Mode Stabilities and Instabilities for Scalar Fields on Kerr Exterior Spacetimes

by

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Abstract

In this thesis we study wave and Klein-Gordon equations on Kerr exterior spacetimes. For the wave equation, we give a quantitative refinement and simple proofs of mode stability type statements on Kerr backgrounds in the full sub-extremal range ($|a| < M$). As an application, we are able to quantitatively control the energy flux along the horizon for solutions to the wave equation in any bounded-frequency regime. This estimate plays a crucial role in the author's recent proof, joint with Mihalis Dafermos and Igor Rodnianski, of boundedness and decay for the solutions to the wave equation on the full range of sub-extremal Kerr spacetimes. For the Klein-Gordon equation, we show that given any Kerr exterior spacetime with non-zero angular momentum, we may find an open family of non-zero Klein-Gordon masses for which there exist smooth, finite energy, and exponentially growing solutions to the corresponding Klein-Gordon equation. If desired, for any non-zero integer m, an exponentially growing solution can be found with mass arbitrarily close to $|am|/2Mr_+$.

Thesis Supervisor: Igor Rodnianski
Title: Professor of Mathematics

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Contents

Chapter 1

Introduction

In the previous decade there has been a tremendous amount of work on the problem of boundedness and decay for solutions to the wave equation

$$\Box_g \psi = 0 \qquad (1.1)$$

on Kerr black hole exterior spacetimes $(\mathcal{M}, g_{a,M})$. Here M denotes the mass of the spacetime, and a denotes the specific angular momentum. We will be interested in sub-extremal Kerr spacetimes which satisfy $|a| < M$. The study of the wave equation (1.1) is of direct mathematical and physical interest, however, an important motivation is the perceived connection between possessing sufficiently robust proofs of decay for solutions to (1.1) and the problem of black hole stability (cf. the role of the work [15] in the proof of the stability of Minkowski space [14]).

In a joint work [21], along with Mihalis Dafermos and Igor Rodnianski, we established boundedness and decay for the wave equations on the full sub-extremal range of Kerr spacetimes. Let us quote the following informal statement of the main result of [21].

Theorem 1. *1. General solutions ψ of* (1.1) *on the exterior of a Kerr black hole background* $(\mathcal{M}, g_{a,M})$ *in the full subextremal range $|a| < M$, arising from bounded initial energy on a suitable Cauchy surface Σ_0, have bounded energy flux through a global foliation Σ_τ of the exterior, bounded energy flux through the event horizon $\mathcal{H}^+$ and null infinity $\mathcal{I}^+$, and satisfy a suitable version of "local integrated energy decay".*

2. Similar statements hold for higher order energies involving time-translation invariant derivatives. This implies immediately uniform pointwise bounds on ψ and all translation-invariant derivatives to arbitrary order, up to and including $\mathcal{H}^+$, in terms of a sufficiently high order initial energy.

Combining Theorem 1 with the "r^p-estimates" of [16] immediately implies pointwise decay statements for solutions to (1.1) arising from localized initial data. These decays rates are sufficiently strong so as to, in principle, be applicable to small data global existence problems for nonlinear wave equations, see e.g. [34], [45], and [46].

A full discussion of the context for and history behind Theorem 1 would lead us too far afield; we direct the interested reader to the discussion in [21] and to the lecture notes [20]. Here we will content ourselves by recalling that the work [21] was preceded by various results concerning the "very slowly rotating case" where $|a| \ll M$. In this case many important difficulties can be treated pertubatively around the case $a = 0$. Under the $|a| \ll M$ assumption, boundedness for solutions to (1.1) was first shown in the work [17] and, subsequently, decay was shown in the independent works [20] and [19], [2], and [41].

There are many ingredients which go into the proof of Theorem 1. In this thesis we will focus on what is perhaps the most exotic of these, at least for the reader who is most familiar with the wave equation on product spacetimes $(\mathbb{R} \times \mathcal{N}, -dt^2 + h)$, with $(\mathcal{N}, h)$ Riemannian. This is the problem of *mode stability*. Mode solutions to the wave equation will be reviewed in Section 1.3; for now, we simply recall that a solution ψ to the wave equation $\Box_g \psi = 0$ is called a mode solution if

$$\psi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} S(\theta) R(r) \text{ with } \omega \in \mathbb{C} \text{ and } m \in \mathbb{Z},$$

where (t, r, θ, ϕ) are Boyer-Lindquist coordinates (defined in Section 1.1) and S and R must satisfy appropriate ordinary differential equations and boundary conditions (given in Section 1.3) so that, among other things, ψ has finite energy along suitable spacelike hypersurfaces.[1] Ruling out the exponentially growing mode solutions corresponding to $\mathrm{Im}(\omega) > 0$ is the content of "mode stability." This was established by Whiting in the ground-breaking [44] (of course, mode stability is a trivial corollary of Theorem 1). In the first main result of this thesis we will extend Whiting's techniques and establish a *quantitative* understanding of the lack of mode solutions with *real* ω.[2] As a byproduct of our methods, we will also be able to simplify the proof of Whiting's original mode stability result.[3] Next, we will show that this "quantitative mode stability on the real axis" can be used to control the "microlocal energy flux" along the event horizon in any "bounded frequency regime." It is this final estimate which is crucially appealed to in [21] during the proof of Theorem 1.

[1]When $\mathrm{Im}(\omega) > 0$ one should take asymptotically flat hypersurfaces connecting the future event horizon and spacelike infinity. When $\mathrm{Im}(\omega) \leq 0$ one should instead consider hyperboloidal hypersurfaces connecting the future event horizon and future null infinity. See the discussion in Section 2.2.3.

[2]See also [27] and [28] which concern solutions to the Cauchy problem of the form $e^{im\phi}\psi_0(t, r, \theta)$ and discuss mode solutions with real ω.

[3]We should note that Whiting's result [44] held for a wide class of equations; our simplification only holds for the case of the wave equation.

We note that a version of these results has appeared already in the work [40].

The second main result of this thesis will complement this stability result for the wave equation with an instability result for the Klein-Gordon equation,

$$\Box_g \psi - \mu^2 \psi = 0. \tag{1.2}$$

The key theorem is the following.

Theorem 2. *Fix a Kerr spacetime $(\mathcal{M}, g_{a,M})$ with $M > 0$ and $0 < |a| < M$. Then there exists an open family of masses μ with $\epsilon_\mu > 0$ and a non-zero, smooth, and finite energy mode solution ψ to the corresponding Klein-Gordon equation*

$$\left(\Box_g - \mu^2\right) \psi = 0$$

such that for every $(t, r, \theta, \phi) \in \mathbb{R} \times (M + \sqrt{M^2 - a^2}, \infty) \times \mathbb{S}^2$

$$e^{\epsilon_\mu t} |\partial^\alpha \psi(0, r, \theta, \phi)| \lesssim_\alpha |\partial^\alpha \psi(t, r, \theta, \phi)| \text{ for all multi-indices } \alpha. \tag{1.3}$$

These statements should be understood with respect to Boyer-Lindquist coordinates. For every non-zero integer m, μ can be chosen arbitrarily close to $\frac{|am|}{2Mr_+}$. In particular, μ can be made arbitrarily small as $a \to 0$.

Remark 1.0.1. *For convenience, we have stated our theorem in Boyer-Lindquist coordinates; however, these coordinates break down on the future event horizon $\mathcal{H}^+$ (see Section 1.1). Nevertheless, it will be easy to see that along $\mathcal{H}^+$ the solutions constructed are also exponentially growing with respect to the regular t^* coordinate; see the discussion again in Section 1.1.*

This theorem suggests that these Kerr spacetimes are non-linearly *unstable* as a solution to the Einstein-Klein-Gordon system.[4] Also, Theorem 2 serves to the emphasize the subtle effects superradiance (see Section 1.2 below) can have on the linear stability problem and helps to "explain" why even establishing the boundedness part of Theorem 1 is so difficult. In particular, since the Klein-Gordon equation decays faster than the wave equation on Minkowski space, one may have expected that the Klein-Gordan equation would be easier to control. However, as $|a| \to 0$ (where superradiance is *weaker* and one expects the problem to get *easier*) we have produced exponentially growing and finite energy solutions with arbitrarily small mass. Thus, any argument used for the wave equation must break down for Klein-Gordon equations with arbitrarily small mass. On a more conceptual level, we see that as one passes into the relativistic world, new obstructions to *boundedness*, not just decay, arise in the superradiant bounded-frequency regime. We note that a version of this result on the Klein-Gordon equation has appeared already in the work [39].

Finally, we observe that the sub-extremal Kerr spacetime is far from the only background on which to study the wave or Klein-Gordon equations; in fact, it is quite interesting to explore how changing the black hole geometry alters the subtle interplay between trapping, superradiance, and the redshift, and leads to various instabilities. In the sequence of works [30], [31], and [32], Holzegel and Holzegel-Smulevici established a logarithmic upper and *lower* bound on the decay rate for the wave and Klein-Gordon equations on non-superradiant Schwarzschild/Kerr-AdS spacetimes.[5] The slow decay rate is

[4] The Einstein-Klein-Gordon system for a spacetime $(\mathcal{M}, g)$ and massive scalar field ψ is

$$\mathrm{Ric}_{\alpha\beta}(g) - \frac{1}{2}\mathrm{R}(g)g_{\alpha\beta} = 8\pi\mathbf{T}_{\alpha\beta}(g,\psi),$$

$$\left(\Box_g - \mu^2\right)\psi = 0.$$

Here $R(g)$ is the scalar curvature, and $\mathbf{T}_{\alpha\beta}$ is the energy-momentum tensor (1.5).

[5] Here non-superradiant means that there exists a global timelike Killing vector field. In

directly traceable to a stable trapping phenomenon. In a series of papers [3], [4], [5], [6], and [7] Aretakis has studied the wave equation on various *extreme* black holes,[6] where there is a loss of the redshift due to the vanishing surface gravity of the horizon. One of the most striking results obtained is that even within the context of *axisymmetric* solutions to the wave equation on extremal Kerr, for which there is no superradiance, second derivatives of the solution blow up along the horizon.[7] Taken together with Theorem 1, these various "instabilities" serve to emphasize the miraculous properties of the wave equation on sub-extremal Kerr.

1.1 The Spacetime

Before proceeding further with the introduction, we give a precise definition of the Kerr exterior and introduce the relevant notation. Our treatment here will be brief; for a true introduction to the Kerr spacetime we recommend [42] and [36].

Fix a pair of parameters (a, M) with $|a| < M$, and define

$$r_+ \doteq M + \sqrt{M^2 - a^2}.$$

Define the underlying manifold $\mathcal{M}$ to be covered by a global[8] "Boyer-Lindquist" coordinate chart

$$(t, r, \theta, \phi) \in \mathbb{R} \times (r_+, \infty) \times \mathbb{S}^2.$$

particular, it is possible to immediately rule out solutions of the type constructed in this paper.

[6]The extreme Kerr spacetime occurs when $|a| = M$.

[7]Interestingly, it was also shown that the solutions itself decays in time.

[8]"Global" is be understood with respect to the usual degeneracy of polar coordinates.

The Kerr metric then takes the form

$$g_{a,M} = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta}dr^2 \tag{1.4}$$

$$+\rho^2 d\theta^2 + \sin^2\theta\frac{\Pi}{\rho^2}d\phi^2,$$

$$r_\pm \doteq M \pm \sqrt{M^2 - a^2},$$

$$\Delta \doteq r^2 - 2Mr + a^2 = (r - r_+)(r - r_-),$$

$$\rho^2 \doteq r^2 + a^2\cos^2\theta,$$

$$\Pi \doteq (r^2 + a^2)^2 - a^2\sin^2\theta\Delta.$$

It is convenient to define an $r^*(r) : (r_+, \infty) \to (-\infty, \infty)$ coordinate up to a constant by

$$\frac{dr^*}{dr} \doteq \frac{r^2 + a^2}{\Delta}.$$

We will often drop the parameters and refer to $g_{a,M}$ as g.

It turns out that the manifold $\mathcal{M}$ can be extended to a manifold $\tilde{\mathcal{M}}$ such that $\partial\tilde{\mathcal{M}}$ is a null hypersurface called the "horizon." Since Boyer-Lindquist coordinates would break down at the horizon, one needs a new coordinate system. The standard choice is "Kerr-star" coordinates (t^*, r, ϕ^*, θ):

$$\frac{d\bar{t}}{dr} \doteq \frac{r^2 + a^2}{\Delta},$$

$$\frac{d\overline{\phi}}{dr} \doteq \frac{a}{\Delta},$$

$$t^*(t, r) \doteq t + \bar{t}(r),$$

$$\phi^*(\phi, r) \doteq \phi + \overline{\phi}(r).$$

In these coordinates the metric becomes

$$g = -\left(1 - \frac{2Mr}{\rho^2}\right)(dt^*)^2 - \frac{4Mar\sin^2\theta}{\rho^2}dt^*d\phi^* + 2dt^*dr +$$

$$\rho^2 d\theta^2 + \sin^2\theta\frac{\Pi}{\rho^2}(d\phi^*)^2 - 2a\sin^2\theta dr d\phi^*.$$

Note that we can now extend the metric to the manifold

$$\tilde{\mathcal{M}} \doteq \left\{(t^*, r, \theta, \phi^*) \in \mathbb{R} \times (0, \infty) \times \mathbb{S}^2\right\}.$$

The (future) event horizon $\mathcal{H}^+$ is defined to be the null hypersurface $\{r = r_+\}$. Lastly, we note that in their common domain, ∂_t in Boyer-Lindquist coordinates is equal to ∂_{t^*} in Kerr-star coordinates. A similar statement applies to ∂_ϕ and ∂_{ϕ^*}.

1.2 Energy Currents, the Ergoregion, and Superradiance

In this section we will briefly review the energy-momentum tensor formalism (see [1] for a proper introduction) and discuss the presence of the ergoregion and superradiance.

Let $(\mathcal{M}, g)$ denote an arbitrary Lorentzian manifold and ∇ denote covariant differentiation. For any smooth function $\psi : \mathcal{M} \to \mathbb{C}$ we define the energy-momentum tensor

$$\mathbf{T}_{\alpha\beta} \doteq \mathrm{Re}\left(\nabla_\alpha\psi\overline{\nabla_\beta\psi}\right) - \frac{1}{2}g_{\alpha\beta}\left(|\nabla\psi|^2 + \mu^2|\psi|^2\right). \tag{1.5}$$

For any vector field X we define a corresponding 1-form, called a "current," by

$$\mathbf{J}^X_\alpha \doteq \mathbf{T}_{\alpha\beta} X^\beta. \tag{1.6}$$

The key identity is

$$\nabla^\alpha \mathbf{J}^X_\alpha = \mathrm{Re}\left(\left(\nabla^\alpha\nabla_\alpha\psi - \mu^2\psi\right)\overline{(X\psi)}\right) + \frac{1}{2}\mathbf{T}_{\alpha\beta}\pi^{\alpha\beta}. \tag{1.7}$$

Here π denotes the deformation tensor of X:

$$\pi^{\alpha\beta} \doteq \nabla^\alpha X^\beta + \nabla^\beta X^\alpha.$$

This vanishes if and only if X is Killing. In particular, if ψ solves the Klein-Gordon equation and X is Killing, we find that $\mathbf{J}^X_\alpha$ is divergence free. In this case, for any two homologous hypersurfaces Ω_1 and Ω_2, the divergence theorem gives a conservation law:

$$\int_{\Omega_1} \mathbf{J}^X_\alpha n^\alpha_{\Omega_1} = \int_{\Omega_2} \mathbf{J}^X_\alpha n^\alpha_{\Omega_2}. \tag{1.8}$$

Here n_{Ω_i} denotes the (future oriented) normal to the hypersurface Ω_i, and the integrals are with respect to the natural volume forms (the ones that make the divergence theorem true). For the identity (1.8) to be useful, we need some positivity of $\mathbf{J}^X_\alpha n^\alpha_{\Omega_i}$. This leads to the following very useful lemma.

Lemma 1.2.1. *Let* $(L, \underline{L}, E_1, E_2)$ *be a null frame, i.e.*

$$g(L,L) = g(\underline{L},\underline{L}) = g(L,E_1) = g(L,E_2) = g(\underline{L},E_1) = g(\underline{L},E_2) = g(E_1,E_2) = 0,$$

$$g(L,\underline{L}) = -2,$$

$$g(E_1, E_1) = g(E_2, E_2) = 1.$$

Then, for all constants c_1, c_2, c_3, and $c_4 \geq 0$, we have

$$\mathbf{J}_\alpha^{c_1 L + c_2 \underline{L}} (c_3 L + c_4 \underline{L})^\alpha = c_1 c_3 |L\psi|^2 + c_2 c_4 |\underline{L}|^2 + (c_2 c_3 + c_1 c_4) \left[|E_1 \psi|^2 + |E_2 \psi|^2 \right].$$

Proof. This is a simple computation using the algebraic properties of the energy-momentum tensor (see [1]). □

Remark 1.2.1. *Observe that an easy Linear Algebra exercise implies that for any two timelike vectors X and Y satisfying $g(X, Y) < 0$, one may find positive constants c_1, c_2, c_3, and c_4 and two null vectors L and $\underline{L}$ satisfying $g(L, \underline{L}) = -2$ such that*

$$X = c_1 L + c_2 \underline{L},$$

$$Y = c_3 L + c_4 \underline{L}.$$

This leads to

Definition 1.2.1. *Let X be a future oriented timelike (non-spacelike) vector field and Σ be a hypersurface with future oriented timelike (non-spacelike) normal n_Σ. We define the non-degenerate (degenerate) energy of ψ with respect to X along Σ by*

$$\int_\Sigma J_\alpha^X n_\Sigma^\alpha \tag{1.9}$$

where the integral is with respect to the induced volume form. We will often use the schematic notation

$$\int_\Sigma |\partial\psi|^2$$

to denote (1.9).

We thus have the following version of the celebrated Noether's Theorem:

Theorem 1.2.1. *(Noether's Theorem) Every Killing vector field X on a spacetime $(\mathcal{M}, g)$ implies the existence of a conservation law for the Klein-Gordon equation. The conservation law is coercive if and only if it is evaluated on a hypersurface with a non-spacelike future directed normal and X is non-spacelike and future directed. The conservation law is positive-definite in the derivatives of ψ if and only if it is evaluated on a hypersurface with a timelike future directed normal and X is timelike and future directed.*

The calculations in the following two lemmas will be convenient later.

Lemma 1.2.2. *Let X and Y be two linearly independent future oriented timelike vectors normalized to have $g(X,X) = g(Y,Y) = -1$. Set $\gamma \doteq -g(X,Y)$. Note that $\gamma > 1$ by the reverse Cauchy-Schwarz inequality. Define*

$$W \doteq \frac{1}{\sqrt{2(\gamma+1)}}(X+Y),$$

$$Z \doteq \frac{1}{\sqrt{2(\gamma-1)}}(X-Y),$$

$$L \doteq W + Z,$$

$$\underline{L} \doteq W - Z.$$

Let E_1 and E_2 be an orthonormal basis in the 2-dimensional subspace orthogonal to the span of X and Y. Then,

$$J^X_\alpha Y^\alpha = \frac{1}{4}\left(|L\psi|^2 + |\underline{L}\psi|^2\right) + \frac{\gamma}{2}\left(|E_1\psi|^2 + |E_2\psi|^2\right).$$

Proof. Observe that

$$g(L,L) = g(\underline{L},\underline{L}) = 0,$$

$$g(L,\underline{L}) = -2,$$

$$X = (1/4)\left(\left(\sqrt{2(\gamma+1)} + \sqrt{2(\gamma-1)}\right)L + \left(\sqrt{2(\gamma+1)} - \sqrt{2(\gamma-1)}\right)\underline{L}\right),$$

$$Y = (1/4)\left(\left(\sqrt{2(\gamma+1)} - \sqrt{2(\gamma-1)}\right)L + \left(\sqrt{2(\gamma+1)} + \sqrt{2(\gamma-1)}\right)\underline{L}\right).$$

The result then follows from Lemma 1.2.1. □

It is also possible to find a convenient expression for $J^X_\alpha X^\alpha$.

Lemma 1.2.3. *Let X be a timelike vector normalized to have $g(X,X) = -1$. Let R be any spacelike vector orthogonal to X, normalized to have size 1. Define*

$$L \doteq X + R,$$

$$\underline{L} \doteq X - R.$$

Let E_1 and E_2 be an orthonormal basis for the subspace orthogonal to the span of X and R. Then

$$J^X_\alpha X^\alpha = |L\psi|^2 + |\underline{L}\psi|^2 + |E_1\psi|^2 + |E_2\psi|^2.$$

Proof. The result then follows from Lemma 1.2.1 with the null frame $(L, \underline{L}, E_1, E_2)$. □

By examining the formula (1.4) for the metric, one easily sees that on the Kerr spacetime the only Killing vector field which is timelike for large r is the time translation vector field ∂_{t^*} (recall that $\partial_t = \partial_{t^*}$). Unfortunately, when $a \neq 0$, then ∂_{t^*} is spacelike on the non-empty set

$$\mathcal{S} = \left\{\Delta - a^2 \sin^2\theta < 0\right\}. \tag{1.10}$$

The set $\mathcal{S}$ is referred to as the *ergoregion*. In particular, ∂_{t^*} is spacelike at every point on the horizon where ∂_{ϕ^*} does not vanish. In fact, noting that the null generator of the horizon $\mathcal{H}^+$ may be taken to be

$$L \doteq \partial_{t^*} + \omega_+ \partial_{\phi^*}$$

where $\omega_+ \doteq \frac{a}{2Mr_+}$ is the "angular velocity" of the black hole, a straightforward computation shows that the energy density along the horizon for a solution ψ to the Klein-Gordon equation is

$$\mathbf{J}^T_\alpha L^\alpha = \mathrm{Re}\left(T\psi\overline{L\psi}\right) = \mathrm{Re}\left(T\psi\overline{\left(T\psi + \frac{a}{2Mr_+}\Phi\psi\right)}\right). \tag{1.11}$$

When the black hole possesses non-zero angular momentum ($a \neq 0$) it is clearly possible for this quantity to be negative, and thus, in principle, energy can radiate out of the black hole.

If energy is extracted through the existence of a negative $\mathbf{J}^T$ flux along the horizon, then we say that the solution ψ exhibits *superradiant amplification.* Theorem 2 shows that for certain Klein-Gordon masses and Kerr spacetimes $(\mathcal{M}, g_{a,M})$, superradiant amplification occurs and leads to an exponential instability. Conversely, Theorem 1 shows that for the wave equation (1.1), the total amount of superradiant amplification is uniformly bounded. Next, we note that one corollary of the scattering theory developed in our very recent work [22], is an explicit construction of solutions ψ to (1.1) which experience superradiant amplification. Thus, even though the level of superradiant amplification is uniformly bounded for solutions to (1.1), it does in fact occur.

1.3 Separating the Wave and Klein-Gordon Equations: Mode Solutions

When $a = 0$, in addition to possessing the Killing vector field ∂_{t^*}, the metric (1.4) is spherically symmetric. Thus, it is immediately clear that the Klein-Gordon equation $\left(\Box_{g_{0,M}} - \mu^2\right)\psi = 0$ is separable. When $a \neq 0$ the only Killing vector fields are ∂_{t^*} and ∂_{ϕ^*}. Nevertheless, as first discovered by Carter [12], the Klein-Gordon equation $\left(\Box_g - \mu^2\right)\psi = 0$ remains separable (in an appropriate coordinate system). Indeed, letting $(\omega, m) \in \mathbb{C} \setminus \{0\} \times \mathbb{Z}$, we have

$$\rho^2 e^{i\omega t} e^{-im\phi}\left(\Box_g - \mu^2\right)\left(e^{-i\omega t}e^{im\phi}\psi_0(r,\theta)\right) =$$

$$\partial_r\left(\Delta\partial_r\right)\psi_0 + \left(\frac{(r^2+a^2)^2\omega^2 - 4Mamr\omega + a^2m^2}{\Delta} - a^2\omega^2 - r^2\mu^2\right)\psi_0 + \qquad (1.12)$$

$$\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right)\psi_0 - \left(\frac{m^2}{\sin^2\theta} - a^2\left(\omega^2 - \mu^2\right)\cos^2\theta\right)\psi_0.$$

In fact, the separability of the Klein-Gordon equation follows from the presence on Kerr of a *Killing tensor* [43].

It is convenient to introduce the following definition.

$$\kappa \doteq a^2\left(\omega^2 - \mu^2\right). \qquad (1.13)$$

We call

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) - \left(\frac{m^2}{\sin^2\theta} - \kappa\cos^2\theta\right)S + \lambda S = 0 \qquad (1.14)$$

the "angular ODE." One can show that when $\omega \in \mathbb{R}$, then (1.14) along with the boundary condition

$$e^{im\phi}S(\theta) \text{ extends smoothly to } \mathbb{S}^2 \qquad (1.15)$$

defines a Sturm-Liouville problem with a corresponding collection of eigenfunctions $\{S_{\kappa ml}\}_{l=|m|}^{\infty}$ and real eigenvalues $\{\lambda_{\kappa ml}\}_{l=|m|}^{\infty}$. These $\{S_{\kappa ml}\}$ are an orthonormal basis of $L^2(\sin\theta d\theta)$ and are called "oblate spheroidal harmonics." When $\kappa = 0$ these are simply spherical harmonics, and we label them in the standard way so that $\lambda_{0ml} = l(l+1)$. The following lemma shows that λ depends smoothly on $\kappa \in \mathbb{R}$ and that for any fixed $\kappa_0 \in \mathbb{R}$, $\lambda_{\kappa ml}$ can be extended to a holomorphic function of $\kappa \in \mathbb{C}$ as long as κ is sufficiently close to κ_0.

Lemma 1.3.1. *Suppose that for some fixed $\kappa_0 \in \mathbb{R}$ we have an eigenvalue λ_0. Then, for $\kappa \in \mathbb{C}$ sufficiently close to κ_0, we can uniquely find a holomorphic curve $\lambda(\kappa)$ of eigenvalues for the angular ODE with parameter κ such that $\lambda_0 = \lambda(\kappa_0)$.*

Proof. Let's change variables to $x \doteq \cos\theta$. Then the angular ODE becomes

$$\frac{d}{dx}\left((1-x^2)\frac{dS}{dx}\right) - \left(\frac{m^2}{1-x^2} - \kappa x^2\right)S + \lambda S = 0 \text{ with } x \in (-1,1).$$

A standard asymptotic analysis (see [35]) at $x = \pm 1$ shows that any solution to the angular ODE must be asymptotic to a linear combination of $(1 \mp x)^{|m|/2}$ and $(1 \mp x)^{-|m|/2}$ as $x \to \pm 1$. If S is an eigenfunction we clearly must have

$$S \sim (1 \mp x)^{|m|/2} \text{ as } x \to \pm 1.$$

For any κ and λ we can uniquely define a solution $S(\theta, \kappa, \lambda)$ by requiring that

$$S(\theta,\kappa,\lambda)(1+x)^{-|m|/2} \text{ is smooth around } x = -1, \tag{1.16}$$

$$\left(S(\cdot,\kappa,\lambda)(1+\cdot)^{-|m|/2}\right)(x = -1) = 1.$$

We then have holomorphic functions $F(\kappa,\lambda)$ and $G(\kappa,\lambda)$ such that

$$S(\theta,\kappa,\lambda) \sim F(\kappa,\lambda)(1-x)^{-|m|/2} + G(\kappa,\lambda)(1-x)^{|m|/2} \text{ as } x \to 1.$$

Since λ_0 is an eigenvalue, we have $F(\kappa_0,\lambda_0) = 0$. We will be able to uniquely define our curve $\lambda(\kappa)$ for κ near κ_0 via an application of the implicit function theorem if we can verify that

$$\frac{\partial F}{\partial \lambda}(\kappa_0,\lambda_0) \neq 0.$$

For the sake of contradiction, assume that

$$\frac{\partial F}{\partial \lambda}(\kappa_0,\lambda_0) = 0.$$

Set

$$S_\lambda \doteq \frac{\partial S}{\partial \lambda}.$$

By differentiating (1.16) and using that F and $\frac{\partial F}{\partial \lambda}$ vanish at (κ_0,λ_0), one may easily check that S_λ still satisfies the boundary conditions of an eigenfunction. It will also satisfy

$$\frac{d}{dx}\left((1-x^2)\frac{dS_\lambda}{dx}\right) - \left(\frac{m^2}{1-x^2} - \kappa_0 x^2\right) S_\lambda + \lambda_0 S_\lambda = -S.$$

Multiplying both sides of this equation by $\overline{S}$, integrating over $(0,\pi)$, and then integrating by parts will imply that

$$\int_0^\pi |S|^2 \sin\theta d\theta = 0.$$

This is clearly a contradiction. □

In particular, it follows from Lemma 1.3.1 that for $\kappa \neq 0$, we can uniquely determine the labeling of $S_{\kappa ml}$ and $\lambda_{\kappa ml}$ by enforcing continuity in κ.

Now we are ready for the main definition of the section.

Definition 1.3.1. *Let $(\mathcal{M}, g)$ be a sub-extremal Kerr spacetime with parameters (a, M). A smooth solution ψ to the Klein-Gordon equation*

$$\left(\Box_g - \mu^2\right)\psi = 0 \tag{1.17}$$

is called a "mode solution" if there exist "parameters" $(\omega, m, l) \in \mathbb{C} \setminus \{0\} \times \mathbb{Z} \times \mathbb{Z}_{\geq m}$ such that

$$\psi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} S_{\kappa ml}(\theta) R(r, \omega, m, l), \tag{1.18}$$

where we recall that $\kappa = a^2(\omega^2 - \mu^2)$, and we require

1. *If $\mu \neq 0$, then $\omega^2 < \mu^2$.*

2. *$S_{\kappa ml}$ satisfies the boundary condition (1.15) and is an eigenfunction with eigenvalue $\lambda_{\kappa ml}$ for the angular ODE (1.14).*

3. *R is a solution to*

$$\partial_r\left(\Delta \partial_r\right) R + \left(\frac{(r^2 + a^2)^2\omega^2 - 4Mamr\omega + a^2m^2}{\Delta} - r^2\mu^2 - \lambda_{\kappa ml} - a^2\omega^2\right) R = 0 \tag{1.19}$$

4.
$$R \sim (r - r_+)^{\frac{i(am - 2Mr_+\omega)}{r_+ - r_-}} \quad at\ r = r_+.^9 \tag{1.20}$$

[9] This notation means that $R(r)(r - r_+)^{\frac{-i(am-2Mr_+\omega)}{r_+ - r_-}}$ is smooth at $r = r_+$.

5. If $\mu = 0$, then

$$R \sim \frac{e^{i\omega r^*}}{r} \text{ at } r = \infty.^{10} \tag{1.21}$$

If $\mu \neq 0$ (and hence $\omega^2 < \mu^2$) then

$$R \sim \frac{e^{-\sqrt{\mu^2-\omega^2}r^*}}{r^{1-\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}}} \text{ at } r = \infty.^{11} \tag{1.22}$$

We will often suppress some of the arguments of $S_{\kappa ml}$ and R and refer to them as $S_{\kappa ml}(\theta)$ and $R(r)$.

During the proof of our instability results, we will often write R's equation as

$$\Delta\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right) - W_\mu R = 0, \tag{1.23}$$

$$W_\mu \doteq -(r^2+a^2)^2\omega^2 + 4Mamr\omega - a^2m^2 + \Delta\left(\lambda_{\kappa ml} + a^2\omega^2 + \mu^2 r^2\right).$$

During the proof of our stability results for the wave equation, instead of considering $R(r)$, it will often be more convenient to work with the function

$$u(r^*) \doteq (r^2+a^2)^{1/2}R(r).$$

Then, letting primes denote r^*-derivatives, equation (1.19) with $\mu^2 = 0$ is equivalent to

$$u'' + \left(\omega^2 - V\right)u = 0, \tag{1.24}$$

[10]This notation means that there exists constants $\{C_i\}_{i=0}^{\infty}$ such that for every $N \geq 1$, $R(r^*) = \frac{e^{i\omega r^*}}{r}\left(\sum_{i=0}^{N}\frac{C_i}{r^i} + O\left((r)^{-N-1}\right)\right)$ for large r.

[11]This notation means that there exists constants $\{C_i\}_{i=0}^{\infty}$ such that for every $N \geq 1$, $R(r^*) = \frac{e^{i\sqrt{\omega^2-\mu^2}r^*}}{r^{1-\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}}}\left(\sum_{i=0}^{N}\frac{C_i}{r^i} + O\left((r)^{-N-1}\right)\right)$ for large r.

$$V \doteq \frac{4Mram\omega - a^2m^2 + \Delta(\lambda_{\kappa ml} + a^2\omega^2)}{(r^2+a^2)^2} + \frac{\Delta}{(r^2+a^2)^4}\left(a^2\Delta + 2Mr(r^2-a^2)\right).$$

Let's check that the boundary conditions at $r = r_+$ exactly guarantee that out mode solutions extend smoothly to the horizon.

Lemma 1.3.2. *Suppose that ψ is a mode solution. Then ψ extends smoothly to the event horizon $\mathcal{H}^+$.*

Proof. Since Boyer-Lindquist coordinates break down at $r = r_+$, in order to investigate the smoothness of ψ there, we will change to Kerr-star coordinates (t^*, r, θ, ϕ^*). In these coordinates we get

$$\psi(t^*, r, \theta, \phi^*) = e^{-i\omega(t^* - \bar{t}(r))} e^{im(\phi^* - \bar{\phi}(r))} S_{\kappa ml}(\theta) R(r).$$

Hence, ψ extends smoothly to $r = r_+$ if and only if

$$R(r) = e^{-i(\omega\bar{t}(r) - m\bar{\phi}(r))} h(r)$$

where h extends smoothly to r_+. However, this is precisely what the boundary condition (1.20) guarantees. □

As we will see in Section 2.2.3, for the wave equation, when $\mathrm{Im}(\omega) > 0$, the boundary conditions given for R and $S_{\kappa ml}$ ((1.20), (1.21), and (1.15)) are uniquely determined by requiring that ψ, given by (1.18), extends smoothly to the horizon and has finite energy along asymptotically flat hypersurfaces. The boundary conditions when $\mathrm{Im}(\omega) = 0$ follow from the requirement that the mode solution has finite energy along hyperboloidal hypersurfaces (see

Section 2.2.3). Furthermore, in Section 2.5.4 we will see these boundary conditions directly arise during the proof of integrated local energy decay for the wave equation. In the case when $\omega^2 < \mu^2$, the boundary conditions guarantee that our mode solution is exponentially decaying along asymptotically flat hypersurfaces and extends smoothly to the horizon (see Lemma 1.3.2 above and Appendix 3.2.2). Though they will only concern us tangentially here, it is worth mentioning that there is a large literature devoted to locating mode solutions with $\mathrm{Im}(\omega) < 0$ (see the review [33]). These are called *quasi-normal modes* and are expected to provide refined information about the decay of scalar fields. We direct the interested reader to the thesis [25].

1.4 Mode Stability Type Statements for the Wave Equation

In this subsection we only consider the case when $\mu^2 = 0$. Ruling out the exponentially growing mode solutions corresponding to $\mathrm{Im}(\omega) > 0$ is the content of "mode stability (in the upper half plane)." This was established by Whiting for the wave equation in 1989 [44]. However, this turns out not to be the full story. Indeed, the existence of mode solutions with $\omega \in \mathbb{R} \setminus \{0\}$ is a serious obstruction both to boundedness and "integrated local energy decay" for the wave equation. We will call the ruling out of these mode solutions "mode stability on the real axis." In Section 2.5 we will show how one can upgrade mode stability on the real axis to a bound for the "microlocal energy" along the horizon for the wave equation in any "bounded-frequency regime." In order for the constant in this estimate to be explicit, however, we will be interested in a quantitative version of mode stability of the real axis.

We turn now to an explanation of "quantitative mode stability." A standard asymptotic analysis of (1.24) (see [35]) allows one to make the following definitions:

Definition 1.4.1. *Let the parameters* $|a| < M$ *be fixed and set* $\mu = 0$. *Then define* $u_{hor}(r^*, \omega, m, l)$ *to be the unique function satisfying*

1. $u''_{hor} + (\omega^2 - V)\, u_{hor} = 0.$

2. $u_{hor} \sim (r - r_+)^{\frac{i(am - 2Mr_+\omega)}{r_+ - r_-}}$ *near* $r^* = -\infty$.

3. $\left|\left((r(r^*) - r_+)^{\frac{-i(am-2Mr_+\omega)}{r_+-r_-}} u_{hor}\right)(-\infty)\right|^2 = 1.$

Definition 1.4.2. *Let the parameters* $|a| < M$ *be fixed and set* $\mu = 0$. *Then define* $u_{out}(r^*, \omega, m, l)$ *to be the unique function satisfying*

1. $u''_{out} + (\omega^2 - V)\, u_{out} = 0.$

2. $u_{out} \sim e^{i\omega r^*}$ *near* $r^* = \infty$.

3. $\left|\left(e^{-i\omega r^*} u_{out}\right)(\infty)\right|^2 = 1.$

Remark 1.4.1. *The "*$\sim$*" has the following explicit meaning:*

1. $(r(r^*) - r_+)^{\frac{-i(am-2Mr_+\omega)}{r_+-r_-}} u_{hor}$ *is a smooth function of* r.

2. *There exists constants* $\{C_i\}_{i=0}^{\infty}$ *such that for every* $N \geq 1$,

$$u(r^*) = e^{i\omega r^*}\left(\sum_{i=0}^{N} \frac{C_i}{r^i} + O\left((r)^{-N-1}\right)\right),$$

for large r.

When there is no risk of confusion, we shall drop some or all of u_{hor}'s and u_{out}'s arguments. Next, recall that the *Wronskian*

$$u'_{\mathrm{out}}(r^*)u_{\mathrm{hor}}(r^*) - u'_{\mathrm{hor}}(r^*)u_{\mathrm{out}}(r^*)$$

is independent of r^*. Hence, we can define

$$W(\omega, m, l) \doteq u'_{\mathrm{out}}(r^*)u_{\mathrm{hor}}(r^*) - u'_{\mathrm{hor}}(r^*)u_{\mathrm{out}}(r^*). \tag{1.25}$$

This will vanish if and only if u_{out} and u_{hor} are linearly dependent, i.e. there exists a non-trivial solution (with the correct boundary conditions) to (1.24) $\Leftrightarrow W = 0 \Leftrightarrow |W^{-1}| = \infty$. "Quantitative mode stability" consists of producing an upper bound for $|W^{-1}|$ with an explicit dependence on a, M, ω, m, and l.

1.5 Precise Statement of Stability Results

In this section we will give the precise version of our stability results. Fix a Kerr spacetime $(\mathcal{M}, g)$ with parameters (a, M) satisfying $|a| < M$, and recall the definition of mode solutions (Definition 1.3.1) and the Wronskian (1.25) given in the previous section.

Our main result about mode solutions is

Theorem 1.5.1. *(Quantitative Mode Stability on the Real Axis) Let*

$$\mathscr{A} \subset \left\{(\omega, m, l) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}\right\}$$

be a set of frequency parameters with

$$C_{\mathscr{A}} \doteq \sup_{(\omega,m,l)\in\mathscr{A}} \left(|\omega| + |\omega|^{-1} + |m| + |l|\right) < \infty.$$

Then

$$\sup_{(\omega,m,l)\in\mathscr{A}} \left|W^{-1}\right| \leq G(C_{\mathscr{A}}, a, M)$$

where the function G can, in principle, be given explicitly.

Along the way we will give simple[12] proofs of

Theorem 1.5.2. *(Mode Stability)(Whiting [44]) There exist no non-trivial mode solutions for the wave equation corresponding to $Im(\omega) > 0$.*

Theorem 1.5.3. *(Mode Stability on the Real Axis) There exist no non-trivial mode solutions for the wave equation corresponding to $\omega \in \mathbb{R} \setminus \{0\}$.*

Before discussing our main application, we need a definition.

Definition 1.5.1. *We will say that a $C^\infty\left(\tilde{\mathcal{M}}\right)$ function $\psi(t,r,\theta,\phi)$ is admissible if for every compact $K \in (r_+,\infty)\times\mathbb{S}^2$ and multi-index α with $|\alpha| \geq 0$, we have*

$$\int_0^\infty \int_K |\partial^\alpha \psi|^2 \sin\theta \; dt \; dr \; d\theta \; d\phi < \infty,$$

where all of these derivatives are Boyer-Lindquist derivatives.

Next, we note that the arguments of Section 5.3 of [21] imply the following lemma.

[12]Using Whiting's integral transformations [44] but avoiding differential transformations or a physical space argument with a new metric.

Lemma 1.5.1. *Let ψ be an admissible function on Kerr that is also a solution to the wave equation $\Box_g\psi = 0$. For every $(\omega, m, l) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}$, set*

$$R(r,\omega,m,l) \doteq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} e^{i\omega t} e^{-im\phi} S_{\kappa ml}(\theta,\kappa)\, \psi \sin\theta \; d\omega \; d\theta \; d\phi.$$

Then R satisfies the boundary conditions so that $e^{-i\omega t} e^{im\phi} S_{\kappa ml} R$ is a mode solution in the sense of Definition 1.3.1.

Next, let Σ_0 denote the hypersurface $\{t^* = 0\}$, and Σ_1 denote $\{t^* = 1\}$. Define a cutoff χ which is 0 in the past of Σ_0 and identically 1 in the future of Σ_1.

Our application of Theorem 1.5.1 will be

Theorem 1.5.4. *(Boundedness of the Microlocal Energy Flux to the Horizon in the Bounded-Frequency Regime) Let ψ be an admissible function on Kerr that is also a solution to the wave equation $\Box_g\psi = 0$ with compactly supported initial data. Set*

$$\psi_{\times} \doteq \chi\psi.$$

Let $\mathscr{B} \subset \mathbb{R}$ and $\mathscr{C} \subset \{(m,l) \in \mathbb{Z} \times \mathbb{Z} : l \geq |m|\}$ be such that

$$C_{\mathscr{B}} \doteq \sup_{\omega \in \mathscr{B}} \left(|\omega| + |\omega|^{-1}\right) < \infty$$

$$C_{\mathscr{C}} \doteq \sup_{m,l \in \mathscr{C}} \left(|m| + |l|\right) < \infty,$$

and set

$$R(r,\omega,m,l) \doteq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} e^{i\omega t} e^{-im\phi} S_{\kappa ml}(\theta,\kappa)\, \psi \sin\theta \; d\omega \; d\theta \; d\phi.$$

Then,

$$\sup_{(\omega,m,l)\in\mathscr{B}\times\mathscr{C}} |R(r_+)|^2 \leq B(r_0, r_1, C_{\mathscr{B}}, C_{\mathscr{C}}, a, M) \int_{\Sigma_0} |\partial\psi|^2 . \tag{1.26}$$

where $|\partial\psi|^2$ *denotes a term proportional to a non-degenerate energy flux of a globally timelike vector field (see Section 1.5). The function* $B(r_0, r_1, C_{\mathscr{B}}, C_{\mathscr{C}}, a, M)$ *can, in principle, be given explicitly.*

Remark 1.5.1. *This estimate is crucially used in [21] during the proof of Theorem 1. For this application, it is very important that the right hand side is at the level of energy and does not posses additional weights in* r.

1.6 Precise Statement of Instability Results

In this section we will give the precise statements of our instability results.

As we have already mentioned in the introduction, we will rigorously construct finite energy solutions to the Klein-Gordon equation

$$\left(\Box_g - \mu^2\right)\psi = 0$$

on sub-extremal Kerr which grow exponentially. These growing solutions will be mode solutions:

$$\psi(t,r,\theta,\phi) \doteq e^{-i\omega t} e^{im\phi} S_{\kappa ml}(\theta) R(r) \tag{1.27}$$

where $\omega \in \mathbb{C}$, $m \in \mathbb{Z}$, $l \in \mathbb{Z}_{\geq |m|}$, $\mathrm{Im}(\omega) > 0$, and $\kappa \doteq a^2(\omega^2 - \mu^2)$. The modes we construct will be such that the boundary conditions for the functions $S_{\kappa ml}$ and R (see Sections 1.3 and 2.2.3) imply that ψ extends smoothly to the horizon (where Boyer-Lindquist coordinates break down) and decays exponentially in r along asymptotically flat Cauchy hypersurfaces. Such solutions are called "unstable modes." We say that these modes "lie in the upper half-plane." We will also consider mode solutions with $\omega \in \mathbb{R}$ which will be called "real modes." We say that these modes "lie on the real axis." It will be convenient to refer to the tuple (ω, m, l, μ) as the "parameters" of the mode. Lastly, we observe that (1.11) implies that a mode solution exhibits superradiance if and only if

$$am\mathrm{Re}\,(\omega) - 2Mr_+ |\omega|^2 > 0. \tag{1.28}$$

We can now state our main result:

Theorem 1.6.1. *Fix a sub-extremal Kerr spacetime with mass M and angular momentum aM. Let $m \in \mathbb{Z}$ and $\omega_R(0) \in \mathbb{R}$ satisfy $am - 2Mr_+\omega_R(0) = 0$ and $am \neq 0$. Then, for each $l \in \mathbb{Z}_{\geq |m|}$ and sufficiently small $\delta > 0$, there exists $\mu(0) > |\omega_R(0)|$, real analytic $\omega_R(\epsilon)$, and real analytic $\mu(\epsilon)$ such that for every $-\delta < \epsilon < \delta$, there exists a mode solution with parameters $(\omega_R(\epsilon) + i\epsilon, m, l, \mu(\epsilon))$. As $l \to \infty$, $\mu(0)$ will converge to $\omega_R(0)$. Lastly, these unstable modes must all be superradiant*[13]

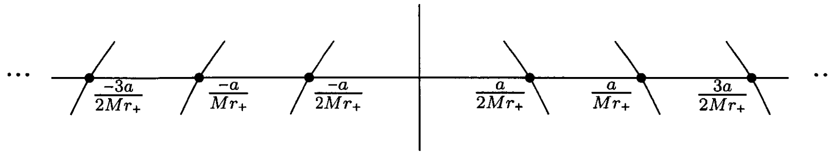
$$|am| - 2Mr_+\sqrt{\omega_R^2(\epsilon) + \epsilon^2} > 0 \tag{1.29}$$

and lose mass as they become unstable

$$\frac{\partial \mu}{\partial \epsilon}(0) < 0.$$

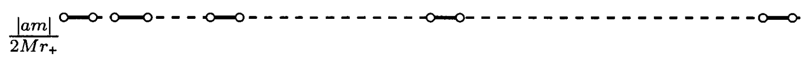
Here is a picture of the values $\{\omega(\epsilon)\} \in \mathbb{C}$ traced out by the various 1-parameter families of modes associated to a fixed l:

[13]One may easily check that (1.29) is stronger than (1.28) via the inequality $\frac{x^2+y^2}{|x|} \geq \sqrt{x^2 + y^2}$.



The reader should keep in mind that we have *not* produced any estimates for the lengths of these curves.

For each choice of $m \in \mathbb{Z} \setminus \{0\}$, there is a countable family of *intervals* of masses μ associated to growing solutions (indexed by l). These intervals will have an accumulation point at $\frac{|am|}{2Mr_+}$. The following picture may be useful for visualization:



Lest the reader be misled, we emphasize that we do *not* have any estimates for how large these intervals are, and (despite the picture) we have *not* proven that we can find $\epsilon > 0$ such that the interval $\left(\frac{|am|}{2Mr_+}, \frac{|am|}{2Mr_+} + \epsilon\right)$ is entirely made up of unstable masses. However, in light of the arguments in Section 3.4.5 we would certainly conjecture that this last statement is true.

The construction of the exponentially growing modes is achieved by perturbing modes corresponding to real ω. Thus, before proving Theorem 1.6.1, we will undertake an analysis of modes corresponding to real ω. For these modes we have two main results. The first is an existence result (already contained in Theorem 1.6.1). The second shows that the assumptions on the frequency parameters from Theorem 1.6.1 are necessary.

Theorem 1.6.2. *Suppose there exists a mode solution with parameters (ω, m, l, μ) such that $\omega \in \mathbb{R}$ and $\mu^2 > \omega^2$. Then the following statements are true.*

1. *We have $am - 2Mr_+\omega = 0$.*

2. We have $am \neq 0$.

3. There exists a function $C(\omega, m, l)$ such that $\omega^2 < \mu^2 < \omega^2 + C(\omega, m, l)$ and

$$\lim_{l\to\infty} C(\omega, m, l) = 0.$$

We will close the section with two remarks. First, we note that we can rephrase the condition $am - 2Mr_+\omega = 0$ more geometrically. Let $L = T + \omega_+\Phi$ be the null generator of the horizon. Then

$$am - 2Mr_+\omega = 0 \Leftrightarrow L\psi = 0 \Leftrightarrow \text{No energy flux along the horizon}$$

Thus, our real mode solutions are simply solutions to the Klein-Gordon equation with exactly vanishing energy flux along the horizon.

Second, by modifying the arguments behind the proof of Theorem 1.5.1 one can establish an appropriately modified version of Theorem 1.6.2 in the case $\omega^2 > \mu^2$, however we will not pursue this here.

1.7 Previous Work on Mode Solutions

By far the most important previous work on mode solutions is Whiting's mode stabilty result from 1989.

Theorem 1.7.1. *(Mode Stability)(Whiting [44]) There exist no non-trivial mode solutions for the wave equation corresponding to $Im(\omega) > 0$.*

Mode stability on the real axis was first explored numerically in [38]. In addition, [38] presented a heuristic argument (rigorously established in [29]) indicating that mode stability on the real axis would imply mode stability in the upper half plane.

Turning now to the Klein-Gordon equation, to the best of the author's knowledge, there are no previous rigorous constructions of growing solutions. However, in [8] Beyer showed that no unstable modes can exist if

$$\mu \geq \frac{|am|}{2Mr_+}\sqrt{1+\frac{2M}{r_+}}.$$

However, if we leave the realm of rigorous mathematics, then there exists a rich heuristic physics literature discussing unstable Klein-Gordon modes: Soon after the discovery of superradiant wave scattering [47], the authors of [37] speculated about placing a mirror around a black hole which would reflect superradiant frequencies. They argued that this would create a positive feedback loop and result in a "black-hole bomb." Naturally, one is led to wonder if this superradiant instability can arise in a more physically natural fashion. A key breakthrough came in 1976 when Damour, Deruelle, and Ruffini observed that a good candidate is the Klein-Gordon equation with non-zero mass [23]. A few years later, Zouros and Eardley [48] and Detweiler [24] developed more involved heuristics, all leading to the same conclusion. In particular, in [48] a connection was drawn between unstable modes for the Klein-Gordon equation and the existence of bound Keplerian orbits outside the ergoregion. Furthermore, they gave some approximations for the instability rates. Various extensions/refinements, numerical and otherwise, of these results continue to appear in the physics literature, see the very recent survey article [9] and the references therein.

We remark that many of the studies of unstable modes in the physics literature rely on the WKB approximation ([24] is an exception). Even if these

WKB arguments were made rigorous, they would only become accurate as $l \to \infty$. Since our techniques are variational, no large parameter is necessary, and we produce a much more complete picture. We also remark that it is expected that "small" Kerr-AdS black holes will exhibit superradiant instabilities [10, 11].

Chapter 2

Mode Stability for the Wave Equation

The main goal of this chapter is to prove Theorems 1.5.2, 1.5.3, 1.5.1, and 1.5.4. The chapter opens with Section 2.1 where we prove a useful ODE "unique continuation from infinity" result. Next, we review the basic aspects of mode solutions in Section 2.2. Then, in Section 2.3, the technical heart of the chapter, we introduce our refinement of Whiting's transformation and explore its important properties. We switch gears in Section 2.4 where we establish a useful estimate for solutions to the inhomogeneous radial ODE. Finally, in Section 2.5 we put everything together and prove Theorems 1.5.2, 1.5.3, 1.5.1, and 1.5.4.

2.1 A Unique Continuation Lemma

The following "unique continuation lemma" will be useful in what follows.

Lemma 2.1.1. *Suppose that we have a solution* $u(r^*) : (-\infty, \infty) \to \mathbb{C}$ *to an*

ODE

$$u'' + \left(\nu^2 - P\right) u = 0$$

such that

1. $\nu \in \mathbb{R} \setminus \{0\}$,

2. $u \in L^\infty$ *and* $\left(|u'|^2 + |u|^2\right)(\infty) = 0$,

3. P *is real,* $P \in L^\infty$, $P = O\left(r^{-1}\right)$ *as* $r \to \infty$, *and* $P' = O\left(r^{-2}\right)$ *as* $r \to \infty$.

Then u *is identically* 0.

Proof. Define

$$y(r^*) \doteq \exp\left(-B \int_{r^*}^{\infty} \zeta(r) dr\right)$$

where B is a large positive constant to be chosen later and ζ is a *fixed* positive function which is identically 1 near $r = -\infty$ and is equal to r^{-2} near $r = \infty$. In particular, we have $y'|_{(-\infty,\infty)} > 0$, $y(-\infty) = 0$, and $y(\infty) = 1$.

Next, set

$$Q^y(r^*) \doteq y\,|u'|^2 + y\left(\nu^2 - P\right)|u|^2.$$

Observe that the hypothesis of the lemma imply that $Q^y(\pm\infty) = 0$. A simple computation gives

$$(Q^y)' = y'\,|u'|^2 + y'\nu^2\,|u|^2 - (yP)'\,|u|^2.$$

Thus, the fundamental theorem of calculus implies

$$\int_{-\infty}^{\infty} \left(y'\,|u'|^2 + y'\nu^2\,|u|^2 - (yP)'\,|u|^2\right) dr^* = 0. \tag{2.1}$$

Let $R \in (1,\infty)$ be a large constant to be chosen later. Then set $\chi(r^*)$ to be a function identically 1 on $(-\infty, R]$ and 0 on $[R+1, \infty)$. We then define

$$P_1 \doteq \chi P,$$

$$P_2 \doteq (1-\chi)P.$$

Of course we have $P = P_1 + P_2$.

We have the following estimate:

$$\left| \int_{-\infty}^{\infty} (yP_1)' |u|^2 \, dr^* \right| = 2 \left| \int_{-\infty}^{\infty} yP_1 \mathrm{Re}\,(u'\overline{u}) \, dr^* \right| \tag{2.2}$$

$$\leq \epsilon \int_{-\infty}^{\infty} y' |u'|^2 \, dr^* + \epsilon^{-1} \int_{-\infty}^{\infty} y' \nu^2 \left(\frac{y^2 P_1^2}{(y')^2 \nu^2} \right) |u|^2 \, dr^* \tag{2.3}$$

$$\leq \epsilon \int_{-\infty}^{\infty} y' |u'|^2 \, dr^* + C\epsilon^{-1}\nu^{-2}B^{-2}R^2 \int_{-\infty}^{\infty} y'\nu^2 |u|^2 \, dr^*. \tag{2.4}$$

Here C is a constant which only depends on ζ and P.

Next we estimate

$$\left| \int_{-\infty}^{\infty} (yP_2)' |u|^2 \, dr^* \right| = \left| \int_{-\infty}^{\infty} (y'P_2 + yP_2') |u|^2 \, dr^* \right| \tag{2.5}$$

$$\leq C \int_{-\infty}^{\infty} \left(R^{-1}\nu^{-2} + B^{-1}\nu^{-2} \right) y'\nu^2 |u|^2 \, dr^*. \tag{2.6}$$

Taking ϵ small, R large, and then B sufficiently large and combining (2.1), (2.4), and (2.6) implies that

$$\frac{1}{2} \int_{-\infty}^{\infty} \left(y' |u'|^2 + y'\nu^2 |u|^2 \right) dr^* = 0.$$

□

2.2 Basic Properties of Mode Solutions to the Wave Equation

In this section we will review various basic properties of mode solutions for the wave equation. In particular, we first discuss how the boundary conditions for a mode imply finite energy along suitable hypersurfaces, and then review the situations where mode solutions can be ruled out with easy arguments based on energy conservation.

2.2.1 The Hypersurfaces

For purposes of exposition we will restrict attention to spacelike hypersurfaces Σ_f which, for sufficiently large R, satisfy

$$\Sigma_f \cap \{r \geq R\} \doteq \{(t, r^*, \theta, \phi) : r \geq R \text{ and } t - f(r^*) = 0\}.$$

In addition to the requirement that Σ_f be spacelike, we also ask that Σ_f intersects the future event horizon and

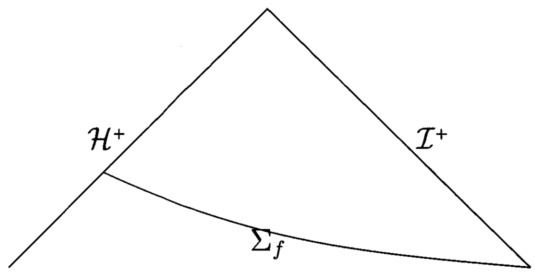
$$f \geq 0 \text{ as } r^* \to \infty.$$

This last requirement implies that Σ_f connects the event horizon $\mathcal{H}^+$ to either spacelike infinity or future null infinity.

Definition 2.2.1. *We will say that Σ_f is asymptotically flat if $f \sim 1$ as $r^* \to \infty$.*[1]

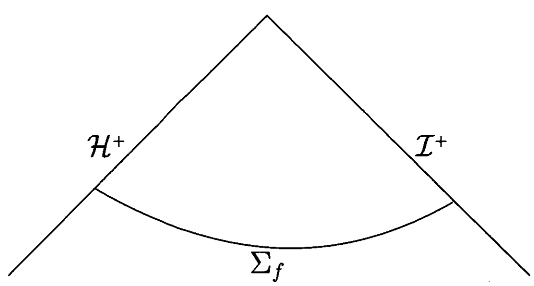
[1] More generally, one could consider any hypersurface which terminates at spacelike infinity, but this extra generality is not particularly useful for the study of mode solutions.

These hypersurfaces converge to spacelike infinity as $r^* \to \infty$. The prototypical example of an asymptotically flat hypersurface is one where f is identically constant for large r. The relevant Penrose diagram is



Definition 2.2.2. *We will say that Σ_f is hyperboloidal if $(f')^2 - 1 = -\frac{C}{r^2} + O(r^{-3})$ as $r^* \to \infty$ for some sufficiently large positive constant C ($C \geq M$ will work).*[2]

These hypersurfaces converge to future null infinity as $r^* \to \infty$. The key examples to keep in mind are hyperbolas in Minkowski space (where $f = \sqrt{C + r^2}$). The relevant Penrose diagram is



[2]In more general contexts one usually says a spacelike hypersurface is hyperboloidal if the induced metric asymptotically approaches a constant negative curvature metric. One could work with this more general definition here; but, since there is not much advantage for the study of mode solutions, we shall spare ourselves the extra work.

2.2.2 Some Useful Calculations

We start by noting that

$$g^{tt} = \frac{a^2 \sin^2 \theta \Delta - (r^2 + a^2)^2}{\rho^2 \Delta},$$

$$g^{\phi\phi} = \frac{\Delta - a^2 \sin^2 \theta}{\sin^2 \theta \Delta},$$

$$g^{t\phi} = -\frac{4Mar}{\rho^2 \Delta}.$$

Then we have

Lemma 2.2.1. *Let* $\mu = 0$, Σ_f *be an asymptotically flat hypersurface,* N *be a future oriented timelike vector field which equals* ∂_t *for large* r, *and* ψ *be a smooth function. Then, for sufficiently large* R, *the energy of* ψ *with respect to* N *along* $\Sigma_f \cap \{r \geq R\}$ *is proportional to*

$$\int_{r \geq R} \int_{\mathbb{S}^2} \left(|\partial_t \psi|^2 + |\partial_r \psi|^2 + r^{-2} \left((\partial_\theta \psi)^2 + \sin^{-2} \theta \, (\partial_\phi \psi)^2 \right) \right) (f(r^*), r, \theta, \phi) \times$$

$$r^2 \sin \theta \; dr \; d\theta \; d\phi.$$

Proof. First, observe that

$$-\nabla t = -g^{tt} \partial_t - g^{t\phi} \partial_\phi = \frac{(r^2 + a^2)^2 - a^2 \sin^2 \theta}{\rho^2 \Delta} \partial_t + \frac{4Mar}{\rho^2 \Delta} \partial_\phi,$$

$$g(\nabla t, \nabla t) = \frac{-(r^2 + a^2)^2 + a^2 \sin^2 \theta \Delta}{\rho^2 \Delta}.$$

In particular, ∇t is timelike. Next, we calculate

$$g(\nabla (t - f(r^*)), \nabla (t - f(r^*))) =$$

$$\left((f')^2 - 1\right)\frac{(r^2+a^2)^2}{\rho^2\Delta} + \frac{a^2\sin^2\theta}{\rho^2} \to -1 \text{ as } r \to \infty.$$

We conclude that the normal to Σ_f satisfies

$$n_{\Sigma_f} = \left(1 + O\left(r^{-1}\right)\right)(-\nabla t) + O\left(r^{-1}\right)\partial_{r^*} \text{ as } r \to \infty.$$

Now, Lemma 1.2.2 implies

$$J^N_\alpha n^\alpha_{\Sigma_f} \approx |\partial_t\psi|^2 + |\partial_{r^*}\psi|^2 + r^{-2}\left((\partial_\theta\psi)^2 + \sin^{-2}\theta\,(\partial_\phi\psi)^2\right) \text{ as } r \to \infty.$$

The volume form on Kerr satisfies

$$dVol = \frac{\Delta\rho^2}{r^2+a^2}\sin\theta\; dt \wedge dr^* \wedge d\theta \wedge d\phi.$$

Thus, the induced volume on Σ_f is given by

$$\left(1 + O(r^{-1})\right) r^2 \sin\theta\; dr^* \wedge d\theta \wedge d\phi + \left(1 + O(r^{-1})\right) r\sin\theta\; dt \wedge\; d\theta \wedge d\phi +$$

$$\left(1 + O(r^{-1})\right) r\sin\theta\; dt \wedge dr^* \wedge d\theta \text{ as } r \to \infty.$$

The lemma follows by writing out the integral (1.9) in the parametrization $(r^*, \theta, \phi) \mapsto (f(r^*), r^*, \theta, \phi)$. □

The analogous lemma in the hyperboloidal case is more subtle since we need to understand precisely how the energy degenerates due to the hypersurface becoming "approximately null."

Lemma 2.2.2. *Let $\mu = 0$, Σ_f be a hyperboloidal hypersurface, N be a future oriented timelike vector field which equals ∂_t for large r, and ψ be a smooth function. Then, for sufficiently large R, the energy of ψ with respect to N*

along $\Sigma_f \cap \{r \geq R\}$ is proportional to

$$\int_{r\geq R}\int_{\mathbb{S}^2}\Bigg(r^{-2}\left|\left(\partial_t-\partial_{r^*}\right)\psi\right|^2+\left|\left(\partial_t+\partial_{r^*}\right)\psi\right|^2+ \\ r^{-2}\left(\left(\partial_\theta\psi\right)^2+\sin^{-2}\theta\left(\partial_\phi\psi\right)^2\right)\Bigg)r^2\sin\theta\;dr\;d\theta\;d\phi$$

where the integrand is evaluated at $(f(r^*), r^*, \theta, \phi)$.

Proof. Let's set

$$\begin{aligned} A_{\Sigma_f} &\doteq \sqrt{-g\left(\nabla\left(t-f(r^*)\right),\nabla\left(t-f(r^*)\right)\right)} \\ &= \sqrt{\left(1-(f')^2\right)\frac{(r^2+a^2)^2}{\rho^2\Delta}-\frac{a^2\sin^2\theta}{\rho^2}} \\ &= O\left(r^{-1}\right) \text{ as } r\to\infty. \end{aligned}$$

The normal n_{Σ_f} thus satisfies

$$n_{\Sigma_f} = A_{\Sigma_f}^{-1}\left(-\nabla t + f'\frac{(r^2+a^2)^2}{\rho^2\Delta}\partial_{r^*}\right).$$

The key difference with the asymptotically flat case is that $A_{\Sigma_f}^{-1} = r + O(1)$ as $r\to\infty$.

Let's apply Lemma 1.2.2 to the vectors $X \doteq \left(-g_{tt}\right)^{-1/2}\partial_t$ and $Y \doteq n_{\Sigma_f}$. We have

$$\gamma \doteq -g\left(X,Y\right) = \left(-g_{tt}\right)^{-1/2}A_{\Sigma_f}^{-1} = r + O(1) \text{ as } r\to\infty.$$

Next, we compute

$$\begin{aligned} W &= \frac{1}{\sqrt{2(\gamma+1)}}(X+Y) \\ &= O\left(r^{-1/2}\right)\partial_t + \left(r^{1/2} + O\left(r^{-1/2}\right)\right)(-\nabla t + \partial_{r^*}) \text{ as } r \to \infty, \end{aligned}$$

$$\begin{aligned} Z &= \frac{1}{\sqrt{2(\gamma-1)}}(X-Y) \\ &= O\left(r^{-1/2}\right)\partial_t - \left(r^{1/2} + O\left(r^{-1/2}\right)\right)(-\nabla t + \partial_{r^*}) \text{ as } r \to \infty, \end{aligned}$$

$$\begin{aligned} L &= W + Z \\ &= O\left(r^{-1/2}\right)(\partial_t + (-\nabla t + \partial_{r^*})) \text{ as } r \to \infty, \end{aligned}$$

$$\begin{aligned} \underline{L} &= W - Z \\ &= O\left(r^{-3/2}\right)\partial_t + 2\left(r^{1/2} + O\left(r^{-1/2}\right)\right)(-\nabla t + \partial_{r^*}) \text{ as } r \to \infty. \end{aligned}$$

Finally, as $r \to \infty$, the induced volume form satisfies

$$\begin{aligned} &\left(A_{\Sigma_f}^{-1}\frac{(r^2+a^2)^2}{\rho^2\Delta} + O\left(r^{-3}\right)\right)\left(\frac{\Delta\rho^2}{r^2+a^2}\right)\sin\theta\; dr^* \wedge d\theta \wedge d\phi \\ &+O(1)\frac{\Delta\rho^2}{r^2+a^2}\sin\theta\; dt \wedge dr^* \wedge d\theta \\ &-\left(A_{\Sigma_f}^{-1} f'\frac{(r^2+a^2)^2}{\rho^2\Delta}\right)\left(\frac{\Delta\rho^2}{r^2+a^2}\right)\sin\theta\; dt \wedge d\theta \wedge d\phi. \end{aligned}$$

The lemma now follows by carefully writing out the integral (1.9) in the parametrization $(r^*,\theta,\phi) \mapsto (f(r^*),r^*,\theta,\phi)$, using $(f')^2 - 1 = -\frac{C}{r^2} + O(r^{-3})$, and appealing to Lemma 1.2.2. □

2.2.3 Finite Energy Hypersurfaces for Mode Solutions

Lemma 2.2.3. *Let $\mu = 0$, Σ_f be an asymptotically flat hypersurface, N be a future oriented timelike vector field which equals ∂_t for large r, and*

$$\psi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} S_{\kappa ml}(\theta) R(r)$$

be a mode solution. If $Im(\omega) > 0$ then ψ has finite energy with respect to N along Σ_f. If $Im(\omega) \leq 0$, then ψ has infinite energy with respect to N along Σ_f.

Proof. In Kerr-star coordinates, it is easy to see that the volume form remains bounded in a compact region of r (including the event horizon). Thus, in order for ψ to have finite energy along $\Sigma_f \cap \{r \leq R\}$ it is sufficient for ψ to be smooth (and hence bounded). Furthermore, ψ is manifestly smooth if $r > r_+$. Since Boyer-Lindquist coordinates break down at $r = r_+$, in order to investigate the smoothness of ψ there, we will change to Kerr-star coordinates (t^*, r, θ, ϕ^*). In these coordinates we get

$$\psi(t^*, r, \theta, \phi^*) = e^{-i\omega\left(t^* - \bar{t}(r)\right)} e^{im\left(\phi^* - \bar{\phi}(r)\right)} S_{\kappa ml}(\theta) R(r).$$

Hence, ψ extends smoothly to $r = r_+$ if and only if

$$R(r) = e^{-i\left(\omega\bar{t}(r) - m\bar{\phi}(r)\right)} h(r)$$

where h extends smoothly to r_+. However, this is precisely what the boundary condition (1.20) guarantees.

For R sufficiently large, Lemma 2.2.1 imply that the energy along $\Sigma_f \cap \{r \geq$

$R\}$ is proportional to

$$\int_{r\geq R}\int_{\mathbb{S}^2}\left(|\partial_t\psi|^2+|\partial_r\psi|^2+r^{-2}\left((\partial_\theta\psi)^2+\sin^{-2}\theta\,(\partial_\phi\psi)^2\right)\right)(f(r^*),r,\theta,\phi)\times$$

$$r^2\sin\theta\;dr\;d\theta\;d\phi.$$

Now, if $\mathrm{Im}\,(\omega)>0$, then the boundary condition (1.21) implies that all of these terms are decaying exponentially as $r\to\infty$, and hence, the integral is finite. If $\mathrm{Im}\,(\omega)=0$, then the first two terms in the integral as proportional to r^{-2}, and hence the integral is infinite. If $\mathrm{Im}\,(\omega)<0$, then all of the terms are exponentially growing in r, and hence the integral is infinite. □

Lemma 2.2.4. *Let $\mu=0$, Σ_f be a hyperboloidal hypersurface, N be a future oriented timelike vector field which equals ∂_t for large r, and*

$$\psi(t,r,\theta,\phi)=e^{-i\omega t}e^{im\phi}S_{\kappa ml}(\theta)R(r)$$

be a mode solution with $Im\,(\omega)\leq 0$. Then ψ has finite energy with respect to N along Σ_f.

Proof. The analysis of ψ for any compact region of r is exactly the same as in the proof of Lemma 2.2.3. In Lemma 2.2.2 we saw that the energy along $\Sigma_f\cap\{r\geq R\}$ is proportional to

$$\int_{r\geq R}\int_{\mathbb{S}^2}\Bigg(r^{-2}\left|(\partial_t-\partial_{r^*})\,\psi\right|^2+\left|(\partial_t+\partial_{r^*})\,\psi\right|^2+$$
$$r^{-2}\left((\partial_\theta\psi)^2+\sin^{-2}\theta\,(\partial_\phi\psi)^2\right)\Bigg)r^2\sin\theta\;dr\;d\theta\;d\phi$$

where the integrand is evaluated at $(f(r^*),r^*,\theta,\phi)$.

When $\mathrm{Im}(\omega) = 0$, then the boundary condition (1.21) exactly implies that $(\partial_t + \partial_{r^*})\psi = O(r^{-2})$. Combining this with the fact that ψ and it's derivatives are all $O(r^{-1})$ shows that the integral is finite.

Now consider the case when $\mathrm{Im}(\omega) < 0$. Using the boundary condition (1.21), we get

$$\begin{aligned}\psi(f(r^*), r^*, \theta, \phi) &= \exp(-i\omega f(r^*)) e^{im\phi} S_{\kappa ml}(\theta) R(r) \\ &= O\left(r^{-1}\exp(-i\omega r^*)\exp(i\omega(r^* - 2M\log r))\right) \text{ as } r \to \infty \\ &= O\left(r^{-1}\right) \text{ as } r \to \infty.\end{aligned}$$

Similarly,

$$\begin{aligned}\partial_t\psi(f(r^*), r^*, \theta, \phi) &= O\left(r^{-1}\right) \text{ as } r \to \infty, \\ \partial_{r^*}\psi(f(r^*), r^*, \theta, \phi) &= O\left(r^{-1}\right) \text{ as } r \to \infty, \\ \partial_\theta\psi(f(r^*), r^*, \theta, \phi) &= O\left(r^{-1}\right) \text{ as } r \to \infty, \\ \partial_\phi\psi(f(r^*), r^*, \theta, \phi) &= O\left(r^{-1}\right) \text{ as } r \to \infty, \\ (\partial_t + \partial_{r^*})\psi(f(r^*), r^*, \theta, \phi) &= O\left(r^{-2}\right) \text{ as } r \to \infty.\end{aligned}$$

Thus, the integral is finite.

□

2.2.4 Modes on Schwarzschild

It is instructive to observe that the counterpart to mode stability in the Riemannian setting[3] is the "automatic" fact that the Laplace-Beltrami operator has no spectrum in the upper half plane. A better way to see the trivial-

[3]This is the case of a product metric $(\mathbb{R} \times N, -dt^2 + g_N)$ with (N, g_N) complete and Riemannian.

ity of Riemannian mode stability is to note that the existence of a uniformly timelike vector field ∂_t immediately implies the uniform boundedness of a non-degenerate energy (see Section 1.2).

Recall that the Schwarzschild spacetime is the Kerr spacetime with vanishing angular momentum ($a = 0$). This is not a product metric; nevertheless, T is a timelike Killing vector field for all $r > r_+$, the associated conserved energy is coercive, and mode stability is immediately established in a similar fashion to the previous paragraph.[4]

Mode stability on the real axis for Schwarzschild is more subtle since, as we have seen in the previous section, real mode solutions for the wave equation have infinite energy along asymptotically flat hypersurfaces. However, this does not preclude physical space methods; one simply observes

1. Lemma 2.2.4 implies that real mode solutions for the wave equation have finite energy along any hyperboloidal hypersurface $\tilde{\Sigma}$.

2. A straightforward computation shows that the energy flux for such real modes along the portion of null infinity in the future of $\tilde{\Sigma}$ must be infinite.

3. The energy identity associated to T (see Section 1.2) implies that the energy flux along the portion of null infinity in the future of $\tilde{\Sigma}$ must be less than or equal to the energy flux along $\tilde{\Sigma}$.

This is a clear contradiction to the existence of real modes.

For later purposes it will be convenient to revisit these arguments from a "microlocal" point of view. In phase space, the analogue of the energy flux is

[4]Of course, T becomes null on the horizon, and thus the conserved energy degenerates as $r \to r_+$. However, a moment's thought shows that this does not affect the argument.

the microlocal energy current:

$$Q_T(r^*) \doteq \operatorname{Im}\left(u'\overline{\omega u}\right).$$

Let us show how the microlocal energy can be used to give a short proof of mode stability. Suppose we have a mode solution to the wave equation with corresponding $u(r^*)$ and $\omega = \omega_R + i\omega_I$ for some $\omega_I > 0$. First, we observe that the boundary conditions (1.20) and (1.21) imply that $Q_T(\pm\infty) = 0$. Next, we compute

$$\begin{aligned} -(Q_T)' &= \omega_I |u'|^2 + \operatorname{Im}\left(\left(\omega^2 - V\right)\overline{\omega}\right)|u|^2 \\ &= \omega_I \left(|u'|^2 + \left(|\omega|^2 + \frac{(r-2M)(rl(l+1)+2M)}{r^4}\right)|u|^2\right). \end{aligned}$$

Since the coefficients of $|u'|^2$ and $|u|^2$ are positive, the fundamental theorem of calculus implies that u is identically 0. Algebraically, we are exploiting the fact that the potential V does not depend on ω and is positive.

Now consider a real mode solution to the wave equation with corresponding $u(r^*)$ and $\omega \in \mathbb{R} \smallsetminus \{0\}$. This time we have "conservation of energy,"

$$(Q_T)' = 0.$$

Integrating gives

$$Q_T(\infty) - Q_T(-\infty) = 0 \Rightarrow$$

$$\omega^2 |u(\infty)|^2 + 2Mr_+\omega^2 |u(-\infty)|^2 = 0.$$

We have used the boundary conditions (1.20) and (1.21) to evaluate the microlocal energy current at $\pm\infty$. Applying the unique continuation lemma from

Section 2.1 immediately implies that u vanishes identically.

2.2.5 Modes on Kerr

On the Kerr spacetime all of these arguments break down.

We have already observed that when $a \neq 0$ there is a region $\mathcal{S}$, the ergoregion, where the Killing vector field T is spacelike. Hence, the associated conserved quantity is no longer coercive and is useless by itself.

At the level of the ODE, we may again define a microlocal energy current:

$$Q_T \doteq \operatorname{Im}\left(u'\overline{\omega u}\right).$$

However,

$$\operatorname{Im}\left(\left(\omega^2 - V\right)\overline{\omega}\right) =$$

$$\omega_I\left(|\omega|^2 - \frac{a^2m^2}{(r^2+a^2)^2} + \frac{\Delta}{(r^2+a^2)^4}\left(a^2\Delta + 2Mr(r^2-a^2)\right)\right) +$$

$$\frac{\Delta}{(r^2+a^2)^2}\operatorname{Im}\left(\left(\lambda_{\kappa ml} + a^2\omega^2\right)\overline{\omega}\right)$$

is no longer always positive. In fact, for $\omega_I > 0$

$$\operatorname{Im}\left(\left(\omega^2 - V\right)\overline{\omega}\right)(-\infty) = \omega_I\left(|\omega|^2 - \frac{a^2m^2}{4M^2r_+^2}\right) < 0 \Leftrightarrow$$

$$|am| - 2Mr_+|\omega| > 0.$$

This troublesome frequency regime also arises if $\omega \in \mathbb{R} \setminus \{0\}$. For such ω we still have "conservation of energy,"

$$(Q_T)' = 0.$$

Integrating and evaluating with the boundary conditions (1.20) and (1.21) gives

Proposition 2.2.1. *(The Microlocal Energy Estimate)*

$$\omega^2 |u(\infty)|^2 - \omega(am - 2Mr_+\omega)|u(-\infty)|^2 = 0.$$

If $\omega(am - 2Mr_+\omega) < 0$, then this gives a successful estimate of the boundary terms $|u(-\infty)|^2$ and $|u(\infty)|^2$. However, if

$$\omega(am - 2Mr_+\omega) \geq 0, \tag{2.7}$$

then Proposition 2.2.1 fails to give an estimate for $|u(-\infty)|^2$ and $|u(\infty)|^2$. In the case of (2.7) we say that our frequency parameters are *superradiant*. The existence of superradiant frequencies is the phase space manifestation of the fact that the physical space energy flux associated to T may be negative along the horizon, i.e. energy can be extracted from a spinning black hole.

Despite these difficulties, in [44] Whiting was able to give a relatively short proof of mode stability for a wide class of equations on sub-extremal Kerr, including the wave equation $\Box_g\psi = 0$, i.e. Theorem 1.5.2. By closely examining the structure of u's and $S_{\kappa ml}$'s equations, Whiting found (appropriately non-degenerate) integral and differential transformations taking u to $\tilde{u}$ and $S_{\kappa ml}$ to $\tilde{S}_{\omega ml}$ such that

$$\tilde{\psi}(t,r,\theta,\phi) \doteq (r^2+a^2)^{-1/2}e^{-i\omega t}e^{im\phi}\tilde{S}_{\omega ml}(\theta)\tilde{u}(r^*(r))$$

satisfied a wave equation $\Box_{\tilde{g}}\tilde{\psi} = 0$ associated to a new metric $\tilde{g}$ for which there was no ergoregion. After this miracle, the proof concluded with a physical space energy argument as in our discussion of Schwarzschild in Section 2.2.4.

2.3 The Whiting Transformation

In this section, which is the technical heart of the proof of our stability results, we will introduce our extension of Whiting's transformation and establish its important properties. We emphasize that throughout this section we will always take $\mu = 0$.

It turns out to be useful to work with the inhomogeneous version of R's and u's equations:

$$\Delta\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right) - \tilde{V}R = \Delta(r^2+a^2)F(r) \doteq \Delta\hat{F}, \tag{2.8}$$

$$\tilde{V} \doteq -(r^2+a^2)^2\omega^2 + 4Mamr\omega - a^2m^2 + \Delta\left(\lambda_{\kappa ml} + a^2\omega^2\right).$$

Here $\hat{F}$ will be assumed to be a C^∞ function compactly supported in (r_+,∞).

Recalling that $u(r^*) = (r^2+a^2)^{1/2}R(r)$, we have

$$u'' + \left(\omega^2 - V\right)u = H, \tag{2.9}$$

$$\begin{aligned} V \doteq & \frac{4Mram\omega - a^2m^2 + \Delta(\lambda_{\kappa ml} + a^2\omega^2)}{(r^2+a^2)^2} \\ & + \frac{\Delta}{(r^2+a^2)^4}\left(a^2\Delta + 2Mr(r^2-a^2)\right), \end{aligned}$$

$$H(r^*) \doteq \frac{\Delta}{(r^2+a^2)^{1/2}}F(r). \tag{2.10}$$

Our starting point is Whiting's integral transformation:

$$\tilde{u}(x^*) \doteq (x^2+a^2)^{1/2}(x-r_+)^{-2iM\omega}e^{-i\omega x}\times \int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+-r_-}(x-r_-)(r-r_-)}(r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}R(r)dr. \tag{2.11}$$

Here η and ξ are given by

$$\eta \doteq \frac{-i(am-2Mr_-\omega)}{r_+-r_-}, \tag{2.12}$$

$$\xi \doteq \frac{i(am-2Mr_+\omega)}{r_+-r_-}. \tag{2.13}$$

In [44] Whiting used the above transformation only on modes satisfying the homogeneous equation with $\text{Im}(\omega)>0$, and the integral was thus absolutely convergent. Since we shall also allow $\omega \in \mathbb{R}\setminus\{0\}$, at first, $\tilde{u}$ only makes sense as an L^2_{loc} function. Nevertheless, we will establish

Proposition 2.3.1. *Let $\mu=0$, $Im(\omega)\geq 0$, $\omega\neq 0$, R solve the inhomogeneous radial ODE (2.8), and R satisfy the boundary conditions from Definition 1.3.1. Define $\tilde{u}$ via Whiting's integral transformation (2.11). Then $\tilde{u}(x)$ is C^∞ on (r_+,∞) and, letting primes denote x^*-derivatives, satisfies*

$$\tilde{u}''+\Phi\tilde{u}=\tilde{H},$$

where

$$\tilde{H}(x^*) \doteq \frac{(x-r_+)(x-r_-)}{(x^2+a^2)^2}\tilde{G}(x), \tag{2.14}$$

$$\tilde{G}(x) \doteq (x^2 + a^2)^{1/2}(x - r_+)^{-2iM\omega} e^{-i\omega x} \times$$
$$\int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+ - r_-}(x - r_-)(r - r_-)} (r - r_-)^{\eta}(r - r_+)^{\xi} e^{-i\omega r} \hat{F}(r)dr,$$

$$\Phi(x^*) \doteq \frac{(x - r_-)\tilde{\Phi}_1(x)}{(x^2 + a^2)^2} - \tilde{\Phi}_2(x),$$ [5]

$$\tilde{\Phi}_1(x) \doteq \omega^2(x - r_+)^2(x - r_-) - \left(4M\omega^2 + \frac{4\omega(am - 2Mr_+\omega)}{r_+ - r_-}\right)(x - r_-)(x - r_+)$$
$$+4M^2\omega^2(x - r_-) + \left(2am\omega - \lambda_{\kappa ml} - a^2\omega^2\right)(x - r_+),$$
$$\tilde{\Phi}_2(x) \doteq \frac{(x - r_+)(x - r_-)}{(x^2 + a^2)^4}\left(a^2(x - r_+)(x - r_-) + 2Mx(x^2 - a^2)\right).$$

Of course, it is important to understand the boundary conditions for $\tilde{u}$. When $\mathrm{Im}(\omega) > 0$, the following quite crude analysis of $\tilde{u}$ is sufficient.

Proposition 2.3.2. *If $Im(\omega) > 0$, then*

1. $\tilde{u} = O\left((x - r_+)^{2M Im(\omega)}\right)$ *as* $x \to r_+$.
2. $\tilde{u}' = O\left((x - r_+)^{2M Im(\omega)}\right)$ *as* $x \to r_+$.
3. $\tilde{u} = O\left(e^{-Im(\omega)x} x^{1+2M Im(\omega)}\right)$ *as* $x \to \infty$.
4. $\tilde{u}' = O\left(e^{-Im(\omega)x} x^{1+2M Im(\omega)}\right)$ *as* $x \to \infty$.

When $\omega \in \mathbb{R} \setminus \{0\}$ we need to be a little more precise.

Proposition 2.3.3. *If $\omega \in \mathbb{R} \setminus \{0\}$, then*

1. *$\tilde{u}$ is uniformly bounded.*
2. $|\tilde{u}(\infty)|^2 = \frac{(r_+ - r_-)^2 |\Gamma(2\xi + 1)|^2}{8M\omega^2 r_+} |u(-\infty)|^2$.
3. *$\tilde{u}'$ is uniformly bounded.*

[5] For mode stability on the real axis, it is only important that Φ is real.

4. $\tilde{u}' - i\omega\tilde{u} = O(x^{-1})$ *at* $x^* = \infty$.

5. $\tilde{u}' + \frac{i\omega(r_+ - r_-)}{r_+}\tilde{u} = O(x - r_+)$ *at* $x^* = -\infty$.

Here

$$\Gamma(z) \doteq \int_0^\infty e^{-t} t^{z-1} dt$$

is the Gamma function. Recall the well known fact that the (extended) Gamma function is meromorphic, never vanishes, and only has poles at 0, −1, −2, ⋯.

We now turn to the proof of Propositions 2.3.1, 2.3.2, and 2.3.3. For clarity of exposition we will restrict ourselves to $\omega \in \mathbb{R} \setminus \{0\}$; indeed, for $\mathrm{Im}(\omega) > 0$ the proofs are much easier and follow from the same sort of reasoning as the real ω case. Furthermore, due to the symmetries of the radial ODE, we may restrict ourselves to $\omega > 0$.

It will be convenient to adopt the notation

$$A \doteq \frac{2i\omega}{r_+ - r_-}.$$

It will also be useful to consider the following functions

$$g(r) \doteq (r - r_+)^{-\xi}(r - r_-)^{-\eta} e^{i\omega r} R(r),$$

$$\tilde{g}(z) \doteq \int_{r_+}^\infty e^{A(z-r_-)(r-r_-)}(r - r_-)^{2\eta}(r - r_+)^{2\xi} e^{-2i\omega r} g(r) dr$$
$$= \int_{r_+}^\infty e^{A(z-r_-)(r-r_-)}(r - r_-)^{\eta}(r - r_+)^{\xi} e^{-i\omega r} R(r) dr. \qquad (2.15)$$

Here $z = x + iy$ with $y \geq 0$. Recall that η and ξ are defined by (2.12) and (2.13) respectively.

We close this introductory section with a brief outline of the arguments. If

$y > 0$ then the integrals and their derivatives are all absolutely convergent; we immediately conclude that $\tilde{g}$ is holomorphic for z in the upper half plane. However, when $y = 0$, then $\tilde{g}(x)$ is, a priori, only an L^2 function. In Section 2.3.1 we will show that nevertheless $\tilde{g}(x)$ is in fact a C^1 function on $[r_+, \infty)$. Then, in Section 2.3.2 we will verify $\tilde{g}$'s equation and show that $\tilde{g}$ is smooth on (r_+, ∞). Finally, in Section 2.3.3 we will carry out an asymptotic analysis of $\tilde{g}(x)$ as $x \to \infty$; in particular, we will identify $\lim_{x\to\infty} |x\tilde{g}(x)|$. Finally, in Section 2.3.4 we will put everything together and conclude the proofs Propositions 2.3.1, 2.3.2, and 2.3.3.

2.3.1 Defining $\tilde{g}$ on the Real Axis

For any $y > 0$ and $\epsilon > 0$, we shall rewrite $\tilde{g}$ in the following way:

Lemma 2.3.1.

$$\tilde{g}(z) = \int_{r_+}^{r_+ + \epsilon} e^{A(z-r_-)(r-r_-)}(r-r_-)^\eta (r-r_+)^\xi e^{-i\omega r} R(r)dr$$

$$-\Bigg((A(z-r_-))^{-1} e^{A(z-r_-)(r_+ - r_- + \epsilon)} (r_+ - r_- + \epsilon)^\eta \times \epsilon^{2\xi} e^{-i\omega(r_+ + \epsilon)} \epsilon^{-\xi} R(r_+ + \epsilon)\Bigg)$$

$$+\Bigg((A(z-r_-))^{-2} e^{A(z-r_-)(r_+ - r_- + \epsilon)} \times \frac{d}{dr}\left((\cdot - r_-)^\eta (\cdot - r_+)^\xi e^{-i\omega \cdot} R(\cdot)\right)(r_+ + \epsilon)\Bigg)$$

$$+\left(A(z-r_-)\right)^{-2}\int_{r_++\epsilon}^{\infty}\left(e^{A(z-r_-)(r-r_-)}\frac{d^2}{dr^2}\left((r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}R(r)\right)\right)dr.$$

Proof. This follows by integrating by parts twice the expression (2.15) in a straightforward manner. ☐

Lemma 2.3.2. *The function $\tilde{g}(x)$ is continuous on $[r_+,\infty)$ and $O(x^{-1})$ as $x\to\infty$.*

Proof. Recall that the boundary conditions for R, (1.20) and (1.21), imply

1. $(r-r_+)^{-\xi}R(r)$ is smooth at r_+.
2. $\frac{d^k}{dr^k}\left(e^{-i\omega r}R(r)\right)=O\left(r^{-k-1}\right)$ as $r\to\infty$.

In particular, the integral in the last line of the formula from (2.3.1) is absolutely convergent even when $y=0$. Thus, even when $y=0$, we may conclude that the right hand side of the formula is continuous in x.

In order to see the decay in x, set $\epsilon=x^{-1}$. By direct inspection one finds that each term is $O(x^{-1})$. Since the right hand side of the formula converges in L^2 as $y\downarrow 0$, by uniqueness of L^2 limits we conclude that $\tilde{g}(x)$ is equal to the formula. The lemma then follows. ☐

Now we turn to $\frac{\partial g}{\partial x}$. We have

Lemma 2.3.3. *For any $y>0$ and $\epsilon>0$ we have*

$$\frac{\partial\tilde{g}}{\partial x}-A(r_+-r_-)\tilde{g}=$$

$$-(z-r_-)^{-1}\int_{r_+}^{r_++\epsilon}e^{A(z-r_-)(r-r_-)}\frac{d}{dr}\left((r-r_-)^{\eta}(r-r_+)^{\xi+1}e^{-i\omega r}R(r)\right)dr+$$

$$\Big(A^{-1}(z-r_-)^{-2}e^{A(z-r_-)(r_+-r_-+\epsilon)}$$
$$\times\frac{d}{dr}\left((r-r_-)^{\eta}(r-r_+)^{\xi+1}e^{-i\omega r}R(r)\right)(r_++\epsilon)\Big)$$

$$-\Big(A^{-2}(z-r_-)^{-3}e^{A(z-r_-)(r_+-r_-+\epsilon)}$$
$$\times\frac{d^2}{dr^2}\left((r-r_-)^{\eta}(r-r_+)^{\xi+1}e^{-i\omega r}R(r)\right)(r_++\epsilon)\Big)$$

$$-A^{-2}(z-r_-)^{-3}\int_{r_++\epsilon}^{\infty}e^{A(z-r_-)(r-r_-)}$$
$$\times\frac{d^3}{dr^3}\left((r-r_-)^{\eta}(r-r_+)^{\xi+1}e^{-i\omega r}R(r)\right)dr.$$

Proof. This follows from a straightforward series of integration by parts on the expression

$$\frac{\partial\tilde{g}}{\partial x}-A(r_+-r_-)\tilde{g}=$$
$$A\int_{r_+}^{\infty}e^{A(z-r_-)(r-r_-)}(r-r_-)^{\eta}(r-r_+)^{\xi+1}e^{-i\omega r}R(r)dr.$$

□

Next, we have

Lemma 2.3.4. $\frac{\partial\tilde{g}}{\partial x}(x)$ *exists and is continuous on* $[r_+,\infty)$. *Furthermore*

$$\frac{\partial\tilde{g}}{\partial x}-A\left(r_+-r_-\right)\tilde{g}=O\left(x^{-2}\right)\ \text{as } x\to\infty.$$

Proof. This follows by setting $\epsilon=x^{-1}$ in Lemma 2.3.3 and then reasoning as in Lemma 2.3.2. □

2.3.2 Verifying the New Equation

In this section we will compute $\tilde{g}$'s new equation.

We say that a function h satisfies a Confluent Heun Equation (CHE) if there are complex parameters γ, δ, p, α, and σ and a function G such that

$$Th \doteq (r-r_+)(r-r_-)\frac{d^2h}{dr^2}+(\gamma(r-r_+)+\delta(r-r_-)+p(r-r_+)(r-r_-))\frac{dh}{dr}+ \tag{2.16}$$

$$(\alpha p(r-r_-)+\sigma)\,h = G.$$

A straightforward calculation shows that g satisfies such a CHE with

$$\gamma = 2\eta + 1 \doteq \gamma_0,$$

$$\delta = 2\xi + 1 \doteq \delta_0,$$

$$p = -2i\omega \doteq p_0,$$

$$\alpha = 1 \doteq \alpha_0,$$

$$\sigma = 2am\omega - 2\omega r_- i - \lambda_{\kappa ml} - a^2\omega^2 \doteq \sigma_0,$$

$$G = (r-r_+)^{-\xi}(r-r_-)^{-\eta}e^{i\omega r}\hat{F} \doteq G_0.$$

We need an integration by parts lemma whose straightforward proof is omitted.

Lemma 2.3.5. *Let T denote a Confluent Heun operator as defined in (2.16). Then*

$$\int_{\beta_1}^{\beta_2}(Tf)\,(r-r_+)^{\delta-1}(r-r_-)^{\gamma-1}e^{pr}h\,dr$$

$$= (r-r_+)^{\delta}(r-r_-)^{\gamma}e^{pr}\left(\frac{df}{dr}h - f\frac{dh}{dr}\right)\Bigg|_{\beta_1}^{\beta_2}$$

$$+\int_{\beta_1}^{\beta_2} (Th)\,(r-r_+)^{\delta-1}(r-r_-)^{\gamma-1}e^{pr}f\,dr.$$

Next we will compute $\tilde{g}$'s equation for $y > 0$.

Lemma 2.3.6. *If $y > 0$ we have*

$$(z-r_+)(z-r_-)\frac{\partial^2\tilde{g}}{\partial x^2}+$$

$$((z-r_+)+(1-4iM\omega)(z-r_-)-2i\omega(z-r_-)(z-r_+))\frac{\partial\tilde{g}}{\partial x}+$$

$$\left(-2i\omega(2\eta+1)(z-r_-)+2am\omega-2\omega r_-i-\lambda_{\kappa ml}-a^2\omega^2\right)\tilde{g}=\tilde{G}$$

where

$$\tilde{G} \doteq \int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+-r_-}(z-r_-)(r-r_-)}(r-r_-)^{2\eta}(r-r_+)^{2\xi}e^{-2i\omega r}G_0(r)dr.$$

Proof. Since the coefficients of the CHE are all holomorphic, we may take the derivatives in the CHE to be complex derivatives. Let L_r denote a Confluent Heun Operator in the r variable with parameters $(\gamma_0,\delta_0,p_0,\alpha_0,\sigma_0)$ and right hand side G_0. Let $\tilde{L}_z$ denote a Confluent Heun operator in the z $(= x+iy)$ variable with, to be determined, tilded parameters.

We wish to determine if

$$\int_{r_+}^{\infty} e^{A(z-r_-)(r-r_-)}(r-r_-)^{2\eta}(r-r_+)^{2\xi}e^{-2i\omega r}g(r)dr$$

is a solution to a CHE with tilded parameters. When $y > 0$ the exponential damping in the integral allows differentiation under the integral sign, and we see from Lemma 2.3.5 that the following two conditions will suffice:

$$\left(\tilde{L}_z - L_r\right)e^{A(z-r_-)(r-r_-)} = 0,$$

$$(r-r_+)^{\delta_0}(r-r_-)^{\gamma_0}e^{p_0 r}e^{A(z-r_-)(r-r_-)}\left(A(z-r_-)g-\frac{dg}{dr}\right)\Bigg|_{r_+}^{\infty}=0$$

$$\forall z \text{ such that } y>0.$$

We have

$$e^{-A(z-r_-)(r-r_-)}\left(\tilde{L}_z-L_r\right)e^{A(z-r_-)(r-r_-)}=$$

$$A\left(A(r_+-r_-)+\tilde{p}\right)(r-r_-)(z-r_-)^2-A\left(A(r_+-r_-)+p_0\right)(r-r_-)^2(z-r_-)$$

$$-A\left(\gamma_0+\delta_0+p_0(r_+-r_-)-\tilde{\gamma}-\tilde{\delta}-\tilde{p}(r_+-r_-)\right)(z-r_-)(r-r_-)$$

$$+\left(A\gamma(r_+-r_-)+\tilde{\alpha}\tilde{p}\right)(z-r_-)-\left(A\tilde{\gamma}(r_+-r_-)+\alpha_0 p_0\right)(r-r_-)+(\tilde{\sigma}-\sigma_0).$$

From this it is clear that we must have

$$A=-p(r_+-r_-)^{-1}=2i\omega(r_+-r_-)^{-1},$$

$$\tilde{p}=p_0=-2i\omega,$$

$$\tilde{\alpha}=\gamma_0,$$

$$\tilde{\gamma}=\alpha_0=1,$$

$$\tilde{\delta}=\gamma_0+\delta_0-\tilde{\gamma}=1-4iM\omega,$$

$$\tilde{\sigma}=\sigma_0.$$

We still need to check that the boundary conditions are satisfied. Since g and $\frac{dg}{dr}$ both decay for large r, the exponential decay from $e^{A(z-r_-)(r-r_-)}$ clearly implies that

$$\left((r-r_+)^{\delta_0}(r-r_-)^{\gamma_0}e^{p_0 r}e^{A(z-r_-)(r-r_-)}\left(A(z-r_-)g-\frac{dg}{dr}\right)\right)(r=\infty)=0$$

for all z with $y > 0$.

Since $\delta_0 = 2\xi + 1$, with ξ purely imaginary, and $|g|$ extends continuously to r_+, we see that

$$\left((r-r_+)^{\delta_0}(r-r_-)^{\gamma_0}e^{p_0 r}e^{A(z-r_-)(r-r_-)}\left(A(z-r_-)g - \frac{dg}{dr}\right)\right)(r=r_+) = 0 \Leftrightarrow$$

$$\frac{dg}{dr^*}(r_+) = 0.$$

If we r^* differentiate the expression defining g, we get

$$\left|\frac{dg}{dr^*}\right|(r_+) = \left|\frac{dR}{dr^*} - \frac{\xi(r_+-r_-)}{2Mr_+}R\right|(r_+) = 0.$$

We conclude that $\tilde{g}$ satisfies $\tilde{L}_z\tilde{g} = 0$. Lastly, since $\tilde{g}$ is holomorphic in the upper half plane, $\frac{d\tilde{g}}{dz} = \frac{\partial\tilde{g}}{\partial x}$. □

Finally, using the analysis from Section 2.3.1 we can upgrade this lemma to

Lemma 2.3.7. *When $y = 0$, $\tilde{g}$ is smooth in (r_+,∞) and we have*

$$(x-r_+)(x-r_-)\frac{\partial^2\tilde{g}}{\partial x^2} + \tag{2.17}$$

$$((x-r_+) + (1-4iM\omega)(x-r_-) - 2i\omega(x-r_-)(x-r_+))\frac{\partial\tilde{g}}{\partial x} +$$

$$\left(-2i\omega(2\eta+1)(x-r_-) + 2am\omega - 2\omega r_- i - \lambda_{\kappa ml} - a^2\omega^2\right)\tilde{g} = \tilde{G}$$

where

$$\tilde{G} \doteq \int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+-r_-}(x-r_-)(r-r_-)}(r-r_-)^{2\eta}(r-r_+)^{2\xi}e^{-2i\omega r}G_0(r)dr.$$

Proof. One consequence of the analysis in Section 2.3.1 is that $\tilde{g}(x+iy)$ converges to $\tilde{g}$ in H^1_x as $y \to 0$. In particular, we may take $y \to 0$ in the weak formulation of the equation from Lemma 2.3.6 to conclude that $\tilde{g}(x)$ is a weak H^1_x solution of (2.17). Since $\tilde{G}$ is smooth,[6] we may then conclude the proof by an appeal to elliptic regularity. □

2.3.3 Asymptotic Analysis of $\tilde{g}$

Recall that in Section 2.3.1 we saw that $\tilde{g} = O(x^{-1})$ as $x \to \infty$. In this section we will carry out the somewhat subtle task of identifying

$$\lim_{x \to \infty} |x\tilde{g}(x)|.$$

We start with

Lemma 2.3.8. *Let h be a smooth function on $[r_+, \infty)$ which vanishes on $[r_+ + 2, \infty)$, and recall that ξ is defined by (2.13). For $\tau \geq 0$ and $\nu > 0$, define*

$$Z(\nu, \tau) \doteq \int_{r_+}^{\infty} e^{i\nu r} (r - r_+ + i\tau)^{2\xi} h(r) dr.$$

Then we have

$$|Z(\nu, \tau)| \lesssim \nu^{-1}$$

where the implied constant does not depend on τ.

---

[6]Recall that

$$\tilde{G}(x) = \int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+ - r_-}(x - r_-)(r - r_-)} (r - r_-)^{\eta} (r - r_+)^{\xi} e^{-i\omega r} \hat{F} dr$$

where $\hat{F}$ is smooth and compactly supported in (r_+, ∞).

Proof. Integrating by parts twice produces the following expression for $Z(\nu,\tau)$:

$$Z(\nu,\tau) = \int_{r_+}^{r_++\nu^{-1}} e^{i\nu r}(r-r_++i\tau)^{2\xi} h(r)dr \tag{2.18}$$

$$-(i\nu)^{-1} e^{i\nu(r_++\nu^{-1})}\left(\nu^{-1}+i\tau\right)^{2\xi} h\left(r_++\nu^{-1}\right) \tag{2.19}$$

$$+(i\nu)^{-2} e^{i\nu(r_++\nu^{-1})}\frac{d}{dr}\left((\cdot-r_++i\tau)^{2\xi} h(\cdot)\right)\left(r_++\nu^{-1}\right) \tag{2.20}$$

$$+(i\nu)^{-2}\int_{r_++\nu^{-1}}^{\infty} e^{i\nu r}\frac{d^2}{dr^2}\left((r-r_++i\tau)^{2\xi} h(r)\right)dr. \tag{2.21}$$

The lemma follows by direct inspection of each term. □

The following lemma is the technical core of our argument. The proof consists of minor adaptions of techniques discussed in the books [13] and [26].

Lemma 2.3.9. *Let h be a smooth function on $[r_+,\infty)$ which vanishes in $[r_+ + 2,\infty)$, and recall that ξ is defined by (2.13).*

For $\nu>0$, define

$$Z(\nu) \doteq Z(\nu,0) = \int_{r_+}^{\infty} e^{i\nu r}(r-r_+)^{2\xi} h(r)dr.$$

Then we have

$$Z(\nu) = \exp\left(\frac{i\pi}{2}(1+2\xi)\right)\Gamma(2\xi+1)\,h(r_+)e^{i\nu r_+}\nu^{-1-2\xi} + O\left(\nu^{-2}\right) \text{ as } \nu\to\infty$$

where

$$\Gamma(z) \doteq \int_0^{\infty} e^{-t}t^{z-1}dt$$

is the Gamma function.

Proof. The key trick is to come up with a clever form of the anti-derivative of $e^{i\nu r}(r-r_+)^{2\xi}$. In order to do this, we extend $e^{i\nu r}(r-r_+)^{2\xi}$ to $s\in\mathbb{C}\setminus\{(-\infty,r_+]\}$

where we are taking the principal branch of $(s - r_+)^{2\xi}$. One may easily check that $(s - r_+)^{2\xi} = \exp(2\xi \log(s - r_+))$ is uniformly bounded in the region

$$\{s : \mathrm{Re}(s) \in [r_+, r_+ + 2)\}.$$

Thus, keeping in the mind the exponential decay from $e^{i\nu s}$ as $\mathrm{Im}(s) \to \infty$ and Cauchy's Theorem, we may unambiguously define

$$l(r,\nu) \doteq -\int_r^{i\infty} e^{i\nu s} (s - r_+)^{2\xi} \, ds$$

whenever $r \in (r_+, r_+ + 2)$. This will satisfy

$$\frac{\partial l}{\partial r} = e^{i\nu r} (r - r_+)^{2\xi}.$$

Now, integrating along the curve $t \mapsto r + it$ implies

$$l(r,\nu) = -ie^{i\nu r} \int_0^\infty e^{-\nu t} (r - r_+ + it)^{2\xi} \, dt. \tag{2.22}$$

Now, keeping in mind that $z^{2\xi} \doteq \exp(2\xi \log z)$, we have

$$\lim_{r \to r_+} l(r,\nu) = -i^{1+2\xi} e^{i\nu r_+} \int_0^\infty e^{-\nu t} t^{2\xi} dt \tag{2.23}$$

$$= -i^{1+2\xi} e^{i\nu r_+} \nu^{-1-2\xi} \Gamma(2\xi + 1). \tag{2.24}$$

More generally, changing variables in 2.22 implies

$$l(r,\nu) = -ie^{i\nu r} \nu^{-1} \int_0^\infty e^{-t} \left(r - r_+ + i\frac{t}{\nu} \right)^{2\xi} dt. \tag{2.25}$$

Now we are ready for an estimate:

$$Z(\nu,\tau) = \int_{r_+}^{\infty} e^{i\nu r}(r-r_+)^{2\xi} h(r)dr \tag{2.26}$$
$$= \int_{r_+}^{\infty} \frac{\partial l}{\partial r}(r,\nu)\, h(r)dr \tag{2.27}$$
$$= i^{1+2\xi}\Gamma(2\xi+1)\, h(r_+)e^{i\nu r_+}\nu^{-1-2\xi} \tag{2.28}$$
$$- \int_{r_+}^{\infty} l(r,\nu)\, h'(r)dr \tag{2.29}$$
$$= i^{1+2\xi}\Gamma(2\xi+1)\, h(r_+)e^{i\nu r_+}\nu^{-1-2\xi} \tag{2.30}$$
$$+ i\nu^{-1}\int_0^{\infty} e^{-t}\left(\int_{r_+}^{\infty} e^{i\nu r}\left(r-r_+ + i\frac{t}{\nu}\right)^{2\xi} h'(r)dr\right)dt. \tag{2.31}$$

We have used (2.25) and Fubini in the last equality.

To conclude the proof we just need to show that

$$\int_0^{\infty} e^{-t}\left(\int_{r_+}^{\infty} e^{i\nu r}\left(r-r_+ + i\frac{t}{\nu}\right)^{2\xi} h'(r)dr\right)dt = O\left(\nu^{-1}\right). \tag{2.32}$$

However, this follows by an application of Lemma 2.3.8 to the inner integral.

□

Let's apply this analysis to $\tilde{g}$.

Lemma 2.3.10. *As $x \to \infty$ we have*

$$\tilde{g}(x) = \Bigg(\exp\left(\frac{i\pi}{2}(1+2\xi)\right)\Gamma(2\xi+1)(r_+-r_-)^{\eta} e^{-A(r_+-r_-)r_-} e^{-i\omega r_+} \tag{2.33}$$
$$\times \left(2\omega(r_+-r_-)^{-1}\right)^{-1-2\xi}\left((\cdot - r_+)^{-\xi} R(\cdot)\right)(r_+) \tag{2.34}$$
$$\times e^{Ax(r_+-r_-)}x^{-1-2\xi}\Bigg) + O\left(x^{-2}\right). \tag{2.35}$$

Proof. Let $\chi(r)$ be a positive smooth function which is identically 1 on $[r_+, r_+ +$

1] and identically 0 on $[r_+ + 2, \infty)$. We may write

$$\tilde{g}(x) = \int_{r_+}^{\infty} e^{A(x-r_-)(r-r_-)}(r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}R(r)\chi(r)dr \tag{2.36}$$

$$+ \int_{r_+}^{\infty} e^{A(x-r_-)(r-r_-)}(r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}R(r)\left(1-\chi(r)\right)dr. \tag{2.37}$$

The second integral satisfies

$$\int_{r_+}^{\infty} e^{A(x-r_-)(r-r_-)}(r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}R(r)\left(1-\chi(r)\right)dr$$

$$= \left(A\left(x-r_-\right)\right)^{-2}\int_{r_+}^{\infty} e^{A(x-r_-)(r-r_-)} \times \frac{d^2}{dr^2}\left((r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}R(r)\left(1-\chi(r)\right)\right)dr$$

$$= O\left(x^{-2}\right).$$

We have used the boundary condition (1.21).

Now we conclude the proof by applying Lemma 2.3.9 (with $\nu = Ax$) to the first integral. □

2.3.4 Putting Everything Together

Now we will prove Propositions 2.3.1 and 2.3.3.

Proof. (Proposition 2.3.1)

Recall the definition of $\tilde{u}$:

$$\tilde{u}(x^*) \doteq (x^2+a^2)^{1/2}(x-r_+)^{-2iM\omega}e^{-i\omega x}\times \tag{2.38}$$

$$\int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+-r_-}(x-r_-)(r-r_-)}(r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}R(r)dr. \tag{2.39}$$

In terms of $\tilde{g}$ we have

$$\tilde{u}(x^*) = (x^2 + a^2)^{1/2}(x - r_+)^{-2iM\omega}e^{-i\omega x}\tilde{g}(x).$$

In particular $\tilde{u}$ is smooth on (r_+, ∞) and Proposition 2.3.1 follows from Lemma 2.17 and a straightforward (if tedious) calculation. □

Proof. (Proposition 2.3.3)

Keeping in mind that

$$\tilde{u}' = \frac{(x - r_+)(x - r_-)}{x^2 + a^2}\frac{\partial \tilde{u}}{\partial x},$$

the lemma follows immediately from

$$\tilde{u}(x^*) = (x^2 + a^2)^{1/2}(x - r_+)^{-2iM\omega}e^{-i\omega x}\tilde{g}(x),$$

the fact that $\tilde{g}$ is C^1 at r_+ (see Section 2.3.1), and Lemma 2.3.10. □

Recall that we are omitting the proof of Proposition 2.3.2 since it is much easier and follows from the same sort of reasoning as the proofs of Propositions 2.3.1 and 2.3.3.

2.4 An Estimate for the Wave Equation Radial ODE

The goal of this section is to prove the following proposition which will be useful later in Section 2.5.

Proposition 2.4.1. *Let*

$$\mathscr{A} \subset \left\{(\omega, m, l) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}\right\}$$

be a set of frequency parameters with

$$C_{\mathscr{A}} \doteq \sup_{(\omega,m,l)\in\mathscr{A}} \left(|\omega| + |\omega|^{-1} + |m| + |l|\right) < \infty.$$

Consider $(\omega, m, l) \in \mathscr{A}$ *and suppose that* $u(r^*) = (r^2 + a^2)^{1/2} R(r)$ *is smooth,* R *satisfies the mode solution boundary conditions*

$$R \sim \frac{e^{i\omega r^*}}{r} \text{ as } r \to \infty,$$

$$R \sim (r - r_+)^{\frac{i(am - 2Mr_+\omega)}{r_+ - r_-}} \text{ as } r \to r_+,$$

and u *solves (2.9) with a smooth, compactly supported right hand side* $H(r^*) = \frac{\Delta}{(r^2+a^2)^{1/2}} F(r)$. *Then we have the estimate*

$$\int_{r_+}^{\infty} |R|^2 \, dr \leq B(C_{\mathscr{A}}) \left[|u(-\infty)|^2 + |u(\infty)|^2 + \int_{r_+}^{\infty} |F(r)|^2 r^4 dr \right].$$

It will sometimes be useful to switch our "perspective" with regards to $-\infty$ and ∞ and write u's equation as

$$u'' + \left(\omega_0^2 - V_0\right) u = H$$

where

$$\omega_0 = \omega - \frac{am}{2Mr_+},$$

$$V_0 = V + \omega_0^2 - \omega^2.$$

For the following estimates the relevant properties of V and V_0 are

1. V is uniformly bounded.

2. $V = O(r^{-2})$ at ∞.

3. $V_0 = O(r - r_+)$.

4. For fixed non-zero a, m, and $M > 0$ there exists a constant $c > 0$ such that $am - 2Mr_+\omega \geq -c\left(\lambda_{\kappa ml} + a^2\omega^2\right) \Rightarrow \frac{dV_0}{dr}(r_+) > 0$.

The last statement is the only non-obvious one, and the relevant computations can be found in [21]. It will also be useful to note that

$$\lambda_{\kappa ml} + a^2\omega^2 \geq |m|\left(|m| + 1\right). \tag{2.40}$$

This follows from the observation that when $a^2\omega^2 = 0$, the $e^{im\phi}S_{\kappa ml}(\theta)$ are simply spherical harmonics with corresponding eigenvalues all larger than $|m|(|m| + 1)$ (see [21]).

Note that the eigenvalue estimate above in particular implies that there exists a constant $b\left(C_{\mathscr{A}}\right)$ depending on $C_{\mathscr{A}}$ so that

$$\omega_0^2 \leq b(C_{\mathscr{A}}) \Rightarrow \frac{dV_0}{dr}(r_+) > 0. \tag{2.41}$$

In the following sections we will introduce the "currents" and then systematically explore various estimates and their realm of applicability. At the end we will show how they can be combined to establish Proposition 2.4.1.

2.4.1 The Currents

Our ODE estimates will be based on the "microlocal viral" and "microlocal redshift" currents. These were first introduced in [18].

Definition 2.4.1. *Let $y(r^*)$ be a continuous and piecewise C^1 function and $u(r^*)$ be a smooth function. Then the corresponding microlocal virial current is*

$$Q^y[u] \doteq y|u'|^2 + y\left(\omega^2 - V\right)|u|^2.$$

Definition 2.4.2. *Let $z(r^*)$ be a continuous and piecewise C^1 function and $u(r^*)$ be a smooth function. Then the corresponding microlocal redshift current is*

$$Q^z_{\text{red}}[u] \doteq z\left|u' + i\omega_0 u\right|^2 - zV_0|u|^2.$$

We will often suppress the $[u]$ in both $Q^y[u]$ and $Q^z_{\text{red}}[u]$.

The following lemma is a straightforward calculation.

Lemma 2.4.1. *Suppose that $u(r^*)$ is a smooth function satisfying the radial ODE*

$$u'' + \left(\omega^2 - V\right)u = H.$$

Then

$$(Q^y)' = y'|u'|^2 + y'\omega^2|u|^2 - (yV)'|u|^2 + 2yRe(H\overline{u}'). \tag{2.42}$$

$$(Q^z_{\text{red}})' = z'\left|u' + i\omega_0 u\right|^2 - (zV_0)'|u|^2 + 2zRe\left((u' + i\omega_0 u)\overline{H}\right). \tag{2.43}$$

2.4.2 Virial Estimate I

The estimate we give in this section will hold for all frequencies $(\omega, m, l) \in \mathscr{A}$ but will degenerate as $r^* \to -\infty$ $(r \to r_+)$.

Lemma 2.4.2. *Let $(\omega, m, l) \in \mathscr{A}$, $u = (r^2+a^2)^{1/2}R$ solve the radial ODE with right hand side $H = \frac{\Delta}{(r^2+a^2)^{1/2}}F(r)$, and suppose that R satisfies the boundary conditions of a arising from a mode solution. Then, for every $r_0 > r_+$, we have*

$$\int_{r_0}^{\infty}\left[|R|^2 + \left|\frac{dR}{dr}\right|^2\right] dr \le B(r_0, C_{\mathscr{A}})\left[|u(\infty)|^2 + \int_{r_+}^{\infty}|F(r)|^2 r^4\, dr\right].$$

Proof. Integrating (2.42) gives

$$\int_{-\infty}^{\infty}\left(y'|u'|^2 + y'\omega^2|u|^2 - (yV)'\,|u|^2\right) dr^* =$$

$$Q^y(\infty) - Q^y(-\infty) - \int_{-\infty}^{\infty} 2y\mathrm{Re}\left(u'\overline{H}\right) dx^*.$$

We want to choose y so that the left hand side controls $|u|^2 + |u'|^2$ (with weights which degenerate as $r^* \to \pm\infty$) and so that the boundary terms are controllable. Let $\zeta(r^*)$ be a non-negative function which is identically 1 near $r^* = -\infty$ and equals r^{-2} near $r^* = \infty$. We set

$$y(r^*) \doteq \exp\left(-C\int_{r^*}^{\infty}\zeta dr^*\right).$$

Here C is a large parameter to be chosen later. We have $y(r_+) = 0$, $y(\infty) = 1$, and $y' = C\zeta y > 0$. We will show that the term

$$-\int_{-\infty}^{\infty}(yV)'\,|u|^2 dr^*$$

which threatens to destroy the coercivity of our estimate can in fact be absorbed into the other two terms. After an integration by parts and the inequality $|ab| \leq \epsilon|a| + (4\epsilon)^{-1}|b|$, we find

$$\left|\int_{-\infty}^{\infty} (yV)' |u|^2 dr^*\right| \leq$$

$$\frac{1}{2}\int_{-\infty}^{\infty} y'|u'|^2 dr^* + 2\int_{-\infty}^{\infty} \left(y'\omega^2\right) \frac{y^2|V|^2}{\omega^2 \left(y'\right)^2}|u|^2 dr^* + \left|\left(yV|u|^2\right)|_{-\infty}^{\infty}\right|.$$

Note that $|V|$ is uniformly bounded, decays like r^{-2}, and that $y/y' \leq C^{-1}r^2$. Also, the boundary terms clearly vanish. Thus, for sufficiently large C (depending on ω), we get

$$\left|\int_{-\infty}^{\infty} (yV)' |u|^2 \, dr^*\right| \leq \frac{1}{2}\int_{-\infty}^{\infty} \left(y'|u'|^2 + y'\omega^2|u|^2\right) dr^*.$$

Lastly, the boundary conditions for R imply that

$$Q^y(\infty) \leq B\left(C_{\mathscr{A}}\right)|u(\infty)|^2,$$

$$Q^y(-\infty) = 0.$$

Thus, we end up with

$$\int_{-\infty}^{\infty} \left(y'|u'|^2 + y'\omega^2|u|^2\right) dr^* B\left(C_{\mathscr{A}}\right)\left[|u(\infty)|^2 - \int_{-\infty}^{\infty} y\mathrm{Re}\left(u'\overline{H}\right) dr^*\right].$$

The usual Cauchy-Schwarz argument then gives

$$\int_{-\infty}^{\infty} \left(y'|u'|^2 + y'\omega^2|u|^2\right) dr^* \leq B\left(C_{\mathscr{A}}\right)\left[|u(\infty)|^2 + \int_{-\infty}^{\infty} y|H|^2 r^2 dr^*\right]. \quad (2.44)$$

The proof concludes by a straightforward change of variables from r^* to r. □

2.4.3 Virial Estimate II

The estimate we give in this will not degenerate at $r^* = -\infty$. However, it will require that ω_0 is bounded away from 0, and thus it does not cover all triples $(\omega, m, l) \in \mathscr{A}$.

Lemma 2.4.3. *Let* $(\omega, m, l) \in \mathscr{A}$ *such that* $\omega_0^2 \doteq \left(\omega - \frac{am}{2MR_+}\right)^2 > 0$*, let* $u = (r^2 + a^2)^{1/2} R$ *solve the radial ODE with right hand side* $H = \frac{\Delta}{(r^2+a^2)^{1/2}} F(r)$*, and suppose that* R *satisfies the boundary conditions of a arising from a mode solution. Then, for every* $r_1 > r_+$*, we have*

$$\int_{r_+}^{r_1} |R|^2 \, dr \leq B\left(r_1, \omega_0^2, C_{\mathscr{A}}\right) \left[|u(-\infty)|^2 + \int_{r_+}^{\infty} |F(r)|^2 r^4 \, dr \right].$$

The constant $B\left(r_1, \omega_0^2, C_{\mathscr{A}}\right)$ *blows up as* $\omega_0^2 \to 0$.

Proof. We rewrite the virial current as

$$Q^y \doteq y|u'|^2 + y\left(\omega_0^2 - V_0\right)|u|^2.$$

Let $\zeta(r)$ be a positive function equal to Δ near $r = r_+$, and equal to 1 near $r = \infty$. Then define

$$y(r^*) \doteq \exp\left(-C \int_{-\infty}^{r^*} \zeta dr^*\right).$$

Integrating the virial current gives

$$\int_{-\infty}^{\infty} \left(-y'|u'|^2 - y'\omega_0^2|u|^2 + (yV_0)'|u|^2\right) dr^* =$$

$$-Q^y(\infty) + Q^y(-\infty) + \int_{-\infty}^{\infty} 2y\mathrm{Re}\left(u'\overline{H}\right) dx^*.$$

We may deal with the $(yV_0)'$ exactly as in the previous section, but the necessary largeness of C now depends on how small ω_0^2 is. We also have

$$Q^y(\infty) = 0$$

$$|Q^y(-\infty)| \le B\left(C_{\mathscr{A}}\right)\left[|u(-\infty)|^2 + |u'(-\infty)|^2\right].$$

We end up with

$$\int_{-\infty}^{\infty} \left(-y'|u'|^2 - y'\omega_0^2|u|^2\right) dr^* \le$$
$$B\left(C_{\mathscr{A}}\right)\left[|u(-\infty)|^2 + |u'(-\infty)|^2 + \int_{-\infty}^{\infty} y\mathrm{Re}\left(u'\overline{H}\right) dr^*\right].$$

As in the previous section, the standard Cauchy-Schwarz argument yields

$$\int_{-\infty}^{\infty} \left(-y'|u'|^2 - y'\omega_0^2|u|^2\right) dr^* \le \qquad (2.45)$$
$$B\left(C_{\mathscr{A}}, \omega_0^2\right)\left[|u(-\infty)|^2 + |u'(-\infty)|^2 + \int_{r_+}^{\infty} |F|^2\, dr\right].$$

The lemma then follows from a straightforward changing of variables from r^* to r. □

2.4.4 The Red-Shift Estimate

The estimate in this section will not degenerate at $r^* \to -\infty$ and will cover the regime when ω_0^2 is small.

The following Poincaré type inequality will be useful.

Lemma 2.4.4. *Suppose h has support in $[r_+, r_+ + \epsilon]$ and has*

$$\left(\left(\cdot - r_+\right)|h|^2(\cdot)\right)(r_+) = 0.$$

Then

$$\int_{r_+}^{\infty} |h|^2 dr \leq C(1+\epsilon^2) \int_{r_+}^{\infty} |h' + i\omega_0 h|^2 \, dr.$$

Proof. Keeping in mind that

$$\frac{dh}{dr^*} = \frac{(r-r_+)(r-r_-)}{r^2+a^2} \frac{dh}{dr},$$

we have

$$\int_{r_+}^{\infty} |h|^2 dr = \int_{r_+}^{\infty} \frac{d}{dr}(r-r_+)\, |h|^2 dr = -\int_{r_+}^{\infty} (r-r_+) \left(\frac{dh}{dr}\overline{h} + h\frac{d\overline{h}}{dr} \right) dr =$$

$$-\int_{r_+}^{\infty} \left(\frac{r^2+a^2}{r-r_-} \right) \left(h'\overline{h} + h\overline{h}' \right) dr =$$

$$-\int_{r_+}^{\infty} \left(\frac{r^2+a^2}{r-r_-} \right) \left((h' + i\omega_0 h)\,\overline{h} + h\left(\overline{h}' - i\omega_0 \overline{h} \right) \right) dr.$$

From here the lemma follows by the usual argument. □

Now we are ready for the estimate.

Lemma 2.4.5. *Let $(\omega, m, l) \in \mathscr{A}$ such that ω_0^2 is sufficiently small, let $u = (r^2+a^2)^{1/2} R$ solve the radial ODE with right hand side $H = \frac{\Delta}{(r^2+a^2)^{1/2}} F(r)$, and suppose that R satisfies the boundary conditions of a arising from a mode solution. Then, there exists $\epsilon > 0$ such that*

$$\int_{r_+}^{r_+ + \epsilon/2} |R|^2 \, dr \leq$$

$$B(C_{\mathscr{A}}) \left[\int_{\epsilon/2}^{2\epsilon} \left(|u(r^*(r))|^2 + |u'(r^*(r))|^2 \right) dr^* + \int_{-\infty}^{\infty} |F(r)|^2 \, dr \right].$$

Proof. We begin by noting that the boundary conditions for R imply that $(u' + i\omega_0 u)(r^*) = O(r - r_+)$ near $r^* = -\infty$. Hence, if we consider a Q^z_{red} current and take z to be a function which blows up at $-\infty$ (at an appropriate rate), the boundary terms will still be finite. With this in mind, let $\zeta(r)$ be a bump function identically 1 on $[r_+, r_+ + \epsilon]$ and vanishing on $[r_+ + 2\epsilon, \infty)$. Here ϵ is a free parameter that we will later take sufficiently small late. Now consider Q^z_{red} with the function z defined by

$$z(r^*) \doteq -\frac{\zeta(r(r^*))}{V_0}.$$

Note that $z' > 0$ near $-\infty$ since $\frac{d}{dr}V_0(-\infty) > 0$ by (2.41). We also have

$$(Q^z_{\text{red}})\big|_{-\infty}^{\infty} = -|u(-\infty)|^2,$$

which has a good sign. For $r \in [r_+, r_+ + \epsilon]$, (2.43) gives

$$(Q^z_{\text{red}})' = z'|u' + i\omega_0 u|^2 + 2z\text{Re}\left((u' + i\omega_0 u)\,\overline{H}\right).$$

Note that we have $z' \sim (r - r_+)^{-1}$ in this region.[7] For $r \in [r_+ + \epsilon, r_+ + 2\epsilon]$ we will treat everything as an error:

$$|(Q^z_{\text{red}})| \leq B\,(C_{\mathscr{A}})\left(|u'|^2 + |u|^2\right) + \left|z\text{Re}\left((u' + i\omega_0 u)\,\overline{H}\right)\right|.$$

Of course for $r \geq r_+ + 2\epsilon$ we have $\left(Q^z_{\text{red}}\right)' = 0$. Thus, putting everything together will produce an estimate for

$$\int_{r_+}^{r_+ + \epsilon} (r - r_+)^{-2}|u'(r^*(r)) + i\omega_0 u(r^*(r))|^2 dr.$$

For ϵ sufficiently small, an application of Lemma 2.4.4 will show that this

[7]Keep in mind that

$$z' = \frac{(r - r_+)(r - r_-)}{r^2 + a^2}\frac{dz}{dr}.$$

controls

$$\int_{r_+}^{r_++\epsilon/2} |u(r^*(r))|^2 dr$$

at the expense of introducing error terms

$$\int_{r_++\epsilon/2}^{r_++\epsilon} \left(|u'(r^*(r))|^2 + |u(r^*(r))|^2\right) dr.$$

We end up with

$$\int_{r_+}^{r_++\epsilon} (r-r_+)^{-2} |u'(r^*(r)) + i\omega_0 u(r^*(r))|^2 dr + \int_{r_+}^{r_++\epsilon/2} |u(r^*(r))|^2 \, dr \le B\left(C_{\mathscr{A}}\right)$$

$$\int_{\epsilon/2}^{2\epsilon} \left(|u|^2 + |u'|^2\right) dr^* + \int_{-\infty}^{\infty} \left|z\mathrm{Re}\left((u' + i\omega_0 u)\overline{H}\right)\right| dr^*. \tag{2.46}$$

As usual, Cauchy-Schwarz implies

$$\int_{r_+}^{r_++\epsilon} (r-r_+)^{-2} |u'(r^*(r)) + i\omega_0 u(r^*(r))|^2 dr + \int_{r_+}^{r_++\epsilon/2} |u(r^*(r))|^2 \, dr \le$$

$$B\left[\int_{\epsilon/2}^{2\epsilon} \left(|u(r^*(r))|^2 + |u'(r^*(r))|^2\right) dr^* + \int_{-\infty}^{\infty} |F(r)|^2 \, dr\right]. \tag{2.47}$$

□

2.4.5 Proof of Proposition 2.4.1

Now we are ready to prove Proposition 2.4.1.

Proof. Let $b_1 \in (r_+, \infty)$ be sufficiently close to r_+ and $b_0 = \frac{1}{2} b_1$. First we apply Lemma 2.4.2 and conclude

$$\int_{b_0}^{\infty} \left(|R|^2 + \left|\frac{dR}{dr}\right|^2\right) dr \le B\left(b_0, C_{\mathscr{A}}\right)\left[|u(\infty)|^2 + \int_{r_+}^{\infty} |F|^2 r^4 dr\right]. \tag{2.48}$$

Now, depending on whether ω_0 is small or large, we either appeal to Lemma 2.4.3 or Lemma 2.4.5, take b_1 sufficiently small and fix its value, and combine with (2.48) to get

$$\int_{r_+}^{\infty} |R|^2 \, dr \leq B(C_{\mathscr{A}}) \left[|u(-\infty)|^2 + |u(\infty)|^2 + \int_{r_+}^{\infty} |F(r)|^2 r^4 dr \right].$$

□

2.5 Proof of the Mode Stability Results and the Microlocal Horizon Energy Flux Bound

In this section we will use our extension of Whiting's integral transformations and Proposition 2.4.1 from the previous section to prove Theorems 1.5.2, 1.5.3, 1.5.1, and 1.5.4.

2.5.1 Mode Stability in the Upper Half Plane

We start with the proof of Theorem 1.5.2.

Proof. (Mode Stability, Theorem 1.5.2) Suppose we have a mode solution with corresponding $(u, S_{\kappa ml}, \lambda_{\kappa ml})$ and $\omega = \omega_R + i\omega_I$ with $\omega_I > 0$. Let $\tilde{u}$ be defined by (2.11), and consider the microlocal energy current associated to $\tilde{u}$:

$$\tilde{Q}_T \doteq \operatorname{Im}\left(\tilde{u}'\overline{\omega\tilde{u}}\right).$$

Proposition 2.3.2 implies that $\tilde{Q}_T(\pm\infty) = 0$. We proceed as in our discussion of Schwarzschild from Section 2.2.4 with $\tilde{u}$ replacing u:

$$0 = -\tilde{Q}_T\big|_{-\infty}^{\infty} = -\int_{-\infty}^{\infty} \left(\tilde{Q}_T\right)' dr^* = \int_{-\infty}^{\infty} \left(\omega_I \left|\tilde{u}'\right|^2 + \mathrm{Im}\left(\Phi\overline{\omega}\right)\left|\tilde{u}\right|^2\right) dr^*.$$

Hence, if we can show that $\mathrm{Im}\left(\Phi\overline{\omega}\right) \geq 0$, we may conclude that $\tilde{u}$ vanishes.

An easy computation using the formula from Proposition 2.3.1 gives

$$\mathrm{Im}\left(\Phi\overline{\omega}\right) = \omega_I \left(\frac{(x-r_-)}{(x^2+a^2)^2}\Psi_0 + \frac{(x-r_+)(x-r_-)}{(x^2+a^2)^4}\Psi_1 \right) -$$

$$\frac{(x-r_+)(x-r_-)}{(x^2+a^2)^2}\mathrm{Im}\left(\lambda_{\kappa ml}\overline{\omega}\right),$$

$$\Psi_0 \doteq |\omega|^2 (x-r_+)^2(x-r_-) + |\omega|^2 \left[\frac{8M^2(x-r_-)}{r_+ - r_-} - a^2\right](x-r_+) + 4M^2|\omega|^2(x-r_-),$$

$$\Psi_1 \doteq a^2(x-r_+)(x-r_-) + 2Mx(x^2-a^2).$$

All of these terms are clearly positive except for $-\mathrm{Im}\left(\lambda_{\kappa ml}\overline{\omega}\right)$. For this term we need to return to $S_{\kappa ml}$'s equation (1.14):

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS_{\kappa ml}}{d\theta}\right) - \left(\frac{m^2}{\sin^2\theta} - a^2\omega^2\cos^2\theta\right)S_{\kappa ml} + \lambda_{\kappa ml}S_{\kappa ml} = 0.$$

Now multiply the equation by $\overline{\omega S_{\kappa ml}}\sin\theta$, integrate by parts, and take the imaginary part. There are no boundary terms due to $S_{\kappa ml}$'s boundary conditions,[8] and we find

$$\omega_I \int_0^{\pi} \left(\left|\frac{dS_{\kappa ml}}{d\theta}\right|^2 + \left(\frac{m^2}{\sin^2\theta} + a^2|\omega|^2\cos^2\theta\right)\left|S_{\kappa ml}\right|^2 \right)\sin\theta d\theta =$$

[8]Recall that the boundary conditions (1.15) required that $e^{im\phi}S_{\kappa ml}(\theta)$ extend smoothly to $\mathbb{S}^2$. More explicitly, let $x \doteq \cos\theta$; then an asymptotic analysis of the angular ODE shows that the boundary condition (1.15) is equivalent to $S_{\kappa ml} \sim (x \pm 1)^{|m|/2}$ as $x \to \mp 1$.

$$- \int_0^\pi \mathrm{Im}\left(\lambda_{\kappa m l}\overline{\omega}\right) \left|S_{\kappa m l}\right|^2 \sin\theta d\theta \Rightarrow$$

$$- \mathrm{Im}\left(\lambda_{\kappa m l}\overline{\omega}\right) \geq 0. \tag{2.49}$$

We conclude that $\mathrm{Im}\left(\Phi\overline{\omega}\right)$ is positive, and hence that $\tilde{u}$ must vanish.

In terms of R, this implies that

$$\tilde{R}(x) \doteq \int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+-r_-}(x-r_-)(r-r_-)}(r-r_-)^\eta(r-r_+)^\xi e^{-i\omega r}R(r)dr$$

vanishes for all $x \in (r_+, \infty)$. To see that this implies that R vanishes, we first extend R by 0 to all of $\mathbb{R}$ and note that the Fourier transform of $(r-r_-)^\eta(r-r_+)^\xi e^{-i\omega r}R(r)$ is, up to a change of variables,

$$\hat{R}(z) \doteq \int_{-\infty}^{\infty} e^{2i|\omega|^2 z(r-r_-)}(r-r_-)^\eta(r-r_+)^\xi e^{-i\omega r}R(r)dr.$$

In view of the support of R, $\hat{R}$ extends to a holomorphic function on the upper half plane. The vanishing of $\tilde{R}$ for $x \in (r_+, \infty)$ implies that $\hat{R}$ vanishes along the line $\{\frac{y}{\overline{\omega}} : y \in (1, \infty)\}$. Analyticity implies that $\hat{R}$ and hence R itself vanishes. □

Note that the above proof occurs completely at the level of $\tilde{u}$ and $S_{\kappa m l}$. In particular, we neither need Whiting's differential transformations of $S_{\kappa m l}$ (see Section IV of [44]) nor a physical space argument with a new metric (see Section VI of [44]).

2.5.2 Mode Stability on the Real Axis

Now we prove Theorem 1.5.3.

Proof. (Mode Stability on the Real Axis, Theorem 1.5.3) Suppose we have a mode solution with corresponding $(u, S_{\kappa ml}, \lambda_{\kappa ml})$ and $\omega \in \mathbb{R} \setminus \{0\}$. Let $\tilde{u}$ be defined by (2.11), and consider the microlocal energy current associated to $\tilde{u}$:

$$\tilde{Q}_T \doteq \operatorname{Im}\left(\tilde{u}'\overline{\omega\tilde{u}}\right).$$

Then, noting that Φ from Proposition 2.3.1 is real, a straightforward computation shows that we have conservation of energy:

$$\left(\tilde{Q}_T\right)' = 0 \Rightarrow$$

$$\tilde{Q}_T(\infty) - \tilde{Q}_T(-\infty) = 0.$$

Now the boundary conditions from Proposition 2.3.3 imply that we get a useful estimate out of this:

$$\tilde{Q}_T(\infty) - \tilde{Q}_T(-\infty) =$$

$$\frac{1}{2}\left(\omega^2|\tilde{u}(\infty)|^2 + |\tilde{u}'(\infty)|^2 + \omega^2\frac{r_+ - r_-}{r_+}|\tilde{u}(-\infty)|^2 + \frac{r_+}{r_+ - r_-}|\tilde{u}'(-\infty)|^2\right).$$

The unique continuation lemma from Section 2.1 implies that $\tilde{u}$ must vanish.

In terms of R, we see that

$$\tilde{R}(y) \doteq \int_{-\infty}^{\infty} e^{2i\omega y(r-r_-)}(r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}R(r)dr$$

vanishes for $y \in (1,\infty)$, where we have extended R by 0 so that it is defined on all of $\mathbb{R}$. However, it is well known that the Fourier transform of a non-trivial function supported in $(0,\infty)$ cannot vanish on an open set.[9]

As an alternative to this argument, one may instead use the fact from

[9]This follows from holomorphically extending to the upper half plane and the Schwarz reflection principle.

Proposition 2.3.3 that

$$|\tilde{u}(\infty)|^2 = \frac{(r_+ - r_-)^2 |\Gamma(2\xi + 1)|^2}{8M\omega^2 r_+} |u(-\infty)|^2$$

to conclude that $u(-\infty)$ must vanish. Proposition 2.2.1 then implies that $u(\infty)$ and hence u vanishes (again using the unique continuation lemma from Section 2.1). □

Note that this proof is even simpler than the proof of mode stability in the upper half plane since we only need to refer to $\tilde{u}$.

2.5.3 Quantitative Mode Stability

To produce quantitative estimates for the Wronskian we shall need to work a little harder than we did for the qualitative statements.

The following proposition and lemma will be useful for the proof of Theorem 1.5.1.

Proposition 2.5.1. *Let $(\omega, m, l) \in \mathscr{A}$, let $u = (r^2 + a^2)^{1/2} R$ solve the radial ODE with right hand side $H = \frac{\Delta}{(r^2+a^2)^{1/2}} F(r)$, and suppose that R satisfies the boundary conditions of a arising from a mode solution. Then*

$$|u(-\infty)|^2 \leq B(C_{\mathscr{A}}) \int_{r_+}^{\infty} |F(r)|^2 r^4 dr. \tag{2.50}$$

Proof. Let $\tilde{u}$ be defined by (2.11), and consider the microlocal energy current associated to $\tilde{u}$:

$$\tilde{Q}_T \doteq \mathrm{Im}\left(\tilde{u}' \overline{\omega \tilde{u}}\right).$$

A straightforward computation (and the fact that Φ is real) yields

$$\left(\tilde{Q}_T\right)' = \omega \mathrm{Im}\left(\tilde{H}\bar{\tilde{u}}\right) \Rightarrow$$

$$\tilde{Q}_T(\infty) - \tilde{Q}_T(-\infty) = \omega \int_{-\infty}^{\infty} \mathrm{Im}\left(\tilde{H}\bar{\tilde{u}}\right) dx^*. \tag{2.51}$$

As in the proof of Theorem 1.5.3, the boundary conditions from Proposition 2.3.3 imply that we get a useful estimate:

$$\tilde{Q}_T(\infty) - \tilde{Q}_T(-\infty) =$$

$$\frac{1}{2}\left(\omega^2|\tilde{u}(\infty)|^2 + |\tilde{u}'(\infty)|^2 + \omega^2\frac{r_+ - r_-}{r_+}|\tilde{u}(-\infty)|^2 + \frac{r_+}{r_+ - r_-}|\tilde{u}'(-\infty)|^2\right). \tag{2.52}$$

Next, note that for any $\epsilon > 0$, changing variables and applying Plancherel implies

$$\int_{-\infty}^{\infty} \mathrm{Im}\left(\tilde{H}\bar{\tilde{u}}\right) dr^* \leq B\left[(4\epsilon)^{-1}\int_{r_+}^{\infty} |F(r)|^2 r^4 dr + \epsilon\int_{r_+}^{\infty} |R(r)|^2 dr\right]. \tag{2.53}$$

Thus, combining (2.51), (2.52), and (2.53) implies

$$|\tilde{u}(\infty)|^2 \leq B\left(C_{\mathscr{A}}\right)\left[(4\epsilon)^{-1}\int_{r_+}^{\infty} |F(r)|^2 r^4 dr + \epsilon\int_{r_+}^{\infty} |R(r)|^2 dr\right]. \tag{2.54}$$

Now, recall that Proposition 2.3.3 gives

$$|\tilde{u}(\infty)|^2 = \frac{(r_+ - r_-)^2\left|\Gamma\left(2\xi + 1\right)\right|^2}{8M\omega^2 r_+}|u(-\infty)|^2.$$

Thus, we obtain

$$|u(-\infty)|^2 \leq B\left(C_{\mathscr{A}}\right)\left[(4\epsilon)^{-1}\int_{r_+}^{\infty} |F(r)|^2 r^4 dr + \epsilon\int_{r_+}^{\infty} |R(r)|^2 dr\right]. \tag{2.55}$$

Next, we consider the microlocal energy current Q_T for u:

$$Q_T = \omega \mathrm{Im}\,(u'\overline{u})\,.$$

This satisfies

$$(Q_T)' = \omega \mathrm{Im}\,(H\overline{u})\,.$$

Integrating and using R's boundary conditions yields

$$\omega^2 |u(\infty)|^2 - \omega\,(am - 2Mr_+\omega)\,|u(-\infty)|^2 \le \\ B\,(C_{\mathscr{A}})\left[(4\epsilon)^{-1}\int_{r_+}^{\infty}|F(r)|^2\,r^4dr + \epsilon\int_{r_+}^{\infty}|R(r)|^2\,dr\right].$$

Combining this with (2.55) yields

$$|u(-\infty)|^2 + |u(\infty)|^2 \le B\,(C_{\mathscr{A}})\left[(4\epsilon)^{-1}\int_{r_+}^{\infty}|F(r)|^2\,r^4dr + \epsilon\int_{r_+}^{\infty}|R(r)|^2\,dr\right]. \tag{2.56}$$

Finally, taking ϵ sufficiently small, and combining (2.56) with Proposition 2.4.1 easily establishes (2.50) and finishes the proof. □

Next, we switch gears a little and directly construct solutions to the inhomogeneous radial ODE via the following lemma.

Lemma 2.5.1. *Let $H(x^*)$ be compactly supported. For any $(\omega, m, l) \in \mathscr{A}$, define*

$$u(r^*) \doteq W^{-1}\left(u_{out}(r^*)\int_{-\infty}^{r^*} u_{hor}(x^*)H(x^*)dx^* \right. \\ \left. + u_{hor}(r^*)\int_{r^*}^{\infty} u_{out}(x^*)H(x^*)dx^*\right).$$

Then

$$u'' + \left(\omega^2 - V\right) u = H,$$

and $R = (r^2 + a^2)^{-1/2} u$ *satisfies the boundary conditions of a mode solution (1.20) and (1.21).*

Proof. This is a simple computation. □

Finally, we can prove Theorem 1.5.1.

Proof. Define $\tilde{u}$ via Lemma 2.5.1. Then we have

$$|u(-\infty)|^2 = |W|^{-2} \left| \int_{-\infty}^{\infty} u_{\text{out}}(x^*) H(x^*) dx^* \right|^2 .$$

Combining this with Proposition 2.4.1 gives

$$|W|^{-2} \leq B \frac{\int_{r_+}^{\infty} \left|(r^2 + a^2)^{1/2} \Delta^{-1} H\right|^2 r^4 dr}{\left|\int_{-\infty}^{\infty} u_{out}(x^*) H(x^*) dx^*\right|^2}$$

Of course, W is independent of H, so it remains to pick any particular compactly supported H we want so that the right hand side is finite. Since for sufficiently large x, $\left|u_{\text{out}} - e^{i\omega x^*}\right| \leq \frac{C}{x}$ for an explicit constant C (see, for example, [35]), it is certainly possible to find such an H. Thus, we have produced a quantitative bound for W^{-1}. □

2.5.4 The Microlocal Horizon Energy Flux Bound

We start by reviewing the notation introduced for the statement of Theorem 1.5.4 and then introduce some more notation. Let ψ be a solution to the wave equation $\Box_g \psi = 0$ arising from compactly supported initial data along Σ_0 such that ψ is "admissible" in the sense of Definition 1.5.1. Let χ be a cutoff function such that χ is 0 in the past of Σ_0 and identically 1 in the future of Σ_1.

We then define

$$\psi_{\times} \doteq \chi\psi,$$

$$E \doteq \left(\left(\Box_g \chi \right) \psi + 2\nabla^\mu \chi \nabla_\mu \psi \right).$$

Next, we let F be the projection onto the oblate spheroidal harmonics of the Fourier transform of $(r^2 + a^2)^{-1}\rho^2 E$, i.e.

$$F \doteq \int_0^\pi \int_0^{2\pi} \int_{-\infty}^{\infty} (r^2 + a^2)^{-1} \rho^2 E e^{i\omega t} e^{-im\phi} S_{\kappa ml} \sin\theta \; dt \; d\phi \; d\theta.$$

Then let $u(r^*)$ similarly be the projection onto the oblate spheroidal harmonics of the Fourier transform of $(r^2 + a^2)^{1/2}\psi$, and

$$H(r^*) = \frac{\Delta}{(r^2 + a^2)^{1/2}} F.$$

We get

$$u'' + \left(\omega^2 - V\right) u = H, \tag{2.57}$$

$$\begin{aligned} V \doteq &\frac{4Mram\omega - a^2m^2 + \Delta(\lambda_{\kappa_\epsilon ml} + a^2\omega)}{(r^2 + a^2)^2} \\ &+ \frac{\Delta}{(r^2 + a^2)^4} \left(a^2\Delta + 2Mr(r^2 - a^2) \right). \end{aligned}$$

Finally, let $\mathscr{B} \subset \mathbb{R}$ and $\mathscr{C} \subset \{(m, l) \in \mathbb{Z} \times \mathbb{Z} : l \geq |m|\}$ be such that

$$C_{\mathscr{B}} \doteq \sup_{\omega \in \mathscr{B}} \left(|\omega| + |\omega|^{-1} \right) < \infty$$

$$C_{\mathscr{C}} \doteq \sup_{m,l \in \mathscr{C}} \left(|m| + |l| \right) < \infty.$$

The define $\mathscr{A} \doteq \mathscr{B} \times \mathscr{C} \subset \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}$. We will have

$$C_{\mathscr{A}} \doteq \sup_{(\omega,m,l)\in\mathscr{A}} \left(|\omega| + |\omega|^{-1} + |m| + |l| \right) < \infty.$$

Next, we observe that the arguments of Section 5.3 of [21] imply the following lemma.

Lemma 2.5.2. *Let ψ be an admissible function on Kerr that is also a solution to the wave equation $\Box_g \psi = 0$ with compactly supported initial along Σ_0. For every $(\omega, m, l) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}$, set*

$$R(r, \omega, m, l) \doteq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} e^{i\omega t} e^{-im\phi} S_{\kappa m l}(\theta, \kappa) \psi \sin\theta \, d\omega \, d\theta \, d\phi.$$

Then R satisfies the boundary conditions so that $e^{-i\omega t} e^{im\phi} S_{\kappa m l} R$ is a mode solution in the sense of Definition 1.3.1.

Now we check that we can control the $L^1(r^*)$ norm of H and that H is smooth in r^*.

Lemma 2.5.3. *We have*

$$\sup_{(\omega,m,l)\in C_{\mathscr{A}}} \int_{-\infty}^{\infty} |H| \, dr^* < \infty,$$

H is smooth as a function of r^*, $\int_{-\infty}^{r^*(r)} H \, dr^*$ is a smooth function of r, and $\int_{r^*(r)}^{\infty} H \, dr^*$ vanishes for large r.

Proof. The standard $L^\infty - L^1$ inequality for the Fourier transform (and changing variables from r^* to r appropriately) imply

$$\sup_{(\omega,m,l)\in C_{\mathscr{A}}} \int_{-\infty}^{\infty} |H| \, dr^* \leq B \int_{\text{supp}(\nabla\chi)} |rF| \, dVol.$$

Since ψ has compact support, it is easy to see that the right hand side of this inequality is finite.

An analogous argument shows that H is smooth as a function of r^* and that $\int_{-\infty}^{r^*(r)} H\, dr^*$ is a smooth function of r. The final statement of the lemma follows from ψ's compact support. □

The following representation formula is a useful starting point.

Lemma 2.5.4. *We have*

$$u(r^*) = W^{-1}\Bigg(u_{out}(r^*) \int_{-\infty}^{r^*} u_{hor}(x^*)H(x^*)dx^* \qquad (2.58)$$
$$+ u_{hor}(r^*) \int_{r^*}^{\infty} u_{out}(x^*)H(x^*)dx^* \Bigg).$$

Proof. Standard ODE theory (see [35]) implies that $u_{\rm hor}$, $u'_{\rm hor}$, $u_{\rm out}$, and $u'_{\rm out}$ all have a finite $L^\infty_{(\omega,m,l)\in\mathscr{A},r^*\in\mathbb{R}}$ norm. Thus, since $H \in L^1(r^*)$ and Theorem 1.5.3 implies that $W \neq 0$, the right hand side of (2.58) is absolutely convergent.

Define $\check{u}(r^*)$ by the right hand side of (2.58). Now, since H is a smooth function of r^*, we can apply the fundamental theorem of calculus and easily check that $\check{u}$ is a solution to

$$\check{u}'' + (\omega^2 - V)\check{u} = H.$$

Next, using Lemma 2.5.3, one sees that $\check{R} = (r^2+a^2)^{-1/2}\check{u}$ satisfies the boundary conditions associated to a mode solution. Thus, $e^{-it\omega}e^{im\phi}S_{ml}(R - \check{R})$ is a real mode solution. Theorem 1.5.3 then implies that $R = \check{R}$ and thus that $u = \check{u}$. □

We are finally ready to prove Theorem 1.5.4.

Proof. Lemma 2.5.4 immediately yields

$$\int_{\mathscr{B}} \sum_{(m,l)\in\mathscr{C}} |u(-\infty)|^2 \, d\omega \le \int_{\mathscr{B}} \sum_{(m,l)\in\mathscr{C}} |W|^{-2} \left| \int_{-\infty}^{\infty} u_{\text{hor}}(r^*) H(r^*) dr^* \right|^2 \, d\omega.$$

Theorem 1.5.1 then yields

$$\int_{\mathscr{B}} \sum_{(m,l)\in\mathscr{C}} |W|^{-2} \left| \int_{-\infty}^{\infty} u_{\text{hor}}(r^*) H(r^*) dr^* \right|^2 \, d\omega \le$$
$$B(C_{\mathscr{A}}) \int_{\mathscr{B}} \sum_{(m,l)\in\mathscr{C}} \left| \int_{-\infty}^{\infty} u_{\text{hor}}(r^*) H(r^*) dr^* \right|^2 \, d\omega.$$

Thus the theorem is reduced to proving

$$\int_{\mathscr{B}} \sum_{(m,l)\in\mathscr{C}} \left| \int_{-\infty}^{\infty} u_{\text{hor}}(r^*) H(r^*) dr^* \right|^2 \, d\omega \le B \int_{\Sigma_0} |\partial\psi|^2 . \tag{2.59}$$

The first thing we observe, is that unfortunately, passing the $|\cdot|^2$ into the integral with a naive Cauchy-Schwarz inequality will produce too many powers of r to establish (2.59) (keep in mind that $H(r^*)dr^* = (r^2+a^2)^{1/2}F(r)dr$). However, if we introduce a cut-off function $\tilde{\chi}(r)$ which is identically 0 for $r \in [r_+, r_+ + 10M]$ and 1 on $[r_+ + 20M, \infty)$, then using the $L^\infty - L^1$ inequality to control the L^∞ norm of H in $(\omega, m, l) \in \mathscr{A}$, we easily obtain via a Cauchy-Schwarz inequality in r

$$\int_{\mathscr{B}} \sum_{(m,l)\in\mathscr{C}} \left| \int_{-\infty}^{\infty} (1-\tilde{\chi})\, u_{\text{hor}}(r^*) H(r^*) dr^* \right|^2 \, d\omega \le B \int_{\text{supp}(\nabla\chi)} \left[|\psi|^2 + |\partial\psi|^2 \right] \, dVol \tag{2.60}$$
$$\le B \int_{\text{supp}(\nabla\chi)} |\partial\psi|^2 \, dVol$$
$$\le B \int_{\Sigma_0} |\partial\psi|^2 .$$

In the second line we appealed to a standard Hardy inequality, which says that for all f which vanish for large r, we have

$$\int_1^\infty |f|^2 \, dr \leq B \int_1^\infty r^2 |\partial_r f|^2 \, dr, \tag{2.61}$$

and in the third line we have used a finite in time energy estimate.

Thus, we have reduced the theorem to establishing

$$\int_{\mathscr{B}} \sum_{(m,l)\in\mathscr{C}} \left| \int_{-\infty}^{\infty} \tilde{\chi} u_{\text{hor}}(r^*) H(r^*) dr^* \right|^2 d\omega \leq B \int_{\Sigma_0} |\partial\psi|^2. \tag{2.62}$$

Next, standard ODE theory implies that $\left|u_{\text{hor}} - e^{i\omega r^*}\right| \leq \frac{B(C_{\mathscr{A}})}{r}$. Note whenever we gain a power of r, then the argument of (2.60) becomes applicable. We conclude that the theorem will follow from the estimate

$$\int_{\mathscr{B}} \sum_{(m,l)\in\mathscr{C}} \left| \int_{-\infty}^{\infty} \tilde{\chi} e^{i\omega r^*} H(r^*) dr^* \right|^2 d\omega \leq B \int_{\Sigma_0} |\partial\psi|^2. \tag{2.63}$$

In order to control this term, we will exploit the oscillation in ω via a suitable application of Plancherel. However, we first need to fully account for the ω dependence of H. Writing out everything explicitly in Boyer-Lindquist coordinates, and applying the easy Cauchy-Schwarz argument from above to any terms which pick up additional decay in r, yields

$$
\begin{aligned}
&\int_{\mathscr{B}} \sum_{(m,l)\in\mathscr{C}} \left| \int_{-\infty}^{\infty} \tilde{\chi} e^{i\omega r^*} H(r^*) dr^* \right|^2 d\omega \leq \\
&B \int_{\mathscr{B}} \sum_{(m,l)\in\mathscr{C}} \left| \int_{\mathbb{S}^2} \int_{-\infty}^{\infty} \int_{r_+}^{\infty} \tilde{\chi} e^{i\omega t} e^{i\omega r^*} \chi' \left(\partial_t \psi\right) e^{-im\phi} S_{\omega ml}(\theta,\omega) r \sin\theta dt\, dr\, d\theta\, d\phi \right|^2 d\omega + \\
&B \int_{\mathscr{B}} \sum_{(m,l)\in\mathscr{C}} \left| \int_{\mathbb{S}^2} \int_{-\infty}^{\infty} \int_{r_+}^{\infty} \tilde{\chi} e^{i\omega t} e^{i\omega r^*} \chi'' \psi e^{-im\phi} S_{\omega ml}(\theta,\omega) r \sin\theta dt\, dr\, d\theta\, d\phi \right|^2 d\omega + \\
&B \int_{\Sigma_0} |\partial\psi|^2 \\
&\doteq B\left[I + II + \int_{\Sigma_0} |\partial\psi|^2 \right].
\end{aligned}
$$

We begin with an analysis of the first term I. We have

$$
\begin{aligned}
&\int_{\mathscr{B}} \sum_{(m,l)\in\mathscr{C}} \left| \int_{\mathbb{S}^2} \int_{-\infty}^{\infty} \int_{r_+}^{\infty} \tilde{\chi} e^{i\omega t} e^{i\omega r^*} \chi' \left(\partial_t \psi\right) e^{-im\phi} S_{\omega ml}(\theta,\omega) r \sin\theta dt\, dr\, d\theta\, d\phi \right|^2 d\omega \\
&\leq B \int_{\mathscr{B}} \int_{\mathbb{S}^2} \left| \int_{-\infty}^{\infty} \int_{r_+}^{\infty} \tilde{\chi} e^{i\omega t} e^{i\omega r^*} \chi' \left(\partial_t \psi\right) r dt\, dr \right|^2 d\omega\ \sin\theta\ d\theta\ d\phi \\
&\leq B \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \int_{\mathscr{B}} \left| \int_{r_+}^{\infty} \tilde{\chi} e^{i\omega r^*} \left(\partial_t \psi\right) r\ dr \right|^2 (\chi')^2\ d\omega\ \ dt\ \sin\theta\ d\theta\ d\phi \\
&\leq B \int_{-\infty}^{\infty} \int_{r_+}^{\infty} \int_{\mathbb{S}^2} \int_{r_+}^{\infty} |\partial_t \psi|^2 r^2 (\chi')^2\ dt\ dr\ \sin\theta\ d\theta\ d\phi \\
&\leq \int_{\Sigma_0} |\partial\psi|^2 .
\end{aligned}
$$

In the first line we used the orthonormality of the $e^{-im\phi} S_{ml}$, in the second line we use that support of χ' is compact in the t-direction, in the third line we used Plancherel in ω, and in the final line we used a finite in time energy estimate.

For the term II, proceeding exactly as we did for term I yields

$$\int_{\mathscr{B}} \sum_{(m,l)\in\mathscr{C}} \left| \int_{\mathbb{S}^2} \int_{-\infty}^{\infty} \int_{r_+}^{\infty} \tilde{\chi} e^{i\omega t} e^{i\omega r^*} \chi'' \psi e^{-im\phi} S_{\omega ml}(\theta,\omega) r \sin\theta dt\, dr\, d\theta\, d\phi \right|^2 d\omega$$
$$\leq B \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \int_{\mathscr{B}} \left| \int_{r_+}^{\infty} \tilde{\chi} e^{i\omega r^*} \psi r\, dr \right|^2 (\chi'')^2\, d\omega \;\; dt\, \sin\theta\, d\theta\, d\phi.$$

However, now write $e^{i\omega r^*} = (i\omega)^{-1}\left(1 + O\left(r^{-1}\right)\right) \frac{\partial}{\partial r}\left(e^{i\omega r^*}\right)$ and integrate by parts in r. We obtain

$$\int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \int_{\mathscr{B}} \left| \int_{r_+}^{\infty} \tilde{\chi} e^{i\omega r^*} \psi r\, dr \right|^2 (\chi'')^2\, d\omega \;\; dt\, \sin\theta\, d\theta\, d\phi$$
$$\leq B\omega^{-1} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \int_{\mathscr{B}} \left| \int_{r_+}^{\infty} \tilde{\chi} e^{i\omega r^*} \partial_r \psi r\, dr \right|^2 (\chi'')^2\, d\omega \;\; dt\, \sin\theta\, d\theta\, d\phi$$
$$+ B\omega^{-1} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \int_{\mathscr{B}} \left| \int_{r_+}^{\infty} (\partial_r \tilde{\chi})\, e^{i\omega r^*} \psi r\, dr \right|^2 (\chi'')^2\, d\omega \;\; dt\, \sin\theta\, d\theta\, d\phi$$
$$+ B\omega^{-1} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \int_{\mathscr{B}} \left| \int_{r_+}^{\infty} \tilde{\chi} e^{i\omega r^*} \psi\, dr \right|^2 (\chi'')^2\, d\omega \;\; dt\, \sin\theta\, d\theta\, d\phi.$$

For the first term on the right hand side of this inequality we can now proceed as we did with term I. For the other two terms, we can also proceed as we did before, we just have to append the Hardy inequality (2.61) at the very end. Adding everything together finishes the proof. □

Chapter 3

Mode Instability for the Klein-Gordon Equation

The goal of this chapter is to prove Theorems 1.6.1 and 1.6.2. We start in Section 3.1 with a review of linear ODE's with regular singularities. Then, in Section 3.2 we apply the analysis of Section 3.1 to the radial ODE. Finally, in Section 3.3 we prove Theorem 1.6.2, and in Section 3.4 we prove Theorem 1.6.1.

3.1 Linear ODE's with Regular Singularities

Let's recall some facts about linear ODEs in the complex plane.

Lemma 3.1.1. *Consider the complex ODE*

$$\frac{d^2H}{dz^2} + f(z,\lambda)\frac{dH}{dz} + g(z,\lambda)H = 0. \tag{3.1}$$

We will assume that there exists $\{f_j(\lambda)\}$, $\{g_j(\lambda)\}$, r, *and open* $U \subset \mathbb{C}$ *such that for* $z \in B_r(z_0)$ *and every compact* $K \subset U$ *there exists* $\left\{F_j^{(K)}\right\}$ *and* $\left\{G_j^{(K)}\right\}$

such that

$$|f_j(\lambda)| \le F_j^{(K)} \text{ and } |g_j(\lambda)| \le G_j^{(K)} \text{ when } \lambda \in K,$$

$$\sum_{j=0}^{\infty} G_j^{(K)}(z-z_0)^j \text{ and } \sum_{j=0}^{\infty} F_j^{(K)}(z-z_0)^j \text{ converge absolutely,}$$

$$\{f_j(\lambda)\} \text{ and } \{g_j(\lambda)\} \text{ are holomorphic in } \lambda \in U,$$

$$(z-z_0)f(z,\lambda) = \sum_{j=0}^{\infty} f_j(\lambda)(z-z_0)^j \text{ and } (z-z_0)^2 g(z) = \sum_{j=0}^{\infty} g_j(\lambda)(z-z_0)^j.$$

If these hypotheses hold we say that z_0 is a regular singularity. Set

$$Q(\alpha,\lambda) \doteq \alpha(\alpha-1) + f_0(\lambda)\alpha + g_0(\lambda).$$

The indicial equation is

$$Q(\alpha,\lambda) = 0.$$

We suppose that a holomorphic $\alpha(\lambda)$ has been chosen such that

$$Q(\alpha(\lambda),\lambda) = 0 \text{ and } \min_{j\in\mathbb{Z}^+} |Q(\alpha(\lambda)+j,\lambda)| = A(\lambda) > 0.$$

Then there exists a unique solution to (3.1) for z near z_0 of the form

$$h(z,\lambda) = (z-z_0)^{\alpha(\lambda)}\rho(z,\lambda)$$

such that $\rho(z_0,\lambda) = 1$. Furthermore, ρ is holomorphic for $z \in B_{r_0}(z_0)$ and $\lambda \in K$ where r_0 is sufficiently small and $K \subset U$ is sufficiently small.

Proof. One can extract a proof of this from the discussion of regular singularities in [35]. For the sake of completeness we will give the needed slight extension here. Without loss of generality we may set $z_0 = 0$. We begin by looking for a formal solution of the form

$$h(z,\lambda) = z^{\alpha(\lambda)} \sum_{j=0}^{\infty} \rho_j(\lambda) z^j$$

where we set $\rho_0(\lambda) = 1$. Formally plugging this into (3.1) we find (see [35])

$$Q\left((\alpha(\lambda),\lambda)\right) = 0,$$

$$Q\left(\alpha(\lambda)+j,\lambda\right)\rho_j(\lambda) = -\sum_{k=0}^{j-1}\left(\left(\alpha(\lambda)+k\right)f_{j-k}(\lambda) + g_{j-k}(\lambda)\right)\rho_k(\lambda) \text{ for } j \geq 1.$$

Since $Q(\alpha(\lambda),\lambda) = 0$ by hypothesis, the first equation is satisfied. Furthermore, by assumption $Q(\alpha(\lambda)+j,\lambda) \neq 0$ for any j. Hence, the second equation determines $\rho_j(\lambda)$ recursively. This establishes the uniqueness of ρ. It remains to check that the series converges appropriately. We will do this by majorizing the series. Let us pick an arbitrary compact set $K \subset U$ sufficiently small and $r_0 < r$ sufficiently small. After applying Cauchy's estimate to the holomorphic functions $\sum_j F_j^{(K)} z^j$ and $\sum_j G_j^{(K)} z^j$, we may find a constant C_K so that

$$|f_j(\lambda)| \leq C_K r_0^{-j} \text{ and } |g_j(\lambda)| \leq C_K r_0^{-j} \text{ for } \lambda \in K.$$

Let $\beta(\lambda)$ be the other root of $Q(\cdot,\lambda)$, and set $n(\lambda) \doteq |\alpha(\lambda)-\beta(\lambda)|$. Since $Q(\alpha(\lambda)+k,\lambda) = k(k+\alpha(\lambda)-\beta(\lambda))$, our hypotheses imply that $\alpha(\lambda)-\beta(\lambda) \notin \mathbb{Z}_{\leq 0}$. Next, define $b_j(\lambda)$ by

$$b_j(\lambda) = |\rho_j(\lambda)| \text{ for } j \leq n,$$

$$j(j-|\alpha(\lambda)-\beta(\lambda)|)b_j(\lambda) = C_K \sum_{k=0}^{j-1}\left(|\alpha(\lambda)|+k+1\right)b_k(\lambda) r_0^{k-j} \text{ for } j > n.$$

It is easy to check by induction that $|\rho_j(\lambda)| \leq b_j(\lambda)$ for all j. For sufficiently large j, one finds that

$$r_0 j(j - |\alpha(\lambda) - \beta(\lambda)|)b_j(\lambda) - (j-1)(j-1-|\alpha(\lambda) - \beta(\lambda)|)b_{j-1}(\lambda) =$$

$$C_K(|\alpha(\lambda)| + j)b_{j-1}(\lambda).$$

Now the ratio test implies that the series $\sum_{j\geq 0}^{\infty} b_j(\lambda)z^j$ converges in the ball of radius r_0. Hence, by the comparison test, $\sum_{j=0}^{\infty} \rho_j(\lambda)z^j$ converges in the same ball. Since r_0 was arbitrary, we find that for every $\lambda \in K$, $\sum_{j=0}^{\infty} \rho_j(\lambda)z^j$ converges and is holomorphic in $z \in B_r(0)$. Next we may freeze $z \in B_r(0)$ and consider $\rho(z,\lambda) = \sum_{j=0}^{\infty} \rho_j(\lambda)z^j$ as a function of λ. For every compact $K \subset U$, our proof has shown that $\rho(z,\cdot)$ is a uniform limit of holomorphic functions. Hence, $\rho(z,\lambda)$ is holomorphic for $\lambda \in U$. □

3.2 Local Theory for the Radial ODE

3.2.1 The Horizon

Let's apply the theory from Section 3.1 to the radial ODE. Recall that we earlier set

$$\xi = \frac{i(am - 2Mr_+\omega)}{r_+ - r_-}.$$

First we consider the case where $am - 2Mr_+\omega \neq 0$. In this case the indicial equation has two distinct roots which do not differ by an integer. Hence a local basis of solutions to the radial ODE around r_+ will be given by

$$\{(r - r_+)^{\xi}\rho_1(r), (r - r_+)^{-\xi}\rho_2(r)\}$$

where each $\rho_i(r)$ is holomorphic near r_+ and is normalized to have $\rho_i(r_+) = 1$. Our boundary conditions require that R must be of the form $A(r-r_+)^{\xi}\rho_1(r)$ for some $A \in \mathbb{C}$. Hence, for every ω and μ so that λ is defined (see (1.14) and Lemma 1.3.1), we consider the unique solution to the radial ODE of the form

$$(r - r_+)^{\xi}\rho(r,\omega,\mu) \tag{3.2}$$

where $\rho(r,\omega,\mu)$ is analytic in r, holomorphic in ω, analytic in μ, and $\rho(r_+,\omega,\mu) = 1$. Let us remark that if a mode solution with real ω and $am - 2Mr_+\omega \neq 0$ vanishes at r_+, it must vanish identically.

Now let's consider what happens if $am - 2Mr_+\omega = 0$. In this case the indicial equation has a double root at $\alpha = 0$ and lemma 3.1.1 only produces one solution near r_+. One must then consider solutions which have a logarithmic singularity at r_+. The standard theory (see [35]) then implies that a local basis of solutions is given by

$$\{\varphi_1(r), \log(r-r_+)\varphi_2(r) + \varphi_3(r)\}$$

where the φ_i are all holomorphic near r_+, $\varphi_1(r_+) = 1$, $\varphi_2(r_+) = 1$, and $\varphi_3(r_+) = 0$. It will be important to note that lemma 3.1.1 implies that φ_1 is embedded in the family of local solutions (3.2).

Lastly, it will be useful for the bound state analysis to note that everything said in this section so far applies verbatim to the equation

$$\Delta\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right) - W_\mu R - \nu\Delta R = 0 \text{ for } \nu \in \mathbb{R}.$$

3.2.2 Infinity

The local existence theorems quoted in this section can be found in chapter 7 of [35]. Let us note that the radial ODE can be written as

$$\frac{dR^2}{dr^2} + \frac{\partial_r \Delta}{\Delta}\frac{dR}{dr} - \frac{W_\mu}{\Delta^2} R = 0 \Rightarrow$$

$$\frac{dR^2}{dr^2} + \left(\frac{2}{r} + O(r^{-2})\right)\frac{dR}{dr} + \left((\omega^2 - \mu^2) + \frac{2M(2\omega^2 - \mu^2)}{r} + O(r^{-2})\right) R = 0.$$

Let's write $\omega = \omega_R + i\omega_I$. We will need to construct a local basis at infinity that depends holomorphically on ω and analytically on μ.

Lemma 3.2.1. *For all ω and μ with $\mu^2 - \omega^2 \notin (-\infty, 0]$ there is a unique $\hat{\rho}_2(r, \omega, \mu)$ which solves the radial ODE and satisfies*

$$\hat{\rho}_2(r, \omega, \mu) = e^{-\sqrt{\mu^2-\omega^2}r} r^{-1-\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}} + O\left(e^{-\sqrt{\mu^2-\omega^2}r} r^{-2-\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}}\right).$$

Furthermore, $\hat{\rho}_2$ depends holomorphically on ω and μ. The square root is defined by making a branch cut along the negative real numbers.

Proof. One can more or less extract a proof of this from the discussion of irregular singularities in Chapter 7 section 2 of [35]. For the sake of completeness we will give the needed slight extension. We let C denote a sufficiently large constant which can be taken holomorphic in μ and ω. One may find a formal solution to the radial ODE of the form

$$L(r, \omega, \mu) \doteq e^{-\sqrt{\mu^2-\omega^2}r} r^{-1-\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}} \sum_{j=0}^{\infty} \frac{a_j(\omega, \mu)}{z^j}$$

where $a_0 = 1$ and the a_j are holomorphic in ω and μ. See Chapter 7 section 1 of [35] for the computations behind this. Let's set

$$L_n(r,\omega,\mu) \doteq e^{-\sqrt{\mu^2-\omega^2}r} r^{-1-\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}} \sum_{j=0}^{n-1} \frac{a_j(\omega,\mu)}{z^j}.$$

Then

$$\frac{d^2L_n}{dr^2} + \frac{\partial_r\Delta}{\Delta}\frac{dL_n}{dr} - \frac{W_\mu}{\Delta^2}L_n = e^{-\sqrt{\mu^2-\omega^2}r} r^{-1-\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}} B_n(r,\omega,\mu)$$

where $B_n(r,\omega,\mu) \le Cr^{-n-1}$. Let's look for a solution $\hat{\rho}_2$ of the form

$$\hat{\rho}_2(r,\omega,\mu) = L_n(r,\omega,\mu) + \epsilon(r,\omega,\mu).$$

We must have

$$\frac{d^2\epsilon}{dr^2} + \frac{\partial_r\Delta}{\Delta}\frac{d\epsilon}{dr} - \frac{W_\mu}{\Delta^2}\epsilon = -e^{-\sqrt{\mu^2-\omega^2}r} r^{-1-\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}} B_n \Leftrightarrow$$

$$\frac{d^2\epsilon}{dr^2} + \left(\omega^2-\mu^2\right)\epsilon =$$

$$-e^{-\sqrt{\mu^2-\omega^2}r} r^{-1-\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}} B_n - \frac{\partial_r\Delta}{\Delta}\frac{d\epsilon}{dr} + \left(\frac{W_\mu}{\Delta^2} + \left(\omega^2-\mu^2\right)\right)\epsilon.$$

Let's set

$$K(r,t) \doteq \frac{e^{\sqrt{\mu^2-\omega^2}(r-t)} - e^{-\sqrt{\mu^2-\omega^2}(r-t)}}{2\sqrt{\mu^2-\omega^2}}.$$

Variation of parameters gives

$$\epsilon(r,\omega,\mu) =$$

$$\int_r^\infty K(r,t)\times$$
$$\left(e^{-\sqrt{\mu^2-\omega^2}t}t^{-1-\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}}B_n(t)-\left(\frac{W_\mu(t)}{\Delta^2(t)}+\omega^2-\mu^2\right)\epsilon(t)+\frac{\partial_t\Delta(t)}{\Delta(t)}\frac{d\epsilon}{dr}(t)\right)dt.$$

We may solve this by iterating in the usual fashion. Set $h_0(r,\omega,\mu)=0$ and

$$h_{j+1}(r,\omega,\mu)=$$

$$\int_r^\infty K(r,t)\times$$
$$\left(e^{-\sqrt{\mu^2-\omega^2}t}t^{-1-\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}}B_n-\left(\frac{W_\mu(t)}{\Delta^2(t)}+\omega^2-\mu^2\right)h_j(t)+\frac{\partial_t\Delta(t)}{\Delta(t)}\frac{dh_j}{dr}(t)\right)dt.$$

It is easy to see that

$$|h_1(r,\omega,\mu)|+\left|\frac{dh_1}{dr}(r,\omega,\mu)\right|\leq$$

$$\frac{Ce^{-\sqrt{\mu^2-\omega^2}r}r^{-1-\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}}}{r^n}\left(n+\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}\right)^{-1}.$$

Then, with induction one can show that

$$|h_{j+1}-h_j|(r,\omega,\mu)+\left|\frac{dh_{j+1}}{dr}-\frac{dh_j}{dr}\right|(r,\omega,\mu)\leq$$

$$\frac{C^je^{-\sqrt{\mu^2-\omega^2}r}r^{-1-\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}}}{r^n}\left(n+\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}\right)^{-j}.$$

For ω and μ in a sufficiently small compact set and sufficiently large n, the $h_j(r,\omega,\mu)$ will converge uniformly in r, ω, and μ. □

It is of course easy to pick a second holomorphic family of solutions $\hat{\rho}_1(r,\omega,\mu)$

that is linearly independent of $\hat{\rho}_2$. One can show (Chapter 7 of [35]) that we must then have

$$\hat{\rho}_2(r,\omega,\mu) \sim e^{-\sqrt{\mu^2-\omega^2}r} r^{-1-\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}},$$

$$\hat{\rho}_1(r,\omega,\mu) \sim e^{\sqrt{\mu^2-\omega^2}r} r^{-1+\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}}.$$

Lastly, we note that a similar discussion can be carried out for the equation

$$\Delta\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right) - W_\mu R + \nu\Delta R = 0.$$

3.2.3 Reflection and Transmission Coefficients

Let's fix some set of frequency parameters with $\mu^2 - \omega_R^2 \notin (-\infty, 0]$. Above we constructed $\rho(r,\omega,\mu)$ holomorphic in ω and μ so that $(r-r_+)^\xi \rho(r,\omega,\mu)$ gives a solution to the radial ODE with the correct boundary condition at r_+. We can then introduce reflection and transmission coefficients $A(\omega,\mu)$ and $B(\omega,\mu)$:

$$R(r,\omega,\mu) \doteq (r-r_+)^\xi \rho(r,\omega,\mu) = A(\omega,\mu)\hat{\rho}_1(r,\omega,\mu) + B(\omega,\mu)\hat{\rho}_2(r,\omega,\mu).$$

Let $W(\cdot,\cdot)$ denote the Wronskian. Then

$$A = \frac{W(R,\hat{\rho}_2)}{W(\hat{\rho}_1,\hat{\rho}_2)}.$$

Thus A is holomorphic in ω and analytic μ. Similarly, B is holomorphic in ω and analytic in μ.

3.3 Proof of Theorem 1.6.2: Restrictions on Mode Solutions Corresponding to Real ω

We will start with the proof of Theorem 1.6.2 since it is simpler than and motivates the hypotheses of Theorem 1.6.1.

3.3.1 Part 1

Let R be a solution to the radial ODE with parameters (ω, m, l, μ) such that $\omega \in \mathbb{R} \setminus \{0\}$, $\mu^2 > \omega^2$, and R satisfies the boundary conditions associated with a mode solution. We wish to show that

$$am - 2Mr_+\omega = 0.$$

Let's define the energy current,

$$\hat{Q}_T \doteq \mathrm{Im}\left(\Delta \frac{dR}{dr}\overline{R}\right).$$

An easy calculation yields

$$\frac{d\hat{Q}_T}{dr} = 0.$$

Since R must decay exponentially at infinity, we have $\hat{Q}_T(\infty) = 0$. Hence, using the horizon boundary condition (1.20), we get

$$0 = Q_T(\infty) = Q_T(r_+) = (2Mr_+)\mathrm{Im}\left(\frac{dR}{dr^*}(r_+)\overline{R(r_+)}\right) = (am - 2Mr_+\omega)\left|R(r_+)\right|^2.$$

Thus, either $am - 2Mr_+\omega = 0$ or $R(r_+) = 0$. However, $R(r_+) = 0$ implies that R is identically 0 (see Section 3.2). We conclude that $am - 2Mr_+\omega = 0$.

3.3.2 Part 2

Again we let R be a solution to the radial ODE with parameters (ω, m, l, μ) such that $\omega \in \mathbb{R} \setminus \{0\}$, $\mu^2 > \omega^2$, and R satisfies the boundary conditions associated to a mode solution. From the previous section we know that we must have

$$am - 2Mr_+\omega = 0.$$

We now wish to show that

$$am \neq 0.$$

Using $am - 2Mr_+\omega = 0$, we may write

$$W_\mu = -(r^2 + a^2)^2\omega^2 + 4M^2\omega^2 r_+(2r - r_+) + \Delta\left(\lambda_{\kappa ml} + a^2\omega^2 + r^2\mu^2\right). \tag{3.3}$$

We now argue by contradiction. If $2Mr_+\omega = am = 0$, then

$$W_\mu = \Delta\left(\lambda_{0ml} + r^2\mu^2\right) = \Delta\left(l\,(l+1) + r^2\mu^2\right) \geq 0.$$

Now consider the function

$$f(r) \doteq \mathrm{Re}\left(\Delta \frac{dR}{dr}\overline{R}\right).$$

Since our mode solution must be exponentially decreasing at infinity, we see that $f(\infty) = 0$. The boundary conditions at the horizon imply that $f(r_+) = 0$. Hence,

$$0 = \int_{r_+}^{\infty} \frac{df}{dr} dr = \int_{r_+}^{\infty} \left(\Delta \left|\frac{dR}{dr}\right|^2 + \frac{W_\mu}{\Delta}|R|^2\right) dr.$$

This contradicts the non-triviality of R.

3.3.3 Part 3

We still let R be a solution to the radial ODE with parameters (ω, m, l, μ) such that $\omega \in \mathbb{R} \setminus \{0\}$, $\mu^2 > \omega^2$, and R satisfies the boundary conditions of a mode. From the previous two sections we know that

$$am - 2Mr_+\omega = 0,$$

$$am \neq 0.$$

We wish to show that there exists a function $C(\omega, m, l)$ such that

$$\omega^2 < \mu^2 < \omega^2 + C(\omega, m, l).$$

Starting from (3.3), using $\omega^2 = \frac{a^2m^2}{4M^2r_+^2}$, and (2.40), one finds

$$\begin{aligned}
\frac{dW_\mu}{dr}(r_+) &= -4r_+\left(r_+^2 + a^2\right)\omega^2 + 8M^2\omega^2 r_+ + (r_+ - r_-)\left(\lambda_{\kappa ml} + a^2\omega^2 + r_+^2\mu^2\right) \\
&= 8Mr_+\omega^2\left(M - r_+\right) + (r_+ - r_-)\left(\lambda_{\kappa ml} + a^2\omega^2 + r_+^2\mu^2\right) \\
&= (r_+ - r_-)\left(-\frac{a^2m^2}{Mr_+} + \lambda_{\kappa ml} + a^2\omega^2 + r_+^2\mu^2\right) \\
&\geq (r_+ - r_-)\left(|m|\left(|m| + 1\right) - \frac{a^2m^2}{Mr_+} + r_+^2\mu^2\right) > 0.
\end{aligned}$$

In the third equality we used that $2(M - r_+) = -(r_+ - r_-)$, and in the last line we used that $a < M < r_+$. Away from r_+, increasing μ strictly increases W_μ, and as long as $\mu^2 > \omega^2$ the potential converge to ∞ as $r \to \infty$; hence, we may conclude that there exists $C(\omega, m, l)$ such that

$$\mu^2 > \omega^2 + C(\omega, m, l) \Rightarrow W_\mu \geq 0.$$

Now the proof concludes exactly as in Section 3.3.2.

In order to establish that

$$\lim_{l \to \infty} C(\omega, m, l) = 0,$$

it suffices to know that

$$\frac{\partial \lambda_{\kappa m l}}{\partial \mu} > 0,$$

$$\lim_{l \to \infty} \lambda_{\kappa m l} = \infty.$$

This second fact follows from standard Sturm-Liouville theory.

Thus, the proof is finished with the following lemma.

Lemma 3.3.1. *When ω is real, we have*

$$\frac{\partial \lambda}{\partial \mu} > 0.$$

Proof. Let

$$S_\mu \doteq \frac{\partial S}{\partial \mu}.$$

We have

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS_\mu}{d\theta}\right) - \left(\frac{m^2}{\sin^2\theta} - a^2(\omega^2 - \mu^2)\cos^2\theta\right)S_\mu + \lambda S_\mu =$$

$$\left(2a^2\mu\cos^2\theta - \frac{\partial \lambda}{\partial \mu}\right)S.$$

One may easily check using the theory from Section 3.2 that S_μ still satisfies the boundary conditions of an eigenfunction. Thus, multiplying the equation by $\overline{S}$, integrating by parts, and taking the real part gives

$$\int_0^\pi \left(2a^2\mu\cos^2\theta - \frac{\partial\lambda}{\partial\mu}\right)|S|^2 \sin\theta d\theta = 0.$$

□

3.4 Proof of Theorem 1.6.1: Construction of Mode Solutions

Now we will prove Theorem 1.6.1.

3.4.1 Outline of Proof

Before beginning the proof we will give a brief outline. As mentioned in the introduction, we start by constructing real mode solutions. The key technical insight is a variational interpretation of real mode solutions. The variational problem will posses a degeneracy, but this will turn out to be a minor technical problem. Next, we will perturb our real mode solution into the upper complex half-plane by slightly varying ω and μ. This argument relies on observing that mode solutions are in a one to one correspondence with zeros of a certain holomorphic function of ω and μ. Given this, an appropriate application of the implicit function will conclude the argument. Lastly, we analyze how a mode in the upper half-plane can cross the real axis. The upshot will be that as long as we are in a bound state regime ($\mu^2 > \omega^2$), a mode must become superradiant (Proposition 3.4.3) and lose mass (Proposition 3.4.5) as it enters the upper half-plane. Putting everything together will conclude the proof of Theorem 1.6.1.

3.4.2 Existence of Real Mode Solutions

We begin with construction of modes corresponding to real ω. In light of Theorem 1.6.2 we shall fix a choice of ω, m, and l such that $am - 2Mr_+\omega = 0$ and $\omega \neq 0$. In the rest of this section all constants may depend on ω, m, and l.

A Variational Interpretation of Real Mode Solutions

First, we shall need to review the local theory for the radial ODE. As recalled in Section 3.2, when $am - 2Mr_+\omega = 0$, a local basis around r_+ of solutions to the radial ODE is given by

$$\{\varphi_1, \log(r - r_+)\varphi_2 + \varphi_3\}$$

where the φ_i are all analytic near r_+, $\varphi_1(r_+) = 1$, $\varphi_2(r_+) = 1$, and $\varphi_3(r_+) = 0$. Our to be constructed solution R should be a non-zero multiple of φ_1. It will be useful to further observe that the analysis of Section 3.2 implies that if a solution of the radial ODE does not satisfy the correct boundary condition, then it is exponentially increasing.

Next, we explore the graph of $\frac{W_\mu}{\Delta}$. Using the formula (3.3) and the assumption $\omega^2 = \frac{a^2m^2}{4M^2r_+^2}$, one may derive

$$\frac{W_\mu}{\Delta} = -\frac{a^2m^2}{4M^2r_+^2}\left(\Delta + 4Mr + \frac{4M^2(r - r_+)}{r - r_-}\right) + \lambda_{\kappa ml} + a^2\omega^2 + r^2\mu^2. \tag{3.4}$$

In particular, combining this with (2.40) and the inequality $a < M < r_+$ gives

$$\frac{W_\mu}{\Delta}(r_+) = -\frac{a^2m^2}{Mr_+} + \lambda_{\kappa ml} + a^2\omega^2 + r_+^2\mu^2 \geq$$

$$-\frac{a^2m^2}{Mr_+} + |m|\left(|m| + 1\right) + r_+^2\mu^2 \geq m^2\left(1 - \frac{a^2}{Mr_+}\right) + r_+^2\mu^2 > 0.$$

Furthermore, it is easy to see that

$$\frac{W_\mu}{\Delta} = r^2\left(\mu^2 - \omega^2\right) + O(r) \text{ as } r \to \infty,$$

Thus, there exists $r_+ < r_A(\mu^2) < r_B(\mu^2) < \infty$ such that W_μ can only be non-positive on (r_A, r_B).[1] Furthermore, we can take r_A increasing in μ^2 and r_B decreasing in μ^2. Below, in Lemma 3.4.1 we will see that for μ^2 sufficiently close to ω^2, W_μ does in fact get very negative in (r_A, r_B). This suggests that we could look for bound states of the radial ODE by minimizing the functional

$$\mathcal{L}_\mu(f) \doteq \int_{r_+}^{\infty}\left(\Delta\left|\frac{df}{dr}\right|^2 + \frac{W_\mu}{\Delta}|f|^2\right)dr$$

over functions of unit L^2 norm. Note that any solution f of the radial ODE with $\mathcal{L}_\mu(f) < \infty$ will automatically satisfy the correct boundary conditions (at $r = r_+$ *and* $r = \infty$). This is the crucial way that the $am - 2Mr_+\omega = 0$ assumption enters the construction. The degeneration of the radial ODE at r_+ poses some difficulties for a direct variational analysis of $\mathcal{L}_\mu$. Nevertheless, we will be able to overcome this by working with regularized versions of $\mathcal{L}_\mu$. In Section 3.4.2 we will prove the following two propositions.

Proposition 3.4.1. *For every μ sufficiently close to but larger than ω, there exists a non-zero f_μ satisfying the boundary conditions of a mode solution and a constant $\nu_\mu \leq 0$ such that*

$$\Delta\frac{d}{dr}\left(\Delta\frac{df_\mu}{dr}\right) - W_\mu f_\mu + \nu_\mu \Delta f_\mu = 0.$$

Furthermore, ν_μ can be taken to be increasing in μ^2.

[1] Note that this structure is absent in a study of the wave equation ($\mu = 0$).

Proposition 3.4.2. *There exists μ_0 and corresponding f_{μ_0} such that $\nu_{\mu_0} = 0$.*

The f_{μ_0} is the solution we seek.

We will close the section with a preparatory lemma. Recall that we have fixed ω, m, and l which are assumed to satisfy $am - 2Mr_+\omega = 0$ and $\omega \in \mathbb{R} \setminus \{0\}$. Define

$$\mathscr{A} \doteq \left\{\mu > 0 : \mu^2 > \omega^2 \text{ and } \exists f \in C_0^\infty \text{ with } \mathcal{L}_\mu(f) < 0\right\}.$$

Lemma 3.4.1. *Let μ be sufficiently close to but larger than ω. Then we will have*

$$\mu \in \mathscr{A}.$$

Proof. For every fixed f, $\mathcal{L}_\mu(f)$ is continuous in μ. Thus, it is sufficient to find a smooth f with compact support such that

$$\mathcal{L}_\omega(f) < 0.$$

First, we note that near infinity

$$\frac{V_\omega}{\Delta} = -2M\omega^2 r + O(1).$$

Hence, for f supported in (A, ∞) with A large, we write

$$\mathcal{L}_\omega(f) = \int_A^\infty \left(\left(r^2 + O(r)\right) \left|\frac{df}{dr}\right|^2 - \left(2M\omega^2 r + O(1)\right)|f|^2 \right) dr.$$

Since $\omega \neq 0$, if we set f to be equal to $r^{-3/4}$ on a sufficiently large compact set K and 0 outside a slight enlargement of K, it is clear that we can make $\mathcal{L}_\omega(f)$ as negative as we please. $\square$

We remark that this lemma is the only place where we shall use the $\omega \neq 0$ hypothesis.

Analysis of the Variational Problem

It will be useful to consider the following regularization of $\mathcal{L}_\mu$:

$$\mathcal{L}_\mu^{(\epsilon)}(f) \doteq \int_{r_+ + \epsilon}^\infty \left(\Delta \left| \frac{df}{dr} \right|^2 + \frac{W_\mu}{\Delta} |f|^2 \right) dr.$$

Lemma 3.4.2. *If $\mu^2 > \omega^2$, then there exists $f_\mu^{(\epsilon)} \in H_0^1(r_+ + \epsilon, \infty)$ with unit $L^2(r_+ + \epsilon, \infty)$ norm such that $\mathcal{L}_\mu^{(\epsilon)}$ achieves its infimum over $H_0^1(r_+ + \epsilon, \infty)$ functions of unit $L^2(r_+ + \epsilon, \infty)$ norm on $f_\mu^{(\epsilon)}$.*

Proof. If omitted, all integration ranges are over $(r_+ + \epsilon, \infty)$. Recall that in Section 3.4.2 we showed that $\frac{W_\mu}{\Delta}$ is increasing in μ^2, is non-negative near r_+, and goes to infinity as $r \to \infty$. More specifically, we established

$$\frac{W_\mu}{\Delta}(r_+) \gtrsim \mu^2,$$

$$\frac{W_\mu}{\Delta} = r^2 \left(\mu^2 - \omega^2 \right) + O(r) \text{ as } r \to \infty.$$

Hence, we can find $r_+ < B_0 < B_1$, $C_0 > 0$, and $C_1 > 0$ only depending[2] on an lower bound for μ^2 such that

$$\int \left(\Delta \left| \frac{df}{dr} \right|^2 + C_0 r^2 1_{[B_0, B_1]^c} \left(\mu^2 - \omega^2 \right) |f|^2 \right) dr \le C_1 \int_{B_0}^{B_1} |f|^2 \, dr + \mathcal{L}_\mu^{(\epsilon)}(f). \quad (3.5)$$

From this it is clear that

$$\nu_\mu^{(\epsilon)} \doteq \inf \left\{ \mathcal{L}_\mu^{(\epsilon)}(f) : f \in C_c^\infty \text{ and } \|f\|_{L^2} = 1 \right\} > -\infty.$$

Let $\left\{ f_{n,\mu}^{(\epsilon)} \right\}_{n=1}^\infty$ be a sequence of smooth functions, compactly supported in

[2]Remember that we have fixed ω, m, and l and that all constants in this section may depend on these.

$(r_+ + \epsilon, \infty)$, with $\left\|f^{(\epsilon)}_{n,\mu}\right\|_{L^2} = 1$, such that

$$\mathcal{L}^{(\epsilon)}_{\mu}\left(f^{(\epsilon)}_{n,\mu}\right) \to \nu^{(\epsilon)}_{\mu}.$$

The bound (3.5) implies that $\left\|f^{(\epsilon)}_{n,\mu}\right\|_{H^1}$ is uniformly bounded. We now apply Rellich compactness to produce a $f^{(\epsilon)}_{\mu} \in H^1_0$ such that a re-labeled subsequence of $\left\{f^{(\epsilon)}_{n,\mu}\right\}$ converges to $f^{(\epsilon)}_{\mu}$ weakly in H^1 and strongly in L^2 on compact subsets of (r_+, ∞).

We claim that no mass is lost in the limit, i.e. $\|f^{(\epsilon)}_{\mu}\|_{L^2} = 1$. Suppose not. Then, for any compact set K, there will exist infinitely many of the $f^{(\epsilon)}_{n,\mu}$'s such that

$$\left\|f^{(\epsilon)}_{n,\mu}\right\|_{L^2([r_+,\infty)\setminus K)} \geq \alpha > 0.$$

It is easy to see from (3.5) that this will give a contradiction if K is sufficiently large.

Using the boundedness of weak limits and the strong L^2 convergence, we then get

$$\nu^{(\epsilon)}_{\mu} \leq \mathcal{L}^{(\epsilon)}_{\mu}\left(f^{(\epsilon)}_{\mu}\right) \leq \liminf_{n\to\infty} \mathcal{L}^{(\epsilon)}_{\mu}\left(f^{(\epsilon)}_{n,\mu}\right) = \nu^{(\epsilon)}_{\mu}.$$

This implies that $\mathcal{L}^{(\epsilon)}_{\mu}$ achieves its minimum on $f^{(\epsilon)}_{\mu}$. □

Now we are ready to prove Proposition 3.4.1.

Proof. First we observe that $\left\{\nu^{(\epsilon)}_{\mu}\right\}_{\epsilon>0}$ is bounded and decreasing in ϵ. Set $\nu_{\mu} = \lim_{\epsilon\to 0} \nu^{(\epsilon)}_{\mu}$. Lemma 3.4.1 implies that $\mu \in \mathscr{A}$ which in turn implies that $\nu_{\mu} < 0$. For any interval $K = \left(r_+ + \frac{1}{n}, n\right)$ with n large, (3.5) implies that

$$\sup_{\epsilon>0} \left\|f^{(\epsilon)}_{\mu}\right\|_{H^1(K)} < \infty,$$

$$\inf_{\epsilon>0} \left\| f_\mu^{(\epsilon)} \right\|_{L^2(K)} > 0.$$

After an application of Rellich compactness and passing to a subsequence, we may find a non-zero $f_\mu \in H^1$ that is a weak H^1 and strong L^2_{loc} limit of $f_\mu^{(\epsilon)}$.

Using the Euler-Lagrange equations associated to $\mathcal{L}_\mu^{(\epsilon)}$, we find

$$\Delta \frac{d}{dr}\left(\Delta \frac{df_\mu}{dr}\right) - W_\mu f_\mu + \nu_\mu \Delta f_\mu = 0.$$

On any compact subset K of (r_+, ∞), boundedness of weak limits and the L^2_{loc} convergence of the $f_\mu^{(\epsilon)}$ imply that

$$\int_K \left(\Delta \left|\frac{df_\mu}{dr}\right|^2 + \frac{W_\mu}{\Delta} |f_\mu|^2\right) \leq \nu_\mu.$$

Hence,

$$\int_{r_+}^\infty \Delta \left|\frac{df_\mu}{dr}\right|^2 dr < \infty. \tag{3.6}$$

Near r_+ the local theory from Section 3.2 implies that

$$f_\mu = A\varphi_1 + B\left(\log(r - r_+)\varphi_2 + \varphi_3\right)$$

for some constants A and B and non-zero analytic functions φ_i. However, if $B \neq 0$, then

$$\int_{r_+}^\infty \Delta \left|\frac{df_\mu}{dr}\right|^2 dr = \infty.$$

Hence, $B = 0$. Near infinity the local theory from Section 3.2 implies that that f_μ is asymptotic to a linear combination of an exponentially growing solution and an exponentially decaying solution. The bound (3.6) clearly implies that f_μ is in fact exponentially decaying. Thus, f_μ satisfies the boundary conditions of a mode solution. □

Finally, we can prove Proposition 3.4.2.

Proof. First we will show that ν_μ is continuous for $\mu \in \mathscr{A}$. Let us normalize each $f_\mu^{(\epsilon)}$ so that $\left\|f_\mu^{(\epsilon)}\right\|_{L^2} = 1$. We have

$$\nu_{\mu_1}^{(\epsilon)} = \mathcal{L}_{\mu_1}^{(\epsilon)}\left(f_{\mu_1}^{(\epsilon)}\right) = \mathcal{L}_{\mu_2}^{(\epsilon)}\left(f_{\mu_1}^{(\epsilon)}\right) + \int_{r_+ + \epsilon}^{\infty} \frac{V_{\mu_1} - V_{\mu_2}}{\Delta}\left|f_{\mu_1}^{(\epsilon)}\right|^2 dr \geq$$

$$\nu_{\mu_2}^{(\epsilon)} - \left|\mu_1^2 - \mu_2^2\right| \int_{r_+ + \epsilon}^{\infty} r^2 \left|f_{\mu_1}^{(\epsilon)}\right|^2 dr.$$

Reversing the roles of μ_1 and μ_2 gives

$$\left|\nu_{\mu_1}^{(\epsilon)} - \nu_{\mu_2}^{(\epsilon)}\right| \leq \left|\mu_1^2 - \mu_2^2\right| \int_{r_+ + \epsilon}^{\infty} r^2 \left(\left|f_{\mu_1}^{(\epsilon)}\right|^2 + \left|f_{\mu_2}^{(\epsilon)}\right|^2\right) dr \leq$$

$$\left|\mu_1^2 - \mu_2^2\right| \left(C + \int_{r_+ + \epsilon}^{\infty} r^2 1_{[B_0, B_1]^c} \left(\left|f_{\mu_1}^{(\epsilon)}\right|^2 + \left|f_{\mu_2}^{(\epsilon)}\right|^2\right) dr\right) \leq$$

$$C' \left|\mu_1^2 - \mu_2^2\right|.$$

In these inequalities we have used (3.5), $\left\|f_\mu^{(\epsilon)}\right\|_{L^2} = 1$, and the fact that the $\nu_{\mu_i}^{(\epsilon)}$ are negative. Since the constant C' is independent of ϵ, we may take ϵ to 0.

By Lemma 3.4.1 $\mathscr{A} \neq \varnothing$. Hence, we may set

$$\mu_0 \doteq \sup \mathscr{A}.$$

It is clear that for any $\mu \in \mathscr{A}$, we cannot have W_μ strictly positive on (r_+, ∞). Thus $\mu_0 < \infty$. Since ν_μ is increasing in μ, we may extend ν_μ continuously so that ν_{μ_0} exists. We will of course have $\nu_{\mu_0} \leq 0$. Suppose that $\nu_{\mu_0} < 0$. Then, one may easily show that $\mu_0 \in \mathscr{A}$, and hence we can run the existence argument above to construct a corresponding $f_{\nu_{\mu_0}}$. Next, by continuity we could slightly increase μ_0 to $\mu_0' \in \mathscr{A}$, run the existence argument again, and conclude that

$\nu_{\mu_0'} < 0$. This of course contradicts the definition of μ_0. We conclude that $\nu_{\mu_0} = 0$.

It remains to show that there exists a corresponding f_{μ_0}. From the local theory in Section 3.2, for every μ and ν, we have a unique solution $\tilde{R}(r,\mu,\nu)$ to

$$\Delta \frac{d}{dr}\left(\Delta \frac{d\tilde{R}}{dr}\right) - W_\mu \tilde{R} + \nu \Delta \tilde{R} = 0$$

which satisfies $\tilde{R}(r_+,\mu,\nu) = 1$. At infinity there will be a local basis of solutions spanned by $\tilde{\rho}_1(r,\mu,\nu)$ and $\tilde{\rho}_2(r,\mu,\nu)$ where $\tilde{\rho}_1$ is exponentially increasing, $\tilde{\rho}_2$ is exponentially decreasing, and both depend analytically on r, μ, and ν. Lastly, we have analytic reflection and transmission coefficients $\tilde{A}(\mu,\nu)$ and $\tilde{B}(\mu,\nu)$ defined by

$$\tilde{R}(r,\mu,\nu) = \tilde{A}(\mu,\nu)\tilde{\rho}_1(r,\mu,\nu) + \tilde{B}(\mu,\nu)\tilde{\rho}_2(r,\mu,\nu).$$

As $\mu \uparrow \mu_0$ we have $\tilde{A}(\mu,\nu_\mu) = 0$. It follows that $\tilde{A}(\mu_0, 0) = 0$. We may then set

$$f_{\mu_0}(r) \doteq \tilde{R}(r,\mu_0,0).$$

□

3.4.3 Construction of the Exponentially Growing Modes

In this section our goal is to perturb the real modes into the complex upper half-plane with an appropriate application of the implicit function theorem. Using the previous section we may start with a solution R to the radial ODE satisfying the boundary conditions of a mode and with frequency parameters $(\omega_R(0), m, l, \mu(0))$ such that $\omega_R(0) \in \mathbb{R}$ and $\mu^2(0) > \omega_R^2(0)$. For any $\omega = \omega_R + i\omega_I$

and μ sufficiently close to $\omega_R(0)$ and $\mu(0)$ respectively, the local theory from Section 3.2 will give us two linearly independent solutions to the radial ODE $\hat{\rho}_1(r,\omega,\mu)$ and $\hat{\rho}_2(r,\omega,\mu)$ such that $\hat{\rho}_1$ is exponentially increasing at infinity, $\hat{\rho}_2$ is exponentially decreasing at infinity, and both depend holomorphically on ω and analytically on μ. Furthermore, the local theory around r_+ tells us that, up to normalizing properly, for each $\omega = \omega_R + i\omega_I$ and μ we have a unique local solution $R(r,\omega,\mu)$ around r_+ satisfying the correct boundary condition. We have

$$R(r,\omega,\mu) = A(\omega,\mu)\hat{\rho}_1(r,\omega,\mu) + B(\omega,\mu)\hat{\rho}_2(r,\omega,\mu). \tag{3.7}$$

As shown in Section 3.2, A and B are holomorphic in ω and μ. Finding a mode solution is equivalent to finding a zero of A. We have picked our parameters so that $A(\omega_R(0),\mu(0)) = 0$. Let's write $A = A_R + iA_I$. Next, we note that an application of the implicit function theorem will produce our unstable modes if we can establish

$$\det\begin{pmatrix} \frac{\partial A_R}{\partial \omega_R} & \frac{\partial A_R}{\partial \mu} \\ \frac{\partial A_I}{\partial \omega_R} & \frac{\partial A_I}{\partial \mu} \end{pmatrix}(\omega_R(0),\mu(0)) \neq 0.$$

In order to do this, we shall return to the energy current

$$Q_T = \operatorname{Im}\left(\Delta\frac{dR}{dr}\overline{R}\right).$$

Recall that in Section 3.3 we saw

$$\frac{dQ_T}{dr} = 0,$$

$$Q_T(r_+) = am - 2Mr_+\omega_R.$$

We have used the normalization $|R(r_+)|^2 = 1$ in the second statement. Next, let's write Q_T in terms of $\hat{\rho}_1$ and $\hat{\rho}_2$.

$$Q_T = |A|^2 \Delta \mathrm{Im}\left(\frac{d\hat{\rho}_1}{dr}\overline{\hat{\rho}_1}\right) + \Delta \mathrm{Im}\left(A\frac{d\hat{\rho}_1}{dr}\overline{B\hat{\rho}_2}\right) +$$

$$\Delta \mathrm{Im}\left(B\frac{d\hat{\rho}_2}{dr}\overline{A\hat{\rho}_1}\right) + |B|^2 \Delta \mathrm{Im}\left(\frac{d\hat{\rho}_2}{dr}\overline{\hat{\rho}_2}\right).$$

Before examining this at infinity, let us note the precise asymptotics of the $\hat{\rho}_i$ as recalled in Section 3.2.

$$\hat{\rho}_1 \sim e^{\sqrt{\mu^2-\omega_R^2}r} r^{-1+\frac{M(2\omega_R^2-\mu^2)}{\sqrt{\mu^2-\omega_R^2}}},$$

$$\hat{\rho}_2 \sim e^{-\sqrt{\mu^2-\omega_R^2}r} r^{-1-\frac{M(2\omega_R^2-\mu^2)}{\sqrt{\mu^2-\omega_R^2}}}.$$

Furthermore, it's easy to see from the construction of the $\hat{\rho}_i$ that they are both real valued. Now let's compute $Q_T(\infty)$. Since the $\hat{\rho}_i$ are real, the first and last terms clearly vanish. The exponential powers cancel in the middle terms, and we find

$$Q_T(\infty) = \sqrt{\mu^2 - \omega_R^2}\mathrm{Im}\left(A\overline{B}\right) - \sqrt{\mu^2 - \omega_R^2}\mathrm{Im}\left(B\overline{A}\right) = 2\sqrt{\mu^2 - \omega_R^2}\mathrm{Im}\left(A\overline{B}\right).$$

We conclude that

$$am - 2Mr_+\omega_R = 2\sqrt{\mu^2 - \omega_R^2}\mathrm{Im}\left(A\overline{B}\right). \tag{3.8}$$

Since $A(\omega_R(0), \mu(0)) = 0$, taking derivatives of (3.8) implies that

$$-2Mr_+ = 2\sqrt{\mu^2(0) - \omega_R^2(0)}\mathrm{Im}\left(\frac{\partial A}{\partial \omega_R}\left(\omega_R(0), \mu(0)\right)\overline{B}\left(\omega_R(0), \mu(0)\right)\right),$$

$$0 = 2\sqrt{\mu^2(0) - \omega_R^2(0)}\mathrm{Im}\left(\frac{\partial A}{\partial \mu}(\omega_R(0), \mu(0))\overline{B}(\omega_R(0), \mu(0))\right).$$

Since $B(\omega_R(0), \mu(0)) \neq 0$, these two equations imply that the vectors $\left(\frac{\partial A_R}{\partial \omega_R}, \frac{\partial A_I}{\partial \omega_R}\right)$ and $\left(\frac{\partial A_R}{\partial \mu}, \frac{\partial A_I}{\partial \mu}\right)$ are linearly independent at $(\omega_R(0), \mu(0))$ if and only if

$$\frac{\partial A}{\partial \mu}(\omega_R(0), \mu(0)) \neq 0,$$

i.e.

$$\det\begin{pmatrix} \frac{\partial A_R}{\partial \omega_R} & \frac{\partial A_R}{\partial \mu} \\ \frac{\partial A_I}{\partial \omega_R} & \frac{\partial A_I}{\partial \mu} \end{pmatrix}(\omega_R(0), \mu(0)) \neq 0 \Leftrightarrow \frac{\partial A}{\partial \mu}(\omega_R(0), \mu(0)) \neq 0.$$

It remains to establish

Lemma 3.4.3.

$$\frac{\partial A}{\partial \mu} \neq 0.$$

Proof. For the sake of contradiction, suppose that $\frac{\partial A}{\partial \mu}(\omega_R(0), \mu(0)) = 0$. Differentiating (3.7) gives

$$\frac{\partial R}{\partial \mu}(r, \omega_R(0), \mu(0)) = \frac{\partial B}{\partial \mu}(\omega_R(0), \mu(0))\hat{\rho}_2(r, \omega_R(0), \mu(0)) +$$

$$B(\omega_R(0), \mu(0))\frac{\partial \hat{\rho}_2}{\partial \mu}(r, \omega_R(0), \mu(0)).$$

This implies that $\frac{\partial R}{\partial \mu}$ is exponentially decreasing at infinity. The analysis from appendices 3.1 and 3.2 implies that $\frac{\partial R}{\partial \mu}$ is smooth at r_+.[3] Differentiating the radial ODE with respect to μ, multiplying by $\overline{R}$, and integrating gives

$$\int_{r_+}^{\infty}\left(\frac{\partial}{\partial r}\left(\Delta\frac{\partial^2 R}{\partial r \partial \mu}\right) - \frac{W_\mu}{\Delta}\frac{\partial R}{\partial \mu}\right)\overline{R}dr = \int_{r_+}^{\infty}\left(2\mu r^2 + \frac{\partial \lambda_{\kappa m l}}{\partial \mu}\right)|R|^2 dr. \quad (3.9)$$

Integrating by parts twice on the left hand side will produce no boundary terms

[3]Recall that, as discussed in Section 3.2, R is smooth at r_+ when $am - 2Mr_+ = 0$.

since both R and $\frac{\partial R}{\partial \mu}$ are exponentially decreasing at infinity and $\Delta(r_+) = 0$. Thus we have

$$\int_{r_+}^{\infty} \left(\frac{\partial}{\partial r} \left(\Delta \frac{\partial^2 R}{\partial r \partial \mu} \right) - \frac{W_\mu}{\Delta} \frac{\partial R}{\partial \mu} \right) \overline{R} dr =$$

$$\int_{r_+}^{\infty} \frac{\partial R}{\partial \mu} \left(\frac{\partial}{\partial r} \left(\Delta \frac{\partial \overline{R}}{\partial r} \right) - \frac{W_\mu}{\Delta} \overline{R} \right) dr = 0.$$

We have used that $\overline{R}$ is a solution of the radial ODE in the last equality. Plugging this into (3.9) then gives

$$\int_{r_+}^{\infty} \left(2\mu r^2 + \frac{\partial \lambda_{\kappa m l}}{\partial \mu} \right) |R|^2 \, dr = 0.$$

Since Lemma 3.3.1 gives us

$$\frac{\partial \lambda_{\kappa m l}}{\partial \mu} > 0,$$

we conclude that R vanishes, a contradiction. □

3.4.4 Modes Crossing the Real Axis

In this section we shall investigate how a mode can "cross" the real axis. Let's introduce a little more notation. From the analysis of the previous section we have a family of solutions $R(r, \epsilon)$ to the radial ODE satisfying the boundary conditions of a mode with parameters $(\omega(\epsilon), m, l, \mu(\epsilon))$ where $\omega(\epsilon) = \omega_R(\epsilon) + i\epsilon$. Implicitly we have also been using the existence of a family $\lambda_{\kappa m l}$ of eigenvalues to the angular ODE (see Lemma 1.3.1). These functions are all defined for $|\epsilon| \ll 1$. In what follows we will often omit the ϵ's and we shall assume $0 < \omega_R(\epsilon) < \mu(\epsilon)$. Using the symmetry of the equations under $(\omega, m) \mapsto (-\omega, -m)$ one may check that this assumption implies no loss of generality. The function R will satisfy

$$\frac{\partial}{\partial r} \left(\Delta \frac{\partial R}{\partial r} \right) - \frac{W_\mu}{\Delta} R = 0,$$

$$R \sim (r - r_+)^{\frac{i(am - 2Mr_+\omega)}{r_+ - r_-}} \text{ at } r_+,$$

$$R \sim e^{-r\sqrt{\mu^2-\omega^2}} r^{-1-\frac{M(2\omega^2-\mu^2)}{\sqrt{\mu^2-\omega^2}}} \text{ at } r = \infty,$$

$$W_\mu \doteq -(r^2 + a^2)^2\omega^2 + 4Mamr\omega - a^2m^2 + \Delta\left(\lambda + a^2\omega^2 + \mu^2r^2\right).$$

We also have

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) - \left(\frac{m^2}{\sin^2\theta} - a^2\left(\omega^2 - \mu^2\right)\cos^2\theta\right)S + \lambda S = 0$$

where $S(\cdot,\epsilon) : \theta \in (0,\pi) \to \mathbb{C}$ is given boundary conditions so that $e^{-im\phi}S_{ml}$ extends smoothly to $\mathbb{S}^2$. Note that we have suppressed the κ, m, and l indices from $S_{\kappa ml}$ and $\lambda_{\kappa ml}$.

From Theorem 1.6.2 we know that

$$\omega_R(0) = \frac{am}{2Mr_+}. \tag{3.10}$$

We wish to investigate the signs of $\frac{\partial\omega_R}{\partial\epsilon}(0)$ and $\frac{\partial\mu}{\partial\epsilon}(0)$. The condition (3.10) corresponds to our mode solution being exactly on the threshold of superradiance. This makes sense because when $\epsilon = 0$ the solution neither grows nor decays with time. For $\epsilon > 0$ the mode solution will grow with time. Hence, we expect the mode to become superradiant (1.28).

This leads us to

Proposition 3.4.3. *If $\epsilon > 0$ we must have*

$$\omega_R^2(\epsilon) + \epsilon^2 < \left(\frac{am}{2Mr_+}\right)^2.$$

In particular

$$\frac{\partial\omega_R}{\partial\epsilon}(0) \le 0.$$

Proof. We now introduce a variant of the microlocal energy current $\hat{Q}_T$:

$$\check{Q}_T \doteq \mathrm{Im}\left(\Delta \frac{\partial R}{\partial r}\overline{\omega R}\right).$$

For $\epsilon > 0$ we have

$$\check{Q}_T(\infty) = \check{Q}_T(r_+) = 0$$

$$\frac{\partial \check{Q}_T}{\partial r} = -\epsilon\Delta \left|\frac{\partial R}{\partial r}\right|^2 + \mathrm{Im}\left(\frac{W_\mu \overline{\omega}}{\Delta}\right)|R|^2 \Rightarrow$$

$$\int_{r_+}^{\infty}\left(\epsilon\Delta\left|\frac{\partial R}{\partial r}\right|^2 - \mathrm{Im}\left(\frac{W_\mu\overline{\omega}}{\Delta}\right)|R|^2\right)dr = 0.$$

We have

$$\begin{aligned}\mathrm{Im}\left(-W_\mu\overline{\omega}\right) &= \epsilon\left((r^2+a^2)^2|\omega|^2 - a^2m^2 + \Delta r^2\mu^2\right) - \Delta\mathrm{Im}\left(\left(\lambda + a^2\omega^2\right)\overline{\omega}\right)\\ &= \epsilon\Big(|\omega|^2\left[\Delta^2 + \left(4Mr - a^2\right)\Delta + 4M^2r^2\right]\\ &\qquad - a^2m^2 + \Delta r^2\mu^2\Big) - \Delta\mathrm{Im}\left(\lambda\overline{\omega}\right).\end{aligned}$$

Appealing to the statement (2.49) (that statement was proved under the assumption $\mu = 0$, but the same proof works if $\mu \neq 0$) now gives

$$-\mathrm{Im}\left(\lambda\overline{\omega}\right) > 0.$$

Furthermore, $\mathrm{Im}\left(-W_\mu\overline{\omega}\right)$ is increasing in r. Thus, $R \neq 0$ implies that

$$\mathrm{Im}\left(-W_\mu\overline{\omega}\right)(r_+) < 0 \Leftrightarrow$$

$$\omega_R^2(\epsilon) + \epsilon^2 < \left(\frac{am}{2Mr_+}\right)^2.$$

□

Proposition 3.4.4.

$$\frac{\partial \mu}{\partial \epsilon}(0) \geq 0 \Rightarrow Re\left(\frac{\partial \lambda}{\partial \epsilon}\right)(0) \geq 0.$$

Proof. Let's set

$$S_\epsilon \doteq \frac{\partial S}{\partial \epsilon}.$$

At $\epsilon = 0$ we have

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S_\epsilon}{\partial\theta}\right) - \left(\frac{m^2}{\sin^2\theta} - a^2\left(\omega^2 - \mu^2\right)\cos^2\theta\right) S_\epsilon + \lambda S_\epsilon =$$

$$-\left(2a^2\cos^2\theta\left(\omega_R\left(\frac{\partial \omega_R}{\partial \epsilon} + i\right) - \mu\frac{\partial \mu}{\partial \epsilon}\right) + \frac{\partial \lambda}{\partial \epsilon}\right) S.$$

Using Section 3.1 one may check that S_ϵ is regular at $\theta = 0, \pi$. Multiplying by $\overline{S}$ and integrating by parts implies

$$\int_0^\pi \left(2a^2\cos^2\theta\left(\omega_R\left(\frac{\partial \omega_R}{\partial \epsilon} + i\right) - \mu\frac{\partial \mu}{\partial \epsilon}\right) + \frac{\partial \lambda}{\partial \epsilon}\right)|S|^2 \sin\theta d\theta = 0.$$

Using Proposition 3.4.3 we conclude that

$$\frac{\partial \mu}{\partial \epsilon}(0) \geq 0 \Rightarrow \mathrm{Re}\left(\frac{\partial \lambda}{\partial \epsilon}\right)(0) \geq 0.$$

□

Finally, we examine $\frac{\partial \mu}{\partial \epsilon}(0)$.

Proposition 3.4.5.

$$\frac{\partial \mu}{\partial \epsilon}(0) < 0.$$

Proof. Let's set

$$R_\epsilon \doteq \frac{\partial R}{\partial \epsilon}.$$

We have

$$\frac{\partial}{\partial r}\left(\Delta\frac{\partial R_\epsilon}{\partial r}\right)-\frac{W_\mu}{\Delta}R_\epsilon=\frac{\partial}{\partial\epsilon}\left(\frac{W_\mu}{\Delta}\right)R. \tag{3.11}$$

Now we want to multiply by $\overline{R}$ and integrate by parts. However, we have to be careful with regards to $R_\epsilon' s$ boundary conditions. At infinity R_ϵ may easily be seen to be exponentially decreasing, but at r_+ the proper condition is more subtle. By construction

$$(r-r_+)^{\frac{-i(am-2Mr_+\omega(\epsilon))}{r_+-r_-}}R(r,\epsilon)\doteq G(r,\epsilon)$$

is analytic in r and ϵ near $(r_+,0)$. At $\epsilon=0$ we have

$$(r-r_+)^{\frac{-i(am-2Mr_+\omega)}{r_+-r_-}}R_\epsilon-$$

$$\frac{2Mr_+}{r_+-r_-}\left(1-i\frac{\partial\omega_R}{\partial\epsilon}\right)(r-r_+)^{\frac{-i(am-2Mr_+\omega)}{r_+-r_-}}\log(r-r_+)R=\frac{\partial G}{\partial\epsilon}\Rightarrow$$

$$R_\epsilon(r,0)=\frac{2Mr_+}{r_+-r_-}\left(1-i\frac{\partial\omega_R}{\partial\epsilon}\right)\log(r-r_+)R(r,0)+\frac{\partial G}{\partial\epsilon}(r,0).$$

Now we multiply (3.11) by $\overline{R}$, take the real part, and integrate by parts. We end up with

$$-2Mr_+|R(r_+)|^2=\int_{r_+}^{\infty}\mathrm{Re}\left(\frac{\partial}{\partial\epsilon}\left(\frac{W_\mu}{\Delta}\right)\right)|R|^2\,dr=$$

$$\int_{r_+}^{\infty}\Delta^{-1}\left(2\omega_R\frac{\partial\omega_R}{\partial\epsilon}\left(-\Delta^2+(a^2-4Mr)\Delta-4M^2r(r-r_+)\right)\right)|R|^2\,dr+$$

$$\int_{r_+}^{\infty}\left(\mathrm{Re}\left(\frac{\partial\lambda}{\partial\epsilon}\right)+2r^2\mu\frac{\partial\mu}{\partial\epsilon}\right)|R|^2\,dr.$$

Now Proposition 3.4.4 finishes the proof. □

3.4.5 Following the Unstable Modes in the Upper Half Plane

Following our construction of unstable modes near the real axis, it is natural to ask if one can continue to decrease μ and produce more unstable modes. We will not explore this in detail in this paper, but we will briefly describe the expected behavior. One believes that one can vary μ and produce a 1-parameter (at least continuous) family of modes with frequency parameters $(\omega(\mu), \mu, m, l)$.[4] As long as these modes are in the upper half plane, Proposition 3.4.3 shows that they will remain superradiant. Hence, if and when they cross the real axis, they will satisfy $|\omega| \leq \frac{|am|}{2Mr_+}$. Now, note that Proposition 3.4.5 implies that in the bound state regime ($\mu^2 > \omega^2$), an unstable mode can cross the real axis only by increasing the mass. Hence, as long as we decrease μ and maintain $\mu > \frac{|am|}{2Mr_+}$, the curve of modes cannot cross the real axis, and, by continuity, we conclude that these modes would have to remain unstable.

[4]A potential approach is to more directly exploit the underlying analyticity, see [29] for ideas along these lines.

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