - c(x - l(x))) l'(x) + R'(x) = 0. \tag{1.26}
\]

Replacing \( R'(x) \) by \( r(l(x))l'(x) + r'(l(x))l(x)l'(x) \), (IC) constraint can be written as

\[
l'(x) [(r(l(x)) + r'(l(x)))l(x) - p + c(x - l(x))] = 0. \tag{1.27}
\]

This equation is the (IC) condition in proposition 1.

Now, we need to derive the first-order condition (FOC) for bank \( L \). Bank \( L \)'s maximization problem is

\[
\max_{R(\cdot), l(\cdot)} \int_0^\infty (pl(x) - R(x)) f(x) dx.
\]

subject to

\[
l'(x) \geq 0 \tag{1.29}
\]

and to the (IC) constraint

\[
- (p - c(x - l(x))) l'(x) + R'(x) = 0. \tag{1.30}
\]
Using the (IC) constraint, we can write

\[
R(x) = \int_0^x (p - c(y - l(y))) l'(y) dy
\]

\[
= pl(x) + \int_0^x c(y - l(y))(1 - l'(y)) dy - \int_0^{x-l(x)} c(y - l(y)) dy
\]

\[
= pl(x) + \int_0^{x-l(l(0))} c(y) dy - \int_0^x c(y - l(y)) dy.
\] (1.31)

Since \(l(0) = 0\), we have

\[
R(x) = pl(x) + \int_0^{x-l(x)} c(y) dy - \int_0^x c(y - l(y)) dy.
\] (1.32)

Then, the maximand for bank \(L\)'s maximization problem is

\[
\int_0^\infty (pl(x) - R(x)) f(x) dx = \int_0^\infty [- \int_0^{x-l(x)} c(y) dy + \int_0^x c(y - l(y)) dy] f(x) dx.
\] (1.33)

Applying integration by parts to \(\int_0^\infty c(y - l(y)) dy f(x) dx\) with respect to \(x\), we have

\[
\int_0^x c(y - l(y)) dy f(x) dx = \left[ \int_0^\infty c(y - l(y)) dy F(x) \right]_{x=0}^{x=\infty} - \int_0^\infty c(x - l(x)) F(x) dx
\]

\[
= \int_0^\infty c(x - l(x)) dx - \int_0^\infty c(x - l(x)) F(x) dx
\]

\[
= \int_0^\infty c(x - l(x)) \frac{1 - F(x)}{f(x)} f(x) dx.
\] (1.34)
Therefore, the maximand for bank L's maximization problem can be written as
\[
\int_0^\infty \left[ - \int_0^{x-l(x)} c(y)dy + c(x - l(x)) \frac{1 - F(x)}{f(x)} \right] f(x)dx. \tag{1.35}
\]

Since the maximand now depends only on the shape of \(l(x)\), we can maximize the integrand
\[
- \int_0^{x-l(x)} c(y)dy + c(x - l(x)) \frac{1 - F(x)}{f(x)} \tag{1.36}
\]
for each \(x\) with respect to \(l\). Taking the derivative of the expression above with respect to \(l\) and setting its value at zero, we have
\[
-c(x - l(x)) + c'(x - l(x)) \frac{1 - F(x)}{f(x)} = 0. \tag{1.37}
\]

Multiplying both sides of the equation by \(-f(x)\), we obtain (FOC):
\[
f(x)c(x - l(x)) - (1 - F(x))c'(x - l(x)) = 0. \tag{1.38}
\]

Since the (FOC) does not guarantee \(l'(x) \geq 0\) by itself, a shadow cost term \(\lambda(x)\) is generally necessary to ensure \(l'(x) \geq 0\).

### 1.7.2 Proof of Proposition 2

Using the assumption \(c(y) = \alpha y^\theta\), we can write (FOC):
\[
f(x)\alpha(x - l(x))^\theta - (1 - F(x))\alpha\theta(x - l(x))^{\theta-1} = 0. \tag{1.39}
\]

\[
\therefore l(x) = x - \theta \frac{1 - F(x)}{f(x)}. \tag{1.40}
\]

Since \(\frac{1 - F(x)}{f(x)}\) is weakly decreasing in \(x\), \(l(x)\) is an increasing function of \(x\). The restriction \(l(0) = 0, l'(x) \geq 0\) means that for \(x - \theta \frac{1 - F(x)}{f(x)} \leq 0\), \(l(x)\) should be 0; let \(x_0\) be the supremum of \(x\) such that \(l(x) = 0\). Since \(l(x)\) is an increasing function of \(x\), \(l(x)\) never violates the restriction \(l'(x) \geq 0\) when \(x - \theta \frac{1 - F(x)}{f(x)} \geq 0\). Therefore, for
\( x > x_0 \), it maximizes bank \( L \)'s profit to set \( l(x) = x - \theta \frac{1 - F(x)}{f(x)} \). More concisely,

\[
l(x) = [x - \theta \frac{1 - F(x)}{f(x)}]^+ \tag{1.41}
\]

is the optimal choice of \( l(x) \) for bank \( L \), where \([\cdot]^+\) denotes the maximum between the expression inside the brackets and 0.

The (IC) constraint is

\[
l'(x)[r(l) + r'(l)l - p + c(x - l)] = 0. \tag{1.42}
\]

For \( x < x_0 \), the (IC) constraint trivially holds because \( l'(x) = 0 \). For \( x \geq x_0 \), \n
\[
r(l) + r'(l)l - p + c(x - l) = 0. \tag{1.43}
\]

Since \( r(l) + r'(l)l = \frac{d}{dl}(r(l)l) \), we have

\[
\frac{d}{dl}(r(l)l) = p - c(x - l) = p - \alpha \left( \theta \frac{1 - F(l^{-1}(l))}{f(l^{-1}(l))} \right)^\theta. \tag{1.44}
\]

\( l^{-1} \) is well defined because we are looking at the region \( x \geq x_0 \).

Integrating from 0 to \( l \), we have

\[
r(l)l = pl - \int_0^l \alpha \left( \theta \frac{1 - F(l^{-1}(l))}{f(l^{-1}(l))} \right)^\theta dl. \tag{1.45}
\]

\[
\therefore r(l) = p \left( 1 - \int_0^l \alpha \left( \theta \frac{1 - F(l^{-1}(l))}{f(l^{-1}(l))} \right)^\theta dl. \right. \tag{1.46}
\]

### 1.7.3 Implications of Simple Changes in Parameters

In this subsection, we show how the changes in parameters described in section 4.2 affect the interest rate schedule \( r(l) \).

First, \( r(l) - p = -\frac{1}{l} \int_0^l \alpha \left( \theta \frac{1 - F(l^{-1}(l))}{f(l^{-1}(l))} \right)^\theta dl. \) Therefore, if we change \( \alpha \) to \( (1 + \Delta)\alpha \), then \( r(l) - p \) will get multiplied by a factor of \( (1 + \Delta) \). Therefore, the new interest
rate schedule $\tilde{r}(l)$ will be related to the old interest rate schedule by

$$\tilde{r}(l) - p = (1 + \Delta)(r(l) - p). \quad (1.47)$$

$$\therefore \tilde{r}(l) = p + (1 + \Delta)(r(l) - p). \quad (1.48)$$

We now instead shift the distribution of $x$ by using $x' = (1 + \Delta)x$ in place of $x$. Let $G$ be the cumulative distribution function of $x'$ and let $g$ be its probability density function. Then, $G((1 + \Delta)x') = F(x')$ and $g((1 + \Delta)x') = (1 + \Delta)^{-1}f(x')$. Let $\tilde{r}(l)$ be the optimal interest rate schedule under the new distribution and $I(x')$ be the loan amount under the new distribution. Then,

$$\tilde{r}(l) = p - \frac{1}{l} \int_0^l \alpha(\theta \frac{1 - G(\tilde{l}^{-1}(l))}{g(\tilde{l}^{-1}(l))})^\theta dl.$$

Also,

$$\tilde{l}(x') = [x' - \theta \frac{1 - G(x')}{g(x')}]^+ = (1 + \Delta)[(1 + \Delta)^{-1}x' - \theta \frac{1 - F((1 + \Delta)^{-1}x')}{f(1 + \Delta)^{-1}x'}]^+ \quad (1.50)$$

$$\therefore \tilde{l}(x') = (1 + \Delta)l((1 + \Delta)^{-1}x'). \quad (1.51)$$

$$\therefore \tilde{l}^{-1}(l) = l^{-1}((1 + \Delta)^{-1}l)(1 + \Delta). \quad (1.52)$$

Replacing $\tilde{l}^{-1}(l)$ by $l^{-1}((1 + \Delta)^{-1}l)(1 + \Delta)$, we have

$$\tilde{r}(l) = p - \frac{1}{l} \int_0^l \alpha(\theta \frac{1 - G(l^{-1}((1 + \Delta)^{-1}l)(1 + \Delta))}{g(l^{-1}((1 + \Delta)^{-1}l)(1 + \Delta))})^\theta dl = p - (1 + \Delta)^\theta \frac{1}{l} \int_0^l \alpha(\theta \frac{1 - F(l^{-1}((1 + \Delta)^{-1}l))}{g(l^{-1}((1 + \Delta)^{-1}l))})^\theta dl = p - (1 + \Delta)^\theta \frac{1}{(1 + \Delta)^{-1}l} \int_{(1 + \Delta)^{-1}l}^{(1 + \Delta)^{-1}l} \alpha(\theta \frac{1 - F(l^{-1}(l'))}{g(l^{-1}(l'))})^\theta dl' = p - (1 + \Delta)^\theta (p - r((1 + \Delta)^{-1}l)). \quad (1.53)$$
Chapter 2

Money Demand for Payments by Banks and the Money Market Rate

2.1 Introduction

Banks hold cash in the form of central bank reserves for various reasons, often in excess of reserve requirements. The cash holdings (or reserve holdings; these terms are used interchangeably in this paper) can represent, for example, precautionary hoarding (Acharya and Merrouche (2012) is an example) or even investment\(^1\) if the central bank pays interest on excess reserves.

This paper is concerned with a specific purpose of holding cash, that of meeting a potential need to make payments to other banks. Banks transfer a large sum of cash between themselves on any given day (Soramaki et al. (2009) describes the interbank payment flows in the United States in detail), and a transfer of cash is commonly referred to as a ‘payment.’ The need to make a payment can arise, for example, due to a transfer instruction from a customer to another bank.

In this paper, we develop a tractable framework in which a bank chooses its optimal level of cash holdings given its potential need to make payments to other banks.

\(^1\)In the United States, for example, banks have accumulated a large amount of excess reserves since the Federal Reserve started paying interest on excess reserves in 2008. The motive to earn income from the interest may be one of the explanations for the accumulation of reserves. Walter and Courtois (2009) discusses the issue.
banks. Since a bank can lend its excess cash or borrow cash from other banks, the bank does not need to hold enough cash to meet all potential payment demands by itself. Rather, the bank should take into account the short-term interest rate at which it can lend or borrow to determine the optimal level of cash.

To characterize this money market and determine a bank’s level of cash holdings, we develop a model of rationing equilibrium. If there is any excess cash in the banking system, some of the banks must end up holding unwanted excess cash. Under a competitive equilibrium, this excess cash drives the short-term interest rate to zero, preventing any variation in interest rate. By specifying a rationing rule to allocate this excess cash across banks and defining the resulting equilibrium as a rationing equilibrium, we develop a model in which different short-term interest rates are possible. Using the model, we analyze how a bank’s choice of cash holdings is affected by the short-term interest rate, providing a connection between the monetary policy and the demand for cash.

Then, using the developed framework, we study how changing the short-term interest rate influences their lending in the economy by affecting the banks’ incentive to hold cash. However, the change will affect some of the banks more strongly than others. We analyze how banks with varying characteristics respond to the change with varying intensity.

In addition to monetary policy, the payment system design and policy affect the level of cash holdings by banks. Even with exactly the same probability distribution of future payments to be made, banks under different payment systems would choose to hold different amounts of cash to meet the payment demands. One reason is that payment systems typically implement liquidity saving mechanisms to reduce the amount of cash that banks need to make payments, and there are different available mechanisms. We discuss how the change in demands for cash that arises from a change in the payment system policy can be incorporated our framework.

The paper stands at the intersection of the literature on payment systems and

\[\text{Guntzer, Jungnickel and Leclerc (1998) and Koponen and Soramaki (1998) compare different mechanisms. Federal Reserve System (2012) describes the overdraft facility currently in use in the United States to facilitate payments by providing cheap intraday credit.} \]
that on monetary policy. Research on payment system design and policy has mostly focused on technical specification of the systems, especially on algorithms that clear offsetting multilateral payments. In many studies, reducing the demand for cash is, by itself, considered an improvement, without clear connections to other economic outcomes such as bank lending. This paper contributes to the literature by developing a model that shows how the payment system policy can potentially affect the way in which banks respond to a monetary policy change.

In our model, the optimal level of cash is determined by the difference in the marginal returns to extra initial cash holdings between a borrowing bank and a lending bank. Since we assume that the payments only occur within the banking system and there is no net payment flow into or out of the banking system, we will see that the supply of lending will exceed the demand for borrowing whenever there is a positive amount of cash held by the banking system. This mismatch between the supply and demand creates a difference in the marginal returns between a lending bank and a borrowing bank.

Therefore, our model of cash holdings is different from Baumol-Tobin types of money demand models (Baumol (1952) and Tobin (1956)) because in those models, the optimal level of cash is determined by the tradeoff between foregone interest and transaction costs. Also, our model is different from models in which banks demand cash because they cannot fully respond to uncertain payment demands (Furfine (1998) is an example); in our model, the banks can always borrow the full amount to meet payment demands after the demands are fully realized.4

The rest of the paper is organized as follows: In section 2, we develop the baseline model, in which bank deposits are supplied infinitely elastically at a fixed interest rate by depositors. In section 3, we slightly modify the model by assuming a non-vertical supply curve for deposits and discuss how a change in the short-term rate affects the

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3 Martin (2004) is an example.
4 Sokolov et al. (2012) is an empirical study of the payment system and the interbank overnight loan market in Australia. The paper empirically finds that the interbank overnight loan market significantly offsets the payment transactions that have occurred during the day, supporting the hypothesis that banks rely on overnight loans to meet unpredictable payment demands, rather than on a pre-existing stock of cash.
banking system. Section 4 concludes.

2.2 Baseline Model

We build a static model of the banking system, with \( n \) banks indexed by \( i = 1, 2, ..., n \). Given the initial level of capital \( C_i \), bank \( i \) decides how much money to raise through deposit \( D_i \). The total liability \( C_i + D_i \) is invested into two types of assets, illiquid investment \( I_i \), which is the lending to businesses, and cash (reserve at the central bank) \( M_i \). Bank \( i \)'s problem is to determine the optimal level of \( I_i, M_i \) and \( D_i \) under the constraint

\[
C_i + D_i = I_i + M_i. \tag{2.1}
\]

In determining the choice variables \( I_i \geq 0, M_i \geq 0 \) and \( D_i \geq 0 \), bank \( i \) considers the possibility that it has to make payments to other banks before the illiquid investment \( I_i \) matures. The amount of cash bank \( i \) has to pay to the other banks is denoted by \( f_i \), which is a random variable. In this paper, we only consider payments occurring within the banking system, in the sense that \( \sum f_i = 0 \). By doing so, we separate the problem of payment uncertainty from the expansion of banking system (\( \sum f_i < 0 \)) or its contraction (\( \sum f_i > 0 \)).

Given this notation, bank \( i \) will have excess cash that it can lend to other banks if \( M_i - f_i > 0 \), but will have to borrow cash from other banks if \( M_i - f_i < 0 \); we assume that banks are not subject to any reserve requirement, and it is enough to maintain a nonzero cash position. We also assume that the interest on positive cash balance is zero, so a bank with an excess cash tries to lend as much of the excess cash as possible. The illiquid investment \( I_i \) cannot be used to meet the payment obligation in any way. Given \( \sum f_i = 0 \), there will always be enough cash to settle all the payments and to allow all the banks to end up with a non-negative cash balance. The expected profit from this ‘money operation,’ which comprises of lending cash to or borrowing cash from other banks, is denoted by \( \pi_i \). The profit \( \pi_i \) will depend on \( M_i \) and other variables, and we will specify the form of \( \pi_i \) later.

\( R_{l,i}(I_i) \) denotes the inverse demand function for bank lending for bank \( i \), which
is decreasing in \( I_i \). More generally, \( R_{I,i} \) should also depend on other banks' behavior to take into account competition among banks, for example. However, for simplicity, we assume that \( R_{I,i} \) is simply a function of \( I_i \).\(^5\)

\( R_{I,i}(I_i) \) denotes the interest rate that the bank can charge to mass \( I_i \) of the most willing borrowers. Assuming that the bank can extract all the surplus from the borrowers, bank \( i \)'s revenue from illiquid investment is \( \int_{0}^{I_i} R_{I,i}(x)dx \).

For simplicity, we assume that the interest rate on deposits is fixed and common to all banks. The rate is denoted by \( R_D \). Also, we assume that there exists \( I_i \) such that \( R_{I,i} < R_D \) for each \( i \) to make sure that every bank \( i \) chooses a finite level of illiquid investments. Then, the expected profit of bank \( i \), \( \pi_i \), is

\[
\pi_i = \int_{0}^{I_i} R_{I,i}(x)dx + \tilde{\pi}_i - R_D D_i. \tag{2.2}
\]

Using the balance sheet identity \( C_i + D_i = I_i + M_i \), we can also write:

\[
\pi_i = \left[ \int_{0}^{I_i} R_{I,i}(x)dx - R_D I_i \right] + \left[ \tilde{\pi}_i - R_D M_i \right] + R_D C_i. \tag{2.3}
\]

The terms inside the first bracket \([\cdot]\) represent the profit from illiquid investment, and those inside the second bracket represent the profit from cash holdings.

### 2.2.1 The Money Market and \( \tilde{\pi}_i \)

To continue with the model, we need to specify how the cash market works to determine the form of \( \tilde{\pi}_i \). Bank \( i \) determines \( I_i, D_i \) and \( M_i \) before the realization of \( f_i \), knowing only the probability distribution of the payment obligation \( f_i \). After the decision, \( f_i \), the payment obligation to other banks in the system, is realized. If \( M_i - f_i < 0 \), the bank needs additional cash, and it can obtain the needed cash only by borrowing from other banks. If \( M_i - f_i > 0 \), the bank has unused cash, and

\(^5\)In this sense, the banks are acting like regional monopolies in different regions of a country, for example.
the bank can lend it to other banks.

Here, only the cash holdings \( M_i \) at the beginning or incoming payment \(-f_i\) can be used to make payments. The illiquid investment \( I_i \) cannot be used in any way, and it is not possible to raise additional deposit. Also, we ignore the problem of gridlocks (or allow free multilateral netting), so the banks for which \( f_i < 0 \) can lend the money immediately, without waiting for the other banks to pay them first.\(^6\) Figure 1 describes this sequence of events for clarification.

We assume that the distribution of \( f_i \) is determined by the level of deposits \( D_1, D_2, ..., D_n \) held by banks. The form of the joint probability density function \( g(f_1, ..., f_n, D_1, ..., D_n) \) of \( f_1, f_2, ..., f_n \) depends on \( D_1, ..., D_n \). The assumption that \( g \) depends on the level of deposits is based on the idea that \( f_i \) represents unpredictable shocks, such as transfer instructions from clients; in such a case, the total size of transfer orders will depend on the size of deposits. Finally, in addition to \( \sum_i f_i = 0 \), we assume that the unconditional distribution of \( f_i \) is symmetric around 0 for every \( i \), and its support is a closed interval, so that there are no ‘holes’ in the distribution.

The money market, or the interbank market, is where the banks trade cash so that every bank can meet payment obligations. We assume that a single short-term interest rate \( R_M \) prevails in the market. We will discuss two types of equilibria, a competitive equilibrium and a rationing equilibrium. The distinction between these two types of equilibria is based on how the money market works.

\[ \text{2.2.2 A Competitive Equilibrium} \]

A competitive equilibrium is defined as a short-term rate \( R_M \) (\( R_M \) is a constant) and a choice of investment \( I_1, I_2, ..., I_n \) and cash holdings \( M_1, M_2, ..., M_n \) such that:

\(^6\)For example, let us assume that there are three banks A, B and C. If bank A needs to pay bank B a dollar, bank B needs to pay bank C a dollar, and bank C needs to pay bank A a dollar, and if the netting of obligations is allowed, no cash is needed to settle the payments; therefore, we only need to keep track of the net position \( M_i - f_i \) of each bank. However, if such netting is not allowed, one of the banks needs to pay first to settle the ‘triangular’ payments between banks A, B and C. We allow free multilateral netting, and it allows us to write the payment obligation simply as \( f_i \), without specifying exactly which of the other banks are receiving payments from bank \( i \). For a detailed discussion of gridlock issues in payment systems, see Rotemberg (2011).
Banks decide how much cash to hold ($M_i$).

Payment need ($f_i$) is determined.

Banks that have excess cash ($M_i - f_i > 0$) lend to banks that need to raise additional cash ($M_i - f_i < 0$).
(Only initial cash holdings and received payments can be used to settle payments. The illiquid investment cannot be used in any way; new deposits cannot be raised either.)

Banks' profits are realized; banks receive income from illiquid investment and from lending to other banks, and pay depositors.
(We do not consider payment needs that arise from this stage.)

Figure 2-1: Sequence of Events
(i) Each bank $i$ chooses $I_i$ and $M_i$ that maximizes its own expected profit. In computing the expected profit, banks assume that the money market will be in a competitive equilibrium with interest rate $R_M$ for any realization of payment demands $f_1, f_2, ..., f_n$. Also, each bank $i$ takes the choices of all the other banks as given.

(ii) For each realization of $f_1, f_2, ..., f_n$, banks determine the amount of lending $l_i$ ($l_i < 0$ represents borrowing) that maximizes its own profit under given interest rate $R_M$, subject to the constraint $M_i - f_i - l_i > 0$. $\pi_i$, the expected profit from money operation, is the expected value of $R_M l_i$.

(iii) For each realization of $f_1, f_2, ..., f_n$, the profit-maximizing choice $l_1, l_2, \ldots, l_n$ must be consistent with market clearing condition $\sum l_i = 0$.

Condition (ii) simply means that with $R_M > 0$, each bank $i$ such that $M_i - f_i \geq 0$ chooses to lend $l_i = M_i - f_i$. Also, each bank $i$ such that $M_i - f_i < 0$ chooses to borrow $f_i - M_i$, so $l_i = -(f_i - M_i) = M_i - f_i$.

Therefore, as long as there is an ‘excess cash’ in the banking system in the sense that $\sum M_i > 0$, the sum $\sum l_i = \sum M_i > 0$, which means that the supply of lending exceeds the demand for borrowing by $\sum M_i$. Therefore, in an equilibrium, the banking system cannot hold any excess cash as long as $R_M > 0$. Also, banks will choose to hold zero cash, $M_i = 0$, only if $R_M \leq R_D$.

Proposition 1 explicitly gives the set of competitive equilibria:

**Proposition 1:** The set of competitive equilibria is given by $0 \leq R_M \leq R_D$ and $M_i = 0$ for every $i$. In addition, $I_i$ is chosen to make $R_{I_i}(I_i) = R_D$ for every $i$.

We already discussed the intuition behind this result. The proof of the proposition is in the appendix.

A competitive equilibrium does not uniquely determine the value of $R_M$. In this paper, we interpret $R_M$ as a policy variable that the central bank determines.

The competitive equilibrium is restrictive because it can occur only if $R_M$ is smaller than $R_D$. Also, every bank has zero cash in a competitive equilibrium, so banks’
cash-holding incentives cannot play an important role in the economy. Therefore, we consider an alternative equilibrium, which we call a rationing equilibrium.

### 2.2.3 A Rationing Equilibrium

For a rationing equilibrium, we do not impose the simple market clearing condition in the money market, $\sum M_i = 0$. Rather, if $\sum M_i > 0$, we assume that every lender will be able to lend the same proportion of its excess cash to satisfy the needs of the borrowers. (If $\sum M_i > 0$, excess cash exactly equals the borrowing demand, so the proportion will be 1.) The exact amount of money that each lender can lend is given by the following rationing rule:

**Rationing Rule:** Bank $i$ can lend $(M_i - f_i) \frac{\sum [f_i - M_i]^+}{\sum [f_i - M_i]^+}$.

The notation $[\cdot]^+$ denotes the maximum between 0 and the expression inside the brackets. This rule is valid because $\sum [M_i - f_i]^+ = \sum [f_i - M_i]^+ + \sum M_i$; the fraction $\frac{\sum [f_i - M_i]^+}{\sum [f_i - M_i]^+}$ is always inside the interval $[0, 1]$. Therefore, no bank $i$ ends up lending more than the free cash that it has, $(M_i - f_i)$.

Also, the total amount of lending under the proposed rationing rule is

$$\sum_{\{i: M_i - f_i > 0\}} (M_i - f_i) \frac{\sum [f_i - M_i]^+}{\sum [f_i - M_i]^+} = \sum_{\{i: M_i - f_i < 0\}} (f_i - M_i),$$

which is the total amount of money that the borrowing banks need to borrow. Therefore, the proposed rationing rule is valid.

With the rationing rule defined, a rationing equilibrium is formally defined by the following conditions:

(i) Each bank $i$ chooses $I_i$ and $M_i$ that maximizes its own expected profit. In computing the expected profit, banks assume that the money market will work under the rationing rule with interest rate $R_M$ for any realization of payment demands $f_1, f_2, ..., f_n$. Also, each bank $i$ takes the choices of all the other banks
(ii) For each realization of \( f_1, f_2, \ldots, f_n \), banks with excess cash \( (M_i - f_i > 0) \) lend a fraction of \( (M_i - f_i) \) according to the rationing rule. The banks that need to borrow \( (M_i - f_i < 0) \) borrow the full amount of needed money, \( f_i - M_i \), from the money market. This trade is possible because the proposed rationing rule matches the amount of lending with the amount of borrowing and makes sure that every bank ends up with a non-negative cash balance. The interest rate in the money market is a given constant, \( R_M \). The expected profit from money operation, \( \pi_i \), is

\[
\begin{align*}
\bar{\pi}_i &= \frac{\sum_i [f_i - M_i]^+}{\sum_i [M_i - f_i]^+} - R_M [f_i - M_i]^+. \\
\end{align*}
\]

It is difficult to exactly characterize the rationing equilibria as the first-order conditions are complicated. The appendix shows the first-order conditions and describes some of the properties of the equilibria.

To describe some of the important properties of the rationing equilibria, we first note that the competitive equilibria are alsorationing equilibria; when the cash holdings in the banking system \( \sum M_i \) is zero, the lending banks can lend the full amount of excess cash they have. Therefore, in that case, the rationing equilibria coincide with the competitive equilibria.

More importantly, under a rationing equilibrium, even \( R_M > R_D \) is possible. Therefore, by using the concept of rationing equilibria, we allow more freedom in the choice of \( R_M \). The reason is that the rationing rule can reduce, from \( R_M \) to \( R_D \), the expected marginal revenue from an increase in \( M_i \) through the rationing rule; if \( \sum M_i > 0 \), the rationing factor \( \frac{\sum_i [f_i - M_i]^+}{\sum_i [M_i - f_i]^+} \) is always smaller than 1. Also, the rationing rule allows \( M_i > 0 \), so banks can actually hold positive amount of cash under a rationing equilibrium, in contrast to a competitive equilibrium.

The exact first-order conditions for a rationing equilibrium are complicated and stated in the appendix. Under simplifying assumptions (which are discussed in the appendix), including \( I_i \gg M_i \) and that \( g(f_1, \ldots, f_n, D_1, \ldots, D_n) \) can be approximated by \( g(f_1, \ldots, f_n, I_1, \ldots, I_n) \), the first-order conditions can be reduced to:
0 = \frac{\partial \pi_i}{\partial I_i} = R_{i,i}(I_i) - R_D. \quad (2.5)

0 = \frac{\partial \pi_i}{\partial M_i} = R_M[\text{Prob}(M_i - f_i < 0) + \text{Prob}(M_i - f_i > 0)]\eta(M_1, ..., M_n, I_1, ..., I_n) - R_D, \quad (2.6)

where \( \eta(M_1, ..., M_n, I_1, ..., I_n) \) is the expected value of the rationing factor, \( \frac{\sum_{[f_i-M_i]}^\theta}{\sum_{[M_i-f_i]}^\theta} \).

As usual, the second condition can be replaced by \( \frac{\partial \pi_i}{\partial M_i} \leq 0 \) if \( M_i = 0 \). We see that in the second condition, the expression for \( \frac{\partial \pi_i}{\partial M_i} \) is different from that in a competitive equilibrium, which is \( R_M - R_D \). The reduction in the return to \( M_i \) from \( R_M \) in a competitive equilibrium to \( R_M[\text{Prob}(M_i - f_i < 0) + \text{Prob}(M_i - f_i > 0)]\eta(M_1, ..., M_n, I_1, ..., I_n) \) happens due to rationing.

With this simplification, the problem of finding an equilibrium is a fixed-point problem defined by \( \frac{\partial \pi_i}{\partial M_i} \) and the definition of the function \( \eta(M_1, ..., M_n, I_1, ..., I_n) \). First, in a slight abuse of notation, we use \( \eta \) to denote a variable, not a function as above. Then, we define a new function \( h(M_1, ..., M_n, I_1, ..., I_n) \), which is indeed the same as the function \( \eta(M_1, ..., M_n, I_1, ..., I_n) \); \( h(M_1, ..., M_n, I_1, ..., I_n) = \int_{(f_1, f_2, ..., f_n)} \sum_{[f_i-M_i]}^\theta df_1 df_2 ... df_n \) is the unconditional expected value of the rationing factor. Finally, we introduce \( F_i \) to denote the cumulative distribution function of \( F_i \), which only depend on \( M_i \) and \( I_1, I_2, ..., I_n \). Since \( I_1, I_2, ..., I_n \) are determined by \( R_{i,i}(I_i) - R_D = 0 \), we can regard \( F_i \) as a fixed distribution that depends only on \( M_i \).

With these new notations, the fixed-point problem is to solve the following \((n+1)\) equations simultaneously to obtain \( M_1, M_2, ..., M_n \) and \( \eta \):

\[ R_M[1 - F_i(M_i)(1 - \eta)] - R_D = 0 \text{ for every } i. \quad (2.7) \]

\[ \eta = h(M_1, ..., M_n, I_1, ..., I_n). \quad (2.8) \]

We can now completely characterize the rationing equilibria.

**Proposition 2:** Under any equilibrium, the level of illiquid investment is simply determined by the equation \( R_{i,i}(I_i) - R_D = 0 \) for every \( i \). The set of rationing
equilibria is:

(i) For each $R_M$ such that $0 \leq R_M \leq R_D$, a unique rationing equilibrium exists: $M_i = 0$ for every $i$ and $\eta = 1$.

(ii) For each $R_M$ such that $R_M > R_D$, a unique rationing equilibrium exists. The solution is obtained by the unique solution $\eta, M_1, ..., M_n$ to the $(n + 1)$ equations:

$$M_i = F_i^{-1}(\frac{1}{1-\eta}(1 - \frac{R_D}{R_M}))$$

for every $i$ and $\eta = h(M_1, ..., M_n, I_1, ..., I_n)$. The solution satisfies $M_i > 0$ for every $i$ and $[2\frac{R_D}{R_M} - 1]^+ < \eta < 1 - \frac{R_D}{R_M}$. Also, for a larger $R_M$, the equilibrium value of $\eta$ is smaller and that of $M_i$ is larger for every $i$.

The proof is given in the appendix. Part (i) of the proposition means that the competitive equilibria are also rationing equilibria; with $R_M$ lower than $R_D$, no bank has an incentive to initially hold cash in either type of equilibria. Part (ii) characterizes the rationing equilibria for $R_M > R_D$; in this case, banks increase their holdings of cash until it is no longer advantageous to do so. Naturally, when $R_M$ is higher, they reach a higher level of cash holdings.

As with the competitive equilibrium, the short-term rate $R_M$ is not determined by the equilibrium. Therefore, we interpret $R_M$ as a policy variable that the central bank can choose at its will.

The advantage of using the rationing equilibrium over the competitive equilibrium is clear: We have a unique equilibrium for every $R_M$, not just for $R_M \leq R_D$.

To summarize, we have developed a framework that relate banks' cash holdings to the short-term rate $R_M$ and the potential demands for payments. Assuming a rationing rule in the money market, we showed that a unique equilibrium exists under any short-term rate $R_M$, and banks hold positive amounts of cash only if $R_M > R_D$.

### 2.2.4 An Example: Normally Distributed Payment Demands

To give a concrete example, we consider a case in which the payment demands $f_i$ follow a normal distribution. We use the simplified rationing model, which is characterized by the following first-order conditions:
\[
0 = \frac{\partial \pi_i}{\partial I_i} = R_{I,i}(I_i) - R_D. \tag{2.9}
\]

\[
0 = \frac{\partial \pi_i}{\partial M_i} = R_M[1 - F_i(M_i, I_1, ..., I_n)(1 - \eta(M_1, ..., M_n, I_1, ..., I_n))] - R_D, \tag{2.10}
\]

First, the level of the investment \( I_i \) is simply determined by the condition \( R_{I,i}(I_i) - R_D = 0 \). Therefore, instead of specifying the function \( R_{I,i} \), we assume that each bank chooses an exogenously given level of illiquid investment \( I_i \).

We now define the joint probability density function \( g(f_1, ..., f_n, I_1, ..., I_n) \). We define a new set of random variables, \( f_{ij} \) for every \( i < j \), such that \( f_{ij} \) follows a normal distribution with mean 0 and variance \( \frac{1}{I} \) where \( I = \sum_k I_k \) and \( v \) is a constant. (We can use any function of \( I_1, ..., I_n \) for variance. Here, we specify the form of the variance to create a numerical example. Also, this specification can be derived from reasonable assumptions on how the underlying payment demands are generated. The assumptions are discussed in the appendix.) All \( f_{ij} \) for \( i < j \) are independent.

For every \( i > j \), we define \( f_{ij} = -f_{ji} \). Then, given these random variables \( f_{ij} \), we define \( f_i \) by \( f_i = \sum_{j \neq i} f_{ij} \) for every \( i \). Here, \( f_{ij} \) can be interpreted as the amount of money that bank \( i \) needs to transfer to bank \( j \).

Since \( f_{ij} + f_{ji} = 0 \) for every \( i \neq j \), the sum \( \sum_i f_i = 0 \).

The unconditional distribution of \( f_i \) is a normal distribution with mean 0 and variance \( v \sum_{j \neq i} I_{ij} \) where \( I = \sum_k I_k \). Therefore, the distribution of \( f_i \) is symmetric around zero and its support is the whole real line, which is a closed interval.

The first-order condition with respect to \( M_i \) can be written as follows:

\[
0 = \frac{\partial \pi_i}{\partial M_i} = R_M[1 - F_i(M_i, I_1, I_2, ..., I_n)(1 - \eta(M_1, ..., M_n, I_1, ..., I_n))] - R_D
= R_M[1 - \Phi(M_i(\sqrt{v I_i(I - I_i)} - 1))](1 - \eta(M_1, ..., M_n, I_1, ..., I_n))] - R_D, \tag{2.11}
\]

where \( \Phi \) denotes the cumulative distribution function of the standard normal variable.

Proposition 2 implies that for \( R_M > R_D \), there exists a unique equilibrium, char-
acterized by $M_1, M_2, \ldots, M_n$ and $\eta^*$ that satisfy the following $(n + 1)$ equations:

$$M_i = \sqrt{\frac{I_i(1 - I_i)}{I}} \Phi^{-1}\left(\frac{1}{1 - \eta^*(1 - \frac{R_D}{R_M})}\right) \text{ for every } i.$$  

(2.12)

$$\eta^* = h(M_1, \ldots, M_n, I_1, \ldots, I_n).$$  

(2.13)

For a numerical example, we assume that there are 101 banks, with $I_1 = 10, I_2 = 10.1, \ldots, I_{101} = 20$ and that $v = 0.01$ and $R_D = 1\%$.

Figure 2 presents the unique equilibrium values of $\eta$ and $\sum_i M_i$ for different $R_M$. As implied by proposition 2, $\eta$ is a decreasing function of $R_M$ and $\sum_i M_i$ is an increasing function of $R_M$.

### 2.3 Economic Effects of a Change in the Short-Term Rate

In the previous section, we have developed a framework that relates the banks’ decision to hold cash to the short-term rate. A higher short-term rate can increase the level of cash holdings $M_i$ and correspondingly, the level of deposits that the banks raise. However, the increased cash holdings $M_i$ with a higher $R_M$ reduce the banks’ profits as a whole. The increased amount of deposit to finance the increased $M_i$ increases the cost to the banks, while the profit from the money market to the banking sector as a whole is always zero.

In this section, we discuss how a change in $R_M$, the short-term rate, affects bank lending. Also, we discuss how payment policy can be incorporated into our framework.

We assume that $n$ banks are in the simplified rationing equilibrium described in the previous section. In addition, we now introduce the inverse supply function for deposits, $R_{D,i}(D_i)$, which represents the interest rate that bank $i$ should pay to the marginal depositor to attract additional deposit. $R_{D,i}(D_i)$ is a strictly increasing function of $D_i$. 

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Figure 2-2: Equilibrium Values of $\eta$ and $\sum M_i$
Then, for each bank, the first-order conditions are

\[ 0 = \frac{\partial \pi_i}{\partial I_i} = R_{I,i}(I_i) - R_{D,i}(D_i). \]  

(2.14)

\[ 0 = \frac{\partial \pi_i}{\partial M_i} = R_M[1 - F_i(M_i)(1 - \eta(M_1, ..., M_n, I_1, ..., I_n))] - R_{D,i}(D_i). \]  

(2.15)

Here, the rates \( R_{D,i}(D_i) \) paid to the marginal depositors are potentially different across banks.\(^7\)

Using the model, we study the response of \( I_i, M_i \) and \( D_i \) to a small change in the short-term rate, \( R_M \), by comparing the equilibrium with another equilibrium under a slightly different \( R_M \). We should note that in the present framework, we ignore the competition for illiquid investment \( I_i \) or deposit \( D_i \) between banks as \( R_{I,i} \) and \( R_{D,i} \) are functions of \( I_i \) and \( D_i \) only. Therefore, the model can be appropriate only for analyzing small changes in the economy.

To look at how the equilibrium values of \( I_i, M_i \) and \( D_i \) change as \( R_M \) changes a little, we denote these choice variables as functions of \( R_M \) and take the derivatives of the first-order conditions. We assume that \( R_M \) is such that for every \( i, M_i > 0 \); banks for which \( M_i < 0 \) would not respond to a small change in \( R_M \) and ignoring them is convenient. Also, we ignore changes to \( \eta \) and to the distribution of \( f_i \); their effects are likely to be much smaller than those of the other terms. (See the appendix for an explanation and an example.)

By differentiating the first-order conditions with respect to \( R_M \), we obtain:

\[ R_{I,i}' \frac{dI_i}{dR_M} - R_{D,i}'(\frac{dI_i}{dR_M} + \frac{dM_i}{dR_M}) = 0. \]  

(2.16)

---

\(^7\)Previously, we have assumed that \( R_D \) is a common constant across the banks. Allowing \( R_{D,i} \) to be constants that are specific to each bank changes the problem little and we still have the existence of equilibria with minimal extra assumptions, as shown in the appendix. However, we do not prove that whether an equilibrium always exists when \( R_{D,i} \) are given as functions of \( D_i \). Here, we assume that an equilibrium exists for the given \( R_M \) and its neighborhood (for differentiation), and studies what happens around the equilibrium.
\[ 1 - (1 - \eta)F_i(M_i) - (1 - \eta)R_MF'_i \frac{dM_i}{dR_M} - R'_{D,i}(\frac{dI_i}{dR_M} + \frac{dM_i}{dR_M}) = \frac{R'_{D,i}}{R_M} - (1 - \eta)R_MF'_i \frac{dM_i}{dR_M} - R'_{D,i}(\frac{dI_i}{dR_M} + \frac{dM_i}{dR_M}) = 0. \] (2.17)

In the second equation, we use the equilibrium condition \( M_i = F_i^{-1}\left(\frac{1}{1-\eta}(1 - \frac{R_{D,i}}{R_M})\right) \) to replace \( 1 - (1 - \eta)F_i(M_i) \) by \( \frac{R_{D,i}}{R_M} \).

The first condition implies

\[ \frac{dI_i}{dR_M} = -\frac{R'_{D,i}}{R'_{D,i} - R'_{I,i}} \frac{dM_i}{dR_M}. \] (2.18)

Since \( R'_{D,i} > 0 \) and \( R'_{I,i} < 0 \), \( \frac{R'_{D,i}}{R'_{D,i} - R'_{I,i}} \) is positive and smaller than 1. As the illiquid investment and cash holdings both 'compete' for deposits, they move in the opposite directions. For example, if \( M_i \) increases due to an increase in \( R_M \) (which makes intuitive sense; also, we show it below), \( D_i \) increases as well to fund the increase in \( M_i \). Then, the marginal deposit rate rises, and becomes higher than the marginal return to illiquid investment. Therefore, bank \( i \) reduces its illiquid investment.

When bank \( i \) reduces its illiquid investment, both \( R_{I,i} \) and \( R_{D,i} \) change at the same time, so \( I_i \) does not need to move as much as \( M_i \) to equalize the marginal return to illiquid investment, \( R_{I,i} \), with the marginal rate of deposit, \( R_{D,i} \).

We can eliminate \( \frac{dI_i}{dR_M} \) from the second condition using the first condition. Then, we have

\[ \frac{dM_i}{dR_M} = \left[ -\frac{R'_{I,i}R'_{D,i}}{R'_{D,i} - R'_{I,i}} + (1 - \eta)R_MF'_i \right]^{-1} \frac{R'_{D,i}}{R_M}. \] (2.19)

Both \( -\frac{R'_{I,i}R'_{D,i}}{R'_{D,i} - R'_{I,i}} \) and \( (1 - \eta)R_MF'_i \) are positive. Therefore \( \frac{dM_i}{dR_M} > 0 \), as expected.

Banks with different characteristics will respond to a change in \( R_M \) with different intensities. The variation in the magnitude of \( \frac{dM_i}{dR_M} \) determines the variation in the intensity of the response across different banks. Below, we discuss some of the relevant characteristics that affect this coefficient:

(i) Variation in \( R'_{D,i} \): A bank that has a larger pool of potential deposits will have
a smaller $R_{D,i}^\prime$, as the marginal deposit rate will respond less to a change in the level of deposit. A smaller $R_{D,i}^\prime$ implies a smaller $\frac{R_{D,i}^\prime R_{D,i}}{R_{D,i}^\prime - R_{I,i}}$ and thus, a larger $\frac{dM_i}{dR_M}$.

This result is intuitive. In response to a change in $R_M$, $M_i$ moves to equalize the marginal return from cash holdings with the marginal cost of deposit. With a smaller $R_{D,i}^\prime$, the marginal cost of deposits does not move as much with $M_i$, so $M_i$ needs to move more.

At the same time, a smaller $R_{D,i}^\prime$ implies a smaller $|\left(\frac{dI_i}{dR_M}\right)\left(\frac{dM_i}{dR_M}\right)|$. With a smaller $R_{D,i}^\prime$, a change in $M_i$ has a smaller effect on the marginal cost of deposit. However, when the level of illiquid investment $I_i$ moves to close the gap between $R_{I,i}$ and $R_{D,i}$, $R_{I,i}$ moves in addition to $R_{D,i}$. With a smaller $R_{D,i}$, $R_{I,i}$ becomes relatively large, so $I_i$ does not need to move as much to offset the move in $M_i$.

This reduced offsetting alone works toward reducing $\frac{dM_i}{dR_M}$, but it is not enough to overcome the direct effect of reduced $R_{D,i}^\prime$ described at the beginning.

(ii) Variation in $R_{I,i}^\prime$: A bank that has a larger pool of potential investment will have a smaller $-R_{I,i}^\prime$, as the marginal investment rate will respond less to a change in the level of investment. A smaller $-R_{I,i}^\prime$ implies a smaller $\frac{R_{I,i}^\prime R_{D,i}}{R_{D,i}^\prime - R_{I,i}}$ and thus, a larger $\frac{dM_i}{dR_M}$.

A smaller $-R_{I,i}^\prime$ implies a larger $|\left(\frac{dI_i}{dR_M}\right)\left(\frac{dM_i}{dR_M}\right)|$; with a smaller $-R_{I,i}^\prime$, $I_i$ needs to move more to equalize $R_{I,i}$ to $R_{D,i}$. In doing so, it offsets $M_i$ more. Due to this greater degree of offsetting movement by $I_i$, $M_i$ moves more in response to a change in $R_M$.

(iii) Variation in $F_{i}'$: Bank $i$ has a smaller $F_{i}'$ generally if the distribution of its payment demands $f_i$ is spread out more widely. A smaller $F_{i}'$ implies a larger $\frac{dM_i}{dR_M}$.

To respond to a change in $R_M$, $M_i$ moves to equalize the marginal return to cash holding $R_M[1 - (1 - \eta)F_i(M_i)]$ with the marginal cost of deposit, $R_{D,i}$. If the distribution of $f_i$ is spread out more widely, $M_i$ needs to move more to move $R_M[1 - (1 - \eta)F_i(M_i)]$ closer to $R_{D,i}$. Since a smaller $F_i'$ corresponds to a more spread-out distribution of $f_i$, a smaller $F_i'$ implies a larger $\frac{dM_i}{dR_M}$. 

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(iv) Variation in size: There is no variable that represents the size of a bank in the model. Instead, it will be reflected through variations in $R'_{I,i}$, $R'_{D,i}$ and $F'_i$. It is reasonable to postulate that a larger bank have a smaller $-R'_{I,i}$, $R'_{D,i}$ and $F'_i$. According to (i), (ii) and (iii), a smaller value for each of the three derivatives implies a larger $\frac{dM_i}{dR_M}$. Therefore, $\frac{dM_i}{dR_M}$ would be larger for a larger bank.

The order of size (measured by the total value of assets or liabilities bank $i$ holds, for example) that $\frac{dM_i}{dR_M}$ depends on will be determined by the order of size that $R'_{I,i}$, $R'_{D,i}$ and $F'_i$ depend on. For example, if $R'_{I,i}$ and $R'_{D,i}$ are proportional to the $\frac{1}{\alpha}$-th order of size and $F'_i$ is proportional to the $\frac{1}{\beta}$-th order of size, the order of size that $\frac{dM_i}{dR_M}$ depends on will be somewhere between $\alpha$ and $\beta$. The exact order will be determined by the parameters of the model and the size of bank $i$.

The model presented in this section describes the effect that the money market rate (short-term rate) has on how banks decide the level of cash to hold in anticipation of payment demands. The change in the level of cash also affects the level of illiquid investment, by affecting the cost of financing from deposit. In addition, the model characterizes how banks with different characteristics would respond differently to a change in the money market rate.

In addition, a change in the payment design or policy changes the amount of cash that a bank needs to meet potential payment demands. Therefore, it can be incorporated into the model by making the distribution of $f_i$ and the functional form of $F_i$ change with payment policy parameters. The model that we have developed implies that under different payment policy parameters, banks will be affected differently by a change in the money market rate because $\frac{dM_i}{dR_M}$ depends on the form of $F_i$. However, as $\eta$ would also change with policy parameters, we could not find any simple characterization of the effect that payment policy has on banks.
2.4 Conclusion

In this paper, we have developed a framework that describes how the money market rate affects banks' decision to hold cash in anticipation of payment demands. When there is a positive amount of cash in the banking system and the payment demands occur within the banking sector, the money market between banks cannot clear through a competitive equilibrium because the supply of lending always exceeds the demand for borrowing. By introducing a rationing rule in the case of supply exceeding demand, we found that the money market rate is not determined by the model, but given a money market rate, the equilibrium is unique. Therefore, it is possible to interpret the money market rate as a monetary policy variable and the banks' choice variables as their response to the monetary policy.

Then, using the developed framework, we studied how banks would respond to a change in the money market rate, and derived a parsimonious characterization of individual banks' responsive intensity. The intensity with which the banks respond to a change in the money market rate depends on several factors, including the depth of the potential pool of illiquid investment and deposit that the banks face and the size of individual banks. In addition, we discussed how the payment system policy can be incorporated into the model through the shape of the probability distribution of payment demands that individual banks face.

The analysis developed in this paper leads to exciting avenues of research. In this paper, we did not discuss welfare, and only studied positive implications of the theory. By more explicitly building welfare into the model, we can have a more serious discussion of monetary policy using our framework.

In addition, we did not consider the effect of interbank competition on illiquid investment or deposit in this paper. A more comprehensive framework that can account for the effect of competition will let us explore how monetary and payment policy interact with the degree or the mode of competition in the banking system.
2.5 Bibliography


Rotemberg, Julio J. 2011 “Minimal Settlement Assets in Economies with Interconnected Financial Obligations.” Journal of Money, Credit and Banking


2.6 Appendix

2.6.1 Proof of Proposition 1

First, let us assume that $R_M > 0$. Then, for any realization $f_1, f_2, ..., f_n$ and for any $i$, (i) if $M_i - f_i \geq 0$, bank $i$ maximizes its profit by lending the full excess cash and chooses $l_i = M_i - f_i$; and (ii) if $M_i - f_i < 0$, bank $i$ maximizes its profit by borrowing the least possible amount of cash and chooses $l_i = M_i - f_i$. Therefore, the market clears if and only if $0 = \sum_i l_i = \sum_i M_i$. Since $M_i \geq 0$, the market can clear if and only if $M_i = 0$ for every $i$.

Under the assumption that the money market will always be in a competitive equilibrium after $f_1, f_2, ..., f_n$ are realized,

$$\pi_i = R_M E[M_i - f_i] = R_M M_i. \quad (2.20)$$

Therefore, the profit $\pi_i$ is

$$\pi_i = \int_0^{l_i} R_{1,i}(x) dx - R_D l_i + [R_M - R_D] M_i + R_D C_i. \quad (2.21)$$

The optimal choice of $M_i$ for bank $i$ is $\infty$ if $R_M > R_D$. It contradicts the previous conclusion that $M_i = 0$. Therefore, no competitive equilibrium exists with $R_M > R_D$.

The optimal choice of $M_i$ for bank $i$ is $0$ if $R_D > R_M > 0$. Therefore, we have a competitive equilibrium if $R_D > R_M > 0$ and $M_i = 0$ for all $i$. For $R_M = R_D$, the form of $\pi_i$ allows any value of $M_i$, but $M_i = 0$ needs to hold to have a competitive equilibrium in the money market with a positive $R_M$.

Finally, if $R_M = 0$, the maximization of $\pi_i$ again implies $M_i = 0$. Then, for any realization of $f_1, f_2, ..., f_n$, $l_i = M_i - f_i$ for all $i$ is the only profit maximizing choice that clears the market; the implicit assumption here is that the banks choose that level of $l_i$, even though they prefer any other choice equally. Therefore, $R_M = 0$ and $M_i = 0$ for all $i$ is also a competitive equilibrium.

We can obtain the first-order condition with respect to $I_i$ by simply solving the
equation \( \frac{\partial \pi_i}{\partial I_i} = 0 \), using the expression for \( \pi_i \) shown above. By doing so, we have \( R_{I,i}(I_i) = R_D \).

### 2.6.2 Characterization of a Rationing Equilibrium

A competitive equilibrium, characterized by \( 0 \leq R_M \leq R_D \) and \( M_i = 0 \) for every \( i \), is also a rationing equilibrium. Given the choice of the other banks, we can consider \( \tilde{\pi}_i \) a function of \( M_i \) and \( D_1, D_2, ..., D_n \), \( \tilde{\pi}_i(M_i, D_1, D_2, ..., D_n) \). If bank \( i \) chooses a positive \( M_i \) while \( M_j = 0 \) for every \( j \neq i \), then \( \tilde{\pi}_i(M_i, D_1, D_2, ..., D_n) < R_M M_i = \tilde{\pi}_i(0, D_1, D_2, ..., D_n) + R_M M_i \) due to the rationing rule. Therefore, the extra revenue from \( M_i \) is smaller than \( R_D M_i \) if \( R_M \leq R_D \), and \( M_i = 0 \) is the optimal choice for bank \( i \).

More generally,

\[
\tilde{\pi}_i = R_M \left[ \int_{f_i > M_i} (M_i - f_i)g(f_1, ..., f_n, D_1, ..., D_n)df_1...df_n \right. \\
+ \left. \int_{f_i < M_i} \sum_i [M_i - f_i]^+ (M_i - f_i)^2g(f_1, ..., f_n, D_1, ..., D_n)df_1...df_n \right].
\] (2.22)

Then, the first-order condition for bank \( i \) with respect to \( I_i \) is:

\[
0 = \frac{\partial \pi_i}{\partial I_i} = R_{I,i}(I_i) - R_D \\
+ R_M \left[ \int_{f_i > M_i} (M_i - f_i) \frac{\partial g}{\partial D_i}df_1...df_n + \int_{f_i < M_i} \sum_i [M_i - f_i]^+ (M_i - f_i) \frac{\partial g}{\partial D_i}df_1...df_n \right].
\] (2.23)

The last term represents the effect that the change in \( D_i \) due to a change in \( I_i \) has on \( \tilde{\pi}_i \).

With respect to \( M_i \), the first-order condition is \( (F_i \) denotes the unconditional
cumulative distribution function of \( f_i \):

\[
0 = \frac{\partial \pi_i}{\partial M_i} = R_M(1 - F_i(M_i)) + \int_{f_i < M_i} \frac{\sum_i [f_i - M_i]^+}{\sum_i [M_i - f_i]^+} g(f_1, ..., f_n, D_1, ..., D_n) df_1 ... df_n
\]

\[
- R_M \int_{f_i < M_i} \frac{(M_i - f_i)}{\sum_i [M_i - f_i]^+} \frac{\sum_i [f_i - M_i]^+}{\sum_i [M_i - f_i]^+} df_1 ... df_n
\]

\[
+ R_M \int_{f_i > M_i} (M_i - f_i) \frac{\partial g}{\partial D_i} df_1 ... df_n
\]

\[
+ \int_{f_i < M_i} (M_i - f_i) \frac{\sum_i [f_i - M_i]^+}{\sum_i [M_i - f_i]^+} \frac{\partial g}{\partial D_i} df_1 ... df_n - R_D.
\]

(2.24)

The right-hand side is the sum of three expressions (each on a different line) and \(-R_D\). The first expression is the value of having more cash in every state, with both the distribution \( g \) and the rationing factor \( \sum_i [f_i - M_i]^+ \sum_i [M_i - f_i]^+ \) fixed; it is smaller than \( R_M \). The second term represents the effect of a higher \( M_i \) reducing the rationing factor, and is always negative. Finally, the third term represents the effect of a change in \( D_i \) due to a change in \( M_i \), which is identical to what we saw in the first-order condition with respect to \( I_i \).

Given the complexity of the expressions, it is difficult to generally characterize the rationing equilibria. One particularly useful approximation is to assume that the effects of the first-order changes in the distribution of \( f_i \) and in the rationing factor are much smaller than the other first-order effects. These assumptions let us simplify the first-order conditions to:

\[
0 = \frac{\partial \pi_i}{\partial I_i} = R_{i,i}(I_i) - R_D.
\]

(2.25)

\[
0 = \frac{\partial \pi_i}{\partial M_i} = R_M(1 - F_i(M_i)) + \int_{f_i < M_i} \frac{\sum_i [f_i - M_i]^+}{\sum_i [M_i - f_i]^+} g(f_1, ..., f_n, D_1, ..., D_n) df_1 ... df_n - R_D.
\]

(2.26)

The term \( \int_{f_i < M_i} \frac{\sum_i [f_i - M_i]^+}{\sum_i [M_i - f_i]^+} g(f_1, ..., f_n, D_1, ..., D_n) df_1 ... df_n \) is the expected value of
the rationing factor conditional on \( f_i < M_i \), multiplied by \( \text{Prob}(f_i < M_i) \). To simplify the problem further, we assume that each bank \( i \) is relatively small compared to the whole banking sector comprised of banks 1, 2, ... \( n \), so that the conditional expectation of the rationing factor is close to its unconditional expectation. With \( \eta(M_1, ..., M_n, D_1, ..., D_n) \) denoting the unconditional expectation of the rationing factor, the first-order condition with respect to \( M_i \) is further simplified to

\[
0 = \frac{\partial \pi_i}{\partial M_i} = R_M[(1 - F_i(M_i)) + F_i(M_i)\eta(M_1, ..., M_n, D_1, ..., D_n)] - R_D. \tag{2.27}
\]

Rationing reduces the extra revenue from holding one more unit of cash, and makes an equilibrium with \( R_M > R_D \) possible.

Finally, we assume that the choice of illiquid assets are much larger than cash holdings, \( I_i \gg M_i \), and consequently, we can approximate \( g(f_1, ..., f_n, D_1, ..., D_n) \) simply by \( g(f_1, ..., f_n, I_1, ..., I_n) \). Then, the function \( \eta \) can be simply written as \( \eta(M_1, ..., M_n, I_1, ..., I_n) \).

From the first-order condition with respect to \( I_i \), we know that \( I_i \) can be determined independently from \( M_i \). Therefore, in solving the first-order condition for \( \frac{\partial \pi_i}{\partial M_i} \), we simply drop the dependence on \( I_i \):

\[
0 = \frac{\partial \pi_i}{\partial M_i} = R_M[(1 - F_i(M_i)) + F_i(M_i)\eta(M_1, ..., M_n)] - R_D. \tag{2.28}
\]

### 2.6.3 Existence of a Unique Rationing Equilibrium

In a slight abuse of notation, we now use \( \eta \) to denote a variable, not a function as above. Then, we define a new function \( h(M_1, ..., M_n, I_1, ..., I_n) \), which is indeed the same as the function \( \eta(M_1, ..., M_n, I_1, ..., I_n) \):

\[
h(M_1, ..., M_n, I_1, ..., I_n) = \int_{(f_1, f_2, ..., f_n)} \frac{\sum [f_i - M_i]^+}{\sum [M_i - f_i]^+} df_1 df_2 ... df_n \tag{2.29}
\]

is the unconditional expected value of the rationing factor.
Then, for each $i$, the optimal choice $M_i$ of each bank can be characterized as a function of $\eta$:

$$M_i = [F_i^{-1}\left(\frac{1}{1 - \eta} (1 - \frac{R_D}{R_M})\right)]^+, \quad (2.30)$$

which is increasing in $\eta$. At the same time, $\frac{\partial h(M_1, \ldots, M_n)}{\partial M_i} < 0$ given the definition of $h$. Therefore, if a solution exists to the fixed-point problem of determining $M_1, \ldots, M_n$ and $\eta$, it is unique.

Now, we can characterize the rationing equilibria:

(i) For each $R_M$ such that $0 < R_M \leq R_D$, a unique rationing equilibrium exists: $M_i = 0$ for every $i$ and $\eta = 1$.

(ii) For each $R_M$ such that $R_M > R_D$, a unique rationing equilibrium exists. The solution is obtained by the unique solution $\eta, M_1, \ldots, M_n$ to the $(n+1)$ equations: $M_i = F_i^{-1}\left(\frac{1}{1 - \eta} (1 - \frac{R_D}{R_M})\right)$ for every $i$ and $\eta = h(M_1, \ldots, M_n, I_1, \ldots, I_n)$. The solution satisfies $M_i > 0$ for every $i$ and $[2\frac{R_D}{R_M} - 1]^+ < \eta < 1 - \frac{R_D}{R_M}$. Also, for a larger $R_M$, the equilibrium value of $\eta$ is smaller and that of $M_i$ is larger for every $i$.

Let us first prove statement (i). If $0 < R_M < R_D$, then $R_M[(1 - F_i(M_i)) + F_i(M_i)\eta(M_1, \ldots, M_n)] < R_D$. Therefore, $M_i = 0$ for every $i$ is the unique equilibrium. With $M_i = 0$ for every $i$, $\eta = h(M_1, \ldots, M_n) = 1$. If $R_M = R_D$, we can show that $M_i = 0$ for every $i$ and $\eta = 1$ is an equilibrium, following the same argument as before. We already proved that an equilibrium, if it exists, is unique.

Now, let us prove statement (ii). Since $R_M > R_D$ and thus, $1 - \frac{R_D}{R_M} > 0$, we have $M_i = [F_i^{-1}\left(\frac{1}{1 - \eta} (1 - \frac{R_D}{R_M})\right)]^+$, which is well-defined for $0 \leq \eta < \frac{R_D}{R_M}$. With $M_i$ now being a function of $\eta$, we can write it as $M_i(\eta_i)$. Then, we can find an equilibrium by finding the solution to $\eta = h(M_1(\eta), \ldots, M_n(\eta))$.

As $\eta \to \frac{R_D}{R_M}$ from below, $F_i(M_i) \to 1$ from below. Therefore, as $\eta \to \frac{R_D}{R_M}$, the probability of $M_i - f_i < 0$ for every $i$ converges to 1, which implies that the probability that the rationing factor is 0 approaches 1. Since the rationing factor is bound from above by 1, the expected value of the rationing factor also approaches 0. Therefore, $h(M_1(\eta), \ldots, M_n(\eta))$ converges to 0 as $\eta \to \frac{R_D}{R_M}$ from below.

If $\eta \leq 2\frac{R_D}{R_M} - 1$, then $\frac{1}{1 - \eta} (1 - \frac{R_D}{R_M}) \leq \frac{1}{2}$ and consequently, given that the distribution
of $f_i$ is symmetric around zero, $M_i = [F_i^{-1}(\frac{1}{1-\eta}(1 - \frac{R_D}{R_M}))]^{+} = 0$ for every $i$. Therefore, $h(M_1(\eta), ..., M_n(\eta)) = 1$.

Finally, if $\frac{R_D}{R_M} > \eta > 2\frac{R_D}{R_M} - 1$, $M_i$ is a strictly increasing function of $M_i$ for every $i$. Therefore, $h(M_1(\eta), ..., M_n(\eta)) = 1$ is a strictly decreasing and continuous function of $\eta$.

Given the shape of the function $h(M_1(\eta), ..., M_n(\eta)) = 1$, there is a unique solution to the equation $\eta = h(M_1(\eta), ..., M_n(\eta))$. Since $h(M_1(0), ..., M_n(0)) > 0$, the solution $\eta^* > 0$. Also, if $2\frac{R_D}{R_M} - 1 \geq 0$, then for $\eta = 2\frac{R_D}{R_M} - 1$, $h(M_1(\eta), ..., M_n(\eta)) = 1$. Since $2\frac{R_D}{R_M} - 1 < 1$, we have $\eta^* > 2\frac{R_D}{R_M} - 1$.

$\eta^* < 1 - \frac{R_D}{R_M}$ because $h(M_1(\eta), ..., M_n(\eta)) \to 0$ as $\eta \to \frac{R_D}{R_M}$ from below.

$\eta^* > 2\frac{R_D}{R_M} - 1$ implies $\frac{1}{1-\eta^*}(1 - \frac{R_D}{R_M}) > \frac{1}{2}$ and $M_i = F_i^{-1}(\eta^*) > 0$.

Now, let us consider a rationing equilibrium in which $R_M > R_D$, with $\eta = \eta^*$ and $M_i = M_i(\eta^*)$ for every $i$. Then, $M_i(\eta^*) > 0$. If we change the money market rate to some $R_M$ such that $R_M > R_M' > R_D$, $M_i(\eta^*) = [F_i^{-1}(\frac{1}{1-\eta^*}(1 - \frac{R_D}{R_M}))]^{+}$ strictly decreases after the change because it was larger than 0 before the change. Therefore, the function $h(M_1(\eta), ..., M_n(\eta))$ strictly increases at $\eta^*$ after the change in the money market rate. Since $h$ is decreasing in $\eta$, this change implies that the new solution $\eta = h(M_1(\eta), ..., M_n(\eta))$ after the change is strictly greater than $\eta^*$.

A larger $\eta$ in the new equilibrium implies that $M_i$ should be strictly smaller in the new equilibrium for some $i$. However, given the formula $M_i = F_i^{-1}(\frac{1}{1-\eta}(1 - \frac{R_D}{R_M}))$ for every $i$, we can see that $M_i$ indeed should be strictly smaller for every $i$ after the change in $R_M$ if it is so for some $i$.

### 2.6.4 Normally Distributed Payment Demands

In subsection 2.4, we assumed that $f_{ij}$ follows a normal distribution with mean 0 and variance $\sigma^2_{ij}$. More generally, let us go back to the original unsimplified case in which the payment demands depend on the level of deposits, $D_1, ..., D_n$. Let us suppose that payment demands occur because the depositors demand their banks to transfer funds to other people’s deposit accounts.

Let us assume that each depositor has a deposit of size $d$, and $d$ is a small number
so that we can think of $D_1, ..., D_n$ as large multiples of $d$. Then, each depositor, independently with probability $p \ll 1$, instructs his bank to transfer a fund of amount $\epsilon$ to another depositor. There are in total $\frac{D}{d}$ depositors, where $D = \sum_i D_i$, and the depositor to receive the fund is chosen randomly among the $\frac{D}{d} - 1$ depositors that exclude the sender.

Let $e_{ij}$ be the random variable that represents the transfer that the depositors of bank $i$ send to the depositors of bank $j$. For each depositor, conditional on the depositor sending out a transfer, the probability that the recipient is in bank $j$ is $\frac{D_j}{D-d}$. Therefore, $e_{ij}$ is the sum of $\frac{D_j}{D-d}$ i. i. d. random variables, which has the value $\epsilon$ with probability $p \frac{D_j}{D-d}$ and the value 0 with probability $1 - p \frac{D_j}{D-d}$.

The mean of $e_{ij}$ is $\frac{\epsilon p D_i D_j}{d (D-d)}$ and its variance is $\frac{\epsilon^2 p D_i D_j}{d (D-d)} (1 - p \frac{D_j}{D-d})$. Assuming $D_i \ll D$ for every $i$ and $p \ll 1$, we approximate the mean by $\frac{\epsilon p D_i D_j}{D-D-d}$ and the variance by $\frac{\epsilon^2 p D_i D_j}{D-D-d}$. Similarly, $e_{ji}$ has exactly the same mean and approximately the same variance.

Therefore, the net flow $f_{ij} = e_{ij} - e_{ji}$ from bank $i$ to bank $j$ has mean 0 and variance $\frac{2\epsilon^2 p D_i D_j}{d D}$. Approximating the distribution of $f_{ij}$ with a normal distribution and replacing $D_i$ by $I_i$, we arrive at the same form of $f_{ij}$ as that in subsection 2. 4.

### 2.6.5 Rationing Equilibria with Bank-Specific Deposit Rates

We have shown that with the same deposit rate, $R_D$, for all banks, there exists a unique equilibrium for any $R_M$. In this subsection, we show that the same is true even if we allow a bank-specific deposit rate, $R_{D,i}$.

The proof is very similar to the one we already saw for the case in which $R_{D,i}$ is the same for every $i$. The first-order condition

$$R_M [1 - F_i(M_i)(1 - \eta)] - R_{D,i} = 0 \tag{2.31}$$

implies

$$M_i = [F_i^{-1}(1 - \eta(1 - R_{D,i} R_M))]^+ \text{ for } \eta < \frac{R_{D,i}}{R_M}.$$

Now, we make an additional assumption that the support of $f_i$ is unbounded (or practically, it is a very broad interval) for some $i$ such that $R_{D,i} \leq R_{D,j}$ for every $j$. Let
to be such \( i' \). Since \( R_{D,i'} \leq R_{D,i} \), \( M_i = [F_i^{-1}(\frac{1}{1-\eta}(1 - \frac{R_{D,i'}}{R_M}))]^+ \) is well-defined for every \( i \) as long as \( \eta < \frac{R_{D,i'}}{R_M} \). Also, \( M_i^* \to \infty \) as \( \eta \to \frac{R_{D,i'}}{R_M} \). Therefore, \( h(M_1, \ldots, M_n) \to 0 \) from above as \( \eta \to \frac{R_{D,i'}}{R_M} \) from below. (Here, we use the same notation as that we used in proving the existence of an equilibrium with a single \( R_D \) for every bank. \( M_i \) is the optimal level of cash holdings for each bank as a function of \( \eta \), and \( h \) is a function that represents the expected value of the rationing factor given the level of cash holdings \( M_1, \ldots, M_n \).)

Each \( \eta \) that satisfies the equation \( h(M_1, \ldots, M_n) = \eta \) completely specifies a unique equilibrium value of the other variables, \( M_1, \ldots, M_n \). (\( I_1, \ldots, I_n \) is determined by the other first-order condition.) Since \( h(M_1, \ldots, M_n) \) is a decreasing function of \( \eta \) that converges to 0 from above as \( \eta \to \frac{R_{D,i'}}{R_M} \) from below, there exists a unique solution to the equation \( h(M_1, \ldots, M_n) = \eta \). Therefore, there exists a unique equilibrium for the given \( R_M \) and \( R_{D,1}, \ldots, R_{D,n} \).

2.6.6 Quantifying the Effects of Changes in \( \eta \) and \( F_i \)

In this subsection, we discuss the choice made in section 3 to approximate the derivative of the expression \( 1 - F_i(M_i)(1 - \eta(M_1, \ldots, M_n, I_1, \ldots, I_n)) \) with respect to \( R_M \) by \((\eta - 1)F_i' \frac{dM_i}{dR_M}\), ignoring any change to \( \eta \) and any change to \( F_i \) working through \( I_i \). The rational for this approximation is that we assumed \( I_i \) to be much larger than \( M_i \) (in section 2), so the change in \( F_i \) caused by the change in \( I_i \) should be very small; we saw that \( |\frac{dF_i}{dR_M}| \) is smaller than \( \frac{dM_i}{dR_M} \). Similarly, any change in \( \eta \) that is realized through \( I_i \) should be small. The change in \( \eta \) that is realized directly though its dependence on \( M_i \) should also be small, given that bank \( i \) is just one of the many banks whose cash positions are summed up to determine the rationing factor.

We use the example from subsection 2.4 to actually compute the various terms in the exact derivative and compare them. Some of the terms contain \( \frac{dF_i}{dR_M} \), and we saw in section 3 that \( |\frac{dF_i}{dR_M}| < \frac{dM_i}{dR_M} \). Since we now consider the change to the distribution of \( f_i \) due to a change in \( I_i \) as well, we denote \( F_i \) as a function of both \( M_i \) and \( I_i \). The four terms that make up the derivative of the expression \( 1 - F_i(M_i, I_1, \ldots, I_n)(1 - \eta(M_1, \ldots, M_n, I_1, \ldots, I_n)) \) are computed for the bank \( i = 201 \) with \( I_i = 20 \):
We can see that the first term, which is the only term in the approximation, is far greater than the other terms.
Chapter 3

Analysis of a Transformation in Housing Rental Contracts in Korea

3.1 Introduction

Under the conventional form of housing rental contracts in Korea, the renter deposits a significant sum of money to the owner for the duration of the contract, instead of paying a monthly rent.1 The renter, at the end of the contract, which typically lasts 2 years, receives back the deposit without any interest.

In a typical example, a renter may deposit $150,000 to rent a house worth $300,000. Alternatively, under a contract based on monthly payments like those found in most other countries, the renter may deposit $20,000 as a security deposit and pay monthly rents of $2,000. Finally, a hybrid contract that is becoming more popular recently features a larger security deposit in exchange for lower monthly rents; in the current example, the renter may deposit $85,000 and pay monthly rents of $1,000.

The first form of rental contracts, which features a large deposit with zero monthly rents, is the most common and accounts for more than 60% of housing rental contracts. However, during recent years, such a traditional form of rental contracts has been losing popularity to the alternatives described above. This phenomenon has sparked

1Cho (2006) describes in detail the nature and the history of this traditional form of rental contracts.
discussions on what caused the changes in the rental market and on whether the government should intervene to reverse this change in the housing rental market.

Generally, the fall in the expected return from residential real estate investment and the fall in the interest rate are considered to be responsible for the decline of the traditional rental contracts. The traditional form of rental contracts allows the owner to borrow cheaply using the property as a collateral. There are some partial arguments that support this analysis. As residential real estate becomes less attractive as a means of investment, the traditional rental contract, which is a tool for an investor to buy a house with leverage, becomes less attractive. At the same time, the fall in the interest rate makes the zero-interest loan provided by the rental deposit relatively less valuable.

In this paper, I present a stylized model of the interaction between an owner and a renter, which illuminates how changes in the real estate market and in more general financial markets affect the preference for different forms of housing rental contracts. More specifically, I show how these changes affect the bargaining positions of the owner and the renter and how the contractual form that maximizes the joint payoff to the owner and the renter changes when market conditions change. In the theoretical analysis, I assume that the contractual form that increases the joint payoff is likely to be chosen by the renter and the owner.

However, I do not consider the possibility that loosening of a significant institutional or physical barrier to collecting monthly rents has caused a shift in the form of rental contracts. A universal banking system with easy and cheap money transfer capabilities has been in place for a long time and has been used for many types of transactions, so the additional cost of monthly payments relative to one-time exchange of deposit is minimal. This issue is discussed in the next section.

Under a contract with a large deposit and zero monthly rent, I view the renter as effectively lending a zero-interest loan to the owner in exchange for the right to live in the house. The idea that the renter is lending a significant amount of money to the owner may seem counter-intuitive given that the renter is renting because he cannot afford to buy a house. However, given the cost of mortgage and the risk of house
price fluctuations, a renter who can afford a house but does not have much wealth may well choose to rent, as discussed in the next section.

The model developed in this paper suggests a statistical relationship between the change in the level of deposits under zero-rent contracts, the change in the price of residential properties, the change in the expected house price appreciation, and the change in the interest rate. Using aggregate data on real estate markets in Korea, I show that the relationship between these market variables is qualitatively consistent with the predictions of the model.

The findings of this paper illustrate that modeling the joint payoff of different parties to a contract is useful in explaining the prevalence of existing forms of contracts in a particular market. There are many rental markets where pricing combines some forms of a lump-sum payment, a deposit and periodic payments, and the findings of the paper can be applied to those markets as well.²

In the next section, I describe the nature of zero-rent contracts and present statistics that describe some of the recent changes in the housing market. In section 3, I present a model of the interaction between the owner of a residential property and the renter. Also, with the model, I discuss how the recent changes in the market would change the preferred contractual form of housing rental contracts. In section 4, I present a statistical relationship between changes in the market variables implied by the model and show that the predictions of the model are broadly consistent with the data. Section 5 concludes.

### 3.2 Background

#### 3.2.1 Nature of Zero-Rent Contracts

The zero-rent housing contract, which features a large amount of deposit with zero monthly rent, is the most common form of housing rental contracts in Korea; among the new housing contracts that were signed in 2014, 63.0% of the contracts had zero

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²For example, Eisfeldt and Rampini (2009) studies leasing of industrial capital.

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monthly rent. The size of the deposit is significant relative to the house price, and was about 62.4% of the house price on average in 2014.

Poorer people tend to rent houses under contracts based on monthly rents rather than on a large deposit, presumably because they find it more difficult to prepare the large deposit required. The left plot on figure 1 shows the proportion of zero-rent contracts among all rental contracts for each decile of house size, based on a database of condominium rental contracts in Seoul in 2014. The plot shows that contracts based on monthly rents are much more common among renters living in smaller houses than in larger houses. This pattern is consistent with the assertion that there is little institutional or physical barrier to collecting monthly rents; if there were such barriers, they would likely be more problematic for renters of smaller houses who tend to be poorer.

The right plot on figure 1 shows the average deposit for zero-rent contracts within each size decile. For all deciles, the size of the deposit is large. For example, in 2013, the per capita GDP in South Korea was around $26,000 and the average size of rental deposits is much larger than that. Given this large size of deposits, it is reasonable to assume that all or a large part of the deposits is provided by the renters' own savings rather than borrowings from banks; it would be very difficult and costly to provide the rental deposit by borrowing from the banks, especially given that the renters typically do not own a house that can be used as collateral. Therefore, the rental deposit represents a genuine loan from the renter to the owner, not some type of indirect payment through the banking system, in which the renter borrows the money for deposit from a bank and pays interest to the bank instead of monthly rents.

Given the high level of rental deposit, the renter should be able to buy the house if he obtains a mortgage. However, it needs not to happen if the renter’s total wealth is not much larger than the level of rental deposit due to the risks involved in housing

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3 The plot is based on a large sample of condominium rental contracts in 2014 in Seoul, which is the capital of South Korea. About one-third of all rental contracts occur within Seoul, and condominium is the most common form of housing in Seoul (and in the country as a whole as well). Living in condominiums is universal across all socioeconomic groups.
price fluctuations. Also, given the interest rate that has to be paid on a mortgage, it is not always an optimal choice to the renter. Figure 2 shows the approximate cost of living in a house priced at $400,000 for a year for a renter who has just enough wealth to pay for the rental deposit under the following two scenarios\(^4\): (i) Rent the house under a zero-deposit contract and (ii) Buy the house by getting a mortgage on the house and sell the house after one year. Neither of these two options dominates the other.

### 3.2.2 Housing Rental Market Trends

During recent years, the traditional form of rental contracts with zero monthly rent has declined in popularity relative to rental contracts that are based on monthly rents. Figure 3 shows the proportion of house rental contracts with zero monthly rents, as a proportion of all rental contracts, for each month from January 2011 to January 2015.

At the same time, the required size of deposits for zero-rent contracts has been

\(^4\)In scenario (i), the cost is the house price multiplied by the average deposit-to-price ratio, again multiplied by 1-year term deposit rate. In scenario (ii), the cost is the cost under scenario (i) plus the interest rate on mortgage minus the actually observed price appreciation.
Figure 3-2: Cost of Housing under Different Scenarios
Figure 3-3: The Proportion Pure Deposit-Based Contracts
increasing steadily, as figure 4 shows. These important changes to the housing market have sparked a discussion on their causes. Generally, market trends that decrease the value of the loan provided by the rental contract can potentially increase the required size of the deposit.

Three broad market trends, which can affect the value of the service provided by the rental deposit as a loan, are often mentioned as drivers of the change: The decline in the expected return from residential real estate investment, the decrease in the interest rate and the improvement in mortgage services. In the rest of this section, I document relevant statistics that describe each of these three trends. Then, in the next section, I discuss how these changes may affect the size of the required rental deposits.

The left plot in figure 5 shows the historical return to residential real estate investment. After 2010, the returns seem to be generally lower than those observed over the decade 2001 - 2010. However, between 1990 and 2000, even lower returns
had been observed.

The right plot in figure 5 shows the historical return to residential real estate investment, along with returns to other readily accessible investment opportunities. The plot suggests that returns to alternative investment devices exhibit a similar decline in the returns after 2010 relative to 2001 - 2010.

Figure 6 shows the historical interest rate to 6-month to 1-year term deposits; the same series is used in figure 5, but reproduced for better visibility.

Finally, as a measure of the cost to borrow using a residential property as a collateral, the plot on the left of figure 7 shows the average household borrowing rate from banks, along with deposit rates. The plot shows that household borrowing rate has been declining, but the deposit rate has been declining as well, which leaves the spread between the two rates relatively stable, as shown by the right plot in figure 7. The current spread is near historically low levels, but similarly low levels have been observed in the past and it is not clear whether there is a definite downward trend.

Mortgage regulations concerning LTV (Loan-To-Value) ratios provide another way to measure the easiness of financing real estate investment through borrowing, which is an alternative to deposit-based rental contracts. According to the figures reported by Chang (2010) and Seo (2014), the regulation on LTV has been eased from around
Figure 3-6: Historical Deposit Rates

Figure 3-7: Historical Borrowing Rates and Spreads
50% in 2010 to around 70% in 2014. However, this high level of allowed LTV has been observed in the past as well, in the early 2000’s.

In summary, during recent years, (i) returns to residential real estate investment have declined; (ii) interest rates have declined; and (iii) borrowing from banks using a house as a collateral has become easier. However, it should be noted that, with the exception of (ii), similarly low levels of investment return or borrowing cost have been observed in the past. In the next section, I study a stylized model that relate these three factors to the level of deposit in rental contracts.

3.2.3 Sources of the Data

Here, I briefly describe the sources of the data used in this paper. I combine three data sources, Bank of Korea (2015), Kookmin Bank (2015) and Ministry of Land Infrastructure and Transportation (2015). The time-series of various interest rates come from Bank of Korea (2015), which is the average of various interest rates reported from individual commercial banks in Korea. The time-series of house prices and the level of deposits for zero-rent contracts come from Kookmin Bank (2015); the time-series of prices are constructed from observations on house purchases and rental transactions. Finally, Ministry of Land Infrastructure and Transportation (2015) contains detailed information on a subsample of house rental contracts that were signed after 2011 and reported to the ministry. The time-series of the composition of different forms of rental contracts is constructed from this database.

3.3 Model

The model presented in this section is a stylized static model of bargaining between the renter and the owner of a house. Five main variables appear in the model: Interest rate $r$, the utility of living in the house $U$, the house price $P$, the deposit $D$ and the rent payment $R$. In the model, $r$, $U$ and $P$ are exogenous variables, and I explore the problem of determining $R$ and $L$ as functions of the exogenous variables. This assumption is reasonable for $r$ and $U$, as these are general market
variables with strong exogenous determinants (monetary policy and physical structures, respectively). However, for $P$, the level of $L$ can affect the valuation of a house, as a higher $L$ makes the house more valuable. In this section, I proceed under the assumption that the level of $L$ is not an important determinant of $P$.

I first consider the payoff (in real money) to the owner of possessing the house, relative to selling it and investing the proceed:

$$
(E_0 \frac{P_1}{1 + \pi_1} - P_0) - r_d P_0 + V_o(L) + R = E_0 [A_0] - r_d P_0 + V_o(L) + R.
$$

(3.1)

$P_1$ is the price of the house next year (or next month, and so on), $\pi_1$ is the inflation between this year and the next year, and $P_0$ is its current price. $A_0 = \frac{P_1}{1 + \pi_1} - P_0$ denotes real price appreciation. $r_d$ is the real interest rate on deposits, and $r_d P_0$ is the outside option of selling the house and investing the proceed as a savings deposit. The owner is risk-neutral and only cares about the difference in the expected return. $R$ is the income from rental payments, such as the usual monthly rents.

$V_o(D)$ is the value of the deposit to the owner, as a loan. I assume that $V_o > 0$ is an increasing and concave function with $V_o(0) = 0$, and it is based on the idea that the owner is using the deposit as a loan to fund real estate investments or other entrepreneurial activities, and that the marginal return to the loan decreases when $D$ increases because, for example, credit constraints are most strongly binding when the owner can raise only a small amount of loan through the rental deposit.

The payoff (in real money) to the renter, relative to his outside option of living in a different house is:

$$
U + V_r(D) - R - U_a.
$$

(3.2)

$U$ is the utility, in money units, of living in the house. $V_r(D) < 0$ is the value (or the negative of the cost) of the deposit to the renter. I assume that $V_r$ is decreasing and concave (or the cost $-V_r$ is convex) with $V_r(0) = 0$, for example, because a higher

---

5 Kim and Shin (2011) develops a general equilibrium model in which the owners use the rental deposit to fund their production.

6 The idea that credit constraint is important in housing market has been applied to different countries under different settings. For example, Duca, Muellbauer and Murphy (2010) applies it to the housing market in the United States.
level of $D$ will make it more difficult for the renter to prepare the deposit. $U_a$ is a constant representing the utility from an available alternative housing.

$D$ and $R$ serve two purposes. First, $D$ can increase the joint payoff to the renter and the owner by working as a loan. Second, they distribute the joint surplus between the renter and the owner, with higher levels of $D$ and $R$ giving a higher utility to the owner at the expense of the renter.

I assume that $V'(0) + V_o'(0) > 0$, so that it is beneficial for the renter and the owner to have at least some level of deposit. Also, given the concavity assumptions, $V'' + V_o'' < 0$. I assume that zero-rent contracts are viable if the renter and the owner can agree on a level of $D$ that creates enough joint surplus, $V_r(D) + V_o(D)$, and that distributes the joint surplus reasonably between the two parties with $R = 0$.

As a benchmark, I assume that, for a given level of deposit $D$, the level of $R$ is chosen to divide the joint surplus between the renter and the owner by a fixed ratio, $\alpha : (1 - \alpha)$. Since the joint surplus is simply the sum of the individual surpluses, $E[A] + U + V_o(D) + V_r(D) - r_dP_0 - U_a$, the value of $R$ that achieves the assumed division of joint surplus is

$$R(D) = (1 - \alpha)(U - U_a) - \alpha(E_0[A] - r_dP_0) + [(1 - \alpha)V_r(D) - \alpha V_o(D)]. \quad (3.3)$$

The rent $R$, given by the expression above, is a decreasing function of $D$. A zero-rent contract is viable, compared to a pure rent-based contract (or a zero-deposit contract), if $V_o(D) + V_r(D) \geq 0$ at the solution $D$ to $R(D) = 0$. Figure 10 illustrates two cases, one in which the zero-rent contract is viable and another in which it is not.

In the next three subsections, I discuss how various changes in some of the variables affect the level of the required deposit $D$ for the zero-rent contract and the attractiveness of the zero-rent contract relative to that of the zero-deposit contract.

### 3.3.1 A Fall in Expected Price Appreciation

A fall in expected price appreciation makes the owner demand a greater compensation for owning the house, which increases the required level of deposit $D$ for the zero-rent
(R, L) that only the owner would accept.
(R, L) that only the renter would accept.
(R, L) that both parties would accept.

Figure 3-8: Comparison between a Zero-Rent Contract and a Zero-Deposit Contract

contract. Algebraically, the zero-rent contract is characterized by

$$(1 - \alpha)V_r(D) - \alpha V_o(D) = -(1 - \alpha)(U - U_a) + \alpha(E[A] - \rho d P_0), \tag{3.4}$$

where the left-hand side is a decreasing function of $D$ and the fall in expected price appreciation, $E[A]$, decreases the right-hand side.

If $D$ is small enough so that $V'_r(D) + V'_o(D)$ is still positive, the change makes the zero-rent contract even more attractive by increasing $V_r(D) + V_o(D)$. However, if $D$ is large enough so that $V'_r(D) + V'_o(D)$ is negative, the change makes the zero-rent contract relatively less attractive compared to the zero-deposit contract.

### 3.3.2 A Fall in the Real Interest Rate

A fall in the interest rate would likely affect $V_r, V_o$ and $P_0$, in addition to $\rho d$. To simplify the analysis, I first assume that $P_0$ is fixed, and that $V_r, V_o$ and $\rho d$ move by the same amount: $V_r(D) = -rD + W_r(D)$, $V_o = rD + W_o(D)$, and $\rho d = r + \delta$, where $r$ is a benchmark interest rate, $\delta$ is a constant, and $W_r$ and $W_o$ are functions that do not depend on $r$.

A fall in $r$ does not affect the joint surplus as the decrease in $V_o$ is offset by the
increase in $V_r$. In other words, the joint surplus from the loan is determined by the
difference in the valuation of the deposit between the renter and the owner, not by
their levels. The increase in $V_r$ with unchanged joint surplus implies that the renter
now needs to pay more to the renter. However, at the same time, the outside option
of the owner, $r_dP_0$, also decreases, which allows the renter to pay less to the owner.

Therefore, the effect of a fall in $r$ on the required level of deposit for the zero-rent
contract is ambiguous, and depends on the sign of $D - \alpha P_0$; if $D - \alpha P_0 > 0$, a fall in
$r$ will increase the required level of deposit $D$.

It is generally believed that a fall in $r$ tends to increase $P_0$, but no generally
accepted relationship between $r$ and $P_0$ is known. The negative relationship between
$r$ and $P_0$ will make it more likely that a fall in $r$ leads to an increase in the required
level of deposit $D$, as it will mitigate or even reverse the change in $r_dP_0$ due to the
change in $r$.

As in the previous subsection, the effect on the relative attractiveness of the two
types of contracts will depend on the sign of $V'_r(D) + V'_o(D)$.

3.3.3 Improvement in the Financial Service

An improvement in the financial service can be viewed in various ways. Here, I
interpret it as a change that reduces the cost of borrowing for households. In the
model, it is represented by a downward shift in $V_o$, without changing $V_r$ or $r_d$.

Such a change increases the required level of deposit $D$ for a zero-rent contract;
the owner needs to be compensated more because the utility that the owner obtains
from the deposit is lower. Algebraically, a downward shift increases the left-hand side

---

7There can be different reasons for this belief, such as the difficulty of financing real estate
investment when interest rate is high, or a dividend-based pricing model (a well-known model is
from Gordon (1962)). Tsatsaronis and Zhu (2004) finds that the empirical relationship between
interest rate and house price is negative, using data from a sample of developed countries.

8The aggregate movement in house prices is generally hard to characterize. Even standard metrics
such as the price-to-rent ratio and the price-to-income ratio have limited ability to predict housing
price movements and the ratios themselves show considerable movements over time. For example,
Himmelberg, Mayer and Sinai (2005) study the nature of the movements in these metrics.
of the equation that characterizes the required level of deposit \( D \),

\[
(1 - \alpha)V_r(D) - \alpha V_o(D) = -(1 - \alpha)(U - U_o) + \alpha(E_o[A] - \tau_d P_o),
\]

which implies that the solution \( D \) to this equation increases as well.

As in the previous subsections, the effect on the relative attractiveness of the two types of contracts will depend on the sign of \( V_r(D) + V_o(D) \).

3.3.4 Discussion

In the stylized model, the change in the value for the owner of possessing the house relative to selling it and the change in the surplus created by the rental deposit explain how various recent macroeconomic trends are qualitatively consistent with the rise in the required level of deposit \( D \) for zero-rent contracts.

The historically low levels of return to residential real estate investment, interest rates and the spread between the household borrowing rate and the deposit rate should, according to the model, imply a high level of required rental deposits, which is also observed in the data, as shown by figure 11.

At the same time, the high level of rental deposits, by itself and compared to various measures of how much households can comfortably (in terms of the model, without reaching large \(-V'\)) commit as rental deposits, makes it likely that the real estate market is in a state where any increase in the level of deposit makes the zero-rent contract less attractive compared to the zero-deposit contact. Figure 11 shows the historical levels of deposits on zero-rent contracts.

Any increase in the level of deposit makes the zero-rent contract relatively less attractive if \( D \) is high enough so that \( V_o'(D) + V_r'(D) < 0 \). When the market is in such a state, the form of the contract that maximizes the joint surplus of the renter and the owner in some cases will be a hybrid contract that features both a moderate periodic (mostly monthly) rent and a moderate deposit. This implication is consistent with the recent rise of such hybrid contracts, which used to be rare but has become more
Each series is normalized to 100 at January 2010.

- Average Deposit (Nominal)
- Average Deposit (Real)
- (Average Deposit) / (Average House Price)
- (Average Deposit) / (Per Capita Bank Deposit)

Figure 3-9: Historical Levels of Deposits on Zero-Rent Contracts
3.4 Empirical Analysis of Zero-Rent Contracts

In this section, I study how the required level of deposit for zero-rent contracts is correlated with movements in house price and interest rate, using data on the aggregate real estate market.

3.4.1 Statistical Framework

First, I simplify the model developed in the previous section to derive an approximate relationship between the real estate market variables. I will show that the resulting relationship is broadly consistent with the data.

In the model developed in the last section, the required level of deposit $D$ for a zero-rent contract is determined by the following equation:

$$ (1 - \alpha)V_r(D) - \alpha V_o(D) = -(1 - \alpha)(U - U_a) + \alpha(E_o[A] - r_d P_0). \tag{3.6} $$

Here, I simplify the equation further by assuming that $V_o(D) = (r + \delta_o)D$, $V_r(D) = -(r + \delta_r)D$ and $r_d = r + \delta_d$, where $r$ is some benchmark real interest rate and $\delta_o, \delta_r$ and $\delta_d$ are positive constants such that $\delta_o > \delta_r$. Then, the equation can be rewritten as follows:

$$ - (r + (1 - \alpha)\delta_r + \alpha\delta_o)D = -(1 - \alpha)(U - U_a) + \alpha(E[A] - r_d P_0). \tag{3.7} $$

$$ \therefore D = \frac{(1 - \alpha)(U - U_a) - \alpha E[A] + \alpha(r + \delta_d)P_0}{r + (1 - \alpha)\delta_r + \alpha\delta_o}. \tag{3.8} $$

Detailed historical statistics are not available on how common such hybrid contracts have been, except for recent few years. However, there are many newspaper articles that report this change in the housing rental market. Lee (2014) is an example.
Given \( D > 0 \),
\[
\begin{align*}
\frac{\partial D}{\partial P_0} &= \frac{\alpha (r + \delta_d)}{r + (1 - \alpha)\delta_r + \alpha \delta_o} > 0 \\
\frac{\partial D}{\partial E[A]} &= \frac{-\alpha}{r + (1 - \alpha)\delta_r + \alpha \delta_o} < 0 \\
\frac{\partial D}{\partial r} &= -\frac{D}{r + (1 - \alpha)\delta_r + \alpha \delta_o} + \frac{\alpha P_0}{r + (1 - \alpha)\delta_r + \alpha \delta_o}.
\end{align*}
\]

The sign of \( \frac{\partial D}{\partial r} \) depends on the values of the parameters and the variables.

To study whether the data show behavior broadly consistent with the model, I perform a linear regression of changes in \( D \) on changes in \( P, E_0[A] \) and \( r \). More specifically, with \( t \) denoting the time index, the linear model is:
\[
\frac{D_t - D_{t-1}}{P_{t-1}} = \beta_0 + \beta_1 \frac{P_t - P_{t-1}}{P_{t-1}} + \beta_2 (\frac{E_t[A]}{P_t} - \frac{E_{t-1}[A_{t-1}]}{P_{t-1}}) + \beta_3 (r_t - r_{t-1}) + \epsilon_t, \quad (3.9)
\]
where \( \epsilon_t \) is an idiosyncratic error term. The variables \( P \) and \( D \) are in real terms. To be consistent with the model, \( \beta_0 \) should be close to zero, \( \beta_1 \) should have the order of 1 and \( \beta_2 \) should have the order of \( \frac{1}{P_0} \). The model does not indicate clearly the order of \( \beta_3 \) as it has terms of opposite signs that offset each other, but its order should not be greater than that of \( \frac{1}{P_0} \), which is the order of \( \frac{1}{r} \).

Finally, the variable \( \frac{E_t[A]}{P_t} - \frac{E_{t-1}[A_{t-1}]}{P_{t-1}} \), which is the change in expected house price appreciation, is not directly observable. In this paper, I use observed acceleration of house price in the past to form the change in expected appreciation. In particular, I form \( \frac{E_t[A]}{P_t} - \frac{E_{t-1}[A_{t-1}]}{P_{t-1}} \) following an \( AR(p) \) model; the price acceleration \( (\frac{P_{t+1} - P_t}{P_t} - \frac{P_t - P_{t-1}}{P_{t-1}}) \) is the single variable of interest, and I use the expected value of \( (\frac{P_{t+1} - P_t}{P_t} - \frac{P_t - P_{t-1}}{P_{t-1}}) \) from the \( AR(p) \) model in place of \( \frac{E_t[A]}{P_t} - \frac{E_{t-1}[A_{t-1}]}{P_{t-1}} \). For each observation, I estimate the \( AR(p) \) model with a fixed number of most recent past observations.\(^{10}\)

\(^{10}\)There is no generally agreed method to determine expected future real estate prices. Using past observations is an extensively studied method in real estate price forecasting. Ghysels, Plazzi, Valkanov and Torous (2013) provides an extensive review.
3.4.2 Estimation Results

Table 1 displays the regression results using about 20 years of monthly data, from July 1996 to January 2015. For $r$, I use 6-month to 1-year average term deposit rate. I use panel data which contain monthly series of house prices and rental deposits for 8 major urban areas in Korea.

The signs of the coefficients on price and on expected appreciation is consistent with the model. An increase in the price is associated with an increase in the required level of deposit, and a rise in the expected price appreciation is associated with a fall in the required level of deposit. The coefficient on $r$ is small and not significantly different from zero.

The intercept is significantly different from zero. Under a deposit-to-price ratio of 0.5, the estimated intercept alone corresponds to about 1.5% rise in $D$ over a year. Also, the intercept is important in size, as it is close to the mean of the explained variable, $\frac{D_t - D_{t-1}}{P_t - P_{t-1}}$.

The coefficient on $\frac{P_t - P_{t-1}}{P_{t-1}}$ has the order of 1, which is consistent with the model. Also, the standard deviation generated by the expected appreciation variable (standard deviation of the variable multiplied by the absolute value of the estimated coefficient) is about 75% of the standard deviation the explained variable, $\frac{D_t - D_{t-1}}{P_{t-1}}$; the price change generates a substantial variation in the explained variable.

Finally, the coefficient on $\frac{E_t[A_t]}{P_t} - \frac{E_{t-1}[A_{t-1}]}{P_{t-1}}$ has the order of 0.1, which is smaller than the order $\frac{1}{r}$ expected from the model. This discrepancy may be explained by the fact that it is generally very difficult to predict house price movements, and the particular method that I employ in this paper to form the expectation has limitations; in the baseline model, the correlation between the constructed value of $\frac{E_t[A_t]}{P_t} - \frac{E_{t-1}[A_{t-1}]}{P_{t-1}}$ through $AR(p)$ and the realized value of $\frac{P_{t+1} - P_t}{P_t} - \frac{P_t - P_{t-1}}{P_{t-1}}$ is 0.27. Given the estimated coefficient, the standard deviation generated by the expected appreciation variable (standard deviation of the variable multiplied by the absolute value of the estimated coefficient) is about 5% of the standard deviation the explained variable, $\frac{D_t - D_{t-1}}{P_{t-1}}$.

Using the estimated coefficients, I analyze the rise in $D$ during recent years, as it
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Clustering

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Numbers in parentheses () are t-values.

$n$ indicates the number of prior months used to form the expectation.

$p$ indicates that $AR(p)$ model is used to form the expectation.

** indicates 5%-significance.

Table 3.1: Regression using Panel Data
has become an important issue together with the decline of zero-rent contracts. From 2011 to 2014, $D$ had increased by 16%. According to the estimated coefficients, the increase in $D$ due to the constant term is 4.8%, the increase in $D$ due to the price change $\frac{P_t - P_{t-1}}{P_{t-1}}$ is 3.4%, and the increase in $D$ due to the change in expected price appreciation $\frac{E_t[A_t]}{P_t} - \frac{E_{t-1}[A_{t-1}]}{P_{t-1}}$ is less than 1%. Therefore, the model fails to find a substantial effect of changes in expected returns to real investment returns on the level of deposits on zero-rent contracts.

3.5 Conclusion

This paper builds a model in which the deposit under a zero-rent rental contract serves as a zero-interest loan lent to the owner of a house from the renter. The required level of deposit rises when the house price rises because the rental agreement needs to compensate the owner for not taking the alternative option of selling the house at a higher price. This mechanism would work for usual rent-based contracts as well.

Interestingly, the required level of deposit also rises when the expected house price appreciation falls. When expected appreciation falls, the owner derives less profit from possessing the house. Therefore, the value of the zero-interest loan that the owner receives from the renter becomes less valuable, and thus, the owner demands a larger loan.

Theoretically, the high level of house price, in combination with falling returns to residential real estate investment, would push the level of deposits to historically highest levels. If the level of deposit becomes too high, the renter will find it costly to secure the required level of deposit and the zero-rent contract will become relatively less attractive. This reasoning is consistent with the recent increase in the popularity of rental contracts based on monthly rents and the rise of a new form of hybrid contracts, which combine a significant level of deposit with a moderate level of monthly rent.

Empirically, I show that the data are qualitatively consistent with the theory that when expected appreciation falls, the required level of deposit rises. However, the
model fails to show that the fall in the expected appreciation is responsible for any substantial rise in the level of deposits for zero-rent contracts during recent years. This result may be caused by the flaws in the process of constructing the expected appreciation variable using the data.

This change in the form of housing rental contracts has become an important policy issue in Korea, as renters perceive rent-based contracts and hybrid contracts to be more expensive than deposit-based contracts. The paper clarifies the nature of the issue and argues that the perceived difficulty by renters is fundamentally caused by the loss of the opportunity to lend profitably to the lenders. The form of the rental contract itself is not an issue, as the shift in the form reflects the decreased return to zero-deposit contracts.

This paper suggests interesting avenues for future research. Though limited in scope, micro-level data on rental contracts are available, and it would be interesting to apply the model to the micro-level data and study how well the model works with them and how houses with different characteristics respond differently to changes in the economic environment. Another interesting expansion would be to consider the effect that the level of deposit has on the level of house price by building a more general model.
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