Submitted to the 2016 IEEE American Control Conference, July 6-8, Boston, MA, USA

Analysis of Slow Convergence Regions in Adaptive Systems*

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Abstract— **We examine convergence properties of errors in a class of adaptive systems that corresponds to adaptive control of linear time-invariant plants with state variables accessible. We demonstrate the existence of a sticking region in the error space where the state errors move with a finite velocity independent of their magnitude. We show that these properties are also exhibited by adaptive systems with closed-loop reference models which have been demonstrated to exhibit improved transient performance as well as those that include an integral control in the inner-loop. Simulation and numerical studies are included to illustrate the size of this sticking region and its dependence on various system parameters.**

I. INTRODUCTION

Stability of adaptive systems corresponding to the control of linear time-invariant plants has been well documented in the literature, with the tracking error converging to zero for any reference input [5]. If in addition, conditions of persistence of excitation are met, these adaptive systems can be shown to be uniformly asymptotically stable (u.a.s.) as well. Recently, in [3], it was shown that for low order plants, these adaptive systems cannot be shown to be exponentially stable, and are at best u.a.s. In this paper, we extend this result and show that for general linear time-invariant plants, u.a.s. holds but not exponential stability. The most important implication of this property is the existence of a *sticking region* in the underlying error-state space where the trajectories move very slowly. This corresponds to places where the overall adaptive system is least robust. As a result, a precise characterization of this sticking region is important and is the main contribution of this paper.

We consider two different types of adaptive controllers, the first of which corresponds to the use of closed-loop reference models [2], [6] (denoted as CRM-adaptive systems), and the second corresponds to the use of integral control for command following [4] (denoted as IC-adaptive systems), and show that in both cases, a sticking region exists. The results are applicable to a general n^{th} order linear time-invariant plant, whose states are accessible. For ease of exposition, it is assumed that the plants are in control canonical form. Simulation results are provided to complement the theoretical derivations.

II. PROBLEM STATEMENT

We consider two classes of adaptive systems to demonstrate the region of slow convergence, the first of which is the CRM-adaptive system and second is the IC-adaptive system. In this section, we present the underlying adaptive systems and state the overall problem. Throughout the paper, it is assumed that the underlying reference input is bounded and smooth.

A. THE CRM-ADAPTIVE SYSTEM

The n^{th} order time-invariant plant differential equation is given by

$$\dot{x}(t) = Ax(t) + \mathbf{b}u(t) \tag{1}$$

where A is a constant $n \times n$ unknown matrix and $\mathbf{b}$ is a known vector of size n. Here $(A,\ \mathbf{b})$ is expressed in control canonical form with

$$A = \begin{bmatrix} 0 & & & \\ \vdots & & I & \\ & & & \\ a_1 & a_2 & \cdots & a_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{2}$$

A state variable feedback controller is defined by

$$u(t) = \Theta(t)x(t) + q^* r(t) \tag{3}$$

where $\Theta(t)$ is the time varying adaptive parameter updated as

$$\dot{\Theta}(t) = -\mathbf{b}^T P e(t) x^T(t). \tag{4}$$

Here $e(t) = x(t) - x_m(t)$ and $x_m(t)$ is the output of a reference model defined by

$$\dot{x}_m(t) = A_m x_m(t) + \mathbf{b}_m r(t) + Le(t) \tag{5}$$

where $\mathbf{b}_m = q^*\mathbf{b}$, q^* is a known scalar and L is a constant $n \times n$ feedback matrix which introduces a closed-loop in the reference model. If $L = 0$, then (5) represents the open-loop reference model, denoted as the ORM adaptive system. With the standard matching condition [5]

$$A + \mathbf{b}\Theta^* = A_m \tag{6}$$

satisfied, the error differential equation is defined by

$$\dot{e}(t) = \left[A_m - L\right] e(t) + \mathbf{b}\widetilde{\Theta}(t)x(t) \tag{7}$$

where $\widetilde{\Theta}(t) = \Theta(t) - \Theta^*$. If $\left[A_m - L\right]$ is Hurwitz, then a symmetric positive definite P exists that solves the well known Lyapunov equation

$$\left[A_m - L\right]^T P + P\left[A_m - L\right] = -Q_0 \tag{8}$$

where Q_0 is a symmetric positive definite matrix. It is well known that the error model in (7) and (4) can be shown to be globally stable at the origin $[0,0]$ and that [5]

$$\lim_{t\to\infty} e(t) = 0. \tag{9}$$

*This work is supported by the Boeing University Strategic Initiative.

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The goal in this paper is to characterize regions in the $[e, \widetilde{\Theta}]$ space where the speed of convergence is slow, i.e. identify the sticking region.

B. THE IC-ADAPTIVE SYSTEM

The n_p^{th} order time-invariant plant differential equation is given by

$$\dot{x}_p(t) = A_p x_p(t) + \mathbf{b}_p u(t) \tag{10}$$

where A_p is a constant $n_p \times n_p$ unknown matrix and $\mathbf{b}_p$ is a known vector of size n_p. The goal is to design a control input $u(t)$ such that the system output

$$y(t) = C_p x_p(t) \tag{11}$$

tracks a time-varying reference signal $r(t)$, where C_p is known and constant. An integral state e_{yI} is proposed as

$$e_{yI}(t) = \int_0^t [y(\tau) - r(\tau)] d\tau. \tag{12}$$

Augmenting (10) with the integrated output tracking error yields the n^{th} order extended plant differential equation given by

$$\dot{x}(t) = Ax(t) + \mathbf{b}u(t) + \mathbf{b}_m r(t) \tag{13}$$

where $x = [e_{yI} \; x_p^T]^T$, $n = n_p + 1$ and

$$A = \begin{bmatrix} 0 & C_p \\ 0_{n_p \times 1} & A_p \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0 \\ \mathbf{b}_p \end{bmatrix} \quad \mathbf{b}_m = \begin{bmatrix} -1 \\ 0_{n_p \times 1} \end{bmatrix}. \tag{14}$$

It is clear that if $C_p = [1 \; 0 \; \cdots \; 0]$ and $(A_p, \; \mathbf{b}_p)$ is represented in control canonical form, then $(A, \; \mathbf{b})$ in (14) is similar to (2) with $a_1 = 0$, as is the case considered in this paper. A state variable feedback controller is defined by

$$u(t) = \Theta(t) x(t) \tag{15}$$

where $\Theta(t)$ is updated as

$$\dot{\Theta}(t) = -\mathbf{b}^T P e(t) x^T(t). \tag{16}$$

Here $e(t)$ is defined as in Section II-A and $x_m(t)$ is the output of a n^{th} reference model defined by

$$\dot{x}_m(t) = A_m x_m(t) + \mathbf{b}_m r(t) + Le(t) \tag{17}$$

where L is again a constant $n \times n$ feedback matrix. The matching condition and error differential equation are given as in (6) and (7) respectively, and the existence of a positive definite P that solves (8) is also guaranteed for a Hurwitz $[A_m - L]$. Using the same arguments as in the CRM-adaptive system, here too we can show that $\lim_{t \to \infty} e(t) = 0$. With this it can be shown that the control goal of interest may be reached [4]. The objective of this paper is to characterize sticking regions in this IC-adaptive system, given by (13) through (17), in addition to those in the CRM-adaptive system.

C. SLOW CONVERGENCE ANALYSIS

From (4) and (16), it can be seen that the time varying adaptive gain $\Theta(t)$ is updated through the plant and reference model states only. This creates the premise for characterizing sticking regions as the update law has no dependence on the adaptive gain $\Theta(t)$ itself.

Our approach will be as follows: Determine a region $\mathbf{S}$ in the $\widetilde{\Theta}$ space, $\mathbf{N}$ in the $\widetilde{x}$ space and $\mathbf{R}$ in the x_m space. Here $\widetilde{x}$ is simply a deviation of the plant state from a fictitious trajectory as will be later defined. We continue our approach by showing that there are some initial conditions for which $\widetilde{\Theta}(t)$ will remain in $\mathbf{S}$, $\widetilde{x}(t)$ in $\mathbf{N}$ and $x_m(t)$ in $\mathbf{R}$, over a certain interval, with $\|\dot{\widetilde{\Theta}}(t)\|$ remaining finite. The combined set

$$\mathscr{S} : \left\{ \left[\widetilde{\Theta} \; \widetilde{x} \; x_m\right] \in \mathbb{R}^{3n} \;\middle|\; \widetilde{\Theta} \in \mathbf{S}, \; \widetilde{x} \in \mathbf{N}, \; x_m \in \mathbf{R} \right\} \tag{18}$$

is defined to be the sticking region. In the following section, we demonstrate the existence of this sticking region.

III. ANALYSIS OF THE STICKING REGION

In order to establish the sticking region, we need to guarantee the existence of a finite Θ_d^* such that

$$\|\dot{\widetilde{\Theta}}(t)\| \leq \Theta_d^* \quad \forall \quad t \in [t_1, t_2] \tag{19}$$

and a t_2 such that

$$t_2 - t_1 \geq \frac{\delta\theta^*}{\Theta_d^*} \tag{20}$$

where $\delta\theta^*$ is a lower bound defined as

$$\|\widetilde{\Theta}(t_2) - \widetilde{\Theta}(t_1)\| \geq \delta\theta^*. \tag{21}$$

The above implies that the parameter error moves slowly for all $t \in [t_1, t_2]$. In order to satisfy (19), we examine (4) and (16) and conditions under which $x(t)$ remains small. This is addressed in Section III-A which follows. A similar procedure is adopted to characterize $x_m(t)$ in Section III-B. With these characterizations, the sticking region $\mathscr{S}$ as defined above, is analyzed in Section III-C.

A. CHARACTERIZATION OF $x(t)$

Using the matching condition in (6) and feedback controllers from (3) and (15), the plant differential equations for the CRM and IC-adaptive systems in (1) and (13), respectively, may be written similarly as

$$\dot{x}(t) = \left[A_m + \mathbf{b}\widetilde{\Theta}(t)\right] x(t) + \mathbf{b}_m r(t) \tag{22}$$

with $\mathbf{b}_m = q^* \mathbf{b}$ for CRM-adaptive system and defined as (14) for the IC-adaptive system. We consider an arbitrary point Θ_0, and a fictitious trajectory $\hat{x}(t)$ and the deviation $\widetilde{x}(t)$ of $x(t)$ from $\hat{x}(t)$. That is, we define

$$\widetilde{\Theta}(t) = \Theta_0 + \delta\widetilde{\Theta}(t) \tag{23}$$

$$\hat{x}(t) = -\left[A_m + \mathbf{b}\Theta_0\right]^{-1} \mathbf{b}_m r(t) \tag{24}$$

$$\widetilde{x}(t) = x(t) - \hat{x}(t). \tag{25}$$

Using equations (22) through (25), a differential equation for the state $\widetilde{x}(t)$ may be expressed as

$$\dot{\widetilde{x}}(t) = \left[A_m + \mathbf{b}\widetilde{\Theta}(t)\right] \widetilde{x}(t) + \mathbf{b}\delta\widetilde{\Theta}(t)\hat{x}(t) - \dot{\hat{x}}(t). \tag{26}$$

If $\hat{A}(\widetilde{\Theta}(t)) = \left[A_m + \mathbf{b}\widetilde{\Theta}(t)\right]$ and $\mathbf{w}(t) = \mathbf{b}\delta\widetilde{\Theta}(t)\hat{x}(t) - \dot{\hat{x}}(t)$, then the following linear time-variant plant differential equation is obtained:

$$\dot{\widetilde{x}}(t) = \hat{A}(\widetilde{\Theta}(t))\widetilde{x}(t) + \mathbf{w}(t). \tag{27}$$

The following energy function of $\widetilde{x}(t)$ will be used to examine the propensity of $\widetilde{x}(t)$ towards 0:

$$V_{\widetilde{x}}(t) = \widetilde{x}^T(t)Y\widetilde{x}(t) > 0 \quad \forall \ \widetilde{x}(t) \neq 0 \tag{28}$$

where Y is a symmetric positive definite matrix. Additionally the sets are defined:

$$\mathbf{S}: \left\{\widetilde{\Theta} \in \mathbb{R}^n \ \middle| \ \left[\hat{A}^T(\widetilde{\Theta})Y + Y\hat{A}(\widetilde{\Theta}) + I\right] < 0 \ \cap \ \|\widetilde{\Theta} - \Theta_0\| \leq \|Y\mathbf{b}\|^{-\alpha}\right\} \tag{29}$$

$$\mathbf{M}: \left\{\widetilde{x} \in \mathbb{R}^n \ \middle| \ \|\widetilde{x}\|^2 \geq 4\beta^2\right\} \tag{30}$$

$$\mathbf{N}: \left\{\widetilde{x} \ \in \mathbb{R}^n \ \middle| \ \widetilde{x}^T Y \widetilde{x} \leq 4\lambda_{max}(Y)\beta^2\right\} \tag{31}$$

where

$$\beta \geq \|(A_m + \mathbf{b}\Theta_0)^{-1}\mathbf{b}_m\|\left(\|Y\mathbf{b}\|^{1-\alpha}r^* + \|Y\|r_d^*\right). \tag{32}$$

Here $|r(t)| \leq r^* \ \forall \ t$, $|\dot{r}(t)| \leq r_d^* \ \forall \ t$ and α is a positive constant chosen as $0 \leq \alpha \leq 1$. Throughout this paper, $\lambda_{min}(B)$ and $\lambda_{max}(B)$ will be used to denote the smallest and largest eigenvalues, respectively, of a matrix B.

It should be noted that $\mathbf{M}$ is an unbounded region in $\mathbb{R}^n$ outside a bounded sphere, while $\mathbf{N}$ is a bounded ellipsoid. $\mathbf{S}$ is a set in $\mathbb{R}^n$ whose existence is yet to be demonstrated.

Lemma 1. *From the definition of* $\mathbf{M}$ *and* $\mathbf{N}$ *in (30) and (31) respectively, it follows that*

$$\mathbf{N}^c \subset \mathbf{M}.$$

Proof. From the definition of $\mathbf{N}$, it is known that

$$\lambda_{max}(Y)\|\widetilde{x}\|^2 \geq \widetilde{x}^T Y \widetilde{x} > 4\lambda_{max}(Y)\beta^2 \quad \forall \ \ \widetilde{x} \in \mathbf{N}^c \tag{33}$$

or simply

$$\|\widetilde{x}\|^2 > 4\beta^2 \quad \forall \ \ \widetilde{x} \in \mathbf{N}^c. \tag{34}$$

The bounds in (33) and (34) are well defined as Y is a symmetric positive definite matrix. From the definition of $\mathbf{M}$, equation (34) implies that $\mathbf{N}^c \subset \mathbf{M}$. □

With the above definitions and properties, we demonstrate the propensity for $\widetilde{x}$ to remain in $\mathbf{N}$ in the following theorem.

Theorem 2. *If (i)* $\widetilde{\Theta}(t) \in \mathbf{S} \ \forall \ t \in [t_1, t_2]$ *where* $t_2 > t_1$, *and (ii)* $\widetilde{x}(t_1) \in \mathbf{N}$, *then*

$$\widetilde{x}(t) \in \mathbf{N} \quad \forall \quad t \in [t_1, t_2].$$

Proof. The time derivative of $V_{\widetilde{x}}(t)$ in (28) is

$$\dot{V}_{\widetilde{x}}(t) = \widetilde{x}^T\left[\hat{A}^T(\widetilde{\Theta}(t))Y + Y\hat{A}(\widetilde{\Theta}(t))\right]\widetilde{x} + 2\mathbf{w}^T(t)Y\widetilde{x}. \tag{35}$$

From condition (i) in Theorem 2, equation (35) leads to the inequality

$$\dot{V}_{\widetilde{x}}(t) < -\widetilde{x}^T\widetilde{x} + 2\mathbf{w}^T(t)Y\widetilde{x} \tag{36}$$

for $t \in [t_1, t_2]$. Equation (36) may be rewritten as

$$\dot{V}_{\widetilde{x}}(t) < -\left(\widetilde{x} - Y\mathbf{w}(t)\right)^T\left(\widetilde{x} - Y\mathbf{w}(t)\right) + \|Y\mathbf{w}(t)\|^2. \tag{37}$$

From condition (i) in Theorem 2 and the definition of $\mathbf{S}$, it follows that

$$\|\delta\widetilde{\Theta}(t)\| \leq \frac{1}{\|Y\mathbf{b}\|^{\alpha}} \quad \forall \ \ t \in [t_1, t_2]. \tag{38}$$

From this, an upper bound on $\|Y\mathbf{w}(t)\|$ is determined for $t \in [t_1, t_2]$:

$$\begin{aligned}\|Y\mathbf{w}(t)\| &\leq \|Y\mathbf{b}\|\|\delta\widetilde{\Theta}(t)\|\|(A_m + \mathbf{b}\Theta_0)^{-1}\mathbf{b}_m\|r^* + \\ &\qquad \|Y\|\|(A_m + \mathbf{b}\Theta_0)^{-1}\mathbf{b}_m\|r_d^* \\ &\leq \|(A_m + \mathbf{b}\Theta_0)^{-1}\mathbf{b}_m\|\left(\|Y\mathbf{b}\|^{1-\alpha}r^* + \|Y\|r_d^*\right) \\ &\leq \beta.\end{aligned} \tag{39}$$

From (37), (39) and the definition of $\mathbf{M}$, it follows that for $t \in [t_1, t_2]$, $\dot{V}_{\widetilde{x}}(t) < 0$ if $\widetilde{x}(t) \in \mathbf{M}$ and condition (i) in Theorem 2 holds. From Lemma 1, this in turn implies that if conditions (i) and (ii) in Theorem 2 hold, then $\widetilde{x}(t) \in \mathbf{N} \ \forall \ t \in [t_1, t_2]$. □

B. CHARACTERIZATION OF $x_m(t)$

The update laws in (4) and (16) are also affected by the reference model and thus it is important to characterize $x_m(t)$. The reference models for the CRM and IC-adaptive controllers may be written as

$$\dot{x}_m(t) = \left[A_m - L\right]x_m(t) + \mathbf{z}(t) \tag{40}$$

where $\mathbf{z}(t) = \mathbf{b}_m r(t) + Lx(t)$. Similar to the approach used to characterize $x(t)$, the following energy function of $x_m(t)$ will be used:

$$V_{x_m}(t) = x_m^T(t)Wx_m(t) > 0 \quad \forall \ \ x_m(t) \neq 0 \tag{41}$$

where

$$\left[A_m - L\right]^T W + W\left[A_m - L\right] = -I. \tag{42}$$

Since $\left[A_m - L\right]$ is Hurwitz, W is a symmetric positive definite matrix. Finally, sets $\mathbf{Q}$ and $\mathbf{R}$ are defined:

$$\mathbf{Q}: \left\{x_m \in \mathbb{R}^n \ \middle| \ \|x_m\|^2 \geq 4\Lambda^2\right\} \tag{43}$$

$$\mathbf{R}: \left\{x_m \ \in \mathbb{R}^n \ \middle| \ x_m^T W x_m \leq 4\lambda_{max}(W)\Lambda^2\right\} \tag{44}$$

where

$$\Lambda > \|W\|\left(\|\mathbf{b}_m\|r^* + \|L\|x^*\right) \tag{45}$$

and

$$x^* = 2\beta\left(\frac{\lambda_{max}(Y)}{\lambda_{min}(Y)}\right)^{\frac{1}{2}} + \|(A_m + \mathbf{b}\Theta_0)^{-1}\mathbf{b}_m\|r^*. \tag{46}$$

Lemma 3. *From the definition of* $\mathbf{Q}$ *and* $\mathbf{R}$ *in (43) and (44) respectively, it follows that*

$$\mathbf{R}^c \subset \mathbf{Q}.$$

Proof. From the definition of $\mathbf{R}$, it is known that

$$\lambda_{max}(W)\|x_m\|^2 \geq x_m^T W x_m > 4\lambda_{max}(W)\Lambda^2 \quad \forall \quad x_m \in \mathbf{R}^c \tag{47}$$

or simply

$$\|x_m\|^2 > 4\Lambda^2 \quad \forall \quad x_m \in \mathbf{R}^c. \tag{48}$$

The bounds in (47) and (48) are well defined as W is a symmetric positive definite matrix. From the definition of $\mathbf{Q}$, equation (47) implies that $\mathbf{R}^c \subset \mathbf{Q}$. □

As in the characterization of $x(t)$, we use the above definitions and properties to demonstrate the propensity for $x_m(t)$ to remain in $\mathbf{R}$ in the following theorem.

Theorem 4. *If (i)* $\widetilde{x}(t) \in \mathbf{N} \quad \forall \quad t \in [t_1,t_2]$ *and (ii)* $x_m(t_1) \in \mathbf{R}$, *then*

$$x_m(t) \in \mathbf{R} \quad \forall \quad t \in [t_1,t_2].$$

Proof. The time derivative of $V_{x_m}(t)$ in (41) is

$$\dot{V}_{x_m}(t) = -x_m^T x_m + 2\mathbf{z}^T(t) W x_m \tag{49}$$

which may also be expressed as

$$\dot{V}_{x_m}(t) = -\big(x_m - W\mathbf{z}(t)\big)^T \big(x_m - W\mathbf{z}(t)\big) + \|W\mathbf{z}(t)\|^2. \tag{50}$$

From condition (i) in Theorem 4 and the definition of $\mathbf{N}$ it follows that

$$\|\widetilde{x}(t)\| \leq 2\beta \left(\frac{\lambda_{max}(Y)}{\lambda_{min}(Y)}\right)^{\frac{1}{2}} \quad \forall \quad t \in [t_1,t_2]. \tag{51}$$

The bound in (51) is well defined as Y is a symmetric positive definite matrix. From the inequality in (51) and the definition of $\widetilde{x}$ in (25), it follows that

$$\|x(t)\| \leq x^* \quad \forall \quad t \in [t_1,t_2]. \tag{52}$$

From this, an upper bound on $\|W\mathbf{z}(t)\|$ is determined for $t \in [t_1,t_2]$:

$$\begin{aligned}\|W\mathbf{z}(t)\| &\leq \|W\|\|\mathbf{b}_m r(t) + Lx(t)\| \\ &\leq \|W\|\big(\|\mathbf{b}_m\| r^* + \|L\| x^*\big) \\ &< \Lambda.\end{aligned} \tag{53}$$

From (50), (53) and the definition of $\mathbf{Q}$, it follows that for $t \in [t_1,t_2]$, $\dot{V}_{x_m}(t) < 0$ if $x_m(t) \in \mathbf{Q}$. From Lemma 3, this in turn implies that if conditions (i) and (ii) in Theorem 4 hold, then $x_m(t) \in \mathbf{R} \quad \forall \quad t \in [t_1,t_2]$. □

C. MAXIMUM RATE OF CONVERGENCE DURING STICKING

Theorems 2 and 4 create the basis for analyzing sticking in the adaptive systems. That is, we determine conditions under which the parameter error $\widetilde{\Theta}(t)$ has a bounded derivative, over a certain time interval. This is presented in the following theorem.

Theorem 5. *Let*

$$\Theta_d^* = \|\mathbf{b}^T P\| \left(x^{*2} + 2x^*\Lambda \left(\frac{\lambda_{max}(W)}{\lambda_{min}(W)}\right)^{\frac{1}{2}}\right). \tag{54}$$

If (i) $\widetilde{\Theta}(t) \in \mathbf{S} \quad \forall \quad t \in [t_1,t_2]$ *where*

$$t_2 = \min\left\{ t \;\middle|\; \widetilde{\Theta}(t_1) \in \mathbf{S} \;,\; \widetilde{\Theta}(t+\delta t) \notin \mathbf{S} \;\forall\; \delta t > 0 \;,\; t > t_1 \right\}, \tag{55}$$

(ii) $\widetilde{x}(t_1) \in \mathbf{N}$ *and (iii)* $x_m(t_1) \in \mathbf{R}$, *then*

$$t_2 - t_1 \geq \frac{\|\widetilde{\Theta}(t_2) - \widetilde{\Theta}(t_1)\|}{\Theta_d^*}. \tag{56}$$

Proof. From Theorem 2 and conditions (i) and (ii) of Theorem 5, it follows that $\widetilde{x}(t) \in \mathbf{N} \quad \forall \quad t \in [t_1,t_2]$. From Theorem 4, this in turn implies that if condition (iii) of Theorem 5 also holds, then $x_m(t) \in \mathbf{R} \quad \forall \quad t \in [t_1,t_2]$.

From the definition of $\mathbf{N}$ and $\mathbf{R}$ it follows that

$$\|\widetilde{x}(t)\| \leq 2\beta \left(\frac{\lambda_{max}(Y)}{\lambda_{min}(Y)}\right)^{\frac{1}{2}} \quad \forall \quad t \in [t_1,t_2] \tag{57}$$

and

$$\|x_m(t)\| \leq 2\Lambda \left(\frac{\lambda_{max}(W)}{\lambda_{min}(W)}\right)^{\frac{1}{2}} \quad \forall \quad t \in [t_1,t_2]. \tag{58}$$

From the inequality in (57) and the definition of $\widetilde{x}$ in (25), it follows that

$$\|x(t)\| \leq x^* \quad \forall \quad t \in [t_1,t_2]. \tag{59}$$

An upper bound on $\dot{\widetilde{\Theta}}(t)$ for $t \in [t_1,t_2]$ can now be determined as

$$\begin{aligned}\|\dot{\widetilde{\Theta}}(t)\| &= \|-\mathbf{b}^T P(x(t) - x_m(t))x^T(t)\| \\ &\leq \|\mathbf{b}^T P\|(\|x(t)\|\|x^T(t)\| + \|x_m(t)\|\|x^T(t)\|) \\ &\leq \|\mathbf{b}^T P\| \left(x^{*2} + 2x^*\Lambda\left(\frac{\lambda_{max}(W)}{\lambda_{min}(W)}\right)^{\frac{1}{2}}\right) \\ &= \Theta_d^*.\end{aligned}$$

This proves Theorem 5. □

Theorem 5 is the main result of this paper. It establishes a lower bound on the duration of the time interval $[t_1,t_2]$ that is dependent on the maximum speed of convergence Θ_d^* and the size of set $\mathbf{S}$. The term "sticking region" was first used in [3] to describe a set in state space where the state rate remained bounded for a minimum time. This implies that the combined set $\mathscr{S}$ in (18) is the *sticking region* with *sticking* occurring over the interval $[t_1,t_2]$ during which $\widetilde{\Theta}(t) \in \mathbf{S}$, $\widetilde{x}(t) \in \mathbf{N}$ and $x_m(t) \in \mathbf{R}$.

The conditions under which the lower bound of t_2 in (56) may be made arbitrarily large are investigated next. In order to determine these conditions, we first argue that $\mathbf{S}$ as defined above exists.

D. EXISTENCE OF SET $\mathbf{S}$

To establish the existence of $\mathbf{S}$, we first choose the symmetric matrix Y defined in (28). For this purpose we define

$$\hat{A}_0 = A_m + \mathbf{b}\Theta_0 \tag{60}$$

where it is assumed that Θ_0 is such that $\hat{A}_0$ is Hurwitz. A symmetric and positive definite matrix $\bar{Y}$ may therefore be defined by

$$\hat{A}_0^T\bar{Y} + \bar{Y}\hat{A}_0 = -I. \tag{61}$$

We now define Y using $\bar{Y}$ in (61) and a positive constant γ^2 as

$$Y = (1+\gamma^2)\bar{Y}. \tag{62}$$

The motivation for this selection of Y will become clear in the following theorem that proves the existence of $\mathbf{S}$:

Theorem 6. *Let*

$$\hat{A}_0 = A_m + \mathbf{b}\Theta_0 \tag{63}$$

be Hurwitz. Then $\mathbf{S}$ *exists and may be defined as*

$$\mathbf{S} : \left\{ \widetilde{\Theta} \in \mathbb{R}^n \;\middle|\; -\delta\Theta^* \leq \widetilde{\Theta} - \Theta_0 \leq \delta\Theta^* \right\} \tag{64}$$

where

$$\delta\Theta^* = [\delta\theta^*,\ \delta\theta^*\ \ldots\ \delta\theta^*] \tag{65}$$

with

$$0 < \delta\theta^* < \min\left\{ \frac{\gamma^2}{2n\|Y\mathbf{b}\|_{max}},\ \frac{1}{n\|Y\mathbf{b}\|^{\alpha}} \right\}. \tag{66}$$

Proof. Let

$$\mathbf{S}_1 : \left\{ \widetilde{\Theta} \in \mathbb{R}^n \;\middle|\; \left[\hat{A}^T(\widetilde{\Theta})Y + Y\hat{A}(\widetilde{\Theta}) + I\right] < 0 \right\} \tag{67}$$

$$\mathbf{S}_2 : \left\{ \widetilde{\Theta} \in \mathbb{R}^n \;\middle|\; \|\widetilde{\Theta} - \Theta_0\| \leq \|Y\mathbf{b}\|^{-\alpha} \right\}. \tag{68}$$

It is easy to note that

$$\mathbf{S} \in \mathbf{S}_1 \cap \mathbf{S}_2. \tag{69}$$

We first show the existence of $\mathbf{S}_1$. Since $\hat{A}(\widetilde{\Theta})$ can be written as

$$\hat{A}(\widetilde{\Theta}) = \hat{A}_0 + \mathbf{b}\delta\widetilde{\Theta} \tag{70}$$

and $\hat{A}_0$ is Hurwitz, we use (61) and (62) to rewrite (67) as

$$\mathbf{S}_1 : \left\{ \widetilde{\Theta} \in \mathbb{R}^n \middle|\ C(\delta\widetilde{\Theta}) < 0 \right\} \tag{71}$$

where

$$C(\delta\widetilde{\Theta}) = [c_{ij}(\delta\widetilde{\Theta})] = \delta\widetilde{\Theta}^T\mathbf{b}^TY + Y\mathbf{b}\delta\widetilde{\Theta} - \gamma^2 I. \tag{72}$$

By considering diagonal dominance, it is known that $C(\delta\widetilde{\Theta}) < 0$ if [1]

$$c_{ii}(\delta\widetilde{\Theta}) < 0 \quad \forall \quad i \tag{73}$$

and

$$|c_{ii}(\delta\widetilde{\Theta})| > \sum_{j\neq i} |c_{ij}(\delta\widetilde{\Theta})| \quad \forall \quad i. \tag{74}$$

We will show that $\mathbf{S}_1$ in (71) exists by demonstrating that the elements of $C(\delta\widetilde{\Theta})$ in (72) satisfy (73) and (74). By defining

$$\begin{aligned} c^* &= 2\|Y\mathbf{b}\|_{max}\|\delta\widetilde{\Theta}\|_{max} \\ &\geq \|\delta\widetilde{\Theta}^T\mathbf{b}^TY + Y\mathbf{b}\delta\widetilde{\Theta}\|_{max} \end{aligned} \tag{75}$$

it is known that

$$c_{ii}(\delta\widetilde{\Theta}) \leq c^* - \gamma^2 \quad \forall \quad i \tag{76}$$

and

$$(n-1)c^* \geq \sum_{j\neq i} |c_{ij}(\delta\widetilde{\Theta})| \quad \forall \quad i. \tag{77}$$

By utilizing inequalities (76) and (77), conditions (73) and (74) become

$$c_{ii}(\delta\widetilde{\Theta}) \leq c^* - \gamma^2 < 0 \quad \forall \quad i \tag{78}$$

and

$$\begin{aligned} |c_{ii}(\delta\widetilde{\Theta})| &> (n-1)c^* \\ &\geq \sum_{j\neq i} |c_{ij}(\delta\widetilde{\Theta})| \quad \forall \quad i. \end{aligned} \tag{79}$$

Both conditions (78) and (79) may be satisfied if

$$c^* < \frac{\gamma^2}{n} \tag{80}$$

or equivalently

$$\|\delta\widetilde{\Theta}\|_{max} \leq \frac{\gamma^2}{2n\|Y\mathbf{b}\|_{max}}. \tag{81}$$

Thus, set $\mathbf{S}_1$ is defined in $\widetilde{\Theta}$ space where (81) is satisfied. We now consider the existence of $\mathbf{S}_2$. Since

$$\|\widetilde{\Theta} - \Theta_0\| \leq n\|\delta\widetilde{\Theta}\|_{max}, \tag{82}$$

the set $\mathbf{S}_2$ is well defined if

$$\|\delta\widetilde{\Theta}\|_{max} \leq n\|Y\mathbf{b}\|^{-\alpha}. \tag{83}$$

The definition of $\mathbf{S}$ in (64) describes an admissible set such that the conditions for set $\mathbf{S}_1$ and $\mathbf{S}_2$ in (81) and (83) respectively, are satisfied for all $\widetilde{\Theta} \in \mathbf{S}$. □

The existence of set $\mathbf{S}$ has now been shown under the condition that $\hat{A}_0$ in (63) is Hurwitz. Each selection of Θ_0 that satisfies this condition describes a certain set $\mathbf{S}$. However, it will become clear from the numerical and simulation results that the selection of Θ_0 greatly affects sticking if $\widetilde{\Theta}(t_1) = \Theta_0$.

IV. SIMULATION EXAMPLE

We carry out a simulation in this section to describe the sticking region $\mathscr{S}$. A second order plant and reference model are chosen as in (1) and (5) with

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{84}$$

$$A_m = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \qquad \mathbf{b}_m = \mathbf{b} \qquad L = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{85}$$

Here the Lyapunov equation in (8) is solved with $Q_0 = I$ and constant reference input is specified with $r(t) = 1$ such that $r^* = 1$ and $r_d^* = 0$. In order to define a set $\mathbf{S}$, we use

Theorem 6 which requires $\hat{A}_0$ in (63) to be Hurwitz. By setting $\Theta_0 = [-24, -24]$, the following eigenvalues of $\hat{A}_0$ are obtained:

$$\lambda_1(\hat{A}_0) = -1 \qquad \lambda_2(\hat{A}_0) \quad = -25.$$

With $\hat{A}_0$ known, we can define $\bar{Y}$ and Y using (61) and (62) with $\gamma = 1$. β and Λ are set to the lower bounds in (32) and (45) respectively such that

$$\beta = 0.04 \qquad \Lambda = 1.71.$$

S is then defined in (64) with $\delta\theta^* = 6.25$ as computed by the upper bound in (66) with $\alpha = 1$. Following the definitions in (31) and (44), the sets **N** and **R** are defined. The sticking region $\mathscr{S}$ is then defined by combined set in (18) with

$$\mathbf{S}: \left\{ \widetilde{\Theta} \in \mathbb{R}^2 \;\middle|\; -\begin{bmatrix} 6.25 \\ 6.25 \end{bmatrix}^T \leq \widetilde{\Theta} + \begin{bmatrix} 24 \\ 24 \end{bmatrix}^T \leq \begin{bmatrix} 6.25 \\ 6.25 \end{bmatrix}^T \right\}$$

$$\mathbf{N}: \left\{ \widetilde{x} \in \mathbb{R}^2 \;\middle|\; \widetilde{x}^T \begin{bmatrix} 2.04 & 0.04 \\ 0.04 & 0.04 \end{bmatrix} \widetilde{x} \leq 0.013 \right\}$$

$$\mathbf{R}: \left\{ x_m \in \mathbb{R}^2 \;\middle|\; x_m^T \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} x_m \leq 19.90 \right\}.$$

We choose the initial conditions at $t_1 = 0$:

$$\widetilde{\Theta}(0) = \Theta_0 \qquad \widetilde{x}(0) = 0 \qquad x_m(0) = -A_m^{-1}\mathbf{b}_m r(0). \tag{86}$$

It is easy to see that the system is initialized in the sticking region and the conditions of Theorem 5 are satisfied. From (54), we compute Θ_d^* to be 3.87. Finally, using (56), we compute the lower bound on t_2 using the values of Θ_d^* and $\delta\theta^*$ as

$$t_2 \geq 1.61. \tag{87}$$

In order to validate this analytical prediction, numerical simulations of the CRM-adaptive system specified by (84), (85) and (86) were carried out, the results of which are shown in Figures 1 to 3. In can be seen from these figures that $\widetilde{x}(t) \in \mathbf{N}$, $\widetilde{\Theta}(t) \in \mathbf{S}$ and $\|\dot{\widetilde{\Theta}}(t)\| \leq \Theta_d^* \; \forall \; t \in [0, 290]$. This confirms (87).

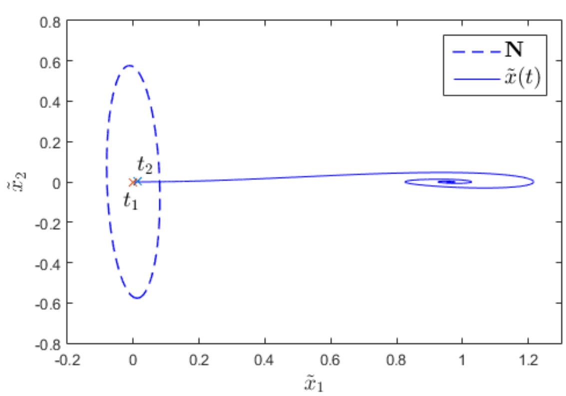

Fig. 1. $\widetilde{x}(t)$ trajectory in **N** with $t_1 = 0$ and $t_2 \approxeq 290$

The lower bound of $t_2 \geq 1.61$ from (87) may mislead the reader into thinking that the sticking may occur only for a short period of time. This is not true; it should be noted that the selection of Θ_0 above was done arbitrarily, insuring only that $\hat{A}_0$ was Hurwitz. Suppose that Θ_0 is chosen such that the eigenvalues of $\hat{A}_0$ are set as follows:

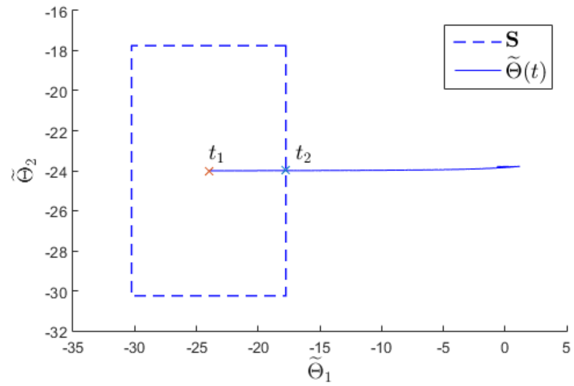

Fig. 2. $\widetilde{\Theta}(t)$ trajectory in **S** with $t_1 = 0$ and $t_2 \approxeq 290$

$$\lambda_1(\hat{A}_0) = -1 \qquad \lambda_2(\hat{A}_0) \quad = -k \;\text{ with }\; k > 1. \tag{88}$$

Repeating the same procedure as above for the $\hat{A}_0$ as in (88), **N** and **S** as well as the lower bound on t_2 can be calculated. These are shown in Figure 4, which clearly illustrates that as initial condition increases in magnitude, the time that the trajectories spend in the sticking region grows as well.

In the following section, we extend the observation in Figure 4 to general n^{th} order systems.

V. NUMERICAL ANALYSIS

In this section, we address the lower bound on t_2, from Theorem 5, and its dependence on initial conditions for general n^{th} order CRM and IC-adaptive sytems.

A. NUMERICAL RESULTS

For a system that is initialized in the sticking region with $\widetilde{\Theta}(t_1) = \Theta_0$, then from Theorem 5 we have

$$t_2 - t_1 \geq \frac{\delta\theta^*}{\Theta_d^*}. \tag{89}$$

For any given adaptive system, it is clear form (45) and (54) that if x^* (defined in (46)) decreases, then Θ_d^* in (89) will decrease provided Λ is always set to the lower bound in (45). Additionally, by considering (66), $\delta\theta^*$ may be increased if the upper bound

$$\delta\theta^*_{max} = \min\left\{ \frac{\gamma^2}{2n\|Y\mathbf{b}\|_{max}}, \; \frac{1}{n\|Y\mathbf{b}\|^{\alpha}} \right\} \tag{90}$$

increases. Therefore, in order to determine the conditions under which the lower bound on t_2 in (89) may be made arbitrarily large, only the quantities x^* and $\delta\theta^*_{max}$ need to be analyzed. This will be the approach used in the remainder of this section.

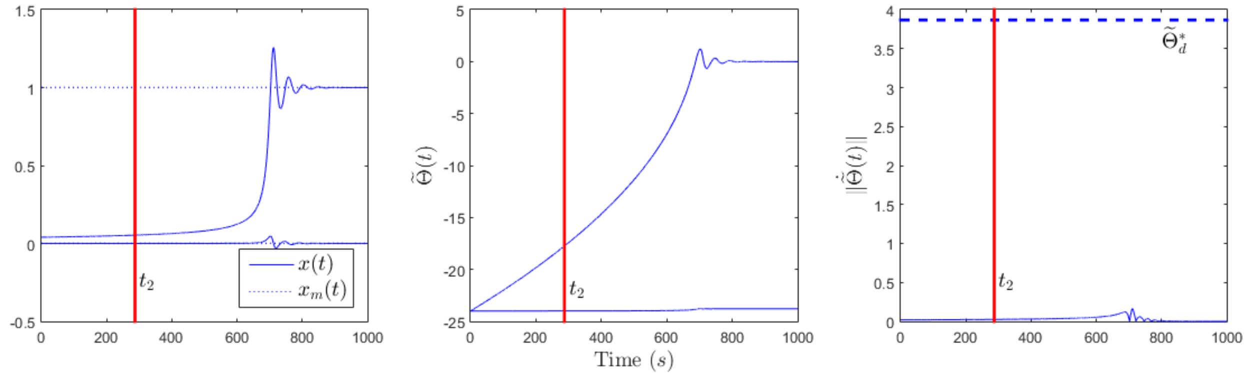

Fig. 3. $x_p(t)$, $x_m(t)$, $\widetilde{\Theta}(t)$ and $\|\dot{\widetilde{\Theta}}(t)\|$ versus time

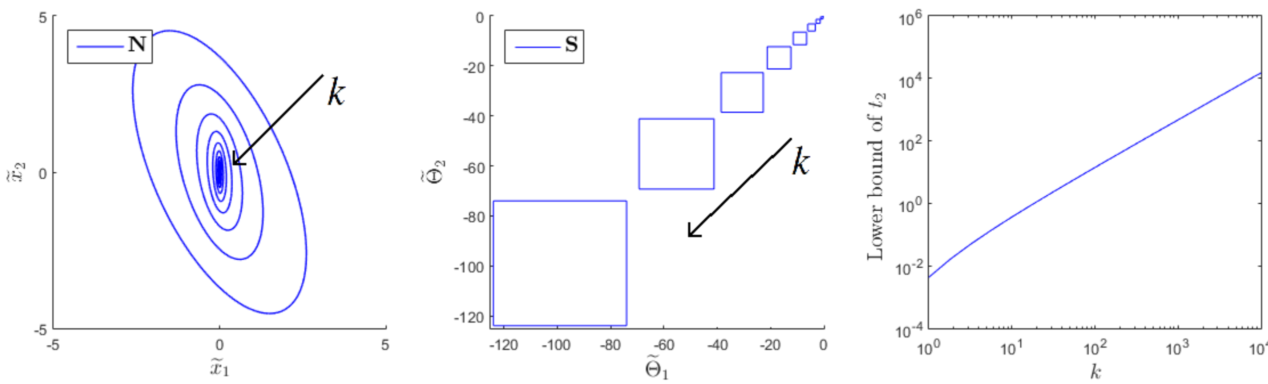

Fig. 4. Set **N**, set **S** and the lower bound of t_2 for varying k

In these numerical results, it is assumed that the reference input is constant such that $r^* = 1$ and $r_d^* = 0$. In order to define a set **S**, we choose Θ_0 in (63) such that

$$\begin{aligned} \lambda_i(\hat{A}_0) &= -1 \quad \text{for} \quad i = 1 \ \dots \ n-1 \\ \lambda_n(\hat{A}_0) &= -k \quad \text{with} \quad k > 1. \end{aligned} \tag{91}$$

We now define $\bar{Y}$ and Y using (61) and (62) with $\gamma = 1$. β is then defined as the lower bound in (32) with $\alpha = 1$ and $\alpha = 0.3$, respectively, for the CRM and IC-adaptive systems. By setting Λ to the lower bound in (45), we show how x^* and $\delta\theta^*_{max}$ vary with k in Figure 5 for the CRM-adaptive system and Figure 6 for the IC-adaptive system. In these figures, results for system orders from two to six are shown.

It is clear from these graphs that the lower bound of t_2 in (89) may be made arbitrarily large for the CRM and IC-adaptive systems. However, the lower bound grows more rapidly for the CRM-adaptive system. Additionally, it can be seen that x^* approaches a non-zero value as k increases for the IC-adaptive system, while x^* approaches zero for the CRM-adaptive system. This means that Θ_d^* may be made arbitrarily small by increasing k for the CRM-adaptive system, but not for the IC-adaptive system.

B. EQUILIBRIUM STATE OF $\widetilde{\Theta}(t)$ IN S

In Theorem 5, t_2 is defined as the time at which $\widetilde{\Theta}(t)$ leaves the set **S**. The purpose of this analysis is to demonstrate that

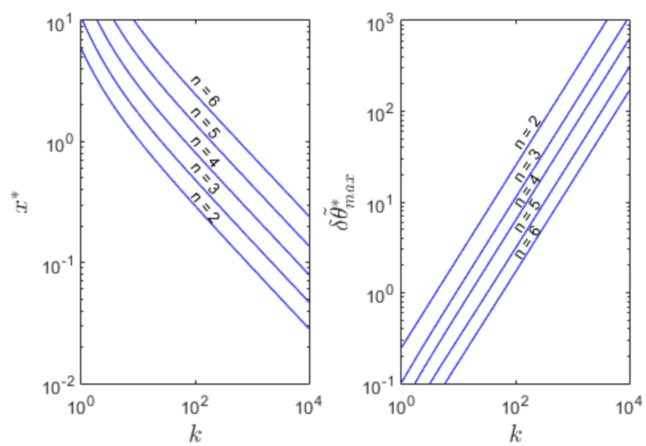

Fig. 5. x^* and $\delta\theta^*_{max}$ for versus k with $n = 2 \rightarrow 6$ in the CRM-adaptive system

t_2 may be made arbitrarily large due to a maximum speed of convergence in the sticking region. However, if $\widetilde{\Theta}(t)$ does not leave **S** (i.e. an equilibrium state exists inside **S**), then t_2 is not finite.

However, the results presented in Figures 5 and 6 demonstrate that the region **S** may be made arbitrarily large as $\delta\theta^*_{max}$ increases with the magnitude of the initial condition from (91). Therefore, regardless of the equilibrium state

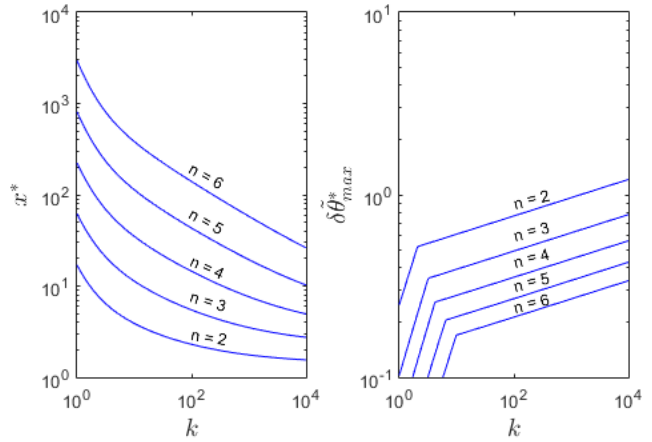

Fig. 6. x^* and $\delta\theta^*_{max}$ for versus k with $n = 2 \to 6$ in the IC-adaptive system

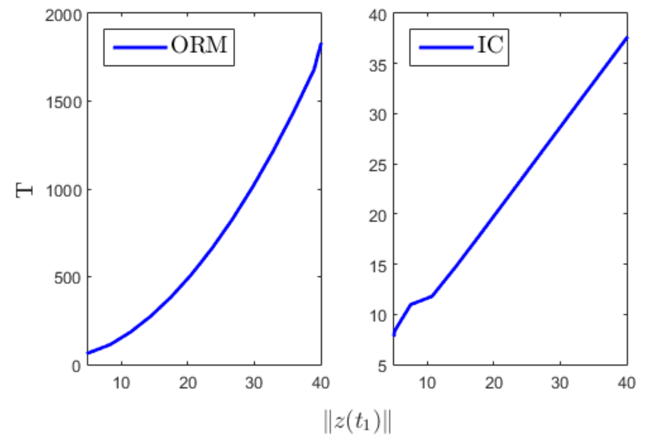

Fig. 7. Settling time T for various initial conditions of the CRM and IC-adaptive systems

of $\widetilde{\Theta}(t)$, this analysis allows for the characterization of an arbitrarily large region **S** in which $\dot{\widetilde{\Theta}}(t)$ is bounded, provided the system is initialized in the sticking region.

C. SIMULATION RESULTS

To illustrate sticking and the significance of Figures 5 and 6, simulations were carried out for a CRM and IC-adaptive system defined by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & -4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{92}$$

$$A_m = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \qquad L = \begin{bmatrix} 0_{3\times 3} \end{bmatrix} \tag{93}$$

with $\mathbf{b}_m = \mathbf{b}$ for CRM-adaptive system and defined in (14) for the IC-adaptive system. For both systems the Lyapunov equation in (8) is solved with $Q_0 = I$ and constant reference input is specified with $r(t) = 1$. The following initial conditions were used:

$$x(t_1) = 0 \qquad x_m(t_1) = 0 \qquad \Theta(t_1) = [\theta_0, \theta_0, \theta_0]. \tag{94}$$

For each simulation, the initial conditions in (94) were used for increasingly negative values of θ_0 while recording the settling time T defined as

$$T = \min\left\{ t \;\middle|\; \frac{\|z(t) - z^*\|}{\|z(t_1) - z^*\|} < \varepsilon \right\} \tag{95}$$

where $z = [x \quad x_m \quad \widetilde{\Theta}]^T$, $z^* = \lim_{t\to\infty} z(t)$ and $\varepsilon = 0.05$. By making θ_0 more negative in (94), the system was initialized further and further into a sticking region (This corresponds similarly to increasing k in (91)). The results of the settling time T are included in Figure 7. Here a decreasing convergence rate for the CRM-adaptive system is observed as θ_0 is made more negative. On the other hand, the IC-adaptive system demonstrates a constant learning rate.

VI. CONCLUSION

In this paper, we have focused on slow convergence properties of errors in a class of adaptive systems that corresponds to adaptive control of linear time-invariant plants with state variables accessible. We prove the existence of a sticking region in the error space where the state errors move with a finite velocity independent of their magnitude. These properties are exhibited by ORM, CRM and IC-adaptive systems. Simulation and numerical studies are included to illustrate the size of this sticking region and its dependence on various system parameters.

REFERENCES

[1] H.E. Bell, "Gerschgorin's Theorem and the Zeros of Polynomials," The American Mathematical Monthly, vol. 72, no. 3, pp. 292-295, 1965.

[2] T.E. Gibson, A.M. Annaswamy, and E. Lavretsky, "On adaptive control with closed-loop reference models: Transients, oscillations, and peaking," IEEE Access, vol. 1, pp. 703-717, 2013.

[3] B.M. Jenkins, T.E. Gibson, A.M. Annaswamy, and E. Lavretsky, "Asymptotic stability and convergence rates in adaptive systems," IFAC Workshop on Adaptation and Learning in Control and Signal Processing, Caen, France, 2013.

[4] E. Lavretsky, and K.A. Wise, "Robust and Adaptive Control With Aerospace Applications," London: Springer-Verlag, 2013, ch. 10.

[5] K.S. Narendra, and A.M. Annaswamy, "Stable Adaptive Systems," New York: Dover, 2005.

[6] V. Stepanyan, and K. Krishnakumar, "MRAC Revisited: Guaranteed Performance with Reference Model Modification," American Control Conference, 2010.