Coherent sheaves on varieties arising in Springer theory, and category O

by

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Abstract

In this thesis, we will study three topics related to Springer theory (specifically, the geometry of the exotic nilpotent cone, and two-block Springer fibers), and stability conditions for category $\mathcal{O}$.

In the first chapter, we will be studying the geometry of the exotic nilpotent cone (which is a variant of the nilpotent cone in type C introduced by Kato). In [13], Bezrukavnikov has established a bijection between Λ^+, the dominant weights for an arbitrary simple algebraic group H, and $\mathbf{O}$, the set of pairs consisting of a nilpotent orbit and a finite-dimensional irreducible representation of the isotropy group of the orbit (as originally conjectured by Lusztig and Vogan). Here we prove an analogous statement for the exotic nilpotent cone.

In the second chapter (which is based on joint work with Rina Anno), we study the exotic t-structure for a two-block Springer fibre (i.e. for a nilpotent matrix of type $(m+n, n)$ in type A). The exotic t-structure has been defined by Bezrukavnikov and Mirkovic for Springer theoretic varieties in order to study representations of Lie algebras in positive characteristic. Using techniques developed by Cautis and Kamnitzer, we show that the irreducible objects in the heart of the exotic t-structure are indexed by crossingless $(m, m+2n)$ matchings. We also show that the resulting Ext algebras resemble Khovanov's arc algebras (but placed on an annulus).

In the third chapter, we study stability conditions on certain sub-quotients of category $\mathcal{O}$. Recently, Anno, Bezrukavnikov and Mirkovic have introduced the notion of a "real variation of stability conditions" (which are related to Bridgeland's stability conditions), and construct an example using categories of coherent sheaves on Springer fibers. Here we construct another example, by studying certain sub-quotients of category $\mathcal{O}$ with a fixed Gelfand-Kirillov dimension. We use the braid group action on the derived category of category $\mathcal{O}$, and certain leading coefficient polynomials coming from translation functors.

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Title: Professor

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Chapter 1

Equivariant coherent sheaves on the nilpotent cone

1.1 Introduction

Let H be a simple algebraic group; denote by $\mathfrak{h}$ its Lie algebra and $\mathcal{N} \subset \mathfrak{h}$ its nilpotent cone. Let $\mathbf{\Lambda}^+$ denote the set of dominant weights for H, and $\mathbf{O}$ denote the set of pairs $(\mathcal{O}, L)$, where $\mathcal{O}$ is a H-orbit on $\mathcal{N}$, and L is a finite-dimensional irreducible representation of the isotropy group G^x of the orbit $\mathcal{O}$, where $x \in \mathcal{O}$. Using geometric methods, in [13], Bezrukavnikov shows that there is a canonical bijection between $\mathbf{\Lambda}^+$ and $\mathbf{O}$ (a result that was previously conjectured by Lusztig and Vogan).

Now let $G = Sp_{2n}(\mathbb{C})$, and $\mathfrak{N}$ be Kato's exotic nilpotent cone, defined as follows:

$$\mathfrak{N} := \mathbb{C}^{2n} \times \{x \in \mathrm{End}(\mathbb{C}^{2n}) \mid x \text{ nilpotent}, \langle xv, v\rangle = 0 \ \forall\ v \in \mathbb{C}^{2n}\}$$

The main purpose is to establish an exotic analogue of Bezrukavnikov's bijection, i.e. a bijection between Λ^+, the dominant weights for G, and $\mathbb{O}$, the set of pairs $(\mathcal{O}, L)$, where $\mathcal{O}$ is a G-orbit on $\mathfrak{N}$, and L is a finite-dimensional irreducible representation of the isotropy group $G^{(v,x)}$ of the orbit $\mathcal{O}$, where $(v, x) \in \mathcal{O}$.

The exotic nilpotent cone was originally introduced by Kato in [36] to study multi-parameter affine Hecke algebras, via the equivariant K-theory of the exotic Steinberg variety (following techiques used by Kazhdan, Lusztig and Ginzburg in the case of one-parameter affine Hecke algebras). The G-orbits on $\mathfrak{N}$ are proven to be in bijection with $\mathcal{Q}_n$, the set of bi-partitions of n (and thus also with irreducible representations of the type C Weyl group). In [37], Kato explicitly realizes this bijection via an exotic Springer correspondence; this correspondence is somewhat cleaner than the type C Springer correspondence since there are no non-trivial local systems on G-orbits in $\mathfrak{N}$.

In [3], Achar and Henderson make precise conjectures describing the intersection cohomology of orbit closures in the exotic nilpotent cone. These have now been proven independently by Shoji-Sorlin (see Theorem 5.7 in [50]); and by Kato (see Theorem A in [38], Theorem A.1.8 in [39] and Remark 5.8 in [50]). The work of Achar, Henderson and Sommers in [5] studies special pieces for $\mathfrak{N}$, which turn out to have the same number of $\mathbb{F}_q$ points as Lusztig's special pieces for the ordinary nilpotent cone. These results all demonstrate a strong connection between the exotic nilpotent cone and the ordinary nilpotent cone of type C; the present work draws another parallel between the geometry of the exotic nilpotent cone $\mathfrak{N}$ and the geometry of the ordinary nilpotent cone $\mathcal{N}$ of type C.

1.1.1 Motivation

One motivation for the Lusztig-Vogan conjecture (in the case of the ordinary nilpotent cone) comes from the theory of two-sided cells in affine Weyl groups. So let $W \subset W_{aff}$ be the finite and affine Weyl groups, respectively. Let us make the following definitions:

$$W^f = \{w \in W_{aff} \mid l(ww') \geq l(w) \; \forall \; w' \in W\}$$
$${}^fW = (W^f)^{-1}$$

Above W^f is the set of minimal length right coset representatives of W in W_{aff}. Note that W^f can be naturally identified with Λ (the set of all integral weights); it can be shown that the intersection $W^f \cap^f W =^f W^f$ corresponds to Λ^+, the subset of dominant weights. For each dominant weight $\lambda \in \Lambda^+$, consider the two-sided $\underline{c}_\lambda$ Kazhdan-Lusztig cell which the corresponding element of ${}^fW^f$ lies in.

In [43], Lusztig associates a nilpotent orbit $e_{\underline{c}}$ to each two-sided cell $\underline{c}$ in W_{aff}. Thus for each dominant weight λ, using Lusztig's construction applied to the cell $\underline{c}_\lambda$, we obtain a nilpotent orbit $\mathcal{O}_\lambda$. The Lusztig-Vogan conjecture now asserts that the fibers of this map $\lambda \to \mathcal{O}_\lambda$ can be naturally identified with irreducible representations of the stabilizer of the orbit $\mathcal{O}_\lambda$.

In fact, we can make the following stronger conjecture. Let $X_{\underline{c}}$ denote the canonical basis in the Grothendieck group of the Springer fiber $K^0(\mathcal{B}_e)$ (where $e = e_{\underline{c}}$), considered as a set. Then $X_{\underline{c}}$ carries an action of the centralizer $Z(e_{\underline{c}})$. Conjecturally, the elements in the two-sided cell $\underline{c}$ are in bijection with pairs consisting of an orbit of $Z(e_{\underline{c}})$ on $X_{\underline{c}} \times X_{\underline{c}}$, and an irreducible representation of the stabilizer $Z(e_{\underline{c}})$.

It can be shown the set $X_{\underline{c}}$ has a distinguished element, x_c, fixed under the action of $Z(e_{\underline{c}})$. Under this bijection, one expects that the elements of $\underline{c} \cap^f W^f$ should be matched up with pairs where the orbit of $Z(e_{\underline{c}})$ on $X_{\underline{c}} \times X_{\underline{c}}$ is (x_c, x_c); thus, the Lusztig-Vogan conjecture would follow as a consequence.

Another motivation for the conjecture comes from the theory of Harish-Chandra modules; see the Introduction to Achar's thesis, [1] for a detailed exposition.

Bezrukavnikov's proof of the Lusztig-Vogan conjecture in [13] involves studying a certain t-structure on the category $D^b(\mathrm{Coh}^G(\mathcal{N}))$. Letting St be the Steinberg variety, it is well-known that the Grothendieck group of $\mathcal{C} = D^b(\mathrm{Coh}^{G\times\mathbb{C}^*}(\mathrm{St}))$ can be identified with the affine Hecke algebra, $\mathbb{H}_{aff}$. There are two natural projections $p_1, p_2 : \mathrm{St} \to \mathcal{N}$; thus we get two natural maps

$$Rp_{1*}, Rp_{2*} : D^b(\mathrm{Coh}^{G\times\mathbb{C}^*}(\mathrm{St})) \to D^b(\mathrm{Coh}^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}}))$$

It can be shown that, on the level of Grothendieck groups, the maps Rp_{1*}, Rp_{2*} are identified with the natural maps $\mathbb{H}_{aff} \to \mathbb{H}_{aff}/\xi\mathbb{H}_{aff}$, and $\mathbb{H}_{aff} \to \mathbb{H}_{aff}/\mathbb{H}_{aff}\xi$, where

$$\xi = \sum_{w\in W} (-1)^{l(w)} T_w$$

We have a natural projection map $\pi : D^b(Coh^G(\widetilde{\mathcal{N}})) \to D^b(Coh^G(\mathcal{N}))$, under which the latter category is identified with $\mathbb{H}_{aff}/(\xi\mathbb{H}_{aff} + \mathbb{H}_{aff}\xi)$. In Corollary 1 of [12], it is proven that the irreducible objects in the heart of the t-structure constructed in [13] categorify the images of certain canonical basis elements in $\mathbb{H}_{aff}$. For a more precise statement, see Proposition 1.5.2.

1.1.2 Summary of results

Below we describe the main results of this chapter in more detail.

Section 2: After recalling some of the basic properties of the exotic nilpotent cone, and some results of Achar, Henderson and Sommers on resolutions of special orbit closures, here we study the cohomology of dominant line bundles on the exotic Springer resolution $\widetilde{\mathfrak{N}}$. We closely follow the methods developed by Broer in [21], [25] that prove analogous results in the case of the ordinary nilpotent cone $\mathcal{N}$. Defining $\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda) := p^*\mathcal{O}_{G/B}(\lambda)$ where $p : \widetilde{\mathfrak{N}} \to G/B$ is the projection, we first prove that $H^i(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)) = 0$ using a theorem of Grauert-Riemenschneider. By using the additivity of the Euler characteristic along with Borel-Weil-Bott, we then compute the structure of $H^0(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda))$ as a G-module.

Section 3: First we recall the theory of quasi-exceptional sets (following Section 2 of [13]). Given a triangulated category $\mathcal{C}$, two ordered set of objects $\nabla = \{\nabla_i | i \in I\}, \Delta = \{\Delta_i | i \in I\}$ are said to constitute a dualizable quasi-exceptional set if they satisfy certain conditions. From a quasi-exceptional set generating a category $\mathcal{C}$, one obtains a t-structure on $\mathcal{C}$ whose heart is a quasi-hereditary category with simple objects also indexed by I. Letting $\mathcal{C} = D^b(Coh^G(\mathfrak{N}))$, define $\nabla_\lambda := R\pi_*\mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda)[\frac{\dim(\mathfrak{N})}{2}]$ where $\pi : \widetilde{\mathfrak{N}} \to \mathfrak{N}$ is the resolution of singularities. Then the main result is that $\nabla = \{\nabla_\lambda | \lambda \in \Lambda^+\}$ and $\Delta = \{\nabla_{w_0\cdot\lambda} | \lambda \in \Lambda^+\}$ constitute a quasi-exceptional set generating $D^b(Coh^G(\mathfrak{N}))$. The results of Section 2 are needed to prove this claim.

Section 4: After recalling some of the theory of perverse coherent sheaves (developed by Deligne and Bezrukavnikov), in this section we compare the t-structure constructed on $D^b(Coh^G(\mathfrak{N}))$ with the perverse coherent t-structure corresponding to the middle perversity. We first prove that ∇_λ is a perverse coherent sheaf for all $\lambda \in \Lambda$ (this essentially follows from the fact that $\pi : \widetilde{\mathfrak{N}} \to \mathfrak{N}$ is semi-small). We then deduce that the quasi-exceptional t-structure coincides with the perverse coherent t-structure. Since the simple perverse coherent sheaves are indexed by $\mathbb{O}$, the bijection between simple objects and co-standard objects in the heart of the t-structure on $D^b(Coh^G(\mathfrak{N}))$ gives the required bijection $\Theta : \mathbb{O} \leftrightarrow \Lambda^+$.

Section 5: We comment on some open questions relating to this construction. First, one may ask for an explicit combinatorial description of the construction Θ. One may also try to extend some results regarding canonical bases in equivariant K-theory, proved by Bezrukavnikov in [12], to the exotic setting.

We remark that there are a few differences between the proof of the bijection $\mathbf{O} \leftrightarrow \mathbf{\Lambda}^+$ for $\mathcal{N}$, and the bijection $\mathbb{O} \leftrightarrow \Lambda^+$ for $\mathfrak{N}$. First, while the canonical line bundle on the Springer resolution $\widetilde{\mathcal{N}}$ is trivial, the canonical line bundle on the exotic Springer resolution $\widetilde{\mathfrak{N}}$ is the anti-dominant line bundle $\mathcal{O}_{\widetilde{\mathfrak{N}}}(-\epsilon_1 - \cdots - \epsilon_n)$. As a result, for the proofs in Section 3, we use the twisted Weyl group action on Λ, $w \cdot \lambda = w(\lambda + \theta) - \theta$ with $\theta = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n)$ instead of the ordinary action.

Second, in the case of the ordinary nilpotent cone $\mathcal{N}$, the proof that the image of the functor $R\pi_* : D^b(Coh^H(\widetilde{\mathcal{N}})) \to D^b(Coh^H(\mathcal{N}))$ generates $D^b(Coh^H(\mathcal{N}))$ (see Lemma 7 in [13]) uses the Jacobson-Morozov resolution of orbit closures in $\mathcal{N}$. In the case of $\mathfrak{N}$, the resolutions of an arbitrary orbit closure $\overline{\mathbb{O}_{\mu,\nu}}$ are (to the best of our knowledge) not known. However, resolutions of special orbit closures have been recently developed in [5], and these are sufficient to prove the corresponding result in the case of the exotic nilpotent cone. However, there may be an easier proof of this result (Proposition 1.3.15) without appealing to the results of [5].

1.2 Geometry of the exotic nilpotent cone

1.2.1 Recollections

Here we recall the definition and basic properties of Kato's exotic nilpotent cone, following [36] and Section 6 of [3].

Definition 1.2.1. Let $\mathbb{V}$ be a $2n$-dimensional $\mathbb{C}$-vector space with a symplectic form $\langle \cdot, \cdot \rangle$, $G = Sp(\mathbb{V})$ be the associated symplectic group, and $\mathfrak{g} = \mathfrak{sp}(\mathbb{V})$ be the associated symplectic Lie algebra. Define $\mathfrak{s} \subset \mathfrak{gl}(\mathbb{V})$ as below (note $\mathfrak{s} \oplus \mathfrak{sp}(\mathbb{V}) = \mathfrak{gl}(\mathbb{V})$):

$$\mathfrak{s} = \{x \in \mathrm{End}(\mathbb{V}) | \langle xv, v \rangle = 0 \ \forall \ v \in \mathbb{V}\}$$

If $\mathcal{N}$ denotes the set of nilpotent endomorphims in $\mathfrak{gl}(\mathbb{V})$, we define the **exotic nilpotent cone** to be $\mathfrak{N} = \mathbb{V} \times (\mathfrak{s} \cap \mathcal{N})$.

Proposition 1.2.2. *The orbits of G on $\mathfrak{N}$ are in bijection with the poset $\mathcal{Q}_n$ of bipartitions of n. Under this correspondence, given $(\mu, \nu) \in \mathcal{Q}_n$, the orbit $\mathbb{O}_{\mu,\nu}$ consists of $(v, x) \in \mathfrak{N}$ such that the Jordan type of x acting on the subspace $E^x v \subset \mathbb{V}$ is $\mu \cup \mu$, and the Jordan type of x acting on the quotient $V/E^x v$ is $\nu \cup \nu$. Here E^x denotes the* *centralizer of x in $End(\mathbb{V})$, and for a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, $\lambda \cup \lambda$ denotes the partition $(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \cdots)$. The closure ordering for G-orbits on $\mathfrak{N}$ corresponds to the natural ordering on $\mathcal{Q}_n$ (i.e. $(\mu, \nu) \geq (\mu', \nu')$ if $\sum_{1\leq i\leq j}(\mu_i + \nu_i) \geq \sum_{1\leq i\leq j}(\mu'_i + \nu'_i)$ and $\mu_{j+1} + \sum_{1\leq i\leq j}(\mu_i + \nu_i) \geq \mu'_{j+1} + \sum_{1\leq i\leq j}(\mu'_i + \nu'_i)$).*

Proof. We refer the reader to Theorem 6.1 of [3] and [36] for the statement regarding orbits; see also Corollary 2.9 in [3] and Theorem 1 in [53]. For the statement regarding orbit closures, see Theorem 6.3 in [3]. □

After fixing a Cartan subgroup $T \subset G$ and a Borel subgroup $B \subset G$, let $(\mathbb{V} \oplus \mathfrak{s})^+ \subset \mathfrak{N}$ denote the sum of the strictly positive weight spaces in the G-module $\mathbb{V} \oplus \mathfrak{s}$; note that it is a B-module.

Definition 1.2.3. Let $\widetilde{\mathfrak{N}} = G \times_B (\mathbb{V} \oplus \mathfrak{s})^+$, and let $\pi : \widetilde{\mathfrak{N}} \to \mathfrak{N}$ be the map given by $\pi(g, (v, s)) = (gv, gsg^{-1})$. Then π is a resolution of singularities; accordingly we call $\widetilde{\mathfrak{N}}$ the exotic Springer resolution.

1.2.2 Resolutions of special orbit closures and some subvarieties of $\widetilde{\mathfrak{N}}$

In this section, for each orbit $\mathbb{O}_{\mu,\nu}$ (with $(\mu, \nu) \in \mathcal{Q}_n$), using resolutions of "special" orbit closures constructed in [5], we will construct a subvariety $\widehat{\mathbb{O}_{\mu,\nu}} \subset \widetilde{\mathfrak{N}}$ with a map $\pi_{\mu,\nu} : \widehat{\mathbb{O}_{\mu,\nu}} \to \mathbb{O}_{\mu,\nu}$ whose fibres are acyclic. The need for this construction is to prove Lemma 1.3.14 (an exotic analogue of Lemma 7 from [13]). We start by recalling Achar-Henderson-Sommer's construction of resolutions for "special" orbit closures from [5]; for more details see Sections 2 and 5 of [5].

Remark 1.2.4. We will use C-distinguished partitions in the following; alternatively one may use B-distinguished partitions (see [5]) instead.

Proposition-Definition 1. *Let $\mathcal{Q}_n^C \subset \mathcal{Q}_n$ denote the subposet consisting of bi-partitions (μ, ν) satisfying $\mu_i \geq \nu_i - 1, \nu_i \geq \mu_{i+1} - 1$. If $\mathcal{P}_{2n}$ is the poset of partitions of $2n$ with the natural order, let $\mathcal{P}_{2n}^C \subset \mathcal{P}_{2n}$ denote the subposet consisting of partitions where each odd part occurs with even multiplicity. Define the map $\Phi^C : \mathcal{Q}_n \to \mathcal{P}_{2n}^C$ by sending the bi-partition (μ, ν) to the partition obtained from the composition $(2\mu_1, 2\nu_1, 2\mu_2, 2\nu_2, \cdots)$ by replacing successive terms $(2s, 2t)$ with $(s + t, s + t)$ if $s < t$. Define also the map $\widehat{\Phi}^C : \mathcal{P}_{2n}^C \to \mathcal{Q}_n^C$: given $\lambda \in \mathcal{P}_{2n}^C$, obtain the composition λ' by first halving any even parts, and replacing any string of odd parts $(2k+1, \cdots, 2k+1)$ by $(k, k+1, \cdots, k, k+1)$; then let $\widehat{\Phi}^C(\lambda') = (\mu, \nu)$ where $\mu = (\lambda'_1, \lambda'_3, \cdots), \nu = (\lambda'_2, \lambda'_4, \cdots)$. Then the maps $\widehat{\Phi}^C$ and $\Phi^C|_{\mathcal{Q}_n^C}$ give an isomorphism of posets $\mathcal{Q}_n^C \simeq \mathcal{P}_{2n}^C$. For $(\mu, \nu) \in \mathcal{Q}_n$, denote $(\mu, \nu)^C = \widehat{\Phi}^C(\Phi^C(\mu, \nu))$.*

Definition 1.2.5. Let $(\mu, \nu) \in \mathcal{Q}_n, \lambda = \Phi^C(\mu, \nu)$. A λ-filtration (which we also refer to as a (μ, ν)-filtration) of $\mathbb{V}$ is a sequence of subspaces $(\mathbb{V}_{\geq a})_{a \in \mathbb{Z}}$, satisfying the following:

- $\mathbb{V}_{\geq a} \subseteq \mathbb{V}_{\geq a-1}$
- $\mathbb{V}_{\geq 1-a}^{\perp} = \mathbb{V}_{\geq a}$
- $\dim \mathbb{V}_{\geq a} = \sum_{i\geq 1} \max(\lceil \frac{\lambda_i - a}{2} \rceil, 0) := \lambda_a$ for $a \geq 1$

For $(v,x) \in \mathfrak{N}$, say that a λ-filtration is "C-adapted" to (v,x) if $v \in \mathbb{V}_{\geq 1}$ and $x(\mathbb{V}_{\geq a}) \subseteq \mathbb{V}_{\geq a+2}$ for all a.

Proposition 1.2.6. *If $(v,x) \in \mathbb{O}_{\mu,\nu}$, then there is a unique (μ,ν)-filtration $(\mathbb{V}_{\geq a})_{a\in\mathbb{Z}}$ that is C-adapted to (v,x).*

Proof. See Theorem 5.5 of [5]. □

Definition 1.2.7. Fix a specific $(v,x) \in \mathbb{O}_{\mu,\nu}$, and the corresponding (μ,ν)-filtration $(\mathbb{V}_{\geq a})_{a\in\mathbb{Z}}$. Let $P \subset G$ be the parabolic subgroup stabilizing this isotropic flag. Define:

$$\mathfrak{s}_{\geq 2} = \{x \in \mathfrak{s} | x(V_{\geq a}) \subseteq V_{\geq a+2}\}$$

With this choice, it follows that $\mathbb{V}_{\geq 1} \oplus \mathfrak{s}_{\geq 2}$ is a P-submodule of $\mathbb{V} \oplus \mathfrak{s}$; let $\widetilde{\mathbb{O}_{(\mu,\nu)^C}} = G \times_P (\mathbb{V}_{\geq 1} \oplus \mathfrak{s}_{\geq 2})$.

Proposition 1.2.8. *The image of the natural map $G \times_P (\mathbb{V}_{\geq 1} \oplus \mathfrak{s}_{\geq 2}) \to \mathfrak{N}$ is $\overline{\mathbb{O}_{(\mu,\nu)^C}}$, and this map $\pi_{(\mu,\nu)^C} : \widetilde{\mathbb{O}_{(\mu,\nu)^C}} \to \overline{\mathbb{O}_{(\mu,\nu)^C}}$ is a resolution of singularities. Further, $\pi_{(\mu,\nu)^C}$ is an isomorphism restricted to $\pi^{-1}_{(\mu,\nu)^C}(\mathbb{O}_{\mu,\nu})$.*

Proof. See Theorem 5.7 in [5]. □

Note that the definition above of $\widetilde{\mathbb{O}_{(\mu,\nu)^C}}$, and the definition in Section 2.1 of $\widetilde{\mathfrak{N}}$ are not entirely canonical (they depend on choices of (v,x), and a Cartan T, respectively). This is slightly inconvenient as we now want to relate the two varieties; the easiest way to fix the problem is via the equivalent, canonical descriptions below.

Lemma 1.2.9. *Let $B' \subset P$ be a Borel subgroup.*

$$\begin{aligned}
&\widetilde{\mathfrak{N}} \simeq \{(0 \subset \mathbb{V}_1 \subset \cdots \subset \mathbb{V}_n \subset \cdots \subset \mathbb{V}_{2n} = \mathbb{V}), (v,s) \in \mathfrak{N} \mid \\
&\quad \mathit{dim}\mathbb{V}_i = i, \mathbb{V}_i = \mathbb{V}_{2n-i}^{\perp}, v \in \mathbb{V}_n, s\mathbb{V}_{i+1} \subseteq \mathbb{V}_i\} \\
&\widetilde{\mathbb{O}_{(\mu,\nu)^C}} \simeq \{(\mathbb{V}_{\geq i}), (v,x) \in \mathfrak{N} \mid (\mathbb{V}_{\geq i}) \text{ is } (\mu,\nu)\text{-adapted to } (v,x)\} \\
&\widetilde{\mathfrak{N}} \supset G \times_{B'} (\mathbb{V}_{\geq 1} \oplus \mathfrak{s}_{\geq 2}) \simeq \{((0 \subset \mathbb{V}_1 \subset \cdots \subset \mathbb{V}_n \subset \cdots \subset \mathbb{V}_{2n} = \mathbb{V}), (v,s)) \in \widetilde{\mathfrak{N}} \mid \\
&\quad (\mathbb{V}_{\lambda_a}) \text{ is } (\mu,\nu)\text{-adapted to } (v,x)\}
\end{aligned}$$

Definition 1.2.10. Let $\widehat{\mathbb{O}_{\mu,\nu}} = G \times_{B'} (\mathbb{V}_{\geq 1} \oplus \mathfrak{s}_{\geq 2}) \cap \pi^{-1}(\overline{\mathbb{O}_{\mu,\nu}})$ be a subvariety of $\widetilde{\mathfrak{N}}$.

Corollary 1.2.11. *Consider the natural map $\theta_{\mu,\nu} : \widehat{\mathbb{O}_{\mu,\nu}} \to \overline{\mathbb{O}_{\mu,\nu}}$. The fibres of $\theta_{\mu,\nu}$ over a point in $\mathbb{O}_{\mu,\nu}$ are flag varieties for the Levi subgroup of P, and consequently have vanishing cohomology in degrees greater than 0.*

Proof. By construction, the fibres of $\theta_{\mu,\nu}$ are the same as the fibres of the composite map $G \times_{B'} (\mathbb{V}_{\geq 1} \oplus \mathfrak{s}_{\geq 2}) \to G \times_P (\mathbb{V}_{\geq 1} \oplus \mathfrak{s}_{\geq 2}) \to \overline{\mathbb{O}_{\mu,\nu}}$. The second map has singleton fibres over $\mathbb{O}_{\mu,\nu}$ by 1.2.8, and the first map has fibres $P/B' \simeq L/L \cap B'$ where L is the Levi subgroup of P. The cohomology vanishing for the structure sheaf follows from Borel-Weil-Bott. □

Example 1.2.12. Let $n = 6$, $\mu = (1^3), \nu = (3)$. Then replacing successive terms $(2s, 2t)$ of the sequence $(2,6,2,0,2,0)$ with $(s+t, s+t)$ if $s < t$, we have $\Phi^C(\mu,\nu) = \lambda = (4,4,2,1,1) \in \mathcal{P}^C_{12}$. Then $\lambda' = (2,2,1,0,1,0)$, so $(\mu,\nu)^C = \widetilde{\Phi}^C(\lambda) = ((2,1^2),(2))$. Using Definition 1.2.5, $\mathbb{V}_{\geq 4} = 0, \dim \mathbb{V}_{\geq 3} = \dim \mathbb{V}_{\geq 2} = 2, \dim \mathbb{V}_{\geq 1} = 5, \dim \mathbb{V}_{\geq 0} = 7, \dim \mathbb{V}_{\geq -1} = \dim \mathbb{V}_{\geq -2} = 10, \mathbb{V}_{\geq -3} = \mathbb{V}$. Thus:

$$\widetilde{\mathbb{O}_{(\mu,\nu)^C}} = G \times_P (\mathbb{V}_{\geq 1} \oplus \mathfrak{s}_{\geq 2}) = \{(0 \subset \mathbb{V}_{\geq 3} \subset \mathbb{V}_{\geq 1} \subset \mathbb{V}_{\geq 0} \subset \mathbb{V}_{\geq -2} \subset \mathbb{V}), (v,x) | \\ v \in \mathbb{V}_{\geq 1}, x\mathbb{V} \subseteq \mathbb{V}_{\geq -2}, x\mathbb{V}_{\geq -2} \subseteq \mathbb{V}_{\geq 1}, x\mathbb{V}_{\geq 0} \subseteq \mathbb{V}_{\geq 3}, x\mathbb{V}_{\geq 3} = 0\}$$

Proposition 1.2.8 now implies that the fibres of the map $\pi_{\mu,\nu^C} : \widetilde{\mathbb{O}_{(\mu,\nu)^C}} \to \mathbb{O}_{(\mu,\nu)^C}$ over $\mathbb{O}_{\mu,\nu}$ are singletons. To check this, it suffices to compute the fibre of π_{μ,ν^C} over the following point $(v,x) \in \mathbb{O}_{\mu,\nu}$:

$$v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, x = \begin{pmatrix} 0 & 1 & & & & & & & & & & \\ & 0 & 1 & & & & & & & & & \\ & & 0 & 1 & & & & & & & & \\ & & & 0 & & & & & & & & \\ & & & & 0 & & & & & & & \\ & & & & & 0 & & & & & & \\ & & & & & & 0 & & & & & \\ & & & & & & & 0 & & & & \\ & & & & & & & & 0 & 1 & & \\ & & & & & & & & & 0 & 1 & \\ & & & & & & & & & & 0 & 1 \\ & & & & & & & & & & & 0 \end{pmatrix}$$

We have $\dim (\operatorname{Im}(x^2) \oplus \mathbb{C}v) = 5$; since $\operatorname{Im}(x^2) \oplus \mathbb{C}v \subseteq \mathbb{V}_{\geq 1}$ it follows $\mathbb{V}_{\geq 1} = \operatorname{Im}(x^2) \oplus \mathbb{C}v, \mathbb{V}_{\geq 0} = (\operatorname{Im}(x^2) \oplus \mathbb{C}v)^{\perp}$. Since $\operatorname{Im}(x^3) \subset \mathbb{V}_{\geq 3}$ and both vector spaces have dimension 2, $V_{\geq 3} = \operatorname{Im}(x^3), V_{\geq -2} = \operatorname{Im}(x^3)^{\perp}$. Hence $\pi^{-1}_{(\mu,\nu)^C}(v,x)$ is a single flag, as expected.

1.2.3 Vanishing higher cohomology of dominant line bundles on $\widetilde{\mathfrak{N}}$

Definition 1.2.13. Denote by $p : \widetilde{\mathfrak{N}} \to G/B$ the natural projection. Let $\Lambda^+ \subset \Lambda$ denote respectively the dominant weights, and weight lattice of G. For $\lambda \in \Lambda^+$, let $\mathbb{C}_\lambda$ be the 1-dimensional representation of B where the torus T acts by λ. Given a B-representation V, denote $\mathcal{L}_{G/B}(V)$ for the sheaf of sections of the vector bundle $G \times_B V$; in particular define $\mathcal{O}_{G/B}(\lambda) = \mathcal{L}_{G/B}(\mathbb{C}^*_\lambda)$. Following Borel-Weil, $H^0(G/B, \mathcal{O}_{G/B}(\lambda)) = V^*_\lambda$ is the dual of the finite dimensional irreducible G-module with highest weight λ. Denote $\mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda) = p^*\mathcal{O}_{G/B}(\lambda)$.

Lemma 1.2.14.

$$V^*_\lambda \simeq V_\lambda$$

Proof. On page 116 of [33], it is proven that V^*_λ is isomorphic to $V(-\sigma\lambda)$ (here $\sigma \in W$ is the unique element of W which sends Δ to $-\Delta$; here Δ is the set of roots). In type C it is easy to check that one can find $\sigma \in W$, such that $\sigma\lambda = -\lambda$ for all $\lambda \in \Lambda$. The conclusion follows. □

Theorem 1.2.15. *For $\lambda \in \Lambda^+, H^i(\widetilde{\mathfrak{N}}, \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda)) = 0$ for $i > 0$.*

Recall the following theorem of Grauert-Riemenschneider in Kempf's version (see [40], Theorem 4, and also [46], Theorem 3.4).

Proposition 1.2.16. *Let U be an algebraic variety, and ω_U be the canonical line bundle. If there is a proper generically finite morphism $U \to X$, where X is an affine variety, then $H^i(U, \omega_U) = 0$ for $i > 0$.*

Lemma 1.2.17. *Let $U = G \times_B ((\mathbb{V} \oplus \mathfrak{s})^+ \oplus \mathbb{C}_{\lambda'})$. Then $H^i(U, \omega_U) = 0$ for $i > 0$.*

Proof. In accordance with 1.2.16, it suffices to find an affine variety X with a proper, generically finite map $U \to X$. Let $X' = (\mathbb{V} \oplus \mathfrak{s})^+ \oplus V_{\lambda'}$, and define $\pi : U \to X'$ by $\pi(g, (x, y)) = (gx, gy)$ where $g \in G, x \in (\mathbb{V} \oplus \mathfrak{s})^+, y \in \mathbb{C}_{\lambda'} \subset V_{\lambda'}$. The map π is proper as we can factorize it as follows (note G/B is projective):

$$\begin{aligned} G \times_B ((\mathbb{V} \oplus \mathfrak{s})^+ \oplus \mathbb{C}_{\lambda'}) \hookrightarrow G \times_B ((\mathbb{V} \oplus \mathfrak{s}) \oplus V_{\lambda'}) \simeq \\ G/B \times ((\mathbb{V} \oplus \mathfrak{s}) \oplus V_{\lambda'}) \to (\mathbb{V} \oplus \mathfrak{s}) \oplus V_{\lambda'} \end{aligned}$$

Let $X = \mathrm{im}(\pi)$; we claim X has the required property. Note that the map $U \to X$ generically has singleton fibres over a point $(s, 0) \in X$, with $s \in (\mathbb{V} \oplus \mathfrak{s})^+$. Since it is also proper, its fibres are generically finite, as required. □

Denote the natural projection $p_U : G \times_B ((\mathbb{V} \oplus \mathfrak{s})^+ \oplus \mathbb{C}_{\lambda'}) \to G/B$, where $\lambda' = \lambda + \epsilon_1 + \cdots + \epsilon_n$.

Lemma 1.2.18. *We have an isomorphism $\omega_U \simeq p_U^*\mathcal{O}_{G/B}(\lambda)$.*

Proof. Since $p_U : U \to G/B$ has fibre $(\mathbb{V} \oplus \mathfrak{s})^+ \oplus \mathbb{C}_{\lambda'}$, we have a short exact sequence of vector bundles on U (noting $\Omega_{G/B} \simeq \mathcal{L}_{G/B}(\mathfrak{u})$):

$$0 \to p_U^*\mathcal{L}_{G/B}(\mathfrak{u}) \to \Omega_U \to p_U^*\mathcal{L}_{G/B}(((\mathbb{V} \oplus \mathfrak{s})^+ \oplus \mathbb{C}_{\lambda'})^*) \to 0$$

Since $\omega_U = \Lambda^{top}(\Omega_U)$, it now follows that $\omega_U \simeq p_U^*\mathcal{L}_{G/B}(V)$ where:

$$V = \Lambda^{top}(\mathfrak{u} \oplus ((\mathbb{V} \oplus \mathfrak{s})^+ \oplus \mathbb{C}_{\lambda'})^*) = \Lambda^{top}(\mathfrak{u}) \otimes \Lambda^{top}((\mathbb{V} \oplus \mathfrak{s})^{+*}) \otimes \mathbb{C}^*_{\lambda'}$$

Noting that, as B-modules, $\mathfrak{u}$ has weights $\epsilon_i - \epsilon_j$ for $i < j$, $\epsilon_i + \epsilon_j$, and $2\epsilon_i$; and $(\mathbb{V} \oplus \mathfrak{s})^+$ has weights $\epsilon_i - \epsilon_j$ for $i < j$, $\epsilon_i + \epsilon_j$, and ϵ_i; we can conclude that $V = \mathbb{C}_{\epsilon_1+\cdots+\epsilon_n} \otimes \mathbb{C}^*_{\lambda'} = \mathbb{C}^*_{\lambda}$, as required. $\square$

Proof of Theorem. Using the above two Lemmas, we compute as follows (for $i > 0$):

$$\begin{aligned} 0 = H^i(U, \omega_U) &= H^i(G \times_B ((\mathbb{V} \oplus \mathfrak{s})^+ \oplus \mathbb{C}_{\lambda'}), p_U^* \mathcal{L}_{G/B}(\mathbb{C}^*_\lambda)) \\ &= H^i(G/B, \mathcal{L}_{G/B}(\mathbb{C}^*_\lambda \otimes S((\mathbb{V} \oplus \mathfrak{s})^{+*} \oplus \mathbb{C}^*_{\lambda'}))) \\ &= \bigoplus_{j,n \geq 0} H^i(G/B, \mathcal{L}_{G/B}(\mathbb{C}^*_\lambda \otimes S^j(\mathbb{V} \oplus \mathfrak{s})^{+*} \otimes S^n(\mathbb{C}^*_{\lambda'}))) \\ &= \bigoplus_{j,n \geq 0} H^i(G/B, \mathcal{L}_{G/B}(\mathbb{C}^*_{\lambda+n\lambda'} \otimes S^j(\mathbb{V} \oplus \mathfrak{s})^{+*})) \\ &= \bigoplus_{n \geq 0} H^i(G \times_B (\mathbb{V} \oplus \mathfrak{s})^+, p^* \mathcal{L}_{G/B}(\mathbb{C}^*_{\lambda+n\lambda'})) \end{aligned}$$

Taking the $n = 0$ summand in the above gives the desired result. $\square$

Corollary 1.2.19. *We have $H^i(\widetilde{\mathfrak{N}}, \omega_{\widetilde{\mathfrak{N}}}) = 0$ for $i > 0$.*

Proof. First we calculate that $\omega_{\widetilde{\mathfrak{N}}} = \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\epsilon_1 - \cdots - \epsilon_n)$. Following the method used in the proof of Lemma 1.2.18, we have the short exact sequence:

$$0 \to p^* \mathcal{L}_{G/B}(\mathfrak{u}) \to \Omega_{\widetilde{\mathfrak{N}}} \to p^* \mathcal{L}_{G/B}((\mathbb{V} \oplus \mathfrak{s})^{+*}) \to 0$$

So it follows that $\omega_{\widetilde{\mathfrak{N}}} = \wedge^{top}(\Omega_{\widetilde{\mathfrak{N}}}) = p^* \mathcal{L}_{G/B}(V)$, where $V = \wedge^{top}(\mathfrak{u}) \otimes \wedge^{top}((\mathbb{V} \oplus \mathfrak{s})^{+*}) = \mathbb{C}_{\epsilon_1+\cdots+\epsilon_n}$ using above calculations. Thus $\omega_{\widetilde{\mathfrak{N}}} \simeq \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\epsilon_1 - \cdots - \epsilon_n)$, as required.

Now note that the proof of Theorem 1.2.15 actually holds under the weaker assumption that $\lambda' = \lambda + \epsilon_1 + \cdots + \epsilon_n$ is dominant; since $\lambda = -\epsilon_1 - \cdots - \epsilon_n$ satisfies this hypothesis, the result follows. $\square$

1.2.4 Global sections of dominant line bundles on $\widetilde{\mathfrak{N}}$

In this section, we will use the results of the previous section to compute the global sections of dominant line bundles on $\widetilde{\mathfrak{N}}$. Given a coherent sheaf $\mathcal{E}$ on G/B, since $H^i(G/B, \mathcal{E})$ acquires the structure of a G-module, we adopt a convenient abuse of notation whereby $H^i(G/B, \mathcal{E})$ denotes the corresponding element of $K(Rep(G))$; denote χ_λ to be the image in $K(Rep(G))$ of the irreducible module with highest weight λ. We will also need the following notation.

Definition 1.2.20. Let $p', p : \Lambda \to \mathbb{Z}$ denote the Kostant partition function in types

B and C respectively, defined as follows:

$$\frac{1}{\prod_{i<j}(1-e^{\epsilon_i-\epsilon_j})(1-e^{\epsilon_i+\epsilon_j})\prod_i(1-e^{\epsilon_i})} = \sum_{\mu\in\Lambda} p'(\mu)e^{\mu}$$
$$\frac{1}{\prod_{i<j}(1-e^{\epsilon_i-\epsilon_j})(1-e^{\epsilon_i+\epsilon_j})\prod_i(1-e^{2\epsilon_i})} = \sum_{\mu\in\Lambda} p(\mu)e^{\mu}$$

Denote $\rho \in \Lambda^+$ to be the half-sum of the positive roots. Also for $\mu \in \Lambda$, denote $\mathrm{conv}(\mu)$ to be the intersection of Λ with the convex hull of the set $\{w\mu \mid w \in W\}$ in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ (here W denotes the Weyl group of G). Denote also $\mathrm{conv}^0(\lambda)$ be the complement of $\{w\lambda | w \in W\}$ in $\mathrm{conv}(\lambda)$.

Theorem 1.2.21. *For $\lambda, \mu \in \Lambda^+$,*

$$\mathit{dim}\ \mathit{Hom}_G(V_\mu, H^0(\widetilde{\mathfrak{N}}, \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda))) = \sum_{w\in W} \mathit{sgn}(w)p'(w(\mu+\rho)-(\lambda+\rho))$$

Proof. Using Theorem 1.2.15 and the additivity of the Euler characteristic, we can compute that:

$$\begin{aligned} H^0(\widetilde{\mathfrak{N}}, \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda)) &= \sum_{i\geq 0}(-1)^i H^i(\widetilde{\mathfrak{N}}, \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda)) = \sum_{i\geq 0}(-1)^i H^i(G/B, \mathcal{L}_{G/B}(\mathbb{C}^*_\lambda \otimes S(\mathbb{V}\oplus\mathfrak{s})^{+*}))) \\ &= \sum_{\mu\in\Lambda} p'(\mu)\sum_{i\geq 0}(-1)^i H^i(G/B, \mathcal{L}_{G/B}(\mathbb{C}^*_{\lambda+\mu})) \end{aligned}$$

Above we have used the filtration of $S((\mathbb{V}\oplus\mathfrak{s})^{+*})$ by 1-dimensional B-modules, where $\mathbb{C}^*_\mu$ occurs with multiplicity $p'(\mu)$. We continue using Borel-Weil-Bott Theorem, which states $\sum_{i\geq 0} H^i(G/B, \mathcal{L}_{G/B}(\mathbb{C}^*_\lambda)) = \mathrm{sgn}(w)\chi_{w(\lambda+\rho)-\rho}$, where $w \in W$ is the unique Weyl group element such that $w(\lambda+\rho)-\rho \in \Lambda^+$; this implies that $\sum_{i\geq 0}(-1)^i H^i(G/B, \mathcal{L}_{G/B}(\mathbb{C}^*_{\lambda+\mu})) = \mathrm{sgn}(w)\chi_\mu$ for $\mu \in \Lambda^+ \Leftrightarrow \mu = w^{-1}(\mu+\rho)-(\lambda+\rho)$. The result now follows:

$$\begin{aligned} H^0(\widetilde{\mathfrak{N}}, \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda))) &= \sum_{w\in W, \mu\in\Lambda^+} \mathrm{sgn}(w)p'(w^{-1}(\mu+\rho)-(\lambda+\rho))\chi_\mu \\ &= \sum_{\mu\in\Lambda^+}\chi_\mu \sum_{w\in W}\mathrm{sgn}(w)p'(w(\mu+\rho)-(\lambda+\rho)) \end{aligned}$$

□

Recall also the Weyl character formulae (see e.g. [33]):

Proposition 1.2.22. *For $\mu \in \Lambda^+$, let m^λ_μ denote the multiplicity of the weight λ in V_μ. Then*

$$m^\lambda_\mu = \sum_{w\in W} \mathit{sgn}(w)p(w(\mu+\rho)-(\lambda+\rho))$$

Proposition 1.2.23. *$Hom_G(V_\mu, H^0(\widetilde{\mathfrak{N}}, \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda))) = 0$ unless $\lambda \in conv(\mu)$.*

Proof. From Definition 1.2.20, first note the following identity:

$$\prod_i (1 + e^{\epsilon_i})(\sum_{\mu \in \Lambda} p(\mu) e^\mu) = \sum_{\mu \in \Lambda} p'(\mu) e^\mu$$

$$p'(\mu) = \sum_{S \subseteq \{1, \cdots, n\}} p(\mu - \sum_{i \in S} \epsilon_i)$$

$$\begin{aligned} \dim \mathrm{Hom}_G(V_\mu, H^0(\widetilde{\mathfrak{N}}, \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda))) &= \sum_{w \in W} \mathrm{sgn}(w) p'(w(\mu + \rho) - (\lambda + \rho)) \\ &= \sum_{w \in W, S \subseteq \{1, \cdots, n\}} \mathrm{sgn}(w) p(w(\mu + \rho) - (\lambda + \sum_{i \in S} \epsilon_i + \rho)) \\ &= \sum_{S \subseteq \{1, \cdots, n\}} m_\mu^{\lambda + \sum_{i \in S} \epsilon_i} \end{aligned}$$

It is well-known that $m_\mu^\lambda = 0$ unless $\lambda \in \mathrm{conv}(\mu)$ (since $m_\mu^\lambda = 0$ unless $\lambda \preccurlyeq \mu$, and $m_\mu^\lambda = m_\mu^{w\lambda}$ for $w \in W$). Since $\lambda \in \Lambda^+$, it then follows that for the above quantity to be nonzero, $\lambda \in \mathrm{conv}(\mu)$. □

1.3 A quasi-exceptional set generating $D^b(Coh^G(\mathfrak{N}))$

1.3.1 Recollections

For the reader's convenience, in this section we give a summary of the results in Section 1 of [13] (see also [47]) regarding quasi-exceptional sets and quasi-hereditary categories that will be relevant to us in the next section.

Let $\mathcal{C}$ be a triangulated category; for $C_1, C_2 \in \mathcal{C}$, denote

$$\mathrm{Hom}^\bullet(C_1, C_2) = \oplus_n \mathrm{Hom}(C_1, C_2[n])$$

(and $\mathrm{Hom}^{<0}(C_1, C_2) = \oplus_n \mathrm{Hom}(C_1, C_2[n])$). Given two sets $\mathcal{S}_1, \mathcal{S}_2$ of objects in $\mathcal{C}$, define $\mathcal{S}_1 \star \mathcal{S}_2$ to be the set of objects X for which there is a distinguished triangle $S_1 \to X \to S_2 \to S_1[1]$; the axioms of a triangulated category imply that this operation is associative. Given a set of objects $\mathcal{S}$ in $\mathcal{C}$, define $\langle \mathcal{S} \rangle = \mathcal{S} \cup \mathcal{S} \star \mathcal{S} \cup \mathcal{S} \star \mathcal{S} \star \mathcal{S} \cup \cdots$. Then $\langle \cup_i \mathcal{S}[i] \rangle$ is the smallest strictly full triangulated subcategory containing $\mathcal{S}$.

Definition 1.3.1. Let I be a totally ordered set. A dualizable quasi-exceptional set in $\mathcal{C}$ consists of a subset $\nabla = \{\nabla_i | i \in I\}$, and its dual $\Delta = \{\Delta_i | i \in I\}$, satisfying the following properties:

- $\mathrm{Hom}^\bullet(\nabla_i, \nabla_j) = 0$ if $i < j$.
- $\mathrm{Hom}^{<0}(\nabla_i, \nabla_i) = 0$ and $\mathrm{Hom}(\nabla_i, \nabla_i) = \mathbb{C}$ for all $i \in I$.

- $\mathrm{Hom}(\Delta_j, \nabla_i) = 0$ if $j > i$
- $\Delta_i \simeq \nabla_i (\mathrm{mod}\ \mathcal{C}/\mathcal{C}_{<i})$, where $\mathcal{C}_{<i}$ is the smallest strictly full triangulated subcategory containing the objects $\{\nabla_j | j < i\}$, and $\mathcal{C}/\mathcal{C}_{<i}$ is the Verdier quotient category.

Remark 1.3.2. In many examples that arise in practice (including the one discussed below), there is a contravariant exact involution "dual" on the category which sends Δ_i to ∇_i. However, this isn't part of the definition.

Proposition 1.3.3. *Suppose that $\{\nabla_i | i \in I\}$ generate $\mathcal{C}$. Then $\mathcal{C}$ has a unique t-structure $(\mathcal{C}^{\geq 0}, \mathcal{C}^{\leq 0})$ such that $\nabla_i \in \mathcal{C}^{\geq 0}, \Delta_i \in \mathcal{C}^{\leq 0}$. It is given by $\mathcal{C}^{\geq 0} = \langle \{\nabla_i[d], i \in I, d \leq 0\} \rangle, \mathcal{C}^{\leq 0} = \langle \{\Delta_i[d], i \in I, d \geq 0\} \rangle$.*

Proof. See Proposition 1 in [4]. □

We define a quasi-hereditary category in preparation for the next result, which shows that the heart of the above t-structure is quasi-hereditary.

Definition 1.3.4. Suppose an abelian category $\mathcal{A}$ has simple objects $\{S_i\}$, indexed by a totally ordered index set I. Then we say that A is quasi-hereditary if it satisfies the following properties:

- For each simple S_i, there is an object A_i with a non-zero morphism $\alpha : A_i \to S_i$, known as its "standard cover", such that:
  - $\ker(\alpha_i) \in \mathcal{A}_{<i}$
  - $\mathrm{Hom}(A_i, S_j) = \mathrm{Ext}^1(A_i, S_j) = 0$ for $j < i$.
- For each simple S_i, there is an object B_i with a non-zero morphism $\beta_i : S_i \to B_i$, known as its "costandard hull", such that
  - $\mathrm{coker}(\beta_i) \in \mathcal{A}_{<i}$.
  - $\mathrm{Hom}(S_j, B_i) = \mathrm{Ext}^1(S_j, B_i) = 0$ for $j < i$.

Denote by $\tau_{\geq 0}, \tau_{\leq 0}$ the truncation functors associated to the above t-structure. Let $\Delta_i' = \tau_{\geq 0}(\Delta_i), \nabla_i' = \tau_{\leq 0}(\nabla_i)$ be objects in the heart of the t-structure.

Proposition 1.3.5. *There exists a morphism $\theta_i : \Delta_i' \to \nabla_i'$, such that $S_i := im(\theta_i)$ is a simple object in $C^{\geq 0} \cap C^{\leq 0}$. Each simple object in $C^{\geq 0} \cap C^{\leq 0}$ is isomorphic to S_i for some i. The map $\Delta_i' \to S_i$ is a standard cover of S_i, and the inclusion $S_i \to \nabla_i'$ is a costandard hull for S_i. Thus $C^{\geq 0} \cap C^{\leq 0}$ is a quasi-hereditary category.*

1.3.2 Main Results

Following the techniques in [13], we may now prove analogous results about $\mathfrak{N}$ by applying the results in Section 1.2. However, we will need to use the following twisted action of W on Λ instead of the usual one:

Definition 1.3.6. Let $\theta = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n)$. Given $w \in W, \lambda \in \Lambda$, defined the twisted action $w \cdot \lambda = w(\lambda + \theta) - \theta$.

Definition 1.3.7. Let $D^b(Coh^G(\mathfrak{N}))$ denote the derived bounded category of G-equivariant coherent sheaves on $\mathfrak{N}$. For $\lambda \in \Lambda$, define $\mathbf{AJ}_\lambda = R\pi_* \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda)[d]$, where $d = \frac{\dim \mathfrak{N}}{2}$. Given $S \subset \Lambda$, denote by $\mathfrak{D}_S$ to be the smallest strictly full triangulated subcategory of $D^b(Coh^G(\mathfrak{N}))$ containing the objects $\mathbf{AJ}_\lambda$ for $\lambda \in S$.

Lemma 1.3.8 (Grothendieck-Serre Duality). *The (equivariant) dualizing sheaf on $\widetilde{\mathfrak{N}}$ is isomorphic to $\omega_{\widetilde{\mathfrak{N}}}[2d]$; the functor $\mathcal{S} : D^b(Coh^G(\mathfrak{N})) \to D^b(Coh^G(\mathfrak{N}))^{op}$ of Grothendieck-Serre duality is an anti-autoequivalence satisfying $\mathcal{S}(\mathbf{AJ}_\lambda) = \mathbf{AJ}_{-\lambda-2\theta}$.*

Proof. Since $H^i(\widetilde{\mathfrak{N}}, \omega_{\widetilde{\mathfrak{N}}}) = 0$ for $i > 0$ using 1.2.19, the equivariant dualizing sheaf on $\widetilde{\mathfrak{N}}$ is isomorphic to $\omega_{\widetilde{\mathfrak{N}}}[2d]$. Since $\omega_{\widetilde{\mathfrak{N}}} \simeq \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\epsilon_1 - \cdots - \epsilon_n)$ (again from 1.2.19), we have

$$R\underline{Hom}(\mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda)[d], \omega_{\widetilde{\mathfrak{N}}}[2d]) = \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\lambda - \epsilon_1 - \cdots - \epsilon_n)[d]$$

Recalling that by definition, $\mathcal{S}(\mathcal{F}) = R\underline{Hom}(\mathcal{F}, \mathbb{D}C)$, where $\mathbb{D}C$ is the equivariant dualizing sheaf; and that Grothendieck-Serre duality commutes with $R\pi_*$ since π is proper, the result follows. $\square$

Definition 1.3.9. For $\mu \in \Lambda$, let $\widetilde{conv}(\mu)$ denote the intersection of the convex hull of the set $\{w \cdot \mu | w \in W\}$ in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ with Λ, and denote by $\widetilde{conv}^0(\mu)$ to be the complement of $\{w \cdot \mu | w \in W\}$ in $\widetilde{conv}(\mu)$.

Proposition 1.3.10. *For $w \in W$, $\mathbf{AJ}_{w \cdot \lambda} \simeq \mathbf{AJ}_\lambda (mod\ \mathfrak{D}_{\widetilde{conv}^0(\lambda)})$.*

Proof. Let α be a simple root for G (i.e. $\alpha \in \{\epsilon_1 - \epsilon_2, \cdots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}$); let $\widetilde{\alpha} = \alpha$ if $\alpha \neq 2\epsilon_n$, and $\widetilde{\alpha} = \epsilon_n$ if $\alpha = 2\epsilon_n$. By induction on the length of the word $w \in W$, it suffices to prove that $\mathbf{AJ}_\lambda \simeq \mathbf{AJ}_{s_\alpha \cdot \lambda} (\text{mod } \mathfrak{D}_{\widetilde{conv}^0(\lambda)})$, for λ satisfying $\langle \lambda, \check{\alpha} \rangle = n > 0$. Let P_α be the minimal parabolic in G corresponding to the root α; denote by p_α the projection $G/B \to G/P_\alpha$. Since G is simply connected, we may choose $\lambda' \in \Lambda$ such that $\langle \lambda', \check{\alpha} \rangle = n - 1$. Define $\mathcal{V}_{\lambda'} = p_{\alpha,*} p_\alpha^* \mathcal{O}_{G/B}(\lambda')$ to be the G-equivariant vector bundle on G/B with a filtration whose subquotients are $\mathcal{O}_{G/B}(\lambda'), \cdots, \mathcal{O}_{G/B}(\lambda' - (n-1)\alpha)$. Let $(\mathbb{V} \oplus \mathfrak{s})^+_\alpha$ be the P_α-submodule of $(\mathbb{V} \oplus \mathfrak{s})^+$ obtained by taking the sum of all weight spaces excluding the weight space corresponding to $\widetilde{\alpha}$. Let $\widetilde{\mathfrak{N}}_\alpha = G \times_B (\mathbb{V} \oplus \mathfrak{s})^+_\alpha$, and $\iota_\alpha : \widetilde{\mathfrak{N}}_\alpha \hookrightarrow \widetilde{\mathfrak{N}}$ be the embedding of this divisor. Let $\pi_\alpha : \widetilde{\mathfrak{N}}_\alpha = G \times_B (\mathbb{V} \oplus \mathfrak{s})^+_\alpha \to G \times_{P_\alpha} (\mathbb{V} \oplus \mathfrak{s})^+_\alpha$ denote the projection, whose fibres are isomorphic to $\mathbb{P}^1$. Since

$\langle \lambda - \lambda' - \alpha, \check{\alpha} \rangle = -1$, the restriction of $p^*\mathcal{V}_{\lambda'}(\lambda - \lambda' - \alpha)$ is isomorphic to the sum of several copies of $\mathcal{O}_{\mathbb{P}^1}(-1)$ when restricted to any fibre of π_α. Thus we have:

$$\pi_{\alpha *}(\mathcal{O}_{\widetilde{\mathfrak{N}}_\alpha} \otimes p^*(\mathcal{V}_{\lambda'})(\lambda - \lambda' - \alpha)) = 0$$
$$\pi_*(\mathcal{O}_{\widetilde{\mathfrak{N}}_\alpha} \otimes p^*(\mathcal{V}_{\lambda'})(\lambda - \lambda' - \alpha)) = 0$$

Under the embedding of the divisor ι_α, $\mathcal{O}(\widetilde{\mathfrak{N}}_\alpha) = \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\widetilde{\alpha})$ (see Lemma 1.3.11 for a proof of this statement), giving us the exact sequence:

$$0 \to \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\widetilde{\alpha}) \to \mathcal{O}_{\widetilde{\mathfrak{N}}} \to \iota_{\alpha *}\mathcal{O}_{\widetilde{\mathfrak{N}}_\alpha} \to 0$$

Since $\pi_*(\mathcal{O}_{\widetilde{\mathfrak{N}}_\alpha} \otimes p^*(\mathcal{V}_{\lambda'})(\lambda - \lambda' - \alpha)) = 0$, tensoring this exact sequence with $p^*\mathcal{V}_{\lambda'}(\lambda - \lambda' - \alpha)$ and taking direct image under π, we have:

$$\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - \alpha + \widetilde{\alpha}) \simeq \pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - \alpha)$$

Case 1: Suppose $\alpha \neq 2\epsilon_n$. Then $\alpha = \widetilde{\alpha}$, so:

$$\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda') \simeq \pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - \alpha)$$

Since $\mathcal{V}_{\lambda'}(\lambda - \lambda')$ has a filtration whose quotients are $\mathcal{O}_{G/B}(\lambda), \cdots, \mathcal{O}_{G/B}(\lambda - (n-1)\alpha)$, we find that $\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda')[d] \in [\mathbf{AJ}_{\lambda-(n-1)\alpha}] * \cdots * [\mathbf{AJ}_\lambda]$. Since $\mathbf{AJ}_{\lambda-(n-1)\alpha}, \cdots, \mathbf{AJ}_{\lambda-\alpha} \in \mathfrak{D}_{\widehat{conv}^0(\lambda)}$, this implies $\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda')[d] \simeq \mathbf{AJ}_\lambda (\text{mod } \mathfrak{D}_{\widehat{conv}^0(\lambda)})$. Similarly, we find that $\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda')[d] \in [\mathbf{AJ}_{\lambda - n\alpha}] * \cdots * [\mathbf{AJ}_{\lambda-\alpha}]$, and hence $\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - \alpha)[d] \simeq \mathbf{AJ}_{\lambda - n\alpha} (\text{mod } \mathfrak{D}_{\widehat{conv}^0(\lambda)})$. Comparing, the required identity follows (note $s_\alpha \cdot \lambda = \lambda - n\alpha$ in this case): $\mathbf{AJ}_{s_\alpha \cdot \lambda} \simeq \mathbf{AJ}_\lambda (\text{mod } \mathfrak{D}_{\widehat{conv}^0(\lambda)})$. **Case 2:** Suppose $\alpha = 2\epsilon_n$; then $\widetilde{\alpha} = \epsilon_n$, so:

$$\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - \epsilon_n) \simeq \pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - 2\epsilon_n)$$

Note that $s_{2\epsilon_n} \cdot (\lambda - \epsilon_n) = \lambda - 2n\epsilon_n$. Similarly to the above, since $\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - \epsilon_n)[d] \in [\mathbf{AJ}_{\lambda-(2n-1)\epsilon_n}] * \cdots * [\mathbf{AJ}_{\lambda - \epsilon_n}]$, we find $\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - \epsilon_n)[d] \simeq \mathbf{AJ}_{\lambda - \epsilon_n} (\text{mod } \mathfrak{D}_{\widehat{conv}^0(\lambda - \epsilon_n)})$. Since $\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - 2\epsilon_n)[d] \in [\mathbf{AJ}_{\lambda - 2n\epsilon_n}] * \cdots * [\mathbf{AJ}_{\lambda - 2\epsilon_n}]$, we find $\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - 2\epsilon_n)[d] \simeq \mathbf{AJ}_{\lambda - 2n\epsilon_n} (\text{mod } \mathfrak{D}_{\widehat{conv}^0(\lambda - \epsilon_n)})$. Comparing, we see that $\mathbf{AJ}_{\lambda - \epsilon_n} \simeq \mathbf{AJ}_{s_{2\epsilon_n} \cdot (\lambda - \epsilon_n)} (\text{mod } \mathfrak{D}_{\widehat{conv}^0(\lambda - \epsilon_n)})$. Replacing $\lambda - \epsilon_n$ by λ, we get the result. $\square$

Lemma 1.3.11. *We have $\mathcal{O}(\widetilde{\mathfrak{N}}_\alpha) = \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\widetilde{\alpha})$.*

Proof. By definition, $\widetilde{\mathfrak{N}}$ is the total space of the vector bundle $\mathcal{L}_{G/B}((\mathbb{V} \oplus \mathfrak{s})^+)$. Consider the pull-back line bundle $\mathcal{E} = p^* \mathcal{L}_{G/B}((\mathbb{V} \oplus \mathfrak{s})^+)$ on $\widetilde{\mathfrak{N}}$; denote by τ the canonical section of $\Gamma(\widetilde{\mathfrak{N}}, \mathcal{E})$. The surjection of B-modules $(\mathbb{V} \oplus \mathfrak{s})^+ \to (\mathbb{V} \oplus \mathfrak{s})^+ / (\mathbb{V} \oplus \mathfrak{s})^+_{\tilde{\alpha}} \simeq \mathbb{C}_{\tilde{\alpha}}$ gives a map $\mathcal{L}_{G/B}(\mathbb{V} \oplus \mathfrak{s})^+ \to \mathcal{O}_{G/B}(-\tilde{\alpha})$, and hence a map $\phi_\alpha : \mathcal{E} \twoheadrightarrow \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\tilde{\alpha})$. Let $\tau_\alpha \in \Gamma(\widetilde{\mathfrak{N}}, \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\tilde{\alpha}))$ denote the image of τ. Then by construction $\widetilde{\mathfrak{N}}_\alpha$ is the zero set of the section τ_α. It follows that $\mathcal{O}_{\widetilde{\mathfrak{N}}}(-\tilde{\alpha}) = \mathcal{O}_{\widetilde{\mathfrak{N}}}(d\widetilde{\mathfrak{N}}_\alpha)$ for some $d \in \mathbb{Z}^+$; but since $\text{Pic}(\widetilde{\mathfrak{N}}) = \Lambda$ we find $d = 1$, as required. $\square$

Proposition 1.3.12. *1. For $\lambda \in \Lambda^+$, Hom$(\mathbf{AJ}_\lambda, \mathbf{AJ}_\lambda) = \mathbb{C}$ and Hom$^{<0}(\mathbf{AJ}_\lambda, \mathbf{AJ}_\lambda) = 0$.*

2. For $\lambda \in \Lambda^+$, we have Hom$^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda) = 0$ if $\lambda \notin \widetilde{conv}(\mu)$.

3. For $\lambda, \mu \in \Lambda^+$, $\lambda \neq \mu$, we have Hom$^\bullet(\mathbf{AJ}_{w_0 \cdot \mu}, \mathbf{AJ}_\lambda) = 0$.

Proof. (1) By Theorem 1.2.15, $\mathbf{AJ}_\lambda$ is concentrated in a single degree, so we have $\mathrm{Hom}^{<0}(\mathbf{AJ}_\lambda, \mathbf{AJ}_\lambda) = 0$. Since $\mathfrak{N}$ has an open, smooth G-orbit, and map $\phi : \mathbf{AJ}_\lambda \to \mathbf{AJ}_\lambda$ is determined by its restriction to the open orbit, where it must be a scalar, it follows that $\mathrm{Hom}(\mathbf{AJ}_\lambda, \mathbf{AJ}_\lambda) = \mathbb{C}$. (2) Fix λ. Assuming that for all $\mu' \in \Lambda^+ \cap \widetilde{conv}^0(\mu)$, $\mathrm{Hom}^\bullet(\mathbf{AJ}_{\mu'}, \mathbf{AJ}_\lambda) = 0$; it suffices to prove that $\mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda) = 0$. First assume $\mu \in \Lambda^+$ also. Noting that $V_\mu \otimes \mathcal{O}_{G/B} \in Coh^G(G/B)$ has a filtration where the subquotient $\mathcal{O}_{G/B}(\lambda)$ occurs with multiplicity m_μ^λ, if the weights of V_μ are $\nu_1, \cdots, \nu_k$ it follows that:

$$V_\mu \otimes \mathcal{O}_{\mathfrak{N}}[d] = \pi_* p^*(V_\mu \otimes \mathcal{O}_{G/B})[d] \in [\mathbf{AJ}_{\nu_1}^{m_\mu^{\nu_1}}] * [\mathbf{AJ}_{\nu_2}^{m_\mu^{\nu_2}}] * \cdots * [\mathbf{AJ}_{\nu_k}^{m_\mu^{\nu_k}}]$$

Given $\mathcal{A}, \mathcal{B} \in D^b(Coh^G(\mathfrak{N}))$, we view $\mathrm{Hom}^\bullet(\mathcal{A}, \mathcal{B})$ as an element of $D(\mathrm{Vect}_{\mathbb{C}})$. It follows now that:

$$\begin{aligned}&\mathrm{Hom}^\bullet(V_\mu \otimes \mathcal{O}_{\mathfrak{N}}[d], \mathbf{AJ}_\lambda)\\ &\in [\mathrm{Hom}^\bullet(\mathbf{AJ}_{\nu_1}^{m_\mu^{\nu_1}}, \mathbf{AJ}_\lambda)] * [\mathrm{Hom}^\bullet(\mathbf{AJ}_{\nu_2}^{m_\mu^{\nu_2}}, \mathbf{AJ}_\lambda)] * \cdots * [\mathrm{Hom}^\bullet(\mathbf{AJ}_{\nu_k}^{m_\mu^{\nu_k}}, \mathbf{AJ}_\lambda)]\end{aligned}$$

All $\nu_i \in \mathrm{conv}(\mu)$; by the induction hypothesis, if $\nu_i \in \mathrm{conv}^0(\mu) \subseteq \widetilde{conv}^0(\mu)$, then $\mathrm{Hom}^\bullet(\mathbf{AJ}_{\nu_i}, \mathbf{AJ}_\lambda) = 0$; thus, for $\mathrm{Hom}^\bullet(\mathbf{AJ}_{\nu_i}^{m_\mu^{\nu_i}}, \mathbf{AJ}_\lambda) \neq 0$, we require $\nu_i = w\mu$ for some $w \in W$ (note $m_\mu^{w\mu} = 1$ in this case). Either $w\mu \in \widetilde{conv}^0(\mu)$, or $w\mu = w \cdot \mu$. In the first case again we have $\mathrm{Hom}^\bullet(\mathbf{AJ}_{\nu_i}, \mathbf{AJ}_\lambda) = 0$ by the induction hypothesis. In the second case, by Proposition 1.3.10, $\mathbf{AJ}_{w\cdot\mu} \simeq \mathbf{AJ}_\mu(\mathrm{mod}\ \mathfrak{D}_{\widetilde{conv}^0(\mu)})$, and by the induction hypothesis, $\mathrm{Hom}^\bullet(\mathcal{A}_\mu, \mathbf{AJ}_\lambda) = 0$ for any $\mathcal{A}_\mu \in \mathfrak{D}_{\widetilde{conv}^0(\mu)}$, we may conclude $\mathrm{Hom}^\bullet(\mathbf{AJ}_{w\cdot\mu}, \mathbf{AJ}_\lambda) = \mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda)$. To summarize, $\mathrm{Hom}^\bullet(\mathbf{AJ}_{\nu_i}^{m_\mu^{\nu_i}}, \mathbf{AJ}_\lambda)$ is either 0 or $\mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda)$. So re-writing the above:

$$\mathrm{Hom}^\bullet(V_\mu \otimes \mathcal{O}_{\mathfrak{N}}[d], \mathbf{AJ}_\lambda) \in \mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda) * \cdots * \mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda)$$

However, using Theorems 1.2.15 and Proposition 1.2.23, note that (since $\lambda \in \widetilde{\mathrm{conv}}(\mu) \to \lambda \in \mathrm{conv}(\mu)$):

$$\begin{aligned}&\mathrm{Hom}^\bullet(V_\mu \otimes \mathcal{O}_{\mathfrak{N}}[d], \mathbf{AJ}_\lambda) = \mathrm{Hom}_G(V_\mu, R^\bullet\Gamma(\mathbf{AJ}_\lambda[-d])) = 0\\ &\Rightarrow 0 \in \mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda) * \cdots * \mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda)\end{aligned}$$

Using Lemma 5, on page 12 of [13], we now conclude that $\mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda) = 0$. Finally, as stated above, $\mathrm{Hom}^\bullet(\mathbf{AJ}_{w\cdot\mu}, \mathbf{AJ}_\lambda) = \mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda) = 0$; since any weight $\mu' \in \Lambda$ is of the form $w \cdot \mu$ for some $\mu \in \Lambda^+$, this removes the initial assumption that $\mu \in \Lambda^+$ and completing the proof of the statement. (3) If $\lambda \notin \widetilde{conv}(\mu)$, then

$\lambda \notin \widetilde{conv}(w_0 \cdot \mu)$, and the result follows from the previous part. So suppose that $\lambda \in \widetilde{conv}(\mu)$, i.e. that $\lambda + \theta \in \text{conv}(\mu + \theta)$; then $\mu \notin \widetilde{conv}(-\lambda - 2\theta)$. Recalling Lemma 1.3.8, from the previous part we deduce the required statement (noting that $w_0 \cdot \mu = w_0(\mu + \theta) - \theta = -\mu - 2\theta$ since $w_0\lambda = -\lambda$):

$$\text{Hom}^\bullet(\mathbf{AJ}_{w_0\cdot\mu}, \mathbf{AJ}_\lambda) = \text{Hom}^\bullet(\mathcal{S}(\mathbf{AJ}_\lambda), \mathcal{S}(\mathbf{AJ}_{w_0\cdot\mu})) = \text{Hom}^\bullet(\mathbf{AJ}_{-\lambda-2\theta}, \mathbf{AJ}_\mu) = 0$$

□

Lemma 1.3.13. *$D^b(Coh^G(\widetilde{\mathfrak{N}}))$ is generated as a triangulated category by the objects $\mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda)$ for $\lambda \in \Lambda$.*

Proof. It is well-known that $\text{Coh}^G(G/B) \simeq \text{Rep}(B)$, and is generated as a category by $\mathcal{O}_{G/B}(\lambda)$. Since $p : \widetilde{\mathfrak{N}} \to G/B$ gives $\widetilde{\mathfrak{N}}$ the structure of a G-equivariant vector bundle over G/B, using Lemma 6 in [13] (see also the last paragraph in page 266 of [29]), the required result follows. □

Lemma 1.3.14. *The image of the functor $R\pi_* : D^b(Coh^G(\widetilde{\mathfrak{N}})) \to D^b(Coh^G(\mathfrak{N}))$ generates $D^b(Coh^G(\mathfrak{N}))$ as a triangulated category.*

Proof. Let $\mathfrak{D}$ denote the triangulated subcategory of $D^b(Coh^G(\mathfrak{N}))$ generated by the image of $R\pi_*$. It suffices to show that any $\mathcal{F} \in Coh^G(\mathfrak{N})$ lies in $\mathfrak{D}$. Given this statement, it will then also follow that $\mathcal{F}[i] \in \mathfrak{D}$; since $\{\mathcal{F}[i] | \mathcal{F} \in Coh^G(\mathfrak{N}), i \in \mathbb{Z}\}$ generates $D^b(Coh^G(\mathfrak{N}))$, it would then follow that $\mathfrak{D} = D^b(Coh^G(\mathfrak{N}))$. We proceed by induction on the dimension of the support to show that an arbitrary $\mathcal{F} \in D^b(Coh^G(\mathfrak{N}))$ lies in $\mathfrak{D}$. It suffices to construct an $\tilde{\mathcal{F}} \in D^b(Coh^G(\widetilde{\mathfrak{N}}))$, and a morphism $\phi : \mathcal{F} \to R\pi_*\tilde{\mathcal{F}}$ such that the cone $\mathcal{G}$ of ϕ has smaller support than $\mathcal{F}$ (since $\mathcal{G} \in \mathfrak{D}$ by induction, and $R\pi_*\tilde{\mathcal{F}} \in \mathfrak{D}$, it would follow that $\mathcal{F} \in \mathfrak{D}$). Suppose that $\mathbb{O}_{\mu,\nu}$ is open in the support of $\mathcal{F}$, for some $(\mu, \nu) \in \mathcal{Q}_n$. Denote the inclusion $\iota_{\mu,\nu} : \widehat{\mathbb{O}_{\mu,\nu}} \hookrightarrow \widetilde{\mathfrak{N}}$. Let $\mathcal{F}' = (\pi \circ \iota_{\mu,\nu})^*\mathcal{F} \in Coh^G(\widehat{\mathbb{O}_{\mu,\nu}})$. We claim that $\tilde{\mathcal{F}} = \iota_{\mu,\nu*}\mathcal{F}'$ has the required property. First construct the map ϕ as the composition of the map $\mathcal{F} \to (\pi \circ \iota_{\mu,\nu})_*\mathcal{F}'$ (coming the adjointness of $(\pi \circ \iota_{\mu,\nu})^*$ and $(\pi \circ \iota_{\mu,\nu})_*$), with the map $(\pi \circ \iota_{\mu,\nu})_*\mathcal{F}' \to R\pi_*(\iota_{\mu,\nu*}\mathcal{F}') = R\pi_*\tilde{\mathcal{F}}$. Since the fibres of the map $\pi \circ \iota_{\mu,\nu} : \widehat{\mathbb{O}_{\mu,\nu}} \to \mathfrak{N}$ over the orbit $\mathbb{O}_{\mu,\nu}$ are acyclic (see Corollary 1.2.11), the projection formula gives that ϕ is an isomorphism when restricted to $\mathbb{O}_{\mu,\nu}$. Thus the cone of ϕ is supported on $\overline{\mathbb{O}_{\mu,\nu}} \setminus \mathbb{O}_{\mu,\nu}$, and has smaller dimension. □

Proposition 1.3.15. *The category $D^b(Coh^G(\mathfrak{N}))$ is generated as a triangulated category by the objects $\{\mathbf{AJ}_\lambda | \lambda \in \Lambda^+\}$.*

Proof. By applying the previous 2 lemmas, we find that (after a shift), $D^b(\text{Coh}^G(\mathfrak{N}))$ is generated by the objects $\{\mathbf{AJ}_\lambda | \lambda \in \Lambda\}$. The result will follow once we establish that $\mathfrak{D}_{\widetilde{conv}(\lambda)\cap\Lambda^+} = \mathfrak{D}_{\widetilde{conv}(\lambda)}$ for each $\lambda \in \Lambda^+$. Suppose that this is true for all $\lambda' \in \Lambda^+ \cap \widetilde{conv}^0(\lambda)$, then it follows that $\mathfrak{D}_{\widetilde{conv}^0(\lambda)\cap\Lambda^+} = \mathfrak{D}_{\widetilde{conv}^0(\lambda)}$. But since $\mathbf{AJ}_{w\cdot\lambda} \simeq \mathbf{AJ}_\lambda(\text{mod } \mathfrak{D}_{\widetilde{conv}^0(\lambda)\cap\Lambda^+})$ by Proposition 1.3.10, it follows that $\mathbf{AJ}_{w\cdot\lambda} \in \mathfrak{D}_{\widetilde{conv}(\lambda)\cap\Lambda^+}$; the result now follows. □

Definition 1.3.16. Denote $\Delta_\lambda = \nabla_{w_0 \cdot \lambda}$.

Theorem 1.3.17. *Fix any total order $\preceq$ on Λ^+, such that $\lambda \in \widetilde{conv}(\mu) \Rightarrow \lambda \preceq \mu$. Then the sets $\nabla = \{\nabla_\lambda | \lambda \in \Lambda^+\}$, and $\Delta = \{\nabla_{w_0 \cdot \lambda} | \lambda \in \Lambda^+\}$, respectively, are a quasi-exceptional set generating $D^b(Coh^G(\mathfrak{N}))$, and a dual quasi-exceptional set.*

Proof. By definition (see section 3.1), we must check the following:

- To check that ∇ is quasi-exceptional, note that $\mathrm{Hom}^\bullet(\nabla_\lambda, \nabla_{\lambda'}) = 0$ if $\lambda \preceq \lambda'$ (and are distinct), since in this case $\lambda' \notin \widetilde{conv}(\lambda)$ and we may apply Proposition 1.3.12; this proposition also implies that $\mathrm{Hom}^{<0}(\nabla_\lambda, \nabla_\lambda) = 0$ and $\mathrm{End}(\nabla_\lambda) = \mathbb{C}$.

- To check that Δ is a dual quasi-exceptional set, note $\mathrm{Hom}^\bullet(\Delta_\lambda, \nabla_{\lambda'}) = 0$ for $\lambda' \preceq \lambda$ using Proposition 1.3.12 (in fact, it is true provided $\lambda \neq \lambda'$); and that $\Delta_\lambda \simeq \nabla_\lambda (\mathrm{mod}\ \mathfrak{D}_{\prec\lambda})$ using Proposition 1.3.10.

- Using Proposition 1.3.15, one notes that ∇_λ generates $D^b(\mathrm{Coh}^G(\mathfrak{N}))$.

$\square$

Definition 1.3.18. Let $\mathfrak{D}^{q,\geq 0}, \mathfrak{D}^{q,\leq 0}$ denote the positive and negative subcategories corresponding to the t-structure on $D^b(Coh^G(\mathfrak{N}))$ given by the quasi-exceptional set $\{\nabla_\lambda | \lambda \in \Lambda^+\}$.

1.4 The perverse coherent t-structure

1.4.1 Recollections

For the reader's convenience, this section is a brief summary of some of the results in [11] (see also [9]) that we will need in the following section. Let X be an algebraic variety with an action of an algebraic group G; we will recount how to construct the "perverse coherent" t-structure on $D^b(Coh^G(X))$.

Definition 1.4.1. Let X^{top} denote the set of generic points of closed G-invariant subschemes in X. For $x \in X^{top}$, let $d(x)$ denote the Krull dimension of the subscheme $\overline{x}$. A perversity function p is a function $X^{top} \to \mathbb{Z}$; associated to p define the dual perversity function p' by $p'(x) = -\dim(x) - p(x)$. The function p is "monotone" (resp. "strictly monotone") if $p(x') \geq p(x)$ (resp $p(x') > p(x)$) $\forall\, x' \in \overline{x}$, and (strictly) "co-monotone" if p' is (strictly) monotone.

Definition 1.4.2. Define $D^{p,\leq 0}, D^{p \geq 0} \subset D^b(Coh^G(X))$ via:

$$\mathcal{F} \in D^{p,\geq 0} \text{ iff } \forall x \in X^{top}, i_x^!(\mathcal{F}) \in D^{\geq p(x)}(\mathcal{O}_x - mod)$$
$$\mathcal{F} \in D^{p,\leq 0} \text{ iff } \forall x \in X^{top}, i_x^*(\mathcal{F}) \in D^{\leq p(x)}(\mathcal{O}_x - mod)$$

Theorem 1.4.3. *If the perversity function p is monotone and co-monotone, then $(D^{p,\leq 0}, D^{p\geq 0})$ defines a t-structure on $D^b(Coh^G(X))$, which we call the perverse coherent t-structure.*

Proof. See Theorem 1 in [11]. $\square$

Proposition 1.4.4. *Suppose that the perversity function is strictly monotone and comonotone and G acts on X with finitely many orbits. Then the irreducible objects in the heart $\mathcal{P} = D^{p,\leq 0} \cap D^{p\geq 0}$ of the perverse t-structure are parametrized by a G-orbit $\mathcal{O}$ on X, and an irreducible G-equivariant vector bundle $\mathcal{L}$ on $\mathcal{O}$; the corresponding object is denoted $j_{!*}\mathcal{L}[p(\mathcal{O})]$. Further, $\mathcal{P}$ is Artinian.*

Proof. See Corollaries 4 and 5 in [11]. $\square$

1.4.2 Comparing the quasi-exceptional and perverse coherent t-structures on $D^b(\mathrm{Coh}^G(\mathfrak{N}))$

Since all orbits in the exotic nilpotent cone have even dimension, we may consider the perverse coherent t-structure on $D^b(\mathrm{Coh}^G(\mathfrak{N}))$ arising from the middle perversity (i.e. for the orbit $\mathbb{O}_{\mu,\nu} \subset \mathfrak{N}, p(\mathbb{O}_{\mu,\nu}) = -\frac{\dim(\mathbb{O}_{\mu,\nu})}{2}$); let $\mathfrak{D}^{p,\geq 0}, \mathfrak{D}^{p,\leq 0}$ denote the negative and positive subcategories as defined above, and let $\mathcal{P} = \mathfrak{D}^{p,\geq 0} \cap \mathfrak{D}^{p,\leq 0}$ denote the core of this t-structure. The goal of this section is to prove that the perverse coherent t-structure on $D^b(\mathrm{Coh}^G(\mathfrak{N}))$ coincides with the t-structure arising from the quasi-exceptional set ∇.

Proposition 1.4.5. *$AJ_\lambda \in \mathcal{P}$ for all $\lambda \in \Lambda$.*

Proof. Using Lemma 1.3.8, $\mathcal{S}(\mathbf{AJ}_\lambda) = \mathbf{AJ}_{-\lambda-\epsilon_1-\cdots-\epsilon_n}$, and the perverse coherent t-structure is self-dual with respect to Grothendieck-Serre duality, it suffices to prove just the first of two conditions defining the perverse coherent t-structure (here $\iota_{gen,\mu,\nu}$: $(\mathbb{O}_{\mu,\nu})_{\mathrm{gen}} \hookrightarrow \mathfrak{N}$ denotes the inclusion of the generic point of the orbit $\mathbb{O}_{\mu,\nu}$):

$$\iota^*_{\mathrm{gen},\mu,\nu}(\mathbf{AJ}_\lambda) \in \mathfrak{D}^{\leq p(\mathbb{O}_{\mu,\nu})}(((\mathbb{O}_{\mu,\nu})_{\mathrm{gen}}) - mod)$$
$$\iota^*_{\mathrm{gen},\mu,\nu}(R\pi_*\mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda)) \in \mathfrak{D}^{\leq p(\mathbb{O}_{\mu,\nu})+d}(((\mathbb{O}_{\mu,\nu})_{\mathrm{gen}}) - mod)$$

Note first that $\pi : \widetilde{\mathfrak{N}} \to \mathfrak{N}$ is a semi-small resolution of singularities, i.e. $\dim(\pi^{-1}(x)) \leq \frac{1}{2}\mathrm{codim}(\mathbb{O}_{\mu,\nu}) = p(\mathbb{O}_{\mu,\nu}) + d$. Note also that from [32], Chapter 3, Corollary 11.2, if $f : X \to Y$ is a projective morphism of Noetherian schemes, such that $\dim(f^{-1}(y)) \leq r$ for all $y \in Y$, then for $i > r$, $R^i f_*\mathcal{F} = 0$ for all $\mathcal{F} \in \mathrm{Coh}(X)$. Since $\iota^*_{\mathrm{gen},\mu,\nu}$ is an exact functor, the required result follows. $\square$

Proposition 1.4.6. *The perverse coherent t-structure on $D^b(Coh^G(\mathfrak{N}))$ coincides with the quasi-exceptional t-structure corresponding to the set $\{\nabla_\lambda | \lambda \in \Lambda\}$.*

Proof. Above we have proven that $\nabla_\lambda \in \mathfrak{D}^{p,\geq 0}$, so it follows that $\nabla_\lambda[d] \in \mathfrak{D}^{p,\geq 0}$ for $d \leq 0$. Since $\mathfrak{D}^{q,\geq 0}$ is generated by the objects $\nabla_\lambda[d]$, it follows that $\mathfrak{D}^{q,\geq 0} \subseteq \mathfrak{D}^{p,\geq 0}$. Since the above proof also gives us $\Delta_\lambda \in \mathfrak{D}^{p,\leq 0}$, it follows that $\Delta_\lambda[d] \in \mathfrak{D}^{p,\leq 0}$ for $d \geq 0$; hence $\mathfrak{D}^{q,\leq 0} \subseteq \mathfrak{D}^{p,\leq 0}$. It now follows from the axioms of a t-structure that we have equality in the above inclusions, i.e. two t-structures coincide. □

Definition 1.4.7. Let $\mathbb{O}$ denote the set of pairs $(\mathbb{O}_{\mu,\nu}, V)$ of an orbit and an irreducible representation V of the isotropy group $G^{v_{\mu,\nu}}$, where $v_{\mu,\nu} \in \mathbb{O}_{\mu,\nu}$.

Proposition 1.4.8. *The irreducibles in $\mathcal{P}$ are indexed by $\mathbb{O}$. The bijection between costandard objects and irreducible objects in $\mathcal{P}$ gives a bijection $\Theta : \lambda \to (\mathbb{O}_{\lambda^{(1)},\lambda^{(2)}}, L_\lambda)$ between Λ^+ and $\mathbb{O}$.*

Proof. The irreducible objects in the heart $\mathcal{P}$ of the perverse coherent t-structure are parametrized by an irreducible G-equivariant vector bundle $\mathcal{L}$ on an orbit $\mathbb{O}_{\mu,\nu}$, with the corresponding perverse coherent sheaf being given by $j_{!*}\mathcal{L}[-\frac{\dim(\mathbb{O}_{\mu,\nu})}{2}]$. If $x \in \mathbb{O}_{\mu,\nu}$, then $\mathbb{O}_{\mu,\nu} \simeq G/G^x$; and the irreducible G-equivariant vector bundles are given by $G \times_{G^x} V$, where V is an irreducible representation of the isotropy group G^x. Let $IC_{\mathbb{O}_{\mu,\nu},V} := j_{!*}(G \times_{G^x} V)[-\frac{\dim(\mathbb{O}_{\mu,\nu})}{2}]$. Thus, using Proposition 1.3.5, we obtain the desired conclusion. □

We deduce from the above that this bijection has the following properties:

Corollary 1.4.9. *The above bijection has the following properties, any of which uniquely characterize it.*

- $Hom(IC_{\mathbb{O}_{\lambda^{(1)},\lambda^{(2)}},L_\lambda}, \nabla_\lambda) \neq 0$
- *There exists a morphism* $IC_{\mathbb{O}_{\lambda^{(1)},\lambda^{(2)}},L_\lambda} \to \nabla_\lambda$ *whose cone lies in* $\mathfrak{D}_{\widetilde{conv^0(\lambda)}}$.
- $Hom(\nabla_{w_0\cdot\lambda}, IC_{\mathbb{O}_{\lambda^{(1)},\lambda^{(2)}},L_\lambda}) \neq 0$.
- *There exists a morphism* $\nabla_{w_0\cdot\lambda} \to IC_{\mathbb{O}_{\lambda^{(1)},\lambda^{(2)}},L_\lambda}$ *whose cone lies in* $\mathfrak{D}_{\widetilde{conv^0(\lambda)}}$.
- $IC_{\mathbb{O}_{\lambda^{(1)},\lambda^{(2)}},L_\lambda} \in \mathfrak{D}_{\widetilde{conv^0(\lambda)}}$.

1.5 Further directions

1.5.1 An explicit combinatorial bijection

Thus far we have constructed an abstract bijection between $\mathbb{O}$ and Λ^+ using geometric techniques. One may ask whether it is possible to give a combinatorial description of this bijection. In the case of the bijection established in [13] for a simple algebraic group H between $\mathbf{O}$ and Λ^+, an explicit combinatorial description has been given by Achar in [2] (see also [1]). While partial progress has been made in [1] to give a combinatorial description of the Bezrukavnikov's bijection in type C, outside of type A a complete explicit description of the bijection is, to the extent of our knowledge, not known.

One may give an explicit description of $\mathbb{O}$ as follows using work of Sun, [52]. Given $(\mu,\nu) \in \mathcal{Q}_n$, pick $(v,x) \in \mathbb{O}_{\mu,\nu}$. Following [52], the Levi component $K^{(v,x)}$ of $G^{(v,x)}$ is given as follows. Following Notation 3 on page 5 of [52], denote $\mu = (j_h^{n_{l_h}}), \nu = (k_h^{n_{l_h}})$ (such that for each h, either $j_h > j_{h+1}, k_h > k_{h+1}$, or both), and define $J = \{h | j_h > j_{h+1}, k_{h-1} > k_h\}$ .Let $n'_{l_h} = n_{l_h}$ if $h \notin J$ and $n'_{l_h} = n_{l_h} - 1$ if $h \in J$. Then Theorem 3.10 in [52] states that:

$$K^{(v,x)} = \prod_{h \in J} Sp_{2n'_{l_h}}(\mathbb{C})$$

Thus we can describe $\mathbb{O}$ as $\{(\mu,\nu) \in \mathcal{Q}_n, \underline{\lambda} | \underline{\lambda} = \{\underline{\lambda}_i\}_{i \in J}, \underline{\lambda}_i = (\underline{\lambda}_i^t)_{1 \le t \le n'_{l_h}}, \underline{\lambda}_i^t \ge \underline{\lambda}_i^{t+1}, \underline{\lambda}_i^t \in \mathbb{Z}_{\ge 0}\}$. It would be interesting to give a combinatorial description of the constructed bijection $\Theta : \mathbb{O} \leftrightarrow \Lambda^+$, following the techniques used by Achar in [1] and [2].

1.5.2 Canonical bases in equivariant K-theory

Consider the setup in the case of the ordinary nilpotent cone, $\mathcal{N}$ for a simple algebraic group H. In [10], an exotic t-structure is constructed on the category $D^b(Coh^G(\tilde{\mathcal{N}}))$ (and also on $D^b(Coh^{G \times \mathbb{C}^\times}(\tilde{\mathcal{N}}))$). Denote by $\boldsymbol{\pi}, \boldsymbol{\Lambda}, \boldsymbol{\Theta}$, etc., the analogues of the π, Λ, Θ, etc, in the above setting (with the nilpotent cone $\mathcal{N}$ of a simple algebraic group H, instead of the exotic nilpotent cone $\mathfrak{N}$). It is proved in Section 2.4 of [10] that:

Proposition 1.5.1. *The functor $R\boldsymbol{\pi}_*$ is t-exact with respect to the exotic t-structure on $D^b(Coh^G(\tilde{\mathcal{N}}))$, and the perverse coherent t-structure on $D^b(Coh^G(\mathcal{N}))$. The irreducible objects in the exotic t-structure on $\mathcal{N}$ are indexed by $\boldsymbol{\Lambda}$; denote them by $\{E_{\boldsymbol{\lambda}} | \boldsymbol{\lambda} \in \boldsymbol{\Lambda}\}$. Then $R\pi_*(E_{\boldsymbol{\lambda}}) = 0$ if $\boldsymbol{\lambda} \notin -\boldsymbol{\Lambda}^+$, and $R\pi_*(E_{\boldsymbol{\lambda}}) = IC_{\boldsymbol{\Theta}(\boldsymbol{w_0\lambda})}$ if $\boldsymbol{\lambda} \in -\boldsymbol{\Lambda}^+$.*

Recall that the Steinberg variety is defined to be $\mathbf{St} := \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$. It is well-known that the equivariant K-theory of the Steinberg variety gives us the affine Hecke algebra $\mathbf{H}$ associated to the simple algebraic group H (i.e. $K^{G \times \mathbb{C}^\times}(\mathbf{St}) = \mathbf{H}$; see Theorem 7.2.5 in [29]). The two projection maps $\boldsymbol{pr}_1, \boldsymbol{pr}_2 : \mathbf{St} = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \to \mathcal{N}$ give maps $\boldsymbol{pr}_{1*}, \boldsymbol{pr}_{2*} : K^{G \times \mathbb{C}^\times}(\mathbf{St}) \to K^{G \times \mathbb{C}^\times}(\tilde{\mathcal{N}})$, while the map $\boldsymbol{\pi} : \tilde{\mathcal{N}} \to \mathcal{N}$ induces the map $\boldsymbol{\pi}_* : K^{G \times \mathbb{C}^\times}(\tilde{\mathcal{N}}) \to K^{G \times \mathbb{C}^\times}(\mathcal{N})$. T'e projection $\boldsymbol{p} : \mathbf{St} \to \mathcal{N}$ now induces a map $\boldsymbol{p}_* : K^{G \times \mathbb{C}^\times}(\mathbf{St}) \to K^{G \times \mathbb{C}^\times}(\mathcal{N})$. In $K^{G \times \mathbb{C}^\times}(\mathcal{N})$, it was conjectured by Ostrik ([45]), that one has a canonical basis consisting of IC coherent sheaves indexed by $\mathbf{O}$, while in $K^{G \times \mathbb{C}^\times}(\mathbf{St}) = \mathbf{H}$ one has the Kazhdan-Lusztig bases $\{C_w | w \in W_{aff}\}$, where W_{aff} is the extended affine Weyl group. This was proven by Bezrukavnikov in [12] and [13]. The following statement is proven in Corollary 1 of [12]:

Proposition 1.5.2. *Let W_{aff}^{fin} denote the set of minimal length representatives of 2-sided cosets of the finite Weyl group W_{fin} in W_{aff}. Then if $w \notin W_{aff}^{fin}$, $p_*(C_w) = 0$,* *while if $w \in W_{aff}^{fin}$, then $p_*(C_w)$ is the class of an irreducible IC coherent sheaf in* $K^{G\times\mathbb{C}^\times}(\mathcal{N})$.

One may ask whether the above story carries over to the set-up of the exotic nilcone, $\mathfrak{N}$. First, we expect that an analogue of Proposition 1.5.1 holds in the exotic setting.

Define the exotic Steinberg variety to be $\mathrm{St} := \widetilde{\mathfrak{N}} \times_{\mathfrak{N}} \widetilde{\mathfrak{N}}$. Denote by $\mathfrak{N}_2$ the Hilbert nullcone of the representation $\mathbb{V}^{\oplus 2} \oplus \mathfrak{s}$, and let $\widetilde{\mathfrak{N}_2} := G \times_B (\mathbb{V}^{\oplus 2} \oplus \mathfrak{s})^+$; then we have a resolution map $\pi_2 : \widetilde{\mathfrak{N}_2} \to \mathfrak{N}_2$. Kato proves in Theorem 2.8 of [36] that $K^{G\times(\mathbb{C}^\times)^2}(\widetilde{\mathfrak{N}_2} \times_{\mathfrak{N}_2} \widetilde{\mathfrak{N}_2}) = \tilde{\mathbb{H}}$, a multi-parameter affine Hecke algebra associated to G (see Definition 2.1 in [36]). Using the techniques employed in Section 2 of [36], we can give a similar description of $K^{G\times\mathbb{C}^\times}(\mathrm{St})$. From Section 4, we have a basis of $K^{G\times\mathbb{C}^\times}(\mathfrak{N})$, indexed by $\mathbb{O}$, consisting of IC coherent sheaves. The natural map $p : \mathrm{St} \to \mathfrak{N}$ now induces a map $p_* : K^{G\times\mathbb{C}^\times}(\mathrm{St}) \to K^{G\times\mathbb{C}^\times}(\mathfrak{N})$. It would be interesting to construct a Kazhdan-Lusztig type basis of $\mathbb{H}$, satisfying the properties similar to that specified in Proposition 1.5.2.

Another interesting question is to see whether Achar's results (see [4]), a version of the result proven in [13] but in positive characteristic, continue to hold for the exotic nilpotent cone.

Chapter 2

Exotic t-structures for two-block Springer fibres

2.1 Introduction

Let G be a semi-simple Lie group, with Lie algebra $\mathfrak{g}$, flag variety $\mathcal{B}$ and nilpotent cone $\mathcal{N}$. It is well-known that there is a natural map $\pi : T^*\mathcal{B} \to \mathcal{N}$ which is a resolution of singularities (known as the Springer resolution). Given $e \in \mathcal{N}$, let $\mathcal{B}_e = \pi^{-1}(e)$; these varieties are known as Springer fibers, and are of special interest in representation theory. For instance, in type A, Springer showed that the top cohomology of a Springer fiber can be equipped with a representation of the Weyl group, and further realizes an irreducible representation.

This special case when $G = SL(m + 2n)$, and the nilpotent e has Jordan type $(m + n, n)$, is easier to understand, and has been studied extensively. In [51], Stroppel and Webster study the geometry and combinatorics of these "two-block Springer fibers" and investigate connections with Khovanov's arc algebras. In [48], Russell studies the topology of these varieties, and describes a certain basis in the Springer representation.

In [16], Bezrukavnikov and Mirkovic introduce "exotic t-structures" on derived categories of coherent sheaves on Springer theoretic varieties, in order to study the modular representation theory of $\mathfrak{g}$. These exotic t-structures are defined using a certain action of the affine braid group $\mathbb{B}_{aff}$ on these categories, which was defined by Bezrukavnikov and Riche (see [18]).

Here we will study exotic t-structures for two-block Springer fibers (ie. for a nilpotent of Jordan type $(m+n, n)$ in type A), and give a description of the irreducible objects in the heart of the t-structure. First we will give some motivation for studying exotic t-structures; then we describe the contents of this chapter in more detail. This chapter is joint work with Rina Anno, and is based on her earlier pre-print [6].

2.1.1 Motivation:

Let $\mathbf{k}$ be an algebraically closed field of characteristic p with $p > h$ (here h is the Coxeter number). Let $\lambda \in \mathfrak{h}_{\mathbf{k}}$ be integral and regular; and let $e \in \mathcal{N}(\mathbf{k})$ be a nilpotent. Let $\mathrm{Mod}_e^{fg,\lambda}(U_{\mathbf{k}})$ be the category of modules with generalized central character (λ, e). Theorem 5.3.1 from [17] (see also Section 1.6.2 from [16]) states that there is an equivalence:

$$D^b(\mathrm{Coh}_{\mathcal{B}_{e,\mathbf{k}}}(\widetilde{\mathfrak{g}}_{\mathbf{k}})) \simeq D^b(\mathrm{Mod}_e^{fg,\lambda}(U_{\mathbf{k}})) \tag{2.1}$$

Further, it is proven that the tautological t-structure on the derived category of modules, corresponds to the exotic t-structure on the derived category of coherent sheaves.

Lusztig's conjecture states that there is a natural identification between $K^0(U\mathfrak{g}-\mathrm{mod}_e^{\lambda})$ and $K^0(\mathrm{Coh}(\mathcal{B}_e))$, under which the classes of the irreducible objects in the former category are mapped to certain canonical bases elements in the latter. To define the canonical basis, one needs to work in the category $K^0(\mathrm{Coh}^{\mathbb{C}^*}(\mathcal{B}_e))$.The canonical bases in question is characterized by the property that it is the unique basis (up to sign) which is invariant under a certain bar involution, and is orthogonal under a certain Ext pairing.

These conjectures are proven (for p sufficiently large) in Section 5 of [16]. A certain equivalence between these categories and perverse sheaves on the affine flag variety for the Langlands dual group plays a vital role here.

2.1.2 Summary

<u>**Two-block Springer fibers:**</u> In this section, we recall the definition and some properties of two-block Springer fibers, and define the categories that we will be studying. Let $m \geq 0$ be fixed; and let $n \in \mathbb{Z}_{\geq 0}$ vary. Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_{m+2n}$, and denote the nilpotent cone of $\mathfrak{sl}_{m+2n}$ (the variety consisting of nilpotent matrices of size $m + 2n$) by $\mathcal{N}_n$. Denote by z_n the standard nilpotent of type $(m + n, n)$:

$$z_n = \begin{pmatrix} & 1 & & & & & & \\ & & \cdots & & & & & \\ & & & 1 & & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ & & & & & 1 & & \\ & & & & & & \cdots & \\ & & & & & & & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Let $\mathcal{B}_n$ be the flag variety for GL_{m+2n}. The Springer resolution is $T^*\mathcal{B}_n$:

$$\mathcal{B}_n = \{(0 \subset V_1 \subset \cdots \subset V_{m+2n} = \mathbb{C}^{m+2n}) \mid \dim V_i = i\}$$
$$T^*\mathcal{B}_n = \{(0 \subset V_1 \subset \cdots \subset V_{m+2n} = \mathbb{C}^{m+2n}, x) \mid x \in \mathfrak{sl}_{m+2n}, xV_i \subset V_{i-1}\}$$

The natural projection $\pi_n : T^*\mathcal{B}_n \to \mathcal{N}_n$ is a resolution of singularities. The two-block Springer fiber is the variety

$$\mathcal{B}_{z_n} = \pi_n^{-1}(z_n) = \{(0 \subset V_1 \subset \cdots \subset V_{m+2n}) \in \mathcal{B}_n \mid z_n V_i \subseteq V_{i-1}\}$$

The Mirkovic-Vybornov transverse slices $S_n \subset \mathfrak{g}$ is a variant of the Slodowy slice. The following variety is of interest, since it is a resolution of $S_n \cap \mathcal{N}$.

$$\begin{aligned} U_n &= \pi_n^{-1}(S_n) \subset T^*\mathcal{B}_n \\ &= \{(0 \subset V_1 \subset \cdots \subset V_{m+2n} = \mathbb{C}^{m+2n}, x) \mid x \in S_n, xV_i \subset V_{i-1}\} \end{aligned}$$

Let $\mathcal{D}_n = D^b(\mathrm{Coh}_{\mathcal{B}_{z_n}}(U_n))$ be the bounded derived category of coherent sheaves on U_n, which are supported on $\mathcal{B}_{z_n}$. These are the categories that we will be studying.

Affine tangles: In this section, we recall the definition, and some properties, of affine tangles.

Definition. Let p, q be positive integers of the same parity. A (p,q) affine tangle is an embedding of $\frac{p+q}{2}$ arcs and a finite number of circles into the region $\{(x,y) \in \mathbb{C} \times \mathbb{R} | 1 \leq |x| \leq 2\}$, such that the end-points of the arcs are:

$$(1,0), (\zeta_p, 0), \cdots, (\zeta_p^{p-1}, 0), (2,0), (2\zeta_q, 0), \cdots, (2\zeta_q^{q-1}, 0)$$

in some order (where $\zeta_k = e^{\frac{2\pi i}{k}}$).

Definition. Let **ATan** be the category with objects $\{k\}$ for $k \in \mathbb{Z}_{\geq 0}$, and the morphisms between p and q consist of all affine (p,q) tangles (up to isotopy).

The above definition is consistent, since a (p,q) affine tangle α, and a (q,r) affine tangle β, then $\beta \circ \alpha$ is a (p,r) affine tangle.

We recall the well-known presentation of this category using generators and relations. The generators consist of "cups", g_n^i, which are $(n-2,n)$ tangles; "caps", f_n^i, which are $(n,n-2)$ tangles, "crossings", $t_n^i(1), t_n^i(2)$ and rotations r_n, r_n', which are (n,n) tangles. The relations are listed in Definition 2.3.7. In this paper we work with the category **AFTan** of affine framed tangles that has additional generators $w_n^i(1)$ and $w_n^i(2)$ that twist the framing of the ith strand.

Functors associated to affine tangles:

Definition. Let $\mathbf{AFTan}_m$ be the full subcategory of **AFTan**, containing the objects $\{m+2n\}$ for $n \in \mathbb{Z}_{\geq 0}$. A "weak representation" of the category $\mathbf{AFTan}_m$ is an assignment of a triangulated category $\mathcal{C}_n$ for each $n \in \mathbb{Z}_{\geq 0}$, and a functor $\Psi(\alpha)$:

$\mathcal{D}_p \to \mathcal{D}_q$ for each affine framed $(m+2p, m+2q)$ tangle α, such that the relations between tangles hold for these functors: i.e. if β is an $(m+2q, m+2r)$-tangle, then there is an isomorphism $\Psi(\beta) \circ \Psi(\alpha) \simeq \Psi(\beta \circ \alpha)$.

The main result of Section is a construction of a weak representation of $\mathbf{AFTan}_m$ using the categories $\mathcal{D}_n$ above. To do this, we mimic the strategy used by Cautis and Kamnitzer in [27], where they construct a weak representation of the category **OTan** of oriented (non-affine) tangles, using slightly larger categories.

The exotic t-structure on $\mathcal{D}_n$: In this section, we recall the definition of exotic t-structures (introduced by Bezrukavnikov and Mirkovic in [16]), and describe how they are related to the action of affine tangles constructed above.

Let $\mathbb{B}_{aff}$ be the affine braid group. As a special case of the construction in Section 1 of [16] (see also Bezrukavnikov-Riche, [18]), we have an action of $\mathbb{B}_{aff}$ on $\mathcal{D}_n$ (ie. for every $b \in \mathbb{B}_{aff}$, there exists a functor $\Psi(b) : \mathcal{D}_n \to \mathcal{D}_n$, and an isomorphism $\Psi(b_1 b_2) \simeq \Psi(b_1) \circ \Psi(b_2)$ for $b_1, b_2 \in \mathbb{B}_{aff}$). It turns out that $\mathbb{B}_{aff}$ can be identified as a subgroup of the monoid of $(m+2n, m+2n)$-tangles; and under this identification, the action of $\mathbb{B}_{aff}$ coincides with the action constructed above.

Let $\mathbb{B}^+_{aff} \subset \mathbb{B}_{aff}$ be the semigroup generated by the lifts of the simple reflections $\tilde{s}_\alpha$ in the Coxeter group W^{Cox}_{aff}. Bezrukavnikov-Mirkovic's construction in [16] specializes to give an exotic t-structure on $\mathcal{D}_n$, which is defined as follows:

$$\mathcal{D}_n^{\geq 0} = \{\mathcal{F} \mid R\Gamma(\Psi(b^{-1})\mathcal{F}) \in D^{\geq 0}(\mathrm{Vect}) \; \forall \; b \in \mathbb{B}^+_{aff}\}$$
$$\mathcal{D}_n^{\leq 0} = \{\mathcal{F} \mid R\Gamma(\Psi(b)\mathcal{F}) \in D^{\leq 0}(\mathrm{Vect}) \; \forall \; b \in \mathbb{B}^+_{aff}\}$$

We also prove that the "cup" functors $\Psi(g_n^i)$ are exact with the exotic t-structures, and send irreducible objects to irreducible objects (Theorem 2.5.6).

Irreducible objects in the heart of the exotic t-structure on $\mathcal{D}_n$: In this section, we give a description of the irreducible objects in the exotic t-structure on $\mathcal{D}_n$, and compute the Ext spaces between them.

Let Cross(n) be the set of affine $(m, m+2n)$ tangles, where the m inner points are not labelled, the $m+2n$ outer points are labelled, and whose vertical projections to $\mathbb{C}$ do not have crossings. For every $\alpha \in Cross(n)$ we have a functor $\Psi(\alpha) : \mathcal{D}_0 \to \mathcal{D}_n$; let $\Psi_\alpha = \Psi(\alpha)(\underline{\mathbb{C}})$ (here $\underline{\mathbb{C}} \in D^b(\mathrm{Vect}) \simeq \mathcal{D}_0$). We show that that $\{\Psi_\alpha \,|\, \alpha \in \mathrm{Cross}(n)\}$ constitute the irreducible objects in $\mathcal{D}_n^0$ (Proposition 2.5.10).

We also prove that for $\beta \in \mathrm{Cross}(n)$, $\mathrm{Ext}^\bullet(\Psi_\alpha, \Psi_\beta)$ is given by the below formula. Here Λ denotes a complex in $D^b(\mathrm{Vect})$ concentrated in degrees 1 and -1; and $\check{\alpha}$ is the $(m+2n, m)$ affine tangle obtained by "inverting" α. An a (m, m) affine tangle γ with no crossings is said to be "good" if it has no cups or caps, and $\omega(\gamma)$ denote the number of circles present. In Theorem 2.5.15, we prove that

$$\mathrm{Ext}^\bullet(\Psi_\alpha, \Psi_\beta) = \begin{cases} \Lambda^{\otimes \omega(\check{\alpha}\circ\beta)}[-n] & \text{if } \check{\alpha}\circ\beta \text{ is good} \\ 0 & \text{otherwise} \end{cases}$$

We also give a conjectural description of the multiplication in the algebra

$$\mathrm{Ext}^\bullet\Big(\bigoplus_{\alpha\in\mathrm{Cross}(n)} \Psi_\alpha\Big)$$

__Further directions:__ In the equivalence (2.1), the heart of the exotic t-structure is identified with an abelian category of modules over $U\mathfrak{g}$ having a fixed central character. Thus the simple objects that we have classified in the heart of the exotic t-structure will correspond to irreducible representations with that fixed central character. In future work, we plan to study these modules (e.g. compute dimensions, and give character formulaes) by using our description of these exotic sheaves.

Using techniques developed by Cautis and Kamnitzer, we can show the Grothendieck group of the category $\mathcal{D}_n$ can be naturally identified with $V_{[m]}^{\otimes m+2n}$, the m-weight space in $V^{\otimes m+2n}$ (here $V = \mathbb{C}^2$, considered as an $\mathfrak{sl}_2$ representation). By looking at the images of the functors $\Psi(\alpha)$ in the Grothendieck group, we obtain a map

$$\hat{\psi} : \{(m+2k, m+2l)\text{-affine tangles}\} \to \mathrm{Hom}(V_{[m]}^{\otimes m+2k}, V_{[m]}^{\otimes m+2l})$$

We expect that this map will coincide with a well-known invariant for affine tangles, and that the images of the irreducible objects Ψ_α in the Grothendieck group will be the canonical basis (or perhaps the dual canonical basis). Inspired by Khovanov's construction in [41] and [28], we also expect that it will be possible to give an alternate categorification of $\hat{\psi}$, using categories of modules over the Ext algebras controlling $\mathcal{D}_n$ (which closely resemble Khovanov's arc algebras).

2.2 Two-block Springer fibres

2.2.1 Transverse slices for two-block nilpotents

Fix $m \geq 0$. For $n \in \mathbb{Z}_{\geq 0}$, let z_n be the standard nilpotent of Jordan type $(m+n, n)$. Let $S_n \subset \mathfrak{sl}_{m+2n}$ denote the Mirkovic-Vybornov transverse slice to the nilpotent z_n (see section 3.3.1 in [44]):

$$S_n = \{z_n + \sum_{1\leq i\leq m+2n} a_i e_{m+n,i} + \sum_{i\in\{1,\cdots,n,m+n+1,\cdots,m+2n\}} b_i e_{m+2n,i}\}$$

Definition 2.2.1. Denote by $\mathcal{N}_n$ the nilpotent cone for $\mathfrak{sl}_{m+2n}$. Let $\mathcal{B}_n$ denote the complete flag variety for $GL_{m+2n}(\mathbb{C})$, and for $0 < k < m+2n$ define the varieties $\mathcal{P}_{k,n}$

as follows:

$$\mathcal{P}_{k,n} = \{(0 \subset V_1 \subset \cdots \subset \widehat{V_k} \subset \cdots \subset V_{m+2n} = \mathbb{C}^{m+2n})\}.$$

Then the varieties $T^*\mathcal{B}_n$, $T^*\mathcal{P}_{k,n}$ can be described as follows:

$$T^*\mathcal{B}_n = \{(0 \subset V_1 \subset \cdots \subset V_{m+2n} = \mathbb{C}^{m+2n}, x) \mid x \in \mathfrak{sl}_{m+2n},\ xV_i \subset V_{i-1}\};$$
$$T^*\mathcal{P}_{k,n} = \{(0 \subset V_1 \subset \cdots \subset \widehat{V_k} \subset \cdots \subset V_{m+2n} = \mathbb{C}^{m+2n}), x \mid$$
$$x \in \mathfrak{sl}_{m+2n},\ xV_{k+1} \subset V_{k-1},\ xV_i \subset V_{i-1} \text{ for } i \neq k, k+1\}.$$

Pick a basis $e_1, \dots, e_{m+n+1}, f_1, \dots, f_{n+1}$ of $\mathbb{C}^{m+2n+2}$ so that $z_{n+1}e_i = e_{i-1}$, $z_{n+1}f_j = f_{j-1}$ (where we set $e_0 = f_0 = 0$).

Lemma 2.2.2. *For any $x \in S_{n+1}$ such that dim(Ker x) $= 2$, we have Ker $x = \mathbb{C}e_1 \oplus \mathbb{C}f_1$, and there is a natural isomorphism $\phi_x : xV_{m+2n+2} \simeq \mathbb{C}^{m+2n}$.*

Proof. By the construction in [44, section 3.3.1] we can assume that $xe_i = e_{i-1} + a_ie_{m+n+1} + c_if_{m+1}$ if $i \leq m+1$, $xe_i = e_{i-1} + a_ie_{m+n+1}$ if $i > m+1$, and $xf_j = f_{j-1} + b_je_{m+n+1} + d_jf_{m+1}$. Then we have:

$$x\left(\sum_{1\leq i\leq m+n+1} \lambda_ie_i + \sum_{1\leq j\leq m+1} \nu_jf_j\right) =$$
$$\sum_{1\leq i\leq m+n} \lambda_{i+1}e_i + \left(\sum_{1\leq i\leq m+n+1} a_i\lambda_i + \sum_{1\leq j\leq m+1} b_j\nu_j\right) e_{m+n+1}$$
$$+ \sum_{1\leq j\leq m} \nu_{j+1}f_j + \left(\sum_{1\leq i\leq m+1} a_i\lambda_i + \sum_{1\leq j\leq m+1} d_j\nu_j\right) f_{m+1}$$

So $xv = x\left(\sum_{1\leq i\leq m+n+1} \lambda_ie_i + \sum_{1\leq j\leq m+1} \nu_jf_j\right) = 0$ implies that $\lambda_i = \nu_j = 0$ for $i,j > 1$, i.e. that $v \in \mathbb{C}e_1 \oplus \mathbb{C}f_1$. If $xv = 0$ it follows that $a_1 = b_1 = c_1 = d_1 = 0$. So:

$$xV_{m+2n+2} = \left\{ \sum_{1\leq i\leq m+n} \lambda_ie_i + \left(\sum_{1\leq i\leq m+n} a_{i+1}\lambda_i + \sum_{1\leq j\leq n} b_{j+1}\nu_j\right) e_{m+n+1} + \right.$$
$$\left. \sum_{1\leq j\leq n} \mu_jf_j + \left(\sum_{1\leq i\leq m} c_{i+1}\lambda_i + \sum_{1\leq j\leq n} d_{j+1}\nu_j\right) f_{n+1}\right\}$$

Let us denote by $\gamma_{m,n}$ the following map:

$$\mathbb{C}^{m+2n+2} = \Big(\bigoplus_{1\leq i\leq m+n} \mathbb{C}e_i \oplus \bigoplus_{1\leq j\leq n} \mathbb{C}f_j\Big) \oplus (\mathbb{C}e_{m+n+1} \oplus \mathbb{C}f_{n+1}) \to \Big(\bigoplus_{1\leq i\leq m+n} \mathbb{C}e_i \oplus \bigoplus_{1\leq j\leq n} \mathbb{C}f_j\Big)$$

denote the natural projection map. The following map is an isomorphism:

$$\phi_x := \gamma_{m,n}|_{xV_{m+2n+2}} : xV_{m+2n+2} \to \bigoplus_{1\leq i\leq m+n} \mathbb{C}e_i \oplus \bigoplus_{1\leq j\leq n} \mathbb{C}f_j$$

$\square$

Proposition 2.2.3. *For every $0 < k < m+2n+2$ we have $S_{n+1} \times_{\mathfrak{sl}_{m+2n+2}} T^*\mathcal{P}_{k,n+1} \simeq S_n \times_{\mathfrak{sl}_{m+2n}} T^*\mathcal{B}_n$.*

Proof. By definition:

$$\begin{aligned} S_{n+1} \times_{\mathfrak{sl}_{m+2n+2}} T^*\mathcal{P}_{k,n+1} =& \{(0 \subset V_1 \subset \cdots \subset \widehat{V_k} \subset \cdots \subset V_{m+2n+2}) \mid \\ & x \in S_{n+1},\ xV_{k+1} \subset V_{k-1},\ xV_i \subset V_{i-1} \text{ for } i \neq k, k+1\} \end{aligned}$$

$$S_n \times_{\mathfrak{sl}_{m+2n}} T^*\mathcal{B}_n = \{(0 \subset W_1 \subset \cdots \subset W_{m+2n} = \mathbb{C}^{m+2n}) | y \in S_n, yW_i \subset W_{i-1}\}$$

Since $x \in S_{n+1}$, the Jordan type of x is a two-block partition, and $\dim(\mathrm{Ker}(x)) \leq 2$; but $xV_{k+1} \subset V_{k-1}$ so we must have $xV_{k+1} = V_{k-1}$. Consider the flag $(0 \subset V_1 \subset \cdots \subset V_{k-1} = xV_{k+1} \subset xV_{k+2} \subset \cdots \subset xV_{m+2n+2})$. Recall the isomorphism $\phi_x : xV_{m+2n+2} \xrightarrow{\sim} \mathbb{C}^{m+2n}$ from Lemma 2.2.2 and denote by $\Phi(x) \in \mathrm{End}(\mathbb{C}^{m+2n})$ the endomorphism induced on $\mathbb{C}^{m+2n}$ by the action of x on xV_{m+2n+2}. Construct a map $\alpha : S_{n+1} \times_{\mathfrak{sl}_{m+2n+2}} T^*\mathcal{P}_{k,n+1} \to T^*\mathcal{B}_n$ as follows:

$$\begin{aligned} &\alpha(0 \subset V_1 \subset \cdots \subset V_{m+2n}, x) = \\ &= ((0 \subset \phi_x(V_1) \subset \cdots \subset \phi_x(V_{k-1}) = \phi_x(xV_{k+1}) \subset \phi_x(xV_{k+2}) \subset \cdots \subset \mathbb{C}^{m+2n}), \Phi(x)) \end{aligned}$$

We claim that α gives the required isomorphism $S_{n+1} \times_{\mathfrak{sl}_{m+2n+2}} T^*\mathcal{P}_{k,n+1} \simeq S_n \times_{\mathfrak{sl}_{m+2n}} T^*\mathcal{B}_n$. First we check that $\Phi(x) \in S_n$. From the argument in Lemma 2.2.2, $\Phi(x)e_i = e_{i-1} + a_{i+1}e_{m+n} + c_{i+1}f_n$ if $i \leq n$, $\Phi(x)e_i = e_{i-1} + a_{i+1}e_{m+n}$ if $i > n$, and $\Phi(x)f_j = f_{j-1} + c_{j+1}e_{m+n} + d_{j+1}f_n$. Thus Φ gives a bijection between $\{x \in S_{n+1} \cap \mathcal{N}_{n+1} | \dim(\mathrm{Ker}\ x) = 2\}$ and $S_n \cap \mathcal{N}_n$. It follows that α has image $S_n \times_{\mathfrak{sl}_{m+2n}} T^*\mathcal{B}_n$ and that α is an isomorphism onto its image, as required. $\square$

Definition 2.2.4. Under the Springer resolution map $\pi_n : T^*\mathcal{B}_n \to \mathcal{N}_n$, let $\mathcal{B}_{z_n} = \pi_n^{-1}(z_n)$. Let $U_n = S_n \times_{\mathfrak{sl}_{m+2n}} T^*\mathcal{B}_n$; define $\mathcal{D}_n = D^b(\mathrm{Coh}_{\mathcal{B}_{z_n}}(U_n))$ to be the bounded derived category of coherent sheaves on U_n supported on $\mathcal{B}_{z_n}$.

Next we will prove an extension of Proposition 2.4 in [27] to the case of two-block nilpotents. Consider a $2(m+2n)$-dimensional vector space $V_{m,n}$ with basis

$$e_1, \ldots, e_{m+2n}, f_1, \ldots, f_{m+2n}$$

and a nilpotent z such that $ze_i = e_{i-1}$, $zf_i = f_{i-1}$. Let $W_{m,n} \subset V_{m,n}$ denote the vector subspace with basis $e_1, \ldots, e_{m+n}, f_1, \ldots, f_n$, so that $z|_{W_{m,n}}$ has Jordan type $(m+n, n)$; we will identify $W_{m,n}$ with V_{m+2n}. Let $P : V_{m,n} \to W_{m,n}$ denote the projection defined by $Pe_i = e_i$ if $i \leq m+n$, $Pe_i = 0$ if $i > m+n$; $Pf_i = f_i$ if $i \leq n$, $Pf_i = f_i$ if $i > n$. Following Section 2 of [27], define:

$$\begin{aligned} Y_{m+2n} &= \{(L_1 \subset \cdots \subset L_{m+2n} \subset V_{m,n}) | \dim L_i = i,\ zL_i \subset L_{i-1}\}; \\ U_{m+2n} &= \{(L_1 \subset \cdots \subset L_{m+2n}) \in Y_{m+2n} | P(L_{m+2n}) = W_{m,n}\}. \end{aligned}$$

Definition 2.2.5.

$$S'_n = \{ \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ a_1 & a_2 & \cdots & a_{m+n} & b_1 & b_2 & \cdots & b_n \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ c_1 & c_2 & \cdots & c_{m+n} & d_1 & d_2 & \cdots & d_n \end{pmatrix} \}.$$

Note that we have $S_n \subset S'_n$, and $U_n \subset S'_n \times_{\mathfrak{sl}_{m+2n}} T^*\mathcal{B}_n$. The following lemma (generalizing Proposition 2.4 in [27]) gives a map $\iota_n : U_n \to Y_{m+2n}$, identifying U_n with a locally closed subvariety of Y_{m+2n}.

Lemma 2.2.6. *There is an isomorphism $U_{m+2n} \simeq S'_n \times_{\mathfrak{sl}_{m+2n}} T^*\mathcal{B}_n$.*

Proof. Given $(L_1 \subset \cdots \subset L_{m+2n}) \in U_{m+2n}$, since $P : L_{m+2n} \to W_{m,n}$ is an isomorphism, we have a nilpotent endomorphism $x = PzP^{-1} \in \mathrm{End}(V_{m+2n})$ (here we identify $W_{m,n}$ and V_{m+2n}). If $P^{-1}e_i = e_i + v'$, where v' lies in the span of $e_{m+n+1}, \cdots, e_{m+2n}, f_{n+1}, \cdots, f_{m+2n}$, then $zP^{-1}e_i = e_{i-1} + v''$ where v'' is in the span of $e_{m+n}, \cdots, e_{m+2n-1}, f_n, \cdots, f_{m+2n-1}$. Hence $PzP^{-1}e_i = xe_i \in \mathrm{span}(e_{i-1}, e_{m+n}, f_n)$, and similarly $xf_i \in \mathrm{span}(f_{i-1}, e_{m+n}, f_n)$; so $x \in S'_n$. Thus we have a map $\alpha : U_{m+2n} \to S'_n \times_{\mathfrak{sl}_{m+2n}} T^*\mathcal{B}_n$ given by $\alpha(L_1, \cdots, L_{m+2n}) = (PzP^{-1}, (P(L_1), P(L_2), \cdots, P(L_{m+2n})))$.

For the converse direction, from the below Lemma 2.2.7 we know that given $x \in S'_n \cap \mathcal{N}_n$ there exists a unique z-stable subspace $L_{m+2n} \subset V_{m,n}$ such that $PL_{m+2n} = W_{m,n}$ and $PzP^{-1} = x$; call this subspace $L_{m+2n} = \Theta(x)$. We have an isomorphism $P : \Theta(x) \simeq W_{m,n}$. Thus given an element $((0 \subset V_1 \subset \cdots \subset V_{m+2n}), x) \in S'_n \times_{\mathfrak{sl}_{m+2n}} T^*\mathcal{B}_n$, let $\beta(x) = (0 \subset P^{-1}V_1 \subset P^{-1}V_2 \subset \cdots \subset \Theta_x)$. It is clear that α and β are inverse to one another. $\square$

Lemma 2.2.7. *Given $x \in S'_n \cap \mathcal{N}_n$, there exists a unique subspace $L_{m+2n} \subset V_{m,n}$, with $PL_{m+2n} = W_{m,n}$, such that $zL_{m+2n} \subset L_{m+2n}$ and $PzP^{-1} = x$.*

Proof. Since $PL_{m+2n} = W_{m,n}$, to specify the subspace L_{m+2n} it suffices to specify

$$\tilde{e}_i := P^{-1}(e_i) = e_i + \sum_{1 \le k \le n} a_i^{(k)} e_{m+n+k} + \sum_{1 \le l \le m+n} c_i^{(l)} f_{n+l}$$
$$\tilde{f}_j := P^{-1}(f_j) = f_j + \sum_{1 \le k \le n} b_j^{(k)} e_{m+n+k} + \sum_{1 \le l \le m+n} d_j^{(l)} f_{n+l}$$

Suppose for $1 \le i \le m+n, 1 \le j \le n$, $xe_i = e_{i-1} + a_i e_{m+n} + c_i f_n, xf_j = f_{j-1} + b_j e_{m+n} + d_j f_n$; then the identity $PzP^{-1} = x$ is equivalent to $a_i^{(1)} = a_i, c_i^{(1)} = c_i, b_j^{(1)} = b_j$ and

$d_j^{(1)} = d_j$. The statement $zL_{m+2n} \subset L_{m+2n}$, i.e. $z\tilde{e}_i, z\tilde{f}_j \in L_{m+2n}$, is equivalent to saying that:

$$z\tilde{e}_i = \tilde{e}_{i-1} + a_i\tilde{e}_{m+n} + c_i\tilde{f}_n$$
$$z\tilde{f}_j = \tilde{f}_{j-1} + b_j\tilde{e}_{m+n} + d_j\tilde{f}_n$$

Expanding the above two equations:

$$\begin{aligned} e_{i-1} + \sum_{1\le k\le n} a_i^{(k)} e_{m+n+k-1} + \sum_{1\le l\le m+n} c_i^{(l)} f_{n+l-1} = e_{i-1} + \sum_{1\le k\le n} a_{i-1}^{(k)} e_{m+n+k} \\ + \sum_{1\le l\le m+n} c_{i-1}^{(l)} f_{n+l} + a_i\left(e_{m+n} + \sum_{1\le k\le n} a_{m+n}^{(k)} e_{m+n+k} + \sum_{1\le l\le m+n} c_{m+n}^{(l)} f_{n+l}\right) \\ + c_i\left(f_n + \sum_{1\le k\le n} b_n^{(k)} e_{m+n+k} + \sum_{1\le l\le m+n} d_n^{(l)} f_{n+l}\right); \end{aligned}$$

$$\begin{aligned} & f_{j-1} + \sum_{1\le k\le n} b_j^{(k)} e_{m+n+k-1} + \sum_{1\le l\le m+n} d_j^{(l)} f_{n+l-1} = \\ & f_{j-1} + \sum_{1\le k\le n} b_{j-1}^{(k)} e_{m+n+k} + \sum_{1\le l\le m+n} d_{j-1}^{(l)} f_{n+l} \\ & + b_j\left(e_{m+n} + \sum_{1\le k\le n} a_{m+n}^{(k)} e_{m+n+k} + \sum_{1\le l\le m+n} c_{m+n}^{(l)} f_{n+l}\right) \\ & + d_j\left(f_n + \sum_{1\le k\le n} b_n^{(k)} e_{m+n+k} + \sum_{1\le l\le m+n} d_n^{(l)} f_{n+l}\right). \end{aligned}$$

Extracting coefficients of e_{m+n+k} and f_{n+l} in the above two equations gives:

$$\begin{aligned} a_i^{(k+1)} &= a_{i-1}^{(k)} + a_i a_{m+n}^{(k)} + c_i b_n^{(k)}, & b_j^{(k+1)} &= b_{j-1}^{(k)} + b_j a_{m+n}^{(k)} + d_j b_n^{(k)} \\ c_i^{(l+1)} &= c_i^{(l)} + a_i c_{m+n}^{(l)} + c_i d_n^{(l)}, & d_j^{(l+1)} &= d_{j-1}^{(l)} + b_j c_{m+n}^{(l)} + d_j d_n^{(l)} \end{aligned}$$

Consider the matrix coefficients $(x^k)_{p,q}$ for $1 \le p, q \le m+2n$. It follows by induction that we have $a_i^{(k)} = (x^k)_{m+n,i}, b_j^{(k)} = (x^k)_{m+n,m+n+j}, c_i^{(l)} = (x^l)_{m+2n,i}, d_j^{(l)} = (x^l)_{m+2n,m+n+j}$. Indeed, the case where $k = l = 1$ is clear; and the induction step follows from expanding the equation $(x^{r+1})_{uv} = \sum_{1\le w\le m+2n}(x^r)_{uw}(x)_{wv}$ for $u = m+n$ and $u = m+2n$.

Using the above recursive definition of $a_i^{(k)}, b_j^{(k)}, c_i^{(l)}, d_j^{(l)}$, it remains to prove that $a_i^{(n+1)} = b_j^{(n+1)} = 0$ and $c_i^{(m+n+1)} = d_j^{(m+n+1)} = 0$. Thus we must show that $(x^{n+1})_{m+n,p} = (x^{m+n+1})_{m+2n,p} = 0$ given $1 \le p \le m+2n$. Using the equation

$(x^{r+1})_{uv} = \sum_{1 \le w \le m+2n} (x)_{uw}(x^r)_{wv}$, we compute that:

$$(x^{n+1})_{m+n,p} = (x^{n+2})_{m+n-1,p} = \cdots = (x^{m+2n})_{1,p} = 0$$
$$(x^{m+n+1})_{m+2n,p} = (x^{m+n+2})_{m+2n-1,p} = \cdots = (x^{m+2n})_{m+n+1,p} = 0$$

This completes the proof of the existence and uniqueness of a z-stable subspace $L_{m+2n} \subset V_{m,n}$ with $PL_{m+2n} = W_{m,n}$ and $PzP^{-1} = x$. $\square$

2.3 Tangles

2.3.1 Affine tangles

Definition 2.3.1. If $p \equiv q \pmod 2$, a (p,q) affine tangle is an embedding of $\frac{p+q}{2}$ arcs and a finite number of circles into the region $\{(x,y) \in \mathbb{C} \times \mathbb{R} | 1 \le |x| \le 2\}$, such that the end-points of the arcs are $(1,0), (\zeta_p, 0), \cdots, (\zeta_p^{p-1}, 0), (2,0), (2\zeta_q, 0), \cdots, (2\zeta_q^{q-1}, 0)$ in some order; here $\zeta_k = e^{\frac{2\pi i}{k}}$.

Remark 2.3.2. Given a (p,q) affine tangle α, and a (q,r) affine tangle β, we can compose them using scaling and concatenation. This composition is associative up to isotopy. The composition $\beta \circ \alpha$ is a (p,r) affine tangle.

Definition 2.3.3. Given $1 \le i \le n$, define the following affine tangles:

- Let g_n^i denote the $(n-2,n)$ tangle with an arc connecting $(2\zeta_n^i, 0)$ to $(2\zeta_n^{i+1}, 0)$. Let other strands connect $(\zeta_{n-2}^k, 0)$ to $(2\zeta_n^k, 0)$ for $1 \le k < i$ and $(\zeta_{n-2}^k, 0)$ to $(2\zeta_n^{k+2}, 0)$ for $i+1 < k \le n-2$.
- Let f_n^i denote the $(n, n-2)$ tangle with an arc connecting $(\zeta_n^i, 0)$ and $(\zeta_n^{i+1}, 0)$. Let other strands connect $(\zeta_n^k, 0)$ to $(2\zeta_{n-2}^k, 0)$ for $1 \le k < i$ and $(\zeta_n^k, 0)$ to $(2\zeta_{n-2}^{k-2}, 0)$ for $i+1 < k \le n-2$.
- Let $t_n^i(1)$ (respectively, $t_n^i(2)$) denote the (n,n) tangle in which a strand connecting $(\zeta_n^i, 0)$ to $(2\zeta_n^{i+1}, 0)$ passes beneath (respectively, above) a strand connecting $(\zeta_n^{i+1}, 0)$ to $(2\zeta_n^i, 0)$. Let other strands connect $(\zeta_n^k, 0)$ to $(2\zeta_n^k, 0)$ for $k \neq i, i+1$.
- Let r_n denote the (n,n) tangle connecting $(\zeta_n^j, 0)$ to $(2\zeta_n^{j+1}, 0)$ for each $1 \le j \le n$, and let r_n' denote the (n,n) tangle connecting $(\zeta_n^j, 0)$ to $(2\zeta_n^{j-1}, 0)$ for each $1 \le j \le n$.

Definition 2.3.4. Define a linear tangle to be an affine tangle that is isotopic to a product of the generators $g_n^i, f_n^i, t_n^i(1)$ and $t_n^i(2)$ for $i \neq n$.

Remark 2.3.5. Linear tangles can be moved away from the half-line $e^{-i\epsilon}\mathbb{R}_{\ge 0}$ where ϵ is a small positive number. If we cut the annulus $1 \le |z| \le 2$ by that line and apply the logarithm map, linear tangles turn into the usual tangles that live between two parallel lines.

Lemma 2.3.6. *Any affine tangle is isotopic to a composition of the above generators.*

Proof. For a curve in $\mathbb{C}$, define its affine critical point as a point where this curve is tangent to a circle with center at 0. We can adjust a tangle within its isotopy class so that its projection onto $\mathbb{C}$ has a finite number of transversal crossings and affine critical points. We can also assume that no two of these points lie on the same circle with center at 0. Cut the projection of the tangle by circles with center at 0 into annuli so that each annulus contains only one crossing or affine critical point. We can further adjust the tangle so that we have a tangle inside each annulus, and by construction these tangles have to be g_n^i, f_n^i, or $t_n^i(p)$, possibly composed with a power of r_n. $\square$

Definition 2.3.7. Let **ATan** (resp. **Tan**) denote the category with objects k for $k \in \mathbb{Z}_{\geq 0}$, and the set of morphisms between p and q consist of all affine (resp. linear) (p,q) tangles.

In the category **ATan** we record the following relations between the above generators; here let $1 \leq i \leq n-1, 1 \leq p, q \leq 2, k \geq 2$:

1. (Reidemeister 0) $f_n^i \circ g_n^{i+1} = f_n^{i+1} \circ g_n^i = \mathrm{id}$
2. (Reidemeister 1) $f_n^i \circ t_n^{i\pm 1}(2) \circ g_n^i = f_n^i \circ t_n^{i\pm 1}(1) \circ g_n^i = \mathrm{id}$
3. (Reidemeister 2) $t_n^i(1) \circ t_n^i(2) = t_n^i(2) \circ t_n^i(1) = id$
4. (Reidemeister 3) $t_n^i(1) \circ t_n^{i+1}(1) \circ t_n^i(1) = t_n^{i+1}(1) \circ t_n^i(1) \circ t_n^{i+1}(1)$.
5. (Cup-cup isotopy) $g_{n+2}^{i+k} \circ g_n^i = g_{n+2}^i \circ g_n^{i+k-2}$
6. (Cap-cap isotopy) $f_n^{i+k-2} \circ f_{n+2}^i = f_n^i \circ f_{n+2}^{i+k}$
7. (Cup-cap isotopy) $g_n^{i+k-2} \circ f_n^i = f_{n+2}^i \circ g_{n+2}^{i+k}, g_n^i \circ f_n^{i+k-2} = f_{n+2}^{i+k} \circ g_{n+2}^i$
8. (Cup-crossing isotopy) $g_n^i \circ t_{n-2}^{i+k-2}(q) = t_n^{i+k}(q) \circ g_n^i, g_n^{i+k} \circ t_{n-2}^i(q) = t_n^i(q) \circ g_n^{i+k}$
9. (Cap-crossing isotopy) $f_n^i \circ t_n^{i+k}(q) = t_{n-2}^{i+k-2}(q) \circ f_n^i, f_n^{i+k} \circ t_n^i(q) = t_{n-2}^i(q) \circ f_n^{i+k}$
10. (Crossing-crossing isotopy) $t_n^i(p) \circ t_n^{i+k}(q) = t_n^{i+k}(q) \circ t_n^i(p)$
11. (Pitchfork move) $t_n^i(1) \circ g_n^{i+1} = t_n^{i+1}(2) \circ g_n^i, t_n^i(2) \circ g_n^{i+1} = t_n^{i+1}(1) \circ g_n^i$.
12. (Rotation) $r_n \circ r_n' = r_n' \circ r_n = id$
13. (Cap rotation) $r_{n-2}' \circ f_n^i \circ r_n = f_n^{i+1}, f_n^{n-1} \circ r_n^2 = f_n^1$
14. (Cup rotation) $r_n' \circ g_n^i \circ r_{n-2} = g_n^{i+1}, r_n'^2 \circ g_n^{n-1} = g_n^1$
15. (Crossing rotation) $r_n' \circ t_n^i(q) \circ r_n = t_n^{i+1}(q), r_n'^2 \circ t_n^{n-1}(q) \circ r_n^2 = t_n^1(q)$.

By Lemma 4.1 from [27], any relation between linear tangles can be expressed as a composition of the relations (1)-(11) above. We can generalize that to affine tangles:

Proposition 2.3.8. *Any relation between affine tangles can be expressed as a composition of the relations* (1)-(15) *above.*

Proof. First, let us reduce any relation to a composition of relations (1)-(11) involving g_n^i, f_n^i, $t_n^i(p)$ for $1 \le i \le n$ (for the definition of g_n^n, f_n^n, $t_n^n(p)$ see the proof of Lemma 2.3.6). Then, we can express the relations (1)-(11) involving g_n^n, f_n^n, $t_n^n(p)$ using relations (1)-(15), by a direct computation.

Let us call an isotopy *linear* if it fixes a segment of the form $[(\zeta, 0), (2\zeta, 0)]$ for some ζ. Note that a linear isotopy is a composition of elementary isotopies (1)-(11) (possibly involving g_n^n, f_n^n, $t_n^n(p)$) since the points where the tangle intersects $[(\zeta, 0), (2\zeta, 0)]$ stay fixed. Now, if two affine tangles are isotopic, then they are also isotopic through a composition of two linear isotopies, which completes the proof. $\square$

For our purposes, it will be more convenient to replace the relations (13)-(15) by the equivalent set of defining relations below.

Definition 2.3.9. Let s_n^i denote the (n, n)-tangle with a strand connecting $(\zeta_j, 0)$ to $(2\zeta_j, 0)$ for each j, and a strand connecting $(\zeta_i, 0)$ to $(2\zeta_i, 0)$ passing clockwise around the circle, beneath all the other strands.

Lemma 2.3.10. *The following relations are equivalent to the relations* (13)-(15) *above.*

- $s_n^n \circ g_n^i = g_n^i \circ s_{n-2}^{n-2}, s_{n-2}^{n-2} \circ f_n^i = f_n^i \circ s_n^n, s_n^n \circ t_n^i(p) = t_n^i(p) \circ s_n^n$
- $f_n^{n-1} \circ s_n^n \circ t_n^{n-1}(2) \circ s_n^n \circ t_n^{n-1}(2) = f_n^{n-1}$
- $s_n^n \circ t_n^{n-1}(2) \circ s_n^n \circ t_n^{n-1}(2) \circ g_n^{n-1} = g_n^{n-1}$

Proof. It is straightforward to check that the above relations are satisfied. It suffices to now use the identity $r_n = s_n^n \circ t_n^{n-1}(2) \circ \cdots \circ t_n^1(2)$, and Proposition 2.3.8, to show that the above three relations imply equations (13)-(15) above (given relations (1)-(11)):

$$\begin{aligned} r'_{n-2} \circ f_n^i \circ r_n &= t_{n-2}^1(1) \circ \cdots \circ t_{n-2}^{n-3}(1) \circ (s_{n-2}^{n-2})^{-1} \circ f_n^i \circ s_n^n \circ t_n^{n-1}(2) \circ \cdots \circ t_n^1(2) \\ &= t_{n-2}^1(1) \circ \cdots \circ t_{n-2}^{n-3}(1) \circ (s_{n-2}^{n-2})^{-1} \circ s_{n-2}^{n-2} \circ f_n^i \circ t_n^{n-1}(2) \circ \cdots \circ t_n^1(2) \\ &= t_{n-2}^1(1) \circ \cdots \circ t_{n-2}^{n-3}(1) \circ f_n^i \circ t_n^{n-1}(2) \circ \cdots \circ t_n^1(2) = f_n^{i+1} \end{aligned}$$

$$\begin{aligned} r'_n \circ g_n^i \circ r_{n-2} &= t_n^1(1) \circ \cdots \circ t_n^{n-1}(1) \circ (s_n^n)^{-1} \circ g_n^i \circ s_{n-2}^{n-2} \circ t_{n-2}^{n-3}(2) \circ \cdots t_{n-2}^1(2) \\ &= t_n^1(1) \circ \cdots \circ t_n^{n-1}(1) \circ (s_n^n)^{-1} \circ s_n^n \circ g_n^i \circ t_{n-2}^{n-3}(2) \circ \cdots t_{n-2}^1(2) \\ &= t_n^1(1) \circ \cdots \circ t_n^{n-1}(1) \circ g_n^i \circ t_{n-2}^{n-3}(2) \circ \cdots t_{n-2}^1(2) = g_n^{i+1} \end{aligned}$$

$$
\begin{aligned}
r'_n \circ t_n^i(q) \circ r_n &= t_n^1(1) \circ \cdots \circ t_n^{n-1}(1) \circ (s_n^n)^{-1} \circ t_n^i(q) \circ s_n^n \circ t_n^{n-1}(2) \circ \cdots \circ t_n^1(2) \\
&= t_n^1(1) \circ \cdots \circ t_n^{n-1}(1) \circ t_n^i(q) \circ t_n^{n-1}(2) \circ \cdots \circ t_n^1(2) = t_n^{i+1}(q)
\end{aligned}
$$

$$
\begin{aligned}
f_n^{n-1} \circ r_n^2 &= f_n^{n-1} \circ s_n^n \circ t_n^{n-1}(2) \circ \cdots \circ t_n^1(2) \circ s_n^n \circ t_n^{n-1}(2) \circ \cdots \circ t_n^1(2) \\
&= f_n^{n-1} \circ s_n^n \circ t_n^{n-1}(2) \circ s_n^n \circ t_n^{n-1}(2) \circ t_n^{n-1}(1) \circ t_n^{n-2}(2) \circ \cdots \circ t_n^1(2) \\
&\quad \circ t_n^{n-1}(2) \circ \cdots \circ t_n^1(2) \\
&= f_n^{n-1} \circ t_n^{n-1}(1) \circ t_n^{n-2}(2) \circ \cdots \circ t_n^1(2) \circ t_n^{n-1}(2) \circ \cdots \circ t_n^1(2) = f_n^1
\end{aligned}
$$

$$
\begin{aligned}
r_n'^2 \circ g_n^{n-1} &= t_n^1(1) \circ \cdots \circ t_n^{n-1}(1) \circ (s_n^n)^{-1} \circ t_n^1(1) \circ \cdots \circ t_n^{n-1}(1) \circ (s_n^n)^{-1} \circ g_n^{n-1} \\
&= t_n^1(1) \circ \cdots \circ t_n^{n-1}(1) \circ t_n^1(1) \circ \cdots \circ t_n^{n-2}(1) \circ t_n^{n-1}(2) \\
&\quad \circ t_n^{n-1}(1) \circ (s_n^n)^{-1} \circ t_n^{n-1}(1) \circ (s_n^n)^{-1} \circ g_n^{n-1} \\
&= t_n^1(1) \circ \cdots \circ t_n^{n-1}(1) \circ t_n^1(1) \circ \cdots \circ t_n^{n-2}(1) \circ t_n^{n-1}(2) \circ g_n^{n-1} = g_n^1
\end{aligned}
$$

$$
\begin{aligned}
r_n'^2 \circ t_n^{n-1}(q) \circ r_n^2 &= (t_n^1(1) \circ \cdots \circ t_n^{n-1}(1) \circ (s_n^n)^{-1})^2 \circ t_n^{n-1}(q) \circ (s_n^n \circ t_n^{n-1}(2) \circ \cdots \circ t_n^1(2))^2 \\
&= (t_n^1(1) \circ \cdots \circ t_n^{n-1}(1))^2 \circ t_n^{n-1}(q) \circ (t_n^{n-1}(2) \circ \cdots \circ t_n^1(2))^2 = t_n^1(q)
\end{aligned}
$$

□

2.3.2 Framed tangles

All preceding constructions may be carried out for framed tangles. Define the generators $\hat{g}_n^i$ (resp. $\hat{f}_n^i$, resp. $\hat{t}_n^i(l)$, resp. $\hat{r}_n^i$) as tangles g_n^i (resp. f_n^i, resp. $t_n^i(l)$, resp. r_n^i) with blackboard framing. Introduce new generators $\hat{w}_n^i(1)$ and $\hat{w}_n^i(2)$, which correspond to positive and negative twists of framing of the ith strand of an (n, n) identity tangle.

Definition 2.3.11. Define a framed linear tangle to be a framed affine tangle that isotopic to a product of the generators $\hat{g}_n^i, \hat{f}_n^i, \hat{t}_n^i(1), \hat{t}_n^i(2)$ for $i \neq n$, and $\hat{w}_n^i(1), \hat{w}_n^i(2)$.

Definition 2.3.12. Consider the category **AFTan** (resp. **FTan**), with objects k for $k \in \mathbb{Z}_{\geq 0}$, and the set of morphisms between p and q consist of all framed affine (resp. framed linear) (p, q) tangles.

The relations for framed tangles are transformed as follows:

1. $\hat{f}_n^i \circ \hat{g}_n^{i+1} = id = \hat{f}_n^{i+1} \circ \hat{g}_n^i$
2. (Reidemeister 1) $\hat{f}_n^i \circ \hat{t}_n^{i\pm1}(l) \circ \hat{g}_n^i = \hat{w}_n^i(l)$
3. $\hat{t}_n^i(2) \circ \hat{t}_n^i(1) = id = \hat{t}_n^i(1) \circ \hat{t}_n^i(2)$
4. $\hat{t}_n^i(l) \circ \hat{t}_n^{i+1}(l) \circ \hat{t}_n^i(l) = \hat{t}_n^{i+1}(l) \circ \hat{t}_n^i(l) \circ \hat{t}_n^{i+1}(l)$

5. $\hat{g}_{n+2}^{i+k} \circ \hat{g}_n^i = \hat{g}_{n+2}^i \circ \hat{g}_n^{i+k-2}$

6. $\hat{f}_n^{i+k-2} \circ \hat{f}_{n+2}^i = \hat{f}_n^i \circ \hat{f}_{n+2}^{i+k}$

7. $\hat{g}_n^{i+k-2} \circ \hat{f}_n^i = \hat{f}_{n+2}^i \circ \hat{g}_{n+2}^{i+k}, \quad \hat{g}_n^i \circ \hat{f}_n^{i+k-2} = \hat{f}_{n+2}^{i+k} \circ \hat{g}_{n+2}^i$

8. $\hat{g}_n^i \circ \hat{t}_{n-2}^{i+k-2}(l) = \hat{t}_n^{i+k}(l) \circ \hat{g}_n^i, \quad \hat{g}_n^{i+k} \circ \hat{t}_{n-2}^i(l) = \hat{t}_n^i(l) \circ \hat{g}_n^{i+k}$

9. $\hat{f}_n^i \circ \hat{t}_n^{i+k}(l) = \hat{t}_{n-2}^{i+k-2}(l) \circ \hat{f}_n^i, \quad \hat{f}_n^{i+k} \circ \hat{t}_n^i(l) = \hat{t}_{n-2}^i(l) \circ \hat{f}_n^{i+k}$

10. $\hat{t}_n^i(l) \circ \hat{t}_n^{i+k}(m) = \hat{t}_n^{i+k}(m) \circ \hat{t}_n^i(l)$

11. $\hat{t}_n^i(1) \circ \hat{g}_n^{i+1} = \hat{t}_n^{i+1}(2) \circ \hat{g}_n^i, \quad \hat{t}_n^i(2) \circ \hat{g}_n^{i+1} = \hat{t}_n^{i+1}(1) \circ \hat{g}_n^i$

12. $\hat{r}_n \circ \hat{r}'_n = id = \hat{r}'_n \circ \hat{r}_n$

13. $\hat{r}'_{n-2} \circ \hat{f}_n^i \circ \hat{r}_n = \hat{f}_n^{i+1}, \ i = 1, \ldots, n-2; \quad \hat{f}_n^{n-1} \circ (\hat{r}_n)^2 = \hat{f}_n^1$

14. $\hat{r}'_n \circ \hat{g}_n^i \circ \hat{r}_{n-2} = \hat{g}_n^{i+1}, \ i = 1, \ldots, n-2; \quad (\hat{r}'_n)^2 \circ \hat{g}_n^{n-1} = \hat{g}_n^1$

15. $\hat{r}'_n \circ \hat{t}_n^i(l) \circ \hat{r}_n = \hat{t}_n^{i+1}(l); \quad (\hat{r}'_n)^2 \circ \hat{t}_n^{n-1}(l) \circ (\hat{r}_n)^2 = \hat{t}_n^1(l)$

   We have the following additional relations for twists:

16. $\hat{w}_n^i(1) \circ \hat{w}_n^i(2) = id, \quad \hat{w}_n^i(l) \circ \hat{w}_n^j(k) = \hat{w}_n^j(k) \circ \hat{w}_n^i(l), \ i \neq j$

17. $\hat{w}_n^i(k) \circ \hat{g}_n^i = \hat{w}_n^{i+1}(k) \circ \hat{g}_n^i, \quad \hat{w}_n^i(k) \circ \hat{g}_n^j = \hat{g}_n^j \circ \hat{w}_n^{i+1\pm1}(k), \ i \neq j, j+1$

18. $\hat{f}_n^i \circ \hat{w}_n^i(k) = \hat{f}_n^i \circ \hat{w}_n^{i+1}(k), \quad \hat{w}_n^i(k) \circ \hat{f}_n^j = \hat{f}_n^j \circ \hat{w}_n^{i-1\pm1}(k), \ i \neq j, j+1$

19. $\hat{w}_n^i(k) \circ \hat{t}_n^i = \hat{w}_n^{i+1}(k) \circ \hat{t}_n^i, \quad \hat{w}_n^i(k) \circ \hat{t}_n^j = \hat{t}_n^j \circ \hat{w}_n^i(k), \ i \neq j, j+1$

20. $\hat{t}_n^i \circ \hat{w}_n^i(k) = \hat{f}_n^i \circ \hat{w}_n^{i+1}(k), \quad \hat{w}_n^i(k) \circ \hat{f}_n^j = \hat{t}_n^j \circ \hat{w}_n^i(k), \ i \neq j, j+1$

21. $\hat{w}_n^i(k) \circ \hat{r}_n = \hat{r}_n \circ \hat{w}_n^{i-1}(k), \quad \hat{w}_n^i(k) \circ \hat{r}'_n = \hat{r}'_n \circ \hat{w}_n^{i+1}(k)$

Note how the Reidemeister 1 move (2) is the only relation between the non-twist generators that differs from the relations in **ATan**.

Proposition 2.3.13. *Any isotopy of affine framed tangles is equivalent to a composition of elementary isotopies* (1)-(21).

Proof. There is a forgetful functor from the 2-category of framed tangles and their isotopies to the 2-category of non-framed tangles and their isotopies, which forgets the framing. Thus, for every isotopy there is a composition of relations (1)-(15) (in **ATan**) which differs only in framing, and that can be ruled out by the commutation laws (16)-(21) (in **AFTan**) of twists with all other generators. □

Lemma 2.3.10 still holds in this context, after replacing s_n^n by its "framed" version $\hat{s}_n^n$.

2.4 Functors associated to affine tangles

Definition 2.4.1. Recall that **AFTan** (resp **Tan**, **FTan**) has objects $\{k\}$ for $k \in \mathbb{Z}_{\geq 0}$, and the set of morphisms between $\{p\}$ and $\{q\}$ consists of all framed affine (resp. framed linear) (p, q) tangles. Define the category $\mathbf{AFTan}_m$ (resp. $\mathbf{Tan}_m, \mathbf{FTan}_m$) to be the full subcategory of **AFTan** (resp. **Tan**, **FTan**) with objects $\{m + 2k\}$ for $k \in \mathbb{Z}_{\geq 0}$.

Definition 2.4.2. A "weak representation" of the category $\mathbf{AFTan}_m$ is an assignment of a triangulated category $\mathcal{C}_k$ for each $k \in \mathbb{Z}_{\geq 0}$, and a functor $\Psi(\alpha) : \mathcal{C}_p \to \mathcal{C}_q$ for each framed affine $(m + 2p, m + 2q)$-tangle, so that the relations between tangles hold for these functors: i.e. if β is an $(m + 2q, m + 2r)$ tangle, then there is an isomorphism $\Psi(\beta) \circ \Psi(\alpha) \simeq \Psi(\beta \circ \alpha)$.

Similarly one can define the notion of a "weak representation" of the categories $\mathbf{Tan}_m, \mathbf{FTan}_m$. The goal of this section is to construct a weak representation of $\mathbf{AFTan}_m$ using the categories $\mathcal{D}_k$.

In [27] Cautis and Kamnitzer construct a weak representation of the category of oriented tangles. We are going to adapt their construction to our setting of framed tangles, and then generalize it to the category $\mathbf{AFTan}_m$ of affine framed tangles. The relations between the generators for oriented tangles are mostly the same as the relations we use here, with a notable exception of Reidemeister I move.

2.4.1 Cautis and Kamnitzer's representation of the oriented tangle calculus

Let $\widetilde{\mathcal{D}}_n = D^b(\mathrm{Coh}(Y_{m+2n}))$. In section 4 of [27], Cautis and Kamnitzer construct a weak representation of the category $\mathbf{OTan}_m$ of oriented tangles using the categories $\widetilde{\mathcal{D}}_n$. In fact, Cautis and Kamnitzer construct a weak representation of the full category **OTan** (which gives a weak representation of the subcategory $\mathbf{OTan}_m$). Also, Cautis and Kamnitzer deal with the $\mathbb{C}^*$-equivariant derived categories; but we will omit this $\mathbb{C}^*$-equivariance as we do not need it. In this subsection we are going to recall their construction, altered so that it becomes a weak representation of **FTan**.

Recall the definition of Fourier-Mukai transforms (see [35] for an extended treatment). Here all pullbacks, pushforwards, Homs and tensor products of sheaves will denote the corresponding derived functors.

Definition 2.4.3. ([35]) Let X, Y be two complex algebraic varieties, and let $\pi_1 : X \times Y \to X, \pi_2 : X \times Y \to Y$ denote the two projections. For an object $\mathcal{T} \in D^b(\mathrm{Coh}(X \times Y))$, define the Fourier-Mukai transform $\Psi_{\mathcal{T}} : D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(Y))$ by $\Psi_{\mathcal{T}}(\mathcal{F}) = \pi_{2*}(\pi_1^*\mathcal{F} \otimes \mathcal{T})$. The object $\mathcal{T}$ is then called the Fourier-Mukai kernel of $\Psi_{\mathcal{T}}$.

Definition 2.4.4. ([27]) Define the subvarieties $X^i_{m+2n} \subset Y_{m+2n}$ and $Z^i_{m+2n} \subset Y_{m+2n} \times Y_{m+2n}$ as follows:

- $X^i_{m+2n} = \{(L_1 \subset L_2 \subset \cdots \subset L_{m+2n}) | \ L_{i+1} = z^{-1}(L_{i-1})\}$.
- $Z^i_{m+2n} = \{(L, L') \in Y_{m+2n} \times Y_{m+2n} | \ L_j = L'_j \ \forall \ j \neq i\}$

The map $j : X^i_{m+2n} \to Y_{m+2n}$ is an embedding of a divisor. Note that there is a natural surjection $p : X^i_{m+2n} \to Y_{m+2n-2}$ defined by $p(L_1, L_2, \cdots, L_{m+2n}) = (L_1, \cdots, L_{i-1}, zL_{i+2}, \cdots, zL_{m+2n})$; thus we may also view X^i_{m+2n} as a subvariety of $Y_{m+2n-2} \times Y_{m+2n}$. Let $\widetilde{\mathcal{V}}_i$ be the tautological vector bundle on Y_{m+2n} corresponding to the vector space L_i; let $\widetilde{\mathcal{E}}_i = \widetilde{\mathcal{V}}_i / \widetilde{\mathcal{V}}_{i-1}$ denote the quotient line bundle.

The following two definitions are based on [27], but not identical to the definitions there:

Definition 2.4.5. Define the following Fourier-Mukai kernels:

$$\begin{aligned}
\widetilde{\mathcal{G}}^i_{m+2n} &= \mathcal{O}_{X^i_{m+2n}} \otimes \pi_2^* \widetilde{\mathcal{E}}_i \in D^b(\mathrm{Coh}(Y_{m+2n-2} \times Y_{m+2n})), \\
\widetilde{\mathcal{F}}^i_{m+2n} &= \mathcal{O}_{X^i_{m+2n}} \otimes \pi_1^* \widetilde{\mathcal{E}}^{-1}_{i+1} \in D^b(\mathrm{Coh}(Y_{m+2n} \times Y_{m+2n-2})) \\
\widetilde{\mathcal{T}}^i_{m+2n}(1) &= \mathcal{O}_{Z^i_{m+2n}} \in D^b(\mathrm{Coh}(Y_{m+2n} \times Y_{m+2n})) \\
\widetilde{\mathcal{T}}^i_{m+2n}(2) &= \mathcal{O}_{Z^i_{m+2n}} \otimes \pi_1^* \widetilde{\mathcal{E}}^{-1}_{i+1} \otimes \pi_2^* \widetilde{\mathcal{E}}_i \in D^b(\mathrm{Coh}(Y_{m+2n} \times Y_{m+2n}))
\end{aligned}$$

Definition 2.4.6. Define the functors

$$\begin{aligned}
\widetilde{G}^i_{m+2n} &= \widetilde{\Psi}(g^i_{m+2n}) = \Psi_{\widetilde{\mathcal{G}}^i_{m+2n}} : \widetilde{\mathcal{D}}_{n-1} \to \widetilde{\mathcal{D}}_n \\
\widetilde{F}^i_{m+2n} &= \widetilde{\Psi}(f^i_{m+2n}) = \Psi_{\widetilde{\mathcal{F}}^i_{m+2n}} : \widetilde{\mathcal{D}}_n \to \widetilde{\mathcal{D}}_{n-1} \\
\widetilde{T}^i_{m+2n}(1) &= \widetilde{\Psi}(t^i_{m+2n}(1)) = \Psi_{\widetilde{\mathcal{T}}^i_{m+2n}(1)} : \widetilde{\mathcal{D}}_n \to \widetilde{\mathcal{D}}_n \\
\widetilde{T}^i_{m+2n}(2) &= \widetilde{\Psi}(t^i_{m+2n}(2)) = \Psi_{\widetilde{\mathcal{T}}^i_{m+2n}(2)} : \widetilde{\mathcal{D}}_n \to \widetilde{\mathcal{D}}_n \\
\widetilde{W}^i_{m+2n}(1) &= \widetilde{\Psi}(w^i_{m+2n}(1)) = [-1] : \widetilde{\mathcal{D}}_n \to \widetilde{\mathcal{D}}_n \\
\widetilde{W}^i_{m+2n}(2) &= \widetilde{\Psi}(w^i_{m+2n}(2)) = [1] : \widetilde{\mathcal{D}}_n \to \widetilde{\mathcal{D}}_n
\end{aligned}$$

Note that the difference with the definition in [27] is that we only use two kinds of twists $\widetilde{T}(1)$ and $\widetilde{T}(2)$ where they use four, and our twists differ from their twists by a shift. The reasons for this change are, first, that there are only two different crossing generators in the category **FTan** while there are four in **OTan**; second, this is the change that turns the oriented tangle relations into the framed tangle relations (see Proposition 2.4.8 below); and third, it gives us the skein relation in a nice form of an exact triangle $Id \to \widetilde{\Psi}(t^i_n(2)) \to \widetilde{\Psi}(g^i_n \circ f^i_n)$ in the spirit of Khovanov's homology construction as described in [42] (see Lemma 2.4.7 below).

It is easy to see that $\widetilde{G}^i_{m+2n} : \widetilde{\mathcal{D}}_{n-1} \to \widetilde{\mathcal{D}}_n$ admits the following alternate description: $\widetilde{G}^i_{m+2n}(\mathcal{F}) = j_*(p^*\mathcal{F} \otimes \widetilde{\mathcal{E}}_i)$ for $\mathcal{F} \in \widetilde{\mathcal{D}}_{n-1}$. Similarly, the functor $\widetilde{F}^i_{m+2n} : \widetilde{\mathcal{D}}_n \to \widetilde{\mathcal{D}}_{n-1}$ admits the following description: $\widetilde{F}^i_{m+2n}(\mathcal{G}) = p_*(j^*\mathcal{G} \otimes \widetilde{\mathcal{E}^{-1}_{i+1}})$ for $\mathcal{G} \in \widetilde{\mathcal{D}}_n$. The following calculation of the left and right adjoints to $\widetilde{G}^i_{m+2n}$, and an alternative description of the functors $\widetilde{T}^i_{m+2n}(1), \widetilde{T}^i_{m+2n}(2)$, from [27] will be of use to us.

Lemma 2.4.7. *We have* $(\widetilde{G}^i_{m+2n})^R = \widetilde{F}^i_{m+2n}[-1]$ *and* $(\widetilde{G}^i_{m+2n})^L = \widetilde{F}^i_{m+2n}[1]$. *Also, for* $\mathcal{F} \in \widetilde{\mathcal{D}}_n$*, there are distinguished triangles* $\widetilde{G}^i_{m+2n}(\widetilde{G}^i_{m+2n})^R\mathcal{F} \to \mathcal{F} \to \widetilde{T}^i_{m+2n}(2)\mathcal{F}$ *and* $\widetilde{T}^i_{m+2n}(1)\mathcal{F} \to \mathcal{F} \to \widetilde{G}^i_{m+2n}(\widetilde{G}^i_{m+2n})^L$.

Proof. This follows from Lemma 4.4, and Theorem 4.6 in [27]. □

Recall that any framed linear tangle can be expressed as a composition of the above generators, and that any relation between linear tangles can be expressed via the relations (1)-(11), (16)-(20) in Definition 2.3.12. Hence defining functors $\Psi(\alpha)$ for each $(m+2p, m+2q)$-tangle α, which are compatible under composition, is equivalent to defining functors for each of the generators, satisfying the relations (1)-(11), (16)-(20) (up to isomorphism).

Proposition 2.4.8. *The functors* $\widetilde{\Psi}(f^i_{m+2n})$, $\widetilde{\Psi}(g^i_{m+2n})$, $\widetilde{\Psi}(t^i_{m+2n}(l))$, $\widetilde{\Psi}(w^i_{m+2n}(l))$ *satisfy the relations* (1)-(11), (16)-(20). *Thus, given a linear* $(m+2p, m+2q)$ *tangle,* α*, written as a product of generators, we can define* $\widetilde{\Psi}(\alpha)$ *by composition (and up to isomorphism, the result does not depend on the choice of decomposition as a product of generators). This gives a weak representation of* $\boldsymbol{FTan}_m$ *using the categories* $\widetilde{\mathcal{D}}_n$.

Proof. By Theorem 4.2 in [27], we know that the functors $\widetilde{G}^i_{m+2n}$, $\widetilde{F}^i_{m+2n}$, $\widetilde{T}^i_{m+2n}(1)[1]$, and $\widetilde{T}^i_{m+2n}(2)[-1]$ satisfy the relations in the category **OTan** that differ slightly from the relations (1)-(11). The relations (1), (3)-(11) are identical for **OTan** and **FTan**, and they hold for the functors $\widetilde{G}^i_{m+2n}$, $\widetilde{F}^i_{m+2n}$, $\widetilde{T}^i_{m+2n}(1)$, $\widetilde{T}^i_{m+2n}(2)$ as well since every relation has the same number of each type of crossings on both sides, so after shifting every type 1 crossing by [1] and every type 2 crossing by $[-1]$ the relations still hold. The oriented Reidemeister move I relation

$$\widetilde{F}^i_{m+2n} \circ \widetilde{T}^{i\pm1}_{m+2n}(1)[1] \circ \widetilde{G}^i_{m+2n} \simeq \mathrm{Id} \simeq \widetilde{F}^i_{m+2n} \circ \widetilde{T}^{i\pm1}_{m+2n}(2)[-1] \circ \widetilde{G}^i_{m+2n}$$

is exactly the relation (2) for $\widetilde{G}^i_{m+2n}$, $\widetilde{F}^i_{m+2n}$, $\widetilde{T}^i_{m+2n}(1)$, $\widetilde{T}^i_{m+2n}(2)$, and $\widetilde{W}^i_{m+2n}(l)$:

$$\widetilde{F}^i_{m+2n} \circ \widetilde{T}^{i\pm1}_{m+2n}(1) \circ \widetilde{G}^i_{m+2n} \simeq [-1] = \widetilde{W}^i_{m+2n}(1)$$
$$\widetilde{F}^i_{m+2n} \circ \widetilde{T}^{i\pm1}_{m+2n}(2) \circ \widetilde{G}^i_{m+2n} \simeq [1] = \widetilde{W}^i_{m+2n}(2)$$

The relations (16)-(20) are straightforward. □

2.4.2 Constructing functors $\Psi(\alpha) : \mathcal{D}_p \to \mathcal{D}_q$ indexed by linear tangles: cups and caps

In the previous section we constructed a weak representation of the category **FTan** of framed tangles using the triangulated categories $\widetilde{\mathcal{D}}_n = D^b(\mathrm{Coh}(Y_{m+2n}))$. Our next goal is to construct a weak representation of the category **AFTan** of affine framed tangles using the categories $\mathcal{D}_n = D^b(\mathrm{Coh}_{\mathcal{B}_{z_n}}(U_n))$. The open embedding

$i_n : U_n \to Y_{m+2n}$ induces a functor $i_{n*} : \mathcal{D}_n \to \widetilde{\mathcal{D}}_n$ for each n, thus one may hope to "lift" the functor $\widetilde{\Psi}(\alpha) : \widetilde{\mathcal{D}}_p \to \widetilde{\mathcal{D}}_q$ to a functor $\Psi(\alpha) : \mathcal{D}_p \to \mathcal{D}_q$. In more precise terms, we aim to construct a functor $\Psi(\alpha)$ such that $i_{q*} \circ \Psi(\alpha) = \widetilde{\Psi}(\alpha) \circ i_{p*}$. Note that this isomorphism together with the isomorphism $\widetilde{\Psi}(\beta \circ \alpha) \simeq \widetilde{\Psi}(\beta) \circ \widetilde{\Psi}(\alpha)$ does not yet imply the isomorphism $\Psi(\beta \circ \alpha) \simeq \Psi(\beta) \circ \Psi(\alpha)$, so we will need to prove the latter separately along with our construction of $\Psi(\alpha)$, employing an argument similar to one in [27].

Define the variety $X_{n,i} := S_n \times_{\mathfrak{sl}_{m+2n}} T^*\mathcal{P}_{i,n} \times_{\mathcal{P}_{i,n}} \mathcal{B}_n$:

$$X_{n,i} = S_n \times_{\mathfrak{sl}_{m+2n}} T^*\mathcal{P}_{i,n} \times_{\mathcal{P}_{i,n}} \mathcal{B}_n = \{(0 \subset V_1 \subset \cdots \subset V_{m+2n}), x \mid \\ x \in S_n, xV_{i+1} \subset V_{i-1}, xV_j \subset V_{j-1} \forall\, j\}$$

We have a $\mathbb{P}^1$-bundle $\pi_{n,i} : X_{n,i} \to S_n \times_{\mathfrak{sl}_{m+2n}} T^*\mathcal{P}_{i,n} \simeq S_{n-1} \times_{\mathfrak{sl}_{m+2n-2}} T^*\mathcal{B}_{n-1} = U_{n-1}$, and the embedding of the divisor $j_{n,i} : X_{n,i} \to S_n \times_{\mathfrak{sl}_{m+2n}} T^*\mathcal{B}_n = U_n$. Thus we can view $X_{n,i}$ can be viewed as a subvariety of $U_{n-1} \times U_n$. Let $\mathcal{V}_k$ denote the tautological vector bundle on $S_n \times_{\mathfrak{sl}_{m+2n}} T^*\mathcal{B}_n$ corresponding to V_k; and define the line bundle $\mathcal{E}_k = \mathcal{V}_k / \mathcal{V}_{k-1}$.

Definition 2.4.9. Define the following Fourier-Mukai kernels:

$$\mathcal{G}^i_{m+2n} = \mathcal{O}_{X_{n,i}} \otimes \pi_2^* \mathcal{E}_i \in D^b(\mathrm{Coh}(U_{n-1} \times U_n)),$$
$$\mathcal{F}^i_{m+2n} = \mathcal{O}_{X_{n,i}} \otimes \pi_1^* \mathcal{E}_{i+1}^{-1} \in D^b(\mathrm{Coh}(U_n \times U_{n-1}))$$

Definition 2.4.10. Define the functors:

$$G^i_{m+2n} = \Psi(g^i_{m+2n}) = \Psi_{\mathcal{G}^i_{m+2n}} : \mathcal{D}_{n-1} \to \mathcal{D}_n$$
$$F^i_{m+2n} = \Psi(f^i_{m+2n}) = \Psi_{\mathcal{F}^i_{m+2n}} : \mathcal{D}_n \to \mathcal{D}_{n-1}$$

Remark 2.4.11. A priori, the functor G^i_{m+2n} maps $D^b(\mathrm{Coh}(U_{n-1}))$ to $D^b(\mathrm{Coh}(U_n))$. However, it is easy to see that G^i_{m+2n} maps the subcategory $\mathcal{D}_{n-1} = D^b(\mathrm{Coh}_{\mathcal{B}_{z_{n-1}}}(U_{n-1}))$ of $D^b(\mathrm{Coh}(U_{n-1}))$ to the subcategory $\mathcal{D}_n = D^b(\mathrm{Coh}_{\mathcal{B}_{z_n}}(U_n))$ of $D^b(\mathrm{Coh}(U_{n-1}))$. Similarly F^i_{m+2n} maps $\mathcal{D}_n$ to $\mathcal{D}_{n-1}$.

It is easy to see that $G^i_{m+2n} : \mathcal{D}_{n-1} \to \mathcal{D}_n$ admits the following alternate description: $G^i_{m+2n}(\mathcal{F}) = j_{n,i*}(\pi^*_{n,i}\mathcal{F} \otimes \mathcal{E}_k)$ for $\mathcal{F} \in \mathcal{D}_{n-1}$. Similarly, the functor $F^i_{m+2n} : \mathcal{D}_n \to \mathcal{D}_{n-1}$ can be expressed as follows: $F^i_{m+2n}(\mathcal{G}) = \pi_{n,i*}(j^*_{n,i}\mathcal{G} \otimes \mathcal{E}^{-1}_{k+1})$ for $\mathcal{G} \in \mathcal{D}_n$. We will define the functors $\Psi(t^i_{m+2n}(1))$ and $\Psi(t^i_{m+2n}(2))$ in the next section, by proving an analogue of Lemma 2.4.7 above.

2.4.3 Constructing functors $\Psi(\alpha) : \mathcal{D}_p \to \mathcal{D}_q$ indexed by linear tangles: crossings and the framing

Recall the definitions of spherical twists and spherical functors from [8]:

Definition 2.4.12. Suppose we have two triangulated categories $\mathcal{C}$ and $\mathcal{D}$, and a functor $S : \mathcal{C} \to \mathcal{D}$, with a left adjoint $L : \mathcal{D} \to \mathcal{C}$ and a right adjoint $R : \mathcal{D} \to \mathcal{C}$. Assume that the categories $\mathcal{C}$ and $\mathcal{D}$ admit DG-enhancements, and the functors S, R, and L descend from DG-functors between those (this holds for Fourier-Mukai transforms between derived categories of coherent sheaves, see [8] Example 4.3). Then the four adjunction maps for (L, S, R) have canonical cones, and we can define these cones to be the twist $T_S(1)$, the dual twist $T_S(2)$, the cotwist $F_S(1)$, and the dual co-twist $F_S(2)$:

$$\begin{aligned} SR \to \mathrm{id} \to T_S(1); \qquad & T_S(2) \to \mathrm{id} \to SL; \\ F_S(1) \to \mathrm{id} \to RS; \qquad & LS \to \mathrm{id} \to F_S(2). \end{aligned}$$

Definition 2.4.13. The functor S is called spherical if the following four conditions hold:

1. $T_S(1)$ and $T_S(2)$ are quasi-inverse autoequivalences of $\mathcal{D}$;
2. $F_S(1)$ and $F_S(2)$ are quasi-inverse autoequivalences of $\mathcal{C}$;
3. The composition $LT_S(1)[-1] \to LSR \to R$ of canonical maps is an isomorphism of functors;
4. The composition $R \to RSL \to F_S(1)L[1]$ of canonical maps is an isomorphism of functors.

Theorem 2.4.14. *([8]) Any two conditions in Definition 2.4.13 imply all four.*

The usual way to prove that a functor is spherical is to use condition (2) and one of the conditions (3) and (4). We are going to focus on functors for which a stronger version of (2) holds:

Definition 2.4.15. A spherical functor $S : \mathcal{C} \to \mathcal{D}$ is called strongly spherical if $F_S(1) = [-3]$.

It turns out that if we use strongly spherical functors and their adjoints and twists to construct weak representations of $\mathbf{FTan}_m$, the only relations we need to check are the Reidemeister 0 move and the commutation relations between non-adjacent cups and caps; all relations involving crossings follow automatically.

Theorem 2.4.16. *Suppose we have a triangulated category $\mathcal{C}_{m+2k}$ for each $k \in \mathbb{Z}_{\geq 0}$; and for each $k \geq 1$, $1 \leq i < m + 2k$, a strongly spherical functor $S^i_{m+2k} : \mathcal{C}_{m+2k-2} \to \mathcal{C}_{m+2k}$. Let L^i_{m+2k} be its left adjoint; R^i_{m+2k} be its right adjoint; $T^i_{m+2k}(1)$ its twist, and $T^i_{m+2k}(2)$ its dual twist. If the following conditions hold:*

1. $S^i_{m+2k} L^{i\pm1}_{m+2k}[-1] \simeq \mathrm{id}$
2. $S^{i+l}_{m+2k+2} S^i_{m+2k} \simeq S^i_{m+2k+2} S^{i+l-2}_{m+2k}$ *for* $l \geq 2$

3. $S^{i+l-2}_{m+2k} \circ L^i_{m+2k} \simeq L^i_{m+2k+2} \circ S^{i+l}_{m+2k+2}$, $S^i_{m+2k} \circ L^{i+l-2}_{m+2k} \simeq L^{i+l}_{m+2k+2} \circ S^i_{m+2k+2}$ *for* $l \geq 2$.

then assign:

- $\Psi(g^i_{m+2k}) \simeq S^i_{m+2k}$, $\Psi(f^i_{m+2k}) = L^i_{m+2k}[-1] \simeq R^i_{m+2k}[1]$
- $\Psi(t^i_{m+2k}(1)) = T^i_{m+2k}(1), \Psi(t^i_{m+2k}(2)) = T^i_{m+2k}(2)$
- $\Psi(w^i_{m+2k}(1)) = [-1], \Psi(w^i_{m+2k}(-1)) = [1]$

These functors will give a weak representation of $\boldsymbol{FTan}_m$.

Proof. Let us check that the relations (1)-(11), (16)-(20) from Definition 2.3.12 hold for the above choice of functors.

The Reidemeister move 0, cup-cup isotopy and cup-cap isotopy relations hold by the assumptions of the theorem, and the cap-cap isotopy relation follows immediately from the cup-cup isotopy relation and the fact that caps are adjoint to cups up to a shift. The cap-crossing isotopy, cup-crossing isotopy and crossing-crossing isotopy relations follow then from the above relations and the definition of a twist. The Reidemeister move II relation $T^i_{m+2k}(1)T^i_{m+2k}(2) \simeq id \simeq T^i_{m+2k}(2)T^i_{m+2k}(1)$ follows from the fact that S^i_{m+2k} are spherical functors, hence $T^i_{m+2k}(l)$ are equivalences of categories. The commutation relations with twists (16)-(20) hold because all exact functors commute with shifts.

The remaining less trivial relations are Reidemeister move I (2), Reidemeister move III (4) and the pitchfork move (8). For simplicity of notation assume that $k = 3$ and denote Υ^i_{m+6} by Υ_i, where Υ stands for L, R, $T(1)$ or $T(2)$.

Reidemeister move I: $L_2T_1(1)S_2[-1] \simeq [1]$. We have an exact triangle

$$L_2S_1R_1S_2 \to L_2S_2 \to L_2T_1(1)S_2$$

by the definition of $T_1(1)$ and another exact triangle

$$\mathrm{id}[2] \to L_2S_2 \to \mathrm{id}$$

since S_2 is a strong spherical functor. Note that the composition of maps $L_2S_1R_1S_2 \to L_2S_2 \to \mathrm{id}$ from these two exact triangles is in fact the adjunction counit for the pair of L_2S_1 and its right adjoint R_1S_2. By the assumptions of the theorem, L_2S_1 is an equivalence, so this composition is an isomorphism. Therefore by the octahedral axiom we have $L_2T_1(1)S_2 \simeq \mathrm{id}[2]$, qed.

Pitchfork move: $T_1(1)S_2 \simeq T_2(2)S_1$. Consider the following diagram:

$$\begin{tikzcd}
S_1R_1S_2 \arrow[r] \arrow[d] & S_2 \arrow[r] \arrow[d] & T_1(1)S_2 \\
S_1[-1] \arrow[r] & S_2L_2S_1[-1] \arrow[r] & T_2(2)S_1
\end{tikzcd}$$

where the rows are exact triangles and the two vertical morphisms are induced by the isomorphisms $R_1S_2[1] \simeq \mathrm{id}$ and its dual $\mathrm{id} \simeq L_2S_1[-1]$. The diagram commutes (again because the adjunction maps for (L_2S_1, R_1S_2) are compositions of adjunction maps for (L_1, S_1, R_1) and (L_2, S_2, R_2)), therefore there is an isomorphism $T_1(1)S_2 \simeq T_2(2)S_1$, qed.

Reidemeister move III: $T_1(1)T_2(1)T_1(1) \simeq T_2(1)T_1(1)T_2(1)$. This follows from [8], Theorem 1.2, since L_iS_i are equivalences of categories, so the maps $L_iS_jR_jS_i \to id$ have zero cones.

□

2.4.4 Checking the tangle relations

We will apply the above Theorem with $\mathcal{C}_{m+2k} = \mathcal{D}_k$, and $S^i_{m+2k} = G^i_{m+2k}$. So we will need to prove that $G^i_{m+2n} : \mathcal{D}_{n-1} \to \mathcal{D}_n$ are spherical functors, and check the 3 relations from Theorem 2.4.16. To do this, we will imitate the techniques that Cautis and Kamnitzer use to prove Theorem 2.4.8.

Lemma 2.4.17. *Recall that we have the inclusion of the divisor $j_{n,i} : X_{n,i} \to U_n$, as well as the $\mathbb{P}^1$-bundle $\pi_{n,i} : X_{n,i} \to U_{n-1}$.*

1. *We have $\mathcal{O}_{U_n}(X_{n,i}) \simeq \mathcal{E}^{-1}_{i+1} \otimes \mathcal{E}_i$*
2. *We have $\omega_{X_{n,i}} \otimes j^*_{n,i}\omega^{-1}_{U_n} \simeq j^*_{n,i}(\mathcal{E}^{-1}_{i+1} \otimes \mathcal{E}_i) \simeq \omega_{X_{n,i}} \otimes \pi^{-1}_{n,i}\omega^{-1}_{U_{n-1}}$*
3. *We have $(G^i_{m+2n})^R = F^i_{m+2n}[-1]$, and $(G^i_{m+2n})^L = F^i_{m+2n}[1]$.*

Proof. For the first two statements, see the argument used in parts (i) and (ii) of Lemma 4.3 in [27]. The third statement follows from the first two using the argument in Lemma 4.4 in [27]. □

For future use, we will need the following Lemma (which is an adaptation of Lemma 5.1 of [27] to our setting).

Lemma 2.4.18. *For $i \neq j$, the varieties $X_{n,i}$ and $X_{n,j}$ intersect transversely inside U_n.*

Proof. We will view U_n (and $X_{n,i}, X_{n,j}$) as a subvariety of $G \times^B \mathfrak{n}$, and compute tangent spaces to $X_{n,i}$ and $X_{n,j}$ at points in $X_{n,i} \cap X_{n,j}$ to show transversality.

Given $(g, x) \in G \times^B \mathfrak{n}$; first we will calculate the tangent space $T_{(g,x)}(G \times^B \mathfrak{n})$. Given $X_1 \in \mathfrak{g}, X_2 \in \mathfrak{n}$, a curve through (g, x) in $G \times \mathfrak{n}$ with tangent direction $(g \cdot X_1, X_2)$ is $(g \cdot \exp(\epsilon X_1), x + \epsilon X_2)$. Infinitesmally, $(g \cdot \exp(\epsilon X_1), x + \epsilon X_2) = (g, x)$ in $G \times^B \mathfrak{n}$ provided that $X_1 \in \mathfrak{b}$ (ie. $\exp(\epsilon X_1) \in B$), and

$$\exp(\epsilon X_1)(x + \epsilon X_2)\exp(-\epsilon X_1) \approx x$$

Discarding non-linear powers of ϵ, the latter translates to $x + \epsilon(X_2 + [X_1, x]) = x$, ie. $X_2 = -[X_1, x]$. Thus the kernel of the map $\mathfrak{g} \oplus \mathfrak{n} = T_{(g,x)}(G \times \mathfrak{n}) \twoheadrightarrow T_{(g,x)}(G \times^B \mathfrak{n})$ is the subspace $\{(X, -[X,x]) | X \in \mathfrak{b}\}$, so:

$$T_{(g,x)}(G \times^B \mathfrak{n}) \simeq \frac{\mathfrak{g} \oplus \mathfrak{n}}{\{(X, -[X,x]) \mid X \in \mathfrak{b}\}}$$

Suppose $(g, x) \in G \times^B \mathfrak{n}$ lies in U_n; or equivalently, that $\tilde{x} := gxg^{-1} \in S_n$. Now given $(X_1, X_2) \in T_{(g,x)}(G \times^B \mathfrak{n})$, we have that $(X_1, X_2) \in T_{(g,x)}(U_n)$ when the curve $(g \cdot \exp(\epsilon X_1), x + \epsilon X_2)$ lies in U_n. precisely when $g \cdot \exp(\epsilon X_1)(x + \epsilon X_2)\exp(-\epsilon X_1) \cdot g^{-1} \in S_n$ (infinitesmally). Discarding non-linear powers of ϵ, this is equivalent to saying that

$$g \cdot (x + \epsilon(X_2 + [X_1, x]) \cdot g^{-1} \in S_n$$

Since $gxg^{-1} \in S_n$, this is equivalent to $X_2 + [X_1, x] \in g^{-1} \cdot C_n \cdot g$ (recall that $S_n = z_n + C_n$ where C_n is a vector subspace). Thus:

$$\begin{aligned} T_{(g,x)}(U_n) &\simeq \frac{\{(X_1, X_2) \in \mathfrak{g} \oplus \mathfrak{n} \mid X_2 + [X_1, x] \in g^{-1} C_n g\}}{\{(X, -[X,x]) \mid X \in \mathfrak{b}\}} \\ &\simeq \frac{\{(X, Y) \in \mathfrak{g} \oplus C_n \mid [X, \tilde{x}] + Y \in g \cdot \mathfrak{n}\}}{g \cdot \mathfrak{b} \oplus 0} \end{aligned}$$

For the last isomorphism, use the substitution $X = -gX_1g^{-1}, Y = g(X_2 + [X_1, x])g^{-1}$. Recall from the discussion in Section 1.4 of [44] that the map $\pi : \mathfrak{g} \oplus C_n \to \mathfrak{g}, \pi(X, Y) = [X, \tilde{x}] + Y$ is surjective. Hence:

$$\dim(T_{g,x}(U_n)) = \dim(\mathfrak{n}) + \dim(C_n) - \dim(\mathfrak{b})$$

In particular, this shows that U_n is smooth. Now suppose that $(g, x) \in X_{n,i} \cap X_{n,j}$. It is clear that $X_{n,i} = U_n \cap (G \times^B \mathfrak{n}^i)$, where $\mathfrak{n}^i \subset \mathfrak{n}$ is the nilradical of the minimal parabolic corresponding to i. The above argument is valid after replacing $\mathfrak{n}$ with $\mathfrak{n}^i$, and we obtain:

$$T_{(g,x)}(X_{n,i}) \simeq \frac{\{(X, Y) \in \mathfrak{g} \oplus C_n \mid [X, \tilde{x}] + Y \in g \cdot \mathfrak{n}^i\}}{g \cdot \mathfrak{b} \oplus 0}$$

$$T_{(g,x)}(X_{n,j}) \simeq \frac{\{(X, Y) \in \mathfrak{g} \oplus C_n \mid [X, \tilde{x}] + Y \in g \cdot \mathfrak{n}^j\}}{g \cdot \mathfrak{b} \oplus 0}$$

Using the surjectivity of π, it is clear that $T_{(g,x)}(X_{n,i})$ and $T_{(g,x)}(X_{n,j})$ are distinct co-dimension 1 subspaces in $T_{(g,x)}(U_n)$. Hence $T_{(g,x)}(X_{n,i}) + T_{(g,x)}(X_{n,j}) = T_{(g,x)}(U_n)$, and $X_{n,i}$ and $X_{n,j}$ intersect transversely in U_n. $\square$

Corollary 2.4.19. *The following intersections are transverse:*

1. $\pi_{12}^{-1}(X_{n,i}) \cap \pi_{23}^{-1}(X_{n,j})$ *inside* $U_{n-1} \times U_n \times U_{n-1}$ *for* $i \neq j$.
2. $\pi_{12}^{-1}(X_{n,i}) \cap \pi_{23}^{-1}(X_{n+1,j})$ *inside* $U_{n-1} \times U_n \times U_{n+1}$.

Proof. Both statements follow using Lemma 5.3 from [27]; for the first, we also need Lemma 2.4.18. □

Now we can show that the functors G^i_{m+2n} satisfy the conditions of Theorem 2.4.16.

Proposition 2.4.20. *We have* $F^i_{m+2n} \circ G^i_{m+2n} \simeq [-1] \oplus [1]$.

Proof. This is an analogue of Corollary 5.10 from [27], and can be proved using the same arguments (see the proofs of Proposition 5.8 and Theorem 5.9 in [27]). □

Corollary 2.4.21. *The functor* $G^i_{m+2n} : \mathcal{D}_{n-1} \to \mathcal{D}_n$ *is a strongly spherical functor.*

Proof. The first condition from Definition 2.4.13 states that we should have a triangle $\mathbf{id} \to (G^i_{m+2n})^R G^i_{m+2n} \to [-2]$, ie. a triangle $\mathbf{id} \to F^i_{m+2n} G^i_{m+2n}[-1] \to [-2]$. This follows from Proposition 2.4.20.

The second condition states that the natural map $(G^i_{m+2n})^R \to (G^i_{m+2n})^L[-2]$ is an isomorphism; this follows from Lemma 2.4.17, since $(G^i_{m+2n})^R = F^i_{m+2n}[-1]$, $(G^i_{m+2n})^L = F^i_{m+2n}[1]$. □

Proposition 2.4.22. $F^i_{m+2n} \circ G^{i+1}_{m+2n} \simeq \boldsymbol{id} \simeq F^{i+1}_{m+2n} \circ G^i_{m+2n}$

Proof. This follows from the argument used in Proposition 5.6 of [27] (here one has to use the first statement from Corollary 2.4.19). □

Proposition 2.4.23. *The following relations hold:*

1. $G^{i+l}_{m+2k+2} \circ G^i_{m+2k} \simeq G^i_{m+2k+2} G^{i+l-2}_{m+2k}$ *for* $l \geq 2$.
2. $G^{i+l-2}_{m+2k} \circ F^i_{m+2k} \simeq F^i_{m+2k+2} G^{i+l}_{m+2k+2}$, $G^i_{m+2k} \circ F^{i+l-2}_{m+2k} \simeq F^{i+l}_{m+2k+2} G^i_{m+2k+2}$ *for* $l \geq 2$

Proof. This follows from the argument used in Proposition 5.16 of [27] (here one has to use the second statement from Corollary 2.4.19). □

Now we have verified the conditions of Theorem 2.4.16, so we introduce the twists $T^i_{m+2n}(l)$ and construct a weak representation of $\mathbf{FTan}_m$.

Definition 2.4.24. Define the functors $T^i_{m+2n}(1)$ and $T^i_{m+2n}(2)$ via the distinguished triangles:

$$G^i_{m+2n}(G^i_{m+2n})^R \to \mathbf{id} \to T^i_{m+2n}(1), \qquad T^i_{m+2n}(2) \to \mathbf{id} \to G^i_{m+2n}(G^i_{m+2n})^L$$

Theorem 2.4.25. *The assignments*

$$\Psi(g^i_{m+2n}) = G^i_{m+2n}, \Psi(f^i_{m+2n}) = F^i_{m+2n}$$

$$\Psi(t^i_{m+2n}(1)) = T^i_{m+2n}(1), \Psi(t^i_{m+2n}(2)) = T^i_{m+2n}(2)$$

$$\Psi(w^i_{m+2n}(1)) = [-1], \Psi(w^i_{m+2n}(-1)) = [1]$$

give rise to a weak representation of $\boldsymbol{FTan}_m$ *using the categories* $\mathcal{D}_k$. □

2.4.5 Functors $\Psi(\alpha) : \mathcal{D}_p \to \mathcal{D}_q$ indexed by affine tangles

At this point, we have constructed a functor $\Psi(\alpha) : \mathcal{D}_p \to \mathcal{D}_q$ for each framed linear $(m+2p, m+2q)$-tangle α. To extend this construction to framed affine tangles, it suffices to construct a functor $\Psi(s^{m+2n}_{m+2n}) : \mathcal{D}_n \to \mathcal{D}_n$ satisfying the relations in Lemma 2.3.10. Define $\mathcal{S}_n(\mathcal{F}) = \mathcal{F} \otimes \mathcal{E}^{-1}_{m+2n}$, and let $\Psi(s^{m+2n}_{m+2n}) := \mathcal{S}_n$. The relations that we must check are the following:

Proposition 2.4.26. *The following identities hold, where $1 \leq i \leq m+2n-2, 1 \leq p \leq 2$:*

- $\mathcal{S}_{n-1} \circ F^i_{m+2n} \simeq F^i_{m+2n} \circ \mathcal{S}_n$
- $\mathcal{S}_n \circ G^i_{m+2n} \simeq G^i_{m+2n} \circ \mathcal{S}_{n-1}$
- $\mathcal{S}_n \circ T^i_{m+2n}(p) \simeq T^i_{m+2n}(p) \circ \mathcal{S}_n$
- $F^{m+2n-1}_{m+2n} \circ \mathcal{S}_n \circ T^{m+2n-1}_{m+2n}(2) \circ \mathcal{S}_n \circ T^{m+2n-1}_{m+2n}(2) \simeq F^{m+2n-1}_{m+2n}$
- $\mathcal{S}_n \circ T^{m+2n-1}_{m+2n}(2) \circ \mathcal{S}_n \circ T^{m+2n-1}_{m+2n}(2) \circ G^{m+2n-1}_{m+2n} \simeq G^{m+2n-1}_{m+2n}$

Proof. The first three identities are clear. To prove the fourth identity, we calculate as follows (the last identity is proved similarly). Let $\mathcal{Q} \in \mathcal{D}_n$; since we have $T^{m+2n-1}_{m+2n}(2)\mathcal{Q} \simeq \{\mathcal{Q} \to G^{m+2n-1}_{m+2n} \circ G^{m+2n-1,L}_{m+2n}(\mathcal{Q})\} \simeq \{\mathcal{Q} \to G^{m+2n-1}_{m+2n} \circ F^{m+2n-1}_{m+2n}(\mathcal{Q})[1]\}$ (by Lemma 2.4.17 part 3), we compute:

$$
\begin{aligned}
&G^{m+2n-1}_{m+2n} \circ F^{m+2n-1}_{m+2n}(\mathcal{Q} \otimes \mathcal{E}^{-1}_{m+2n}) \\
&\simeq G^{m+2n-1}_{m+2n} \circ (\pi_{n,m+2n-1*}(i^*_{n,m+2n-1}(\mathcal{Q} \otimes \mathcal{E}^{-1}_{m+2n}) \otimes \mathcal{E}^{-1}_{m+2n})) \\
&\simeq i_{n,m+2n-1*}[\mathcal{E}_{m+2n-1} \otimes \pi^*_{n,m+2n-1}\pi_{n,m+2n-1*}(i^*_{n,m+2n-1}\mathcal{Q} \otimes \mathcal{E}^{-2}_{m+2n})]
\end{aligned}
$$

$$T^{m+2n-1}_{m+2n}(2) \circ \mathcal{S}_n(\mathcal{Q}) \simeq \{\mathcal{Q} \otimes \mathcal{E}^{-1}_{m+2n} \to G^{m+2n-1}_{m+2n} \circ F^{m+2n-1}_{m+2n}(\mathcal{Q} \otimes \mathcal{E}^{-1}_{m+2n})[1]\}$$

$$
\begin{aligned}
&F^{m+2n-1}_{m+2n} \circ S_{m+2n} \circ T^{m+2n-1}_{m+2n}(2) \circ S_{m+2n}(\mathcal{Q}) \simeq \{\pi_{n,m+2n-1*}(\mathcal{E}^{-3}_{m+2n} \otimes i^*_{n,m+2n-1}\mathcal{Q}) \to \\
&\pi_{n,m+2n-1*}(\mathcal{E}^{-2}_{m+2n} \otimes i^*_{n,m+2n-1} i_{n,m+2n-1*} \\
&(\mathcal{E}_{m+2n-1} \otimes \pi^*_{n,m+2n-1}\pi_{n,m+2n-1*}(\mathcal{E}^{-2}_{m+2n} \otimes i^*_{n,m+2n-1}\mathcal{Q})))[1]\}
\end{aligned}
$$

Note that we have an exact triangle $\mathcal{O}_{U_n}(-X_{n,m+2n-1})[-1] \otimes \mathcal{F} \to i^*_{n,m+2n-1} i_{n,m+2n-1*}\mathcal{F} \to \mathcal{F}$, and that $\mathcal{O}_{U_n}(-X_{n,m+2n-1}) \simeq \mathcal{E}^{-1}_{m+2n-1} \otimes \mathcal{E}_{m+2n}$. This gives us the following exact triangle, where we abbreviate $\mathcal{R} = \pi_{n,m+2n-1*}(\mathcal{E}^{-2}_{m+2n} \otimes i^*_{n,m+2n-1}\mathcal{Q})$:

$$
\begin{aligned}
&\pi_{n,m+2n-1*}(\mathcal{E}^{-1}_{m+2n} \otimes \pi^*_{n,m+2n-1}\mathcal{R}) \\
&\to \pi_{n,m+2n-1*}(\mathcal{E}^{-2}_{m+2n} \otimes i^*_{n,m+2n-1} i_{n,m+2n-1*}(\mathcal{E}_{m+2n-1} \otimes \pi^*_{n,m+2n-1}\mathcal{R})) \\
&\to \pi_{n,m+2n-1*}(\mathcal{E}^{-2}_{m+2n} \otimes \mathcal{E}_{m+2n-1} \otimes \pi^*_{n,m+2n-1}\mathcal{R})
\end{aligned}
$$

We claim that $\pi_{n,m+2n-1*}(\mathcal{E}^{-1}_{m+2n}\otimes\pi^*_{n,m+2n-1}\mathcal{R})\simeq 0$. This implies that the second and third terms in the above triangle are isomorphic; so continuing the above computation (and using the isomorphism $\pi_{n,m+2n-1*}(\mathcal{A}\otimes\pi^*_{n,m+2n-1}\mathcal{B})\simeq\mathcal{B}\otimes\pi_{n,m+2n-1*}\mathcal{A}$):

$$\begin{aligned}
&F^{m+2n-1}_{m+2n}\circ S_{m+2n}\circ T^{m+2n-1}_{m+2n}(2)\circ S_{m+2n}(\mathcal{Q})\simeq\{\pi_{n,m+2n-1*}(\mathcal{E}^{-3}_{m+2n}\otimes i^*_{n,m+2n-1}\mathcal{Q})\to\\
&\pi_{n,m+2n-1*}(\mathcal{E}^{-2}_{m+2n}\otimes\mathcal{E}_{m+2n-1}\otimes\pi^*_{n,m+2n-1}\pi_{n,m+2n-1*}(\mathcal{E}^{-2}_{m+2n}\otimes i^*_{n,m+2n-1}\mathcal{Q}))[1]\}\\
&\simeq\{\pi_{n,m+2n-1*}(\mathcal{E}^{-3}_{m+2n}\otimes i^*_{n,m+2n-1}\mathcal{Q})\to\pi_{n,m+2n-1*}(\mathcal{E}^{-2}_{m+2n}\otimes\mathcal{E}_{m+2n-1})\\
&\otimes\pi_{n,m+2n-1*}(\mathcal{E}^{-2}_{m+2n}\otimes i^*_{n,m+2n-1}\mathcal{Q})[1]\}
\end{aligned}$$

Here note that since $\pi_{n,m+2n-1}$ is proper, $\pi_{n,m+2n-1*}$ commutes with the functors of Grothendieck-Serre duality on the categories $D^b(\mathrm{Coh}(X_{n,m+2n-1}))$ and $D^b(\mathrm{Coh}(U_{n-1}))$. We have $\omega_{X_{n,m+2n-1}}\simeq\mathcal{E}_{m+2n-1}\otimes\mathcal{E}^{-1}_{m+2n}$, and $\omega_{U_{n-1}}\simeq\mathcal{O}_{U_{n-1}}$ since U_{n-1} is symplectic. Thus (noting $\dim(X_{n,m+2n-1})=\dim(U_{n-1})+1$):

$$\begin{aligned}
&\pi_{n,m+2n-1*}(\mathcal{E}^{-2}_{m+2n}\otimes\mathcal{E}_{m+2n-1})[1]\simeq\pi_{n,m+2n-1*}(\mathcal{RH}om(\mathcal{E}_{m+2n},\omega_{X_{n,m+2n-1}}))[1]\\
&\simeq\mathcal{RH}om(\pi_{n,m+2n-1*}(\mathcal{E}_{m+2n}),\omega_{U_{n-1}})\simeq\pi_{n,m+2n-1*}(\mathcal{E}_{m+2n})^\vee
\end{aligned}$$

Consider the locally free sheaf $\mathcal{E}$ on $U_{n-1}=S_n\times_{\mathfrak{sl}_{m+2n}}T^*\mathcal{P}_{m+2n-1,m+2n}$ corresponding to $\mathrm{Ker}(z)$; it is clear that the associated $\mathbb{P}^1$ bundle is $X_{n,m+2n-1}$ (i.e. $\mathbb{P}(\mathcal{E})\simeq X_{n,m+2n-1}$). Under this identification, the vector bundle $\mathcal{E}_{m+2n}$ on U_{n-1} is isomorphic to the relative $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E})$. Therefore, $\pi_{n,m+2n-1*}(\mathcal{E}_{m+2n})\simeq\mathcal{E}$. Returning to the above computation:

$$\begin{gathered}
F^{m+2n-1}_{m+2n}\circ S_{m+2n}\circ T^{m+2n-1}_{m+2n}(2)\circ S_{m+2n}(\mathcal{Q})\simeq\\
\{\pi_{n,m+2n-1*}(\mathcal{E}^{-3}_{m+2n}\otimes i^*_{n,m+2n-1}\mathcal{Q})\to\pi_{n,m+2n-1*}(\mathcal{E}^{-2}_{m+2n}\otimes i^*_{n,m+2n-1}\mathcal{Q})\otimes\mathrm{Ker}(z)^\vee\}
\end{gathered}$$

Note that from the exact sequence $0\to\mathcal{E}^{-1}_{m+2n}\to\pi^*_{n,m+2n-1}\mathrm{Ker}(z)^\vee\to\mathcal{E}^{-1}_{m+2n-1}\to 0$, we have an exact triangle $\mathcal{E}^{-3}_{m+2n}\otimes i^*_{n,m+2n-1}\mathcal{Q}\to\mathcal{E}^{-2}_{m+2n}\otimes\pi^*_{n,m+2n-1}(\mathrm{Ker}(z))^\vee\otimes i^*_{n,m+2n-1}\mathcal{Q}\to\mathcal{E}^{-1}_{m+2n-1}\otimes\mathcal{E}^{-2}_{m+2n}\otimes i^*_{n,m+2n-1}\mathcal{Q}$. Taking the push-forward under $\pi_{n,m+2n-1}$ gives us the exact triangle $\pi_{n,m+2n-1*}(\mathcal{E}^{-3}_{m+2n}\otimes i^*_{n,m+2n-1}\mathcal{Q})\to\pi_{n,m+2n-1*}(\mathcal{E}^{-2}_{m+2n}\otimes i^*_{n,m+2n-1}\mathcal{Q})\otimes(\mathrm{Ker}(z))^\vee\to\pi_{n,m+2n-1*}(\mathcal{E}^{-1}_{m+2n-1}\otimes\mathcal{E}^{-2}_{m+2n}\otimes i^*_{n,m+2n-1}\mathcal{Q})$. Thus:

$$\begin{aligned}
&\{\pi_{n,m+2n-1*}(\mathcal{E}^{-3}_{m+2n}\otimes i^*_{n,m+2n-1}\mathcal{Q})\to\pi_{n,m+2n-1*}(\mathcal{E}^{-2}_{m+2n}\otimes i^*_{n,m+2n-1}\mathcal{Q})\otimes\mathrm{Ker}(z)^\vee\}\\
&\simeq\pi_{n,m+2n-1*}(\mathcal{E}^{-1}_{m+2n-1}\otimes\mathcal{E}^{-2}_{m+2n}\otimes i^*_{n,m+2n-1}\mathcal{Q})\simeq\pi_{n,m+2n-1*}(\mathcal{E}^{-1}_{m+2n}\otimes i^*_{n,m+2n-1}\mathcal{Q})\\
&\simeq F^{m+2n-1}_{m+2n}\mathcal{Q}
\end{aligned}$$

In the second equality above we have used the fact that the line bundle $\mathcal{E}^{-1}_{m+2n-1}\otimes\mathcal{E}^{-1}_{m+2n}$ is trivial when restricted to the divisor $X_{n,m+2n-1}$. Using the fact that $F^{m+2n-1}_{m+2n}\simeq F^{m+2n-1}_{m+2n}T^{m+2n-1}_{m+2n}[1]$ (Reidemeister I move), we conclude that $F^{m+2n-1}_{m+2n}\circ\mathcal{S}_n\circ T^{m+2n-1}_{m+2n}(2)\circ\mathcal{S}_n\simeq F^{m+2n-1}_{m+2n}\circ T^{m+2n-1}_{m+2n}(1)$, as desired.

The proof of the last relation is similar. □

To summarize:

Theorem 2.4.27. *The assignments*

$$\begin{aligned}\Psi(g^i_{m+2n}) &= G^i_{m+2n}, \Psi(f^i_{m+2n}) = F^i_{m+2n}\\ \Psi(t^i_{m+2n}(1)) &= T^i_{m+2n}(1), \Psi(t^i_{m+2n}(2)) = T^i_{m+2n}(2)\\ \Psi(w^i_{m+2n}(1)) &= [-1], \Psi(w^i_{m+2n}(-1)) = [1]\\ \Psi(s^{m+2n}_{m+2n}) &= \mathcal{S}_n\end{aligned}$$

give rise to a weak representation of $\boldsymbol{AFTan}_m$ *using the categories* $\mathcal{D}_k$. □

2.5 The exotic t-structure on $\mathcal{D}_n$

2.5.1

First we recall that the construction of the exotic t-structure on $\mathcal{D}_n$ from [16] is given by the following. Let $\mathbb{B}_{aff}$ denotes the braid group attached to the affine Weyl group $W_{aff} = W \ltimes \Lambda$, where W is the Weyl group of $\mathfrak{g} = \mathfrak{sl}_{m+2n}$, and Λ is the weight lattice. Let $\mathbb{B}^{\mathrm{Cox}}_{aff} \subset \mathbb{B}_{aff}$ denote the braid group attached to the $W^{\mathrm{Cox}}_{aff} = W \ltimes Q$ where Q is the root lattice. Denote by $\mathbb{B}^{+}_{aff} \subset \mathbb{B}^{\mathrm{Cox}}_{aff}$ denote the semigroup generated by the lifts of the simple reflections $\tilde{s}_\alpha$ in the Coxeter group W^{Cox}_{aff}.

Using Bezrukavnikov and Mirkovic's construction (see Sections 1.1.1 and 1.3.2 of [16]), there exists a weak action of the affine braid group $\mathbb{B}_{aff}$ on $\mathcal{D}_n$ (i.e. for every $b \in \mathbb{B}_{aff}$, there exists a functor $\Psi(b) : \mathcal{D}_n \to \mathcal{D}_n$, such that $\Psi(b_1 b_2) \simeq \Psi(b_1) \circ \Psi(b_2)$). This action is related to that from the previous section using the following result, which is easy to check.

Lemma 2.5.1. $\mathbb{B}_{aff}$ *can be identified with the group of all bijective* $(m+2n, m+2n)$ *affine tangles (i.e. where each strand connects a point in the inner circle with a point in the outer circle). Under this identification, the action of* $\mathbb{B}_{aff}$ *on* $\mathcal{D}_n$ *coincides with the action coming from Theorem 2.4.27.* □

Following Bezrukavnikov and Mirkovic (see section 1.5 of [16]), the exotic t-structure on $\mathcal{D}_n$ is defined as follows:

$$\mathcal{D}^{\geq 0}_n = \{\mathcal{F} \mid R\Gamma(\Psi(b^{-1})\mathcal{F}) \in D^{\geq 0}(\mathrm{Vect})\ \forall\ b \in \mathbb{B}^{+}_{aff}\}$$
$$\mathcal{D}^{\leq 0}_n = \{\mathcal{F} \mid R\Gamma(\Psi(b)\mathcal{F}) \in D^{\leq 0}(\mathrm{Vect})\ \forall\ b \in \mathbb{B}^{+}_{aff}\}$$

By definition, the functors that correspond to positive braids are left t-exact, and the functors that correspond to negative braids are right t-exact. In particular, the functor R_n that corresponds to the braid r_n that is both positive and negative (since it has no crossings) is t-exact.

Proposition 2.5.2. *The functor $G^i_{m+2n} : \mathcal{D}_{n-1} \to \mathcal{D}_n$ is t-exact with respect to the exotic t-structures on the two categories.*

To prove this, we will need the following two lemmas:

Lemma 2.5.3. *Given $b \in \mathbb{B}'^{+}_{aff}$ considered as a bijective $(m+2n-2, m+2n-2)$-tangle, there exists bijective $(m+2n-2, m+2n-2)$ tangles $\epsilon_i(b), \eta_i(b) \in \mathbb{B}^+_{aff}$ such that $b \circ f^i_{m+2n} = f^i_{m+2n} \circ \epsilon_i(b), b^{-1} \circ f^i_{m+2n} = f^i_{m+2n} \circ \eta_i(b)^{-1}$.*

Proof. Using the cap-crossing isotopy relation (9), we may define $\epsilon_i(t^j_{m+2n-2}) = \eta_i(t^j_{m+2n-2}) = t^{j+2}_{m+2n}$ if $j \geq i$, and $\epsilon_i(t^j_{m+2n-2}) = \eta_i(t^j_{m+2n-2}) = t^j_{m+2n}$ if $j \leq i-2$. Also define $\epsilon_i(t^i_{m+2n-2}) = \eta_i(t^i_{m+2n-2})$ to be the tangle with a strand connecting $(\zeta_k, 0)$ to $(2\zeta_k, 0)$ for $k \neq i-1, i$, and a strand connecting $(\zeta_{i-1}, 0)$ to $(2\zeta_{i+2}, 0)$ that passes beneath a strand connecting $(\zeta_{i+2}, 0)$ to $(2\zeta_{i-1}, 0)$. It is straightforward to check that with this definition, $\epsilon_i(t^j_{m+2n-2}) = \eta_i(t^j_{m+2n-2}) \in \mathbb{B}^+_{aff}$.

Given $b \in \mathbb{B}'^{+}_{aff}$, choose a decomposition $b = t^{i_1}_{m+2n-2}(1) \circ t^{i_2}_{m+2n-2}(1) \circ \cdots \circ t^{i_k}_{m+2n-2}(1)$; clearly $\epsilon_i(b) = \epsilon_i(t^{i_1}_{m+2n-2}(1)) \circ \cdots \circ \epsilon_i(t^{i_k}_{m+2n-2}(1))$ and $\eta_i(b) = \eta_i(t^{i_k}_{m+2n-2}(1)) \circ \cdots \circ \eta_i(t^{i_1}_{m+2n-2}(1))$ satisfy the required condition. □

Lemma 2.5.4. *Given $\mathcal{F} \in \mathcal{D}^{\geq 0}_n, \mathcal{G} \in \mathcal{D}^{\leq 0}_n$, we have $R\Gamma((G^{m+2n-1}_{m+2n})^R \mathcal{F}) \in \mathcal{D}^{\geq 0}(Vect)$ and $R\Gamma((G^1_{m+2n})^L \mathcal{G}) \in \mathcal{D}^{\leq 0}(Vect)$.*

Proof. Let $\mathcal{F} \in \mathcal{D}^{\geq 0}_n$. Since:

$$G^{m+2n-1L}_{m+2n}(\mathcal{E}_{m+2n}[-1]) \simeq F^{m+2n-1}_{m+2n}\mathcal{E}_{m+2n} \simeq \pi_{n,m+2n-1*}(i^*_{n,m+2n-1}\mathcal{O}_{U_n}) \simeq \mathcal{O}_{U_{n-1}}$$

we have that:

$$\begin{aligned} \mathrm{R}\Gamma((G^{m+2n-1}_{m+2n})^R\mathcal{F}) &\simeq \mathrm{RHom}((G^{m+2n-1}_{m+2n})^L\mathcal{E}_{m+2n}[-1], (G^{m+2n-1}_{m+2n})^R\mathcal{F}) \\ &\simeq \mathrm{RHom}(\mathcal{E}_{m+2n}[-1], G^{m+2n-1}_{m+2n}(G^{m+2n-1}_{m+2n})^R\mathcal{F}) \\ &\simeq \mathrm{R}\Gamma(\mathcal{E}^{-1}_{m+2n}[1] \otimes G^{m+2n-1}_{m+2n}(G^{m+2n-1}_{m+2n})^R\mathcal{F}) \end{aligned}$$

We will prove that $\mathcal{E}^{-1}_{m+2n}[1] \otimes G^{m+2n-1}_{m+2n}(G^{m+2n-1}_{m+2n})^R\mathcal{F} \in \mathcal{D}^{\geq 0}_n$, which will imply the first statement of the lemma. Since G^{m+2n-1}_{m+2n} is strongly spherical and $(G^{m+2n-1}_{m+2n})^L \simeq (G^{m+2n-1}_{m+2n})^R[2]$, it suffices to prove that $\mathcal{E}^{-1}_{m+2n} \otimes G^{m+2n-1}_{m+2n}(G^{m+2n-1}_{m+2n})^L\mathcal{F} \simeq \mathcal{E}^{-1}_{m+2n} \otimes G^{m+2n-1}_{m+2n}(G^{m+2n-1}_{m+2n})^R\mathcal{F}[2] \in \mathcal{D}^{\geq -1}_n$. We have a distinguished triangle $T^{m+2n-1}_{m+2n}(2)\mathcal{F} \to \mathcal{F} \to G^{m+2n-1}_{m+2n}(G^{m+2n-1}_{m+2n})^L\mathcal{F}$; and hence a distinguished triangle $\mathcal{E}^{-1}_{m+2n} \otimes T^{m+2n-1}_{m+2n}(2)\mathcal{F} \to \mathcal{F} \to \mathcal{E}^{-1}_{m+2n} \otimes G^{m+2n-1}_{m+2n}(G^{m+2n-1}_{m+2n})^L\mathcal{F}$. The functor $\mathcal{E}^{-1}_{m+2n} \otimes \cdot$ corresponds to the braid s^{m+2n}_{m+2n} which leaves the first $m+2n-1$ vertices in place, and winds the last vertex counter-clockwise around the circle underneath the other strands. Using the identity $r^{m+2n}_{m+2n} = s^{m+2n}_{m+2n} \circ t^{m+2n-1}_{m+2n}(2) \circ \cdots t^1_{m+2n}(2)$, we see that the functors $\mathcal{E}^{-1}_{m+2n} \otimes \cdot$ and $\mathcal{E}^{-1}_{m+2n} \otimes T^{m+2n-1}_{m+2n}(2)$ are left exact since they correspond to negative braids. Thus $\mathcal{E}^{-1}_{m+2n} \otimes T^{m+2n-1}_{m+2n}(2)\mathcal{F}, \mathcal{E}^{-1}_{m+2n} \otimes \mathcal{F} \in \mathcal{D}^{\geq 0}_n$, using the long exact sequence of cohomology we obtain that $\mathcal{E}^{-1}_{m+2n} \otimes G^{m+2n-1}_{m+2n}(G^{m+2n-1}_{m+2n})^L\mathcal{F} \in \mathcal{D}^{\geq -1}_n$, as required.

The proof of the second half of the lemma follows the same logic. Let $\mathcal{G} \in \mathcal{D}_n^{\leq 0}$. First we show that $(G^1_{m+2n})^L\mathcal{E}_1 \simeq \mathcal{O}_{U_{n-1}}$. By definition, $F^1_{m+2n}\mathcal{E}_1 \simeq \pi_{n,1*}(i^*_{n,1}\mathcal{E}_1 \otimes \mathcal{E}_2^{-1})$. Since the map $\pi_{n,1} : X_{n,1} \to U_{n-1}$ is a $\mathbb{P}^1$ fibre bundle, we have $\pi_{n,1*}\omega_{X_{n,1}}[\dim X_{n,1}] \simeq \omega_{U_{n-1}}[\dim U_{n-1}]$. Since U_{n-1} is a symplectic variety, $\omega_{U_{n-1}} \simeq \mathcal{O}_{U_{n-1}}$; so $\pi_{n,1*}\omega_{X_{n,1}} \simeq \mathcal{O}_{U_{n-1}}[-1]$. We will show that $\omega_{X_{n,1}} \simeq i^*_{n,1}\mathcal{E}_1 \otimes \mathcal{E}_2^{-1}$; it will then follow that $(G^1_{m+2n})^L\mathcal{E}_1 \simeq F^1_{m+2n}\mathcal{E}_1[1] \simeq \pi_{n,1*}\omega_{X_{n,1}}[1] \simeq \mathcal{O}_{U_{n-1}}$, as claimed. Under the embedding of the divisor $i_{n,1} : X_{n,1} \hookrightarrow U_n$, the adjunction formula gives $\omega_{X_{n,1}} \simeq i^*_{n,1}(\omega_{U_n} \otimes \mathcal{O}_{U_n}(X_{n,1})) \simeq i^*_{n,1}\mathcal{O}_{U_n}(X_{n,1})$.

It then follows that:

$$\begin{aligned}
\mathrm{R}\Gamma((G^1_{m+2n})^L\mathcal{G}) &\simeq \mathrm{RHom}((G^1_{m+2n})^L\mathcal{E}_1, (G^1_{m+2n})^L\mathcal{G}) \\
&\simeq \mathrm{RHom}(\mathcal{E}_1, G^1_{m+2n}(G^1_{m+2n})^L\mathcal{G}) \\
&\simeq \mathrm{R}\Gamma(\mathcal{E}_1^{-1} \otimes G^1_{m+2n}(G^{m+2n-1}_{m+2n})^L\mathcal{G})
\end{aligned}$$

The functor $\mathcal{E}_1^{-1} \otimes \cdot$ is right exact as it corresponds to the positive braid s^1_{m+2n} that leaves the last $m+2n-1$ vertices in place, and winds the first vertex counterclockwise around the circle underneath the other strands. Using the exact triangle $T^1_{m+2n}(2)\mathcal{F} \to \mathcal{F} \to G^1_{m+2n}(G^{m+2n-1}_{m+2n})^L\mathcal{F}$, we deduce $G^1_{m+2n}(G^{m+2n-1}_{m+2n})^L$ is right exact since $T^1_{m+2n}(2)$ is right exact. Thus $\mathcal{E}_1^{-1} \otimes G^1_{m+2n}(G^{m+2n-1}_{m+2n})^L\mathcal{G} \in \mathcal{D}_n^{\leq 0}$, as required. □

Now we are ready to prove Proposition 2.5.2.

Proof. The functors G^i_{m+2n} are conjugate by the t-exact, invertible functor R_n; thus it suffices to prove that G^1_{m+2n} is left t-exact, and that G^{m+2n-1}_{m+2n} is right t-exact, or equivalently that $(G^1_{m+2n})^L$ is right t-exact and $(G^{m+2n-1}_{m+2n})^R$ is left t-exact

Let $\mathcal{F} \in \mathcal{D}_n^{\leq 0}$. To prove that $(G^1_{m+2n})^L$ is right t-exact, we must show that $(G^1_{m+2n})^L\mathcal{F} \in \mathcal{D}_{n-1}^{\leq 0}$:

$$R\Gamma(\Psi(b)(G^1_{m+2n})^L\mathcal{F}) \in \mathcal{D}^{\leq 0}(\mathrm{Vect}) \ \forall \ b \in \mathbb{B}'^{+}_{aff} \tag{2.2}$$

By Lemma 2.4.17 we have $(G^1_{m+2n})^L \simeq F^1_{m+2n}[1]$ and by Lemma 2.5.3 we know that for any positive braid b there is a positive braid $\epsilon_i(b)$ such that $\Psi(b)F^1_{m+2n} \simeq F^1_{m+2n}\Psi(\epsilon_i(b))$, so (2.2) is equivalent to

$$R\Gamma((G^1_{m+2n})^L\Psi(\epsilon_i(b))\mathcal{F}) \in \mathcal{D}^{\leq 0}(\mathrm{Vect}) \ \forall \ b \in \mathbb{B}'^{+}_{aff} \tag{2.3}$$

The braid $\epsilon_i(b)$ is positive, so $\Psi(\epsilon_i(b))\mathcal{F} \in \mathcal{D}^{\leq 0}$, and (2.3) follows from Lemma 2.5.4.

The proof that $(G^{m+2n-1}_{m+2n})^R$ is left t-exact follows analogously from Lemmas 2.5.3 and 2.5.4. □

Definition 2.5.5. Let $\mathcal{D}_n^0$ denote the heart of the exotic t-structure on $\mathcal{D}_n$.

The following theorem is the main result of this section:

Theorem 2.5.6. *The functor G_n^i sends irreducible objects in $\mathcal{D}_{n-1}^0$ to irreducible objects in $\mathcal{D}_n^0$.*

Proof. This follows from Proposition 2.5.2 and the Theorem in Section 4.2 of [16]. □

2.5.2 Irreducible objects in the heart of the exotic t-structure on $\mathcal{D}_n$

Definition 2.5.7. Let an affine crossingless $(m, m+2n)$ matching be an affine $(m, m+2n)$-tangle whose vertical projection to $\mathbb{C}$ has no crossings, with the blackboard framing. Let an unlabelled affine crossingless matching $(m, m+2n)$-matching be an affine crossingless matching where the m inner points are not labelled. Let $\mathrm{Cross}(n)$ be the set of all unlabelled affine crossingless matchings.

We will describe the irreducible objects in the heart of the exotic t-structure on $\mathcal{D}_n$ using the functors constructed in the previous section.

Lemma 2.5.8. *We have $|Cross(n)| = \binom{m+2n}{n}$.*

Proof. It suffices to construct a bijection between unlabelled affine $(m, m+2n)$ crossingless matchings and assignments of $m+n$ plus signs and n minus to $m+2n$ labelled points on a circle. Given such an assignment of pluses and minuses to the points $(2,0), (2\zeta_{m+2n}, 0), \cdots, (2\zeta_{m+2n}^{m+2n-1}, 0)$, for each minus, move anti-clockwise around the circle and connect the minus to the first plus such that the number of pluses and minuses between these two points is equal. After connecting the m remaining pluses on the outer circle to the m unlabelled points on the inner circle without crossings, we have our desired unlabelled affine $(m, m+2n)$ crossingless matching. □

Lemma 2.5.9. *Let $\widetilde{\alpha}$ be any affine $(m, m+2n)$-tangle, from which we obtain α by forgetting the labelling on the inner circle. Then the isomorphism class of the functor $\Psi(\widetilde{\alpha})$ depends only on α.*

Proof. It is easy to check $\Psi(r_m)$ is isomorphic to the identity on $\mathcal{D}_0$; the result follows. □

Thus given $\alpha \in \mathrm{Cross}(n)$, we obtain a functor $\Psi(\alpha) : \mathcal{D}_0 \to \mathcal{D}_n$. Let $\Psi_\alpha = \Psi(\alpha)\underline{v}$ denote the image of the 1-dimensional vector space $\underline{v}$ in $\mathcal{D}_0 \simeq D^b(\mathrm{Vect})$ under the functor $\Psi(\alpha)$.

Proposition 2.5.10. *The irreducible objects in the heart of the exotic t-structure on $\mathcal{D}_n$ are precisely given by Ψ_α, as α ranges across the $\binom{m+2n}{n}$ unlabelled affine $(m, m+2n)$ crossingless matchings.*

Proof. It follows by induction that every affine crossingless $(m, m+2n)$ matching can be expressed as a product $g^{i_n}_{m+2n} \circ \cdots \circ g^{i_2}_{m+4} \circ g^{i_1}_{m+2}$. Thus $\Psi_\alpha \simeq G^{i_n}_{m+2n} \circ \cdots \circ G^{i_2}_{m+4} \circ G^{i_1}_{m+2}\underline{v}$; by Proposition 2.5.6, Ψ_α is an irreducible object in the heart of the exotic t-structure. It will follow from the arguments in the next section that these irreducible objects are distinct. Since $K^0(\mathcal{D}^{\geq 0}_n \cap \mathcal{D}^{\leq 0}_n) \simeq K^0(\mathcal{D}_n) \simeq K^0(\mathrm{Coh}(\mathcal{B}_{z_n}))$, and $K^0(\mathrm{Coh}(\mathcal{B}_{z_n}))$ has rank $\binom{m+2n}{n}$, these constitute all the irreducible objects in the heart of the exotic t-structure on $\mathcal{D}_n$. □

2.5.3 The Ext algebra

The goal of this section is to describe the space $\mathrm{Ext}^\bullet(\bigoplus_{\alpha \in \mathrm{Cross}(n)} \Psi_\alpha)$. Let $m > 0$ (the $m = 0$ is handled in [6]). Note the following description of the right adjoint functor to $\Psi(\gamma)$, where γ is an affine $(m+2p, m+2q)$ tangle. Denote by $\check{\gamma}$ the affine $(m+2q, m+2p)$ tangle obtained by inverting γ.

Lemma 2.5.11. *The right adjoint to $\Psi(\gamma)$ is $\Psi(\check{\gamma})[p-q]$.*

Proof. From Section 2.3, the right adjoint to $\Psi(\gamma)$ is $\Psi(\check{\gamma})[p-q]$, when $\gamma = g^k_{m+2n}$, f^k_{m+2n}, $t^k_{m+2n}(1)$ or $t^k_{m+2n}(2)$. The result follows, since any tangle is the composition of these tangles, and if $\gamma = \gamma_1 \circ \gamma_2$, $\check{\gamma} = \check{\gamma}_2 \circ \check{\gamma}_1$. □

Now we will compute $\mathrm{Ext}^\bullet(\Psi_\alpha, \Psi_\beta)$ as a vector space, where $\alpha, \beta \in \mathrm{Cross}(n)$.

$$\begin{aligned}
\mathrm{Ext}^\bullet(\Psi_\alpha, \Psi_\beta) &= \mathrm{Ext}^\bullet(\Psi(\alpha)\underline{v}, \Psi(\beta)\underline{v}) \\
&\simeq \mathrm{Ext}^\bullet(\underline{v}, \Psi(\check{\alpha} \circ \beta)[-n]\underline{v}) \\
&\simeq \Psi(\check{\alpha} \circ \beta)[-n]\underline{v}
\end{aligned}$$

Let $\Lambda \in D^b(\mathrm{Vect})$ be a complex concentrated in degrees 1 and -1, with dimension 1 in those degrees.

Lemma 2.5.12.

$$\Psi(f^i_{m+2} \circ g^j_{m+2})\underline{v} \simeq \begin{cases} \Lambda & \text{if } i = j \\ \underline{v} & \text{if } |i-j| = 1 \text{ or } \{i,j\} = \{1, m+2\} \\ 0 & \text{otherwise} \end{cases}$$

Proof. Firstly if $i = m+2$ or $j = m+2$ and $m > 0$ we can conjugate $f^i_{m+2} \circ g^j_{m+2}$ by a power of r_m, so it is safe to assume that $1 \leq i, j \leq m+1$. From Section 2.3, $\Psi(g^j_{m+2})\underline{v} \simeq i_{1,j*}(\mathcal{E}_j \otimes \mathcal{O}_{X_{1,j}}) \simeq i_{1,j*}(\mathcal{E}_j)$, since $\pi_{1,j}$ maps $X_{1,j}$ to the point $S_0 \times_{\mathfrak{sl}_m} T^*\mathcal{B}_0$.

$$\begin{aligned}
&\Psi(f^i_{m+2} \circ g^j_{m+2})\underline{v} \simeq \Psi(f^i_{m+2})(i_{1,j*}(\mathcal{E}_j)) \\
&\simeq \pi_{1,i*}[i^*_{1,i}(i_{1,j*}(\mathcal{E}_j) \otimes \mathcal{E}^{-1}_{i+1})] \simeq \pi_{1,i*}[i^*_{1,i}(i_{1,j*}(\mathcal{E}_j)) \otimes \mathcal{E}^{-1}_{i+1}]
\end{aligned}$$

Since the below conditions defining $X_{1,i}$ force $x = z_1$ (the standard nilpotent of type $(m+1,1)$ acting on the Jordan basis $\{e_1, \cdots, e_{m+1}, f_1\}$), $V_k = \langle e_1, \cdots, e_k \rangle$ for $1 \leq k \leq i-1$, and $V_k = \langle e_1, \cdots, e_{k-1}, f_1 \rangle$ for $k \geq i+1$, we have:

$$X_{1,i} = \{(0 \subset V_1 \subset \cdots \subset V_{i-1} \subset V_i \subset V_{i+1} \subset \cdots \subset V_{m+2}), x| \\ x \in S_1, xV_{i+1} \subset V_{i-1}, xV_k \subset V_{k-1}\} = \mathbb{P}(V_{i+1}/V_{i-1})$$

Thus the intersection $X_{1,i} \cap X_{1,i+1}$ consists of the single point $\{(0 \subset V_1 \subset \cdots \subset V_{m+2}), z_1\}$, where $V_k = \langle e_1, \cdots, e_k \rangle$ for $1 \leq k \leq i$, and $V_k = \langle e_1, \cdots, e_{k-1}, f_1 \rangle$ for $k \geq i+1$; while $X_{1,i} \cap X_{1,j} = \emptyset$ if $|i-j| > 1$.

So in the above equation, if $i = j$ since we have an exact triangle $\mathcal{F} \otimes \mathcal{O}_{U_1}(-X_{1,i})[1] \to i_{1,i}^* i_{1,i*} \mathcal{F} \to \mathcal{F}$ for $\mathcal{F} \in \mathrm{Coh}(X_{1,i})$; and that $\mathcal{O}_{U_1}(-X_{1,i}) \simeq \mathcal{E}_i^{-1} \otimes \mathcal{E}_{i+1}$. Thus we have an exact triangles $\mathcal{E}_{i+1}[1] \to i_{1,i}^* i_{1,i*} \mathcal{E}_i \to \mathcal{E}_i$, and hence $\mathcal{O}_{U_1}[1] \to (i_{1,i}^* i_{1,i*} \mathcal{E}_i) \otimes \mathcal{E}_{i+1}^{-1} \to \mathcal{E}_i \otimes \mathcal{E}_{i+1}^{-1}$. Noting that $\mathcal{E}_i \otimes \mathcal{E}_{i+1}^{-1}$ is $\mathcal{O}(-2)$, and using the long exact sequence of cohomology:

$$\cdots \to R\Gamma^i(\mathcal{O}_{U_1}[1]) \to R\Gamma^i((i_{1,i}^* i_{1,i*} \mathcal{E}_i) \otimes \mathcal{E}_{i+1}^{-1}) \to R\Gamma^i(\mathcal{E}_i \otimes \mathcal{E}_{i+1}^{-1}) \to R\Gamma^{i+1}(\mathcal{O}_{U_1}[1]) \to \cdots$$

Thus $R\Gamma^i((i_{1,i}^* i_{1,i*} \mathcal{E}_i) \otimes \mathcal{E}_{i+1}^{-1}) = 0$ if $i \neq 1, 0, -1$; computing the terms in the above exact sequence we get

$$0 \to \mathbb{C} \to R\Gamma^{-1}((i_{1,i}^* i_{1,i*} \mathcal{E}_i) \otimes \mathcal{E}_{i+1}^{-1}) \to 0 \to 0 \to \\ \to R\Gamma^0((i_{1,i}^* i_{1,i*} \mathcal{E}_i) \otimes \mathcal{E}_{i+1}^{-1}) \to 0 \to 0 \to R\Gamma^1((i_{1,i}^* i_{1,i*} \mathcal{E}_i) \otimes \mathcal{E}_{i+1}^{-1}) \to \mathbb{C} \to 0$$

Thus $\Psi(f_{m+2}^i \circ g_{m+2}^i)\underline{v} \simeq \Lambda$.

If $|i-j| > 1$, since $i_{1,i}^* i_{1,j*} \mathcal{F} = 0$ for any $\mathcal{F} \in \mathrm{Coh}(X_{1,i})$ (as $X_{1,i} \cap X_{1,j} = \emptyset$), we obtain $\Psi(f_{m+2}^i \circ g_{m+2}^j)\underline{v} = 0$.

If $|i-j| = 1$, denote by $X_{1,i,j}$ the point $X_{1,i} \cap X_{1,j}$; denote $i'_{1,i} : X_{1,i,j} \hookrightarrow X_{1,j}, i'_{1,j} : X_{1,i,j} \hookrightarrow X_{1,i}$. Then $i_{1,i}^*(i_{1,j*} \mathcal{E}_j) \simeq i'_{1,j*}(i'^*_{1,i} \mathcal{E}_j) \simeq i'_{1,j*} \mathcal{O}_{X_{1,i,j}}$, so $\Psi(f_{m+2}^i \circ g_{m+2}^j)\underline{v} \simeq \pi_{1,i*}(i'_{1,j*} \mathcal{O}_{X_{1,i,j}} \otimes \mathcal{E}_{i+1}^{-1}) \simeq \underline{v}$. □

Definition 2.5.13. Define an m-link as an affine crossingless (m,m)-tangle, where both the m inner points, and the m outer points are unlabelled. If each of the m points in the inner circle are joined to m points in the outer circle, say that the m-link is "good" , and otherwise (i.e. if there are cups and caps) say that the m-link is "bad". If an m-link γ is good, denote by $\omega(\gamma)$ the number of loops contained in γ.

Note that if $m > 0$ no loop can enclose the origin and a good m-link γ is determined up to an isotopy by the number $\omega(\gamma)$. Given unlabelled affine crossingless $(m, m+2n)$ matchings α, β, we can construct an m-link $\check{\alpha} \circ \beta$; furthermore, any m-link γ corresponds to a functor $\Psi(\gamma) : \mathcal{D}_0 \to \mathcal{D}_0$. It follows from Lemma 2.5.12 that:

Proposition 2.5.14. *Let $m > 0$. If the m-link γ is bad, then $\Psi(\gamma)$ is zero. If the m-link γ is good, then $\Psi(\gamma)$ is isomorphic to tensor multiplication by $(\Lambda)^{\otimes \omega_1(\gamma)}$.*

Proof. The m-link $f^i_{m+2} \circ g^i_{m+2}$ is good, with $\omega(f^i_{m+2} \circ g^i_{m+2}) = 1$; and $\Psi(f^i_{m+2} \circ g^i_{m+2})$ corresponds to multiplication by Λ. If $|i-j| = 1$, the m-link $f^i_{m+2} \circ g^j_{m+2}$ is good, with $\omega(f^i_{m+2} \circ g^j_{m+2}) = 0$; and $\Psi(f^i_{m+2} \circ g^j_{m+2})$ is the identity functor $\mathcal{D}_0 \to \mathcal{D}_0$. If $|i-j| > 1$, the m-link $f^i_{m+2} \circ g^i_{m+2}$ is bad, and $\Psi(f^i_{m+2} \circ g^i_{m+2})\underline{v} = 0$ for any $w \in \mathcal{D}_0$. Call an m-link "basic" if it is of the form $f^i_{m+2} \circ g^j_{m+2}$; then any m-link γ can be written as a composition of basic m-links. The conclusion then follows from our knowledge of $\Psi(\gamma)$ for basic m-links γ, and the fact that m-link γ is bad iff each expression of γ in terms of basic m-links contains at least one bad basic m-link. $\square$

Note that the $\mathbb{Z}$-graded algebra $\mathrm{Ext}^\bullet(\Psi_{g^i_{m+2}}) = \Lambda[-1]$ is isomorphic to $\mathbb{C}[x]/(x^2)$, where x has degree 2. Indeed, this is the only possible algebra structure on $\Lambda[-1]$ which respects its grading. Thus:

Theorem 2.5.15. *Let $m > 0$. For any $\alpha, \beta \in Cross(n)$ we have an isomorphism of vector spaces:*

$$Ext^\bullet(\Psi_\alpha, \Psi_\beta) \simeq \begin{cases} \Lambda^{\otimes\omega(\check{\alpha}\circ\beta)}[-n] & \text{if } \check{\alpha}\circ\beta \text{ is good} \\ 0 & \text{otherwise} \end{cases}$$

$\square$

Now we will give a conjectural description of the multiplication in the Ext algebra. It suffices to describe the map

$$\begin{gathered} \mathrm{Ext}^\bullet(\Psi_\alpha, \Psi_\beta) \times \mathrm{Ext}^\bullet(\Psi_\beta, \Psi_\gamma) \to \mathrm{Ext}^\bullet(\Psi_\alpha, \Psi_\gamma) \\ \Lambda^{\omega(\check{\alpha}\circ\beta)}[-n] \times \Lambda^{\omega(\check{\beta}\circ\gamma)}[-n] \to 8\Lambda^{\omega(\check{\alpha}\circ\gamma)}[-n] \end{gathered}$$

Consider a sequence of links, the first of which is a disjoint union of $\check{\alpha}\circ\beta$ and $\check{\beta}\circ\gamma$, and the last of which is $\check{\alpha}\circ\gamma$. Each subsequent link is obtained from the previous link by performing a "surgery": pick two strands between i and j in $\beta\circ\check{\beta}$, and replace them with two radial lines at i and j, so that the resulting tangle has no crossings. It is clear that at each step, there is at least one way of performing a surgery operation. Depending on whether the strands between i and j are part of a circle, a line, or an arc (i.e. a cup or a cap), then diagrammatically each surgery operation corresponds to one of the following. We also write down a corresponding map between the Ext spaces (note there is a unique map, up to scalar, which respects the grading).

- Merging a line and a circle, into a line: $\mathbb{C} \otimes \Lambda[-1] \to \mathbb{C}$
- Splitting a line, into a line and a circle: $\mathbb{C}[-1] \to \mathbb{C} \otimes \Lambda$
- Merging two circles into one circle: $\Lambda \otimes \Lambda[-1] \to \Lambda$
- Splitting one circle into two circles: $\Lambda[-1] \to \Lambda \otimes \Lambda$
- Merging two lines into two arcs: $\underline{\mathbb{C}} \to 0$

- Merging two arcs into two lines: $0 \to \underline{\mathbb{C}}$
- Merging a line and an arc, into (a different) line and an arc: $0 \to 0$
- Merging a circle and an arc, into an arc: $0 \to 0$
- Splitting an arc, into a circle and an arc: $0 \to 0$

By iteratively applying these "surgery" operations (so that all the cups and caps in β and $\check{\beta}$ are replaced by radial lines), we arrive at the desired map.

2.6 Further directions

2.6.1 Decategorification

Denote by V the 2-dimensional representation of $\mathfrak{sl}_2$. In section 6 (see Theorem 6.2 and Section 6.4) of [27], Cautis and Kamnitzer prove that:

$$K^0(\mathrm{Coh}(Y_{m+2n})) \simeq V^{\otimes m+2n}$$

Recall also that there is a map, which is compatible under composition (see Section 6.1 of [27] for an explicit description).

$$\psi : \{(k,l)\text{-tangles}\} \to \mathrm{Hom}_{U(\mathfrak{sl}_2)}(V^{\otimes k}, V^{\otimes l})$$

In Section 6 of [27], it is proven that for each $(m+2p, m+2q)$-tangle α, the functor $\widetilde{\Psi}(\alpha)$ corresponds to $\psi(\alpha)$ on the level of the Grothendieck group. In fact, Cautis and Kamnitzer work with a q-deformation of this picture (using $\mathbb{C}^*$-equivariant sheaves, and representations of $U_q(\mathfrak{sl}_2)$); however we will not need to work in this generality.

Under the natural embedding $U_n \to Y_{m+2n}$ (constructed in Section 2.1), we have a natural map:

$$K^0(\mathcal{D}_n^0) = K^0(\mathrm{Coh}_{\mathcal{B}_{zn}}(U_n)) \to K^0(\mathrm{Coh}(Y_{m+2n})) \simeq V^{\otimes m+2n}$$

It can be shown that under this map, $K^0(\mathcal{D}_n^0)$ is identified with the m weight space in $V^{\otimes m+2n}$ (which we shall denote $V_{[m]}^{\otimes m+2n}$). After taking the image of the maps $\Psi(\alpha)$ in the Grothendieck group (where α is an affine tangle), we obtain:

$$\hat{\psi} : \{(m+2k, m+2l)\text{-affine tangles}\} \to \mathrm{Hom}(V_{[m]}^{\otimes m+2k}, V_{[m]}^{\otimes m+2l})$$

Now the images of the irreducible objects $\Psi(\alpha)$ in $\mathcal{D}_n^0$ will form a basis in the $(m+n, n)$ weight space in $V^{\otimes m+2n}$. We expect that this will coincide with the canonical (or perhaps the dual canonical) basis introduced by Lusztig; this is the subject of work in progress. This should follow once we have an explicit description of the map $\hat{\psi}$.

2.6.2 Applications to modular representation theory

Recall, from the introduction that, Theorem 5.3.1 from [17] (see also Section 1.6.2 from [16]) states that there is an equivalence:

$$D^b(\mathrm{Coh}_{\mathcal{B}_{e,\mathbf{k}}}(\widetilde{\mathfrak{g}}_{\mathbf{k}})) \simeq D^b(\mathrm{Mod}_e^{fg,\lambda}(U_{\mathbf{k}}))$$

Further, the tautological t-structure on the right hand side corresponds to the exotic t-structure on the left hand side. Thus, by studying the irreducible objects in the heart of the exotic t-structure on the other side, one may derive information about irreducible objects in $\mathrm{Mod}_e^{fg,\lambda}(U_{\mathbf{k}})$. In the case where e is a two-block nilpotent, our results give a fairly explicit description of the irreducible objects in the former category (by repeatedly applying the functors G^i_{m+2n}). By studying what these functors look like on the other side of the above equivalence, we expect that it will be possible to give an explicit, recursive construction of the irreducible representations lying in $\mathrm{Mod}_e^{fg,\lambda}(U_{\mathbf{k}})$, along with formulae for their dimensions and characters; this is the subject of work in progress.

More precisely, the dimension of the modules should be related to computing the Euler characteristic of the corresponding exotic sheaves (after tensoring by a line bundle); and the characters should correspond to computing the Euler characteristic in the equivariant category (where the group acting is a maximal torus inside the centralizer of the nilpotent). Computing these Euler characteristics is related to computing the image of the irreducible objects in the Grothendieck group (the problem discussed in the previous section). It would also be interesting to describe the projective covers of the irreducibles, and give a description of $\mathrm{Mod}_e^{fg,\lambda}(U_{\mathbf{k}})$ as modules over a diagram algebra; this would be related to computing the Koszul dual of the arc algebra described above.

2.6.3 Categorifying invariants for affine tangles

From the discussion in the above subsection, we have constructed a map:

$$\hat{\psi} : \{(m+2k, m+2l)\text{-affine tangles}\} \to \mathrm{Hom}(V_{[m]}^{\otimes m+2k}, V_{[m]}^{\otimes m+2l})$$

This map is categorified by the functors $\Psi(\alpha) : \mathcal{D}_p \to \mathcal{D}_q$ between categories of coherent sheaves on Springer fibers.

In [41] and [28], Khovanov and Chen construct a categorification of the invariant $\psi(\alpha) : V^{\otimes m} \to V^{\otimes n}$ using categories of modules over certain diagram algebras; the functors which categorify the action of the generators g^i_n, f^i_n and $t^i_n(1), (2)$ correspond to tensoring with certain (complexes of) bi-modules. These diagram algebras used there are very similar in nature to the Ext algebras we have described; however, the crossingless matchings that appear are drawn on a line (instead of a circle).

We expect that it will be possible to categorify the maps $\hat{\psi}_\alpha : V_m^{\otimes m+2p} \to V_m^{\otimes m+2q}$ by using categories of modules over our diagram algebras. We know that these categories of modules categorify the weight spaces; and it remains to show that the maps $g_n^i, f_n^i, t_n^i(1), (2)$ and r_n correspond to tensoring with certain (complexes of) bimodules. This categorification should be equivalent to the one constructed above using coherent sheaves (after looking at the Koszul dual picture); however, it should be possible to develop the theory independently, without reference to the theory developed here.

Chapter 3

Stability conditions for subquotients of category $\mathcal{O}$

3.1 Introduction

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Then the BGG category $\mathcal{O}$ is the category of all finitely generated $U(\mathfrak{g})$-modules that are $\mathfrak{h}$-diagonalizable and locally $U(\mathfrak{n}_+)$-nilpotent. This category splits into blocks in accordance with the action of the center $Z(U(\mathfrak{g}))$ of the enveloping algebra; we will be interested primarily in the principal block $\mathcal{O}_0$.

Consider the natural filtration on $U(\mathfrak{g})$ (where $x_1 x_2 \cdots x_i \in U(\mathfrak{g})^{\leq j}$ if $x_1, \cdots, x_i \in \mathfrak{g}$ and $i \leq j$). Given a module $M \in \mathcal{O}$, pick a finite-dimensional vector space M_0 which generates it, and let $M_i = (U(\mathfrak{g}))^{\leq i} M_0$. Then it can be shown that for i sufficiently large, the function $p(i) = \dim(M_i)$ is polynomial, and that its leading term does not depend on the choice of filtration. The degree of the polynomial $p(i)$, which we denote by $\mathrm{GK}(M)$, is known as the Gelfand-Kirillov dimension of M. We denote the leading term of p by $\underline{\mathrm{LC}}(M)$; in fact, we will be more interested in the quantity $\mathrm{LC}(M)$, which a variant of $\underline{\mathrm{LC}}(M)$ introduced in Section 3.2.1.

Given an integer d, let $\mathcal{O}_0^{\leq d}$ (resp. $\mathcal{O}_0^{<d}$) be the subcategories of $\mathcal{O}_0$ consisting of modules with Gelfand-Kirillov dimension at most d (resp. strictly less than d). Let $\mathcal{O}_0^d$ be the Serre sub-quotient $\mathcal{O}_0^{\leq d}/\mathcal{O}_0^{<d}$. It is known that there the braid group $\mathbb{B}_W$ acts on the derived category $D^b(\mathcal{O}_0)$; here the simple reflections act via wall-crossing functors. It can be shown that the braid group action factors through to the quotient categories $\mathcal{O}_0^{\leq d}/\mathcal{O}_0^{<d}$. Here we will use this datum to construct an example of "real variations of stability conditions" (which essentially amounts to checking a number of compatibilities between the braid group action and the leading coefficient functions); now we proceed to give some more precise definitions.

3.1.1 Real variations of stability conditions

Inspired by Bridgeland's theory of stability conditions, in [7], Anno, Bezrukavnikov and Mirkovic define the notion of a "real variation of stability conditions" on a triangulated category. They then give an example of this construction, using exotic t-structures on the derived category of coherent sheaves on a Springer fiber. We briefly recall the main definition (see Section 1.4 of [7] for more details).

Definition 3.1.1. Let $\mathcal{C}$ be a finite type triangulated category, and let Σ be a discrete collection of affine hyperplanes in a finite-dimensional, real vector space V. Let **Alc** (the set of "alcoves") be the connected components of $V^0 = V\backslash\Sigma$. For each affine hyperplane in Σ, consider the parallel hyperplane passes through 0, and let Σ_{lin} be the set of those hyperplanes. Fix a component V^+ of $V\backslash\Sigma_{lin}$. Given two alcoves $A, A' \in$ **Alc** which share a co-dimension 1 face, we say that A' is above A if, after we shift the hyperplane so that it passes through 0, then A' lies on the same side of the hyperplane as V^+.

A "real variation of stability conditions" on $\mathcal{C}$ consist of a polynomial map $Z : V \to (K^0(\mathcal{C}) \otimes \mathbb{R})^*$ (known as "the central charge"), and a map τ from **Alc** to the set of bounded t-structures on $\mathcal{C}$, such that:

- Let $A \in$ **Alc**, and let M be a non-zero object in the heart $\mathcal{A}$ of $\tau(A)$. Then $\langle Z(x), [M]\rangle > 0$ for $x \in A$.
- Let $A' \in$ **Alc** be another alcove sharing a co-dimension one face H with A, and lying above A. Let $\mathcal{A}_n$ be the Serre subcategory consisting of objects M such that the polynomial function $x \to \langle Z(x), [M]\rangle$ has a zero of order at least n on H. Also define $\mathcal{C}_n = \{C \in \mathcal{C} \mid H^i_{\tau(A)}(C) \in \mathcal{A}_n\}$. Then the truncation functors for $\tau(A')$ preserves the filtration by $\mathcal{C}_n$, and the two t-structures on $\mathcal{C}_n/\mathcal{C}_{n+1}$ induced by $\tau(A)$ and $\tau(A')$ differ by a shift of $[n]$.

Anno, Bezrukavnikov and Mirkovic construct an example with $\mathcal{C} = D^b(\mathrm{Coh}_{\mathcal{B}_e}(\widetilde{S}))$ in [7]. Here e is a nilpotent, $\mathcal{B}_e$ is the corresponding Springer fiber, and $\widetilde{S}$ is the pre-image to the Slodowy slice at e under the Springer map. The hyperplane arrangement Σ in question is the set of affine co-root hyperplanes in $V = \mathfrak{h}^*_{\mathbb{R}}$. The central charge $Z : \mathfrak{h}^* \to (K^0(\mathcal{C}) \otimes \mathbb{R})^*$ is defined to be the unique polynomial map such that given $\mathcal{F} \in \mathrm{Coh}_{\mathcal{B}_e}(\widetilde{S})$, $\lambda \in \Lambda^+$, $Z(\lambda)[F]$ is the Euler characteristic of $\mathcal{F} \otimes \mathcal{O}(\lambda)$. The map τ from **Alc** to the set of bounded t-structures on $D^b(\mathrm{Coh}_{\mathcal{B}_e}(\widetilde{S}))$ is constructed in Section 1.8.2 of [16]; in Proposition 1 of [7], it is proven that the axioms for a real variation of stability conditions are satisfied.

In [7], the notion of a real variation of stability conditions were introduced in order to study the space of Bridgeland stability conditions for the category $\mathcal{C}$. For a detailed exposition of Bridgeland stability conditions, we refer the reader to Bridgeland's expository paper, [22]; see also [23] and [24] for some related work.

3.1.2 Summary

In our example, the hyperplane arrangement Σ will be the set of linear co-root hyperplanes in $V = \mathfrak{h}_{\mathbb{R}}^*$. Given $\lambda \in \Lambda^+, M \in \mathcal{O}_0^{\mathbf{d}}$, the central charge map Z is defined by letting $Z(\lambda)[M]$ to be equal to $\mathrm{LC}(T_{0\to\lambda}M)$ (here $T_{0\to\lambda}$ denotes the translation functor); it takes some work to show that this gives rise to a well-defined polynomial map. Recalling that Alc is in bijection with W, the map τ can be defined using the action of the braid group $\mathbb{B}_W$ on $\mathcal{C} = D^b(\mathcal{O}_0^{\mathbf{d}})$: for each $w \in W$, let $\tau(w)$ be the image of the natural t-structure on $\mathcal{C}$ under the automorphism $\Phi(\tilde{w})$ (here $\tilde{w}$ is the lift of w to the braid group). We claim that this data satisfies the axioms for a real variation of stability conditions.

In Section 3.2, we start by rigorously defining the categories involved, the quantity $\mathrm{LC}(M)$ and the action of the braid group $\mathbb{B}_W$. We then state the main result (which was sketched briefly in the last paragraph). We conclude the section by proving that the function $Z(\lambda)[M]$ (defined above when λ is dominant) can be extended to a polynomial map. In order to do this, we give a way of computing the leading coefficient $\mathrm{LC}(M)$ by studying the Taylor expansion of the character of M. As a result, we are also able to show a certain compatibility relation between the braid group action and these leading coefficient polynomials.

In Section 3.3, we prove that the two conditions stated above are satisfied. The first condition almost follows from the results of Section 3.2; however, we need some additional machinery to show that $\langle Z(x), [M]\rangle$ is strictly positive (and not just non-negative). To show the second condition, we examine how translation to the wall interacts with these leading coefficient polynomials. We appeal to the theory of harmonic polynomials to the show the categories $\mathcal{A}_n$ and $\mathcal{C}_n$ are empty if $n \geq 2$.

3.2 Subquotients of category $\mathcal{O}$.

3.2.1 Gelfand-Kirillov dimension and leading coefficients.

Let $\mathcal{O}_0$ be the principal block of category $\mathcal{O}$. Given a module $M \in \mathcal{O}$, recall that its Gelfand-Kirillov dimension is defined as follows. Consider the natural filtration on $U(\mathfrak{g})$, where $U(\mathfrak{g})^{\leq i}$ denotes the subspace of $U(\mathfrak{g})$ spanned by products $x_1x_2\cdots x_k$ with $k \leq i$ and $x_j \in \mathfrak{g}$.

Let $\underline{M}_0 \subseteq M$ be a vector sub-space which generates M as a $U(\mathfrak{g})$-module, and let $\underline{M}_i = U(\mathfrak{g})^{\leq i} \cdot \underline{M}_0$. Then:

Proposition 3.2.1. *There exists a polynomial p such that: for all i sufficiently large, $\dim(\underline{M}_i) = p(i)$. The leading term of this polynomial p does not depend on the choice of subspace M_0.*

Proof. With this grading, the associated graded of $U(\mathfrak{g})$ is $S(\mathfrak{g})$ (where all elements of $\mathfrak{g}$ have degree 1); let $\underline{M} = \mathrm{gr}(M)$. Then by the theory of Hilbert polynomials, there exists a polynomial q such that:

$$\dim(\underline{M}_i)t^i = \frac{P(t)}{(1-t)^{\dim(\mathfrak{g})}}$$

It follows that there exists a polynomial p, such that $\dim(\underline{M}_i) = p(i)$ for i sufficiently large. Suppose now that we pick a different subspace M_0', which gives rise to a filtration $\underline{M}_i'$ with dimension polynomial p'. Then $\underline{M}_0' \subseteq \underline{M}_k$ for some k, so $\underline{M}_i' \subseteq \underline{M}_{i+k}$, and $p'(i) \leq p(i+k)$ for i large. Similarly, for some l, $p(i) \leq p'(i+l)$, provided that i large. These two inequalities imply that p and p' have the same leading term, i.e. that the leading term doesn't depend on the choice of subspace $\underline{M}_0$. □

Definition 3.2.2. The degree of the polynomial p, $\mathrm{GK}(M)$, is known as the Gelfand-Kirillov dimension of M; denote the leading coefficient of p by $\underline{\mathrm{LC}}(M)$.

From the following Lemma, we deduce that the set of all $M \in \mathcal{O}_0$ with Gelfand-Kirillov dimension at most d (for some $d \in \mathbb{Z}_{\geq 0}$) forms a Serre sub-category. The following Lemma is self-evident:

Lemma 3.2.3. *Given an exact sequence* $0 \to A \to B \to C \to 0$, *then*

$$max\{GK(A), GK(C)\} = GK(B)$$

Definition 3.2.4. Denote by $\mathcal{O}_0^{\leq d}$ (resp., $\mathcal{O}_0^{<d}$) be the Serre sub-category of $\mathcal{O}_0$ consisting of objects M with Gelfand-Kirillov dimension at most d (resp., strictly less than d). Let $\mathcal{O}_0^d$ denote the Serre quotient category $\mathcal{O}_0^{\leq d}/\mathcal{O}_0^{<d}$.

Lemma 3.2.5. *The Verma module* $\Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ *has Gelfand-Kirillov dimension* $|\Delta^+|$. *More generally, given a parabolic sub-algebra* $\mathfrak{p} \supseteq \mathfrak{b}$ *and a finite-dimensional irreducible representation* V_λ *of* $\mathfrak{p}$ *(which factors through to the Levi sub-algebra* $\mathfrak{l}$*), then the parabolic Verma module* $\Delta_\mathfrak{p}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_\lambda$ *has Gelfand-Kirillov dimension* $|\Delta^+| - N(\mathfrak{p})$, *where* $N(\mathfrak{p}) = |\alpha \in \Delta^+ : F_\alpha \in \mathfrak{p}|$.

Proof. Pick a basis $\{v_1, \cdots, v_k\}$ for V_λ, where $k = \dim(V_\lambda)$. Using the PBW theorem, the parabolic Verma has basis

$$\prod_{\alpha \in \Delta_\mathfrak{p}^+, 1 \leq i \leq k} F_\alpha^{n_\alpha} v_i$$

Here we have fixed an order on the set $\Delta_\mathfrak{p}^+ = \{\alpha \in \Delta^+ : F_\alpha \in \mathfrak{p}\}$; and the n_α-s are arbitrary positive integers. Picking $\underline{M}_0 = V_\lambda$, the above product lies in $\underline{M}_n$, where $n = \sum_{\alpha \in \Delta^+} n_\alpha$. Thus:

$$\dim\ (\underline{M}_n) = \dim\ (V_\lambda)|\{n_\alpha\} : \sum_{\alpha \in \Delta_\mathfrak{p}^+} n_\alpha \leq n|$$

The result now follows. □

Example 3.2.6. Let us consider the example with $\mathfrak{g} = \mathfrak{sl}_3$, and calculate the Gelfand-Kirillov dimensions of the simple objects.

The simple objects in $\mathcal{O}_0$ are $L(w \cdot 0)$ with $w \in S_3$. When $w = 1$, $L(w \cdot 0)$ is the trivial 1-dimensional module, and clearly has Gelfand-Kirillov dimension 0. When $w = w_0 = s_1 s_2 s_1$, $L(w \cdot 0) = \Delta(w \cdot 0)$ since the Verma module is irreducible; and has Gelfand-Kirillov dimension 3 using the above Lemma.

Let $\mathfrak{p}_1$ (resp. $\mathfrak{p}_2$) be the parabolic sub-algebra containing F_α, where $\alpha = \epsilon_1 - \epsilon_2$ (resp. where $\alpha = \epsilon_2 - \epsilon_3$). Consider the corresponding parabolic sub-categories $\mathcal{O}_0^{\mathfrak{p}_1}, \mathcal{O}_0^{\mathfrak{p}_2} \subset \mathcal{O}_0$. From Chapter 9 of [34], we know that $L(w \cdot 0) \in \mathcal{O}_0^{\mathfrak{p}_I}$ (where $\mathfrak{p}_I$ is the parabolic subalgebra corresponding to a subset I of the set of simple roots) precisely when $\langle w \cdot 0, \check{\alpha} \rangle > 0$ for all $\alpha \in I$. Thus $L((s_2 s_1) \cdot 0), L(s_2 \cdot 0) \in \mathcal{O}_0^{\mathfrak{p}_1}$, and $L((s_1 s_2) \cdot 0), L(s_1 \cdot 0) \in \mathcal{O}_0^{\mathfrak{p}_2}$.

In fact, one can prove that the parabolic Verma modules $\Delta_{\mathfrak{p}_1}((s_2 s_1) \cdot 0)$ and $\Delta_{\mathfrak{p}_2}((s_1 s_2) \cdot 0)$ are irreducible (either directly, or by using the criterion in Section 9.12 of [34]); thus $L((s_2 s_1) \cdot 0) = \Delta_{\mathfrak{p}_1}((s_2 s_1) \cdot 0)$ and $L((s_1 s_2) \cdot 0) = \Delta_{\mathfrak{p}_2}((s_1 s_2) \cdot 0)$. Further, one can prove that we have exact sequences:

$$0 \to \Delta_{\mathfrak{p}_1}((s_2 s_1) \cdot 0) \to \Delta_{\mathfrak{p}_1}(s_2 \cdot 0) \to L(s_2 \cdot 0) \to 0$$
$$0 \to \Delta_{\mathfrak{p}_2}((s_1 s_2) \cdot 0) \to \Delta_{\mathfrak{p}_2}(s_1 \cdot 0) \to L(s_1 \cdot 0) \to 0$$

Using the above Lemma, it is now easy to check that the four simples $L((s_1 s_2) \cdot 0), L((s_2 s_1) \cdot 0)$, $L(s_1 \cdot 0), L(s_1 \cdot 0)$ have Gelfand-Kirillov dimension 2. ■

For our purposes it will be more convenient to modify the definition of leading coefficients. To this end, note that $M = U(\mathfrak{n}_-) M_0$, and define a different grading on $U(\mathfrak{n}_-)$ by setting $\deg(F_\alpha) = \langle \check{\rho}, \alpha \rangle$ (so, in particular, $\deg(F_\alpha) = 1$ when α is a simple root). This gives a filtration on $U(\mathfrak{n}_-)$, where

$$U(\mathfrak{n}_-)^{\leq i} = \{\mathrm{span}(\prod_{\alpha \in \Delta^+} F_\alpha^{n_\alpha}) \mid \sum_{\alpha \in \Delta^+} n_\alpha \deg(\alpha) \leq i\}$$

Define $M_i = U(\mathfrak{n}_-)^{\leq i} \cdot M_0$. We will see in the next example that dim (M_i) is no longer a polynomial in i; however, we will prove that the weaker statement below does hold.

Definition 3.2.7. We say that a function $q : \mathbb{Z} \to \mathbb{Z}$ is "quasi-polynomial", if there exists an integer k, and polynomials $q_0, q_1, \cdots, q_{k-1}$ with the same degree and leading coefficient, such that $q(n) = q_i(n)$ if $n \equiv i \pmod{k}$.

Proposition 3.2.8. *There exists a quasi-polynomial function q, such that for i sufficiently large,*

$$\mathit{dim}(M_i / M_{i-1}) = q(i)$$

Further, $\deg(q) = GK(M)$, and the leading coefficient of q does not depend on the choice of M_0.

Proof. It is clear that the associated graded algebra of $U(\mathfrak{n}_-)$ with respect to the grading described above, is $S(\mathfrak{n}_-)$ (where the corresponding elements have the same grading); let $\overline{M} = \mathrm{gr}(M)$. From the general theory of Hilbert polynomials, we deduce there exists a polynomial q with the following property (more generally, this statement is true with $\langle \alpha, \check{\rho} \rangle$ being replaced by the degrees of the generators).

$$\sum_{i \geq 0} \dim(M_i) t^i = \frac{P(t)}{\prod_{\alpha \in \Delta^+} (1 - t^{\langle \alpha, \check{\rho} \rangle})}$$

Using the above formula, and inducting on the number of generators, it follows that there exist an integer k, and polynomials $q_0, q_1, \cdots, q_{k-1}$ such that $\dim(M_n) = q_i(n)$ if $n \equiv i \pmod{k}$ for n large. It remains to prove that the these polynomials have the same degree and leading coefficient.

Using primary decomposition for modules, we obtain a filtration $0 = \overline{M}_0 \subset \overline{M}_1 \subset \cdots \subset \overline{M}_{k-1} \subset \overline{M}_k = \overline{M}$, such that for each i, $\mathrm{Ann}(\overline{M}_j / \overline{M}_{j-1}) = \mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$; and $R/\mathfrak{p}_i$ acts injectively. It is sufficient to prove the above statement for each of the sub-quotients $\overline{M}_j / \overline{M}_{j-1}$. So we may assume that $\mathrm{Ann}(\overline{M}) = \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, and $R/\mathfrak{p}$ acts injectively on $\overline{M}$. The support of M is contained in $(\mathfrak{n}_-)^*$; we will identify $(\mathfrak{n}_-)^*$ with $\mathfrak{n}_+$ via the Killing form.

It follows using Lemma 3.2.9 below, that there exists an element t of degree 1, such that t acts injectively on M. This means that we have injective map from M_i to M_{i+1}, given by multiplication by t; thus $\dim(M_i) \leq \dim(M_{i+1})$, and $q_i(n) \leq q_{i+1}(n+1) \leq q_i(n+k)$ if $0 \leq i \leq k-2, n \equiv i \pmod{k}$. This implies that the polynomials q_i and q_{i+1} have the same degree and leading coefficient, as required. $\square$

Lemma 3.2.9. *The support of the module $\overline{M}$ is not contained inside the subvariety $[\mathfrak{n}^+, \mathfrak{n}^+]$ of $\mathfrak{n}^+$.*

Proof. Recall that an orbital variety is an irreducible component of the intersection $\mathbb{O} \cap \mathfrak{n}^+$, where $\mathbb{O}$ is a nilpotent orbit. Since M is a module in category $\mathcal{O}$, it is well-known that the support of $\overline{M}$, is a union of orbital varieties (see Joseph, [30], and Borho-Brylinski, [19], for a proof). We will show that no orbital variety is contained inside $[\mathfrak{n}^+, \mathfrak{n}^+]$.

Pick $e \in \mathbb{O}$, and let $\mathcal{B}_e$ be the Springer fiber:

$$\mathcal{B}_e = \{\mathfrak{b} \in \mathcal{B} \mid e \in \mathfrak{b}\}$$

In Claim 6.5.8 of [29], it is proven that the irreducible components of $\mathbb{O} \cap \mathfrak{n}^+$ are in bijection with $C^\circ(e)$-orbits on $\mathcal{B}_e$ (here $C^\circ(e)$ denotes the set of connected components of the centralizer of e). Suppose an orbital variety Y is contained in $[\mathfrak{n}^+, \mathfrak{n}^+]$, and pick one of the corresponding components X of $\mathcal{B}_e$. Define:

$$\mathcal{B}_e^\circ = \{\mathfrak{b} \in \mathcal{B} \mid e \in [\mathfrak{n}^+, \mathfrak{n}^+]\}$$

Since $Y \subseteq [\mathfrak{n}^+, \mathfrak{n}^+]$, from the bijection sketched in Claim 6.5.8 it is clear that $X \subseteq \mathcal{B}_e^\circ$.

For each positive root $\alpha \in \Delta^+$, let $P_\alpha \supseteq B$ be the corresponding minimal parabolic and let $\pi_\alpha : G/B \to G/P_\alpha$ be the natural projection map. Since $X \subseteq \mathcal{B}_e^\circ$, it is easy to see that $X \subseteq \pi_\alpha^{-1}(\pi_\alpha(X)) \subseteq \mathcal{B}_e$. Since $\pi_\alpha^{-1}(\pi_\alpha(X))$ is irreducible, and X is one of the irreducible components of $\mathcal{B}_e$, in fact we have that $X = \pi_\alpha^{-1}(\pi_\alpha(X))$ for each α.

Define an equivalence relation on points in $\mathcal{B}$ as the transitive closure of the following relation: if $x, y \in \mathcal{B}$, define $x \sim y$ if $\pi_\alpha(x) = \pi_\alpha(y)$ for some simple root α. It is well-known, that in fact $x \sim y$ for any two points $x, y \in \mathcal{B}$ (see Spaltenstein's paper [49] for a reference). Now pick any point $x \in X$; since $X = \pi_\alpha^{-1}(\pi_\alpha(X))$, any other point in the same equivalence class as x is also in X; it follows that $X = \mathcal{B}$. This is only possible when $e = 0$, and in this case one easily checks that the orbital variety Y is not contained in $[\mathfrak{n}^+, \mathfrak{n}^+]$. Thus we have reached a contradiction, and so no orbital variety is contained in $[\mathfrak{n}^+, \mathfrak{n}^+]$. $\square$

Remark 3.2.10. Above we have used some non-trivial facts about the support of modules in category $\mathcal{O}$; it is possible that the above proof can be simplified, and that the statement holds in greater generality.

Definition 3.2.11. Define $\mathrm{LC}(M)$ to be the leading coefficient of the quasi-polynomial q from Proposition 3.2.8.

In fact, we conjecture that the two quantities $\mathrm{LC}(M)$ and $\underline{\mathrm{LC}}(M)$ differ by a constant:

Conjecture: There exists a constant C, depending only on the Lie algebra $\mathfrak{g}$ and d, such that for all $M \in \mathcal{O}_\lambda^d$, we have:

$$\underline{\mathrm{LC}}(M) = C \cdot \mathrm{LC}(M)$$

Example 3.2.12. Returning to Example 3.2.6, let us compute $\mathrm{LC}(M)$ and $\underline{\mathrm{LC}}(M)$ when M is a simple modules $L(w \cdot \lambda)$ lying in $\mathcal{O}_\lambda^1$ for $\mathfrak{g} = \mathfrak{sl}_3$. First consider $M = L((s_2 s_1) \cdot \lambda)$ and recall that:

$$L((s_2 s_1) \cdot \lambda) \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_1)} V_{(s_2 s_1) \cdot \lambda}$$

Pick a weight basis of $V_{(s_2 s_1)\cdot\lambda}$: $v_1, \cdots, v_k$ where $k = \dim(V_{(s_2 s_1)\cdot\lambda})$. Then a basis for $L((s_2 s_1) \cdot \lambda)$ is given by $E_{31}^i E_{32}^j v_l$ where $i, j \geq 0, 1 \leq k \leq l$; and a basis for M_n / M_{n-1} is given by $E_{31}^i E_{32}^j v_l$ where $2i + j = n$. Thus:

$$\begin{aligned}
\dim(M_n/M_{n-1}) &= |\{(i,j) \mid 2i + j = n\}| \\
&= \begin{cases} k(\frac{n}{2} + 1) & \text{if } n \equiv 0 \\ k(\frac{n+1}{2}) & \text{if } n \equiv 1 \end{cases} \pmod 2 \\
\mathrm{LC}(M) &= \frac{1}{2}\dim(V_{(s_2 s_1)\cdot\lambda}) = \frac{1}{2}\langle \lambda + \rho, \check{\alpha_2} \rangle
\end{aligned}$$

Similarly, one may compute that $\underline{\mathrm{LC}}(M) = \langle \lambda + \rho, \check{\alpha_2} \rangle$. Using the descriptions of

$L((s_1s_2)\cdot\lambda), L(s_2\cdot\lambda)$ and $L(s_1\cdot\lambda)$ given in Example 3.2.6, we may also deduce that:

$$\mathrm{LC}(L((s_1s_2)\cdot\lambda)) = \frac{1}{2}\langle\lambda+\rho,\check{\alpha_1}\rangle,\ \underline{\mathrm{LC}}(L((s_1s_2)\cdot\lambda)) = \frac{1}{2}\langle\lambda+\rho,\check{\alpha_1}\rangle$$
$$\mathrm{LC}(L(s_1\cdot\lambda)) = \frac{1}{2}\langle\lambda+\rho,\check{\alpha_2}\rangle,\ \underline{\mathrm{LC}}(L(s_1\cdot\lambda)) = \frac{1}{2}\langle\lambda+\rho,\check{\alpha_2}\rangle$$
$$\mathrm{LC}(L(s_2\cdot\lambda)) = \frac{1}{2}\langle\lambda+\rho,\check{\alpha_1}\rangle,\ \underline{\mathrm{LC}}(L(s_2\cdot\lambda)) = \frac{1}{2}\langle\lambda+\rho,\check{\alpha_1}\rangle \qquad \blacksquare$$

3.2.2 Braid group action on derived category of $\mathcal{O}_0$.

Let $\mathbb{B}_W$ denote the braid group associated to the Weyl group W. Here we briefly recall the action of $\mathbb{B}_W$ on the derived category $D^b(\mathcal{O}_0)$

Given a simple root $\alpha\in\Delta^+$, and $M\in D^b(\mathcal{O}_0)$, we will define the action of $\tilde{s}_\alpha$ on M as follows.

Recall that the wall-crossing functor $R_\alpha:\mathcal{O}_0\to\mathcal{O}_0$ is defined as follows. First pick μ so that:

$$\langle\mu+\rho,\check{\alpha}\rangle = 0, \langle\mu+\rho,\check{\beta}\rangle > 0 \text{ if } \beta\in\Delta^+,\beta\neq\alpha$$

Then define $R_\alpha = T_{\mu\to 0}T_{0\to\mu}$. It can be shown that R_α does not depend on the choice of μ. Now define:

$$\Phi(\tilde{s}_\alpha)M = \mathrm{Cone}(M\to R_\alpha M)$$

Theorem 3.2.13. *The above action gives rise to an action of the braid group $\mathbb{B}_W$ on $D^b(\mathcal{O}_0)$.*

Proof. See the proof of Corollary 9.6 in [15]. □

However, since we are dealing the sub-quotients $\mathcal{O}_0^d$, we would like to have an action of the braid group $\mathbb{B}_W$ on $D^b(\mathcal{O}_0^d)$. Given $M\in D^b(\mathcal{O}_0)$, define

$$\mathrm{GK}(M) = \max_{i\in\mathbb{Z}}\mathrm{GK}(H^i(M))$$

Proposition 3.2.14. *Given $w\in\mathbb{B}_W$, then $GK(M) = GK(\Phi(w)\cdot M)$. In particular, the action of $\mathbb{B}_W$ on $D^b(\mathcal{O}_0)$ induces an action of $\mathbb{B}_W$ on $D^b(\mathcal{O}_0^d)$.*

Proof. Since the simple reflections $\tilde{s}_\alpha$ and their inverses generate the braid group, it is sufficient to show that M and $\Phi(\tilde{s}_\alpha)M$ have the same Gelfand-Kirillov dimension. First, $T_{\mu\to 0}T_{0\to\mu}M$ has Gelfand-Kirillov dimension at most equal to that of M (since tensoring by a finite-dimensional does not increase the Gelfand-Kirillov dimension). Therefore:

$$\mathrm{GK}(\Phi(\tilde{s}_\alpha)M)\leq\mathrm{GK}(M)$$

Since $\Phi(\tilde{s}_\alpha^{-1})M = \mathrm{Cone}(T_{\mu\to 0}T_{0\to\mu}(M)\to M)[-1]$, by a similar argument the reverse inequality also holds, and the conclusion follows. In fact, the stronger statement that the braid group action preserves the support of a module is true (i.e. $\mathrm{supp}(M)$ and $\mathrm{supp}(\Phi(w)\cdot M)$); see Joseph ([31]) for a more detailed discussion. □

The induced action of the braid group on the Grothendieck group $K^0(\mathcal{O}_0)$ is particularly simple to describe; we record it for later use (see [15] for a proof).

Lemma 3.2.15. *In the Grothendieck group of $\mathcal{O}_0$, we have the equality $[\Phi(\tilde{s}_\alpha)\Delta_{w\cdot 0}] = [\Delta_{(s_\alpha w)\cdot 0}]$.*

3.2.3 The central charge map

Given $M \in \mathcal{O}_0^d$, and $\lambda \in \Lambda^+$ dominant, we have $T_{0\to\lambda}M \in \mathcal{O}_\lambda^d$.

Proposition 3.2.16. *There exists a unique polynomial function $Z : \mathfrak{h}^* \to (K^0(C) \otimes \mathbb{R})^*$ such that:*

- $Z(\lambda)([M]) = LC(T_{0\to\lambda}M)$ *for* $\lambda \in \Lambda^+$, $M \in \mathcal{O}_0^d$.
- $Z(y^{-1}\cdot\lambda)([\Phi(\tilde{y})M]) = Z(\lambda)([M])$ *for* $\lambda \in \mathfrak{h}^*$, $M \in D^b(\mathcal{O}_0^d)$, $y \in W$.

Example 3.2.17. Before proving this proposition, let us return to the example with $\mathfrak{g} = \mathfrak{sl}_3$, $d = 1$ and verify the first part of the above Proposition by calculating the function Z. More precisely, we will calculate $Z(\lambda)[M]$ when M is one of the four simple objects in $\mathcal{O}_0^d$.

First let us calculate $Z(\lambda)[M_1]$ for $M_1 = L((s_2s_1)\cdot 0) = \Delta_{\mathfrak{p}_1}((s_2s_1)\cdot 0)$. In this case, using the computations from Example 3.2.12, we have:

$$T_{0\to\lambda}M_1 \simeq L((s_2s_1)\cdot\lambda) \simeq \Delta_{\mathfrak{p}_1}((s_2s_1)\cdot\lambda)$$
$$Z(\lambda)[M_1] = \frac{1}{2}\langle\lambda+\rho,\check{\alpha}_2\rangle$$

Similarly, we compute that:

$$Z(\lambda)[L((s_1s_2)\cdot 0)] = \frac{1}{2}\langle\lambda+\rho,\check{\alpha}_1\rangle$$
$$Z(\lambda)[L(s_1\cdot 0)] = \frac{1}{2}\langle\lambda+\rho,\check{\alpha}_2\rangle$$
$$Z(\lambda)[L(s_2\cdot 0)] = \frac{1}{2}\langle\lambda+\rho,\check{\alpha}_1\rangle \qquad \blacksquare$$

3.2.4 Main result

Now we are ready to state the main result.

Theorem 3.2.18. *The following datum constitutes an example of "real variations of stability conditions":*

- *Let $V = \mathfrak{h}^*$, and let Σ consist of the co-root hyperplanes $\langle\lambda+\rho,\check{\alpha}\rangle = 0$ (where $\check{\alpha} \in \Delta^+$).*

- *The set of alcoves,* **Alc** *are naturally identified with the Weyl group W; denote by $\underline{w}$ the alcove consisting of λ, with $\langle w^{-1}(\lambda+\rho), \check{\alpha}\rangle \geq 0$ for all $\alpha \in \Delta^+$.*

- *Let V^+ be the alcove $\underline{1}$.*

- *Let $\mathcal{C} = D^b(\mathcal{O}_0^d)$.*

- *Let the central charge $Z : \mathfrak{h}^* \to (K^0(\mathcal{C}) \otimes \mathbb{R})^*$ be the map constructed in Proposition 3.2.16.*

- *Given $w \in W$, let $\tilde{w}$ be its lift to the braid group $\mathbb{B}_W$. Let $\tau(\underline{w})$ be the image of the tautological t-structure on $\mathcal{C}$ under the automorphism $\Phi(\tilde{w})$.*

3.2.5 Describing leading coefficients in terms of the character

Given $M \in \mathcal{O}_\lambda$, for each $\mu \in \mathfrak{h}^*$ denote by M_μ the corresponding weight space. Recall that

$$\mathrm{ch}(M) = \sum_{\mu\in\mathfrak{h}^*} (\dim\, M_\mu) e^\mu$$

Using the PBW theorem, $\Delta(w\cdot\lambda)$

$$\mathrm{ch}(\Delta(w\cdot\lambda)) = \frac{e^{w\cdot\lambda}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})} = \frac{e^{w\rho}}{\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})}$$

Given $M \in \mathcal{O}_\lambda$, since the Verma modules $\{\Delta(w\cdot\lambda)\}_{w\in W}$ form a basis of the Grothendieck group $K^0(\mathcal{O}_\lambda)$, we have

$$[M] = \sum_{w\in W} a_w[\Delta(w\cdot\lambda)]$$

for some $a_w \in \mathbb{Q}$. Then we can express

$$\mathrm{ch}(M) = \frac{\sum_{w\in W} a_w e^{w\rho}}{\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})}$$

In order to prove the above proposition, we will make use of the following Lemma. Given a module $M \in \mathcal{O}_\lambda^d$, the following Lemma tells us how to deduce $\mathrm{GK}(M)$ and $\mathrm{LC}(M)$, knowing the character of M.

Proposition 3.2.19. *Suppose*

$$ch(M) = \frac{\sum_{w\in W} a_w e^{w(\rho+\lambda)}}{\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})}$$

Expand the numerator as a Taylor series, so that we obtain a function, f_M on $\mathfrak{h}^*$. Let k be minimal, such that the degree k component, f_M^k, of this polynomial does not *vanish. Then, there exists a constant c (depending on $\mathfrak{g}$ and d) such that (here $\check{\rho}$ is the half-sum of the positive roots):*

$$GK(M) = |\Delta^+| - k$$
$$LC(M) = cf_M^k(\check{\rho})$$

Example 3.2.20. Suppose $M = L(\lambda)$ is the finite-dimensional irreducible module with highest weight λ, so that by the Weyl character formula

$$\mathrm{ch}(M) = \frac{\sum_{w \in W} \mathrm{sgn}(w) e^{w(\rho+\lambda)}}{\prod_{\alpha \in \Delta^+}(e^{\alpha/2} - e^{-\alpha/2})}$$

It can be shown that for $k < |\Delta^+|$:

$$\sum_{w \in W} \mathrm{sgn}(w)\{w(\rho+\lambda)\}^k = 0$$

Thus $k = |\Delta^+|$ is the smallest k for which $f_M^k \neq 0$. Clearly M has Gelfand-Kirillov dimension 0, so this is consistent with the first claim in Proposition 3.2.19.

On the other end of the spectrum, suppose instead that M is the Verma module $\Delta(\lambda)$ for some $\lambda \in \mathfrak{h}^*$. The function f_M is the Taylor expansion of $e^{\lambda+\rho}$, and $k = 0$ (since the degree 0 component of f_M is non-zero). By Lemma 3.2.5, M has Gelfand-Kirillov dimension $|\Delta^+|$; again, this is consistent with the first claim in Proposition 3.2.19. ■

Example 3.2.21. Now let us verify Proposition 3.2.19 for the 4 simple modules lying in $\mathcal{O}_0^1$ for $\mathfrak{g} = \mathfrak{sl}_3$. From the discussion in Example 3.2.6, it follows that:

$$\begin{aligned}
\mathrm{ch}\ L((s_1s_2)\cdot\lambda) &= \mathrm{ch}\ \Delta((s_1s_2)\cdot\lambda) - \mathrm{ch}\ \Delta((s_1s_2s_1)\cdot 0)\\
\mathrm{ch}\ L((s_2s_1)\cdot\lambda) &= \mathrm{ch}\ \Delta((s_2s_1)\cdot\lambda) - \mathrm{ch}\ \Delta((s_1s_2s_1)\cdot\lambda)\\
\mathrm{ch}\ L(s_1\cdot\lambda) &= \mathrm{ch}\ \Delta(s_1\cdot\lambda) - \mathrm{ch}\ \Delta((s_1s_2)\cdot\lambda) - \mathrm{ch}\ \Delta((s_2s_1)\cdot\lambda) + \mathrm{ch}\ \Delta((s_1s_2s_1)\cdot\lambda)\\
\mathrm{ch}\ L(s_2\cdot\lambda) &= \mathrm{ch}\ \Delta(s_2\cdot\lambda) - \mathrm{ch}\ \Delta((s_1s_2)\cdot\lambda) - \mathrm{ch}\ \Delta((s_2s_1)\cdot\lambda) + \mathrm{ch}\ \Delta((s_1s_2s_1)\cdot\lambda)
\end{aligned}$$

For each of these modules M, clearly $f_M^0 = 0$, and f_M^1 is as follows:

$$\begin{aligned}
M &= L((s_2s_1)\cdot 0), f_M^1 = \langle\lambda+\rho, \check{\alpha_2}\rangle\alpha_1\\
M &= L((s_1s_2)\cdot 0), f_M^1 = \langle\lambda+\rho, \check{\alpha_1}\rangle\alpha_2\\
M &= L(s_1\cdot 0), f_M^1 = \langle\lambda+\rho, \check{\alpha_2}\rangle\alpha_2\\
M &= L(s_2\cdot 0), f_M^1 = \langle\lambda+\rho, \check{\alpha_1}\rangle\alpha_1
\end{aligned}$$

Using the calculations in Example 3.2.17, it is clear these that these computations are consistent with Proposition 3.2.19. ■

Proof of Proposition 3.2.19. We may make the assumption that M is a highest weight module, i.e. it is a quotient of $\Delta(\lambda)$ for some λ. To see this, note that the functions $M \to \mathrm{LC}(M)$ and $M \to f^k_M(\check{\rho})$ are additive on exact triangles $0 \to M_1 \to M \to M_2 \to 0$ such that $\mathrm{GK}(M_1) = \mathrm{GK}(M_2) = \mathrm{GK}(M)$. Thus we can choose a Jordan-Holder filtration of M, where each simple sub-quotient has the same Gelfand-Kirillov dimension; the conclusion would then follow if we knew it to be true for all highest weight modules, since all simples fall into this category.

So let $M_0 = \mathbb{C}v_\lambda$, for some highest weight vector $v_\lambda \in M$. Given $\mu \in \mathfrak{h}^*$ such that $M_\mu \neq 0$, define $d(\mu) = \langle \check{\rho}, \lambda - \mu \rangle$; alternatively $d(\mu) = \sum_{i \in I} d_i$ if $\lambda - \mu = \sum_{i \in I} d_i \alpha_i$ (note that $d_i \geq 0$, since M is a quotient of $\Delta(\lambda)$). Then:

$$M_i = \bigoplus_{d(\mu) \leq i} M_\mu$$
$$\dim M_i = \sum_{d(\mu) \leq i} \dim M_\mu$$

To prove this proposition, we will explicitly compute both sides of the below equality. Let us start with the LHS.

$$\Big(\sum_{\mu \in \Lambda^+} (\dim M_\mu) e^\mu\Big)(e^{t\check{\rho}}) \prod_{\alpha \in \Delta^+} (e^{\frac{\alpha}{2}} - e^{\frac{-\alpha}{2}})(e^{t\check{\rho}}) = \sum_{w \in W} d_w e^{w(\lambda+\rho)}(e^{t\check{\rho}})$$
$$\Big(\sum_{\mu \in \Lambda^+} (\dim M_\mu) e^\mu\Big) e^{t\check{\rho}} = e^{t\langle \lambda, \check{\rho} \rangle} \sum_{n \geq 0} \dim(M_n/M_{n-1}) e^{-nt}$$

For n sufficiently large, we have $\dim M_n = p_M(n)$ for some polynomial p_M; thus $\dim(M_n/M_{n-1}) = q_M(n)$, where $q_M(x) = p_M(x) - p_M(x-1)$. By differentiating the identity $1 + s + s^2 + \cdots = (1-s)^{-1}$ repeatedly, we obtain that:

$$\sum_n n(n-1)\cdots(n-k+1)s^{n-k} = k!(1-s)^{-k-1}$$

By taking linear combinations of the above identity, we deduce that there exists a polynomial $\tilde{q}_M$ with degree d, leading coefficient $d!\mathrm{LC}(M)$ and no constant term, satisfying the following. Then we will evaluate at $s = e^{-t}$, and continue with the computation of the left hand side (here the polynomial r_M accounts for the fact that $\dim M_n \neq p(n)$ at finitely many values, and C is some constant):

$$\sum_n q_M(n)s^n = \tilde{q}_M((1-s)^{-1})$$
$$\sum_{n \geq 0} \dim(M_n/M_{n-1}) e^{-nt} = \tilde{q}_M((1-e^{-t})^{-1}) + r_M(e^{-t})$$

$$\prod_{\alpha\in\Delta^+}(e^{\frac{\alpha}{2}}-e^{\frac{-\alpha}{2}})(e^{t\check{\rho}}) = \prod_{\alpha\in\Delta^+}(e^{t\langle\check{\rho},\frac{\alpha}{2}\rangle}-e^{-t\langle\check{\rho},\frac{\alpha}{2}\rangle})$$
$$= \prod_{\alpha\in\Delta^+}(t\langle\check{\rho},\alpha\rangle+\frac{t^3\langle\check{\rho},\alpha\rangle^3}{24}+\cdots)$$
$$= \{\prod_{\alpha\in\Delta^+}\langle\check{\rho},\alpha\rangle\}t^{|\Delta^+|}(1+t^2C+\cdots)$$

$$\text{LHS} = e^{t\langle\lambda,\check{\rho}\rangle}\{\tilde{q}_M((1-e^{-t})^{-1})+r_M(e^{-t})\}\{\prod_{\alpha\in\Delta^+}\langle\check{\rho},\alpha\rangle\}t^{|\Delta^+|}(1+t^2C+\cdots)$$
$$= \{d!\ \text{LC(M)}t^{-d}(1+\cdots)\}\{\prod_{\alpha\in\Delta^+}\langle\check{\rho},\alpha\rangle\}t^{|\Delta^+|}(1+t\langle\lambda,\check{\rho}\rangle+\cdots)$$
$$= d!\ \text{LC(M)}\prod_{\alpha\in\Delta^+}\langle\check{\rho},\alpha\rangle\ t^{|\Delta^+|-d}(1+\cdots)$$

Above, we have used the Taylor expansion $(1-e^{-t})^{-1} = t^{-1}(1+\frac{t}{2}+\cdots)$ to obtain the leading coefficient of the Taylor expansion of $\tilde{q}_M((1-e^{-t})^{-1})$.

$$\sum_{w\in W} d_w e^{w(\lambda+\rho)}(e^{t\check{\rho}}) = \sum_{w\in W} d_w e^{\langle w(\lambda+\rho),t\check{\rho}\rangle}$$
$$= \sum_{w\in W,n\geq 0} d_w\frac{t^n}{n!}\langle w(\lambda+\rho),\check{\rho}\rangle^n$$

Comparing the Taylor expansions of the LHS and the RHS, it follows that $\text{LC}(M) = cf^k_M(\check{\rho})$ for some constant c and $k = |\Delta^+|-d$; and further $f^i_M(\check{\rho}) = 0$ for $i<k$. Now pick $\check{\rho}'$ arbitrary satisfying $\langle\check{\rho}',\alpha\rangle\in\mathbb{Z}_{>0}$ for simple roots α. Repeating the whole argument, we deduce that $f^i_M(\check{\rho}') = 0$ for any such $\check{\rho}'$. It then follows that $f^i_M = 0$ for $i<k$, completing the proof. □

Proof of Proposition 3.2.16. Suppose that

$$\text{ch}(M) = \frac{\sum_{w\in W} a_w e^{w\rho}}{\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})}$$

Then, since $T_{0\to\lambda}\Delta(w\cdot 0) = \Delta(w\cdot\lambda)$:

$$\text{ch}(T_{0\to\lambda}M) = \frac{\sum_{w\in W} a_w e^{w(\rho+\lambda)}}{\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})}$$

It is clear that $\text{GK}(T_{0\to\lambda}M) = \text{GK}(M) = d$. Let us define the polynomial

$$Z(\lambda)([M]) = c\sum_{w\in W} a_w\{(w(\rho+\lambda)(\check{\rho})\}^{|\Delta^+|-d}$$

Using Proposition 3.2.19, it follows that $\text{LC}(T_{0\to\lambda}M) = Z(\lambda)([M])$ for $\lambda\in\Lambda^+$, proving the first claim. The uniqueness of the polynomial is clear once its values on the lattice Λ^+ are specified.

Next, using Lemma 3.2.15, we have:

$$\begin{aligned}
\mathrm{ch}(\Phi(\widetilde{y})M) &= \frac{\sum_{w\in W} a_w e^{(wy)\rho}}{\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})} \\
&= \frac{\sum_{w\in W} a_{wy^{-1}} e^{w\rho}}{\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})} \\
Z(y^{-1}\cdot\lambda)([\Phi(\widetilde{y})M]) &= c\sum_{w\in W} a_{wy^{-1}}\{w(\rho+y^{-1}\cdot\lambda)(\check{\rho})\}^{|\Delta^+|-d} \\
&= c\sum_{w\in W} a_{wy^{-1}}\{wy^{-1}(\lambda+\rho)(\check{\rho})\}^{|\Delta^+|-d} \\
&= c\sum_{w\in W} a_w\{w(\lambda+\rho)(\check{\rho})\}^{|\Delta^+|-d} = Z(\lambda)([M])
\end{aligned}$$

This proves the statement of the second claim. □

Remark 3.2.22. At this point, we are almost ready to check the first condition involving real variations of stability conditions. Suppose M lies in the heart of the t-structure $\tau(w)$ (i.e. $M = \Phi(\tilde{w})(M')$ for some $M' \in \mathcal{O}_d^0$), and that λ lies inside the alcove $\underline{w}$ (i.e. $\lambda = w\cdot\lambda'$ for some λ' lying inside the alcove $\underline{1}$). Then, using Proposition 3.2.16:

$$\begin{aligned}
\langle Z(\lambda)[M]\rangle &= Z(w\cdot\lambda')[\Phi(\tilde{w})(M')] \\
&= Z(\lambda')[M']
\end{aligned}$$

Now if $\lambda' \in \Lambda^+$, then $Z(\lambda')[M'] = \mathrm{LC}(T_{0\to\lambda'}M') \geq 0$ (by definition of leading coefficient). However, to rigorously show that this statement is true for arbitrary λ' in the fundamental Weyl chamber, we will need more machinery.

3.3 Filtration of the heart corresponding to adjacent alcoves

In this section we will check that the second condition from the above Definition is satisfied.

Suppose $\underline{w}$ and $\underline{w'}$ are two adjacent alcoves separated by a hyperplane H (i.e. $w' = s_\alpha w$, where α is a simple root); and suppose that $\underline{w'}$ lies above $\underline{w}$. Recall that $\mathcal{A}_{\underline{w}}$ denotes the heart of the t-structure $\tau(\underline{w})$. Denote by $\mathcal{A}^n_{\underline{w},\underline{w'}}$ (resp. $\mathcal{A}^n_{\underline{w'},\underline{w}}$) the sub-category consisting of objects $M \in \mathcal{A}_{\underline{w}}$ (resp. $M \in \mathcal{A}_{\underline{w'}}$) such that the function $f_M : \mathfrak{h}^* \to \mathbb{C}$ defined by $f_M(x) = \langle Z(x), [M]\rangle$ has a zero of order at least n on H. The following Lemma allows us to reduce to the case where $w = 1$, and the two propositions give a very concrete descriptions of these sub-categories in that case.

Lemma 3.3.1. *1.* $\mathcal{A}^k_{\underline{w'},\underline{w}} = \Phi(\widetilde{w})\mathcal{A}^k_{\underline{s_\alpha},\underline{1}}$, *and* $\mathcal{A}^k_{\underline{w},\underline{w'}} = \Phi(\widetilde{w})\mathcal{A}^k_{\underline{1},\underline{s_\alpha}}$

2. $\mathcal{A}^k_{\underline{w},\underline{w'}} = \Phi(\widetilde{s_\alpha})\mathcal{A}^k_{\underline{w'},\underline{w}}$ *(in particular,* $\mathcal{A}^k_{\underline{s_\alpha},\underline{1}} = \Phi(\widetilde{s_\alpha})\mathcal{A}^1_{\underline{k},\underline{s_\alpha}}$*)*

Proposition 3.3.2. *The category* $\mathcal{A}^1_{\underline{1},\underline{s_\alpha}} \subset \mathcal{A}_{\underline{1}} = \mathcal{O}^d_0$ *consists of those objects* $M \in \mathcal{O}^d_0$ *which possess a filtration, with each quotient being a simple modules* $L(w \cdot 0)$ *with* $l(ws_\alpha) = l(w) + 1$.

Proposition 3.3.3. *For* $n \geq 2$, *the categories* $\mathcal{A}^n_{\underline{w},\underline{w'}} = \{0\}$.

Example 3.3.4. Before proving the above two propositions, let us re-visit our running example (with $\mathfrak{g} = \mathfrak{sl}_3$ and $d = 1$) and verify them by hand in that case. From the calculations in Example 3.2.17, we deduce that the category $\mathcal{A}^1_{\underline{1},\underline{s_1}}$ consists of modules in $\mathcal{O}^1_0$ which have a filtration whose sub-quotients are $L((s_1s_2) \cdot 0)$ or $L(s_2 \cdot 0)$; and that the category $\mathcal{A}^1_{\underline{1},\underline{s_2}}$ consists of modules in $\mathcal{O}^1_0$ which have a filtration whose sub-quotients are $L((s_2s_1) \cdot 0)$ or $L(s_1 \cdot 0)$. This is consistent with Proposition 3.3.2. For each $M \in \mathcal{O}^1_0$, $f_M : \mathfrak{h}^* \to \mathbb{C}$ is a linear function, and hence cannot have a double zero on any hyperplane (unless $M = 0$); this is consistent with Proposition 3.3.3. ■

Proof of Lemma 3.3.1. First let us prove (1). Given an object $M \in \mathcal{C}$, recall that f_M denotes the function on $\mathfrak{h}^*$ defined by $f_M(x) = \langle Z(x), [M] \rangle$. Denote by $\deg(f_M|_H)$ the order of vanishing of the polynomial f_M on the hyperplane H.

$$\begin{aligned}
\mathcal{A}^k_{\underline{w},\underline{w'}} &= \{M \in \mathcal{A}_w \mid \deg(f_M|_H) \geq k\} \\
&= \{\Phi(\widetilde{w})N, N \in \mathcal{O}^d_0 \mid \deg(f_{\Phi(\widetilde{w})N}|_H) \geq k\} \\
&= \{\Phi(\widetilde{w})N, N \in \mathcal{O}^d_0 \mid \deg(f_N|_{H^\alpha}) \geq k\} \\
&= \Phi(\widetilde{w})\mathcal{A}^k_{\underline{1},\underline{s_\alpha}}
\end{aligned}$$

Here H^α denotes the wall $\langle \lambda + \rho, \check{\alpha} \rangle = 0$ separating Λ^+ and $s_\alpha \cdot \Lambda^+$. Above we have used the fact that $\deg(f_{\Phi(\widetilde{w})N}|_H) = \deg(f_N|_{H^\alpha})$, which follows from W-equivariance. Similarly, it follows that $\mathcal{A}^k_{\underline{w'},\underline{w}} = \Phi(\widetilde{w})\mathcal{A}^k_{\underline{s_\alpha},\underline{1}}$

Using (1), in order to show (2) it suffices to prove that $\mathcal{A}^k_{\underline{s_\alpha},\underline{1}} = \Phi(\widetilde{s_\alpha})\mathcal{A}^1_{\underline{1},\underline{s_\alpha}}$. This is clear, since:

$$\begin{aligned}
\mathcal{A}^k_{\underline{s_\alpha},\underline{1}} &= \{\Phi(\widetilde{s_\alpha})C, C \in \mathcal{O}^d_0 \mid \deg(f_{\Phi(\widetilde{s_\alpha})C}|_{H^\alpha}) \geq k\} \\
&= \{\Phi(\widetilde{s_\alpha})C, C \in \mathcal{O}^d_0 \mid \deg(f_C|_{H^\alpha}) \geq k\} \\
&= \Phi(\widetilde{s_\alpha})\mathcal{A}^1_{\underline{1},\underline{s_\alpha}}
\end{aligned}$$

Above, $\deg(f_{\Phi(\widetilde{s_\alpha})C}|_{H^\alpha}) = \deg(f_C|_{H^\alpha})$ using W-equivariance (since s_α acts via reflecting about the hyperplane H^α). □

Lemma 3.3.5. *Suppose* M *has a filtration, with each successive quotient being a simple module* $L(w \cdot 0)$ *with* $l(ws_\alpha) = l(w) + 1$. *Then* $\Phi(\widetilde{s_\alpha})M = M[1]$

Proof. Fix μ, such that $\langle \mu+\rho, \check{\alpha} \rangle = 0, \langle \mu+\rho, \check{\beta} \rangle = 0$. By definition, we need to show that $\mathrm{Cone}(M \to R_\alpha M) = M[1]$ (where $R_\alpha = T_{\mu\to 0}T_{0\to\mu}$). So it suffices to show that $R_\alpha M = 0$. We will show the stronger statement that $T_{0\to\mu}M = 0$.

Suppose that $l(ws_\alpha) = l(w)+1$. Then we have a sequence of maps:

$$\Delta(ws_\alpha \cdot 0) \xrightarrow{i} \Delta(w \cdot 0) \xrightarrow{p} L(w \cdot 0)$$

The map i is injective, and its existence follows using Proposition 1.4 of [34]; the map p is clearly surjective. Since the image of i lands inside the maximal submodule of $\Delta(w\cdot 0)$, the composition of these two maps is 0. Applying the translation functor $T_{0\to\mu}$ to this triangle, we get:

$$\Delta(ws_\alpha \cdot \mu) \xrightarrow{i_T} \Delta(w \cdot \mu) \xrightarrow{p_T} T_{0\to\mu}L(w \cdot 0)$$

However, since $\langle \mu+\rho, \check{\alpha}\rangle = 0$, it follows that $s_\alpha \cdot \mu = \mu$; and hence the map i_T is an isomorphism. Since the composition of the two maps is 0, and p_T is surjective, it follows that $T_{0\to\lambda}L(w\cdot 0) = 0$.

Now suppose M has a filtration by such modules $L(w\cdot 0)$. By using the exact-ness of the functor $T_{0\to\lambda}$, and inducting on the length $l(M)$ of this filtration, it follows that $T_{0\to\lambda}M = 0$. □

Proof of Proposition 3.3.2. First we prove that any module M, with such a filtration lies inside $\mathcal{A}^1_{\underline{1,s_\alpha}}$. Suppose $\langle x+\rho, \alpha\rangle = 0$; then $s_\alpha \cdot x = x$, so:

$$\begin{aligned}\langle Z(x), [M]\rangle &= \langle Z(s_\alpha\cdot x), [\Phi(s_\alpha)M]\rangle \\ &= \langle Z(x), [M[1]]\rangle \\ &= -\langle Z(x), [M]\rangle \\ \Rightarrow \langle Z(x), [M]\rangle &= 0\end{aligned}$$

Then we have that, $\langle Z(x), [M]\rangle = 0$ if $\langle x+\rho, \check{\alpha}\rangle = 0$, and so $M \in \mathcal{A}^1_{\underline{1,s_\alpha}}$ by definition.

Next let us prove that any module $M \in \mathcal{A}^1_{\underline{1,s_\alpha}}$ possesses such a filtration. Suppose the Jordan-Holder filtration of M contains the simple module $L(w\cdot 0)$ with multiplicity m_w, so that

$$\begin{aligned}\langle Z(x), [M]\rangle &= \langle Z(x), \sum_{w\in W} m_w[L(w\cdot 0)]\rangle \\ &= \sum_{w\in W} m_w\langle Z(x), [L(w\cdot 0)]\rangle\end{aligned}$$

Now pick λ to be integral such that $\langle \lambda+\rho, \check{\alpha}\rangle = 0$, but $\langle \lambda+\rho, \check{\beta}\rangle > 0$ for all simple roots $\beta \neq \alpha$ (so λ lies in the closure of the dominant alcove). Then we claim that $\langle Z(\lambda), L(w\cdot 0)\rangle > 0$ if $l(ws_\alpha) = l(w)-1$. The desired result would follow.

It is known (for instance, see Section 7.7 of [34]) that $T_{0\to\lambda}$ sends an irreducible module either to 0, or to another irreducible; by counting the number of irreducibles it follows that $T_{0\to\lambda}L(w\cdot 0) = L(w\cdot\lambda)$. By using the techniques employed in the proof of Proposition 3.2.19, it follows that $L(w\cdot\lambda)$ and $L(w\cdot 0)$ have the same Gelfand-Kirillov dimension. Further, we deduce that $\langle Z(\lambda), L(w\cdot 0)\rangle = \mathrm{LC}(L(w\cdot\lambda)) > 0$, as required. □

In order to prove Proposition 3.3.3, we appeal to the theory of harmonic polynomials. For a detailed exposition, we refer the reader to Section 6.3 and 6.4 of [29].

Definition 3.3.6. A polynomial function $f : \mathfrak{h}^* \to \mathbb{C}$ is a "harmonic polynomial"' if for every $\partial \in \mathcal{D}(\mathfrak{h}^*)^W_+$, $\partial f = 0$.

Proposition 3.3.7. *Fix an object $M \in \mathcal{C}$. The function $f_M : \mathfrak{h}^* \to \mathbb{C}$, given by $f_M(x) = \langle Z(x), [M]\rangle$ is a harmonic polynomial.*

Proposition 3.3.8. *Any non-zero harmonic polynomial cannot have a double zero on a co-root hyperplane.*

Example 3.3.9. Before proving these two propositions, let us revisit the example $\mathfrak{g} = \mathfrak{sl}_3$. First let us calculate the invariant differential operators, and the harmonic polynomials in this case. Define $X_1, X_2 : \mathfrak{h}^* \to \mathbb{C}$ by setting $X_1(\lambda) = \langle\lambda, \check{\alpha}_1\rangle, X_2(\lambda) = \langle\lambda, \check{\alpha}_2\rangle$; then the set of polynomial functions from $\mathfrak{h}^*$ to $\mathbb{C}$ can be naturally identified by $\mathbb{C}[X_1, X_2]$. We compute that:

$$s_1\frac{\partial}{\partial X_1} = -\frac{\partial}{\partial X_1} + \frac{\partial}{\partial X_2}, \qquad s_2\frac{\partial}{\partial X_1} = \frac{\partial}{\partial X_1}$$
$$s_1\frac{\partial}{\partial X_2} = \frac{\partial}{\partial X_2}, \qquad s_2\frac{\partial}{\partial X_2} = \frac{\partial}{\partial X_1} - \frac{\partial}{\partial X_2}$$
$$\mathcal{D}(\mathfrak{h}^*)^W_+ = \langle(\frac{\partial}{\partial X_1})^2 + (\frac{\partial}{\partial X_2})^2 - \frac{\partial}{\partial X_1}\frac{\partial}{\partial X_2}, (\frac{\partial}{\partial X_1})^2\frac{\partial}{\partial X_2} - \frac{\partial}{\partial X_1}(\frac{\partial}{\partial X_2})^2\rangle$$

So the space $\mathcal{H}$ of harmonic polynomials are those annihilated by those two polynomials:

$$\mathcal{H} = \mathbb{C}\{X_1^2X_2 + X_1X_2^2, X_1^2 + 2X_1X_2, X_2^2 + 2X_1X_2, X_1, X_2, 1\}$$

This is consistent with Proposition 3.3.7, since when M is a simple module in $\mathcal{O}_0^1$, $f_M(x) = X_1 + 1$ or $f_M(x) = X_2 + 1$; in either case $f_M \in \mathcal{H}$. It is easy to verify that no element of $\mathcal{H}$ is divisible by $(X_1+1)^2$ or $(X_2+1)^2$; this is consistent with Proposition 3.3.8. ■

Proof of Proposition 3.3.7. This follows from Proposition 3.2.19, combined with the following Lemma. □

Lemma 3.3.10. *Given a collection $\{a_w\}_{w\in W}$ of complex numbers, let d be minimal such that*

$$R_d := \sum_{w\in W} a_w \frac{w(\rho+\lambda)^d}{d!}$$

is a non-zero function. Then R_d is a harmonic polynomial.

Proof. From Proposition 6.4.4 in [29], it follows that $R_d(\lambda - \rho)$ is a harmonic polynomial. The result now follows using the well-known fact that any harmonic polynomial is stable under shifts. □

Proof of Proposition 3.3.8. See the last paragraph of Proposition 1 in [7] (on page 9). □

Proof of Proposition 3.3.3. This follows using Propositions 3.3.7 and 3.3.8. □

Before returning to the proof of the Main Theorem, we will need the following three Lemmas (the first of which is a strengthening of Lemma 3.3.5).

Lemma 3.3.11. *If $A \in \mathcal{C}$ satisfies $H^n(A) \in \mathcal{A}^1_{\underline{1},\underline{s}_\alpha}$ $\forall\, n \in \mathbb{Z}$, then $\Phi(\tilde{s}_\alpha)(A) \simeq A[1]$.*

Proof. Given an object $A \in \mathcal{C}$, define the length $l(A) = |\{i \in \mathbb{Z}, H^i(A) \neq 0\}|$. Let us proceed by induction on $l(A)$. If $l(A) = 1$, the statement follows from Proposition 3.1 and Lemma 3.4.

Now suppose that $l(A) = i$. We can pick j so that $l(\tau_{\leq j}A), l(\tau_{\geq j+1}A) < i$ (here τ denotes truncation with respect to the standard t-structure on $\mathcal{C}$). Applying the automorphism $\Phi(\tilde{s}_\alpha)(A)$, and using the induction hypothesis, we have distinguished triangles:

$$\tau_{\leq j}A \to A \to \tau_{\geq j+1}A$$
$$\Phi(\tilde{s}_\alpha)\tau_{\leq j}A \to \Phi(\tilde{s}_\alpha)A \to \Phi(\tilde{s}_\alpha)\tau_{\geq j+1}A$$
$$\tau_{\leq j}A[1] \to \Phi(\tilde{s}_\alpha)A \to \tau_{\geq j+1}A[1]$$

Now using the axioms of a triangulated category, it follows that $\Phi(\tilde{s}_\alpha)A \simeq A[1]$, as required. □

Lemma 3.3.12. *Any $A \in \mathcal{C}$ satisfies $\Phi(\tilde{s}_\alpha)A \simeq A$ (mod $\mathcal{C}^1_{\underline{1},\underline{s}_\alpha}$)*

Proof. First let us prove the statement when A is a simple module. Recall that $\Phi(\tilde{s}_\alpha)A = \mathrm{Cone}(A \to T_{\mu\to 0}T_{0\to\mu}A)$, where μ satisfies

$$\langle \mu + \rho, \check{\alpha} \rangle = 0, \langle \mu + \rho, \check{\beta} \rangle > 0 \text{ if } \beta \in \Delta^+, \beta \neq \alpha$$

Since the functors $T_{\mu\to 0}$ and $T_{0\to\mu}$ are bi-adjoint, we have natural maps $T_{\mu\to 0}T_{0\to\mu}A \to A$ and $A \to T_{\mu\to 0}T_{0\to\mu}A$. We claim that the composition of the two $A \to T_{0\to\mu}T_{\mu\to 0}A \to A$ is zero. Suppose that it isn't; then the composite map is an isomorphism. Further, neither of the maps can be an isomorphism, since the action of the braid group element $\Phi(\tilde{s}_\alpha)$ is invertible. Thus $M = T_{0\to\mu}T_{\mu\to 0}A$ contains A as a direct summand. But using adjointness, $\mathrm{Hom}(M, L)$ is 1-dimensional if $L \simeq A$, and is 0 otherwise; this is a contradiction.

Since the composition of the two maps is 0, we have a map $\Phi(\tilde{s}_\alpha)A \to A$. From Corollary 7.12 in [34], we have:

$$T_{0\to\mu}T_{\mu\to 0}(T_{0\to\mu}A) \simeq T_{0\to\mu}A \oplus T_{0\to\mu}A$$

It follows that $T_{0\to\mu}[\Phi(\tilde{s}_\alpha)A] = T_{0\to\mu}A$, and hence $T_{0\to\mu}[\Phi(\tilde{s}_\alpha)A \to A] = 0$.

From Lemma 3.3.11, we deduce that if $T_{0\to\mu}C = 0$ then $C \in \mathcal{C}^1_{\underline{1},s_\alpha}$. Thus the statement is true when A is a simple module; the general case now follows using the argument used in Lemma 3.3.11. □

Lemma 3.3.13. *Suppose $p \in \mathbb{R}[x_1, \cdots, x_n]$ is a homogeneous polynomial, such that $p(x_1, x_2, \cdots, x_n) \geq 0$ if $(x_1, \cdots, x_n) \in \mathbb{Z}^n_{\geq 0}$. Then $p(x_1, x_2, \cdots, x_n) \geq 0$ if $(x_1, \cdots, x_n) \in \mathbb{R}^n_{\geq 0}$.*

Proof. Assume to the contrary that $p(x_1, x_2, \cdots, x_n) < 0$ for some $(x_1, \cdots, x_n) \in \mathbb{R}^n_{\geq 0}$. By continuity, we can pick a small open ball B containing $(x_1, \cdots, x_n)$, such that $p(x_1, x_2, \cdots, x_n) < 0$ for all $(x_1, \cdots, x_n) \in B$. Let $\widetilde{B} = \{t \cdot x \mid x \in B, t \in \mathbb{R}_{\geq 0}\}$; since p is a homogeneous polynomial, it follows that $p(x_1, x_2, \cdots, x_n) < 0$ for all $(x_1, \cdots, x_n) \in \widetilde{B}$. However, it is clear that $\widetilde{B}$ contains points in $\mathbb{Z}^n_{\geq 0}$; this contradicts our initial assumption. □

Proof of Main Theorem. First let us complete the proof of the first condition in Definition 3.1.1 (which we started in Remark 3.2.22). From the discussion in that remark, it suffices to prove that $Z(\lambda)[M] \geq 0$, where $\lambda \in \underline{1}$, and $M \in \mathcal{O}^0_d$. We know that $Z(\lambda)[M] = \mathrm{LC}(T_{0\to\lambda}M) \geq 0$ if $\lambda \in \Lambda^+$; from the discussion in the proof of Proposition 3.3.2, we know that that this statement also holds if λ is an integral weight lying inside the closure of the alcove $\underline{1}$.

Let $\{\check{\alpha}_1, \cdots, \check{\alpha}_r\}$ are the simple roots, $\{\Lambda_1, \cdots, \Lambda_r\}$ the corresponding fundamental weights, and suppose that $w^{-1}\check{\alpha}_i = \sum_{1\leq j\leq r} c^w_{i,j}\check{\alpha}_j$. By Proposition 3.2.19, if $[M] = \sum_{w\in W} a_w[\Delta(w\cdot 0)]$ and $k = |\Delta^+| - d$, then:

$$\begin{aligned}
Z(\lambda)[M] &= \frac{c}{k!}\sum_{w\in W} a_w\{w(\rho+\lambda)(\check{\rho})\}^k \\
&= \frac{c}{k!}\sum_{w\in W} a_w\{\sum_{1\leq j\leq r}\langle w(\lambda+\rho), \check{\alpha}_j\rangle \Lambda_j(\check{\rho})\}^k \\
&= \frac{c}{k!}\sum_{w\in W} a_w\{\sum_{1\leq j\leq r}\langle \lambda+\rho, w^{-1}\check{\alpha}_j\rangle \Lambda_j(\check{\rho})\}^k \\
&= \frac{c}{k!}\sum_{w\in W} a_w\{\sum_{1\leq l\leq r}\langle \lambda+\rho, \check{\alpha}_l\rangle (\sum_{1\leq j\leq r} c^w_{j,l}\Lambda_j(\check{\rho}))\}^k
\end{aligned}$$

So it is clear that $Z(\lambda)[M]$ is a homogeneous polynomial of degree k in the variables $\langle\lambda+\rho, \check{\alpha}_1\rangle, \cdots, \langle\lambda+\rho, \check{\alpha}_r\rangle$. Thus by applying Lemma 3.3.13, it follows that $Z(\lambda)[M] \geq 0$ when λ lies inside the closure of the alcove $\underline{1}$. We will now prove the stronger statement that $Z(\lambda)[M] > 0$ when λ lies inside $\underline{1}$. Suppose instead that $Z(\lambda)[M] = 0$. From Proposition 3.3.7, and Proposition 6.3.25 of [29], we have that, for any μ,

$$Z(\lambda)[M] = \frac{1}{|W|} \sum_{w \in W} Z(\lambda + w\mu)[M]$$

Pick μ sufficiently small so that $\lambda + w\mu$ lies in the alcove $\underline{1}$ for all $w \in W$; then $Z(\lambda + w\mu)[M] = 0$ for all $w \in W$. Since μ was arbitrary, this means that $Z(\lambda)[M] = 0$ on some neighbourhood of λ. Hence $Z(\lambda)[M] = 0$, contradicting the fact that M was a non-zero object. This completes the proof of the first condition from Definition 3.1.1.

Now we will check the second condition in Definition 3.1.1. In keeping with the notation used there, recall that

$$\mathcal{C}^n_{\underline{w},\underline{w'}} = \{C \in \mathcal{C} \mid H^i_{\tau(\underline{w})}(C) \in \mathcal{A}^n_{\underline{w},\underline{w'}} \ \forall\, i \ \in \mathbb{Z}\}$$

First we need to show that the filtration $\{0\} = \mathcal{C}^2_{\underline{w},\underline{w'}} \subset \mathcal{C}^1_{\underline{w},\underline{w'}} \subset \mathcal{C}$ is stable under the truncation functors for the t-structure $\tau(\underline{w'})$. Using Lemma 3.3.1, we may reduce to the case where $w' = 1, w = s_\alpha$ (where α is a simple root). Thus we have:

$$\begin{aligned}
H^n_{\tau(\underline{s_\alpha})}(C) &= \Phi(\widetilde{s_\alpha}) H^n(\Phi(\widetilde{s_\alpha}^{-1})C) \\
\mathcal{C}^1_{\underline{s_\alpha},\underline{1}} &= \{C \in \mathcal{C} \mid H^i_{\tau(\underline{s_\alpha})}(C) \in \mathcal{A}^1_{\underline{s_\alpha},\underline{1}}\} \\
&= \{C \in \mathcal{C} \mid \Phi(\widetilde{s_\alpha}) H^n(\Phi(\widetilde{s_\alpha}^{-1})C) \in \Phi(\widetilde{s_\alpha})\mathcal{A}^1_{\underline{1},\underline{s_\alpha}}\} \\
&= \{C \in \mathcal{C} \mid H^n(\Phi(\widetilde{s_\alpha}^{-1})C) \in \mathcal{A}^1_{\underline{1},\underline{s_\alpha}}\}
\end{aligned}$$

We wish to show that if $C \in \mathcal{C}^1_{\underline{s_\alpha},\underline{1}}$, then $\tau_{\leq i}C, \tau_{\geq i}C \in \mathcal{C}^1_{\underline{s_\alpha},\underline{1}}$ for $i \in \mathbb{Z}$.

Now $A = \Phi(\widetilde{s}_\alpha^{-1})C$ satisfies the hypothesis of Lemma 3.3.11, so:

$$\begin{aligned}
&\Phi(\widetilde{s}_\alpha)(A) = A[1] \Rightarrow \Phi(\widetilde{s}_\alpha^{-1})C = C[-1] \\
&\therefore H^j(C) \in \mathcal{A}^1_{\underline{1},\underline{s_\alpha}} \forall i \in \mathbb{Z} \\
&H^j(\tau_{\geq i}C) = \begin{cases} H^j(C) & \text{if } j \geq i \\ 0 & \text{if } j < i \end{cases}, H^j(\tau_{\leq i}C) = \begin{cases} H^j(C) & \text{if } j \leq i \\ 0 & \text{if } j > i \end{cases} \\
&\Rightarrow H^j(\tau_{\geq i}C), H^j(\tau_{\leq i}C) \in \mathcal{A}^1_{\underline{1},\underline{s_\alpha}} \forall i \in \mathbb{Z}
\end{aligned}$$

Applying Lemma 3.3.11 again, we get that:

$$\begin{aligned}
\Phi(\tilde{s}_\alpha)(\tau_{\geq i}C) = \tau_{\geq i}C[1], \Phi(\tilde{s}_\alpha)(\tau_{\leq i}C) = \tau_{\leq i}C[1] \\
\Phi(\tilde{s}_\alpha^{-1})(\tau_{\geq i}C) = \tau_{\geq i}C[-1], \Phi(\tilde{s}_\alpha^{-1})(\tau_{\leq i}C) = \tau_{\leq i}C[-1] \\
\Rightarrow H^i(\Phi(\tilde{s}_\alpha^{-1})(\tau_{\geq i}C)), H^i(\Phi(\tilde{s}_\alpha^{-1})(\tau_{\leq i}C)) \in \mathcal{A}^1_{\underline{1},\underline{s_\alpha}}
\end{aligned}$$

It follows that if $C \in \mathcal{C}^1_{\underline{s_\alpha},\underline{1}}$, then $\tau_{\leq i}C, \tau_{\geq i}C \in \mathcal{C}^1_{\underline{s_\alpha},\underline{1}}$ for $i \in \mathbb{Z}$.

We also need to show that the two t-structures on the quotient $\mathcal{C}^n_{\underline{s_\alpha},\underline{1}}/\mathcal{C}^{n+1}_{\underline{s_\alpha},\underline{1}}$ induced by $\tau(\underline{1})$ and $\tau(\underline{s_\alpha})$ differ by a shift of $[n]$, for $n = 0, 1$. For $n = 0$, this follows from Lemma 3.3.12; for $n = 1$, this follows directly follow Lemma 3.3.11. $\square$

Bibliography

[1] P. Achar, Equivariant coherent sheaves on the nilpotent cone for complex reductive Lie groups, Ph.D. thesis, Massachusetts Institute of Technology, 2001

[2] P. Achar, On the equivariant K-theory of the nilpotent cone for the general linear group, Represent. Theory 8 (2004), 180–211

[3] P. Achar, A. Henderson, Orbit closures in the enhanced nilpotent cone, Advances in Mathematics 219 (2008), 27-62.

[4] P. Achar, Perverse coherent sheaves on the nilpotent cone in good characteristic, "Recent developments in lie algebras, groups and representation theory", 1-23, Proc. Sympos. Pure Math.

[5] P. Achar, A. Henderson, E. Sommers, Pieces of nilpotent cones for classical groups, Represent. Theory 15 (2011), 584-616.

[6] R. Anno, Affine tangles and irreducible exotic sheaves, arXiv:0802.1070

[7] R. Anno, R. Bezrukavnikov, I. Mirkovic, A thin stringy moduli space for Slodowy slices, arXiv:1108.1563.1

[8] R. Anno, T. Logvinenko, Spherical DG-Functors, arXiv:1309.5035

[9] D. Arinkin, R. Bezrukavnikov, Perverse coherent sheaves, Mosc. Math. J., 10 (1):3–29, 271, 2010.

[10] R. Bezrukavnikov, Cohomology of tilting modules over quantum groups and t-structures on derived categories of coherent sheaves , Invent. Math. 166 (2006), 327–357

[11] R. Bezrukavnikov, Perverse coherent sheaves, after Deligne, math.AG/0005152.

[12] R. Bezrukavnikov, Perverse sheaves on affine flags and nilpotent cone of the Langlands dual group, Israel J. Math. 170 (2009), 185–206.

[13] R. Bezrukavnikov, Quasi-exceptional sets and equivariant coherent sheaves on the nilpotent cone, Represent. Theory 7 (2003), 1-18.

[14] T. Bridgeland, Spaces of stability conditions, math.AG/0611510

[15] R. Bezrukavnikov, T. Kanstrup, Notes on canonical bases and representation categories

[16] R. Bezrukavnikov, I. Mirkovic, Representations of Semi-simple Lie algebras in prime characteristic and non-commutative Springer resolution, http://arxiv.org/abs/1001.2562

[17] R. Bezrukavnikov, I. Mirkovic, D. Rumynin, Localization of modules for a semisimple Lie algebra in prime characteristic

[18] R. Bezrukavnikov, S. Riche, Affine braid group actions on derived categories of Springer resolutions, arxiv:1101.3702, Annales sci. de lâĂŹEcole normale supÂťerieure, sĂľrie 4 45, fascicule 4

[19] W. Borho, Jean-Luc Brylinski, Differential operators on homogeneous spaces, Invent. math. 69

[20] W. Bosma, J. Cannon, C. Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput., 24 (1997), 235-265.

[21] A. Broer, Line bundles on the cotangent bundle of the flag variety, Invent. Math. 113 (1993), 1-20.

[22] T. Bridgeland, Spaces of stability conditions, math.AG/0611510

[23] T. Bridgeland, Stability conditions on K3 surfaces, math.AG/0307164

[24] T. Bridgeland, Stability conditions and Kleinian singularities, math.AG/0508257

[25] A. Broer, Normality of Some Nilpotent Varieties and Cohomology of Line Bundles on the Cotangent Bundle of the Flag Variety, Lie Theory and Geometry (Boston), Progr. Math., 123, Birkhauser, Boston, 1994, pp. 1-19.

[26] R.K. Brylinski, Limits of weight spaces, Lusztig's q-analogs, and fiberings of adjoint orbits, J. Am. Math. Soc. 2, 517-533 (1989)

[27] S. Cautis, J. Kamnitzer, Knot homology via derived categories of coherent sheaves. I. The $\mathfrak{sl}_2$-case, Duke Math. J. 142 (2008), no. 3, 511–588.

[28] Y. Chen, M. Khovanov, An invariant of tangle cobordisms via subquotients of arc rings

[29] N.Chriss, V.Ginzburg, Representation Theory and Complex Geometry, Birkhauser, Boston-Basel-Berlin, 1997.

[30] A. Joseph, On the Variety of a Highest Weight Module, Journal of Algebra

[31] A. Joseph, On the Variety of a Primitive Ideal, Journal of Algebra

[32] R. Hartshorne, Algebraic geometry Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977

[33] J. E. Humphreys, Introduction to Lie algebras and representation theory, Graduate Texts in Mathematics, Vol. 9. Springer-Verlag, New York-Berlin (1972)

[34] J. Humphreys, Representations of Semisimple Lie Algebras in BGG Category O

[35] D. Huybrechts, Fourier-Mukai Transforms in Algebraic Geometry, Oxford University Press, 2006.

[36] S. Kato, An exotic Deligne-Langlands correspondence for symplectic groups, arXiv: math.RT/0601155.

[37] S. Kato, An exotic Springer correspondence for symplectic groups, arXiv:math.RT/0607478.

[38] S. Kato, A homological study of Green polynomials, arXiv:1111.4640v4.

[39] S. Kato, An algebraic study of extension algebras, arXiv:1207.4640.

[40] G. Kempf, On the collapsing of homogeneous vector bundles, Invent. Math. 37(1976), 229-239.

[41] M. Khovanov, Functor-valued invariants of tangles

[42] M. Khovanov, Link homology and categorification, Proceedings of the ICM-2006, Madrid, vol.2 989–999.

[43] G. Lusztig, Cells in affine Weyl groups. IV., J. Fac. Sci. Univ. Tokyo Sect. IA Math. 36 (1989), no. 2, 297âĂŞ328.

[44] I. Mirkovic, M. Vybornov, "On quiver varieties and affine Grassmanians of type A", Comptes Rendus Mathematique 336.3 (2003): 207-212, http://arxiv.org/pdf/0712.4160v2.pdf

[45] V. Ostrik, On the equivariant K-theory of the nilpotent cone, Represent. Theory 4 (2000), 296-305

[46] D. Panyushev, Generalised Kostka-Foulkes polynomials and cohomology of line bundles on homogeneous vector bundles, Selecta Math., 16 (2010), 315-342

[47] B. Parshall, L. Scott, Derived categories, quasi-hereditary algebras, and algebraic groups, preprint, 1987.

[48] H. Russell, A topological construction for all two-row Springer varieties, arXiv:1007.0611

[49] N. Spaltenstein, On the fixed point set of a unipotent element on the variety of Borel subgroups

[50] T. Shoji, K. Sorlin, Exotic symmetric space over a finite field, II

[51] C. Stroppel, B. Webster, 2-block Springer fibers: convolution algebras and coherent sheaves, arXiv:0802.1943

[52] M. Sun, Point stabilizers for the enhanced and exotic nilpotent cones, J. Group Theory 14 (2011), 825-839

[53] R. Travkin, Mirabolic Robinson-Schensted-Knuth correspondence, Selecta Math. (N.S.) 14 (2009), no. 3-4, 727-758.