ANGLES OF MULTIVARIABLE ROOT LOCI*

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ABSTRACT

The generalized eigenvalue problem can be used to compute angles of multivariable root loci. This is most useful for computing angles of arrival. The results extend to multivariable optimal root loci.

I. INTRODUCTION

The classical root locus has proven to be a valuable analysis and design tool for single input single output linear control systems. Research is currently underway to extend root locus techniques to multi-input multi-output linear control systems. We contribute to this body of research by showing that the generalized eigenvalue problem can be used to compute angles of the multivariable root locus, and we show this method to be particularly useful for computing angles of arrival to finite transmission zeros. The generalized eigenvalue problem can also be used to compute sensitivities of the multivariable root locus, as well as angles and sensitivities of the multivariable optimal root locus.

Previous work on angles and sensitivities is contained in [1,2,3,4]. Our work follows most closely [1], where the standard eigenvalue problem is used to compute angles.

II. The Multivariable Root Loci

We consider the linear time invariant output feedback problem:

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n, \qquad u \in \mathbb{R}^m \qquad (1)$$

$$y = Cx \qquad y \in \mathbb{R}^m \qquad (2)$$

$$u = -kKy \quad . \qquad (3)$$

The closed loop system matrix and its eigenvalues, right eigenvectors, and left eigenvectors are defined by:

$$A_{c\ell} \triangleq A - kBKC \qquad (4)$$

$$(A_{cl} - s_i I)x_i = 0 \qquad i = 1,\ldots, n \qquad (5)$$

$$y_i^H(A_{cl} - s_i I) = 0 \qquad i = 1,\ldots, n \quad . \qquad (6)$$

Several assumptions are made about the system. We assume (A,B) is controllable, (C,A) is observable, and K is invertable. We assume the number of inputs and outputs are equal. We assume that at any point of the root locus where angles and sensitivities are computed that the closed loop eigenvalues are distinct. Finally, we assume that the system is not degenerate in the sense that A,B, and C do not conspire in such a way that $P(s)$ loses rank for all s in the complex plane, where the polynomial system matrix $P(s)$ is defined as

$$P(s) \triangleq \begin{bmatrix} sI-A & B \\ -C & 0 \end{bmatrix} \quad . \qquad (7)$$

As the gain term k is varied from 0 to infinity the closed loop poles trace out a root locus. At $k=0$ the n branches of the root locus start at the open loop eigenvalues. As $k \to \infty$, some number $p \leq n-m$ of these branches

approach finite transmission zeros, which are defined to be the finite values of s which reduce the rank of P(s). Also as k→∞, the remaining n-p branches group into m patterns and approach infinity.

At any point on the root locus an angle can be defined. Consider the closed loop eigenvalue s_i which is computed for some value of k. If k is perturbed to k+Δk then s_i will be perturbed to s+Δs_i. As Δk→0 then $\Delta s_i/\Delta k$ approaches the constant ds_i/dk (if this limit exists). The angle of the root locus at point s_i is defined to be

$$\phi_i \overset{\Delta}{=} \arg\left(\frac{ds_i}{dk}\right) , \tag{8}$$

where "arg" is the argument of a complex number. The angles of the root locus at the open loop eigenvalues are the angles of departure, and the angles at the finite transmission zeros are the angles of arrival. Figure 1 illustrates these definitions.

At any point on the root locus the sensitivity is defined to be

$$S_i \overset{\Delta}{=} \left|\frac{ds_i}{dk}\right| \tag{9}$$

The sensitivities are used to approximately determine how far a closed loop eigenvalue moves in response to a gain change. Suppose the gain changes from k to k+Δk. Then the closed loop eigenvalue s_i will move (approximately) a distance Δk S_i in the direction ϕ_i.
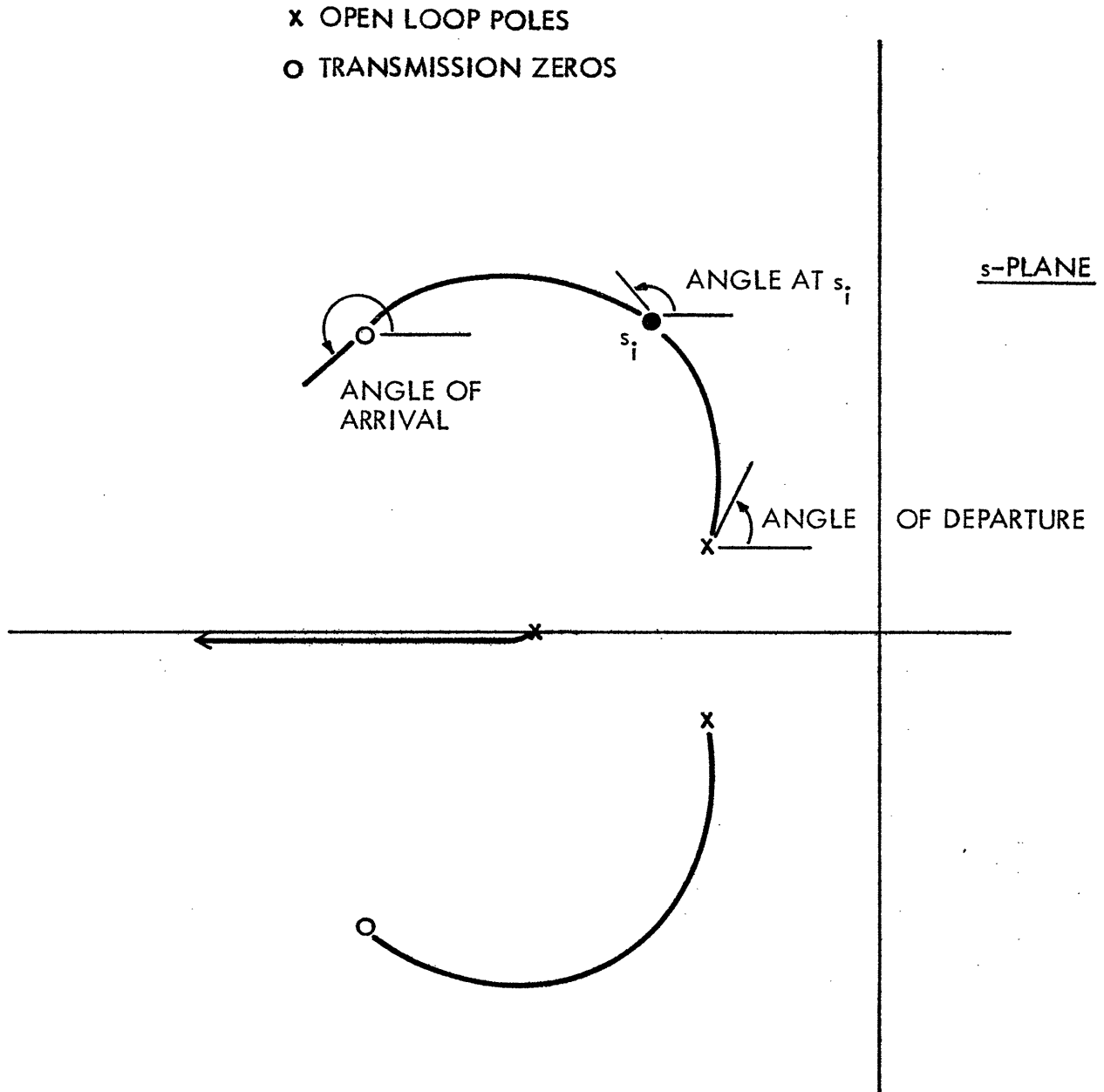
Figure 1 Definition of Angles

III. The Generalized Eigenvalue Problem

The generalized eigenvalue problem is to find all finite λ and their associated eigenvectors v which satisfy

$$Lv = \lambda M v, \tag{10}$$

where L and M are real valued rxr matrices which are not necessarily full rank. If M is full rank then it is invertible, and premultiplication by M^{-1} changes the generalized eigenvalue problem into a standard eigenvalue problem, for which there are exactly r solutions. In general there are 0 to r finite solutions, except for the degenerate case when all λ in the complex plane are solutions. Reliable FORTRAN subroutines based on stable numerical algorithms exist in EISPACK [5] to solve the generalized eigenvalue problem. See [6] for the application of this software to a related class of problems. Also, see [7] for additional information on the solution of the generalized eigenvalue problem.

Our first application of the generalized eigenvalue problem is to compute the closed loop eigenstructure of a system. This has been done before [12] but without specific mention of the generalized eigenvalue problem. The more standard approach to computing the closed loop eigenstructure is to use a standard eigenvalue problem, never-the-less it is instructive to show that an alternative approach exists.

Lemma 1. The s_i, x_i, and y_i^H are solutions of the generalized eigenvalue problems

$$\begin{bmatrix} A - s_i I & B \\ -C & -(kK)^{-1} \end{bmatrix} \begin{bmatrix} x_i \\ v_i \end{bmatrix} = 0 \qquad i = 1, \ldots, p \tag{11}$$

$$[y_i^H \ \eta_i^H] \begin{bmatrix} A-s_i I & B \\ -C & -(kK)^{-1} \end{bmatrix} = 0 \qquad i = 1, \ldots, p \ . \qquad (12)$$

Proof. From (11) we see that

$$(A-s_i I)x_i - kBKCx_i = 0, \qquad (15)$$

which is the same as (5), the defining equation for the closed loop eigenvalues and right eigenvectors. In a similar way (12) can be reduced to (6), the defining equation for the left eigenvectors. This completes the proof.

The generalized eigenvalue problem cannot be used to compute the open loop eigenstructure (k=0), because the lower right block of the matrices in (11) and (12) would be infinite. When k is in the range $0<k<\infty$ then the number of finite solutions is p=n. When k is infinite (more appropriately when $1/k = 0$) then the number of finite solutions is in the range $0 \leq p \leq n-m$. The finite solutions (when k=∞) are the transmission zeros, and the x_i and y_i^H vectors are the right and left zero directions. From Lemma 1 it is clear that as $k \to \infty$ the finite closed loop eigenvalues approach transmission zeros, and the associated eigenvectors approach zero directions.

The solutions of the generalized eigenvalue problems contain two vectors ν_i and η_i^H which do not appear in the solutions of standard eigenvalue problems. The importance of the ν_i vectors can be explained as follows [8,9]. The closed loop right eigenvector x_i is constrained to lie in the m dimensional subspace of $\mathbb{R}^n$ spanned by the columns of $(s_i I-A)^{-1}B$. Exactly where x_i lies in this subspace is determined by ν_i, via $x_i = (s_i I-A)^{-1}B\nu_i$. This follows from the top part of (11). If

the state of the closed loop system at time t=0 is x_0 = ax_i, then the state

trajectory for time t>0 is x(t) = ax_i exp(s_it), and the control action

is u(t) = $a\nu_i$ exp(s_it). This follows from the bottom part of (11). The

η_i^H vectors play an analogous role in the dual system with matrices

$S(-A^T, C^T, B^T)$.

For our purposes, however, the vectors ν_i and η_i^H are significant

because they can be used to compute angles of the root locus. This is

shown in the next section.

IV. Angles

In theorem 1 we show how to compute angles on the root locus. The eigenvalue problem is used for angles of departure, the generalized eigenvalue problem for angles of arrival, and either for intermediate angles. For the intermediate angles the eigenvalue problem is preferable, since it is n^{th} order instead of $n+m^{th}$ order. However, when k is very large but not infinite, then the generalized eigenvalue problem has better numerical properties [6].

Theorem 1. The angles of the root locus, for $0 < k \leq \infty$ and for distinct s_i, are found by

$$\phi_i = \arg \left(\frac{-y_i^H BKCx_i}{y_i^H x_i} \right) \quad 0 < k \leq \infty \qquad i=1,\ldots,p \tag{16}$$

$$\phi_i = \arg \left(\frac{\eta_i^H K^{-1} \nu_i}{y_i^H x_i} \right) \quad 0 < k \leq \infty \qquad i=1,\ldots,p \quad . \tag{17}$$

Remark. The angles of departure are found using (16) with $k=0$, the angles of approach are found using (17) with $k=\infty$. For $k<\infty$, $p=n$; and for $k=\infty$, $0 \leq p \leq n-m$.

Proof. The proof of (16) is found in [1]. The proof of (17) is similar, but uses the generalized rather than the standard eigenvalue problem. First we show that

$$\frac{ds_i}{dk} = \frac{\eta_i^H K^{-1} \nu_i}{k^2 y_i^H x_i} \quad i=1,\ldots, p \quad . \tag{18}$$

Rewrite the generalized eigenvalue problem (11) as

$$(L-s_i M)v_i = 0 \qquad i = 1,\ldots,p \qquad\qquad (19)$$

where

$$L = \begin{bmatrix} A & B \\ -C & -(kK)^{-1} \end{bmatrix} \qquad M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \qquad v_i = \begin{bmatrix} x_i \\ \nu_i \end{bmatrix}$$

Also, let

$$u_i^H = [y_i^H \; \eta_i^H] \quad .$$

Differentiate (19) with respect to k to get

$$\left[\frac{d}{dk}(L-s_i M)\right]v_i + (L-s_i M)\frac{dv_i}{dk} = 0. \qquad\qquad (20)$$

Multiply (20) on the left by u_i^H to get

$$u_i^H \left[\frac{d}{dk}(L-s_i M)\right]v_i = 0. \qquad\qquad (21)$$

Subsitute for L and M, differentiate, and perform some algebra to arrive at (18). The angle is the argument of the left hand side of (18), and since $\arg(k^2) = 0$, the result is (17). This completes the proof.

The following identities, which are obtained from (11) and (12), can be used to pass back and forth from (17) and (18):

$$Cx_i = -(kK)^{-1}\nu_i \qquad\qquad (22)$$

$$y_i^H B = \eta_i^H(kK)^{-1} \quad . \qquad\qquad (23)$$

We see that when $k=\infty$ then $Cx_i = 0$ and $y_i^H B = 0$, which verifies that (16) cannot be used to compute angles of arrival, since $\phi_i = \arg(0)$ is not defined.

In [1] a limiting argument as $k \to \infty$ is used to derive alternate equations for angles of arrival. These equations are more complicated because the rank of CB must be determined. Using the generalized eigenvalue problem eliminates the need to determine rank. We note that [1] contains some errors that are pointed out in [3].

V. Sensitivity

The ν_i and η_i^H vectors are also useful for the calculation of eigenvalue sensitivites. This is shown in Lemma 2. A separate proof of this Lemma is not needed, since it follows from intermediate steps in the proof of Theorem 1 (equation (25) follows from (18)).

Lemma 2. The sensitivities of distinct closed loop eigenvalues to changes in k, for $0 \leq k \leq \infty$, are found by

$$S_i = \left| \frac{y_i^H BKCx_i}{y_i^H x_i} \right| \qquad 0 \leq k < \infty \qquad i = 1,\ldots, p \qquad (24)$$

$$S_i = \frac{1}{k^2} \left| \frac{\eta_i^H K\nu_i}{y_i^H x_i} \right| \qquad 0 < k \leq \infty \qquad i = 1,\ldots,p \ . \qquad (25)$$

Equations (24) and (25) give the same answers for $0 < k < \infty$. Even though k appears only in (25), actually both (24) and (25) are dependent on k, since y_i^H, x_i, η_i^H, and ν_i are all dependent on k.

VI. Extensions to the Multivariable Optimal Root Locus

Our attention now shifts from the linear output feedback problem to the linear state feedback problem with a quadratic cost function. As in [10, 11], we show that the optimal root locus for this problem is a special case of the ordinary output feedback root locus. We then show how to compute angles and sensitivities.

The linear optimal state feedback problem is

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n, \qquad u \in \mathbb{R}^m \qquad (26)$$

$$u = F(x) \quad . \qquad (27)$$

The optimal control is required to be a function of the state and to minimize the infinite time qudadratic cost function

$$J = \int_0^\infty (x^T Q x + \rho u^T R u)\, dt, \qquad (28)$$

where

$$Q = Q^T \geq 0$$

$$R = R^T > 0$$

$$0 \leq \rho \leq \infty \quad .$$

As usually done for this problem we assume that (A,B) is controllable and $(Q^{1/2}, A)$ is observable.

Kalman [12] has shown (for $\rho > 0$) that the optimal control is a linear function of the state

$$u = -Fx \quad , \qquad (29)$$

where

$$F = \frac{1}{\rho} R^{-1}B^{T}P \quad , \tag{30}$$

and P is the solution of the Riccati equation

$$0 = Q + A^{T}P + PA - \frac{1}{\rho} PBR^{-1}B^{T}P \quad . \tag{31}$$

The closed loop system matrix is

$$A_{c\ell} = A - BF \quad . \tag{32}$$

As ρ is varied from infinity down to zero the closed loop eigenvalues

trace out an optimal root locus.

To study the optimal root locus we define a linear output feedback

problem with 2n states, m inputs, and m outputs.

$$\tilde{A} = \begin{bmatrix} A & 0 \\ -Q & -A^{T} \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} 0 & B^{T} \end{bmatrix} \qquad \tilde{K} = R^{-1} \quad .$$

The closed loop matrix of this augmented system is

$$Z = \tilde{A} - \frac{1}{\rho} \tilde{B} \tilde{K} \tilde{C} = \begin{bmatrix} A & -\frac{1}{\rho} BR^{-1}B^{T} \\ -Q & -A^{T} \end{bmatrix} \quad , \tag{33}$$

which is often referred to as the Hamiltonian matrix. Its 2n eigenvalues

are symmetric about the imaginary axis, and those in the left half plane

(LHP) are the same as the eigenvalues of $A_{c\ell}$ in (32).

Define the closed loop eigenvalues, right and left eigenvectors,

for i=1,...,2n, respectively as s_i, z_i, and w_i^H. They can be computed using

an eigenvalue decomposition of Z. Alternatively, using Lemma 1, they are

solutions of the following generalized eigenvalue problems:

$$\begin{bmatrix} \tilde{A} - s_i I & \tilde{B} \\ -\tilde{C} & -\rho\tilde{K}^{-1} \end{bmatrix} \begin{bmatrix} z_i \\ v_i \end{bmatrix} = 0 \qquad i = 1,\ldots, 2p \qquad (34)$$

$$\begin{bmatrix} w_i^H & \eta_i^H \end{bmatrix} \begin{bmatrix} \tilde{A} - s_i I & \tilde{B} \\ -\tilde{C} & -\rho\tilde{K}^{-1} \end{bmatrix} = 0 \qquad i = 1,\ldots,2p \qquad (35)$$

The number of finite generalized eigenvalues is $2p = 2n$ if $\rho > 0$, and is in the range $0 \leq 2p \leq 2(n-m)$ if $\rho = 0$.

The optimal root locus is the LHP portion of the regular root locus of the Hamiltonian system. At $\rho = \infty$ the n branches of the optimal root locus start at the LHP eigenvalues of A, or the mirror image about the imaginary axis of the RHP eigenvalues of A. As $\rho \to 0$, p of these branches remain finite, where $0 \leq p \leq n-m$. The remaining $n-p$ branches group into m Butterworth patterns and approach infinity. Those branches that remain finite approach transmission zeros, which are the finite LHP solutions of (34) with $\rho = 0$.

The angles and sensitivities of the optimal root locus can be found by applying Theorem 1 and Lemma 2 to the Hamiltonian system. The results are the following:

<u>Theorem 2</u>. The angles on the optimal root locus, for $0 \leq \rho \leq \infty$ and for distinct s_i, are found by

$$\phi_i = \arg \left\{ \frac{1}{w_i^H z_i} w_i^H \begin{bmatrix} 0 & BR^{-1}B^T \\ 0 & 0 \end{bmatrix} z_i \right\} \qquad \begin{array}{l} 0 < \rho \leq \infty \\ i = 1,\ldots,p \end{array} \qquad (36)$$

$$\phi_i = \arg \left\{ \frac{\eta_i^H R v_i}{w_i^H z_i} \right\} \qquad 0 \leq \rho < \infty \quad i = 1,\ldots,p \quad . \qquad (37)$$

Remark. The angles of departure are found using (36) with $\rho=\infty$, and the angles of approach by using (37) with $\rho=0$. For $\rho>0$, $p = n$; and for $\rho = 0$, $0 \leq p \leq n-m$.

Lemma 3: The sensitivities of distinct closed loop eigenvalues to changes in ρ, for $0 \leq \rho \leq \infty$, are found by

$$S_i = \frac{1}{\rho^2} \left| \frac{1}{w_i^H z_i} \ w_i^H \begin{bmatrix} 0 & BR^{-1}B^T \\ 0 & 0 \end{bmatrix} z_i \right| \quad \begin{array}{l} 0 < \rho \leq \infty \\ i = 1,\ldots, p \end{array} \tag{38}$$

$$S_i = \left| \frac{\eta_i^H R \nu_i}{w_i^H z_i} \right| \quad 0 \leq \rho < \infty \quad i = 1,\ldots, p \quad . \tag{39}$$

Remark. The computations for (36-39) can be reduced by using the following identifies. First, from (34) and (35), it can be shown that $\nu_i = \eta_i$. Second, let $\overline{s}_i$ be the RHP mirror image about the imaginary axis of s_i, and let $\overline{z}_i = (\overline{x}_i^H, \ \overline{\xi}_i^H)^H$ be the right eigenvector associated with $\overline{s}_i$. Then the left eigenvector associated with s_i is $w_i^H = (-\overline{\xi}_i^H, \ \overline{x}_i^H)$.

VII. Example

To illustrate Theorm 1 we define a system S(A,B,C) and plot root loci for each of 3 output feedback matrices K. The system matrices are:

$$
A = \begin{bmatrix} -4 & 7 & -1 & 13 \\ 0 & 3 & 0 & 2 \\ 4 & 7 & -4 & 8 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \\ -2 & 0 \end{bmatrix}
$$

$$
C = \begin{bmatrix} 0 & -5 & 2 & -2 \\ 8 & -14 & 0 & 2 \end{bmatrix}
$$

The output feedback matrices matrices are

Case #1

$$
K = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}
$$

Case #2

$$
K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

Case #3

$$
K = \begin{bmatrix} 1 & 0 \\ 0 & 50 \end{bmatrix}
$$

Case #2 is the same as used in [1]. The root loci are shown in Figure 2. The angles of departure and approach were computed and are listed in Table 1.

The system has two open loop unstable modes that are attracted to unstable transmission zeroes, so for all values of k the system is unstable. The system has two open loop stable modes that are attracted to -∞ along the negative real axis. One of the branches first goes to the right along the negative real axis and then turns around. The turn around point is called a branch point. The root locus can be thought of a being plotted on a Riemann surface, and the branch points are points at which the root locus moves between different sheets of the

Riemann surface [14].

TABLE 1

Angles of Departure and Approach for Example 1

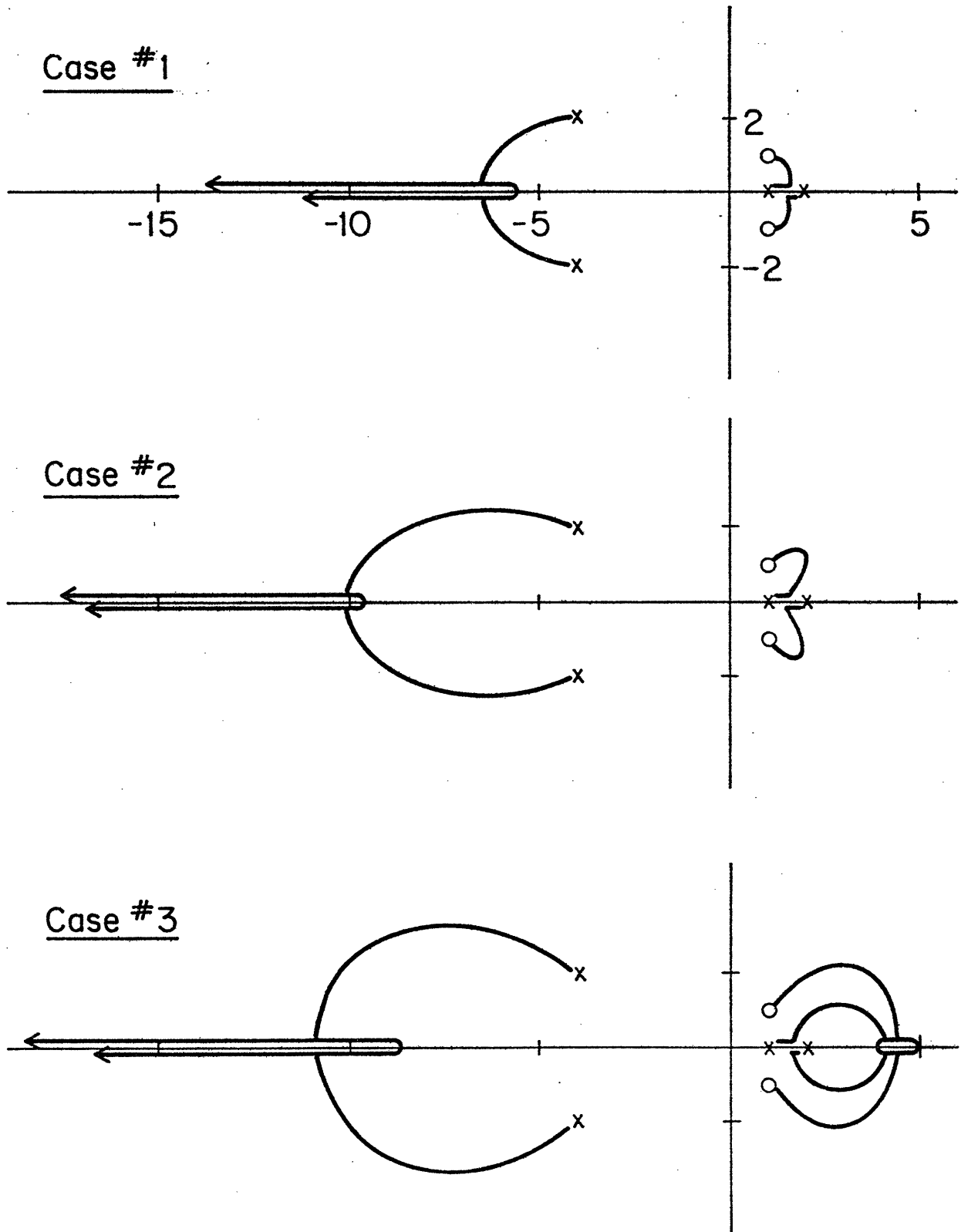| Case	Angles of Departure			Angles of Approach
	$-4 \pm 2i$	1	2	$1 \pm i$
1	$\pm 173°$	$0°$	$180°$	$\mp 170°$
2	± 149	0	180	∓ 121
3	$+ 135$	0	180	∓ 144

Figure 2   Root Loci of a Linear System with Output Feedback

VIII. <u>Conclusion</u>

The multivariable root locus has been a rich source of interesting research problems. Using the generalized eigenvalue problem to compute angles is one example of such a research problem. The ultimate value of the multivariable root locus as a design tool, however, has yet to be determined.

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