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ROBUSTNESS TESTS UTILIZING THE STRUCTURE OF MODELLING ERROR

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ABSTRACT

The results on robustness theory presented here are extensions of those given in [1]. The basic innovation in these new results is that they utilize minimal additional information about the structure of the modelling error as well as its magnitude to assess the robustness of feedback systems for which robustness tests based on the magnitude of modelling error alone are inconclusive.

I. INTRODUCTION

Briefly, the issue of robustness in feedback conrol system design may be summarized as follows. Any mathematical model can only approximate the behavior of a physical system. In designing a feedback compensator, one nominal model must be selected, from a class models that approximate the physical system's behavior. Once a nominal model has been selected an associated class of modelling errors is defined implicitly by the deviation of any model (in the class of models that approximate the physical system's behavior) from the nominal design model. When a compensator is designed using this nominal model, the resulting feedback system is said to be robust with respect to the class of modelling errors if it remains stable when the nominal model is replaced by any other model in the class of models that represents the physical system. Otherwise, the feedback system is not robust.

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Determining the robustness of a given feedback control system can be logically divided into two distinct questions: (1) how near instability is the feedback system and (2) given the class of model errors for which the feedback system is stable, does this class include the model errors that can be reasonably expected for this particular system? The first question can be answered exactly by appropriate mathematical analysis once a suitable notion of "nearness to instability" is defined. The second question is, however, a question that requires engineering judgment in the definition of what constitutes a reasonable modelling error. The role of mathematical analysis with respect to question (2) is that of providing a simple characterization of a sufficiently large subclass of modelling errors that do not destabilize the feedback system. Without a simple characterization of this subclass of model errors even the best engineering judgment may not be adequate to answer question (2). Nevertheless, very simple characterizations of model errors that are not destabilizing often lead to results that are not very useful practically because they are too restrictive and the associated subclass of nondestabilizing model errors too small. Therefore, a compromise between the simplicity of the characterization and the extent of the subclass of nondestabilizing model errors that can be considered is necessary. The main result of this paper will propose one such compromise.

The results presented in this paper are essentially extensions of those presented in [1] on the robustness of multivariable linear time invariant feedback control systems. The work in [1] is based on a multivariable version of Nyquist's theorem from which several robustness theorems were derived In this paper, a slightly more general approach based on Nyquist's theorem is given in a fundamental robustness theorem from which various robustness tests may be obtained. These robustness tests all have the following form. The magnitude or norm of the modelling error or uncertainty in the frequency domain is characterized by a nonnegative frequency dependent scalar. The measure of robustness or margin of stability is also characterized by a nonnegative frequency dependent scalar that represents the mininum norm or magnitude of the modelling error required to destabilize the feedback system. The robustness test consists in comparing these two quantities versus frequency. If the norm of the modelling error is less than the minimum error norm required to destabilize the feedback system at all the frequencies then, obviously stability is guaranteed in the face of this modelling error. However, if the norm of the model error at some frequency exceeds the minimum error norm required to destabilize the test is inconclusive. Additional information about the structure of the modelling error must be used to determine if it will destabilize the feedback system. This additional information about the model error structure is obtained by examining the projection of the error matrix onto the one dimensional subspace spanned by the outerproduct of the left and right singular vectors corresponding to the minimum singular value of the return difference matrix or a related matrix quantity. A corollary of the main result is that the minimum "size" (i.e. norm) of the modelling error required to destabilize a feedback system is equal to the geometric mean of the two smallest singular values of the return difference transfer matrix (or a related matrix quantity) provided the error matrix has no projection onto the one dimensional subspace spanned by the outer product of the left and right singular vectors associated with the smallest singular value. Thus, the feedback system will tolerate an error of this type of possibly much larger magnitude than an artitrarily structured model error. Of course, in order to guarantee that the error matrix has no such projection, engineering judgment based on what class of models gives a reasonable approximation to the behaviour of the physical system is required.

The development of the results on the use of model error structure will proceed first by presenting in Section II a generalized version of a fundamental robustness theorem found in [1] based on the idea of deforming the multivariable Nyquist locus to account for model error without making the return difference matrix singular. Section III gives a brief review of the singular value decomposition and related notions that will be used. It then gives the basic results from matrix theory that will be used in Section IV. Section IV gives a classification of various robustness tests that have appeared previously in the literature as well as a new one that has not, according to the type of model error they guard against. All these tests have the same basic form and therefore may all be generalized to use model error structure as well as magnitude information via the results of Sections III. Section V shows how the results of Section III may be used along with the fundamental robustness theorem to generalize the robustness theorems of Section IV that utilize only error magnitude information. Also, an example is given demonstrating the results.

All proofs are omitted due to space considerations but may be found in [2].

II. FUNDAMENTAL CHARACTERIZATION OF ROBUSTNESS.

The basic system under consideration is given in Figure 1, where $G(s)$, the loop transfer function matrix, incorporates the open loop plant dynamics as well as any compensation employed.

Due to modelling error or uncertainty the actual loop transfer function matrix is $\tilde{G}(s)$, a perturbed version of $G(s)$. For the purposes of this paper the perturbed (or actual) system is assumed to have the form given by

$$\tilde{G}(s) = G(s)L(s) \tag{2.1}$$

where $L(s)$ is a multiplicative factor used to account for model error or uncertainty. Furthermore we assume that both $G(s)$ and $\tilde{G}(s)$ have state space representations given respectively by the triples (A,B,C) and $(\tilde{A},\tilde{B},\tilde{C})$ (i.e., $G(s) = C(Is-A)^{-1}B$ and $\tilde{G}(s) = \tilde{C}(Is-\tilde{A})^{-1}\tilde{B}$). Associated with the state space representation of $G(s)$ are the open and closed loop characteristic polynominals, respectively $\phi_{OL}(s)$ and $\phi_{CL}(s)$ defined by

$$\phi_{OL}(s) = \det(sI-A) \tag{2.2}$$

$$\phi_{CL}(s) = \det(sI-A+BC) \tag{2.3}$$

The polynominals $\tilde{\phi}_{OL}(s)$ and $\tilde{\phi}_{CL}(s)$ associated with $(\tilde{A},\tilde{B},\tilde{C})$ are analogously defined.

The following theorem generalizes Theorem 2.2 of [1] and is based on the idea of continuously deforming the multivariable Nyquist diagram for $G(s)$ into the one corresponding to $\tilde{G}(s)$ without passing the locus through the critical point. If this can be done and the number of encirclements of the critical point required for stability by $G(s)$ and $\tilde{G}(s)$ are the same then this perturbation of $G(s)$ will not induce instability. In this theorem we will let D_R denote the Nyquist contour (shown in Figure 2) along which $\det(I+G(s))$ is evaluated and define $G(s,\varepsilon)$ as a matrix of rational transfer functions continuous in ε for ε in $[0,1]$ and for all s in D_R that also satisfies the following conditions

$$G(s,0) = G(s) \tag{2.4}$$

and

$$G(s,1) = \tilde{G}(s) \tag{2.5}$$

Theorem 1: The polynominal $\tilde{\phi}_{CL}(s)$ has no CRHP (closed-right-half-plane) zeros if the following conditions hold:

1. (a) $\phi_{OL}(s)$ and $\tilde{\phi}_{OL}(s)$ have the same number of CRHP zeros.

   (b) if $\tilde{\phi}_{OL}(j\omega_o) = 0$ then $\phi_{OL}(j\omega_o) = 0$

   (c) $\phi_{CL}(s)$ has no CRHP zeros

2. $\det[I+G(s,\varepsilon)] \neq 0$ for all ε in $[0,1]$ and for all $s \in D_R$ with R sufficiently large.

Theorem 1 forms the basis for the derivation of all subsequent robustness results. We will subsequently assume that the radius R of the contour D_R is taken sufficiently large so that Theorem 1 may be applied.

Theorem 1, condition 2 provides the complete characterization of the class of modelling errors that do not destabilize the feedback system (under the restrictions given in condition 1). However, this characterization of the class of nondestabilizing errors is so complex as to be practically useless. A simple "small gain" type of characterization of a subclass of nondestabilizing model errors is those for which a $G(s,\varepsilon)$ may be constructed with $\|G(s,\varepsilon)\|_2 < 1$ for (s,ε) on $D_R \times [0,1]$.
This simple characterization of the "small gain" subclass of nondestabilizing modelling errors does not cover many systems or modelling errors of interest because of the requirements that $\|G(s)\|_2<1$ and $\|\tilde{G}(s)\|_2<1$ for all $s \varepsilon D_R$.

Notice, that if $\|G(s,\varepsilon)\|_2 \to 0$ as $|s| \to \infty$ for all ε in $[0,1]$, then condition 2 of Theorem 1 need only be verifed for (s,ε) in $\Omega_R \times [0,1]$ where Ω_R is defined as

$$\Omega_R=\{s \mid s \varepsilon D_R \text{ and } Re(s)\leq 0\}. \qquad (2.6)$$

This will be the case when $G(s,\varepsilon)$ is defined in Section IV because both $\|G(s)\|_2 \to 0$ and $\|\tilde{G}(s)\|_2 \to 0$ as $|s| \to \infty$. The development of robustness tests from Theorem 1 involves the construction of inequalities that can guarantee the nonsingularity of $I+G(s,\varepsilon)$ as in condition 2. Therefore, section III will develop general matrix theory results that test for singularity of the sum of two matrices

III. MATRIX THEORY

The purpose of this section is to introduce important tools from matrix theory and present some results that form the backbone of the robustness results of section V. The specific problem considered in this section is the following. Given a nonsingular complex matrix A, find the nearest (in some sense) singular matrix $\tilde{A}$ which belongs to a certain class of matrices. If the error matrix E is defined as $E = \tilde{A} - A$ then the problem may be stated in the following form. Given a nonsingular complex matrix A find the matrix E of minimum norm that makes A + E singular when E is constrained to belong to a certain class of matrices.

Essential use of the singular value decomposition is made in the solution of this problem and therefore is reviewed next.

A. Singular Values and the Singular Value Decomposition

The singular values of a complex nxm matrix A, denoted $\sigma_i(A)$ are the k largest nonnegative square roots of the eigenvalues of A^HA or AA^H where A^H is the complex conjugate transpose of A and k = min(n,m) that is

$$\sigma_i(A) = \lambda_i^{1/2}(A^HA) \quad i = 1,2,..,k \qquad (3.1)$$

where we assume that σ_i are ordered such that $\sigma_i \geq \sigma_{i+1}$. The maximum and minimum singular values may alternatively be defined by

$$\sigma_{max}(A) = \max_{\underline{x}\neq 0} \frac{\|A\underline{x}\|_2}{\|\underline{x}\|_2} = \|A\|_2 \qquad (3.2)$$

$$\sigma_{min}(A) = \min_{\underline{x}\neq 0} \frac{\|A\underline{x}\|_2}{\|\underline{x}\|_2} = \|A^{-1}\|^{-1} \text{ if } A^{-1} \text{ exists} \qquad (3.3)$$

The smallest singular value $\sigma_{min}(A)$ measures how near the matrix A is to being singular or rank deficient (a matrix is rank deficient if both its rows and columns are linearly dependent). To see this consider finding a matrix E of minimum spectral norm that makes A+E rank deficient. Since A+E must be rank deficient there exists a nonzero vector $\underline{x}$ such that $\|\underline{x}\|_2 = 1$ and $(A+E)\underline{x} = 0$ and thus by (3.2) and (3.3)

$$\sigma_{min}(A)\leq\|A\underline{x}\|_2\leq\|E\underline{x}\|_2\leq\|E\|_2=\sigma_{max}(E) \qquad (3.4)$$

Therefore, E must have spectral norm of at least $\sigma_{min}(A)$ otherwise A+E cannot be rank deficient. The property that

$$\sigma_{min}(A) > \sigma_{max}(E) \qquad (3.5)$$

implies that A+E is nonsingular (assuming square matrices) and will be a basic inequality used in the formulation of various robustness tests.

A convenient way of representing a matrix that exposes its internal structure is known as the singular value decomposition (SVD). For an nxm matrix A, the SVD of A is given by

$$A = U\Sigma V^H = \sum_{i=1}^{k} \sigma_i(A)\underline{u}_i\underline{v}_i^H \qquad (3.6)$$

where U and V are unitary matrices with column vectors denoted by

$$U = [\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n] \qquad (3.7)$$

$$V = [\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_m] \qquad (3.8)$$

and Σ contains a diagonal nonnegative definite matrix Σ_1 of singular values arranged in descending order as in

$$\Sigma = \begin{cases} \begin{bmatrix} \Sigma_1 \\ \hline 0 \end{bmatrix}, & n \geq m \\ \begin{bmatrix} \Sigma_1 & 0 \end{bmatrix}, & n \leq m \end{cases} \qquad (3.9)$$

and (3.10)

$$\Sigma_1 = \text{diag}[\sigma_1, \sigma_2, \ldots, \sigma_k] \quad ; \quad k = \min(m,n).$$

The columns of V and U are unit eigenvectors of A^HA and AA^H respectively and are known as right and left singular vectors of the matrix A.

B Projections and Orthonormal Bases

Any unitary matrices, such as the U and V produced by computing the SVD of a matrix, can be used to generate an orthonormal basis in which to express an arbitrary matrix E. Let U and V be nxn unitary matrices with columns as in (3.7,8) and express E as

$$E = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \underline{u}_i \underline{v}_j^H, E \rangle \underline{u}_i \underline{v}_j^H \tag{3.11}$$

where the innerproduct for matrices is defined by

$$\langle A,B \rangle = tr(A^HB) \tag{3.12}$$

for complex matrices A and B. Note that with this innerproduct the n^2 rank one matrices $\underline{u}_i\underline{v}_j^H$ are orthogonal to each other and have unit spectral and Euclidean norms and thus form an orthonormal basis. The matrix $\langle \underline{u}_i \underline{v}_j, E\rangle \underline{u}_i\underline{v}_j^H$ is simply the projection of the matrix E onto the one dimensional subspace spanned by $\underline{u}_i\underline{v}_j^H$. If the elements of $\underline{u}_i\underline{v}_j^H$ are formed into a n^2 length vector $\underline{x}$ by stacking the n rows of $\underline{u}_i\underline{v}_j^H$ and the same procedure is used to reduce the matrix E to a vector $\underline{y}$ then $\langle \underline{u}_i\underline{v}_j^H, E\rangle$ is equal to the usual $\underline{x}^H\underline{y}$ innerproduct between these n^2 length vectors. This makes it clear that $\langle \underline{u}_i\underline{v}_j^H, E\rangle \underline{u}_i\underline{v}_j^H$ can be rearranged into a vector $(\underline{x}^H\underline{y})\underline{x}$ which is just the projection of $\underline{y}$ in the direction of the vector $\underline{x}$. Also, if all the matrices $\underline{u}_i\underline{v}_j^H$ are formed into vectors, they will be orthogonal to each other and have unit Euclidean length. We will thus think of the n^2 rank one matrices as representing n^2 orthogonal directions and refer to $\langle \underline{u}_i\underline{v}_j^H, E\rangle \underline{u}_i\underline{v}_j^H$ as the projection of E along the direction $\underline{u}_i\underline{v}_j^H$. This type of perspective is useful in studying the structure of the error matrix $E = \tilde{A} - A$.

C. Error Matrix Structure

In this section we will use the tools developed in earlier sections to solve the problem of finding a singular matrix $\tilde{A}$ nearest to a given matrix. This can be formulated more precisely as a mathematical optimization problem:

Problem A:

$$\min_{E} \|E\|_2 \quad \text{s.t.} \quad \det(A+E) = 0 \tag{3.13}$$

In this formulation the matrix $\tilde{A}$ is simply A+E, where we refer to E as the error matrix. This is the simplest problem to solve since E is unconstrained. In what follows we make the following technical assumption.

Assumption 1: The matrix A is nxn nonsingular and has distinct singular values.

The assumption of nonsingularity of A assures us of a nontrivial problem otherwise E is identically zero when A is singular. The assumption of distinct singular values is a technical one which allows us to avoid some combinatoric problems associated with multiple solutions but it is not difficult to remove this assumption.

Solution to Problem A:

Suppose that A has the SVD given by

$$A = U\Sigma V^H \tag{3.14}$$

where

$$\Sigma = \text{diag}[\sigma_1, \sigma_2, \cdots, \sigma_n] \; ; \; \sigma_k > \sigma_{k+1} \tag{3.15}$$

$$U = [\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_n] \tag{3.16}$$

$$V = [\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n]. \tag{3.17}$$

then we can characterize the form that all solutions to Problem A must have, namely

$$E = U \left[\begin{array}{c|c} P_s & \underline{0} \\ \hline \underline{0}^T & -\sigma_n \end{array} \right] V^H \tag{3.18}$$

where P_s is (n-1) x (n 1) and

$$\|P_s\|_2 \le \sigma_n = \|E\|_2 \tag{3.19}$$

but is otherwise arbitrary.

Recall from equation (3.11) the interpretation of $\langle \underline{u}_n\underline{v}_n^H, E\rangle \underline{u}_n\underline{v}_n^H$ as the projection of E onto the direction $\underline{u}_n\underline{v}_n^H$. From (3.18) we see that all solutions to Problem A have the same projection in the direction $\underline{u}_n\underline{v}_n^H$ which we shall call the most sensitive direction since this is the direction it is "easiest" to make A singular by changing its elements the "least". Note also the additional conditions for any two solutions to Problem A say E_1 and E_2 that

$$\langle \underline{u}_n\underline{v}_j^H, E_1\rangle = \langle \underline{u}_n\underline{v}_j^H, E_2\rangle = 0, \; j \neq n \tag{3.20}$$

and

$$\langle \underline{u}_j\underline{v}_n^H, E_1\rangle = \langle \underline{u}_j\underline{v}_n^H, E_2\rangle = 0, \; j \neq n \tag{3.21}$$

requiring the projections of E_1 and E_2 to be equal along any direction $\underline{u}_j\underline{v}_n^H$ and $\underline{u}_n\underline{v}_j^H$ where j = 1,2,...,n. In fact, the matrix P given by

$$P = U^HEV \tag{3.22}$$

is just the matrix of projections onto each of the n^2 directions $u_i v_j^H$ (slightly abusing the notion of projection to mean $\langle u_i v_j^H, E\rangle$ instead of $\langle u_i v_j^H, E\rangle\, u_i v_j^H$) that is,

$$p_{ij} = \langle u_i v_j^H , E \rangle. \tag{3.23}$$

Now suppose that we construct a constraint set for E so that E cannot have a projection of magnitude σ_n in the most sensitive direction $u_n v_n^H$. This means that the matrix A+E cannot become singular along the direction $u_n v_n^H$ and thus $\|E\|_2$ must increase if A+E is to be singular. To find out just how much larger $\|E\|_2$ must become we formulate the constrained optimization problem:

Problem B:

$$\min_E \|E\|_2$$
$$\text{s.t. } \det(A+E) = 0 \tag{3.24}$$
$$|\langle u_n v_n^H, E \rangle| \leq \phi < \sigma_n$$

Solution to Problem B:

The error matrix E is given by

$$E = U \left[\begin{array}{c|cc} P_s & 0 & \\ \hline & \phi & \gamma \\ 0 & \gamma^* & -\phi \end{array}\right] V^H \tag{3.25}$$

where P_s arbitrary and

$$\|P_s\| \leq \sqrt{\sigma_n \sigma_{n-1} + \phi(\sigma_n - \sigma_{n-1})} = \|E\|_2, \tag{3.26}$$

where γ is given by

$$\gamma = \sqrt{(\phi + \sigma_{n-1})(\sigma_n - \phi)}\, e^{j\theta}, \ \theta \text{ arbitrary} \tag{3.27}$$

and A has the SVD

$$A = U \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \\ & & & \sigma_n \end{bmatrix} V^H, \quad \sigma_i > \sigma_{i+1} \tag{3.28}$$

The following theorem follows trivially from the solutions of Problems A and B.

Theorem 2: For square matrices A and E, A+E is nonsingular if

$$\sigma_{max}(E) = \|E\|_2 < \sqrt{\sigma_n \sigma_{n-1} + \phi(\sigma_n - \sigma_{n-1})} \tag{3.29}$$

and

$$|\langle u_n v_n^H, E\rangle| \leq \phi < \sigma_n \tag{3.30}$$

where $\sigma_{n-1} \geq \sigma_n > 0$ are the two smallest singular values of A and u_n and v_n are respectively the left and right singular vectors of A corresponding to σ_n. □

Corollary 1: For square matrices A and E, det (A+E) $\neq 0$ if

$$\sigma_{max}(E) = \|E\|_2 < \sqrt{\sigma_n \sigma_{n-1}} \tag{3.31}$$

and

$$\langle u_n v_n^H, E\rangle = 0 \quad \square \tag{3.32}$$

Theorem 2 is the key to making use of model error structure in the subsequent robustness tests. Corollary 1 has a very pleasing geometrical interpretation that will be discussed next.

D. Geometric Interpretation

The nature of the solution to Problems A and B becomes apparent when the SVD is used to transform the A matrix into a positive definite diagonal matrix. This is accomplished with the following simple lemma.

Lemma 1. If the SVD of A is given by

$$A = U \Sigma V^H \tag{3.33}$$

with U and V^H unitary and Σ and diagonal then A+E is singular if and only if Σ+P is singular where

$$P = U^H E V \tag{3.34}$$

and furthermore $\|P\|_2 = \|E\|_2$. □

Thus, one may work with Σ and P rather than A and E. Therefore, in the subsequent discussion we will make the assumption that the matrix A is now diagonal and positive definite.

The matrix A is now given by

$$A = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n \end{bmatrix} \tag{3.35}$$

where $\sigma_i > \sigma_{i+1}$. If the columns of the matrix A are thought of as a set of n orthogonal vectors of lengths σ_i then Corollary 1 can be interpreted geometrically in the 2x2 case as the problem of aligning two orthogonal vectors with minimum "effort" without decreasing the length of the shortest vector. Here the "effort" required to align the two vectors is equal to $\|E\| = \sigma_{max}(E)$ where E makes A+E singular. Corollary 1 states that the minimum "effort" required to align the two vectors is equal to the geometric mean of their lengths. Figure 3 graphically illustrates Corollary 1 in the 2x2 case and displays the columns of A and $\tilde{A}=A+E$ where $\tilde{A}$ is singular and $\|E\|_2$ a minimum. When the number of orthogonal vectors (i.e. columns of A) is greater than 2, Corollary 1 states that it requires the minimum "effort" to align the two shortest vectors in the set.

Using these observations Problems A and B can be generalized to accommodate additional constraints on the matrix E. One additional constraint that may be added is the condition that $\langle \underline{u}_n \underline{v}_{n-1}^H, E\rangle\langle \underline{u}_{n-1}\underline{v}_n^H, E\rangle = 0$, where the vectors $\underline{u}_i$ and $\underline{v}_i$ are the appropriate singular vectors taken from the SVD of the matrix A. This effectively, rules out the form of solutions to Problems A and B given in (3.18) and (3.25) and thus $\|E\|_2$ must again increase. In general, if constraints of the form $\langle \underline{u}_i \underline{v}_j^H, E\rangle \langle \underline{u}_j \underline{v}_i^H, E\rangle = 0$ for all $(i,j) \in M$ for some index set M, are imposed on the matrix E, where $\underline{u}_i$ and $\underline{v}_i$ are the left and right singular vectors of A, then $\|E\|_2 \geq \sqrt{\sigma_k \sigma_l}$ where $\sigma_k \sigma_l = \min \sigma_i \sigma_j$ for $(i,j) \notin M$, if A+E is to be singular.

E. Examples

To make these results clearer we will illustrate the solutions to the problem of finding the matrices E of minimum spectral norm that make A+E singular under various constraints on the E matrix.

Examples:

Let A be given by

$$A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3.35)$$

and consider the various constraints on E.

Unconstrained Case:

$$E = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{E_s} & 0 \\ & & 0 \\ \hline 0 & 0 & -1 \end{array}\right] \qquad (3.36)$$

where $\|E_s\|_2 \leq 1$ but otherwise E_s is arbitrary.

$e_{33} = 0$ Case:

$$E = \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & 0 & 2e^{j\theta} \\ 0 & 2e^{-j\theta} & 0 \end{bmatrix} \qquad (3.37)$$

where $|e_{11}| \leq 2$ and otherwise e_{11} and θ are arbitrary.

$e_{23} = e_{33} = 0$ Case:

$$E = \begin{bmatrix} 0 & 0 & 3e^{j\theta} \\ 0 & e_{22} & 0 \\ 3e^{-j\theta} & 0 & 0 \end{bmatrix} \qquad (3.38)$$

where $|e_{22}| \leq 3$ and otherwise e_{22} and θ are arbitrary.

$e_{13} = e_{23} = e_{33} = 0$ Case:

$$E = \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & -4 & 0 \\ e_{31} & 0 & 0 \end{bmatrix} \qquad (3.39)$$

where

$$\sqrt{|e_{11}|^2 + |e_{31}|^2} \leq 4 = \|E\|_2 \qquad (3.40)$$

but otherwise e_{11} and e_{31} are arbitrary.

$|e_{33}| \leq 1/2$ Case:

$$E = \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & 1/2 & 3/2\, e^{j\theta} \\ 0 & 3/2\, e^{-j\theta} & -1/2 \end{bmatrix} \qquad (3.41)$$

where

$$|e_{11}| \leq \|E\|_2 = \frac{\sqrt{10}}{2} \cong 1.58 \qquad (3.42)$$

and e_{11} and θ are otherwise arbitrary.

It is important to point out that we have limited ourselves to constraints on E of a very special form and in general arbitrary constraints on the form of E lead to a mathematical nonlinear programming problem that does not in general have a closed form solution. However, these special form of constraints on E will be useful in obtaining robustness results of section V.

IV. ROBUSTNESS TESTS AND UNSTRUCTURED MODEL ERROR

In this section, we present theorems that guarantee the stability of the perturbed closed-loop system for different characterizations of model uncertainty (i.e., different types of model error). This is done via Theorem 1 by using a specific error criterion to construct a transfer matrix $G(s,\varepsilon)$ continuous in ε on $D_R \times [0,1]$ that satisfies (2.4) and (2.5). Then a simple test bounding the magnitude of the error is devised which guarantees that condition 2 of Theorem 1 is satisfied. This procedure is carried out for four different types of errors. These tests use only the magnitude of the modelling error and do not exploit any other characteristics or structure of the model error and hence are based on the unstructured part of the model error. These different types of model errors will emphasize different aspects of the difference between the nominal $G(s)$ and $\tilde{G}(s)$ and thus under certain circumstances will give essentially different assessments of the robustness or margin of stability of the feedback control system.

A. Robustness Tests Using Different Error Criteria

Probably the most familiar types of errors are those of absolute and relative errors. Absolute errors are additive in nature whereas relative errors are multiplicative in nature. One can use both types of errors to derive robustnes theorems. However, the familiar notions of gain and phase margins are associated only with relative type of error since these margins are multiplicative in nature.

If we let the matrix $E(s)$ generically denote the particular modelling error under consideration, the absolute error is obviously given by

$$E(s) = \tilde{G}(s) - G(s) \qquad (4.1)$$

and the relative error, in a matrix sense, by

$$E(s) = G^{-1}(s)[\tilde{G}(s)-G(s)]. \qquad (4.2)$$

In (4.2) $G^{-1}(s)$ could post-multiply the absolute error and serve as an alternative relative error in the matrix sense but all subsequent results will still hold with trivial modifications. Two robustness theorems using these errors will be given. However, first $G(s,\varepsilon)$ must be constructed.

Using (4.1) and (4.2) we can define $G(s,\varepsilon)$ by replacing $\tilde{G}(s)$ in (4.1) and (4.2) by $G(s,\varepsilon)$ and $E(s)$ by $\varepsilon E(s)$ and solve for $G(s,\varepsilon)$. If we do this we obtain

$$G(s,\varepsilon) = G(s) + \varepsilon E(s) \qquad (4.3)$$

where $E(s)$ is the absolute error given by (4.1) or

$$G(s,\varepsilon) = G(s)[I+\varepsilon E(s)] \qquad (4.4)$$

where $E(s)$ is the relative error given by (4.2). Both (4.3) and (4.4) imply the same $G(s,\varepsilon)$ although they employ different types of errors to arrive at $G(s,\varepsilon)$. In either (4.3) or (4.4) $G(s,\varepsilon)$ is simply given by

$$G(s,\varepsilon)=(1-\varepsilon)G(s)+\varepsilon\tilde{G}(s) \qquad (4.5)$$

showing that $G(s,\varepsilon)$ is continuous in ε for ε on $[0,1]$ and for all $s \in D_R$ and that $G(s,\varepsilon)$ satisfies (2.4) and (2.5)

In deriving stability margins based on theorems using different error criteria, it is useful to define a multiplicative uncertainty matrix $L(s)$ to account for modelling errors in the open-loop plant. The perturbed or actual system $G(s)$ in this case is given by

$$\tilde{G}(s) = G(s)L(s) \qquad (4.6)$$

which implicitly defines $L(s)$. Notice that for the relative error criteria that $L(s)$ is very simply given by

$$L(s) = (I+E(s)) \qquad (4.7)$$

where $E(s)$ is given by (4.2). However, as we will be shown later (4.7) is not the only description of $L(s)$; there are other types of relative errors in which the relationship between $L(s)$ and the generic $E(s)$ is not so simply given by (4.7). We will use both $L(s)$ as defined implicitly in (4.6) and a variety of error matrices denoted by $E(s)$ in stating the subsequent robustness theorems.

Two robustness theorems based on the preceedings definitions of absolute and relative errors in (4.1) and (4.2) respectively are the following.

Theorem 3 [4,5]: The polynominal $\tilde{\phi}_{CL}(s)$ has no CRHP zeros and hence the perturbed feedback system is stable if the following conditions hold:

1. condition 1 of Theorem 1 holds
2. $\sigma_{min}[I+G(s)] > \sigma_{max}[E(s)]$ for all $s \in \Omega_R$ where $E(s)$ is given by (4.1), and Ω_R was defined by (2.6). □

Theorem 4 [3,4,5]: The polynominal $\tilde{\phi}_{CL}(s)$ has no CRHP zeros and hence the perturbed feedback system is stable if the following conditions hold:

1. condition 1 of Theorem 1 holds
2. $\sigma_{min}[I+G^{-1}(s)] > \sigma_{max}[E(s)]$ $s\in\Omega_R$ where $E(s)$ is given by (4.2)

Theorem 4 was first proved by Doyle [3] using singular values and Nyquist's theorem but under the slightly stronger condition that E(s) be stable. An operator version of Theorem 3 is due to Sandell [4] who was the first to consider additive perturbations. Laub [5] provides further numerical insights to the relationship of Theorems 3 and 4.

Suppose that instead of measuring the absolute relative errors between $\tilde{G}(s)$ and G(s), we measure the absolutes and relative errors between $\tilde{G}^{-1}(s)$ and $G^{-1}(s)$. In the SISO case, this would correspond to measuring the absolute and relative errors between the nominal and perturbed systems on an inverse Nyquist diagram in which the inverse loop transfer functions $g^{-1}(s)$ and $\tilde{g}^{-1}(s)$ are plotted. (The inverse Nyquist diagram can also be used to determine stability by counting encirclements of the critical points (0,0) and (-1,0) in the complex plane.) Therefore, it is natural to define the absolute and relative errors between the nominal and perturbed systems as

$$E(s) = \tilde{G}^{-1}(s) - G^{-1}(s) \tag{4.8}$$

for the absolute error and

$$E(s)=[\tilde{G}^{-1}(s)-G^{-1}(s)]G(s) \tag{4.9}$$

for the relative error. Using (4.8) and (4.9) we may define a $G(s,\varepsilon)$, again by replacing $\tilde{G}(s)$ by $G(s,\varepsilon)$ and E(s) by $\varepsilon E(s)$ in (4.8) and (4.9), and then solving for $G(s,\varepsilon)$. If this is done, we obtain

$$G(s,\varepsilon)=[G^{-1}(s)+\varepsilon E(s)]^{-1} \tag{4.10}$$

where E(s) is given by (4.8) and

$$G(s,\varepsilon) = G(s)\,[I+\varepsilon E(s)]^{-1} \tag{4.11}$$

where E(s) is given by (4.9). Both (4.10) and (4.11) give the same $G(s,\varepsilon)$ which written in terms of G(s) and $\tilde{G}(s)$ is

$$G(s,\varepsilon)=[(1-\varepsilon)G^{-1}(s)+\varepsilon\tilde{G}^{-1}(s)]^{-1} \tag{4.12}$$

where now we see that ε enters nonlinearly and it is not clear that $G(s,\varepsilon)$ is continuous in ε in [0,1] for all $s \in D_R$ but is clear that it does satisfy (2.4) and (2.5). The type of $G(s,\varepsilon)$ in (4.12) could be replaced by the one in (4.5) and theorems worked out in terms of the errors described by (4.8) and (4.9). This approach was taken by Lehtomaki, Sandell and Athans [1] and led to more restrictive and complicated conditions to check than the approach using (4.12).

Since (4.10) and (4.11) and (4.12) are all equivalent in that they give rise to the same $G(s,\varepsilon)$ we may work with any one of them to prove assertions about the continuity of $G(s,\varepsilon)$ required by Theorem 1. If $G^{-1}(s)$ and $\tilde{G}^{-1}(s)$ exist, so that E(s) in (4.9) is well-defined, then for $G(s,\varepsilon)$ to be continuous in ε for $(s,\varepsilon) \in D_R \times [0,1]$ all that is required is that $[I+\varepsilon E(s)]$ be nonsingular. Notice that in this case L(s) is simply

$$L(s) = [I+E(s)]^{-1} \tag{4.13}$$

and that $[I+\varepsilon E(s)]$ is nonsingular for all ε in [0,1] if L(s) defined by (4.6) has no zero or strictly negative eigenvalues. This is true since if L(s) has no zero or negative eigenvalues, neither does I+E(s) and thus E(s) cannot have eigenvalues in the interval $(-\infty,-1]$ so that $\varepsilon E(s)$ never has eigenvalues of -1. Therefore, with these restrictions $G(s,\varepsilon)$ is continuous in ε on $D_R \times [0,1]$. We also see from (4.11) that if L(s) has no zero or negative eigenvalues that $\|G(s,\varepsilon)\|_2 \to 0$ as $|s| \to \infty$ for any ε in [0,1]. This allows us to check for the nonsingularity of $I+G(s,\varepsilon)$ only on $\Omega_R \times [0,1]$ in Theorem 1. We may now state the theorems analogous to Theorems 3 and 4.

Theorem 5: The polynominal $\tilde{\phi}_{CL}(s)$ has no CRHP zeros and hence the perturbed feedback system is stable if the following conditions hold[1]:

1. condition 1 of Theorem 1 holds
2. L(s) of (4.6) has no zero or strictly negative real eigenvalues for any $s \in D_R$
3. $\sigma_{min}[I+G^{-1}(s)] > \sigma_{max}[E(s)]$ for all $s \in \Omega_R$ where E(s) is given by (4.8). □

The next theorem works with the relative error between $G^{-1}(s)$ and $\tilde{G}^{-1}(s)$ and plays a fundamental role in establishing the properties of LQ (linear-quadratic) state feedback regulators and is an improved version of a theorem found in [1].

Theorem 6: The polynominal $\tilde{\phi}_{CL}(s)$ has no CRHP zeros and hence the perturbed feedback system is stable if the following conditions hold:

1. condition 1 of Theorem 1 holds
2. L(s) of (4.6) has no zero or strictly negative real eigenvalues
3. $\sigma_{min}[I+G(s)] > \sigma_{max}[E(s)]$ for all $s \in \Omega_R$ where E(s) is given by (4.9) □

Remark: If condition 3 is satisifed and $\sigma_{min}[I+G(s)] \leq 1$ then it can be easily shown via (4.13) that condition 2 is automatically satisfied.

Observation: The condition that L(s) have no strictly real and negative eigenvalues or be singular can be interpreted in terms of a phase reversal of certain signals between the nominal

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[1] In the proof of Theorem 5 use of the fact that G(s) and $\tilde{G}(s)$ are both invertible on D_R is made.

and perturbed systems or as the introduction of transmission zeros by the modelling error. To make this precise, suppose that for some ω_0 that $L(j\omega_o)\underline{x} = \lambda\underline{x}$ for some complex nonzero vector $\underline{x}$ and some real $\lambda < 0$. Then there exists a vector $\underline{u}(t)$ of input sinusoids of various phasing and at frequency ω_0 which when applied to the nominal system produces an output $\underline{y}(t)$ and produces an output $\lambda\underline{y}(t)$ when applied to the perturbed system. This is depicted in Figure 4.

Thus when λ is negative the phase difference between the sinusoidal outputs of the nominal and perturbed systems is 180^0. If $\lambda=0$ then the perturbed system has transmission zeros at $\pm j\omega_0$.

This fact is significant since Theorems 5 and 6 can never guarantee stability with respect to model uncertainty when the phase of the system outputs is completely uncertain above some frequency or with respect to sensor or actuator failures in the feedback channels.

B. Interpretations of Robustness Tests Error Criteria

Up to this point, it is probably unclear what the significance of the various error criteria are and how they are related. This can be partly clarified by an understanding of how each error enters into the structure of the perturbed system from a block diagram perspective. This is done in Figure 5 where a very pleasing symmetry occurs that corresponds to the four basic arithmetic operations of addition, subtraction, multiplication and division. As can be seen from Figure 5 the absolute type of errors correspond to addition and subtraction whereas the relative errors correspond to multiplication and division. Other types of errors can be represented as combinations of these basic types of errors. One such combination of the two basic relative errors given in (4.2) and (4.9) occurs in connection with Barrett's generalization of the passivity theorem [6] for linear-time invariant systems. One statement of his theorem is given in Theorem 7.

Theorem 7 [6]: The polynominal $\tilde{\phi}_{CL}(s)$ has no CRHP zeros and hence the perturbed feedback system is stable if the following conditions hold:

1. condition 1 of Theorem 1 holds
2. $\lambda(L(s)) \notin (-\infty,-1]$ for all $s \in \Omega_R$
3. $\sigma_{min}[(I-G(s))^{-1}(I+G(s))] > \sigma_{max}(E(s))$ for all $s \in \Omega_R$

where

$$E(s) = [\tilde{G}(s)+G(s)]^{-1} [\tilde{G}(s)-G(s)] \tag{4.14}$$

□

The block diagram (suppressing s dependence) of Figure 6 depicts the corresponding perturbed model $\tilde{G}$.

From Figure 6 the nature of the combination of the two types of relative errors given in (4.2) and (4.9) is readily apparent. Algebraically, if E_1 and E_2 denote

$$E_1 = G^{-1}(\tilde{G}-G) \tag{4.15}$$

and

$$E_2 = -[\tilde{G}^{-1}-G^{-1}]G \tag{4.16}$$

then E of (4.14) is given by

$$E^{-1} = E_1^{-1} + E_2^{-1} \tag{4.17}$$

Therefore, E is a "parallel-resistive" type sum of errors E_1 and E_2. This particular criteria is pleasing in that it produces logarithmically symmetrical stability margins.

All the preceeding robustness tests guarantee that stability is preserved by ensuring that the magnitude of the model error (according to some particular error criteria) is sufficiently small. In these tests the model error is unconstrained in its structure and therefore these tests guard against any type of model error structure. If all types of model error structure are not possible then these robustness tests may be conservative and methods such as those developed in the next section must be employed to take advantage of some particular aspect of the structure of the model error.

V. ROBUSTNESS ANALYSIS FOR LINEAR SYSTEMS WITH STRUCTURED MODEL ERROR

In this section, the robustness tests of Section IV are refined to distinguish between those model errors which do not destabilize the feedback system and those that do, but both of which have magnitudes larger than the MIMO generalization of the "distance to the critical (-1,0) point". To do this it is necessary to be able to distinguish between model errors that increase the margin of stability for the feedback system and those that decrease it. This cannot be done on the basis of the magnitude of the model error. Therefore, it must be done on the basis of the structure of the model error.

The structure of the model error, in general terms, is simply the numerical relationship of the elements of the error matrix E(s), representing the difference between the nominal and the perturbed loop transfer matrices. In other words, the structure of the model is specified by magnitude and phase relationships between the $e_{ij}(s)$ elements of E(s). In this section the structure of E(s) which is necessary to determine the stability of the perturbed feedback system is extracted using the results of Section III and the singular value decomposition (SVD), to generate an orthonormal basis for the expansion of E(s). It will be shown that the projections of E(s) on only certain elements of the basis need be known precisely to extract the information relevant for stability analysis. Thus, only a partial characterization of the modelling error is necessary and its structure is constructively produced by the method of analysis used in Section III.

In order to make a practical use of these results that utilize the structure of the model error, it is necessary to determine if the model error of minimum magnitude that will destabilize the feedback system can be guaranteed not to occur. This assessment must be made on the basis of engineering judgement about the type of model uncertainties that are reasonable for the nominal design model representing the physical system. For discussions on how to practically determine what constitutes a reasonable modelling error, the reader is referred to [7] for a discussion of model errors in an automative engine control system and [8] for a similar discussion with regard to power system models.

A. Robustness Tests Utilizing Model Error Structure

In the robustness theorems of Section IV, the key conditions ensuring the stability of the perturbed closed-loop system were inequalities of the form

$$\sigma_{max}[E(s)]<\sigma_{min}[h(G(s))] \qquad (5.1)$$

where $h(\cdot)$ is some bilinear fractional transformation (i.e., $I+G$, $I+G^{-1}$, $(I-G)^{-1}(I+G)$) and where (5.1) must hold for all $s\epsilon\Omega_R$. This condition assures that the model error is sufficiently small so that a closed-loop system designed on the basis of $G(s)$ will remain stable when it is replaced by $\tilde{G}(s)$. However, the approach used to develop these robustness theorems neglects the fact that there are perturbations or modelling errors for which (5.1) does not hold, i.e., the model error is not small, and yet the closed-loop system remains stable. These Section IV theorems are conservative if one restricts the allowable type of model error structure because they guard against absolutely all types of structure in linear model errors.
One way to reduce this conservatism is to obtain additional conditions that distinguish between modelling errors that do not destabilize the feedback system but violate the test of (5.1), and those that violate the test of (5.1) but also destabilize the feedback system. Or better yet, obtain some conditions that discriminate between modelling errors, that violate (5.1), between those that increase and those that decrease the margin of stability of the feedback system.

The problem is illustrated in Figure 7 for SISO systems where two different perturbed systems $\tilde{g}_1(s)$ and $\tilde{g}_2(s)$ produce exactly the same size of relative error on the Nyquist diagram. As can be seen from Figure 7, the difference between the perturbed systems $\tilde{g}_1(s)$ and $\tilde{g}_2(s)$ cannot be determined from the magnitude of the error alone. Clearly, $\tilde{g}_2(s)$ has a smaller margin of stability than the nominal system $g(s)$, and $\tilde{g}_1(s)$ has a larger margin of stability than the nominal $g(s)$. Since this is a scalar system the only additional information about the error needed to distinguish between $\tilde{g}_1(s)$ and $\tilde{g}_2(s)$ is the phase of the error. Thus, in the SISO case this gives us a complete characterization of the error.

In the MIMO case, the problem is not so simple because for an nxn system G(s) the error matrix E(s) has $2n^2$ degrees of freedom, two for each element of E(s) (i.e., gain and phase or real and imaginary part). Thus, if a single degree of freedom is eliminated from E(s), by information in addition to the norm of E(s), there are still $2n^2-2$ degrees of freedom left. Therefore, it is important that exactly the right additional information about E(s) is obtained so that only a partial characterization of E(s) is necessary to distinguish between modelling errors that increase or decrease the margin of stability of the feedback system. In order to do this it is necessary to examine the structure of the smallest error that destabilizes the feedback loop. We will call this error the worst error. In the SISO case, the worst error is illustraed in the Nyquist diagram of Figure 8.

At point A, in Figure 8, the Nyquist locus of g(s) is nearest the critical -1 point and thus the worst error simply moves point A to A' by "stretching" the Nyquist locus at that particular frequency to just pick up an extra encirclement of the -1 point (the point A' is infinitesimally close to -1). It is important to point out that this type of perturbation could be applied to g(s) in any frequency range but that it need happen only at one particular frequency, ω_0 near A, in order to induce instability. Thus we will speak of the worst error at a particular value of $s\epsilon\Omega_R$.

Notice also that there are any number of curves that we could pass through A' representing perturbations of the original Nyquist diagram of g(s) as depicted by $\tilde{g}_1(s)$ in Figure 8, that induce instability and are identical to the worst error at the frequency of point A but differ at other frequencies. However, these curves will also be considered as worst errors since it is really their nature at a single frequency that is important in distinguishing them from other curves.

One other point must be emphasized. A casual perturbed system $\tilde{g}(s)$ of the type in Figure 8 may be constructed quite simply by finding a continuous stable $\ell(s) = \tilde{g}(s)/g(s)$ that meets a closely as desired the ideal specifications given by

$$\ell_{ideal}(s) = \begin{cases} -g^{-1}(j\omega_0), & s=j\omega_0 \\ 1, & s\neq j\omega_0 \end{cases} \qquad (5.2)$$

where ω_0 is the frequency corresponding to point A in Figure 8. For example, one continuous, stable $\ell(s)$ that approximates ℓ_{ideal} in (5 2) can be generated simply by taking $\ell(s)$ to be of the form

$$\ell(s) = 1 - q(s)\left|1 + g^{-1}(j\omega_0)\right| \qquad (5.3)$$

where

$$q(s) = \frac{2\rho}{s^2+2\rho\,\omega_0 s+\omega_0^2}\left(\frac{s-\alpha}{s+\alpha}\right)c \qquad (5.4)$$

To approximate $\ell_{ideal}(s)$ closely, $\rho>0$ in (5.4) must be very small so that $|q(s)|$ is as small as desired whenever $|s-j\omega_0|>\varepsilon$ for a given ε. The constants $\alpha\geq 0$ and $c=\pm 1$ in (5.4) are used to adjust the phase of $q(s)$ without affecting $|q(s)|$ so that

$$q(j\omega_0)=\exp[j\{\arg(1+g^{-1}(j\omega_0)\}]. \qquad (5.5)$$

This selection of ρ, α and c in (5.4) makes $q(s)$ essentially zero everywhere except in a suitably small frequency range near ω_0 where it has the value given in (5.5). Thus $\ell(s)$ is as close as desired to the specifications in (5.2) but is still continuous in s and stable. The $\ell(s)$ determined by (5.3), (5.4) and (5.5) produces a $\tilde{g}(s)$ essentially like the one of Figure 8.

Returning to the MIMO case, we can make all the analogous statements to those concerning Figure 8, once we have specified the worst error. Then similarities between the SISO and MIMO cases can be easily demonstrated using the ideas of Section III developed in Problems A and B and by use of the SVD on the matrix $h(G(s))$ of (5.1). Suppose that the SVD of $h(G(s))$ is given by

$$h(C(s)) = U(s)\Sigma(s)V^H(s) \qquad (5.6)$$

where

$$U(s)=[\underline{u}_1(s),\underline{u}_2(s),\ldots,\underline{u}_n(s)] \qquad (5.7)$$

$$V(s)=[\underline{v}_1(s),\underline{v}_2(s),\ldots,\underline{v}_n(s)]] \qquad (5.8)$$

$$\Sigma(s)=\mathrm{diag}[\sigma_1,\sigma_2(s),\ldots,\sigma_n(s)]$$

$$\sigma_i(s) > \sigma_{i+1}(s)>0 \qquad (5.10)$$

where the singular values $\sigma_1(s) = \sigma_{max}(s)$ and $\sigma_n(s) = \sigma_{min}(s)$. Recall from (3.18) that the error matrix $E(s)$ of smallest norm that will make $h(G(s)) + E(s)$ singular is given by

$$E(s) = U(s)\left[\begin{array}{c|c} E_0(s) & \underline{0} \\ \hline \underline{0}^T & -\sigma_n(s)\end{array}\right]V(s)^H \qquad (5.11)$$

where $\|E_0(s)\|\leq\sigma_n(s)$ but is otherwise arbitrary.[1] Provided the norm of the matrix $E_0(s)$ is bounded by $\sigma_n(s)$, its structure is completely unimportant information for the test determining the singularity or nonsingularity of $h(G(s)) + E(s)$. Therefore, $E_0(s)$ will be taken as identically zero in the following discussion and thus, $E(s)$ given by (5.11) reduces to

$$E(s)=-\sigma_n(s)\underline{u}_n(s)\underline{v}_n^H(s) \qquad (5.12)$$

The $E(s)$ given by (5.12) will be called the essential structure of the more general form of $E(s)$ given by (5.11) when $E_0(s)\neq 0$. The quantity $-\sigma_n(s)\underline{u}_n(s)\underline{v}_n^H(s)$ is the component of $E(s)$ given by (5.11) that alone must be exactly known if it is to be ascertained whether or not the matrix $h(G(s)) + E(s)$ is singular. Hence, the description of the $E(s)$ given by (5.12) as the essential structure of $E(s)$ given by (5.11) is justified.

Again, as in the SISO case, the error given by (5.12) need only occur at one particular complex frequency s_0 to destabilize the feedback system. That is, we may construct a perturbed $\tilde{G}(s)$ having the same number of unstable poles as the nominal $G(s)$ that has the property that $E(s_0)$ satisfies (5.11) arbitrarily closely and hence destabilizes the feedback system. The MIMO error matrix $E(s_0)=-\sigma_n(s_0)\underline{u}_n(s_0)\underline{v}_n^H(s_0)$ is the generalization of the model errors that produce the $\tilde{g}(s)$ and $\tilde{g}_1(s)$ of Figure 8 passing through point A' just picking up an extra encirclement of the critical point (-1,0). From (5.12) we see that for an arbitrary error matrix $E(s)$ that the projection, $\langle\underline{u}_n(s)\underline{v}_n^H(s),E(s)\rangle\underline{u}_n(s)\underline{v}_n^H(s)$, of $E(s)$ onto the one dimensional subspace spanned by $\underline{u}_n(s)\underline{v}_n^H(s)$ can be used to determine if the component of modelling error in the most sensitive direction $\underline{u}_n(s)\underline{v}_n^H(s)$ will move the multivariable Nyquist diagram of the nominal system nearer or farther from the critical point (0,0) in the complex plane. The direction of this movement of the MIMO Nyquist diagram is simply ascertained by determining if $\langle\underline{u}_n(s)\underline{v}_n^H(s),E(s)\rangle$ is nearer or farther than a distance of $\sigma_n(s)$ from the point $(-\sigma_n(s),0)$ in the complex plane. However, the quantity $\langle\underline{u}_n(s)\underline{v}_n^H(s),E(s)\rangle$ merely determines the effect of one component of the model error and does not take into account the effect of the other components of the model error (i.e., the projections $\langle\underline{u}_i(s)\underline{v}_j^H(s),E(s)\rangle\underline{u}_i(s)\underline{v}_j^H(s))$ have on the multivariable Nyquist diagram. Therefore, some restrictions on these other model error components must be placed if their effect on the stability of the closed-loop system is to be easily predicted.

Suppose now that we restrict the component of modelling error in the most sensitive or worst direction $\underline{u}_n(s)\underline{v}_n^H(s)$ to be exactly zero

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[1] Of course it must also be such that $\tilde{G}(s)$ satisfies condition 1 of Theorem 1.

(i.e., $\langle \underline{u}_n(s)\underline{v}_n^H(s),E(s)\rangle=0$) so that it has no effect on the multivariable Nyquist diagram. Naturally, for this class of modelling errors, one expects that the magnitude of the error required to destabilize the feedback system should increase since the worst possible type of errror has been ruled out and indeed this is the case. The elimination of this type of error can only be done using engineering judgement about what type of error can occur in the physical system. The next theorem assumes that the worst model error can be ruled out and extends Theorems 3,4,5,6 and 7, by allowing them to deal with errors of larger magnitudes than previously allowable.

Theorem 8: The polynominal $\tilde{\phi}_{CL}(s)$ has no CRHP zeros and hence the perturbed feedback system is stable if the following four conditions hold:

1. (a) $\phi_{OL}(s)$ and $\tilde{\phi}_{OL}(s)$ have the same number of CRHP zeros.

   (b) if $\tilde{\phi}_{OL}(j\omega_0)=0$, then $\phi_{OL}(j\omega_0)=0$

   (c) $\phi_{CL}(s)$ has no CRHP zeros

2. $h(G(s))$ is of the form:

   (a) $h(G(s)) = I+G(s)$, $\lambda(L(s))\not\in(-\infty,0]$ and $E(s) = [\tilde{G}^{-1}(s)-G^{-1}(s)]G(s)$ or

   $E(s) = \tilde{G}(s)-G(s)$ for all $s\varepsilon\Omega_R$

   or (b) $h(G(s))=(I+G(s))(I-G(s))^{-1}$, $\lambda(L(s))\not\in(-\infty,-1]$

   and $E(s) = [\tilde{G}(s)+G(s)]^{-1}[\tilde{G}(s)-G(s)]$ for all $s\varepsilon\Omega_R$

   or

   (c) $h(G(s)) = I+G^{-1}(s)$ and $E(s) = G^{-1}(s)[\tilde{G}(s)-G(s)]$ or $E(s) = [\tilde{G}^{-1}(s)-G^{-1}(s)]$ and $\lambda(L(s))\not\in(-\infty,0]$ for all $s\varepsilon\Omega_R$.

3. $\sigma_{max}[E(s)]<[\sigma_n(s)\sigma_{n-1}(s)]^{1/2}$ for all $s\varepsilon\Omega_R$ where $\sigma_n(s)$ and $\sigma_{n-1}(s)$ are the two smallest singular values (assumed to be distinct) of $h(G(s))$

4. $\langle \underline{u}_n(s)\underline{v}_n^H(s),E(s)\rangle = 0$

   for all $s\varepsilon\Omega_R$ where $\underline{u}_n(s)$ and $\underline{v}_n(s)$ are the left and right singular vectors of $h(G(s))$ associated with $\sigma_n(s)$. □

Note that in Theorem 8, conditions 3 and 4 are required to hold for all $s\varepsilon\Omega_R$ even though they need only be used in the frequency range where the sufficient conditions (all given by (5.1) of Theorems 3 and 7) are violated.

The significance of Theorem 8 is that by requiring very little information (condition 4) in addition to the magnitude of the model error, the worst type of modelling error that could destabilize the feedback system (and whose exclusion might be justified on physical grounds) is effectively eliminated. Hence, the "size" of the error necessary to destabilize the system may increase significantly if $\sigma_{n-1}(s)\gg\sigma_n(s)$.

Thus, the conservatism of the Section IV theorems for this class of modelling errors is reduced. The essential structure of the next worst error (i.e., next smallest error) that destabilizes the system in this restricted class of modelling errors is given by (from (3.25) with $\phi=0$ because $\langle \underline{u}_n(s)\underline{v}_n^H(s),E(s)\rangle=0$)

$$E(s) = \sqrt{\sigma_n(s)\sigma_{n-1}(s)}\cdot\left[\underline{u}_n(s)\underline{v}_{n-1}^H(s)e^{j\theta(s)}+\underline{u}_{n-1}(s)\underline{v}_n^H(s)e^{-j\theta(s)}\right]. \tag{5.13}$$

where (a) $\Theta(s)$ is real and arbitrary and (b) the vectors $\underline{u}_{n-1}(s),\underline{u}_n(s)$, $\underline{v}_{n-1}(s)$ and $\underline{v}_n(s)$ are the left and right singular vectors of $h(G(s))$ corresponding to $\sigma_{n-1}(s)$ and $\sigma_n(s)$ respectively. The spectral norm of the matrix $E(s)$ in (5.13) is precisely $\sqrt{\sigma_n(s)\sigma_{n-1}(s)}$.

However, it must be pointed out, that it is extremely unlikely that condition 4 of Theorem 8 will hold exactly for a realistic modelling error since the model error in the particular direction $\underline{u}_n(s)\underline{v}_n^H(s)$ will rarely be exactly zero. A more likely expectation is that this component of the error not be exactly zero but sufficiently small in magnitude. By requiring only that the model error in the direction $\underline{u}_n(s)\underline{v}_n^H(s)$ be sufficiently small, Theorem 8 may be modified so that the essential nature of its results are still valid when the class of model errors considered is characterized by

$$|\langle \underline{u}_n(s)\underline{v}_n^H(s),E(s)\rangle| \le c(s) < \sigma_n(s) = \sigma_{min}(s). \tag{5.14}$$

The positive scalar $c(s)$ in (5.14) bounds the magnitude of the worst modelling error as a function of frequency to be less than $\sigma_{min}(s)$, the minimum magnitude of the smallest destabilizing error required to destabilize the feedback system. Therefore, the magnitude of the model error in the most sensitive or worst direction $\underline{u}_n(s)\underline{v}_n^H(s)$ is not large enough by itself to destabilize the feedback system.

In order to destabilize the feedback system when the model errors satisfy (5.14), other model error components, besides the model error component in the worst direction, must contribute to the movement of the MIMO Nyquist diagram through the critical point (0,0). This is stated formally in the next theorem.

Theorem 9: The polynominal $\tilde{\phi}_{CL}(s)$ has no CRHP zeros and hence the perturbed feedback system is stable if the following conditions hold:

1. conditions 1 and 2 of Theorem 8 hold
2. $\sigma_{max}[E(s)]<[\sigma_n(s)\sigma_{n-1}(s)+c(s)[\sigma_n(s)-\sigma_{n-1}(s)]]^{1/2}$ for all $s\in\Omega_R$
3. $|<\underline{u}_n(s)\underline{v}_n^H(s),E(s)>|\leq c(s)<\sigma_n(s)$ for all $s\in\Omega_R$. □

The essential structure of the next worst perturbation that does not violate condition 3 but destabilizes the feedback system is given by (from 3.25)

$$E(s) = [c(s)\underline{u}_{n-1}(s)\underline{v}_{n-1}^H(s)-c(s)\underline{u}_n(s)\underline{v}_n^H(s)+ + \gamma(s)\underline{u}_{n-1}(s)\underline{v}_n^H(s)+\gamma^*(s)\underline{u}_n(s)\underline{v}_{n-1}^H(s)] \tag{5.15}$$

where

$$\gamma(s) = \Big[[\sigma_n(s)-c(s)][c(s)+\sigma_{n-1}(s)]\Big]^{1/2}e^{j\phi(s)} \tag{5.16}$$

with $\phi(s)$ being arbitrary but real. Note that as $c\to 0$, in condition 3 and in (5.15) and (5.16) that we recover the results of Theorem 8. To make the meaning of the result of Theorem 9 clearer, the following example is given.

Example 1: Suppose that we wish to determine stability robustness of a 2x2 control system which actually has a loop transfer function matrix $\tilde{G}(s)$ but is represented by the nominal diagonal loop transfer matrix G(s) given by

$$G(s) = \begin{bmatrix} g_{11}(s) & 0 \\ 0 & g_{22}(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s+7.5} & 0 \\ 0 & \frac{1}{s+0.5} \end{bmatrix} \tag{5.17}$$

so that the nominal closed-loop system has poles at -8.5 and -1.5. If we use the relative error criterion

$$E(s) = G^{-1}(s)[\tilde{G}(s)-G(s)] = \begin{bmatrix} \frac{\tilde{g}_{11}(s)-g_{11}(s)}{g_{11}(s)} & \frac{\tilde{g}_{12}(s)}{g_{11}(s)} \\ \frac{\tilde{g}_{21}(s)}{g_{22}(s)} & \frac{\tilde{g}_{22}(s)-g_{22}(s)}{g_{22}(s)} \end{bmatrix} \tag{5.18}$$

then the multiplicative uncertainty factor matrix L(s) is given by

$$L(s) = I+E(s) = \begin{bmatrix} \frac{\tilde{g}_{11}(s)}{g_{11}(s)} & \frac{\tilde{g}_{12}(s)}{g_{11}(s)} \\ \frac{\tilde{g}_{21}(s)}{g_{22}(s)} & \frac{\tilde{g}_{22}(s)}{g_{22}(s)} \end{bmatrix} \tag{5.19}$$

First, we compute $\sigma_{min}(I+G^{-1}(j\omega))$ to determine the magnitude of the smallest destabilizing model error E(s). This is simply given by

$$\sigma_{min}(I+G^{-1}(j\omega)) = \Big|1.5+j\omega\Big| = \sqrt{(1.5)^2+\omega^2}\geq 1.5 \tag{5.20}$$

because

$$I+G^{-1}(s) = \begin{bmatrix} s+8.5 & 0 \\ 0 & s+1.5 \end{bmatrix}. \tag{5.21}$$

Now suppose that the error in the loop gain of each loop of the feedback system is known within $\pm 50\%$ of the nominal loop gain, that is

$$0.5 \leq \left|\frac{\tilde{g}_{11}(j\omega)}{g_{11}(j\omega)}\right| = \left|\ell_{11}(j\omega)\right| \leq 1.5 \tag{5.22}$$

and

$$0.5 \leq \left|\frac{\tilde{g}_{22}(j\omega)}{g_{22}(j\omega)}\right| = \left|\ell_{22}(j\omega)\right| \leq 1.5 . \tag{5.23}$$

Next, suppose that we are more uncertain about the channel crossfeeds in the sense that we can only assert that

$$\left|e_{12}(j\omega)\right| = \left|\ell_{12}(j\omega)\right| = \left|\frac{\tilde{g}_{12}(j\omega)}{g_{11}(j\omega)}\right| \leq 2 \tag{5.24}$$

and that

$$\left|e_{21}(j\omega)\right| = \left|\ell_{21}(j\omega)\right| = \left|\frac{\tilde{g}_{21}(j\omega)}{g_{22}(j\omega)}\right| \leq 2 . \tag{5.25}$$

It follows from (5.22) and (5.23) that we can bound $|e_{11}(j\omega)|$ and $|e_{22}(j\omega)|$ by 1/2 and thus, by (5.24) and (5.25), we can only conclude that

$$\|E(j\omega)\|_2 = \sigma_{max}[E(j\omega)] \le 2.5 \tag{5.26}$$

From (5.26) and (5.20) it is clearly possible to have

$$\sigma_{max}[E(j\omega)] > \sigma_{min}[I+G^{-1}(j\omega)]. \tag{5.27}$$

Therefore, Theorem 4 does not apply. However, we can use Theorem 9 to ensure the stability of the perturbed feedback system. To see this, note that the SVD of $I+G^{-1}(j\omega)$ is given by

$$I+G^{-1}(j\omega) = \begin{bmatrix} e^{j\theta_1(\omega)} & 0 \\ 0 & e^{j\theta_2(\omega)} \end{bmatrix} \cdot \begin{bmatrix} |j\omega+8.5| & 0 \\ 0 & |j\omega+1.5| \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= U(j\omega)\Sigma(j\omega)V^H(j\omega) \tag{5.28}$$

where

$$\theta_1(\omega) = \arg[j\omega+8.5] \tag{5.29}$$

and

$$\theta_2(\omega) = \arg[j\omega+1.5] \tag{5.30}$$

Note that condition 3 of Theorem 9 can be satisfied with $c(j\omega)=1/2$ since from (5.28) defining $\underline{u}_2(j\omega)$ and $\underline{v}_2(j\omega)$ and from (5.23) bounding $\ell_{22}(j\omega)$ and thus $e_{22}(j\omega)$ we have that for all ω

$$|\langle \underline{u}_2(j\omega)\underline{v}_2^H(j\omega), E(j\omega)\rangle| = |\underline{u}_2^H(j\omega)E(j\omega)\underline{v}_2(j\omega)| = |e_{22}(j\omega)| \le 1/2 . \tag{5.31}$$

Thus, by (5.31) and (5.20) we have

$$\sigma_2(j\omega) \ge 1.5 > 1/2 \ge |\langle \underline{u}_2(j\omega)\underline{v}_2^H(j\omega), E(j\omega)\rangle| .$$

Next, we calculate the right-hand-side of condition 2 of Theorem 9 and a lower bound as follows

$$\left[\sigma_1(j\omega)\sigma_2(j\omega)+c(j\omega)[\sigma_2(j\omega)-\sigma_1(j\omega)]\right]^{1/2} = \left[|j\omega+8.5||j\omega+1.5| + 1/2[|j\omega+1.5|-|j\omega+8.5|]\right]^{1/2} \ge (8.5)(1.5)+\left(\frac{-7}{2}\right) \ge 3. \tag{5.33}$$

Therefore, using (5.26) we have that

$$\sigma_{max}[E(j\omega)] \le 2.5 < 3 \le \left[\sigma_1(j\omega)\sigma_2(j\omega)+c(j\omega)[\sigma_2(j\omega)-\sigma_1(j\omega)]\right]^{1/2} \tag{5.34}$$

and so condition 2 of Theorem 9 holds. Assuming condition 1 of Theorem 9 holds we have shown that the perturbed feedback system is stable. The next smallest destabilizing error can be calculated from (5.15) and (5.16) with $\phi(j\omega)=0$ and $\omega=0$ since $\sigma_{min}(I+G^{-1}(j\omega)) \ge \sigma_{min}(I+G^{-1}(0))=1.5$ and is given by

$$E(0) = \begin{bmatrix} 1/2 & 3 \\ 3 & -1/2 \end{bmatrix} \tag{5.35}$$

which means that L(s) may be taken as the constant matrix L given by

$$L = \begin{bmatrix} 3/2 & 3 \\ 3 & 1/2 \end{bmatrix} \tag{5.36}$$

Thus, we see that (refer to Figs. 9 and 10) crossfeed gain errors of magnitude 3 and loop gain changes of ±50% are required to destabilize the feedback system if we insist that (5.22) and (5.23) must hold.

<u>Remark:</u> One possible exception, to the form of E(s) given in (5.13) or (5.15) occurs when E(s) is such that at least, one of the eigenvalues of L(s) is real and negative. In Theorem 8 and 9,

condition 2 places restrictions on the eigenvalues of L(s) which may be violated when at least one of the eigenvalues of L(s) is real and negative. In this case, Theorems 8 and 9 may not apply and there may exist a smaller error that destabilizes the feedback system but yet conditions 4 and 3, of Theorems 8 and 9 respectively, still hold. However, when the matrices U(s) and V(s) of the SVD of h(G(s)) are complex it is very unlikely that L(s) determined by the E(s) given in (5.13) or (5.15) will even have real eigenvalues.

We can now consider placing additional constraints on the modelling and further restrict the class of allowable modelling errors in the manner of Problem B in section III and derive the next theorem.

Theorem 10: The polynominal $\tilde{\Phi}_{CL}(s)$ has no CRHP zeros and hence the perturbed feedback system is stable if the following conditions hold:

1. Conditions 1 and 2 of Theorem 8 hold
2. E(s) is of the form

$$E(s) = U(s)\left[\begin{array}{c|c} E_1(s) & \underline{e}_2(s) \\ \hline \underline{e}_3^T & 0 \end{array}\right] V^H(s) \quad (5.37)$$

   where $\underline{e}_2(s)$ and $\underline{e}_3(s)$ are vectors whose last component is identically zero and where U(s) and V(s) are defined in (5.6).

3. $\sigma_{max}(E(s)) < \sqrt{\sigma_k(s)\sigma_\ell(s)}$

   where $$\sigma_k(s)\sigma_\ell(s) = \min_{(i,j)\notin M} \sigma_i(s)\sigma_j(s) \quad (5.38)$$

   and $$M = \{(n,n),(n-1,n),(n,n-1)\} \quad (5.39)$$

Theorem 10 allows us to determine the next larger magnitude of the "next, next worst model error" required to product instability when the smallest destabilizing model error and next smallest destabilizing model error considered in Theorem 8 and given by (5.13) are completely eliminated from consideration. Theorem 10 eliminates these types of errors by requiring zero model error projections in the worst direction $\underline{u}_n(s)\underline{v}_n^H(s)$ and the next worst pair of directions $\underline{u}_n(s)\underline{v}_{n-1}^H(s)$ and $\underline{u}_{n-1}(s)\underline{v}_n^H(s)$. The process of eliminating each "successively worst direction" could obviously be continued and larger magnitudes of these classes of errors would then be necessary to destabilize the feedback system.

VI. SUMMARY AND CONCLUSIONS

This paper has addressed the following problem. Given a finite-dimensional, linear-time-invariant feedback control system designed using an inaccurate nominal model of the open-loop plant, how much and what kind of model error can the feedback system tolerate without becoming unstable? Thus, this paper deals primarily with the evaluation of the robustness of stability of a feedback control system. This robustness evaluation is absolutely essential since all models of physical processes are only approximations to the actual relationship between the system inputs and outputs. In the single-input, single-output (SISO) case, this evaluation is readily accomplished using frequency domain plots, (e.g., using a Bode diagram) to display the behavior and characteristics of the feedback system. However, in the multiple-input, multiple-output (MIMO) case, many generalizations of the SISO methods have proven inadequate because they have not dealt with the MIMO system as a whole but as a sequence of SISO systems.

This paper has avoided this deficiency by utilizing standard matrix theory concepts and methods appropriate for dealing with the MIMO case, namely the singular value decomposition (SVD) and properties of special types of matrices. These were discussed in Section III, where the main problem solved was the determination of the nearest singular matrix, $\tilde{A}$, to a given nonsingular matrix, A, under certain constraints on $\tilde{A}-A$. The solution to this problem (given in Problems A and B) is fundamental to the control system robustness results of Section IV and V.

The basic formulation of the control system robustness problem was considered in Section II via a multivariable version of Nyquist's stability theorem. There, a fundamental robustness theorem (Theorem 1 was presented that implicitly characterized the class of perturbed models that would not destabilize the control system, in terms of the nonsingularity of the return difference matrix. Various robustness tests Theorems 3 and 7), were then derived which can be used to test the nonsingularity of the return difference matrix for several types of model error criteria.
Section V heavily utilizes the results of Section III in determining what types of model error will destabilize a given feedback system. Model errors that tend to destabilize the feedback system are distinguished from those that tend to stabilize the feedback system by examining their structure as well as their magnitude. The key results, contained in Theorems 8, 9, and 10, show that the magnitude of the model error necessary to destabilize the feedback system may greatly increase if the class of model errors that can plausibly occur does not include model errors that are essentially alike in structure to the model error of minimum size that will destabilize the feedback system. This provides an important partial characterization of the model errors that are important in feedback system design. However, the degree to which the the partial characterization of the model error demanded by this approach correlates with one's understanding of modelling errors in the physical system will undoubtably be the key factor in making practical use of these results.

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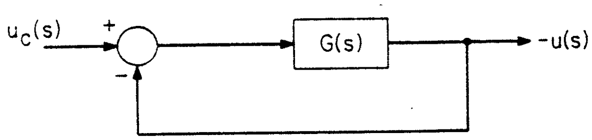


Figure 1: Control system under consideration.

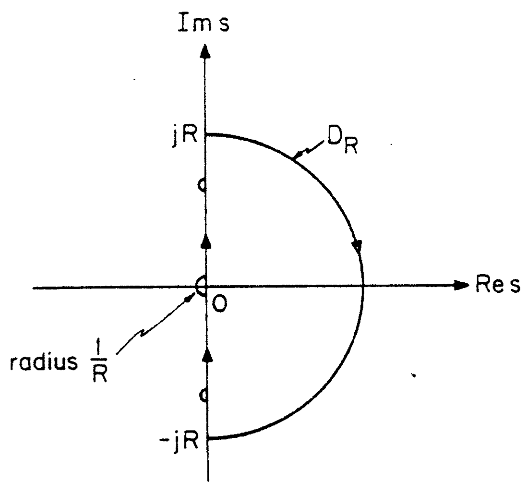


Figure 2: Nyquist contour D_R which avoids $j\omega$-axis zeros of $\phi_{OL}(s)$ by 1/R radius indentations.

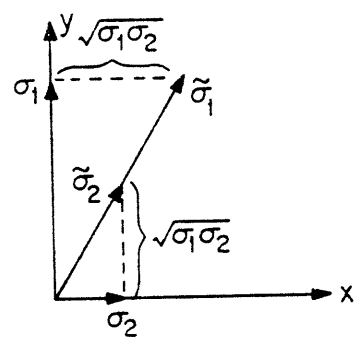


Figure 3: Columns of A and $\tilde{A}$=A+E depicted as vectors aligned with minimum effort.

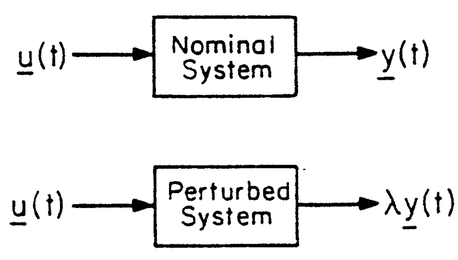
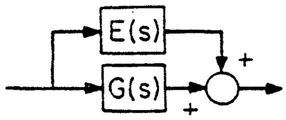
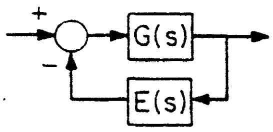
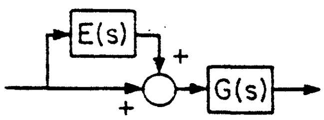
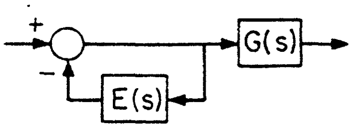


Figure 4: Relationship between nominal and perturbed systems for special input $\underline{u}(t)$ when $L(j\omega_0)$ has eigenvalue λ.

<table>
<tr><th>Block Diagram
of Perturbed System</th><th>Error Criterion
Perturbed Systems and
Stability Test</th></tr>
<tr><td>

Feedforward
(Addition)</td><td>$E(s) = \tilde{G}(s) - G(s)$
$\tilde{G}(s) = G(s) + E(s)$
$\sigma_{min}(I+G(s)) > \sigma_{max}(E(s))$</td></tr>
<tr><td>

Feedback
(subtraction)</td><td>$E(s) = \tilde{G}^{-1}(s) - G^{-1}(s)$
$\tilde{G}(s) = (G^{-1}(s) + E(s))^{-1}$
$\sigma_{min}(I+G^{-1}(s)) > \sigma_{max}(E(s))$</td></tr>
<tr><td>

(Multiplication)</td><td>$E(s) = G^{-1}(s)[\tilde{G}(s)-G(s)]$
$\tilde{G}(s) = G(s)(I+E(s))$
$\sigma_{min}(I+G^{-1}(s)) > \sigma_{max}(E(s))$</td></tr>
<tr><td>

(Division)</td><td>$E(s) = [\tilde{G}^{-1}(s) - G^{-1}(s)]G(s)$
$\tilde{G}(s) = G(s)(I+E(s))^{-1}$
$\sigma_{min}(I+G(s)) > \sigma_{max}(E(s))$</td></tr>
</table>

Figure 5: Block diagrams of perturbed models corresponding to various error criteria and associated stability tests.

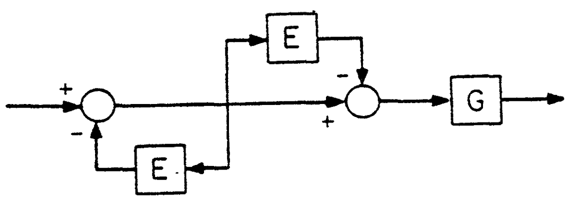


Figure 6: Block diagram of $\tilde{G}$ associated with model error criteria of (4.14).

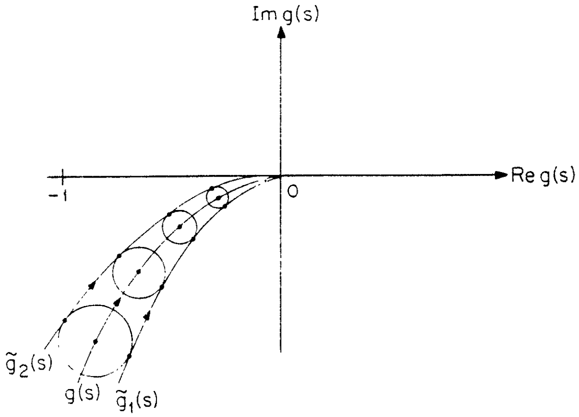


Figure 7: Two different perturbed models with the same relative error magnitude on a SISO Nyquist diagram.

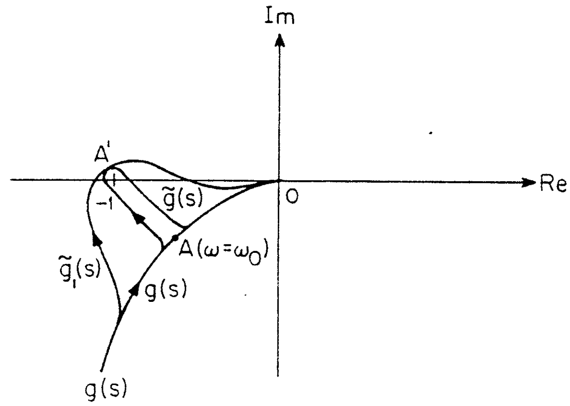


Figure 8: Illustration of worst type of error in SISO case on a Nyquist diagram.

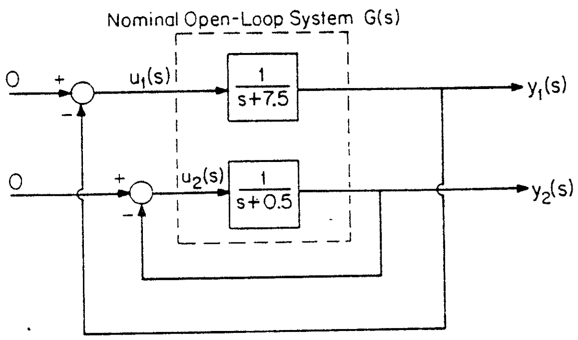


Figure 9: Nominal feedback system (stable).

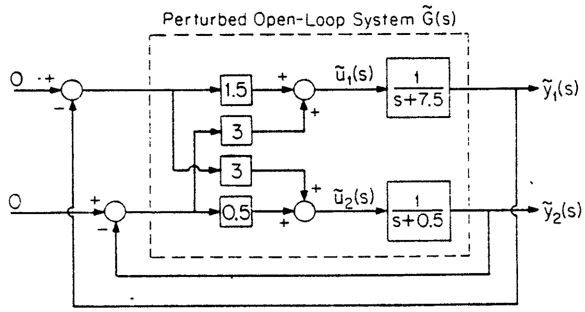


Figure 10: Perturbed feedback system (unstable).