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**Citation:** Bleichrodt, Han, Amit Kothiyal, Drazen Prelec, and Peter P. Wakker. "Compound Invariance Implies Prospect Theory for Simple Prospects." *Journal of Mathematical Psychology* 57, no. 3–4 (June 2013): 68–77.

**As Published:** <http://dx.doi.org/10.1016/j.jmp.2013.04.002>

**Publisher:** Elsevier

**Persistent URL:** <http://hdl.handle.net/1721.1/99122>

**Version:** Original manuscript: author's manuscript prior to formal peer review

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# Compound Invariance Implies Prospect Theory for Simple Prospects

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March, 2013

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**ABSTRACT.** Behavioral conditions such as compound invariance for risky choice and constant decreasing relative impatience for intertemporal choice have surprising implications for the underlying decision model. They imply a multiplicative separability of outcomes and either probability or time. Hence the underlying model must be prospect theory or discounted utility on the domain of prospects with one nonzero outcome. We indicate implications for richer domains with multiple outcomes, and with both risk and time involved.

**KEYWORDS:** compound invariance, prospect theory, discounted utility, hexagon condition, hyperbolic discounting

## 1. Introduction

Expected utility (EU; von Neumann & Morgenstern 1944) and constant discounted utility (CDU; Samuelson 1937) are the normative standards for decision under risk and intertemporal choice, respectively. Alternative behavioral models, such

as prospect theory (Tversky & Kahneman 1992) and hyperbolic discounting (Laibson 1997; Loewenstein & Prelec 1992) were motivated by well known empirical violations of these normative standards. Thus people exhibit risk aversion in ways that expected utility cannot capture (Allais 1953), and discounting decreases because impatience decreases over time (Strotz 1956). The alternative models combine a generalized decision model, establishing representations more flexible than EU and CDU, with auxiliary psychophysical axioms yielding tractable parametric functional forms. This article shows that these psychophysical axioms determine the underlying decision model itself in a number of important cases, something that has gone unnoticed as yet. In particular, we will show that Prelec's (1998) compound invariance condition, usually added to a presupposed prospect theory, by itself implies prospect theory on the domain considered in his study. In the same way, similar preference conditions usually added to a presupposed discounted utility, by themselves imply discounted utility for intertemporal choice. The substantive message is that simple psychophysical principles give rise to the leading behavioral models for risk and intertemporal choice.

We will derive the aforementioned implications using two technical tools that are of interest in their own right. The first tool is an operation that derives endogenous midpoints for probability weighting and time discounting from preferences. It clarifies the formal and intuitive content of preference conditions such as compound invariance. The second tool is a generalized method to analyze functional equations, summarized in Figure D.1 in Appendix D. This appendix will show that several classical characterizations in decision theory concern the same mathematical theorem and differ by no more than a simple substitution of variables. For example, Prelec's (1998) characterization of compound invariance probability weighting under prospect theory becomes the same mathematical theorem as the classical characterizations of CARA (constant absolute risk aversion, or linear-exponential) and CRRA (constant relative risk averse, or log-power) utility under expected utility. Other "identical" theorems concern conditional invariance probability weighting (Prelec 1998) and the constant relative decreasing impatient (renamed unit invariant in this paper) time discount functions of Bleichrodt, Rohde, & Wakker (2009) and Ebert & Prelec (2007). Thus we provide the simplest proofs of these results presently available in the literature, and we extend them to domains more general than considered before. Figure D.1 is of independent interest. It puts all the results on functional equations

together, and can be of use to any application of the concerned families of functional equations. Remark D.2 lists many well-known theorems in the literature that are immediate corollaries of Figure D.1.

The outline of this paper is as follows. Section 2 states the basic problems of this paper for risk and time jointly, showing what these problems have in common. It also defines the domain analyzed in our main results, being simple prospects with only one nonzero outcome. Section 3 presents the specific result of most interest for risk, and §4 does so for time. In the latter section we focus on unit invariant discounting, which has more promising empirical and analytical properties than the commonly used hyperbolic discounting. Section 5 discusses implications of our results. Because virtually all theories for risk and time agree on the domain considered in our main results (only one nonzero outcome), our results are relevant for all those theories. We demonstrate implications for richer domains of prospects with multiple nonzero outcomes. §6 concludes. Examples are in Appendix A. Appendixes B and C prepare for the proofs, and Appendix D completes the proofs.

## 2. Statement of the problem

In the risk domain, with  $(x,p)$  denoting the (risky) *prospect* of receiving outcome  $x$  with probability  $p$  and nothing otherwise, one might observe

$$(x,p) > (y,q) \text{ but } (x,rp) < (y,rq) \text{ for } 0 < r < 1, \quad (2.1)$$

in violation of expected utility. Allais (1953) gave convincing examples of this pattern, which were later confirmed empirically. In the time domain, if  $(x,t)$  denotes the (temporal) *prospect* of receiving outcome  $x$  at time  $t$ , then one might observe

$$(x,t) > (y,s) \text{ but } (x,r+t) < (y,r+t) \quad (2.2)$$

(Ainslie 1975), in violation of constant discounted utility. While these empirical phenomena can be captured by a variety of different models, all currently popular models for risky and intertemporal decisions assume the same, multiplicative, underlying decision model

$$V(x,p) = w(p)U(x) \text{ or } V(x,t) = f(t)U(x) \quad (2.3)$$

for prospects with one nonzero outcome. This entails independent scaling (or separability) of outcomes and probability or time. Besides expected utility (the case where  $w(p) = p$ ), the best-known case of Eq. 2.3 for risk is prospect theory.

To obtain useful functional forms of  $w$  and  $f$ , plausible restrictions have been imposed on the propagation of EU and CDU violations (explained in Eqs. 2.4 and 2.7 below). For risk, EU violations can, for instance, be preserved under compounding if an Allais-violation of the form

$$(x,p) \sim (y,q) \text{ and } (x,rp) \sim (y,rs) \text{ for } s \neq q^1$$

propagates such that

$$(x',p^\lambda) \sim (y',q^\lambda) \text{ implies } (x',(rp)^\lambda) \sim (y',(rs)^\lambda). \quad (2.4)$$

Here  $p^\lambda$  is the  $\lambda$  fold compound of  $p$ . This condition is Prelec's (1998) compound invariance condition. The propagation of EU violations continues and, by repeated application, we get Eq. 2.4 with  $n\lambda$  instead of  $\lambda$  for any natural  $n$ . Within a multiplicative representation (2.3), such a propagation of Allais-violations implies that the weighting function is of the form:

$$w(p) = \left( \exp(-(-\ln p)^\alpha) \right)^\beta \quad (2.5)$$

(Prelec 1998, Proposition 1). A natural question that arises is how compound invariance can be combined with more general non-multiplicative representations, i.e. with forms deviating from Eq. 2.3. For example, one could allow the  $\alpha$  parameter in Eq. 2.5 to depend on outcome magnitude:

$$V(x,p) = \left( \exp(-(-\ln p)^{\alpha(x)}) \right)^\beta U(x) \quad (2.6)$$

in order to accommodate the finding that probability weighting is more nonlinear at high stakes.<sup>2</sup>

A parallel situation obtains for time. The corresponding propagation axiom asserts that CDU violations will be preserved if a violation of the form

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<sup>1</sup> This is equivalent to Eq. 2.1 (increase  $q$  and take  $s < q$ ).

<sup>2</sup> This phenomenon was found in empirical studies by Bosch-Domènech & Silvestre (2006), Camerer (1989 p. 94), Etchart (2004, 2009), Fehr-Duda et al. (2010), Rottenstreich & Hsee (2001), Sutter & Poitras (2010), and Tversky & Kahneman (1992 p. 317).

$(x,t) \sim (y,s)$  and  $(x,r+t) \sim (y,r+q)$  for  $q \neq s$

propagates such that

$$(x',\lambda t) \sim (y', \lambda s) \text{ implies } (x',(\lambda(r+t))) \sim (y',(\lambda(r+q))). \quad (2.7)$$

This propagation of CDU violations was called constant relative decreasing impatience by Prelec (1998 Definition 4) and by Bleichrodt, Rohde, & Wakker (2009). We will refer to it as unit invariance. Within a multiplicative representation (Eq. 2.3) unit invariance implies that the discount function is of the form:

$$f(t) = k \times \exp(-rt^c) \quad (2.8)$$

(Ebert & Prelec 2007; Bleichrodt, Rohde, & Wakker 2009). Again, a natural question is how unit invariance can be combined with more general non-multiplicative representations, deviating from Eq. 2.3. A non-multiplicative generalization of Eq 2.8 results if the time-discount parameter  $r$  depends on the magnitude of the discounted outcome  $x^3$ :

$$V(x,t) = k \times \exp(-r(x)t^c)U(x). \quad (2.9)$$

Theoretical models for our domain (prospects with one nonzero outcome) that incorporate interactions deviation from Eq. 2.3 include Baucells & Heukamp (2012 Theorem 1), Noor (2009, 2011), and Takeuchi (2011 p. 457). Baucells & Heukamp's (2012) probability-time tradeoff theory considers one nonzero outcome but with both time and risk involved, and allows for many kinds of interactions, including the ones discussed here. Blavatskyy (2013) presents an appealing direct empirical test of the separability of Eq. 2.3, finding it violated.<sup>4</sup> Models that give up separability between outcomes and time/risk in richer domains with multiple nonzero outcomes include Chew & Wakker (1996), Chiu (1996), Green & Jullien (1988), and Karni (2003: state-dependent utility). Section 5 will discuss the implications of preference conditions such as unit invariance on such richer domains.

All previous papers on the propagation of violations have stated the multiplicative decision model of Eq. 2.3 as an extra assumption (Aczél & Luce 2007; Al-Nowaihi &

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<sup>3</sup> This phenomenon, the magnitude effect, has been found in many empirical studies (Benzion, Rapoport, & Yagil 1989; Thaler 1981; reviewed by Attema 2012 §3.4).

<sup>4</sup> Blavatskyy in fact tests the well-known Thomsen condition, a necessary and sufficient condition that by itself is somewhat stronger than the hexagon condition that we will use later.

Dhami 2006; Luce 2001). This raises the question what the propagations imply for the more general decision models referenced above, which relax the separability of Eq. 2.3. The surprising conclusion will be that the propagations exclude all these more general models and, hence, cannot be considered in these models. That is, as we will show, both compound invariance and unit invariance by themselves imply the multiplicative forms of Eq. 2.3. It is remarkable that no-one, including the authors of the present paper (Bleichrodt, Rohde, & Wakker 2009; Ebert & Prelec 2007; Prelec 1998), has noticed these implications before, so that all the papers up to now contained a redundant assumption.

### 3. Compound invariance probability weighting for multiplicative preference representations under risk

#### 3.1. Elementary definitions for multiplicative representations

We consider (*risky*) *prospects*  $(x,p)$  yielding *outcome*  $x$  with *probability*  $p$  and outcome  $0$  with probability  $1 - p$ . With  $X$  a nonpoint interval in  $\mathbb{R}^+$  containing  $0$ , the set of all prospects considered is  $X \times [0,1]$ .<sup>5</sup> By  $\succsim$  we denote a *preference relation* over prospects, with  $>$ ,  $\sim$ ,  $\leq$ , and  $<$  as usual. We identify all  $(0,p)$  and  $(x,0)$  with  $(0,0)$ . That is,  $(0,p) \sim (x,0) \sim (0,0)$  for all  $x$  and  $p$ . *Strict stochastic dominance* holds if  $[(x,p) > (y,p)$  whenever  $x > y$  and  $p > 0]$ , and further  $[(x,p) > (x,q)$  whenever  $x > 0$  and  $p > q]$ .

For the preference relation  $\succsim$ ,  $V$  is *representing* if  $V$  maps the domain of preference ( $X \times [0,1]$  in this case) to the reals and  $[(x,p) \succsim (y,q) \Leftrightarrow V(x,p) \geq V(y,q)]$ . If a representing  $V$  exists then  $\succsim$  is a *weak order*: it is transitive and *complete* ( $(x,p) \succsim (y,q)$  or  $(y,q) \succsim (x,p)$  for all prospects). Continuity of  $V$  implies *continuity* of  $\succsim$ :  $\{(y,q): (y,q) \succsim (x,p)\}$  and  $\{(y,q): (y,q) \leq (x,p)\}$  are closed subsets of the prospect space  $X \times [0,1]$  for each prospect  $(x,p)$ .

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<sup>5</sup> Prelec (1998) assumed a set of outcomes  $[x^-, x^+]$  with  $x^- < 0 < x^+$ , so that both gains and losses were involved. Following prospect theory, the preferences for losses  $[x^-, 0]$  were treated separately from those for gains. For simplicity, we will only consider gains. Our results for gains can be applied to losses as well.

DEFINITION 3.1. The representation  $V$  for decision under risk is *multiplicative* if there exist functions  $w: [0,1] \rightarrow \mathbb{R}$  and  $U: X \rightarrow \mathbb{R}$  such that:

$$V(x,p) = w(p)U(x). \quad (3.1)$$

$$U(0)=0 \text{ and } U \text{ is continuous and strictly increasing.} \quad (3.2)$$

$$\begin{aligned} &w \text{ is continuous and strictly increasing,} \\ &\text{and } w(0)=0. \end{aligned} \quad (3.3)$$

We call  $U$  the *utility function* and  $w$  the *weighting function*.  $\square$

In what follows, we can always normalize  $w(1) = 1$ . Multiplicative representability implies strict stochastic dominance.

A clarifying interpretation of the conditions presented later in this paper can be obtained by adapting an endogenous midpoint operation, proposed by Köbberling and Wakker (2003 p. 408) for additive representations, to our multiplicative representation. We call  $q$  the *endogenous midpoint* between  $p$  and  $r$ , denoted  $m(p,r) = q$ , if there exist outcomes  $x$  and  $y$  such that

$$\begin{aligned} (x,p) &\sim (y,q) \quad \& \\ (x,q) &\sim (y,r) \end{aligned} \quad (3.4)$$

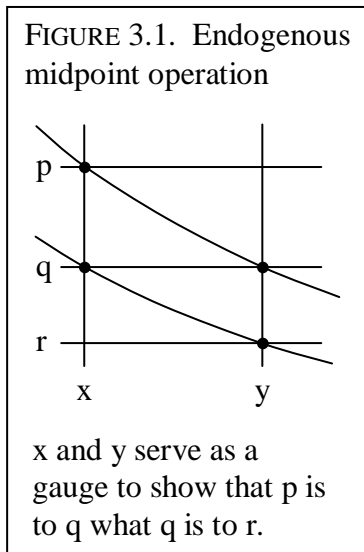
where all outcomes and probabilities differ from 0 (Figure 3.1). In the indifferences (3.4),  $p$  and  $q$  offset the same outcome difference as  $q$  and  $r$ . We call  $x$  and  $y$  the *gauge outcomes*. The following lemma shows that an endogenous midpoint is a geometric midpoint in terms of  $w$ -units in our multiplicative model. The result holds irrespective of the utility function. For an alternative way to obtain results on discounting independently of utility, a detailed discussion, and a review see Takeuchi (2011).

LEMMA 3.2. Assume a multiplicative representation  $w(p)U(x)$ . Then

$$q = m(p,r) \Leftrightarrow w(q) = \sqrt{w(p)w(r)} .$$

$\square$





### 3.2. Compound invariance and the hexagon condition

We will examine the following family of weighting functions (Prelec 1998), as in Eq. 2.5:

DEFINITION 3.3.  $w(p)$  is *compound invariant* if there exist  $\alpha > 0$  and  $\beta > 0$  such that  $w(p) = (\exp(-(-\ln p)^\alpha))^\beta$ .  $\square$

We consider the following weakening of Prelec's (1998) compound invariance preference condition. For  $\lambda > 0$ ,  $\lambda$ -*compound invariance* holds if

$$\begin{aligned} (x,p) \sim (y,q) \quad \& \quad (y,p^\lambda) \sim (z,q^\lambda) \quad \& \\ (x,q) \sim (y,r) & \\ \Rightarrow (y,q^\lambda) \sim (z,r^\lambda) & \end{aligned} \quad (3.5)$$

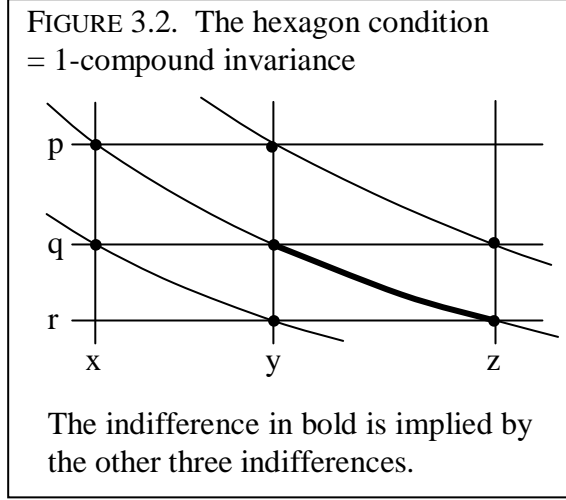
for all nonzero outcomes  $x$ ,  $y$ , and  $z$  and nonzero probabilities  $p$ ,  $q$ , and  $r$  for which also  $p^\lambda$ ,  $q^\lambda$ , and  $r^\lambda$  are contained in  $[0,1]$ .<sup>6</sup> To interpret the condition, we display 1-compound invariance (Figure 3.2):

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<sup>6</sup>  $\lambda$ -compound invariance is weaker than Prelec's (1998) compound invariance in two respects. First, compound invariance requires Eq. 3.5 for all natural numbers  $\lambda$ , whereas we will impose it only for particular  $\lambda$ . Second, in the notation of §2, we restrict attention to the case of  $x' = y$  and  $rp = q$ . This subpart of compound invariance allows for an interpretation in terms of midpoints presented later.

$$\begin{aligned}
& (x,p) \sim (y,q) \quad \& \quad (y,p) \sim (z,q) \quad \& \\
& (x,q) \sim (y,r) \\
& \qquad \qquad \qquad \Rightarrow \quad (y,q) \sim (z,r) \qquad \qquad \qquad (3.6)
\end{aligned}$$

for all nonzero outcomes  $x$ ,  $y$ , and  $z$  and nonzero probabilities  $p$ ,  $q$ , and  $r$ .



1-compound invariance is identical to the *hexagon condition* for nonzero outcomes and probabilities. The hexagon condition (Karni & Safra 1998) is a consistency requirement for endogenous midpoints. The left two indifferences in Eq. 3.6 imply that the midpoint of  $p$  and  $r$  is  $q$  when the gauges are  $x$  and  $y$ . The right two indifferences confirm this for gauges  $y$  and  $z$ . Hence,  $\lambda$ -compound invariance (Eq. 3.5) implies that endogenous midpoints are preserved under  $\lambda$ -power taking. The following lemma expresses this implication in terms of  $w$ .

LEMMA 3.4 [Invariance of geometric  $w$ -midpoints under  $\lambda$ -th power]. Under multiplicative representation,  $\lambda$ -compound invariance (Eq. 3.5) is equivalent to

$$w(q) = \sqrt{w(p)w(r)} \Rightarrow w(q^\lambda) = \sqrt{w(p^\lambda)w(r^\lambda)}. \quad (3.7)$$

□

### 3.3. Multiplicative representation and the main result

The hexagon condition is necessary and sufficient for the existence of an additive representation  $Z(x) + W(p)$  under common regularity conditions (Karni & Safra 1998). That is, the hexagon condition is the two-dimensional analog of separability. At this stage, additive representability  $Z(x) + W(p)$  amounts to multiplicative representability

when excluding outcome 0 and probability 0: By taking the exponent of  $Z(x) + W(p)$  we obtain a positive multiplicative representation  $e^{Z(x)+W(p)} = e^{Z(x)}e^{W(p)}$ . We then define  $w(p) = e^{W(p)}$  and  $U(x) = e^{Z(x)}$ . By assigning 0 utility to the 0 outcome and 0 weight to the 0 probability, respectively, we obtain the multiplicative representation on the whole of  $X \times [0,1]$ .

OBSERVATION 3.5. The following two statements are equivalent for  $\succsim$  on  $X \times [0,1]$ :

- (i) A multiplicative representation  $w(p)U(x)$  (Definition 3.1) exists.
- (ii)  $\succsim$  is a continuous weak order that satisfies strict stochastic dominance and 1-compound invariance (which is the hexagon condition for nonzero outcomes and probabilities), and  $(0,p) \sim (x,0) \sim (0,0)$  for all  $p,x$ .

In Statement (i),  $w$  and  $U$  are unique up to separate positive factors and a joint positive power. This implies that we can take a positive factor for  $w$  such that  $w(1) = 1$ .  $\square$

The multiplicative representation need not be presupposed when analyzing compound invariance because it is automatically implied. It leads to the following efficient preference foundation of prospect theory with compound invariant probability weighting for single nonzero outcomes.

THEOREM 3.6. The following two statements are equivalent for the binary relation  $\succsim$  on prospects  $(x,p)$  from  $X \times [0,1]$  with  $X \subset \mathbb{R}^+$  a nonpoint interval containing 0:

- (i) A multiplicative representation  $w(p)U(x)$  (Definition 3.1) exists with  $w$  from the compound invariance family (Definition 3.3).
- (ii)  $\succsim$  satisfies: weak ordering, continuity, strict stochastic dominance,  $\lambda$ -compound invariance for  $\lambda = 1, 2, \text{ and } 3$ , and  $(0,p) \sim (x,0) \sim (0,0)$  for all  $p,x$ .

In Statement (i),  $U$  and  $w$  are unique up to a joint positive power, and  $U$  is unique up to a separate positive factor. This implies that  $\alpha$  in Eq. 2.5 is uniquely determined from preference, but that  $\beta > 0$  there can be chosen arbitrarily (jointly with the power of  $U$ ). In (ii),  $\lambda$ -compound invariance can be imposed for all  $\lambda > 0$ .  $\square$

In the compound invariant representation,  $w(0)=0$  and  $w(1)=1$  are automatically satisfied.

## 4. Invariant discount functions for multiplicative intertemporal preference representations

### 4.1. Elementary definitions for multiplicative representations

(Intertemporal) prospects  $(x,t)$  yield an outcome  $x$  at time point  $t$  and outcome 0 at all other time points.  $X \times T$  is the set of all prospects considered, with both  $X$  and  $T$  nonpoint intervals within  $\mathbb{R}^+$  and  $X$  containing 0.<sup>7</sup>  $\succcurlyeq$ ,  $\succ$ ,  $\sim$ ,  $\preccurlyeq$ , and  $<$  are as before. We have  $(0,s) \sim (0,t)$  for all  $s,t$ . That is, we identify all  $(0,t)$  (never receiving anything). Time monotonicity holds if  $[(x,s) \succ (x,t)$  whenever  $x > 0$  and  $s < t]$  and  $[(x,t) \succ (y,t)$  whenever  $x > y]$  (the latter also holds if  $t=0$ ). Thus the decision maker is *impatient*: outcomes received sooner are more desirable and the monotonicity regarding time reverses the natural ordering, and  $t=0$ , if contained in  $T$ , is most preferred. Impatience is constant [decreasing, increasing] if  $(x,s) \sim (y,t) \Leftrightarrow (x,s+\varepsilon) \sim (y,t+\varepsilon)$  [ $(x,s+\varepsilon) < (y,t+\varepsilon)$ ,  $(x,s+\varepsilon) > (y,t+\varepsilon)$ ] for all outcomes  $x,y$ ,  $\varepsilon > 0$  and time points  $s, t$  such that  $s+\varepsilon$  and  $t+\varepsilon$  are also in  $T$ . Representing functions  $V$  are defined as before.

DEFINITION 4.1.  $V$  is *multiplicative* if there exist functions  $f: T \rightarrow \mathbb{R}$  and  $U: X \rightarrow \mathbb{R}$  such that:

$$V(x,t) = f(t)U(x). \quad (4.1)$$

$$U(0)=0 \text{ and } U \text{ is continuous and strictly increasing.} \quad (4.2)$$

$$f \text{ is continuous, strictly decreasing, and positive.} \quad (4.3)$$

We again call  $U$  the *utility function*, and  $f$  is the *discount function*.  $\square$

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<sup>7</sup> All following results remain true if we only assume about  $X$  that, with  $U$  defined below,  $U(X) = (0,\varepsilon)$  for an  $\varepsilon > 0$  so that it does not contain 0. This assumption was made by Bleichrodt, Rohde, & Wakker (2009).

These conditions imply time-monotonicity. In what follows, we can always normalize  $f(0) = 1$ .

## 4.2. Constant relative decreasing impatience and the hexagon condition

The most popular discount functions today are the quasi-hyperbolic family (Laibson 1997; Phelps & Pollak 1968) and the generalized hyperbolic family (Loewenstein & Prelec 1992). These have the drawback that they can only accommodate decreasing impatience. It is desirable to have more flexible families, especially for data fitting at the individual level, because there are always some individuals who will exhibit increasing impatience (Epper, Fehr-Duda, and Bruhin 2011). The following family can serve this purpose, extending Eq. 2.8 by incorporating cases where  $t=0$  is not contained in the time domain  $T$ .

DEFINITION 4.2.  $w(t)$  is *unit invariant* if there exist  $k > 0$ ,  $r > 0$ , and  $c \in \mathbb{R}$  such that

$$\text{for } c < 0, w(t) = ke^{rt^c} \text{ (only if } 0 \notin T); \quad (4.4)$$

$$\text{for } c = 0, w(t) = kt^{-r} \text{ (only if } 0 \notin T); \quad (4.5)$$

$$\text{for } c > 0, w(t) = ke^{-rt^c}. \quad (4.6)$$

□

The unit invariance family results from taking the exponent of the well known CRRI (logpower) functions. It appeared as the conditional invariance family in Prelec (1998 Eq. 4.2) for probability weighting<sup>8</sup>, and is characterized by the following preference invariance condition:  $\lambda$ -unit invariance holds if

$$\begin{aligned} (x,r) \sim (y,s) \quad \& \quad (y,\lambda r) \sim (z,\lambda s) \quad \& \\ (x,s) \sim (y,t) & \\ \Rightarrow (y,\lambda s) \sim (z,\lambda t) & \end{aligned} \quad (4.7)$$

for all nonzero outcomes  $x$ ,  $y$ , and  $z$ , and time-points  $r$ ,  $s$ , and  $t$  for which also  $\lambda r$ ,  $\lambda s$ , and  $\lambda t$  are contained in  $T$ . Thus, if  $r$  is to  $s$  what  $s$  is to  $t$ , then this relationship is

maintained under a change of time unit. The condition is weaker than Prelec's (1998) conditional invariance<sup>9</sup> because it is imposed on an arbitrary subinterval  $T$  of  $\mathbb{R}^+$  rather than on  $[0,1]$ . It is also weaker than Bleichrodt, Rohde, & Wakker's (2009) condition because they imposed it for all positive  $\lambda$ , whereas we will impose it only for  $0 < \lambda \leq 1$ . Contrarily to the previous section where we assumed the whole probability domain  $[0,1]$ , we assume a general time interval  $T$  here rather than the whole time axis  $[0,\infty)$ . We do this to generalize Bleichrodt, Rohde, & Wakker (2009), who also assumed a general time interval  $T$ . It will imply that we cannot restrict  $\lambda$  to three values as we could in the previous section.

### 4.3. Multiplicative representation, midpoint interpretation, and the main result

Like 1-compound invariance, 1-unit invariance is equivalent to the hexagon condition for all nonzero outcomes. Contrarily to 1-compound invariance where probability 0 was excluded, all timepoints  $t \in T$ , also  $t=0$ , are included.

OBSERVATION 4.2. The following two statements are equivalent for  $\succsim$  on  $X \times T$ :

- (i) A multiplicative representation  $f(t)U(x)$  (Definition 4.1) exists.
- (ii)  $(0,s) \sim (0,t)$  for all  $s,t$ , and  $\succsim$  is a continuous weak order that satisfies time-monotonicity and 1-unit invariance (the hexagon condition for nonzero outcomes).

In Statement (i),  $f$  and  $U$  are unique up to separate positive factors and a joint power. This implies that we can take a positive factor for  $f$  such that  $f(0) = 1$ .  $\square$

$\lambda$ -unit invariance (Eq. 4.7) can be interpreted as invariance of endogenous midpoints under multiplication by  $\lambda$ . The following lemma expresses this idea in terms of  $f$ .

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<sup>8</sup> Because it was applied to decision under risk, the family was increasing rather than decreasing ( $r < 0$ ) in Prelec (1998).

<sup>9</sup> Prelec assumed the condition only for all  $0 < \lambda < 1$  (whereas we will impose it also for  $\lambda = 1$ ), but then the condition for  $\lambda = 1$  follows from continuity. We will ignore this difference in what follows. Further, in Prelec's notation, we only consider the case of  $r = q$  and  $y = x'$ . Ebert & Prelec (2007) derived Eq. 4.6 from a stronger condition, *scale invariance*, which states (in preference terms) that if  $r$  is to  $s$  what  $q$  is to  $t$ , then this relationship is maintained under a change of time unit on the interval  $(0,1]$ .

LEMMA 4.3 [Invariance of geometric w-midpoints under  $\lambda$ -th power]. Under multiplicative representation,  $\lambda$ -unit invariance is equivalent to

$$f(s) = \sqrt{f(t) f(r)} \Rightarrow f(\lambda s) = \sqrt{f(\lambda t) f(\lambda r)}. \quad (4.8)$$

□

We now obtain the following preference foundation of discounted utility with unit invariant discounting.

THEOREM 4.4. The following two statements are equivalent for the binary relation  $\succsim$  on prospects  $(x,t)$  from  $X \times T$  with  $X$  and  $T$  nonpoint subintervals of  $\mathbb{R}^+$ :

- (i) A multiplicative representation  $f(t)U(x)$  (Definition 4.1) exists with  $f$  from the unit invariance family (Definition 4.2).
- (ii)  $(0,s) \sim (0,t)$  for all  $s,t$ , and  $\succsim$  satisfies: weak ordering, continuity, time-monotonicity, and  $\lambda$ -unit invariance for  $0 < \lambda \leq 1$  (Eq. 4.7).

In Statement (i),  $f$  and  $U$  are unique up to separate positive factors and a joint power. This implies that the parameter  $c$  in Eqs. 4.4-4.6 is uniquely determined from preference, but that  $k$  can be chosen arbitrarily, and so can  $r > 0$  (jointly with the power of  $U$ ). In (ii),  $\lambda$ -unit invariance can be imposed for all  $\lambda > 0$ . □

A necessary and sufficient condition for constant discounting is that  $m(p,r) = (p+r)/2$  (Fishburn & Rubinstein 1982). Prelec (1998) and Bleichrodt, Rohde, & Wakker (2009) also considered a CADI family for  $f$ , where endogenous midpoints are invariant under addition of a constant. This condition also implies the hexagon condition (let the constant added be 0), and the above results can be extended to this case. However, the CADI family seems to fit data for intertemporal choice worse than the unit invariance family (Abdellaoui, Bleichrodt, & l'Haridon 2013) and we will not give the extension in this paper.

## 5. Implications for richer domains

Because virtually all existing theories for risk agree on our domain of one nonzero outcome, and differ only in their extensions to multiple outcomes, our results are relevant for all those theories.<sup>10</sup> Most studies of intertemporal choice have focused on the same domain as we considered. However, for decision under risk, richer domains with more nonzero outcomes have usually been considered. Our domain suffices to determine the utility of outcomes, and the weighting of probabilities, up to a common power. Thus, for rank-dependent utility (which is prospect theory for gains) the preferences considered in this paper determine the preferences over the whole domain up to this one parameter. The common power can be inferred from one indifference involving more than one nonzero outcome. An example is the certainty equivalence  $y \sim (p:x, 1-p:z)$ . This indifference can be chosen carefully so as to give maximally reliable results. Because prospects with one nonzero outcome are easy to understand, restricting most measurements to this domain will enhance reliability.

Baucells & Heukamp (2012) introduced a model where both time and risk are present. In their model prospects are triples  $(x,p,t)$ , meaning that outcome  $x$  is received at time  $t$  with probability  $p$ . They axiomatize a form

$$w(pe^{-rx})U(x), \quad (5.1)$$

based on a separable treatment of probability and time, aggregated into psychological distance, at each fixed level of outcome (their Axiom A3) and a separable treatment of probability and outcomes at time  $t=0$  (their Axiom A5). In their §6 they propose a subfamily that is promising for empirical purposes:

$$e^{(-(-\ln p+rx))^\alpha}U(x). \quad (5.2)$$

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<sup>10</sup> Other models include Allais (1953; several models), Birnbaum's (2008) RAM and TAX models, Edwards (1962), Kahneman & Tversky (1979), Miyamoto (1988), Quiggin (1982), Tversky & Kahneman (1992), and Viscusi (1989). Some betweenness models are also included: Gul (1991) and Routledge & Zin (2010). A betweenness model not included is Chew's weighted utility (Chew & Tan 2005; Chew & Waller 1986). Also excluded are quadratic forms (Chew, Epstein, & Segal 1991; Machina 1982 Eq. 6, see Example 3.7). Obviously, theories that violate transitivity, such as regret theory (Loomes and Sugden 1982), are also excluded. A direct intransitive generalization of Eq. 2.1 is in Dubra (2009) and Ok & Masatlioglu (2007).



In Eq. 5.2, both probability (satisfying compound invariance) and time (satisfying unit invariance, generalized in an outcome dependent manner) are evaluated as in our Theorems 3.6 and 4.4. We can, therefore, immediately axiomatize Baucells and Heukamp's proposal by replacing their central axiom (their A5) on tradeoffs between probabilities and outcomes at time 0 by compound invariance.<sup>11</sup> We can similarly apply our techniques to the tradeoffs between time and probability.<sup>12</sup> Further extensions of our results to general multiattribute settings, and other implications of our results for richer domains, are left to future studies.

We have derived theorems for pairs  $(x,p)$  or  $(x,t)$  where we focused on functional forms evaluating the second coordinate as probability or time. One could reinterpret the second coordinate as outcome (in some cases to be properly normalized) and the first coordinate as time or probability (or something else), again, after proper rescaling. Then one would obtain similar representations as provided in this paper, but obtaining specific forms for utility functions. We did not follow this route because the functional forms obtained are of primary interest for probability and time, and not for utility. But results in the same spirit may be conceivable for alternative functional forms that are of interest for utilities, and this is a topic for future research. Of course, we could also impose our axioms on both dimensions simultaneously, restricting both utility and discounting or probability weighting.

## 6. Conclusion

Using an endogenous midpoint operation for probability and time perception, we have clarified the empirical and logical status of popular psychophysical preference conditions, including Prelec's (1998) compound invariance for risk and unit invariance for time. These conditions for probability and time imply a multiplicative representation in which outcomes can be scaled independently from risk or time. All

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<sup>11</sup> It implies their A5 and additionally imposes the desired functional forms. In particular, because of their A3, it also implies what both those authors and we consider to be the most plausible family for intertemporal discounting.

<sup>12</sup> We then replace their axiom A3 by, for instance, unit invariance and thus generalize the linear treatment of probability.

previous papers imposed these conditions on top of underlying decision models. This paper has shown that the conditions by themselves imply the underlying decision models, so that assuming these models separately was redundant.

## Appendix A. Examples

EXAMPLE A.1 [Multiplicative model with hexagon condition and compound invariance satisfied, implying that endogenous midpoints are consistent and invariant under power taking]. Assume Eqs. 2.3 and 2.5 with  $\alpha = 0.65$ ,  $\beta = 1$ , and  $U(x) = x^{0.88}$ . We discuss the following preferences and indifferences, which follow from substitution of the preference functional:<sup>13</sup>

$$\begin{aligned} (30, 0.90) \sim (70, 0.38) & \quad (70, 0.90) \sim (163, 0.38) & \quad (10, 0.729) \sim (57, 0.055) \\ (30, 0.38) \sim (70, 0.10) & & \quad (10, 0.055) \sim (57, 0.001) \\ & \quad (70, 0.38) \sim (163, 0.10). \end{aligned}$$

The left two indifferences use the gauge outcomes 30 and 70 to elicit that 0.38 is the endogenous midpoint between 0.10 and 0.90. The middle two indifferences use different gauge outcomes, 70 and 163, but confirm the elicitation. Consequently, the left four indifferences corroborate the hexagon condition. The right two indifferences use third powers of probabilities,  $0.90^3 = 0.729$ ,  $0.38^3 = 0.055$ , and  $0.1^3 = 0.001$ . They show that  $0.055 = m(0.729, 0.001)$  (modulo rounding), implying that endogenous midpoints are invariant with respect to third power taking. This corroborates compound invariance.  $\square$

EXAMPLE A.2 [Multiplicative model with hexagon condition satisfied but compound invariance violated; endogenous midpoints are consistent but they are not invariant under power taking]. Assume Eq. 2.3 with  $w(p) = \frac{p^{0.61}}{(p^{0.61} + (1-p)^{0.61})^{1/0.61}}$  and  $U(x) = x^{0.88}$ , the values found by Tversky & Kahneman (1992). We discuss the following preferences and indifferences:

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<sup>13</sup> Below, indifference probabilities are rounded by two digits.

$$\begin{array}{lll}
(30, 0.90) \sim (64, 0.39) & (64, 0.90) \sim (137, 0.39) & (10, 0.729) \sim (79, 0.024) \\
(30, 0.39) \sim (64, 0.10) & & (10, 0.024) \sim (79, 0.001) \\
& & (64, 0.39) \sim (137, 0.10).
\end{array}$$

The left two indifferences use the gauge outcomes 30 and 64 to elicit  $0.39 = m(0.10, 0.90)$ . The middle two indifferences use different gauge outcomes, 64 and 137, but confirm the elicitation. Consequently, the left four indifferences corroborate the hexagon condition. The right two indifferences test compound invariance, by considering third powers of probabilities. Compound invariance is violated, because  $0.0586 = 0.39^3 \neq m(0.90^3, 0.1^3) = m(0.729, 0.001)$ . Instead, that midpoint is 0.024, which is considerably smaller. Thus this multiplicative decision model does satisfy the hexagon condition, but not compound invariance.  $\square$

EXAMPLE A.3 [No multiplicative model; the hexagon condition is violated; no consistent endogenous midpoints; compound invariance is violated a fortiori].

Machina (1982, Eq. 6) proposed a representation  $EU_1 + (EU_2)^2/2$ , with  $EU_2$  a different expected utility model than  $EU_1$ , suggesting that this was about the simplest deviation from expected utility conceivable. We use this model with expected value ( $U_1(x) = x$ ) for  $EU_1$  and with utility function  $5x^{1/5}$  for  $EU_2$  (the multiplication of the function by 5 constitutes the most common scaling). We discuss the following preferences and indifferences, which follow from substitution:<sup>14</sup>

$$\begin{array}{lll}
(10, 0.90) \sim (71, 0.36) & (71, 0.90) \sim (289, 0.36) & (71, 0.90) \sim (310, 0.34) \\
(10, 0.36) \sim (71, 0.10) & & (71, 0.34) \sim (310, 0.10) \\
& & (71, 0.36) \succ (289, 0.10).
\end{array}$$

The left two indifferences use the gauge outcomes 10 and 71 to suggest that  $0.36 = m(0.90, 0.10)$ . The left four indifferences, however, show that the hexagon condition is violated. It confirms that the decision model is not multiplicative, and cannot be ordinally transformed into a multiplicative form.

The middle two indifferences suggest that 0.36 is the endogenous midpoint of 0.90 and a probability exceeding 0.10. Hence endogenous midpoint measurements

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<sup>14</sup> As throughout, indifference probabilities are rounded by two digits. In the strict preference, the probabilities differ by more than 0.01 from the indifference probabilities, with even  $(71, 0.35) \succ (289, 0.11)$ .

will run into inconsistencies. The right two indifferences show that taking the gauge outcomes 71 and 310 suggests that  $0.34 = m(0.10, 0.90)$ , which is inconsistent with the endogenous midpoint 0.36 suggested by the left two indifferences.  $\square$

## Appendix B. Some proofs for Section 3 and preparatory results

PROOF OF LEMMA 3.2. The left two indifferences in Eq. 3.5 imply  $w(p)/w(q) = U(y)/U(x) = w(q)/w(r)$ , and then  $w(q) = \sqrt{w(p)w(r)}$ . Next assume the latter equality. We show that  $q = m(p,r)$ , by finding appropriate gauges. Because  $U(0) = 0$  and  $U$  is continuous and strictly increasing, there exist nonzero outcomes  $x$  and  $y$  with  $w(p)/w(q) = U(y)/U(x)$ , implying  $(x,p) \sim (y,q)$ . We also have  $w(q)/w(r) = U(y)/U(x)$ , implying  $(x,q) \sim (y,r)$ . The two indifferences imply that  $q = m(p,r)$ . A similar way to derive geometric  $w$  midpoints from preferences can be recognized in Zank (2010, Proof of Lemma 3).  $\square$

PROOF OF LEMMA 3.4. By substitution, the implication in the lemma implies  $\lambda$ -compound invariance. Next assume  $\lambda$ -compound invariance, and  $w(q) = \sqrt{w(p)w(r)}$ . The  $w$  condition is vacuously satisfied if one of the probabilities is 0. Hence we assume that all probabilities are positive. Assume  $0 < p < q < r$  (the other case,  $0 < r < q < p$ , is similar). We take any  $x > 0$ , and find, by continuity (and  $U(0) = 0$ ), smaller outcomes  $y > z > 0$  such that the upper two indifferences in Eq. 3.5 hold. The assumed  $w$  equality implies the left lower indifference in Eq. 3.5. The right two indifferences, where the last one is implied by the compound invariance preference condition, imply  $w(q^\lambda) = \sqrt{w(p^\lambda)w(r^\lambda)}$ .  $\square$

PROOF OF OBSERVATION 3.5. The implication (i)  $\Rightarrow$  (ii) is left to the reader. In the rest of this proof, we assume (ii) and derive (i) and the uniqueness results. We first restrict attention to positive probabilities and outcomes. By Wakker (1989 Theorem III.6.6),

there exist continuous functions  $W$  and  $Z$  such that  $Z(x) + W(p)$  represents preferences.<sup>15</sup> By strict stochastic dominance,  $W$  and  $Z$  are strictly increasing.

Defining  $w(p) = e^{W(p)}$  and  $U(x) = e^{Z(p)}$  leads to the multiplicative representation  $U(x)w(p)$  for all lotteries except  $(0,0)$ . To ensure that we can take  $w(0) = 0$ , we take any  $x > y > 0$ . We take any probability  $1 > q_1 > 0$  and define probabilities  $q_i$  inductively: By continuity of preference and  $(x,0) < (y,q_i) < (x,q_i)$ , there exists a  $0 < q_{i+1} < q_i$  such that  $(x,q_{i+1}) \sim (y,q_i)$ , so that  $w(q_{i+1})/w(q_i) = U(y)/U(x)$ . The latter ratio is a constant  $r < 1$ . Hence  $w(q_i) \leq r^{i-1}$ , which tends to 0. We get  $\lim_{p \downarrow 0} w(p) = 0$ . We define  $w(0) = 0$ , and  $w$  is continuous. A dual reasoning shows that  $\lim_{x \downarrow 0} U(x) = 0$ . We define  $U(0) = 0$ , and  $U$  is continuous. The function  $(x,p) \mapsto w(p)U(x)$  represents preferences also for  $p = 0$  and  $x = 0$ , because  $(0,0)$  is the worst prospect and it also has the lowest evaluation. The representation  $w(p)U(x)$  satisfies all requirements.

For uniqueness, we first consider the domain of positive outcomes and probabilities. Wakker (1989 Observation III.6.6') showed that the additive representation  $Z(x) + W(p) = \ln(U(x)) + \ln(w(p))$  is unique up to location and common unit. It implies that  $U(x)$  and  $w(p)$  are unique up to unit and a joint power. For zero outcomes and probabilities, we have  $w(0) = 0$  and  $U(0) = 0$ , which does not affect the uniqueness result obtained.  $\square$

**LEMMA B.1.** Under multiplicative representation, for each  $\lambda > 0$ ,  $\lambda$ -compound invariance implies  $1/\lambda$ -compound invariance.

**PROOF.** We use Lemma 3.4. We show that  $w(q^{1/\lambda}) \neq \sqrt{w(p^{1/\lambda})w(r^{1/\lambda})}$  implies  $w(q) \neq \sqrt{w(p)w(r)}$ . Assume the former inequality. Say  $w(q^{1/\lambda}) > \sqrt{w(p^{1/\lambda})w(r^{1/\lambda})}$ . Assume  $0 < p < q < r$  (the case  $0 < r < q < p$  is similar). By continuity, we can increase  $p$  into  $\tilde{p}$  with  $q > \tilde{p} > p$  such that  $w(q^{1/\lambda}) = \sqrt{w(\tilde{p}^{1/\lambda})w(r^{1/\lambda})}$ .  $\lambda$ -compound invariance and Lemma 3.4 imply  $w(q) = \sqrt{w(\tilde{p})w(r)}$ , implying  $w(q) > \sqrt{w(p)w(r)}$ . Similarly,  $w(q^{1/\lambda}) < \sqrt{w(p^{1/\lambda})w(r^{1/\lambda})}$  implies  $w(q) < \sqrt{w(p)w(r)}$  by decreasing  $r$ .  $\square$

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<sup>15</sup> Given that we only consider nonzero probabilities and outcomes, Wakker's (1989, Theorem III.6.6) coordinate independence is implied by strict stochastic dominance.

LEMMA B.2. Under multiplicative representation,  $\lambda$ -compound invariance for  $\lambda = 1, 2,$  and  $3$  implies the condition for all  $\lambda > 0$ .

PROOF. Lemma B.1, and repeated application of Lemmas 3.2 and 3.4, implies that  $\lambda$ -compound invariance holds for all  $\lambda = 2^k 3^\ell$  with  $k$  and  $\ell$  integers. This set is dense in  $\mathbb{R}^+$ . By continuity of  $w(p^\lambda)$  in  $\lambda$  and Lemma 3.4,  $\lambda$ -compound invariance holds for all positive  $\lambda$ .  $\square$

## Appendix C. Preparatory results and some proofs for Section 4

PROOF OF OBSERVATION 4.2. This is virtually the same as the proof of Observation 3.5. One difference is that  $f$  is strictly decreasing, corresponding with the reversal in monotonicity. The other difference is that  $t=0$  is not degenerate unlike  $p=0$  so that  $t=0$  is part of the domain of  $W$  in the additive representation, and its  $f$  value is positive.  $\square$

PROOF OF LEMMA 4.3. By substitution, the implication in the lemma implies  $\lambda$ -unit invariance. Next assume  $\lambda$ -unit invariance, and  $f(s) = \sqrt{f(t)f(r)}$ . Assume  $0 < t < s < r$  (the other case,  $0 < r < s < t$ , is similar). We take any  $x > 0$ , and find, by continuity (and  $U(0)=0$ ), smaller  $y > z > 0$  such that the upper two indifferences in Eq. 4.7 hold. The assumed  $f$  equality implies the left lower indifference in Eq. 4.7. The right two indifferences, where the last one is implied by the unit invariance preference condition, imply  $f(\lambda s) = \sqrt{f(\lambda t)f(\lambda r)}$ .  $\square$

LEMMA C.1. Under multiplicative representation, for each  $\lambda > 0$ ,  $\lambda$ -unit invariance implies  $1/\lambda$ -unit invariance.

PROOF. We use Lemma 4.3. We show that  $f(s/\lambda) \neq \sqrt{f(t/\lambda)f(r/\lambda)}$  implies  $f(s) \neq \sqrt{f(t)f(r)}$ . Assume the former inequality. Say  $f(s/\lambda) < \sqrt{f(t/\lambda)f(r/\lambda)}$ . Assume  $0 < t < s < r$

(the case  $0 < r < s < t$  is similar). By continuity, we can replace  $t$  by  $\tilde{t} > t$  with  $s > \tilde{t} > t$  such that  $f(s/\lambda) = \sqrt{f(\tilde{t}/\lambda)f(r/\lambda)}$ .  $\lambda$ -unit invariance and Lemma 4.3 imply  $f(s) = \sqrt{f(\tilde{t})f(r)}$ , implying  $f(s) < \sqrt{f(t)f(r)}$ . Similarly,  $f(s/\lambda) > \sqrt{f(t/\lambda)f(r/\lambda)}$  implies  $f(s) > \sqrt{f(t)f(r)}$  by replacing  $r$  by  $\tilde{r} < r$  to give equality, and so on. QED

The following result immediately follows from Lemma C.1.

COROLLARY C.2. Under multiplicative representation,  $\lambda$ -compound invariance for  $0 < \lambda \leq 1$  implies it for all  $\lambda > 0$ .  $\square$

## Appendix D. Proofs using useful functional equations and their relations

This appendix presents some functional equations and shows how they can be used to prove our results. We present this part of the proof in a separate appendix because the functional equations that we give, and their relations and applications, are of interest on their own. They are considerably simpler and more accessible than many related proofs published before. For an alternative general technique for deriving similar functional equations, see Harvey (1990).

We show in particular that a number of well-known characterizations of functional forms, derived independently in the literature, all amount to the same mathematical theorem. They can all immediately be obtained from one another through simple substitutions of variables, and all follow from Figure D.1 below. They are discussed in Remark D.2.

Figure D.1 depicts the functional equations and their relations. In the figure, each of the six cases within dashed rectangles presents a logical equivalence, regarding functions  $f: F \rightarrow \mathbb{R}$  or  $g: G \rightarrow \mathbb{R}$ , where  $F$  and  $G$  can be any interval as described, and with wiggles or bars added as appropriate. Throughout, we follow the convention that

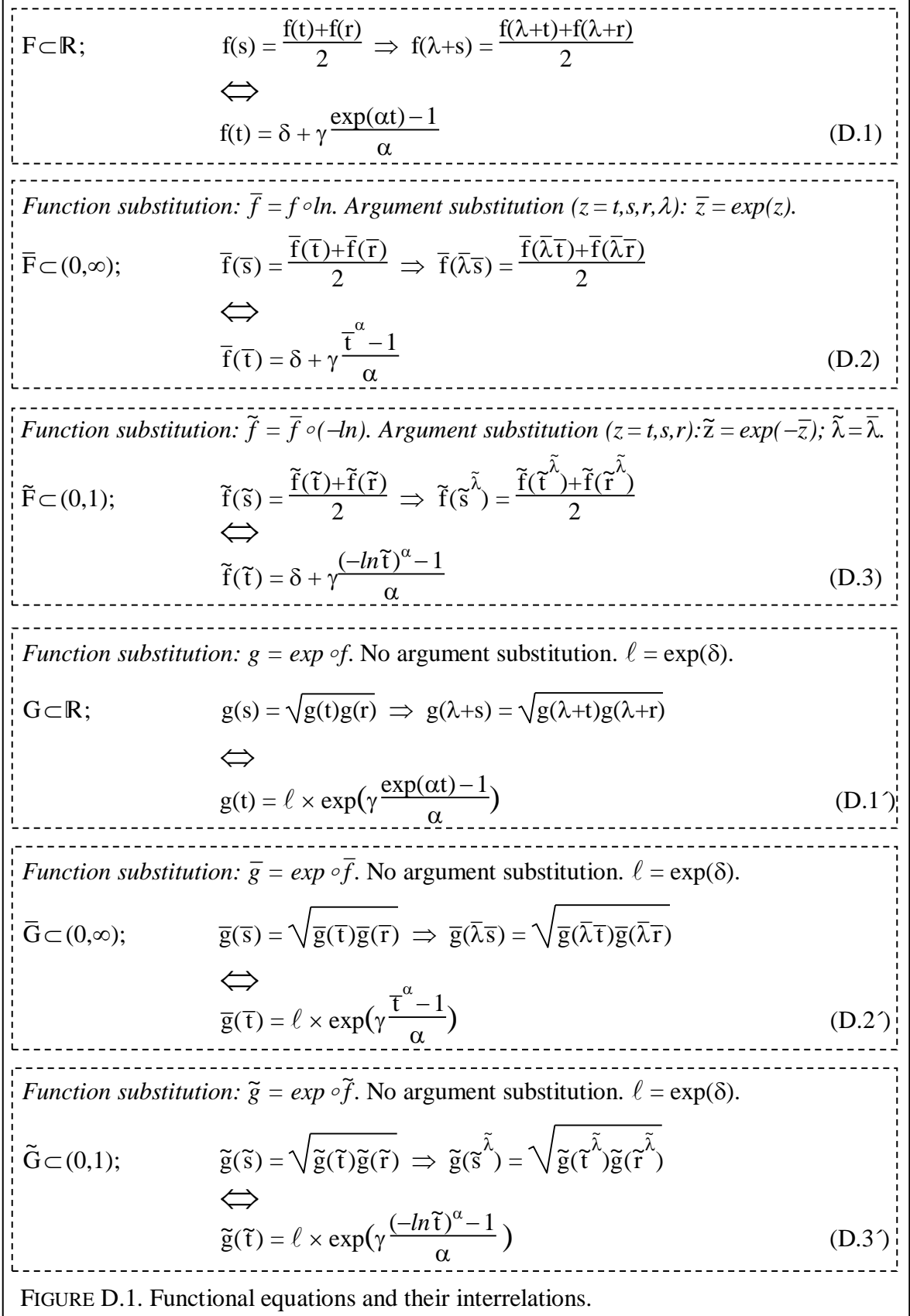
for  $\alpha=0$ ,  $\frac{\exp(\alpha t)-1}{\alpha} = \lim_{\alpha \rightarrow 0} \frac{\exp(\alpha t)-1}{\alpha} = t$ . It implies, by substitutions for  $t$ ,

$$\frac{t^\alpha - 1}{\alpha} = \ln t, \text{ and } \frac{(-\ln \tilde{t})^\alpha - 1}{\alpha} = \ln(-\ln \tilde{t}) \text{ for } \alpha=0.$$

Regarding quantifiers, the functional equations (the upper implications) should hold throughout for all  $s, t, r$ , and  $\lambda$  such that all arguments used in the  $f$  and  $g$  functions are contained in the domain ( $F$  or  $G$ ), again with wiggles and bars added as appropriate. For the forms of the functions  $f$  and  $g$  (the lower equations in all the dashed rectangles), there should exist real  $\alpha, \gamma$ , and  $\delta$  (or  $\ell = \exp(\delta) > 0$ ) and then it should hold for all arguments  $t$ .

The first result (Eq. D.1) gives the well known characterization of CARA (linear-exponential) utility. To see this point, interpret  $f$  as the utility function in expected utility. The premise equality in the upper implication entails that receiving  $s$  for sure is equally preferable as a 50-50 prospect of receiving  $t$  or  $r$ . The consequent equality entails that the indifference is preserved if a constant amount  $\lambda$  is added to all outcomes. This is constant absolute risk aversion for certainty equivalents of 50-50 prospects, which is enough to imply that  $f$  is exponential (CARA) (Aczél 1966 p. 153 Theorem 2; Harvey 1990 Theorem 1b; Miyamoto 1983 Lemma 1). All the other results in the figure immediately follow from the italicized substitutions described in the first lines of the dashed rectangles. The transformation for Eq. D.3 was used by Ebert & Prelec (2007, Appendix). The equations and their implications are discussed further in Remark D.2.





REMARK D.1. The following additional results hold for Figure D.1.

The functions  $f$ ,  $\bar{f}$ ,  $g$ , and  $\bar{g}$  are strictly increasing if and only if  $\gamma > 0$ , and strictly decreasing if and only if  $\gamma < 0$ . For the functions  $\tilde{f}$  and  $\tilde{g}$  it is reversed, and they are strictly increasing if and only if  $\gamma < 0$  and strictly decreasing if and only if  $\gamma > 0$ .

For  $\bar{f}$  (D.2) and  $\bar{g}$  (D.2') we can add 0 to the domain if and only if  $\alpha > 0$  (alternatively,  $\alpha \leq 0$  can be allowed if we allow for function values  $\infty$  or  $-\infty$  at  $t=0$ ). Correspondingly, for  $\tilde{f}$  (D.3) and  $\tilde{g}$  (D.3') we can add 1 to the domain if and only if  $\alpha > 0$  (alternatively,  $\alpha \leq 0$  can be allowed if we allow for function values  $\infty$  or  $-\infty$  at  $t=1$ ).

For  $\tilde{f}$  (D.3) we can add 0 to the domain if we allow for function values  $-\infty$  ( $\alpha > 0$ ) or  $\infty$  ( $\alpha \leq 0$ ) at  $t=0$ . Correspondingly, for  $\tilde{g}$  (D.3') we can add 0 to the domain if  $\alpha > 0$  (then  $\tilde{g}(0) = \exp(-\infty) = 0$ ) or if we allow for the value  $\infty$  ( $= \exp(\infty)$ ) at  $t=0$  ( $\alpha \leq 0$ ).  $\square$

REMARK D.2. We discuss related literature. Equations D.1-D.3' refer to Figure D.1, and all theorems cited below follow from Figure D.1:

- Eq. D.1: As explained above, this is related to CARA (constant absolute risk averse, or linear-exponential) utility functions. Axiomatizations include Aczél (1966 Theorem 3.1.3.2), Harvey (1990 Theorem 1b), Miyamoto (1983 Lemma 1), Nagumo (1930 p. 78 and proof), and Pfanzagl (1959 Theorem 3, p. 290).
- Eq. D.2: CRRA (constant relative risk averse, or log-power) utility functions. Now indifferences are preserved under multiplication by a positive factor. Axiomatizations include Hardy, Littlewood, & Pòlya (1934, Theorem 84), Aczél (1966 Theorem 3.1.3.2) and Miyamoto (1983 Lemma 1).
- Eq. D.3: Prelec's (1998) projection invariance, axiomatized in his Proposition 5.<sup>16</sup>
- Eq. D.1': Bleichrodt, Rohde, & Wakker's (2009) CADI (constant absolute decreasing impatience), axiomatized in their Theorem 5.3.

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<sup>16</sup> Note here that his expression  $1 - \alpha \ln(p)$  can be rewritten as  $-\alpha \ln(p/\exp(1-\alpha))$ . That is, it involves a rescaling of the argument  $p$ .

- Eq. D.2': Prelec's (1998) conditional invariance, axiomatized in his Proposition 4, and Bleichrodt, Rohde, & Wakker's (2009) CRDI (unit invariance), axiomatized in their Theorem 6.3.
- Eq. D.3': Compound invariant probability weighting. Axiomatizations include Luce (2001), Al-Nowaihi & Dhimi (2006), and Prelec (1998 Proposition 1).  $\square$

PROOF OF THEOREM 3.6 (derived from Eq. D.3'). Statement (ii) follows from Statement (i) through substitution. We, hence, assume Statement (ii) and derive Statement (i) and the uniqueness results. We will use Eq. D.3' and Remark D.1 for the case  $\tilde{G} = [0,1]$ . We assume the multiplicative representation of Observation 3.5, and turn to the shape of  $w(p) = \tilde{g}(\tilde{t})$ . By Lemma B.2,  $\lambda$ -compound invariance follows for all  $\lambda > 0$ . By Lemma 3.4,  $\lambda$ -compound invariance is equivalent to the upper implication in Eq. D.3'. This is equivalent to  $w = \tilde{g}$  being of the form in Eq. D.3'. To obtain Eq. 2.5 from Eq. D.3' we substitute, with  $p = \tilde{t}$ :

$$\ell = \exp\left(\frac{\gamma}{\alpha}\right); \beta = -\gamma/\alpha,$$

where further  $\alpha > 0$  so as to include the endpoints of the domain  $[0,1]$ , and  $\gamma < 0$  so as to have the function increasing. The uniqueness results at the end of the theorem follow from the uniqueness in Observation 3.5.  $\square$

PROOF OF THEOREM 4.4 (derived from Eq. D.2'). Statement (ii) follows from Statement (i) through substitution. We, hence, assume Statement (ii) and derive Statement (i) and the uniqueness results. We will use Eq. D.2' and Remark D.1 for a general  $T = \bar{G} \subset [0,\infty)$ . We assume the multiplicative representation of Observation 4.2 and turn to the shape of  $f(t) = \bar{g}(\bar{t})$ . By Lemma C.2,  $\lambda$ -unit invariance follows for all  $\lambda > 0$ . By Lemma 4.3,  $\lambda$ -unit invariance is equivalent to the upper implication in Eq. D.2'. This is equivalent to  $f = \bar{g}$  being of the form in Eq. D.2'.

To obtain Eqs. 4.4-4.6 from Eq. D.2' we substitute, besides  $t = \bar{t}$ , and with  $\gamma < 0$  so as to have the function decreasing:

$$\text{If } \alpha > 0 \text{ (} c > 0 \text{): } k = \ell \times \exp\left(\gamma \frac{-1}{\alpha}\right), r = -\gamma/\alpha, \text{ and } c = \alpha.$$

If  $\alpha = 0$  ( $c = 0$ ):  $k = \ell$ ,  $r = -\gamma$ , and  $c = \alpha$ .

If  $\alpha < 0$  ( $c < 0$ ):  $k = \ell \times \exp(\gamma \frac{-1}{\alpha})$ ,  $r = \gamma/\alpha$ , and  $c = \alpha$ .

The uniqueness results at the end of the theorem follow from the uniqueness in Observation 4.2.  $\square$

## Acknowledgment

Vitalie Spinu and two anonymous referees made many helpful comments.

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