

## MIT Open Access Articles

*A new error bound for reduced basis approximation  
of parabolic partial differential equations*

The MIT Faculty has made this article openly available. **Please share**  
how this access benefits you. Your story matters.

**Citation:** Urban, Karsten, and Anthony T. Patera. "A New Error Bound for Reduced Basis Approximation of Parabolic Partial Differential Equations." *Comptes Rendus Mathematique* 350, no. 3–4 (February 2012): 203–207.

**As Published:** <http://dx.doi.org/10.1016/j.crma.2012.01.026>

**Publisher:** Elsevier

**Persistent URL:** <http://hdl.handle.net/1721.1/99384>

**Version:** Author's final manuscript: final author's manuscript post peer review, without publisher's formatting or copy editing

**Terms of use:** Creative Commons Attribution-Noncommercial-NoDerivatives



# A New Error Bound for Reduced Basis Approximation of Parabolic Partial Differential Equations

Karsten Urban

*University of Ulm, Institute for Numerical Mathematics, Helmholtzstr. 18,  
89081 Ulm (Germany), karsten.urban@uni-ulm.de*

Anthony T. Patera

*Mechanical Engineering Department, Massachusetts Institute of Technology,  
77 Massachusetts Ave., Cambridge, MA 02139-4307 (USA), patera@mit.edu*

---

## Abstract

We consider a space-time variational formulation for linear parabolic partial differential equations. We introduce an associated Petrov-Galerkin truth finite element discretization with favorable discrete inf-sup constant  $\beta_\delta$ :  $\beta_\delta$  is unity for the heat equation;  $\beta_\delta$  grows only linearly in time for non-coercive (but asymptotically stable) convection operators. The latter in turn permits effective long-time *a posteriori* error bounds for reduced basis approximations, in sharp contrast to classical (pessimistic) exponentially growing energy estimates.

## Résumé

Nous considérons une formulation variationnelle espace-temps pour les équations différentielles paraboliques linéaires. Nous y associons une discrétisation par éléments finis de Petrov-Galerkin pour laquelle la constante de stabilité inf-sup  $\beta_\delta$  possède des propriétés agréables :  $\beta_\delta$  est unité pour l'équation de la chaleur;  $\beta_\delta$  a une croissance seulement linéaire en temps pour des opérateurs de convection non-coercifs (mais asymptotiquement stables). Dans le cadre des approximations par bases réduites, cette dernière propriété permet d'obtenir des bornes efficaces pour l'erreur *a posteriori* en temps long, en net contraste avec les estimateurs d'erreur en énergie classiques (pessimistes) qui présentent une croissance exponentielle.

*Key words:* parabolic equations, space-time formulation, inf-sup stability, reduced basis, a posteriori estimation  
*1991 MSC:* 35K15, 65M15, 65M60

---

## Version française abrégée

Soit l'équation aux dérivées partielles parabolique (1) et sa formulation variationnelle espace-temps :  $u \in \mathcal{X} := L_2(I \equiv (0, T]; V) \cap H_{(0)}^1(I; V')$ ,  $H_{(0)}^1(I; V') := \{v \in H^1(I; V') : v(0) = 0\}$ , vérifie (2) pour l'espace des fonctions test  $\mathcal{Y} := L_2(I; V)$ ; ici  $b(w, v) := \int_I [\langle \dot{w}(t), v(t) \rangle_{V' \times V} + a(w(t), v(t))] dt$  et  $f(v) :=$

$\int_I \langle g(t), v(t) \rangle_{V' \times V} dt$ . Les normes sur  $\mathcal{X}$  et  $\mathcal{Y}$  sont définies par  $\|w\|_{\mathcal{X}}^2 := \|w\|_{L_2(I;V)}^2 + \|\dot{w}\|_{L_2(I;V')}^2 + \|w(T)\|_H^2$  et  $\|v\|_{\mathcal{Y}} := \|v\|_{L_2(I;V)}$ . On considère  $V = H_0^1(\Omega)$  (opérateurs spatiaux de 2e ordre), où  $\Omega$  est le domaine spatial. On montre (Proposition 1 basée sur [6]) que pour des problèmes coercifs, la constante inf-sup  $\beta$  de  $b$  est positive et bornée inférieurement.

Nous introduisons ensuite une discrétisation par éléments finis de Petrov-Galerkin : soit  $u_\delta \in \mathcal{X}_\delta$  vérifiant  $b(u_\delta, v_\delta) = f(v_\delta), \forall v_\delta \in \mathcal{Y}_\delta$ ; avec  $\mathcal{X}_\delta := S_{\Delta t} \otimes V_h$  et  $\mathcal{Y}_\delta = Q_{\Delta t} \otimes V_h$ ,  $\delta = (\Delta t, h)$ , où  $S_{\Delta t}, V_h$  et  $Q_{\Delta t}$  sont respectivement des éléments finis linéaire par morceaux en temps (de pas  $\Delta t$ ) et en espace (de diamètre  $h$ ), et constants en temps par morceaux. Cette méthode coïncide avec une discrétisation de Crank-Nicolson en temps, ainsi nos résultats s'appliquent directement à ce schéma standard. Nous définissons une norme modifiée sur  $\mathcal{X}$ ,  $\|\cdot\|_{\mathcal{X},\delta}$ , où la partie  $\int_I \|w\|_V^2 dt$  de  $\|\cdot\|_{\mathcal{X}}$  est remplacée par une somme des moyennes de  $w$  sur chaque élément temporel. Nous montrons (Proposition 3) que pour l'équation de la chaleur, les constantes inf-sup discrète  $\beta_\delta$  et continuité discrète  $\gamma_\delta$  sont unités.

Nous considérons maintenant une approximation par bases réduites  $u_N$ , comme dans [2]. Nous pouvons construire à la Proposition 5 des bornes sur l'erreur *a posteriori* pour le champ solution  $u_N$  et pour la sortie scalaire  $s_N$ . L'utilité de ces bornes est très liée à la dépendance de  $\beta_\delta$  aux paramètres du problème et au temps final  $T$ . La procédure de calcul de ces bornes est similaire à la décomposition hors ligne-en ligne des bases réduites standard.

Enfin, nous présentons des résultats numériques pour la constante inf-sup discrète  $\beta_\delta$ . Pour des opérateurs *non-coercifs* mais asymptotiquement stables (e.g. convection), on observe que le paramètre inf-sup décroît seulement en  $(\mu_1 T)^{-1}$ , où  $\mu_1$  est la vitesse de déplacement et  $T$  le temps final. L'exemple est un domaine spatial unidimensionnel  $(0, 1)$  avec l'opérateur  $\mu_1(x - 1/2)u_x - u_{xx}$  pour  $\mu_1 > 2\pi^2$  ( $\mu_1 < 2\pi^2$  entraîne un problème coercif). Les estimateurs associés sont beaucoup plus précis que ceux classiques en énergie qui prédisent une croissance exponentielle en  $e^{\mu_1 T}$  et qui sont donc inutilisables en pratique.

## 1. Space-time formulation

We first formulate a general linear parabolic equation. Consider Hilbert spaces  $V \hookrightarrow H \hookrightarrow V'$  and an operator  $A \in \mathcal{L}(V, V')$ ,  $\langle A\phi, \psi \rangle_{V' \times V} = a(\phi, \psi)$  for  $\phi, \psi \in V$ . Setting  $I := (0, T]$ ,  $T > 0$  and given  $g \in L_2(I; V')$ , we look for  $u$  such that

$$\dot{u}(t) + A u(t) = g(t) \text{ in } V', \quad t \in I; \quad u(0) = 0, \quad (1)$$

and an associated output of the form  $s := \int_I \ell(u(t)) dt$  for some  $\ell \in V'$ . We restrict ourselves to LTI systems even though some of our results can be extended to a more general situation.

For the derivation of the space-time variational form of (1), we introduce the trial space  $\mathcal{X} := L_2(I; V) \cap H_{(0)}^1(I; V')$ , where  $H_{(0)}^1(I; V') := \{v \in H^1(I; V') : v(0) = 0\}$  with norm  $\|w\|_{\mathcal{X}}^2 := \|w\|_{L_2(I;V)}^2 + \|\dot{w}\|_{L_2(I;V')}^2 + \|w(T)\|_H^2$  and the test space  $\mathcal{Y} := L_2(I; V)$  with norm  $\|v\|_{\mathcal{Y}} := \|v\|_{L_2(I;V)}$ . Note that  $\mathcal{X} = (L_2(I) \otimes V) \cap (H_{(0)}^1(I) \otimes V')$  and  $\mathcal{Y} = L_2(I) \otimes V$  which will allow for a tensor product discretization. Then definitions  $b(w, v) := \int_I [\langle \dot{w}(t), v(t) \rangle_{V' \times V} + a(w(t), v(t))] dt$  and  $f(v) := \int_I \langle g(t), v(t) \rangle_{V' \times V} dt$  yield the space-time variational formulation

$$u \in \mathcal{X} : \quad b(u, v) = f(v), \quad \forall v \in \mathcal{Y}. \quad (2)$$

The well-posedness of (2) has been shown (under suitable assumptions) in [6, Theorem 5.1].

The approach of [6] can also yield an estimate for the inf-sup constant  $\beta := \inf_{w \in \mathcal{X}} \sup_{v \in \mathcal{Y}} \frac{b(w, v)}{\|w\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}}$ . We define  $\varrho := \sup_{0 \neq \phi \in V} \frac{\|\phi\|_H}{\|\phi\|_V}$  and  $\beta_a^* := \inf_{\phi \in V} \sup_{\psi \in V} \frac{a(\psi, \phi)}{\|\phi\|_V \|\psi\|_V}$ ; we then claim

**Proposition 1** *Assume that there exist  $M_a < \infty$ ,  $\alpha > 0$ , and  $\lambda \geq 0$  such that  $|a(\psi, \phi)| \leq M_a \|\psi\|_V \|\phi\|_V$  (continuity) and  $a(\phi, \phi) + \lambda \|\phi\|_H^2 \geq \alpha \|\phi\|_V^2$  (Gårding) for  $v, w \in V$ . Then, we obtain the inf-sup lower bound  $\beta \geq \beta^{LB} := \frac{\min\{1, (\alpha - \lambda \varrho^2)\} \min\{1, M_a^{-2}\}}{\max\{1, (\beta_a^*)^{-1}\} \sqrt{2}}$ .*

*Proof.* (Sketch) Let  $0 \neq w \in \mathcal{X}$ . Set  $z_w := (A^*)^{-1} \dot{w}$  (where  $A^*: V \rightarrow V'$  denotes the adjoint of  $V$ ) and set  $v_w := z_w + w \in \mathcal{Y}$ . Then,  $\|v_w\|_{L_2(I; V)}^2 \leq 2 \max\{1, (\beta_a^*)^{-2}\} \|w\|_{\mathcal{X}}^2$ . Using the estimates  $\|\dot{w}(t)\|_{V'} \leq M_a \|z_w(t)\|_V$ ,  $\langle \dot{w}(t), z_w(t) \rangle_{V' \times V} \geq (\alpha - \lambda \varrho^2) M_a^{-2} \|\dot{w}(t)\|_{V'}^2$ , and  $a(w(t), z_w(t)) = \frac{1}{2} \frac{d}{dt} \|w(t)\|_H^2$  we arrive at  $b(w, v_w) \geq (\alpha - \lambda \varrho^2) (M_a^{-2} \|\dot{w}\|_{L_2(I; V')}^2 + \|w\|_{L_2(I; V)}^2) + \|w(T)\|_H^2 \geq \min\{(\alpha - \lambda \varrho^2) \min\{1, M_a^{-2}\}, 1\} \|w\|_{\mathcal{X}}^2 \geq \beta^{LB} \|w\|_{\mathcal{X}} \|v_w\|_{\mathcal{Y}}$ , which proves the claim.  $\square$

**Remark 2** *Note that  $\beta^{LB}$  does not depend on the final time. However, the estimate is only meaningful if  $\alpha \geq \lambda \varrho^2$ , i.e., if the system is coercive. In the non-coercive case, (1) is often transformed via  $\hat{u}(t) := e^{-\lambda t} u(t)$  to obtain a coercive problem; however, this yields an inf-sup bound that behaves as  $e^{-\lambda T}$  — often extremely pessimistic and clearly unsuitable for error estimation in long-time integration.*

## 2. Petrov-Galerkin Truth Approximation

Let  $\mathcal{X}_\delta \subset \mathcal{X}$ ,  $\mathcal{Y}_\delta \subset \mathcal{Y}$  be finite dimensional subspaces and  $u_\delta \in \mathcal{X}_\delta$  the discrete approximation of (2), i.e.,  $b(u_\delta, v_\delta) = f(v_\delta)$ ,  $\forall v_\delta \in \mathcal{Y}_\delta$ ,  $s_\delta = \int_0^T \ell(u_\delta(t)) dt$ . Henceforth, we concentrate on the case  $H = L_2(\Omega)$ ,  $V = H_0^1(\Omega)$ . Let  $\mathcal{X}_\delta = S_{\Delta t} \otimes V_h$ ,  $\mathcal{Y}_\delta = Q_{\Delta t} \otimes V_h$ ,  $\delta = (\Delta t, h)$ , where  $S_{\Delta t}$ ,  $V_h$  are piecewise linear and  $Q_{\Delta t}$  piecewise constant finite elements with respect to triangulations  $\mathcal{T}_h^{\text{space}}$  in space and  $\mathcal{T}_{\Delta t}^{\text{time}} \equiv \{t^{i-1} \equiv (i-1)\Delta t < t \leq i\Delta t \equiv t^i, 1 \leq i \leq r\}$  in time for  $\Delta t := T/r$ . This coincides with the Crank–Nicolson (CN) scheme if a trapezoidal approximation of the right-hand side temporal integration is used; hence, we can derive error bounds for the CN scheme via our space-time formulation.

We introduce a different norm on  $\mathcal{X}$ : For  $w \in \mathcal{X}$  and  $I^i := (t^{i-1}, t^i)$ , set  $\bar{w}^i := (\Delta t)^{-1} \int_{I^i} w(t) dt \in V$  and  $\bar{w} := \sum_{i=1}^r \chi_{I^i} \otimes \bar{w}^i \in L_2(I; V)$ , where  $\chi_{I^i}$  is the characteristic function on  $I^i$ ; then, set  $\|w\|_{\mathcal{X}, \delta}^2 := \|\dot{w}\|_{L_2(I; V')}^2 + \|\bar{w}\|_{L_2(I; V)}^2 + \|w(T)\|_H^2$  and the inf-sup parameter  $\beta_\delta := \inf_{w_\delta \in \mathcal{X}_\delta} \sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta, v_\delta)}{\|w_\delta\|_{\mathcal{X}, \delta} \|v_\delta\|_{\mathcal{Y}}}$  as well as the stability parameter  $\gamma_\delta := \sup_{w_\delta \in \mathcal{X}_\delta} \sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta, v_\delta)}{\|w_\delta\|_{\mathcal{X}, \delta} \|v_\delta\|_{\mathcal{Y}}}$ .

**Proposition 3** *Let  $a(\cdot, \cdot)$  be symmetric, bounded and coercive and set  $\|\phi\|_V^2 := a(\phi, \phi)$ ,  $\phi \in V$ ; then we have  $\beta_\delta = \gamma_\delta = 1$ .*

*Proof.* (Sketch) Since  $v_\delta \in \mathcal{Y}_\delta$  is piecewise constant in time, we have  $\int_I a(w_\delta, v_\delta) dt = \int_I a(\bar{w}_\delta, v_\delta) dt$  for all  $w_\delta \in \mathcal{X}_\delta$ . Hence,  $b(w_\delta, v_\delta) = \int_I a(A_h^{-1} \dot{w}_\delta + \bar{w}_\delta, v_\delta) dt$ , where  $z_\delta := A_h^{-1} \dot{w}_\delta$  is defined by  $a(z_\delta, \phi_h) = \langle \dot{w}_\delta, \phi_h \rangle_{V' \times V}$ ,  $\forall \phi_h \in V_h$ . Note that  $\|A_h^{-1} \dot{w}_\delta\|_V = \|\dot{w}_\delta\|_{V'}$ . We may prove that for any  $v_\delta \in \mathcal{Y}_\delta$  there exists a unique  $z_\delta \in \mathcal{X}_\delta$  such that  $\int_I a(A_h^{-1} \dot{z}_\delta + \bar{z}_\delta, q_\delta) dt = \int_I a(v_\delta, q_\delta) dt$  for all  $q_\delta \in \mathcal{Y}_\delta$ . Note that  $v_\delta := A_h^{-1} \dot{z}_\delta + \bar{z}_\delta \in \mathcal{Y}_\delta$  for  $z_\delta \in \mathcal{X}_\delta$  and we obtain  $b(w_\delta, v_\delta) = \int_I a(A_h^{-1} \dot{w}_\delta + \bar{w}_\delta, A_h^{-1} \dot{z}_\delta + \bar{z}_\delta) dt$  so that  $\sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta, v_\delta)}{\|v_\delta\|_{\mathcal{Y}}} = \sup_{v_\delta \in \mathcal{Y}_\delta} (\int_I a(A_h^{-1} \dot{w}_\delta + \bar{w}_\delta, A_h^{-1} \dot{z}_\delta + \bar{z}_\delta) dt) / \|A_h^{-1} \dot{z}_\delta + \bar{z}_\delta\|_{\mathcal{Y}} = \|A_h^{-1} \dot{w}_\delta + \bar{w}_\delta\|_{\mathcal{Y}}$ . The fact that  $\|A_h^{-1} \dot{w}_\delta + \bar{w}_\delta\|_{\mathcal{Y}}^2 = \|A_h^{-1} \dot{w}_\delta\|_{L_2(I; V)}^2 + \|\bar{w}_\delta\|_{L_2(I; V)}^2 + 2 \int_I \langle \dot{w}_\delta, \bar{w}_\delta \rangle_{V' \times V} dt = \|\dot{w}_\delta\|_{L_2(I; V')}^2 + \|\bar{w}_\delta\|_{L_2(I; V)}^2 + \|w_\delta(T)\|_H^2 = \|w_\delta\|_{\mathcal{X}, \delta}^2$  proves the assertion.  $\square$

**Remark 4** *Proposition 3 also shows the well-posedness of the discrete problem with continuity and inf-sup constant being unity.*

### 3. The Reduced Basis Method (RBM)

Now, let  $\mu = (\mu_1, \mu_2) \in \mathcal{D} := \mathbb{R}^2$  be a parameter vector and  $A = A(\mu) := -\Delta u + \mu_1 \boldsymbol{\beta}(x) \cdot \nabla u + \mu_2 u$ , i.e., a diffusion-convection-reaction operator with convection field  $\boldsymbol{\beta}$ . We then introduce a standard Reduced Basis (RB) approximation [1,4,5] for the Crank–Nicolson interpretation of our discrete problem,  $u_N(\mu) \in \mathcal{X}_{\Delta t, N} = S_{\Delta t} \otimes V_N$ ; here  $V_N \subset V_h$  is an RB space provided for example by the POD-Greedy procedure of [2]. The RB output is then given by  $s_N = \int_I \ell(u_N(t)) dt (= \int_I \ell(\bar{u}_N(t)) dt)$ . (It is possible, alternatively, to consider a space–time RB approximation as well [7].) It is then simple [5] to demonstrate

**Proposition 5** *The RB error satisfies  $\| \|u_\delta - u_N\| \|_{\mathcal{X}, \delta} \leq \|r_N\|_{\mathcal{Y}'} / \beta_\delta^{LB}$ , where  $r_N(v) := f(v) - b(u_N, v)$ ,  $\forall v \in \mathcal{Y}$ , and  $\beta_\delta^{LB}$  is a lower bound for the  $\beta_\delta$  defined in Proposition 3. Furthermore,  $|s_\delta - s_N| \leq \|\ell\|_{V'} \sqrt{T} \|r_N\|_{\mathcal{Y}'} / \beta_\delta^{LB}$ .*

The utility of these *a posteriori* error bounds is critically dependent on the dependence of  $\beta_\delta$  as a function of the parameter  $\mu$  and final time  $T$ ,  $\beta_\delta(\mu; T)$ .

We briefly comment on the computational implications of this space-time error bound. As always, our model problem (as is the case here) must honor the “affine-in-functions-of-parameter” condition [5] to permit effective offline–online computation of the reduced basis approximation and *a posteriori* error bounds. The current formulation introduces the further complication of the space–time norms. In fact, calculation of the dual norm  $\|\cdot\|_{\mathcal{Y}'}$  (through the corresponding Riesz representation [5]) does not couple different time steps/elements, and hence the additional online difficulty is relatively minor. (The offline stage for computation of the inf-sup lower bound by the Successive Constraint Method [5,8] does require special treatment, in particular due to the norm  $\|\cdot\|_{\mathcal{X}, \delta}$ .)

### 4. Numerical Results

We report numerical results for the Crank–Nicolson scheme for various choices of the parameters  $\mu_1, \mu_2$  as well as for different time steps  $\Delta t$  and uniform mesh sizes  $h$ . For simplicity, we consider the univariate case (in space)  $\Omega = (0, 1)$  and choose  $\boldsymbol{\beta}(x) = x - \frac{1}{2}$ . Let us denote by  $\beta_\delta(\mu; T)$ ,  $\gamma_\delta(\mu; T)$  the numerical values for the truth inf-sup and continuity constants, respectively, corresponding to parameter  $\mu$  and final time  $T$ . We start by confirming Proposition 3. Thus, we choose  $\mu_1 = \mu_2 = 0$ ; for several values of  $T$ ,  $h$ , and  $\Delta t$  we invariantly obtain 1.000 for both  $\beta_\delta(\mu; T)$  and  $\gamma_\delta(\mu; T)$ , as must be the case.

Next, we investigate the case of convection,  $\mu_2 = 0$ , in which case  $a$  is coercive only for  $\mu_1 < 2\pi^2$ . We obtain  $\beta_\delta((50, 0); 1.0) = 0.050$ ,  $\beta_\delta((50, 0); 2.0) = 0.027$  ( $\gamma_\delta = 2.60$ ) and  $\beta_\delta((100, 0); 1.0) = 0.019$ ,  $\beta_\delta((100, 0); 2.0) = 0.010$  ( $\gamma_\delta = 5.6$ ). These numbers are relatively invariant for sufficiently small  $h$  and  $\Delta t$ . We observe numerically an overall behavior of  $\beta_\delta((\mu_1, 0); T) \sim (\mu_1 T)^{-1}$  and  $\gamma_\delta((\mu_1, 0); T) \sim \mu_1$  (the latter is readily proven, but not the former). Note  $T = \mathcal{O}(1)$  is effectively a “long time” in convective units,  $1/\mu_1$ . We emphasize that although the problem is non-coercive, the problem is asymptotically stable in the sense that all eigenvalues  $\sigma$  of  $-a(\psi, \phi) = \sigma \langle \psi, \phi \rangle_{V' \times V}$  lie in the left-hand plane; this stability is reflected in the inf-sup behavior. In contrast, a standard energy approach [3] gives effective inf-sup constants on the order of  $e^{-\mu_1 T}$  (here about  $10^{-8}$ ). Hence, the traditional method fails to provide useful results, whereas our new approach, which reflects the true time-coupled properties of the system, yields relatively sharp error bounds.

Finally, we consider the case  $\mu_1 = 0$  which gives rise to an asymptotically unstable (and non-coercive) system for  $\mu_2 < -\pi^2$ . This means that any error estimate *must* grow exponentially with the final time  $T$ . We observe this for our estimator as well, e.g.  $\beta_\delta((0, -20); 1.0) = 3.93 \cdot 10^{-5}$  (order of  $e^{\mu_2 T}$ ).

**Acknowledgements** A.T.P. was supported by OSD/AFOSR/MURI Grant FA9550-09-1-0613 and by ONR Grant N00014-11-1-0713; K.U. by the Deutsche Forschungsgemeinschaft (DFG) under Ur-63/9 and GrK1100. The authors are grateful to Sylvain Vallaghé for helpful comments. This paper was written while K.U. was Visiting Professor at MIT.

## References

- [1] M. Grepl and A.T. Patera. A posteriori error bounds for reduced-basis approximations of parametrized parabolic partial differential equations. *M2AN Math. Model. Numer. Anal.* **39** (2005), no. 1, 157-181.
- [2] B. Haasdonk and M. Ohlberger. Reduced basis method for finite volume approximations of parametrized linear evolution equations. *M2AN Math. Model. Numer. Anal.* **42** (2008), 277-302.
- [3] D.J. Knezevic, N.C. Nguyen, and A.T. Patera. Reduced basis approximation and a posteriori error estimation for the parametrized unsteady Boussinesq equations. *Math. Mod. Meth. Appl. Sci.*, **21** no. 7 (2011), 1415-1442.
- [4] D.V. Rovas, L. Machiels, and Y. Maday. Reduced-basis output bound methods for parabolic problems. *IMA J. Numer. Anal.* **26** no. 3 (2006), 423-445.
- [5] G. Rozza, D.B.P. Huynh, and A.T. Patera. Reduced basis approximation and a posteriori error estimation for affinely parametrized elliptic coercive partial differential equations — Application to transport and continuum mechanics. *Arch. Comp. Meth. Eng.*, **15** no. 3 (2008), 229-275.
- [6] C. Schwab and R. Stevenson. Space-time adaptive wavelet methods for parabolic evolution problems. *Math. Comp.* **78** (2009), 1293-1318.
- [7] K. Steih and K. Urban. Space-time reduced basis methods for time-periodic parabolic problems. University of Ulm, Preprint 2012. [www.uni-ulm.de/mawi/fakultaet/forschung/preprint-server.html](http://www.uni-ulm.de/mawi/fakultaet/forschung/preprint-server.html)
- [8] S. Vallaghé, A. Le-Hyarić, M. Fouquemberg, and C. Prud'homme. A successive constraint method with minimal offline constraints for lower bounds of parametric coercivity constant. Preprint: hal-00609212, [hal.archives-ouvertes.fr](http://hal.archives-ouvertes.fr).