Traffic Models From a Velocity Point of View and Implementation of Traffic Conditions in Excess Fuel Consumption Estimates

by

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Submitted to the Department of Civil and Environmental Engineering in partial fulfillment of the requirements for the degree of Master of Science in Civil and Environmental Engineering at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Abstract

The estimation of excess fuel consumption due to pavement - vehicle interactions (PVI) is receiving an increasing amount of attention, for it has major economic and ecological implications for the design and maintenance of road networks. As any dissipative mechanisms, the ones we find in the interaction of the pavement and the vehicles are highly dependent on the velocity of the agents in the system, hence the travelling speed of the vehicles. In this work, we propose a coupling of mechanistic dissipative models with evaluation of velocity profiles which shall enable a higher accuracy in the prediction of excess fuel consumption. We focus our attention on the simulation of the probability density functions associated with the velocity of vehicles on a single lane road, for agent-based and cellular automata models. While our approach neglects the influence of lane interactions on the velocity probability distribution, this simulation-based method enables us to obtain predictions of the dissipation resulting from both deflection and roughness of the pavement on any highway segment as a function of the traffic flow. It is shown that taking into account the presence of traffic jams instead of assuming all vehicles traveling at maximum velocity, as is done to obtain usual estimates, increases the deflection-induced dissipation per vehicle per traveled length and reduces the roughness-induced one. This difference is, however, shown to lose in magnitude when the temperature of the environment increases.

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# Contents

1 Introduction .......................................................... 21
   1.1 Industrial Context ............................................... 21
   1.2 Research Objectives and Approach ............................. 22
   1.3 Thesis Outline ................................................. 22
   1.4 Expected Benefits and Impacts ............................... 23

2 Agent-Based Vehicular Dynamics Model ............................. 25
   2.1 Considerations for a Single Agent: Microscopic Properties .... 25
      2.1.1 Geometrical Properties .................................... 26
      2.1.2 Force Separation Principle ................................ 26
      2.1.3 The Driving Force ......................................... 29
      2.1.4 Interaction Forces ........................................ 30
      2.1.5 Two Dimensional Extension: Balance of Moments ...... 33
   2.2 Macroscopic Considerations .................................... 36
      2.2.1 Derivation of the Fundamental Diagram ................... 36
      2.2.2 Linear Stability Analysis .................................. 39
      2.2.3 Local Stability Analysis ................................... 40
      2.2.4 String Stability Analysis .................................. 42
      2.2.5 Convective Stability ....................................... 44
   2.3 Application: Fundamental Diagram and Stability Analysis of Some Common Agent-Based Models .......... 45
      2.3.1 Follow-the-Leader Models ................................. 45
      2.3.2 Optimal Velocity Models ................................. 47
      2.3.3 Helbing's Force-Based Model ................................ 49
3 Velocity Distribution and Phase Transition in Cellular Automata Models 59

3.1 The Nagel-Schreckenberg (NS) Model 60
   3.1.1 Definition of the Model and Macroscopic Quantities 60

3.2 Phase Transition and Order Parameter 64
   3.2.1 Influence of Initial Conditions and Simulation Time 65
   3.2.2 Order Parameter 66
   3.2.3 Correlation Function 69
   3.2.4 Jamming Time 72
   3.2.5 First Interaction Time 73

3.3 The Velocity Distributions 76
   3.3.1 Separation of the Distributions 76
   3.3.2 Link between Headway and Velocity Distributions 79

3.4 Extended Nagel-Schreckenberg Model 88
   3.4.1 Extension Description 88
   3.4.2 Phase Transition and Velocity Distribution 92
   3.4.3 Two- and Three-Body Interactions 93
   3.4.4 Continuum Limit 98

3.5 Chapter Summary 99

4 Link with Energy Dissipation 101

4.1 Deflection Model 102
   4.1.1 Model Adaption and Scaling Relations 102
   4.1.2 Exact Solution 104
   4.1.3 Temperature Dependence of the System 109
   4.1.4 Dimensional Analysis of the Pavements 109
   4.1.5 Deflection-Induced Dissipation Estimates for Typical Pavements 110

4.2 Roughness Model 113
   4.2.1 Mechanistic Model 113
   4.2.2 Link with HDM-4 Model 116
   4.2.3 Model Calibration 117

4.3 Excess Fuel Consumption Estimation 118
4.3.1 Traffic Data and Simulations ........................................ 119
4.3.2 Single Speed Approximation and Velocity Distribution Function .......................... 120
4.3.3 Dissipation as a Function of Occupancy ........................................ 122
4.3.4 Dissipation as a Function of the Mean Velocity ........................................ 123
4.3.5 Dissipation as a Function of Average Flow .......................................... 125
4.4 Chapter Summary ............................................................ 128

5 Conclusion ........................................................................... 131
5.1 Summary and Main Findings ................................................ 131
5.2 Limitations and Possible Future Perspectives ......................... 132

A Nomenclature of Chapter 2 .................................................. 135

B Nomenclature of Chapter 3 .................................................... 139

C Nomenclature of Chapter 4 ................................................... 141

D Stability Calculations .......................................................... 143
D.1 String Stability ................................................................. 143
D.2 Convective Stability .......................................................... 144
D.2.1 Case 1: $\Delta_r > 0$ ......................................................... 146
D.2.2 Case 2: $\Delta_r < 0$ ......................................................... 147
D.3 Case $a_{uv} = 0$ ............................................................... 149
D.3.1 Case $\Delta_r > 0$ ............................................................ 150
D.3.2 Case $\Delta_r < 0$ ............................................................ 150
D.3.3 Summary ................................................................. 152

E Three-Body Interaction Master Equation for the Extended NS Model 155
List of Figures

2-1 Convergence of the sequence \( (v_n) \) towards a velocity inferior to the speed limit of the road in the presence of air drag as a dissipative mechanical force ........................................... 28

2-2 Acceleration profiles for \( \tau = 10 \) s and different initial velocities ........................................... 30

2-3 Empirical data for the fundamental diagram linking flow and occupancy (equivalent to density). Every point corresponds to detector measurements averaged over 5 minutes on a Canadian highway (from [24]) ........................................... 37

2-4 Local stability (left panel): the system is initialized in the steady state, where all vehicles are separated by the same distance \( \Delta x_s \), and travel at the same velocity \( v_s \) (top image). We then perturb the system by only changing the velocity and position of one vehicle by a small increment, respectively \( \nu \) ans \( \delta \) and study the evolution of the time response. String stability (right panel): the system is also initialized in the steady state but we perturb the system by changing the velocity of the vehicles \( m \) and \( n \) by a small increment \( \nu_m \) and \( \nu_n \) as well as the spacing by an increment \( \delta \) (bottom image). ........................................... 41

2-5 Fundamental diagram for the optimal velocity model defined by Eq. (2.88). It presents a discontinuity and does not correspond to realistic expectations, for which the flux should continuously decrease with the density after a certain point. ........................................... 48

2-6 Equilibrium line between the density and the velocity of the vehicles for the Helbing force-based model with fitted parameters. This line is monotonic, the velocity tends to the speed limit of the road at low density, and to zero at high density. This behavior is coherent with heuristic considerations of traffic behavior. ........................................... 52
2-7 Fundamental diagram for the Helbing force-based model, using the fitted parameters. It presents the main characteristics that are expected from heuristic considerations: a linear increase of the flux with the density until a critical density is reached, after which the flux decreases with the increase of density. The non unicity of the flux as a function of density translates into the existence of two regimes: a free flow regime which corresponds to the left part of the diagram, and a congested regime where the vehicle interactions become dominant.

2-8 String stability function for the fitted Helbing-based model (see Section (2.3.3)) (left panel) and a set that presents an instability (see Eqn. (2.110)) (right panel). The positivity of the string stability function is the criteria that determines the stability ranges.

2-9 Time evolution of vehicle positions for the fitted set of parameters at an arbitrary density (left panel) and for the unstable set of parameters at a density higher than the critical one (right panel). Every line corresponds to a single vehicle, its position is given by the Y axis while the time evolution is represented on the X axis. At time $t = 0$ the system is initialized in the equilibrium state, and a perturbation is applied to the vehicle positioned at the origin. The perturbation propagates and grows in time until it reaches a steady state with a traffic jam pattern in the unstable set (right panel) or remains stable for the fitted set of parameters (left panel).

2-10 Fundamental diagrams of the Helbing model, for the stable set of parameters of Section 2.3.3 (left panel) and the unstable set Eq. (2.110) (right panel). In red is the stable part of the diagram according to linear stability analysis and in green the unstable one. The black crosses are the averaged flux over all vehicles and every time step for a simulation of a road of $10^5$ meters with periodic boundary conditions, for a simulation time of $10^5$ seconds. The deviation of the simulation from the equilibrium curve shows the appearance of jams patterns.
2-11 Velocity probability distribution function for the stable set of parameters proposed by Helbing et al. (see Section 2.3.3) (left panel) and the unstable set of parameters Eq. (2.110) (right panel). Every line is a different density, and the density increases from the left to the right. No matter what the density is, the system is always stable and the probability density function is peaked around the mean velocity for the fitted parameters. Concerning the unstable set of parameters, the probability density function is not peaked around the mean velocity, and the probability to find a vehicle at a velocity equal to zero is non null, which is a signature of the formation of traffic jams.

3-1 Fundamental diagrams (averaged flux as a function of density) for the NS model for different values of the stochastic parameter $p$ and speed limit set to $v_{\text{max}} = 5$. Although the value of $p$ influences the shape of the diagram, it remains coherent with empirical observation. The flux increases linearly with the density until a critical density after which the flux decreases.

3-2 Vehicles trajectories. Every line represents a vehicle, the $y$ axis represents the position of the vehicles while the $x$ axis the time. The right panel, where we can note the appearance and dissolution of traffic jams with time, as well as the jammed front travelling upstream, corresponds to a stochasticity of $p = 0.5$. In the left panel ($p = 0$) one can note absence of traffic jams and all the vehicles travelling at the same velocity and equally spaced. Both simulations are at a density higher than the critical one and a maximum velocity $v_{\text{max}} = 10$. This shows the importance of the stochastic parameter $p$ in the simulations of jams.

3-3 Probability distribution functions for $v_{\text{max}} = 5$, $p = 0.5$ and various densities.

3-4 Sketch of the road discretization used in the original NS model. Every cell can be occupied by only a single vehicle and has a characteristic size of about four meters.
3-5 Velocity probability distribution functions for an average density of 0.11 vehicles per cell, for a simulation time of $10^5$ iteration steps (left panel) and $10^6$ iteration steps (right panel) for three different initial conditions. "MJ" corresponds to the megajam initial conditions, "FF" to the free flow initial conditions and "ES" to the equally spaced and stopped initial conditions. The three curves are different for the shorter simulation time (top) but merge for the higher one, implying that $10^5$ iteration steps is too short to obtain results independent of the initial conditions but $10^6$ is sufficient.

3-6 Fundamental diagram for the NS model with $v_{max} = 10$ and $p = 0.5$ for the "ES" initial conditions. The linear growth at low occupancies corresponds to the free flow state, and the decrease after a critical density is the congested phase.

3-7 Velocity distribution for different occupancies and $v_{max} = 10$, $p = 0.5$, and the "ES" initial conditions. The probability to be at zero velocity is non zero after a certain critical occupancy, which is the signature of the congested state.

3-8 Plot of the order parameter (probability to find a stopped vehicle) as a function of the occupancy for $v_{max} = 10$, $p = 0.5$ and the "ES" initial conditions, close to the critical occupancy, measured from numerical data. The critical occupancy is found to be about $\rho_c \approx 0.036$.

3-9 Metastable states as a function of occupancy.

3-10 Velocity-velocity correlation function for low densities (left), close to the critical density (center) and high densities (right). The free flow case (left) is completely uncorrelated and we find a correlation length of about 4 vehicles for the congested case (right) and a diverging correlation length close to the critical density.

3-11 Log plot of the velocity-velocity correlation function as a function of the number of preceding vehicles $r$, for an occupancy $\rho = 0.21$ (corresponding to the right panel of Fig. 3-10). The correlation number is found to be $n_c = \frac{1}{0.26} \approx 4$ vehicles.

3-12 Log plot of the jamming time as a function of the occupancy, from free flow initial conditions at occupancies higher than the critical density $\rho_c \approx 0.036$. The linear dependence goes on for four decades of the jamming time.
3-13 Velocity distribution for a density higher than the critical density for $v_{\text{max}} = 10$, $p = 0.5$, and the "ES" initial conditions. Three parts can be noted: the number of stopped vehicles (a), the vehicles in the transitioning zone (b) and the vehicles in the free flow phase (c). .................................................. 77

3-14 Jammed velocity distribution function for three different densities, all higher than the critical one, for $v_{\text{max}} = 10$, $p = 0.5$, and the "ES" initial conditions. .................................................. 78

3-15 Jammed velocity (left panel) and headway (right panel) distribution function for three different densities, all higher than the critical one, for $v_{\text{max}} = 10$, $p = 0.5$, and the "ES" initial conditions. .................................................. 79

3-16 Jammed velocity (left panel) and headway (right panel) distribution function for three different densities, all lower than the critical one, for $v_{\text{max}} = 10$, $p = 0.5$, and the "ES" initial conditions. .................................................. 80

3-17 Initial conditions of the two body interactions. The two vehicles start spaced from a distance $d_0$ and the leader remains stopped at all times. The follower has a velocity $v = 0$ with a probability 1 at time $t = 0$. At $t > 0$ the follower freely accelerates and eventually stops behind the leader. We can distinguish two zones, the free acceleration zone and the interaction zone (in red), that corresponds to the $v_{\text{max}}$ cells behind the leading vehicle. ................................. 81

3-18 Time evolution of the velocity distribution function for the following vehicle for $d_0 = 500 >> v_{\text{max}}$. We can distinguish two zones: a free acceleration (a) where the following vehicle goes from being stopped to the free flow regime (b) in which the velocity distribution is given by Eq. (3.5). The third zone (c) is the interaction zone, where the vehicle decelerates until it stays completely still behind the leader (d) .................................................. 82

3-19 Velocity (left panel) and headway (right panel) probability distribution functions from the analytical solution of the two body interaction, with $d_0 = 500 >> v_{\text{max}}$ .................................................. 83

3-20 Velocity distribution for the three body interactions for both followers and their mean value at $d_0 >> v_{\text{max}}$ .................................................. 86
3-21 Normalized transitioning zone of the velocity distribution, found by simulation for $v_{\text{max}} = 10$, $p = 0.5$, and the "ES" initial conditions (red), and the four different approximations. The fourth approximation, corresponding to $\alpha = 0.95$ in Eq. (3.38) is in very good agreement with the simulation result. This suggests that a jammed system can be modelled as a succession of many successive stopped vehicles, with a certain distribution of gaps in front of the leader.

3-22 Sketch of the road discretization used in the extended Nagel-Schreckenberg model. Every cell can be occupied by only a fraction of vehicle but every vehicle occupies several cells.

3-23 Fundamental diagram for the extended NS model, with $v_{\text{max}} = 100$, $\alpha = \sigma = \lambda = 11$, "ES" initial conditions, and a uniform deceleration distribution.

3-24 Velocity distribution function for low, medium and high occupancies for $v_{\text{max}} = 100$, $\alpha = \sigma = \lambda = 11$, "ES" initial conditions, and a uniform deceleration distribution. The simulations with high and medium occupancies are beyond the critical occupancy. The distribution can be separated in three parts, the fraction of stopped vehicles (a), the transitioning zone (b) and free flow fraction (c).

3-25 Velocity distribution of the moving vehicle for the two-body interaction, for a large initial distance $d_0$. Similarly to the classic NS model, the transitioning zone is very flat, except for a small bump very close to zero. This bump recovers the one observed with simulation distribution, but the rest of the transitioning zone is very flat, which is not consistent with numerical observations in Fig. 3-24.

3-26 Time evolution of the probability of every velocity for the two-body interaction, for a large initial distance $d_0$. Similarly to the classic NS model, we can observe three phases. An accelerating phase (a) where the vehicle freely accelerates, until it reaches the free flow phase (b) where all the velocities in the interval $[v_{\text{max}} - \sigma ; v_{\text{max}}]$ are equiprobable, and eventually decelerates (c) when it comes close enough to the still vehicle until it reaches complete rest with probability 1. The blue curve corresponds to $P(v = 0, t)$. 

3-27 Transitioning velocity distribution function obtained from the two-body interaction (black) and from the simulation with $v_{max} = 100$, $\alpha = \sigma = \lambda = 11$, "ES" initial conditions, and a uniform deceleration distribution (red).

4-1 Part of the different resistive forces acting on a travelling vehicle in the global resistive force, taken from [51] (where it was adapted from [11])

4-2 Uniform load traveling at constant velocity over a visco-elastic beam supported by an elastic layer, adapted from [39]

4-3 Dimensionless plot of the deflection, the normalized position is $\frac{X}{L}$ and the normalized deflection $\frac{w}{b}$. The center of the load sees a positive slope, that translates into a wheel being constantly going uphill.

4-4 Deflection-induced excess fuel consumption per vehicle per traveled length as a function of the velocity, for different temperature values, normalized by the deflection-induced excess fuel consumption per traveled length at $T = 0^\circ C$ and $v = 35$ m/s. The left panel corresponds to a flexible pavement (asphalt) and the right one to concrete.

4-5 Deflection-induced excess fuel consumption per traveled vehicle as a function of the characteristic length $l_0$, for different temperature values, normalized by the deflection-induced excess fuel consumption per traveled meter at $T = 0^\circ C$ and $v = 35$ m/s. The left panel corresponds to a flexible pavement (asphalt) and the right one to concrete.

4-6 Load distribution on an articulated truck, taken from [39]

4-7 Two DOF quarter-car model, taken from [40]

4-8 Roughness-induced dissipation normalized by the dissipation at $v = 35$ m/s as a function of velocity for two types of pavements: asphalt (AC) and concrete (PCC).

4-9 Mean velocity as a function of the occupancy. $\langle v \rangle < \langle v \rangle_{FF}$ corresponds to the congested phase, and the mean velocity can serve as an order parameter.

4-10 Normalized dissipation as a function of the occupancy, for the deflection phenomenon (left) and the roughness phenomenon (right) for an articulated truck on a concrete pavement at a temperature of $20^\circ C$. The dissipation is normalized by the dissipation in the free flow phase, for $\sigma < \sigma_c$.
4-11 Normalized dissipation as a function of the mean velocity, for the deflection phenomenon (left) and the roughness phenomenon (right) for an articulated truck on a concrete pavement at a temperature of 20°C. The dissipation is normalized by the dissipation in the free flow phase, for $O < O_c$.

4-12 Normalized dissipation as a function of the mean flow, for the deflection phenomenon (left) and the roughness phenomenon (right) for an articulated truck on a concrete pavement at a temperature of 20°C. The dissipation is normalized by the dissipation in the free flow phase, for $O < O_c$.

4-13 Normalized deflection-induced dissipation as a function of the mean flow, for various temperatures for an asphalt pavement (top) and a concrete pavement (bottom). The axis and colors are the same as Fig. 4-12 (left panel) and the scale in every figure is the same and goes from 1 to 10 for the normalized dissipation and 0.3 to 1 for the normalized flow.

4-14 Normalized roughness-induced dissipation as a function of the mean flow, for a asphalt pavement (left panel) and an concrete pavement (right panel). The scale in every figure is the same and goes from 0 to 1 for the normalized dissipation and 0.3 to 1 for the normalized flow.
List of Tables

4.1 Dimensionless parameters range for real pavement conditions . . . . . . . . 110
4.2 Mechanical properties of different classes of vehicles, taken from [40] . . . . . 117
4.3 Mechanical properties of different classes of pavement: AC for asphalt and
   PCC for concrete, taken from [40] . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 119
Chapter 1

Introduction

1.1 Industrial Context

As industrial development is booming all around the globe, reducing greenhouse gas emissions and improving air quality concerns every industrial field and has taken a central place within the context of the sustainable development of our societies. Among other sources of emission, road transportation accounts for up to 27% of all greenhouse gas emissions in the United States according to the Environmental Protection Agency (EPA 2012). Reducing the emissions arising from fuel consumptions therefore has a non negligible potential ecological impact, as well as important economical implications. One of the levers Engineers can act upon is focusing on Pavement - Vehicle Interactions (PVI) to reduce the energy losses they produce. More precisely, pavement - vehicle interactions produce a rolling resistance resulting in energy dissipation, that needs to be compensated by excess fuel consumption. For now this rolling resistance has been broken down into several phenomena, [39] [38] [40] and models have been developed to quantify how the energy dissipation they induce is affected by the pavement the tire properties, as well as by vehicle velocity and weight. There is a strong dependence of the different sources of dissipation on the traveling velocity of the vehicles [39] [40]; but it is not straightforward to obtain an accurate estimate of the energy dissipation on a road network, or even just a single interstate lane, because of the highly non linear nature of vehicular traffic. This shortcoming calls for a new approach, one that couples the mechanistic approach with traffic simulation models in order to take into account the distribution of vehicle velocity.
1.2 Research Objectives and Approach

In this thesis, we propose to couple these two worlds, namely mechanical studies of energy dissipation and traffic simulations, by means of velocity probability distribution functions. Since we are interested in capturing the variations of velocity of every single vehicle, we choose to focus our study on microscopic descriptions of traffic phenomena. As mentioned in recent review books on traffic modelling [57] [62], these models usually fall into two categories: force-based models and cellular automata models. Force-based models are discrete in time and deterministic. In these models one solves using behavioral forces balance a Newtonian scheme, much akin to classical molecular dynamics. Cellular automata models use a discrete description of the road, which they divide into cells of the size of a single vehicle. They have been introduced for traffic simulations in the early 1990's and the most common example of them is the Nagel Schreckenberg model [47] [58].

Surprisingly, none of these models seem to have been the subject of precise studies in terms of velocity distributions. We, therefore, investigate both class of models from a statistical point of view: we focus on a single lane road sufficiently long to avoid finite size effects and we let the simulations run for a long time to obtain reliable statistics. Our overall goal is to target a model which gives velocity distributions that are precise enough to obtain good estimations, with the constraint that it is relatively straightforward to fit and not more parameters than needed to simulate generic behavior. By investigating both class of models, we sort through the literature to find which is the most suited for our purpose. This model is used to obtain energy dissipations on a highway.

1.3 Thesis Outline

In order to reach these goals, we first investigate force-based models in Chapter 2. We describe the mathematical framework used in this class of models, and use this framework to introduce the main elements of traffic science. We start from a bottom-up approach, and focus on the local behavior of the interaction forces to identify their domain of existence. We then switch to a top-down approach and look at global behaviors such as the fundamental diagram and linear instabilities that are the signature of traffic jams, to obtain local properties the interaction forces must verify in order to reproduce these phenomena. A literature review of the existing models shows that this class of models is not convenient to reproduce
traffic jams and accurate mean values, unless a tremendous number of fitting parameters is used. By way of illustration, we show that the most advanced model, developed by Helbing [27], does not reproduce jams with the fitted parameters given in [27].

In Chapter 3 we move on to the second class of traffic models, the cellular automata models. We show that despite the few of fitting parameters, the Nagel Schreckenberg model (NS) is capable of reproducing both instabilities and coherent mean values. We investigate the properties of the model from the point of view of velocity distribution, with a first focus on phase transition that occurs in this model. After making sure that the simulations are well defined and not influenced by the initial conditions, we show that the phase transition presents some similarities with droplet nucleation as an activated process, although it is an out of equilibrium system that is ill-defined in terms of thermodynamics. We then focus our attention on the velocity distributions strictly speaking, and describe an analytical three-body interaction model that enables us to make a link with the headway distribution, and that gives us some insight into the internal structure of the traffic. Finally we extend the NS-model into a more appropriate dicretized version. While this extended model recovers the main properties of the classic NS-model, we show that it provides finer distributions.

Chapter 4 bridges the gap between traffic simulations and the energy dissipation arising from Pavement-Vehicle-Interactions, focusing on deflection-induced [39] and roughness-induced [40] PVI. We first describe both phenomena and the mechanistic models that have been developed to quantify the excess fuel consumption which results from energy dissipation. Once the handshake with traffic simulations is made we study how field measurements can be linked with simulation results, and we compare the estimations arising from the incorporation of velocity distribution with the current analysis, which assumes that all the vehicles travel at the same maximum speed (typically defined by the speed limit).

Finally Chapter 5 summarizes the results and specifies the limitations and possible extensions of our approach.

1.4 Expected Benefits and Impacts

Current life cycle analysis of pavement only accounts for the PVI-induced dissipation by making the assumption that all vehicles travel at maximum speed. Obtaining velocity distributions for various traffic conditions and using them, together with mechanistic models in
the way described in this thesis, shows to hold different estimates of excess fuel consumption. This difference may not be much at the scale of a single vehicle, but once placed into the context of a whole network, it can scale to have a significant influence. Thus, the systematic link between traffic condition and dissipation phenomenon is a more realistic way to estimate the amount of excess fuel consumption arising from pavement-vehicle interactions, and implementing this information into a complete roadway network would provide decision makers with more precise information about the impacts of pavement conditions on vehicle fuel consumption. In this way, coupling traffic flow and mechanistic models can become completely integrated in the operation and maintenance of current road networks, as well as in the construction of future ones.
Chapter 2

Agent-Based Vehicular Dynamics Model

In this first part, we consider the class of so-called agent-based models for traffic flow simulations. This approach, somehow intuitive, is extremely similar to molecular dynamics [19]: every vehicle is considered as an independent particle, commonly called "agent" [62], and interactions with other vehicles are defined by means of social forces [27]. This method is frequently used to model other collective systems composed of individual elements whose interactions are not purely physical but have a social component, such as human crowds [26] or schools of fish or birds [13]. It is also in many ways very similar to the field of active matter [43] [17] [6] [8], which studies the collective motion of groups of self propelled particles. We will start this chapter by defining the general framework of such force-based models. This enables us to introduce some of the main concepts used in traffic simulations, such as the fundamental diagram. We will also present stability analysis concepts, that are intimately linked to the formation of traffic jams in simulations. We will apply these concepts to the study of the properties present of some of the popular models in the open literature, such as the car-following model [28], [54] and the generalized force-based models [27] [64], with an emphasis on the model developed by Helbing [27].

2.1 Considerations for a Single Agent: Microscopic Properties

In the terminology presented in Ref. [57], the models we describe in this chapter correspond to the class of microscopic, deterministic, Lagrangian, low fidelity, force-based models, or for
simplicity agent-based models. In order to recover global traffic properties from agent-based models, it is necessary to describe the local properties of every agent. It is worthy to note that vehicles are not classical particles. They are non-Newtonian in the sense that they do not obey the actio-reactio principle [27] and the overall energy is not conserved. This is somewhat intuitive: if you are in your car and you slow down because of a standing vehicle in front of you, you are not storing potential energy, nor is the energy transferred to the other vehicle as you approach it. Otherwise said, the other vehicle does not feel a repulsive force as you approach it. Even if we pay attention to the vehicles behind us, the interactions are mostly with the vehicles in front of us. This property is in fact only true for the behavioral forces between drivers. Clearly, it technically does not extend to the mechanical forces between the vehicles and the environment, since the actio-reactio principle remains valid as vehicles exert a force on the pavement. But in these class of models the environment is excluded from the overall system. At best, the environment can be considered as external forces that dissipate energy outside of the system.

2.1.1 Geometrical Properties

To translate these microscopic properties into geometrical properties, the agents only interact with what they see in front of them, defined by a preferential direction. Hence, an agent $i$ is defined in space by:

- The position $(x_i, y_i) \in \mathbb{R}^2$, which is the position of the vehicle’s center of gravity,
- The vehicle dimensions (width $w_i$, length $A_i$)
- The driving direction $e_{r,i}$, or equivalently the preferential angle $\theta_i$, that verifies $e_{r,i} = \cos(\theta_i)e_x + \sin(\theta_i)e_y$.

We assume, very naturally, that this vehicle $i$ can move only along its preferential direction. All the forces are thus projected on vector $e_{r,i}$. Computing the motion of the vehicles requires us to solve both a balance of forces, in order to get the change in velocity, and a balance of moments, which provides us with the change in orientation of the vehicles.

2.1.2 Force Separation Principle

The forces acting on the vehicle can be divided into two categories: behavioral and mechanical forces. The behavioral, or social forces [27] do not have a physical meaning but are meant
to reproduce as closely as possible the driver’s behavior. The main difficulty that arises here is that the expression of these behavioral forces - if such an expression even exists - is a priori unknown and has to be constructed from reasonable considerations. We denote three of them: the driving force - that propels the agent forwards, the vehicle-vehicle interaction forces and the road-vehicle interaction forces (so that the driver stays on the road). The mechanical forces are the ones that physically act on the vehicle and the non conservative ones are responsible for energy dissipation, which translates into fuel consumption. They include air drag, pavement-vehicle interactions (PVI) and engine dissipation. We will here make the case to exclude the mechanical forces from the social force balance, basing our reasoning on a simple example: a vehicle travelling alone on a straight road.

Consider the vehicle subjected to behavioral forces, $F^b$, and mechanical forces, $F^m$. The balance of forces is given by:

$$\sum F = F^b + F^m$$

(2.1)

If we suppose that the vehicle is only subjected to air drag, the dissipative force on the vehicle has the well known form

$$F^m = \frac{\rho a C v^2 S}{2}$$

(2.2)

with $\rho$ the air density, $C$ the air drag coefficient, $v$ the velocity of the vehicle and $S$ a typical surface (usually the projected area of the vehicle in a plane orthogonal to the travelling direction). The behavioral forces acting on the vehicle with no interactions are restricted to a driving force, that makes the vehicle accelerate until the speed limit is reached. The simplest form we can think of is

$$F^b = f^d \left(1 - \frac{v}{V_{lim}}\right)$$

(2.3)

with $f^d$ a constant representing the maximum value of the acceleration force. If we now write Eq. (2.1) using a discrete form such as a Leap-Frog scheme [19], we have:

$$\forall n \in \mathbb{N}, \quad m \frac{v_{n+1} - v_n}{\Delta t} = f^d \left(1 - \frac{v_n}{V_{lim}}\right) - \frac{\rho a C (v_n)^2 S}{2}$$

(2.4)

It is easily shown that the sequence $(v_n)$ defined by Eq. (2.4) converges to a velocity that is smaller than the speed limit $V_{lim}$, and it, therefore, does not reproduce the expected behavior of the driver. This example is depicted in Fig. 2-1. The reason for this is that
we decoupled in Eq. (2.1) the resistive forces acting on the vehicle and the social forces; whereas a more realistic approach would need to implement the reaction of the driver to the mechanical forces acting on the vehicle. In fact, in normal operational conditions, drivers are easily capable to compensate for the mechanical forces acting on their vehicles, in order to reach the speed limit of the road:

\[ \sum F = (F^b - F^m) + F^m = F^b \] (2.5)

This is why from now on, we shall focus on the implementation of only the behavioral forces in our model. Once we obtain the velocities and positions of the vehicles at every time step, we can still, however, compute the values of the mechanical dissipative forces on the vehicle in order to obtain the dissipative energy resulting from their action. We now separate the forces into a driving force \( F^d \), that tends to make the vehicle accelerate until
the speed limit is reached, and vehicle-vehicle interaction forces \( F^v \) that prevent accidents.

\[
\sum F = F^b + \sum_j F^v_j
\]  

(2.6)

Let us now discuss the properties of each of these forces.

### 2.1.3 The Driving Force

The driving force reflects the behavior of a driver when there are no other vehicles to interact with. We can rationally assume the acceleration of the driver to be independent of the position in space, and to be only a function of the vehicle velocity \( F^d = F^d(v) \). We consider that the driver will accelerate when his velocity is below a desired velocity, \( V_{lim} \), and decelerate if he is above:

\[
\begin{align*}
F^d(v) & \geq 0 \quad \text{for } [0, V_{lim}] \\
F^d(v) & < 0 \quad \text{for } v > V_{lim}
\end{align*}
\]  

(2.7)

This is particularly relevant on a highway with light traffic, where interactions are negligible, when vehicles are equipped with cruise control. The value of \( V_{lim} \) then depends on the individual preference of each driver. A poll or field measurements would enable us to determine the precise distribution around the speed limit of the road. We could, therefore, introduce a stochastic component in our model with a probability distribution around the value of \( V_{lim} \). If this case is the most obvious one, one could think about other driving forces, when the driver’s goal is not to reach a certain velocity. For instance, for a vehicle that is able to give some data to the driver such as the gas consumption, the driver could reasonably attempt to minimize it. Further we would need to re-write the driving force in another form. We do not consider this case, and the simplest form of the driving force takes the form [27] [28] [54]:

\[
\frac{F^d(v)}{m} = \frac{1}{\tau} (V_{lim} - v)
\]  

(2.8)

with \( \tau \) an empirical time constant that has to be chosen considering realistic acceleration profiles of vehicles. Eq. (2.8) simply gives an exponential decay of the acceleration with a relaxation time \( \tau \). Some acceleration profiles are presented in Fig. 2-2 for three different initial velocities and a relaxation time \( \tau = 10 \) s.
2.1.4 Interaction Forces

The second social force to be considered in the force balance relates to interaction forces. Their choice is non unique but subject to certain rules. We will consider several different hypothesis concerning their general properties, and verify which ones are compatible with traffic simulation. This will be similar to the approach used by Caussin [7] to classify the interactions between self propelled particles moving at a constant velocity.

We will start by presenting the case of interaction forces deriving from a potential and show, using energetic considerations, that they are not suitable for the modeling of traffic flow. We will then show why symmetric interaction forces do not suit traffic description either. This will enable us to derive the correct form of vehicle-vehicle interaction forces and we will derive some mean properties, such as the fundamental diagram.
Potential forces

As a first hypothesis, it seems natural to consider interaction between vehicles in terms of potential forces. If we assume that the interaction forces derive from a potential $U(x)$, we can write:

$$\forall i \in [1, N], \ m_i a_i = F^d_i - \sum_{j=1; j \neq i}^{N} \frac{\partial U^{i,j}}{\partial x^i}$$

(2.9)

where $N$ is the total number of vehicles, and $a_i$ the acceleration of vehicle $i$. The conservation of energy gives us, from time $t$ to $t + \delta t$

$$E_c(t + \delta t) + E_p(t + \delta t) = E_c(t) + E_p(t) + \delta E$$

(2.10)

with

$$\delta E = \sum_{i=1}^{N} \int_{t}^{t+\delta t} F^d_i(v_i(t))v_i(t)dt$$

(2.11)

If we recall from Eq. (2.7) that one of the properties of the driving force $F^d$ is that it is driving the vehicle forward when its velocity is smaller than the speed limit of the road, and slowing it down when the speed exceeds the speed limit, we obtain a change in total energy for every vehicle in the form:

$$\delta E_i = \int_{t}^{t+\delta t} F^d_i(v_i(t))v_i(t)dt > 0 \text{ if } v_i \in [0, V_{lim}]$$

(2.12)

Considering that the velocity of the vehicles is always positive leads to the conclusion that the total energy of the system will always increase due to this dissipative term, until all the velocities are equal to the speed limit of the road. Once this point is reached we obtain $\delta E_i = 0$, which implies that the total energy then remains constant. At this point we have to anticipate on the coming Section 2.2 and mention an essential macroscopic characteristic that we expect our models to exhibit. It is well known from traffic theory that the flow of vehicles, namely the number of vehicles passing a fixed section in a given time, depends non linearly on the traffic density (number of vehicles per meter). As will be explained in Section 2.2 (see Fig. 2-3), the observed behaviour is an increase in average flow with increasing density, followed by a decrease in flow after a critical density is reached. In the case of potential forces here described, the velocity of every vehicles is always equal to the speed limit, independently of vehicle density. As a consequence, the flow constantly increases.
with density and the choice to derive the vehicle-vehicle interaction forces from an elastic potential cannot properly reflect the behavior of vehicles. It proves that the interaction forces need to have a dissipative part, one that takes into account the lost energy during the breaking of vehicles. This dissipative part can be introduced either by breaking the symmetry of actio - reactio, or by introducing a velocity dependence, or both.

**Symmetric Forces**

In order to restrict slightly more the possible choices of social forces, we will now consider interaction forces that obey the third law of Newton:

\[ F_{i \rightarrow j}^v = -F_{j \rightarrow i}^v \]  \hspace{1cm} (2.13)

It is then straightforward to see that there is a possible equilibrium state when all the vehicles have the same velocity and are separated by the same distance. In this case, on every vehicle, the forces cancel out two by two, so that for vehicle \( i \),

\[ F_{i \rightarrow i+k}^v = -F_{i \rightarrow i-k}^v \]  \hspace{1cm} (2.14)

with \( k \) an integer representing the vehicle spaced by \( k \) positions of vehicle \( i \). In this context, for a sufficient number of vehicles.

\[ m_i \frac{dv_i}{dt} = F^d_i \]  \hspace{1cm} (2.15)

In the same fashion as earlier, all the vehicles travel at the speed limit of the road, no matter what the density is. Since there exists a critical density beyond which traffic jams have been shown to form spontaneously (without the presence of a bottleneck) [57] [60], we need to incorporate an asymmetry in the interaction forces.

**Properties of the Vehicle-Vehicle Interaction Forces**

We will therefore consider the case where the third equation of Newton, the actio-reactio principle, is not respected. We will even go further by considering only interactions between a vehicle and the one preceding it. It has been shown in [5] that the interactions between vehicles mostly depend on the distance separating two vehicles, \( \Delta x \), the vehicle velocity \( v \), and the velocity difference between two vehicles \( \Delta v \). This comes to consider interaction
forces of the form

\[ F_{i ightarrow j}^v = F_{i ightarrow j}^v(\Delta x_{i,j}, v_i, \Delta v_{i,j}) \]  

(2.16)

where \( v_i, \Delta v_{i,j} \) and \( \Delta x_{i,j} \) are the velocity of vehicle \( i \) and the difference in the velocities and positions of vehicles \( i \) and \( j \):

\[
\begin{align*}
\Delta v_{i,j} &= v_i - v_j \\
\Delta x_{i,j} &= |x_j - x_i|
\end{align*}
\]  

(2.17)

As with the driving force, we can have a look at the signs of the derivatives of the interaction forces. It is natural to assume that when two vehicles get closer to each other, the interaction force increases. Similarly, the greater the speed or the speed difference the stronger the interaction. This means:

\[
\begin{align*}
\frac{\partial F_{i ightarrow j}^v}{\partial \Delta x_{i,j}} &\leq 0 \\
\frac{\partial F_{i ightarrow j}^v}{\partial v_i} &\geq 0 \\
\frac{\partial F_{i ightarrow j}^v}{\partial \Delta v_{i,j}} &\geq 0
\end{align*}
\]  

(2.18)

2.1.5 Two Dimensional Extension: Balance of Moments

If we want to consider roads that are curved instead of straight, an additional equation relative to the moment equilibrium needs to be considered in order to enable the vehicles to turn. We then define other interaction forces, which are the social interactions between the road and the vehicles. For now we assume that only the road interaction acts on the vehicle’s direction. In this section we describe a way to define appropriate road forces, so that the vehicles stay on the road. For this purpose we use a leap frog algorithm and consider the following moment balance:

\[
\forall i \in [1, N], \quad M_i = M_i^w - M_i^d \]  

(2.19)

where \( M_i^w \) is the moment acting on vehicle \( i \) due to the walls and \( M_i^d \) is a damping term, that will enable equations to converge to a fixed value, instead of oscillating around an equilibrium position. Using a leap frog algorithm we have:

\[
\forall t > 0 \quad \begin{cases}
\theta(t + \Delta t) = \theta(t) + \Delta t \dot{\theta}(t + \frac{\Delta t}{2}) \\
\dot{\theta}(t + \frac{\Delta t}{2}) = \dot{\theta}(t - \frac{\Delta t}{2}) + \Delta t \frac{M(t)}{m(t)} \\
M(t) = M(\theta(t), \dot{\theta}(t - \frac{\Delta t}{2}))
\end{cases}
\]  

(2.20)
with

$$\dot{\theta} = \frac{M}{ml^2} \quad (2.21)$$

where \( l \) is the vehicle length, and \( m \) the vehicle mass. We choose to define the damping term as

$$\frac{M^d}{ml^2} = \frac{\dot{\theta}}{\tau} \quad (2.22)$$

with \( \tau \) a time constant that will be determined later.

In order to characterize the wall potential moment, we proceed as follows: we consider two lengths \( \delta_{front} \) and \( \delta_{rear} \), both smaller than the vehicle's length. Then compute the moment due to the wall forces at points \( x + \delta_{front}e_r \) and \( x - \delta_{rear}e_r \). In other words, we compute the moments due to the forces that act slightly in front of the center of gravity of the car, and the ones that act slightly behind it.

Consider then a car, with \( x \) being the position of its center of gravity, and \( \theta \) its directional angle. Assuming that the force from the wall can be derived from a potential \( U^w \), the tangential part of the force that acts on the vehicle is:

$$f_t^w(x) = -\nabla U^w(x) \cdot e_r = - \left[ - \frac{\partial U^w(x)}{\partial x} \sin(\theta) + \frac{\partial U^w(x)}{\partial y} \cos(\theta) \right] \quad (2.23)$$

Now let us drop the \( w \) index and consider a Taylor expansion:

$$f_t(x + \delta e_r) = f_t(x) + \delta \frac{\partial f_t(x)}{\partial x} + o(\delta) \quad (2.24)$$

with \( \delta \) small enough, and \( \frac{\partial}{\partial x_r} \) being the derivative along \( e_r \). We then write:

$$\frac{\partial f_t(x)}{\partial x_r} = \frac{\partial f_t(x)}{\partial x} \frac{\partial x}{\partial x_r} + \frac{\partial f_t(x)}{\partial y} \frac{\partial y}{\partial x_r} \quad (2.25)$$

and

$$dx_r = \cos(\theta)dx + \sin(\theta)dy = \frac{\partial x}{\partial x_r} dx + \frac{\partial y}{\partial x_r} dy \quad (2.26)$$

which gives

$$\frac{\partial f_t(x)}{\partial x_r} = \frac{\partial f_t(x)}{\partial x} \cos(\theta) + \frac{\partial f_t(x)}{\partial y} \sin(\theta) \quad (2.27)$$
Incorporating Eq. (2.23) into Eq. (2.24), we obtain:

\[
\frac{\partial f_t(x)}{\partial x_r} = - \left[ \left( -\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \frac{1}{2} \sin(2\theta) + \frac{\partial^2 U}{\partial x \partial y} \cos(2\theta) \right] \cos(\theta) \tag{2.28}
\]

Using Schwarz's equation and trigonometric formulas, the previous expression is rewritten in the form:

\[
\frac{\partial f_t(x)}{\partial x_r} = - \left[ \left( -\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \frac{1}{2} \sin(2\theta) + \frac{\partial^2 U}{\partial x \partial y} \cos(2\theta) \right] \cos(\theta) \tag{2.29}
\]

We easily verify that this equation holds for the special cases \( \theta = 0 \) (which implies \( \varepsilon_r = \varepsilon_x \) and \( \varepsilon_\theta = \varepsilon_y \)) and \( \theta = \frac{\pi}{2} \) (\( \varepsilon_r = \varepsilon_y \) and \( \varepsilon_\theta = -\varepsilon_x \)). We can then use the previous expression in the moment equation:

\[
M(x) = \delta_{\text{front}} f_t(x + \delta_{\text{front}} \varepsilon_r) - \delta_{\text{rear}} f_t(x - \delta_{\text{rear}} \varepsilon_r) \tag{2.30}
\]

Assuming that \( \delta_{\text{front}} \) and \( \delta_{\text{rear}} \) are small and of the same order of magnitude as \( \delta = \frac{\delta_{\text{front}} + \delta_{\text{rear}}}{2} \), we obtain:

\[
M(x) = \delta_{\text{front}} \left( f_t(x) + \delta_{\text{front}} \frac{\partial f_t(x)}{x_r} \right) - \delta_{\text{rear}} \left( f_t(x) - \delta_{\text{rear}} \frac{\partial f_t(x)}{x_r} \right) + o(\delta) \tag{2.31}
\]

or equivalently:

\[
M(x) = (\delta_{\text{front}} - \delta_{\text{rear}}) f_t(x) + \left( (\delta_{\text{front}})^2 + (\delta_{\text{rear}})^2 \right) \frac{\partial f_t(x)}{x_r} + o(\delta) \tag{2.32}
\]

The moment can thus be obtained from any potential, provided an appropriate adjustment of the \( \delta \) values, and of the damping time \( \tau \) to enable the system to converge. It is emphasized that the choice of the constants \( \delta_{\text{front}} \) and \( \delta_{\text{rear}} \) will be critical in the behavior of the vehicle. The choice \( \delta_{\text{front}} = \delta_{\text{rear}} \) may appeal as the most natural. But this choice is erroneous for the following reason: the symmetry of the moment would enable the vehicle to be in equilibrium without being at the bottom of the potential wells, and it would be then able to "climb" the walls. It is in fact preferable to choose \( \delta_{\text{front}} > \delta_{\text{rear}} \), so that the vehicle behaves as if its center of gravity was ahead of its actual geometric center of gravity. This choice ensures that the vehicle is not stable unless it is at the bottom of the well.
2.2 Macroscopic Considerations

In the previous section we used microscopic considerations to derive properties of the interaction forces. We will now focus on the macroscopic quantities that define agent-based models.

2.2.1 Derivation of the Fundamental Diagram

One of the most important characteristics of traffic flow is the relationship between the density of vehicles, $\rho$, and the outgoing flux, $\Phi$. Optimizing the flux is in fact a crucial issue in traffic regulation. The mainly observed behaviors are well described in Refs. [57] and [62] and can be summarized as follows: the flux increases linearly with the density until a critical density $\rho_c$ is reached. The slope then decreases and becomes negative at the maximum flux $\Phi_{\text{max}}$. Beyond this point the experiments give a cloud of possible flux values for higher densities as shown in Fig. 2-3, representing highly non linear phenomena such as jamming. In this section we show how to obtain this relationship for any given social force. This will enable us to restrict again the range of admissible social forces.

To this end, let us consider a road of length $L$ on which $N$ vehicles travel. The relation between the average flux $\Phi$ and the density $\rho = \frac{N}{L}$ has to take into account the average velocity of the vehicles [62] [57], that is:

$$\Phi = <\rho v>$$  \hspace{1cm} (2.33)

Now, following developments in Ref. [8] we assume that all vehicles travel at the same velocity, $v_s$. Considering the road with periodic boundary conditions, we have:

$$\Phi = \rho v_s$$ \hspace{1cm} (2.34)

Considering steady state conditions we make the assumption that all vehicles, travelling at the same velocity, are equally spaced by a distance $\Delta x_s$. The average density, which is the number of vehicle per unit length, can then be written as

$$\rho = \frac{1}{\Delta x_s}$$ \hspace{1cm} (2.35)
Figure 2-3: Empirical data for the fundamental diagram linking flow and occupancy (equivalent to density). Every point corresponds to detector measurements averaged over 5 minutes on a Canadian highway (from [24])

The flux is now given by the relation between $v_s$ and $\Delta x_s$. If all the vehicles are travelling at constant speed, then the balance of forces on every vehicle is zero:

$$\forall i \in [1, N], \; F^d_i + F^v_{(i+1)\rightarrow i} = 0$$  \hspace{1cm} (2.36)

To simplify the notations we rewrite this expression in the form:

$$F^d(v) + F^v(v, \Delta x) = F(v, \Delta x) = 0$$  \hspace{1cm} (2.37)

Let $S$ be the range of solutions to this equation:

$$S = \{(\Delta x, v) \in (\mathbb{R}^+ \times \mathbb{R}^+) | F(v, \Delta x) = 0\}$$  \hspace{1cm} (2.38)

This ensemble represents all the admissible couples of inter-vehicular spacing and velocity.
for which the system admits an equilibrium solution. We can restrict this ensemble by only considering the one that verify some elementary driver behaviors concerning the reaction to a perturbation. If the vehicle's velocity changes by an increment $\delta v$, the forces should have the sign of the velocity change, in order to return to the equilibrium line. Similarly, if the spacing changes by an increment $\delta x$, the sign of the forces should be such that the vehicle returns to equilibrium. In other words, the equilibrium state has to be locally stable. We will denote by $\mathcal{E}$ the ensemble of locally stable equilibrium solutions, which is defined as:

$$\mathcal{E} = \{(\Delta x, v) \in \mathcal{S}, F(v + \delta v, \Delta x).\delta v < 0 \text{ and } F(v, \Delta x + \delta x).\delta x > 0\}$$ (2.39)

or equivalently, using convexity:

$$\mathcal{E} = \left\{ (\Delta x, v) \in \mathcal{S}, \frac{\partial^2 F(v, \Delta x)}{\partial v^2} > 0 \text{ and } \frac{\partial^2 F(v, \Delta x)}{\partial \Delta x^2} > 0 \right\}$$ (2.40)

Now, rewrite the equilibrium line as a function of density

$$\left(\frac{1}{\rho}, v_s\right) \in \mathcal{E} \iff G(v_s, \rho) = F\left(v_s, \frac{1}{\rho}\right) = 0$$ (2.41)

and assume that we can solve this expression to obtain the density as a function of velocity on the equilibrium line

$$\left(\frac{1}{\rho}, v_s\right) \in \mathcal{E} \iff \rho = f(v_s), v_s = g(\rho)$$ (2.42)

where $f$ is a bijection from the space of admissible velocities $[0, V_{lim}]$ to the space of admissible densities $[0, \rho_{max}]$, and $g$ its inverse function. $f$ is therefore a strictly monotonic function, and it is quite intuitive to consider that the equilibrium traveling velocity of the vehicles will decrease when the density of vehicles is increased.

$$\frac{dg}{d\rho}(\rho) \leq 0$$ (2.43)

With this in mind, we can express the average flux in the equilibrium state as

$$\Phi = vf(v) = \rho g(\rho)$$ (2.44)
By deriving this expression with respect to the density, we straightforwardly obtain:

\[
\frac{d\Phi}{d\rho}(\rho) = \rho \frac{dg}{d\rho}(\rho) + g(\rho) \tag{2.45}
\]

Furthermore, Eq. (2.43) implies the existence of a critical density \(\rho_c\) for which,

\[- \frac{dg}{d\rho}(\rho_c) = \frac{g(\rho_c)}{\rho_c} \iff \frac{d\Phi}{d\rho}(\rho_c) = 0 \tag{2.46}\]

These considerations are consistent with the experimental observations as the one depicted in Fig. 2-3. Indeed, we just showed that the balance of forces arising from any force-based model can be translated into an equilibrium line. This equilibrium line corresponds to what is commonly called the fundamental diagram. If the forces are asymmetric and depend on the velocity of the vehicles, there can be an extrema for the steady-state flux, when expressed as a function of the density.

**Occupancy**

It is important to note that everything can be derived equivalently from the density or the occupation \(O\), which is the space fraction of the road occupied by vehicles:

\[O = \frac{N\Lambda}{L} = \Lambda \rho \tag{2.47}\]

with \(\Lambda\) the length of a vehicle. The advantage of using occupancy instead of density is in the simplicity of its graphical presentation: occupancy varies from zero to one, which is not the case for density. However, the notations are simpler with the density and for this reason it will be preferably used in the calculations, but both terms can be used interchangeably during the discussion.

**2.2.2 Linear Stability Analysis**

In the previous section we have studied the equilibrium line for any force-based model and mentioned that, in order to have a physical meaning, the line had to be locally stable, i.e. convex around the equilibrium couples for both velocity \(v\) and spacing \(\Delta x\). It is, however, possible to find a range of instabilities on this equilibrium line. In order for a model to reproduce phenomena such as traffic jam, it is crucial that it presents a range of
instabilities. This idea is similar to the one employed for pattern formation in chemical [63] or mechanical [59] systems. In our case, the patterns correspond to traffic jams, considered as an out-of-equilibrium state of the system. We focus on two types of linear instability, namely local stability and string stability. As described in Ref. [62], the easiest way to consider linear stability analysis is by focusing on a single vehicle and by studying its behavior when displaced from equilibrium. We therefore consider a system of length $L$, $N$ vehicles, periodic boundary conditions and in the steady state, that is:

$$
\forall m \in [1, N] \; (v_m, \Delta x_{mn}) = (v_s, \Delta x_s) \in E ; \; \Delta v_{mn} = 0
$$

(2.48)

with $n = m - 1$. The periodic boundary conditions imply that $(v_0, x_0) = (v_N, x_N)$. For every vehicle, the accelerations verify:

$$
a(\Delta x_s, v_s, 0) = 0
$$

(2.49)

where $a$ is the behavioral acceleration (we drop the mass term to simplify the equations). We consider small perturbations around the equilibrium state

$$
\begin{cases}
\Delta x_{mn} = \Delta x_s + \delta_{mn} \\
v_n = v_s + \nu_n \\
\Delta v_{mn} = \nu_m - \nu_n
\end{cases}
$$

(2.50)

### 2.2.3 Local Stability Analysis

For the local stability analysis, we only perturb velocity and positions of a specific vehicle $n$, and assume that all the other vehicles remain on the equilibrium line, as depicted in Fig. 2-4 (left panel); hence:

$$
\begin{cases}
\Delta x_{mn} = \Delta x_s + \delta_{mn} \\
v_m = v_s + \nu_m \\
\Delta v_{mn} = \nu_m
\end{cases}
$$

(2.51)

We will drop the indices for simplification, and linearly expand the acceleration around these perturbations

$$
a(\Delta x, v, -\nu) = (\partial_{\Delta x} a)\delta + (\partial_v a - \partial_{\Delta x} a)\nu
$$

(2.52)
Figure 2-4: Local stability (left panel): the system is initialized in the steady state, where all vehicles are separated by the same distance $\Delta x_s$, and travel at the same velocity $v_s$ (top image). We then perturb the system by only changing the velocity and position of one vehicle by a small increment, respectively $\nu$ and $\delta$ and study the evolution of the time response. String stability (right panel): the system is also initialized in the steady state but we perturb the system by changing the velocity of the vehicles $m$ and $n$ by a small increment $\nu_m$ and $\nu_n$ as well as the spacing by an increment $\delta$ (bottom image).

With the notation $\partial_\alpha = \frac{d}{d\alpha}$. We obtain two differential equations:

\[
\begin{align*}
\partial_t \delta &= -\nu \\
\partial_t \nu &= a(\delta x, v, -\nu)
\end{align*}
\] (2.53)

That can usefully be combined into a single one

\[
\ddot{\nu} - (\partial_x a - \partial_\nu a)\dot{\nu} + (\partial_{\Delta x} a)\nu = 0
\] (2.54)

or equivalently:

\[
\ddot{\nu} - b\dot{\nu} + cv = 0
\] (2.55)
with $b = (\partial_v a - \partial_{\Delta v} a)$ and $c = (\partial_{\Delta v} a)$. The solution $\nu(t)$ is of the exponential form and is only stable if:

$$\begin{aligned}
\Delta &= b^2 - 4c < 0 \\
\Re(\lambda \pm) &< 0 \\
\text{or} \\
\Delta &\geq 0 \\
\lambda^+ &> 0
\end{aligned} \quad (2.56)$$

with $\lambda \pm$ the solutions of the quadratic equation

$$x^2 - bx + c = 0 \quad (2.57)$$

This definition of stability can be compared with the simple phenomena of growth [63]. If a growing rate is proportional to its size, it satisfies the simple differential equation:

$$\dot{y} = \frac{y}{\tau} \quad (2.58)$$

with $\tau$ a relaxation time. The solution is of the form

$$y(t) = e^{\frac{t}{\tau}} \quad (2.59)$$

and increases exponentially with time, until it is damped by another phenomenon, such as lack of resources. In our case, by definition

$$\begin{aligned}
\Delta > 0 &\implies \Re(\lambda) = \frac{b}{2} \\
\Delta \geq 0 &\implies 2\lambda^+ = b + \sqrt{\Delta}
\end{aligned} \quad (2.60)$$

and the stability condition reads:

$$\partial_{\Delta v} a > \partial_v a \quad (2.61)$$

### 2.2.4 String Stability Analysis

A string stability analysis consists of initializing the system in the steady state, as in the local stability analysis; but we now perturb both a specific vehicle and the vehicle in front, as depicted in Fig. 2-4 (right panel). The perturbation equations are given by Eqn. (2.50).
The acceleration of vehicle \(m\) verifies:

\[
a(\Delta x_{mn}, v_m, \nu_n - \nu_m) = (\partial_{\Delta x} a) \delta_{mn} + (\partial_v a - \partial_{\Delta v} a) \nu_m + \nu_n \partial_{\Delta v} a
\]  

(2.62)

The perturbation increments is expressed as:

\[
\begin{cases}
\partial_t \delta_m = \nu_n - \nu_m \\
\partial_t \nu_m = a(\Delta x_{mn}, v_m, \nu_n - \nu_m)
\end{cases}
\]  

(2.63)

In order to find a solution to the time evolution of the system, we will use a Fourier Ansatz [62] [63]:

\[
\begin{pmatrix}
\delta_m \\
\nu_m
\end{pmatrix} = \begin{pmatrix}
\dot{\delta} \\
\dot{\nu}
\end{pmatrix} \exp(\lambda t + i k)
\]  

(2.64)

with \(k\) the dimensionless wave number, and \(\lambda(k) = \sigma(k) + i(k)\) the complex growth rate.

The solution will be stable in time if and only if:

\[
\forall k \in \mathbb{R}, \quad \sigma(k) \leq 0
\]  

(2.65)

Indeed, if there exists one wave number for which the exponential growth has a positive real part, the amplitude of the response associated with this wave number will grow indefinitely, and will eventually become dominant. It is again the same phenomenon as what happens in the formation of Turing patterns [63]. Let us now find the conditions of string stability.

For every wavelength, the Fourier Ansatz verifies:

\[
\begin{pmatrix}
\lambda & 1 - e^{ik} \\
-a_{\Delta x} & \{\lambda - a_v + a_{\Delta v}(1 - e^{ik})\}
\end{pmatrix}
\begin{pmatrix}
\dot{\delta} \\
\dot{\nu}
\end{pmatrix} = M
\begin{pmatrix}
\dot{\delta} \\
\dot{\nu}
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]  

(2.66)

where we lightened the notations by writing \(\partial_{\alpha} a = a_\alpha\). We only have non trivial solutions if,

\[
\text{det}(M) = 0 \iff \lambda^2(k) + p(k)\lambda + q(k) = 0
\]  

(2.67)

with the notations:

\[
\begin{cases}
p(k) = -a_v + a_{\Delta v}(1 - e^{-ik}) \\
q(k) = a_{\Delta x}(1 - e^{-ik})
\end{cases}
\]  

(2.68)
Following assumptions made in Ref. [62], we consider that the first instability arises for small wave numbers (or long wavelengths). We place ourselves in the case \( k \to 0 \). In this particular case,

\[
\begin{align*}
\left\{ \begin{array}{l}
p(k) = p_0 + p_1 k + \mathcal{O}(k^2) \\
q(k) = q_1 k + q_2 k^2 + \mathcal{O}(k^3)
\end{array} \right.
\quad (2.69)
\]

with

\[
\begin{align*}
p_0 &= -a_v ;
p_1 = i a_{\Delta x} \\
q_1 &= i a_{\Delta x} ;
q_2 = \frac{a_{\Delta x}^2}{2}
\end{align*}
\quad (2.70)
\]

The solution of Eq. (2.67) reads:

\[
2 \lambda \pm = -p(k) \pm \sqrt{p(k)^2 - 4q(k)} = -p(k) \left( 1 \mp \sqrt{1 - \frac{4q(k)}{p(k)^2}} \right)
\quad (2.71)
\]

Since we need \( \Re(\lambda \pm) \) to be negative, we will only focus on \( \lambda^+ \), since the negativity of \( \Re(\lambda^+) \) will ensure the one of \( \Re(\lambda^-) \). Expanding \( \lambda^+ \) around small values of \( k \), we can prove (see Appendix D for more details) that:

\[
\lambda^+ = i \frac{a_{\Delta x}}{a_v} k + \left( \frac{a_v}{2a_{\Delta x}} - \left( \frac{a_{\Delta x}}{a_v} + \frac{1}{a_v} \right) \right) \left( \frac{a_{\Delta x}}{a_v} \right)^2 k^2 + \mathcal{O}(k^3)
\quad (2.72)
\]

The stability condition \( \Re(\lambda^+) < 0 \) thus entails:

\[
\Re(\lambda^+) < 0 \iff \frac{a_v}{2a_{\Delta x}} - \left( \frac{a_{\Delta x}}{a_v} + \frac{1}{a_v} \right) < 0
\quad (2.73)
\]

### 2.2.5 Convective Stability

The last stability analysis investigates the convective stability, for which we have the same perturbation as in the string stability analysis. We thus consider the same Fourier Ansatz as in Eq. (2.64), but we do not make the small \( k \) approximation. Instead, we decide to express the instability conditions in terms of \( Y = 1 - \cos(k) \in [0 ; 2] \). We obtain several inequalities and the existence of a value of \( k \) that enables \( Y \) to verify these inequalities. It ensures that, after a sufficient long time, instabilities will arise and the dominant mode will be the one that is associated with the highest real value of \( \lambda(k) \). As shown in Appendix
D.2, instabilities arise in the system under specific condition:

\[
    \Delta_r > 0 \text{ and } (2 - Y)(a_{\Delta x}^2 - p_r(Y)a_{\Delta x}a_{\Delta v}) - p_r(Y)^2a_{\Delta x} > 0
\]

\[
    \Delta_r < 0 \text{ and } (-a_v + a_{\Delta v}Y)^2 \left\{ 2(a_{\Delta v}^2(2 - Y) + a_{\Delta x} + a_v a_{\Delta v})Y - a_v^2 \right\} + 2a_{\Delta x}Y(2 - Y) \{a_{\Delta x} + a_{\Delta v}(a_v - a_{\Delta v}Y)\} > 0
\]

\[
    \Delta_r < 0 \text{ and } 3p_r^2 - p_i^2 - 4q_r < 0
\]

with

\[
\begin{align*}
    p_r &= -a_v + a_{\Delta v}Y \\
p_i &= a_{\Delta v}\sqrt{Y(2 - Y)} \\
q_r &= a_{\Delta x}Y \\
q_i &= a_{\Delta x}\sqrt{Y(2 - Y)}
\end{align*}
\]  

(2.75)

Now that we established equations to study the linear stability of force-based traffic models, we focus on some common models found in the open literature.

2.3 Application: Fundamental Diagram and Stability Analysis of Some Common Agent-Based Models

The different models we consider can be found in the literature and all obey the following characteristics, in order to locally reproduce the vehicle response to a small perturbation from the equilibrium state:

\[
\begin{align*}
    a_{\Delta x} &\geq 0 \\
a_v &\leq 0 \\
a_{\Delta v} &\geq 0
\end{align*}
\]  

(2.76)

For every model studied hereafter, we start by finding the equilibrium line, and then look at the first order derivatives around this line and use the calculations presented in Appendix D to determine the instability ranges, if they exist.

2.3.1 Follow-the-Leader Models

Follow-the-leader models [50][29][20] are probably among the most simple traffic models in the literature. The underlying assumption of these models is that every vehicle tends to reproduce the behavior of the vehicle just in front of it, and especially match its speed. In the simplest case, the acceleration function of a vehicle is proportional to the velocity
difference between this vehicle and the one preceding it, that is:

\[ a(\Delta x, v, \Delta v) = \frac{\Delta v}{\tau} \]  

(2.77)

with \( \tau \) a relaxation time. This very simple model cannot give us a fundamental diagram since the acceleration does not depend on density. The local stability analysis is straightforward and gives us:

\[ a_{\Delta v} = \frac{1}{\tau} > a_v = 0 \]  

(2.78)

proving that the system is locally stable. Concerning the string stability, the condition expressed in Eq. (2.67) becomes here:

\[ \lambda \left( \lambda + \frac{1 - e^{-ik}}{\tau} \right) = 0 \]  

(2.79)

and the real part of the solutions are:

\[ \left\{ \begin{array}{l}
\sigma_+ = 0 \\
\sigma_- = \frac{\cos(k)-1}{\tau} \leq 0
\end{array} \right. \]  

(2.80)

The equilibrium state is linearly stable. An extension has been made by Denos et al. (Refs. [20] and [21]) by adding a density dependence as follows:

\[ a(\Delta x, v, \Delta v) = \kappa \frac{\Delta v}{\Delta x} \]  

(2.81)

Here again, since the acceleration is independent of the vehicle’s own acceleration and solely defined by the difference with the preceding vehicle’s velocity, the equilibrium line cannot be reached. In return, the stability analysis gives:

\[ \left\{ \begin{array}{l}
a_{\Delta x} = 0 \\
a_{\Delta v} = \frac{\kappa}{\Delta x_{eq}}
\end{array} \right. \]  

(2.82)

and we obtain the same stability results as previously. As a conclusion, simple follow-the-leader models as just depicted, are not able to reproduce non linear phenomena such as traffic jams.
2.3.2 Optimal Velocity Models

Another class of commonly used models are those in which every vehicle tends to reach an optimal velocity [2] [1] [3]. They are classically written in the form:

\[ a(\Delta x, v, \Delta v) = \frac{V_{\text{opt}}(\Delta x) - v}{\tau} \]  

(2.83)

with \( V_{\text{opt}} \) the velocity that the vehicles want to reach. This velocity reduces when the spacing is decreased:

\[ V_{\text{opt}}' = \frac{dV_{\text{opt}}}{d\Delta x}(\Delta x) \geq 0 \]  

(2.84)

The equilibrium is here given by the equation:

\[ v_s = V_{\text{opt}}(\Delta x) \]  

(2.85)

The fundamental diagram reads here:

\[ \Phi = \rho V_{\text{opt}} \left( \frac{1}{\rho} \right) \]  

(2.86)

The useful quantities for the stability analysis are

\[
\begin{align*}
    a_{\Delta x} & = \frac{V_{\text{opt}}'(\Delta x)}{\tau} \\
    a_v & = -\frac{1}{\tau} \\
    a_{\Delta v} & = 0
\end{align*}
\]  

(2.87)

There exists many ways to characterize the optimal velocity function, but we will focus on the one depicted in Refs. [48] and [49], which is defined in the following way:

\[ V_{\text{opt}}(\Delta x) = \begin{cases} 
0 & \text{for } \Delta x \in [0 \ ; \ \Delta x_A[ \\
\Delta x & \text{for } \Delta x \in [\Delta x_A ; \ \Delta x_B] \\
V_{\lim} & \text{for } \Delta x > \Delta x_B 
\end{cases} \]  

(2.88)

The derivative of Eq. (2.88) reads:

\[ V_{\text{opt}}'(\Delta x) = \begin{cases} 
0 & \text{for } \Delta x \in [0 \ ; \ \Delta x_A[ \Delta x_B ; \ \infty[ \\
f & \text{for } \Delta x \in [\Delta x_A ; \ \Delta x_B] 
\end{cases} \]  

(2.89)
Figure 2-5: Fundamental diagram for the optimal velocity model defined by Eq. (2.88). It presents a discontinuity and does not correspond to realistic expectations, for which the flux should continuously decrease with the density after a certain point.

with $f = \frac{V_{\text{lim}}}{\Delta t_B}$ to ensure continuity. The fundamental diagram is piecewise defined as

$$
\Phi(\rho) = \begin{cases} 
0 & \text{for } \rho \in (\rho_A; +\infty) \\
f & \text{for } \rho \in (\rho_B; \rho_A] \\
\rho V_{\text{lim}} & \text{for } \rho \in [0; \rho_B[ 
\end{cases} 
$$

(2.90)

with $\rho_\alpha = \frac{1}{\Delta x_A}$. As depicted in Fig. 2-5, the flux presents a discontinuity at the density $\rho_A$. It increases linearly with the density as is expected in the free flow regime, until a critical density $\rho_B$ is reached. It is then constant and equal to $f$ up to $\rho_A$, where it drops to zero. Concerning the stability analysis, it is easy to see from Eq. (2.87) that the local stability is always attained. The definition of the optimal velocity and its derivative Eq. (2.89) shows that the instability will be defined by segment, since $V_{\text{opt}}'$ is defined as a constant function by part. Using the development of Section 2.3.2, the string stability condition described by
Eqn. (2.73) reads here:

\[ R(A_t) < 0 \iff \frac{-1}{2V_{opt}'(\Delta x)} + \tau < 0 \iff V_{opt}'(\Delta x) < \frac{2}{\tau} \]  \hspace{1cm} (2.91)

Hence, in the three domains, this translates as:

\[
\begin{cases}
R(\lambda^+) < 0 \text{ for } \Delta x \in [0; \Delta x_A[ \cup ] \Delta x_B; +\infty[ \\
R(\lambda^+) > 0 \iff \tau f > 2 \text{ for } \Delta x \in [\Delta x_A; \Delta x_B]
\end{cases}
\]  \hspace{1cm} (2.92)

The system is only unstable in the domain \([\rho_b; \rho_a]\). As a consequence, even though this part of the fundamental diagram is not realistic, this branch still presents linear instabilities, and can therefore reproduce traffic jams. Let us now have a look at the convective instability, using the notations defined in Appendix D.3. The key parameter is

\[ \xi = \frac{a_x^2}{a_{\Delta x}} = \begin{cases} +\infty \text{ for } \Delta x \in [0; \Delta x_A[ \cup ] \Delta x_B; +\infty[ \\
\frac{1}{\tau f} \text{ for } \Delta x \in [\Delta x_A; \Delta x_B]
\end{cases} \]  \hspace{1cm} (2.93)

The instability conditions developed in Appendix D.3.3 increase the range of instability to

\[
\begin{cases}
\text{Convectively stable for } \Delta x \in [0; \Delta x_A[ \cup ] \Delta x_B; +\infty[ \\
\text{Convectively unstable } \iff \tau f > \frac{1}{2} \text{ for } \Delta x \in [\Delta x_A; \Delta x_B]
\end{cases}
\]  \hspace{1cm} (2.94)

which is consistent with the string stability analysis. In a nutshell, this optimal velocity model is unable to give appropriate mean quantities such as the fundamental diagram, but it reproduces non linear phenomena such as traffic jam due to the presence of instabilities for a range of densities.

### 2.3.3 Helbing’s Force-Based Model

Another interesting model is the one developed by Helbing et al. [27]. It is a generalization of optimal velocity models, but it adds a dependence on the velocity difference between a vehicle and its preceding one, and uses less fitting parameters than most optimal velocity models. Instead of considering an optimal velocity, this model is based on the assumption that there exists an optimal distance between two vehicles, and that this distance depends on the reaction time of the driver and the velocity at which he travels. It is in fact what is being taught to driver apprentices: in order to avoid collisions one should keep a safe distance with
the preceding vehicle, taking into account one’s reaction time, speed and braking distance. This optimal distance is defined as

\[ l_{opt}(v) = d + Tv \]  

(2.95)

where \( T \) is a combination of the reaction time and the breaking time; \( d \) is the minimal possible distance between two vehicles (depending on their dimensions); and \( v \) is the travelling velocity. These parameters can be different for every vehicle. The acceleration of a vehicle is then defined as the sum of a driving force and an interaction force. The driving force is defined as in Eq. (2.8) and the interaction force between two vehicles \( i \) and \( j \) reads as:

\[
\frac{F_{j \rightarrow i}^{v}}{m_i} = \begin{cases} 
-\frac{V_{lim}}{\tau_a} e^{-\frac{\Delta R^{i,j}_a}{R_a}} - \frac{\Delta v^{i,j}}{\tau_d} e^{-\frac{\Delta R^{i,j}_d}{R_d}} & \text{if } \Delta v_{i,j} > 0 \\
-\frac{V_{lim}}{\tau_a} e^{-\frac{\Delta R^{i,j}_a}{R_a}} & \text{if } \Delta v^{i,j} \leq 0 
\end{cases}
\]  

(2.96)

The interaction is defined piecewise and can be separated into an acceleration part and a deceleration part. The deceleration is only non zero when the vehicles get closer to each other, i.e. \( \Delta v \geq 0 \). The parameters \( R_a, \tau_a, R_d \) and \( \tau_d \) are length and time constants, for the acceleration and deceleration parts respectively, and \( \tau_a \) is the same constant as the one defined for the driving force (see Eq. (2.8)). They have to be fitted in order to reproduce realistic values. The parameter \( \Delta R^{i,j} \) is defined as the difference between the vehicle-vehicle distance and the optimal distance between two vehicles

\[
\Delta R^{i,j} = \Delta x^{i,j} - l_{opt}(v^i) 
\]  

(2.97)

In [27] the following set of parameters have been fitted by minimizing the difference with field data:

- \( V_{lim} = 16.98 \) m/s
- \( \tau_a = 2.45 \) s
- \( d = 1.38 \) m
- \( T = 0.74 \) s
- \( \tau_d = 0.77 \) s
- \( R_a = 5.59 \) m

- \( R_d = 98.78 \) m

Assuming that every vehicle only interacts with the one in front of it, the addition of the behavioral forces gives the total acceleration function:

\[
a(\Delta x, v, \Delta v) = \begin{cases} \\
\frac{1}{\tau_a} \left( V_{lim} \left( 1 - e^{-\frac{\Delta R}{R_a}} \right) - v \right) - \frac{\Delta v}{R_d} e^{-\frac{\Delta R}{R_d}} & \text{if } \Delta v > 0 \\
\frac{1}{\tau_a} \left( V_{lim} \left( 1 - e^{-\frac{\Delta R}{R_a}} \right) - v \right) & \text{if } \Delta v \leq 0 
\end{cases}
\]  

with

\[
\Delta R = \Delta x - l_{opt}(v)
\]

We can thus derive the equilibrium curve:

\[
\begin{align*}
a(\Delta x, v, 0) &= 0 \iff v = V_{lim} \left( 1 - e^{-\frac{\Delta R}{R_a}} \right) \\
a(\Delta x, v, 0) &= 0 \iff e^{-\frac{\Delta R}{R_a}} = 1 - \frac{v}{V_{lim}} \\
a(\Delta x, v, 0) &= 0 \iff \Delta R = -R_a \ln \left( 1 - \frac{v}{V_{lim}} \right) \\
a(\Delta x, v, 0) &= 0 \iff \Delta x = d + Tv - R_a \ln \left( 1 - \frac{v}{V_{lim}} \right)
\end{align*}
\]

We therefore have the equilibrium line equation

\[
\Delta x_{eq} = f(v_{eq})d + Tv_{eq} - R_a \ln \left( 1 - \frac{v_{eq}}{V_{lim}} \right)
\]

It is straightforward to notice from Eq. (2.101) that at equilibrium, the vehicles are equally spaced by a distance that is in fact larger than the prescribed optimal distance by a quantity 
\(-R_a \ln \left( 1 - \frac{v_{eq}}{V_{lim}} \right)\) which vanishes as \(v_{eq}\) goes to zero and blows up when the vehicles travel at the speed limit of the road. The singularity at the speed limit ensures that the vehicles do not travel faster than this limit. If we derive the equilibrium line function \(f(v)\), one obtains:

\[
f'(v) = T + \frac{R_a}{V_{lim} - v_{eq}} > 0 \quad \forall v \in [0, V_{lim}]
\]

which proves that \(f\) is strictly monotonic, and therefore invertible. This can be visually understood as shown in Fig. 2-6, where the equilibrium density is plotted as a function of the equilibrium velocity, using the fitted parameters from Section 2.3.3. Defining the inverse
Equilibrium line for the Hebing force based model

Figure 2-6: Equilibrium line between the density and the velocity of the vehicles for the Helbing force-based model with fitted parameters. This line is monotonic, the velocity tends to the speed limit of the road at low density, and to zero at high density. This behavior is coherent with heuristic considerations of traffic behavior.

function,
\[ v_{eq} = g(\Delta x_{eq}) = f^{-1}(f(v_{eq})) \]  

(2.103)

We can obtain the fundamental diagram, in the form
\[ \Phi = \rho g \left( \frac{1}{\rho} \right) \]  

(2.104)

As shown in Fig. 2-7, the fundamental diagram presents the main characteristics of traffic. From \( \rho = 0 \) to a critical density, the flux increases linearly with \( \rho \), and decreases afterwards.
Figure 2-7: Fundamental diagram for the Helbing force-based model, using the fitted parameters. It presents the main characteristics that are expected from heuristic considerations: a linear increase of the flux with the density until a critical density is reached, after which the flux decreases with the increase of density. The non unicity of the flux as a function of density translates into the existence of two regimes: a free flow regime which corresponds to the left part of the diagram, and a congested regime where the vehicle interactions become dominant.

Considering the stability analysis of this model, using Eq. (2.98) we can derive:

\[
\begin{align*}
    a_{\Delta x} &= \frac{v_0}{R_a \tau_a} \left( 1 - \frac{v_a}{v_0} \right) \\
    a_v &= -\frac{1}{\tau_a} \left( 1 + \frac{v_0 T}{R_a} \left( 1 - \frac{v_a}{v_0} \right) \right) \\
    a_{\Delta v} &= \frac{1}{\tau_d} \left( 1 - \frac{v_a}{v_0} \right)
\end{align*}
\] (2.105)
Introducing the dimensionless numbers

\[
\begin{align*}
\alpha_e &= 1 - \frac{v_{eq}}{v_{lim}} \in [0 ; 1] \\
\bar{R} &= \frac{R_a}{R_d} > 0 \\
\bar{T} &= \frac{T_v}{R_v} \geq 0 \\
\Pi &= \frac{V_{lim}T}{R_a} \geq 0 \\
\bar{\tau} &= \frac{\tau_a}{\tau_d} \geq 0
\end{align*}
\]

Eq. (2.105) becomes:

\[
\begin{align*}
a_{Ax} &= \frac{1}{\tau_d}\Pi\bar{T}\alpha_e \\
\bar{a}_v &= -\frac{1}{\tau_a} (1 + \Pi\alpha_e) \\
\bar{a}_{\Delta v} &= \frac{1}{\tau_a}\bar{\tau}\alpha_e
\end{align*}
\]

Now, the local stability condition from Eq. (2.61) is equivalent to

\[
\frac{1}{\tau_d}\bar{\tau}\alpha_e > -\frac{1}{\tau_a} (1 + \Pi\alpha_e)
\]

and is always verified. This model is therefore locally stable everywhere. Concerning the string stability, the stability condition Eq.(2.73) becomes

\[
\Re \lambda_+ < 0 \iff F(\alpha_e) = (1 + \Pi\alpha_e) \left(\frac{1}{2} + \Pi\alpha_e + \bar{\tau}(\alpha_e)\bar{R}\right) - \Pi\bar{T}\alpha_e > 0
\]

As shown in Fig. 2-8 (left panel), the fitted parameters given in Ref. [27] result in a system that is always string stable. This is in contradiction with the claims made in [27] that the model with the fitted parameters can reproduce traffic jams and out-of-equilibrium phenomena. We have investigated this observation by simulations in the following way. We notice that for a particular choice of parameters, namely:

\[
\begin{align*}
\bar{R} &= \frac{R_a}{R_d} = 1 \\
\bar{T} &= \frac{T_v}{R_v} = 5 \\
\Pi &= \frac{V_{lim}T}{R_a} = 1 \\
\bar{\tau} &= \frac{\tau_a}{\tau_d} = 1
\end{align*}
\]

Eq. (2.109) presents an instability, as shown in Fig. 2-8 (right panel). We then simulate the Helbing force-based model for the two sets of parameters presented in Section 2.3.3 and given
Figure 2-8: String stability function for the fitted Helbing-based model (see Section 2.3.3) (left panel) and a set that presents an instability (see Eqn. (2.110)) (right panel). The positivity of the string stability function is the criteria that determines the stability ranges.

by Eq. (2.110) on a straight line with periodic boundary conditions. The system in each case is initialized in the equilibrium state and a small perturbation is applied for a single time step to one vehicle. For the fitted parameters of Section 2.3.3 the perturbation always vanished, no matter what the imposed density was (see Fig. 2-9, left panel). However, for the model with the parameters fitted to present an instability, after a certain density the perturbation propagates and creates waves in the temporal evolution of the vehicles position as shown in Fig. 2-9 (right panel). After conducting a convective stability analysis, we found the same results as the string stability, namely a perfect stability for the fitted set of parameters of Section 2.3.3 and a range of instability for the unstable set given by Eq. (2.110). Simulations for a long road (100 km, to avoid finite size effects) with periodic boundary conditions over a large time ($10^5$ seconds) for many different densities are in very good agreement with the analytical stability results, as shown in Fig. 2-10. For the stable set of parameters of Section 2.3.3 (left panel), the fundamental diagram matches the equilibrium line and all the vehicles travel at the same velocity, irrespective of the chosen density, as shown in figure 2-11 (left panel). The unstable parameter set Eq. (2.110) presents a critical density, below which the fundamental diagram matches the equilibrium line and all the vehicles travel at the same
Figure 2-9: Time evolution of vehicle positions for the fitted set of parameters at an arbitrary density (left panel) and for the unstable set of parameters at a density higher than the critical one (right panel). Every line corresponds to a single vehicle, its position is given by the Y axis while the time evolution is represented on the X axis. At time $t = 0$ the system is initialized in the equilibrium state, and a perturbation is applied to the vehicle positioned at the origin. The perturbation propagates and grows in time until it reaches a steady state with a traffic jam pattern in the unstable set (right panel) or remains stable for the fitted set of parameters (left panel).

velocity. Above this critical density the fundamental diagram deviates from the equilibrium line, the velocity probability density function is not peaked around a single value, and the probability to find a stopped vehicle is strictly greater than zero (Fig 2-11, right panel). This result highlights not only that simulation and linear stability theory are in extremely good agreement, but also that it is not technically possible to simulate traffic jams with the fitted set of parameters of Section 2.3.3, unless there exists a way to initialize the system in a steady state with jams; but this is highly unlikely. However, since there exists a range of parameters which allows the presence of jams as instabilities, one can fit this model within this range, if a sufficient amount of field data are available to do so.
Figure 2-10: Fundamental diagrams of the Helbing model, for the stable set of parameters of Section 2.3.3 (left panel) and the unstable set Eq. (2.110) (right panel). In red is the stable part of the diagram according to linear stability analysis and in green the unstable one. The black crosses are the averaged flux over all vehicles and every time step for a simulation of a road of $10^5$ meters with periodic boundary conditions, for a simulation time of $10^5$ seconds. The deviation of the simulation from the equilibrium curve shows the appearance of jams patterns.

2.4 Chapter Summary

We presented in this Chapter the mathematical framework and the necessary tools to study the fundamental diagram and the stability of typical force-based models. We applied these tools to some popular models found in the literature, we showed that none of them were capable of reproducing simultaneously the basic traffic behaviors at the microscopic and macroscopic scales. Specifically, the studied optimal velocity models studied exhibit linear instabilities which correspond to a correct heuristic description of the microscopic behavior of traffic at high densities. But these models fail at giving a proper macroscopic description of the system in form of the fundamental diagram. The Helbing force-based model gives the opposite result: its fundamental diagram is in agreement with observations of traffic behavior but the fitting given in Ref. [27] fails at reproducing instability conditions at any traffic densities. It is, however, possible to find a set of parameters for which instabilities are present, as well as a proper fundamental diagram; but the high number of fitting parameters
Figure 2-11: Velocity probability distribution function for the stable set of parameters proposed by Helbing et al. (see Section 2.3.3) (left panel) and the unstable set of parameters Eq. (2.110) (right panel). Every line is a different density, and the density increases from the left to the right. No matter what the density is, the system is always stable and the probability density function is peaked around the mean velocity for the fitted parameters. Concerning the unstable set of parameters, the probability density function is not peaked around the mean velocity, and the probability to find a vehicle at a velocity equal to zero is non null, which is a signature of the formation of traffic jams.

makes it practically unusable without a fitting through comparison with real traffic data.

There exists many more models in the literature, such as the one exposed in Ref. [64]. However, they do not necessarily give better results and the number of fitting parameters increases faster than the accuracy. For this reason we will now focus on another class of models, namely cellular-automa models, which contain very few fitting parameters and can nonetheless reproduce basic traffic phenomena at both the micro and macro scales.
Chapter 3

Velocity Distribution and Phase Transition in Cellular Automata Models

Following our investigation of force-based models in which we showed that they are not well suited for our study because they fail to reproduce both traffic jams and the fundamental diagram with a small number of fitting parameters, we focus now on the most popular cellular automata model: the Nagel Schreckenberg model [47] [58]. The cellular automata models use a discrete description of traffic, both in space and time, together with some heuristic considerations to reproduce generic traffic behavior. We will see that despite the small number of fitting parameters, these models are capable of reproducing basic traffic phenomena both at the micro and macro scales. We start by introducing the Nagel Schreckenberg model and its main parameters. We then focus on the nature of the phase transition from a velocity point of view, which to the best of our knowledge has not been accomplished in the literature. An in-depth study of the velocity distribution functions follows, with the description of an analytical three-body interaction model that allows us to link velocity and inter-vehicular distance (or headway) distributions. Eventually, and in order to have more precise velocity distributions, we propose a finer discretized version of the Nagel Schreckenberg model, and study the main characteristics of this model.
3.1 The Nagel - Schreckenberg (NS) Model

3.1.1 Definition of the Model and Macroscopic Quantities

Update rules

Introduced in Refs. [47] and [58], the NS model considers a road as a succession of cells, each of which contains at most one vehicle. The positions and velocities of every vehicle are updated at each time step according to the following rules:

1) Acceleration: \( v_n = \min(v_n + 1, v_{\text{max}}) \)
2) Deceleration: \( v_n = \min(v_n, d_n) \)
3) Randomization: \( v_n = \max(v_n - 1, 0) \) with probability \( p \)
4) Movement: \( x_n = x_n + v_n \)

where \( v_n \) and \( x_n \) stand for the velocity and position of the vehicle \( n \), whereas \( d_n \) is the distance between vehicle \( n \) and its preceding vehicle. Compared with force-based models, the model relies on very few parameters. These are:

\[
\begin{align*}
  v_{\text{max}} & : \text{speed limit;} \\
  p & : \text{stochasticity parameter;} \\
  L & : \text{road length;} \\
  T & : \text{simulation time;} \\
  \rho & : \text{vehicle density.}
\end{align*}
\]  

A main difference with the force-based models is the presence of a stochastic parameter \( p \). It represents the probability for a vehicle to randomly slow down at any moment and is what enables this model to reproduce various traffic patterns while remaining quite simple.

Fundamental Diagram and Instabilities

The specific case \( v_{\text{max}} = 1 \) reduces to the totally asymmetric simple exclusion process (TASEP) [57] [35] [12] and is exactly solvable. However, despite it’s apparent simplicity the NS model does not have an analytical solution for \( v_{\text{max}} > 1 \) except for limit cases. The study we here perform is therefore mostly simulation based, considering a single lane road with periodic boundary conditions. All the macro scale quantities such as the density \( \rho \)
Figure 3-1: Fundamental diagrams (averaged flux as a function of density) for the NS model for different values of the stochastic parameter $p$ and speed limit set to $v_{\text{max}} = 5$. Although the value of $p$ influences the shape of the diagram, it remains coherent with empirical observation. The flux increases linearly with the density until a critical density after which the flux decreases.

(number of vehicles per cell), the occupancy $\mathcal{O}$ (fraction of cells occupied by vehicles), the mean velocity $\langle v \rangle$ and the flow $\Phi = \rho \times \langle v \rangle$ (number of vehicles crossing a section per unit of time) are averaged over all vehicles and total simulation times (for precise definitions of the macroscopic quantities, see Section 2.2). The fundamental diagram so obtained presents, for any value of the parameter $p$, a linear growth with the density followed by a decrease of the flux at higher densities, as shown in Fig. 3-1. This macroscopic description is coherent with empirical observations of traffic. Concerning the microscopic behavior of the vehicles, simulations show that at high densities traffic jams form and dissolve, as shown in Fig. 3-2 (right panel). We therefore can quite easily recover empirical observations through model-based simulations, despite the small number of fitting parameters. It is interesting to note that if one turns the stochasticity off by setting $p = 0$, the system becomes deterministic.
Figure 3-2: Vehicles trajectories. Every line represents a vehicle, the y axis represents the position of the vehicles while the x axis the time. The right panel, where we can note the appearance and dissolution of traffic jams with time, as well as the jammed front travelling upstream, corresponds to a stochasticity of $p = 0.5$. In the left panel ($p = 0$) one can note absence of traffic jams and all the vehicles travelling at the same velocity and equally spaced. Both simulations are at a density higher than the critical one and a maximum velocity $v_{\text{max}} = 10$. This shows the importance of the stochastic parameter $p$ in the simulations of jams.

and we observe behaviors similar to the Helbing force-based model as obtained in Section 2.3.3. For this deterministic case an analytical solution can be found in Refs. [46] and [57]. In this particular case, an equilibrium state is reached in which all the vehicles travel at the same velocity and are equally spaced, independently of the vehicle density. Some vehicle trajectories obtained from simulations are depicted in Fig. 3-2 (left panel). The macro scale is therefore empirically correct but the micro scale fails at reproducing instability phenomena.

Considerations for Velocity Distributions

The main quantity we are interested in, in this thesis, is the velocity probability distribution function. It is a function that describes the probability for any vehicle to be found at a certain velocity on a road, depending on the density (for periodic boundary conditions). Practically, these functions can be obtained analytically through mean field theories for some
limiting cases such as \( p = 0 \) or \( v_{\text{max}} = 1 \). Unfortunately there are currently no analytical solutions for more general cases, and they must be accessed numerically.

To conduct numerical simulations, we choose to look at our system from a statistical physics point of view. Every vehicle is an element that can be defined at any time by its state \((v, d)\), where \( v \) is the vehicle's velocity and \( d \) is the distance between the vehicle and the one ahead (also named the headway). We name \( \mathcal{A} \) the ensemble of possible configurations attainable by all the vehicles

\[
\mathcal{A} = \{(v_1, d_1), \ldots (v_N, d_N)\} \in ([0, v_{\text{max}}] \times [0, L - N])^N
\]

where \( N \) is the number of vehicles on the road. \( \mathcal{A} \) is also called the phase space of the simulation.

Consider thus a single lane road with periodic boundary conditions, with a fixed number of vehicles, and apply the NS rules for a sufficiently long time so that the whole phase space of possibly attainable coordinates \((v, d)\) is explored. The occurrence of every velocity is averaged so that the probability distribution function \( P(v) \) verifies the properties

\[
\begin{align*}
    v &\in [0 ; v_{\text{max}}] ; P(v) \geq 0 \\
    \sum_{v=0}^{v_{\text{max}}} P(v) &\leq 1
\end{align*}
\]

Some typical probability distributions are shown in Fig. 3-3 for different densities. Using the NS model, the large size of cells (about the length of a vehicle) limits the value of the speed limit to \( v_{\text{max}} = 8 \) cells per initialization step. Indeed, if one assumes that a vehicle's length is about four meters, one has a cell size of \( \Delta x = 4 \) m. Using a time step \( \Delta t \) of one second (about the average reaction time of a driver), and considering the speed limit of the road to be \( V_{\text{lim}} = 120 \text{ km/h} \approx 32 \text{ m/s} \) gives \( v_{\text{max}} = \frac{V_{\text{lim}}}{\Delta x / \Delta t} = 8 \) cells. This gives us only 8 possible velocity values and is not sufficiently refined. For this reason we will extend the NS model by a finer discretization of the velocity in the next section.

In this section we study the original NS model from the angle of the velocity distribution. This will enable us to define an intuitive order parameter that characterizes the phase transition, and to observe a diverging correlation length close to the critical density. We will make a parallel between the phase transition and activated processes such as nucleation by looking at the time needed for the system to transition from a state to the other. This

63
paraxial is not explained by the probability of first interaction in the free flow state. We then use a simple three-body interaction that we solve analytically to link the headway (distance separating two successive vehicles) distribution with the velocity distribution function. These concepts belong to the domain of statistical physics and will be briefly introduced later on.

3.2 Phase Transition and Order Parameter

These were the general considerations for the NS model. We now chose to study the effect of density change on velocity distribution, keeping all the other parameters fixed. This comes to investigate the effect of the number of vehicles on the road for a set of known parameters for some specific traffic conditions. This set of parameters includes a road length of $L = 10^5$ m with periodic boundary conditions, a discrete maximum speed of $v_{max} = 10$ cells per time step, an iteration time of $\tau = 1$ s, and a stochastic parameter $p = 0.5$. This set corresponds

Figure 3-3: Probability distribution functions for $v_{max} = 5$, $p = 0.5$ and various densities.
Figure 3-4: Sketch of the road discretization used in the original NS model. Every cell can be occupied by only a single vehicle and has a characteristic size of about four meters.

to a high velocity relevant for highways, about 140 kilometres per hour if we consider a car size of slightly less than 4 meters. While the relevance of this choice of parameters is debatable, the precise value of \( v_{\text{max}} \) does not have a significant influence on the global behaviour as long as the characteristic length \( l = v_{\text{max}} \times \tau \) is considerably smaller than the road length. This high velocity has the advantage of giving us a finer description of the velocity distribution function compared to smaller values of \( v_{\text{max}} \).

### 3.2.1 Influence of Initial Conditions and Simulation Time

It is stated in many papers that the results of simulations are independent of the initial conditions [57], [22], [36]. This is rather intuitive for a stochastic model, but one has however, to make sure that the simulation time is long enough. The greater the value of \( v_{\text{max}} \), the larger is the size of the phase space and the simulation has to be long enough so that the whole phase space is explored. For our specific case, the simulation time has to be at least equal to \( 10^6 \) iteration steps to obtain a result independent of the initial conditions. This is illustrated in Fig. 3-5. We have run simulations for three different initial conditions. The first one is the most commonly used in the literature [57] and corresponds to placing the vehicles equidistant from each other with initial speed equal to zero. We call this the equally spaced and stopped initial condition (ES). The second one has been used in Ref. [22] to analytically compute a critical density value. It amounts to placing all vehicles in a completely jammed state: every vehicle starts with a velocity and headway equal to zero (except for the first vehicle in line). We call this case the megajam (MJ) initial state. The last one is the so-called free flow (FF) initial condition: all vehicles start equally spaced from each other, but their initial velocity is equal to \( v_{\text{max}} \). As shown in Fig. 3-5, the initial conditions seem to have an influence on the velocity distribution unless the simulation time is taken to be large enough, here \( 10^6 \) iteration steps.
Figure 3-5: Velocity probability distribution functions for an average density of 0.11 vehicles per cell, for a simulation time of $10^5$ iteration steps (left panel) and $10^6$ iteration steps (right panel) for three different initial conditions. "MJ" corresponds to the megajam initial conditions, "FF" to the free flow initial conditions and "ES" to the equally spaced and stopped initial conditions. The three curves are different for the shorter simulation time (top) but merge for the higher one, implying that $10^5$ iteration steps is too short to obtain results independent of the initial conditions but $10^6$ is sufficient.

3.2.2 Order Parameter

From the fundamental diagram (see Fig. 3-6), one can distinguish two different regimes: a linear regime where the flow increases linearly with density, and another regime where the flow decreases with density. If one focuses on the velocity distributions at different densities, one can observe that there is a change in the shape below and above the critical density. Below the critical density, the velocity distribution peaks around the maximum velocity $v_{max}$ with the distribution

$$
\begin{align*}
    P(V = v_{max}) &= 1 - p \\
    P(V = v_{max} - 1) &= p
\end{align*}
$$

This corresponds to the free flow case, when the vehicles do not interact and their velocity is exactly given by the update rules in Eq. (3.1), by considering $d \to \infty$. However, for densities higher than the critical density, the probability to find a vehicle at zero velocity remains finite (see Fig. 3-7). This implies that jams form and persist, and the system is in a congested state.
This result suggests the system exhibits a phase transition at a critical density, $\rho_c$. As is usually done in statistical physics, it is important to qualify this phase transition. An important parameter to do so is the so-called order parameter. The order parameter is usually, to quote Ref. [31], "a thermodynamic function that is different in each phase, and hence can be used to distinguish between them." It is thus a quantity that exhibits different values depending on the state of the system, and is then a sort of "phase indicator". It is a macroscopic quantity whose change in value from a phase to the other arises from the breaking of a certain symmetry in the system, such as a translation symmetry in the liquid phase which is broken in the crystal phase. In liquid-gas transitions of water, to give another example, a common order parameter is the density: measuring the local density is sufficient to know whether we are observing water in its liquid or gas state. The way the order parameter varies around the transition can be used to distinguish between what
we call first-order and second-order phase transitions. In a first order phase transition, the order parameter shows a discontinuity at the critical point, whereas in a second order phase transition the order parameter varies continuously from a phase to the other. Let us come back to the liquid-gas transition of water as an illustration. If one heats a given volume of water in a pot to its boiling temperature, one can easily observe a layer of water in its gas form just above the liquid. If one then looks at the gradient of density along a line perpendicular to the water surface, one will see that the density shows a jump from one phase to the other. The order parameter therefore presents a discontinuity and this phase transition is a first-order phase one.

In our system it is not straightforward to identify what symmetry is broken, but it is natural from the point of view of the velocity distributions to choose the probability to find a vehicle at zero velocity as our order parameter. One can find other choices for this
parameter in the literature, such as the number of jammed vehicles [41] [22] [57], the density of nearest neighbour pairs [18], but the velocity distribution approach we follow here makes our choice intuitive and natural. In the literature the order parameter is usually described as continuously changing from zero to a non-zero value (which is only exactly known for the deterministic form of the NS model \(p = 0\) and for \(v_{\text{max}} = 1\), see Refs. [41] [22] [57] [18]), defining a phase transition of second-order from the free flow state to the congested one. With our choice of order parameter, plotted in Fig. 3-8 the order of the transition remains unclear. We can assume that we would observe a second-order transition, provided that our simulations are accurate enough around the critical state. We indeed see a gap in Fig. 3-8, at the critical density, but nothing proves that this jump would not disappear if the simulation times close to the critical density were longer. The exact value of the critical density is then difficult to determine but we estimate that, for our simulation conditions, \(\rho_c \approx 0.036\).

A possible explanation is the following: the system presents two stationary states, with a transition from one to the other at the critical occupancy. It may be misleading to evoke here the notion of equilibrium, since the system we are studying is, by definition, out of equilibrium. It is in fact not rigorously possible to define a thermodynamic ensemble because neither an energy nor a temperature is defined properly. However, if we had to define a thermodynamic ensemble, it would be a \((N,V)\) ensemble, the volume \(V\) (road length) and number \(N\) of vehicles being conserved. In return, we can make an analogy by means of stationary states. At low densities it then appears that the free flow state is more stable, whereas at high densities the congested state is more stable. However, near the critical occupancy we recognize the presence of two metastable states, and the simulation time needed to converge to the most stable one diverges next to the critical density [22]. This is the reason why, next to the critical occupancy and at simulation times that are not long enough, different initial conditions give us different results. This is illustrated in Fig. 3-9: the MJ and FF initial conditions are both in one of the stationary states, whereas the ES initial condition is somewhere between them.

3.2.3 Correlation Function

Having two stationary states with different spatial structure (one presenting jams and free flow, the other one uniquely free flow), an important parameter to describe the structure
Figure 3-8: Plot of the order parameter (probability to find a stopped vehicle) as a function of the occupancy for $v_{\text{max}} = 10$, $p = 0.5$ and the "ES" initial conditions, close to the critical occupancy, measured from numerical data. The critical occupancy is found to be about $\rho_c \approx 0.036$.

is the velocity - velocity correlation function. Every vehicle has a certain velocity and this function enables us to know how many successive vehicles (represented by the index $r$) it takes so that the information about the velocity is lost. In other words, it measures how the velocities between a given vehicle and the $r^{th}$ vehicle ahead of it are correlated. This function is defined as:

$$G_v(r) = \frac{1}{NT} \sum_{i=1}^{T} \sum_{i=1}^{N} v(i, t)v(i + r, t) - \langle v \rangle^2$$  \hspace{1cm} (3.6)$$

with $N$ the number of vehicles, $T$ the simulation time, while $\langle . \rangle$ defines the average over time and vehicles,

$$\langle v \rangle = \frac{1}{NT} \sum_{i=1}^{N} \sum_{i=1}^{N} v(i, t)$$  \hspace{1cm} (3.7)$$
Figure 3-9: Metastable states as a function of occupancy

Figure 3-10: Velocity-velocity correlation function for low densities (left), close to the critical density (center) and high densities (right). The free flow case (left) is completely uncorrelated and we find a correlation length of about 4 vehicles for the congested case (right) and a diverging correlation length close to the critical density

with $v(i, t)$ corresponding to the velocity of vehicle $i$ at time $t$. The correlation functions for different densities are shown in Fig. 3-10. The left panel corresponds to the free flow stationary state and shows no velocity correlation between any two vehicles. This is consistent with the fact that there are no interactions and the vehicles behave completely independent from each other. The right panel corresponds to the congested state and shows an exponential form:

$$ G_v(r) \propto e^{-\frac{r}{n_c}} $$

(3.8)

with $n_c$ the correlation number (or equivalently the correlation length). The exponential dependence of a correlation function is typical of short-range interactions and is well known
in systems such as polymer models [15]. We numerically obtain a correlation number of about 4 vehicles (by linear fitting) for a occupancy \( \mathcal{O} = 0.21 \), as shown in Fig. 3-11, which corresponds to the presence of jams involving many vehicles and hence short range interactions. The center panel depicts the correlation function at a density right above the transition. The precise form of the function slightly differs for the different initial conditions. But it shows strong correlation for a long distance, implying a divergence of the correlation length close to the transition. This is also consistent with a second-order phase transition. Our system presents here interesting similarities with equilibrium phase transition [52].

### 3.2.4 Jamming Time

It is then interesting to have a look at the way the system evolves from one stationary state to the other. The parallel with droplets condensation has been made in Ref. [42] [25]. Here
we focus on formation of jams, which is similar in principle to a nucleation process, when initiating the system in the free flow state, at an occupancy larger than the critical one. It is in a way also comparable to a supercooled fluid [61], which is at a temperature smaller than the solidification one, but still is in the liquid state. We then compute the average time needed for the first vehicle to stop. It is important to mention that jams can form and dissolve, and the formation of a single jam does not guarantee that the system will stay in the congested state; but it still provides interesting information. This time is plotted in Fig. 3-12. We observe a linear dependence between the logarithm of the jamming time and the occupancy. This suggests that there is a relationship between the jamming time and the occupancy of the form,

\[ T \propto e^{\Delta \rho \lambda} \]  

(3.9)

with \( \lambda \) a characteristic length scale which plays the role of an inverse "temperature", and \( \Delta \rho \) the difference between the density and the critical density \( \rho_c = 0.036 \) (as found in Fig. 3-8). We then obtain a jam formation rate of the form

\[ \nu \propto e^{-\Delta \rho \lambda} \]  

(3.10)

The previous expression implies that somehow the jamming formation from the free flow state is an activated process, similar to nucleation or the drying in hydrophobic nanopores [23], or creep in disordered systems [10]. This result is surprising due to the out-of-equilibrium nature of our system; but the linear dependence is valid over four decades, and therefore quite robust. Equation (3.9) suggests that the jamming formation time does not diverge when we approach the critical density. However, the time we measured is the time of first jamming formation, and nothing guarantees the stability of the jams which can dissolve.

### 3.2.5 First Interaction Time

It is instructive to compare this result with the probability of first interaction. Starting from free flow conditions, the FF initial conditions imply that all vehicles start equally spaced, from a distance \( d_i \) depending on the average density,

\[ d_i = \frac{1}{\rho} - 1 \]  

(3.11)
Figure 3-12: Log plot of the jamming time as a function of the occupancy, from free flow initial conditions at occupancies higher than the critical density $\rho_c \approx 0.036$. The linear dependence goes on for four decades of the jamming time.

where $d_i$ is the initial number of cells between two vehicles and $\rho$ is here expressed in terms of number of vehicles per cell ($d_i$ and $\rho$ are then are both dimensionless). Since the critical density occurs such that $d_i > (v_{max} \times \tau)$, we can fairly assume that the vehicles start without interacting. As the system evolves with time, interactions will occur and if the density is locally sufficiently high, jams will form. We can find a very simple solution to have an estimate of the probability to have two vehicles interacting after a certain simulation time. Without interactions, the velocity of every vehicle follows the simple rules:

$$v(t) = \begin{cases} v_{max} \text{ with probability } (1 - p) \\ (v_{max} - 1) \text{ with probability } p \end{cases}$$

(3.12)
Isolating two vehicles, the distance between them then evolves as,

\[
d(t + 1) = \begin{cases} 
(d(t) - 1) & \text{with probability } p(1 - p) \\
(d(t)) & \text{with probability } p^2 + (1 - p)^2 \\
(d(t) + 1) & \text{with probability } p(1 - p)
\end{cases} \tag{3.13}
\]

We then have an unbiased random walk for the evolution of the spacing of two vehicles. The density probability function is then given by Ref. [16]:

\[
p(t, x) = \frac{1}{\sqrt{4\pi p(1 - p)t}} \exp \left( -\frac{x^2}{4p(1 - p)t} \right) \tag{3.14}
\]

with \( x = d - d_i \). The probability that the vehicles encounter at least one interaction at a certain time \( T \) is then given by:

\[
P(T, d_i) = \int_{-\infty}^{\infty} p(T, x) \, dx + \int_{-\infty}^{-x_{\text{int}}} p(T, x) \, dx \tag{3.15}
\]

The distance \( x_{\text{int}} \) that has to be covered for an interaction to occur is given by:

\[
x_{\text{int}} = d_i - (v_{\text{max}} - 1) = \frac{1}{p} - v_{\text{max}} \tag{3.16}
\]

Rewriting Eq. (3.15), and using the symmetry of the probability distribution function Eq. (3.14), we obtain:

\[
P(T, d_i) = 1 - \int_{-x_{\text{int}}}^{x_{\text{int}}} p(T, x) \, dx \tag{3.17}
\]

or equivalently, using parity and a change or variables,

\[
P(T, d_i) = 1 - \frac{2}{\sqrt{4\pi p(1 - p)T}} \int_0^{x_{\text{int}}} \exp \left( -\frac{x^2}{4p(1 - p)T} \right) \, dx = 1 - \frac{2}{\sqrt{\pi}} \int_0^{y_{\text{int}}} e^{-y^2} \, dy \tag{3.18}
\]

with \( y_{\text{int}} = \frac{x_{\text{int}}}{\sqrt{4p(1 - p)T}} \). Using the error function,

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \tag{3.19}
\]

we thus have:

\[
P(T, x) = 1 - \text{erf} \left( \frac{x_{\text{int}}}{\sqrt{4p(1 - p)T}} \right) \tag{3.20}
\]
Now, if we define a critical probability \( P_c \) associated with at least one interaction, the time needed to obtain this probability is exactly given by

\[
T = \frac{1}{4p(1-p)} \left( \frac{x_{\text{int}}}{\text{erf}^{-1}(1-P_c)} \right)^2
\]  

(3.21)

The dependence of the first interaction time and the density is simply of the order of,

\[
T \propto \rho^{-2}
\]  

(3.22)

which is different from the jamming time dependency found in Eq. (3.9). We can note that this first interaction time goes to infinity if we chose the model to be deterministic (\( p = 0 \) or \( p = 1 \)).

In a nutshell, the order parameter we defined indicates a second-order phase transition between two stationary states: a congested state and a free flow state. Despite the fact that the system we study is out of equilibrium, we find several similarities with common phase transitions. Close to the critical density, the correlation length diverges, and we find that going from the free flow state to the congested state at densities higher than the critical one shows a behavior similar to an activated process, which cannot be explained by the first interaction time for two vehicles. The jamming formation, therefore, arises from different dynamics than the first interaction time for our case (\( v_{\text{max}} = 10 \)), but we expect that they will coincide for \( v_{\text{max}} = 1 \), since any interaction is equivalent to a jam in this particular case.

### 3.3 The Velocity Distributions

#### 3.3.1 Separation of the Distributions

If we have a closer look at the velocity distribution function corresponding to the presence of jams in the simulations, we notice that we can separate it in three regions, as shown in Fig. 3-13: one of them is the free flow region (c), analytically described by Eq. (3.5), one is the region of stopped vehicles (a), corresponding to the order parameter, and in between there is a transition zone, corresponding to vehicles accelerating or going from one jam to another. It is tempting to make the assumption, as it has been made in Ref. [36], that the distribution can be separated into two (or three) distributions, one corresponding to the free
flow part, the other to the jammed part, each of them weighted by the fraction of vehicles in each "phase". Assuming that the fraction of jammed vehicles is $\omega$, this would be equivalent to write:

$$P(v) = \omega P_J(v) + (1 - \omega) P_{FF}(v)$$  (3.23)

with $P_J$ the distribution in the jammed phase, and $P_{FF}$ the one in the free flow phase. Defining what makes a vehicle jammed or not is to some extent arbitrary and many different choices have been made throughout the literature, such as a vehicle with velocity smaller than $v_{max}/2$, or $v_{max} - 2$, but the choice should not matter too much [36]. We chose here the second option, namely a vehicle is in the free flow phase if and only if its velocity is equal to $v_{max}$ or $v_{max} - 1$, and jammed otherwise. The reason for this choice is that the distribution function for the free flow phase is exactly known and given by Eq. (3.5). However, we observe
from our simulations (see Fig. 3-14) that the jammed distribution depends on the density, contrary to what one could initially assume. It is, however, not straightforward to isolate the free flow headway distribution from the jammed one because it is not precisely known, unlike the velocity distribution. We can go even further by looking at the distribution at very high densities (when the free flow phase disappears and the system is only one big jam) and very low densities (when the whole system is in the free flow phase). In the case of very high densities shown in Fig. 3-15, the probability distribution functions should be equal to the jammed function. Similarly, in the case of very low densities depicted in Fig. 3-16, they should be equal to the free flow distribution. However, we notice that except for the free flow velocity distribution presented in Fig. 3-16 (left), all distributions change with density. Therefore, the nature of the system, jammed or in free flow, is not a unique characterization of the shape of the probability distribution functions we obtain, except for the case of the velocity distribution in free flow, for which the nature is exactly determined by the choice.
of the stochasticity parameter of the model.

### 3.3.2 Link between Headway and Velocity Distributions

As shown in Section 3.3.1, the transition velocity distribution and the jammed headway distribution are not unique and are dependent on the density. This suggests the existence of an internal structure that is density dependent. This result is somehow intuitive once it is stated: the density impacts the structure of the flow, whether the system is in complete free flow or completely jammed. It is not as straightforward when both phases coexist, but it is nonetheless the case, as shown in Fig. 3-14, for intermediate densities, and when both states are present. The distribution of headways in the NS model has been investigated in the literature [36], but has not been linked to the velocity distribution. We attempt to model this internal structure and investigate if a link can be established between the headway distribution and the velocity distribution. Starting from the observation that in the congested state the probability to find a stopped vehicle is non zero (see Section 3.2.2), we make the assumption that at any time $t$, if we take a snapshot of the system, we will observe a repartition of jams (understand successive stopped vehicles), with vehicles moving between them. In this section, we present an analytical solution to the velocity and headway distribution.
distribution of the vehicles moving from one jam to the other (the latter being trivial for vehicles inside the jams) and use it to link the headway and velocity distributions in the congested state.

Two Body Interaction

When the system is in the free flow stationary state, the solution for the velocity distribution is independent of the headway distribution; it is simply given by Eq. (3.5). In the congested state, however, the solution is not trivial; but we can use the fact that the probability to find a stopped vehicle is non zero. We can then make the assumption that the velocity distribution function can be studied by looking at the behavior of vehicles starting from a zero velocity and accelerating, then decelerating when arriving at another jam. We go further and assume that everything happens as if vehicles go from a stopped position to another stopped vehicle located at a certain distance $d_0$, as depicted in Fig. 3-17. We define $w(d, v, t|d_0)$ as the probability to find the following vehicle at a distance $d$ from the leading one and velocity $v$ at time $t$, knowing $d_0$. This probability can be analytically computed in an iterative way. If we let $w_{d,v}^t = w(d + v, v, t|d_0)$, the transition from a time step to the
Figure 3-17: Initial conditions of the two body interactions. The two vehicles start spaced from a distance $d_0$ and the leader remains stopped at all times. The follower has a velocity $v = 0$ with a probability 1 at time $t = 0$. At $t > 0$ the follower freely accelerates and eventually stops behind the leader. We can distinguish two zones, the free acceleration zone and the interaction zone (in red), that corresponds to the $v_{max}$ cells behind the leading vehicle.

Following one out of the interaction zone (ie when $d > v_{max}$) is given by:

$$
\begin{align*}
    w(d, v < v_{max} - 1, t | d_0) &= qw^{t-1}_{d,v-1} + pw^{t-1}_{d,v} \\
    w(d, v = v_{max} - 1, t | d_0) &= qw^{t-1}_{d,v-1} + p(w^{t-1}_{d,v} + w^{t-1}_{d,v+1}) \\
    w(d, v = v_{max}, t | d_0) &= qw^{t-1}_{d,v-1} + qw^{t-1}_{d,v}
\end{align*}
$$

with $q = 1 - p$, and in the interaction zone by:

$$
\begin{align*}
    w(d, v = d, t) &= \sum_{v=d}^{v_{max}} w^{t-1}_{d,v} + qw^{t-1}_{d,d-1} \\
    w(d, v = d - 1, t) &= p \sum_{v=d-1}^{v_{max}} w^{t-1}_{d,v} + qw^{t-1}_{d,d-2} \\
    w(d, v < d - 1, t) &= qw^{t-1}_{d,v-1} + pw^{t-1}_{d,v} \\
    w(d = 0, v = 0, t) &= \sum_{v=d}^{v_{max}} w^{t-1}_{d,v}
\end{align*}
$$

Solving this set of equations iteratively and defining the velocity distribution function at a time $t$ for any given spacing $d_0$ as,

$$
P(v, t, d_0) = \sum_{d=0}^{d_0} w(d, v, t)
$$

and the headway distribution as,

$$
Q(d, t, d_0) = \sum_{v=0}^{v_{max}} w(d, v, t)
$$
Figure 3-18: Time evolution of the velocity distribution function for the following vehicle for \( d_0 = 500 >> v_{max} \). We can distinguish two zones: a free acceleration (a) where the following vehicle goes from being stopped to the free flow regime (b) in which the velocity distribution is given by Eq. (3.5). The third zone (c) is the interaction zone, where the vehicle decelerates until it stays completely still behind the leader (d).

we obtain the velocity and headway distributions in the form:

\[
P(v, d_0) = \frac{1}{T} \sum_{t=1}^{T} P(v, t, d_0) ; \quad Q(d, d_0) = \frac{1}{T} \sum_{t=1}^{T} Q(d, t, d_0)
\]  

(3.28)

The solution for \( d_0 = 500 >> v_{max} \) is plotted in Fig. 3-18, and the corresponding velocity and headway probability distribution functions in Fig. 3-19. Under this condition, increasing the value of \( d_0 \) only makes the free flow part longer, and necessitates a longer time to reach the stopped final state.

The velocity distribution function that derives from the two body interaction reproduces well the free flow and the stopped regions; but it fails to represent accurately the transition zone: a free acceleration with no interaction leads to a uniform probability distribution in the transitioning zone. This motivates us to go a step further, and investigate the three-body
Three Body Interaction

Instead of considering one vehicle initially at rest, we now assume that there are two vehicles stopped at $t = 0$. This increases the complexity of our system, but it is more realistic: when a vehicle is stopped, there is usually another stopped vehicle just ahead of it, which made it stop initially. We now have to solve for the four-parameters joint probability distribution function $r(d_1, v_1, d_2, v_2, t|d_0)$, which is the probability to find both the follower 1 at a distance $d_1$ from the leader, and at a velocity $v_1$, and the follower 2 at a distance $d_2$ from the follower 1 at a velocity $v_2$ at time $t$, knowing $d_0$. Identically to the two-body interaction, this can be solved analytically. We distinguish four possible cases concerning the interaction zones: (i) both followers are out of their respective interaction zones; (ii) follower 1 is in interaction with the leader and follower 2 is not interacting with follower 1; (iii) follower 1 is not in interaction with the leader and follower 2 is interacting with follower 1; and, finally, (iv) follower 1 and 2 are both in interaction with their respective leader (being follower 1 for follower 2). The use of the term "interaction" here is not extremely rigorous since only the
leader acts on the follower but not the opposite. We write:

\[ r_{(\alpha, \beta)} = r(d_1 + \alpha, \alpha, d_2 + \beta - \alpha, \beta, t - 1|d_0) \]  

(3.29)

We thus have the following equations for the time evolution of the joint distribution: in the first zone (i) (no interactions, ie \( d_1 > v_{\text{max}} \) and \( d_2 > v_{\text{max}} \)),

\[ r(d_1, v_1, d_2, v_2, t|d_0) = ppr(v_1, v_2) + pqr(v_1, v_2 - 1) + qpr(v_1 - 1, v_2) + qqr(v_1 - 1, v_2 - 1) \]  

(3.30)

In the second zone (ii) (only follower 1 in interaction zone, ie \( d_1 \leq v_{\text{max}} \) and \( d_2 > v_{\text{max}} \)), the case \( v_1 = d_1 \) writes:

\[ r(d_1, v_1 = d_1, d_2, v_2, t|d_0) = \sum_{v_1 = d_1}^{v_{\text{max}}} (pr(v_1, v_2) + qr(v_1, v_2 - 1)) + qpr(d_1 - 1, v_2) + qqr(d_1 - 1, v_2 - 1) \]  

(3.31)

In the third zone (iii) (only follower 2 in interaction zone, ie \( d_1 > v_{\text{max}} \) and \( d_2 \leq v_{\text{max}} \)), the case \( v_2 = d_2 \) writes:

\[ r(d_1, v_1, d_2, v_2 = d_2, t|d_0) = \sum_{v_2 = d_2}^{v_{\text{max}}} (pr(v_1, v_2) + qr(v_1 - 1, v_2)) + pqr(v_1, d_2 - 1) + qqr(v_1 - 1, d_2 - 1) \]  

(3.32)

In the last zone (iv) (both in the interaction zones, ie \( d_1 \leq v_{\text{max}} \) and \( d_2 \leq v_{\text{max}} \)), the case \( v_1 = d_1 \) and \( v_2 = d_2 \) writes:

\[ r(d_1, v_1 = d_1, d_2, v_2 = d_2, t|d_0) = q \sum_{v_1 = d_1}^{v_{\text{max}}} \sum_{v_2 = d_2}^{v_{\text{max}}} r(v_1, v_2) + qq \sum_{v_1 = d_1}^{v_{\text{max}}} \sum_{v_2 = d_2}^{v_{\text{max}}} r(v_1, d_2 - 1) + qq \sum_{v_2 = d_2}^{v_{\text{max}}} \sum_{v_1 = d_1}^{v_{\text{max}}} r(d_1 - 1, v_2) + qqr(v_1 - 1, d_2 - 1) \]  

(3.33)

The other cases, such as ((i) and \( (v_1 \text{ or } v_2) = (v_{\text{max}} \text{ or } v_{\text{max}} - 1) \)) and ((ii) and \( v_1 < d_1 \)), are omitted to simplify the discussion, but they do not pose any specific issue. Now we obtain two velocity distribution functions:

\[
\begin{align*}
\frac{P_1(v_1, t|d_0)}{P_2(v_2, t|d_0)} &= \sum_{d_1=0}^{d_0} \sum_{d_2=0}^{d_0} \sum_{v_1=0}^{v_{\text{max}}} r(d_1, v_1, d_2, v_2, t|d_0) ; \\
\frac{P_2(v_2, t|d_0)}{P_2(v_2, t|d_0)} &= \sum_{d_2=0}^{d_0} \sum_{d_1=0}^{d_0} \sum_{v_1=0}^{v_{\text{max}}} r(d_1, v_1, d_2, v_2, t|d_0) .
\end{align*}
\]  

(3.34)
and the total velocity distribution:

$$P(v|d_0) = \frac{1}{T} \sum_{t=1}^{T} \left( \frac{P_1(v, t|d_0) + P_2(v, t|d_0)}{2} \right)$$  \hspace{1cm} (3.35)$$

Plotting this total velocity distribution in Fig. 3-20 we notice that the difference with the two body interaction is very small, and cannot explain the shape of the transition zone. This result has two implications. Either the three-body interaction model is not sufficient and we need to consider many more vehicles to have a realistic description, or we need to incorporate a distribution of $d_0$, or more probably a combination of both. While the consideration of more than three body interactions is theoretically possible, it would be extremely tedious. The number of possible zones grows with the square of the number of vehicles, and the necessary memory to compute it grows as $(d_0 \times v_{\text{max}} \times n)$ with $n$ the number of followers. This comes from the fact that since we consider an analytical solution on a relatively short time, the probability density functions are not independent. If they were then we could write, for two body interaction,

$$Q(d, t) = Q(d + v, t - 1)P(v)(t - 1)$$  \hspace{1cm} (3.36)$$

But one realizes that summing the $Q(d, t)$ for all the possible distances does not normalize to unity. For this reason we will keep the three body interaction and consider the distribution of $d_0$.

**Link with Headway Distribution**

Now that we have an analytical solution for the distributions of velocities and headways of two successive vehicles leaving a jam to arrive at the next one (which can be numerically computed), we use it to link the headway distribution with the velocity distribution in the whole system. Assuming that the congested regime can be modelled as a distribution $Q_0$ of gap distances $(d_0)$ for successively stopped vehicles, we obtain the velocity distribution:

$$P(v) = \sum_{d_0=0}^{+\infty} P(v|d_0)Q_0(d_0)$$  \hspace{1cm} (3.37)$$
We then need to determine the distribution of gaps, and for this purpose we will use the distribution of headways $Q^a(d)$, obtained from the simulations. For practical purposes, we define a cut off headway, $d^*$, as the distance after which the headway distribution function vanishes. Making the additional assumption that the fraction of stopped vehicles with another one immediately stopped in front of them is equal to $\alpha \in [0 ; 1]$, we define the analytical headway distribution from the three body interaction as:

$$q(d|d_0) = (1 - \alpha)q_1(d|d_0) + \alpha q_2(d|d_0)$$

with

$$q_1(d_1|d_0) = \frac{1}{T} \sum_{t=1}^{T} \sum_{v_1=0}^{v_{\text{max}}} \sum_{d_2=0}^{d_0} \sum_{v_2=0}^{v_{\text{max}}} r(d_1, v_1, d_2, v_2, t|d_0)$$

and

$$q_2(d_2|d_0) = \frac{1}{T} \sum_{t=1}^{T} \sum_{v_2=0}^{v_{\text{max}}} \sum_{d_1=0}^{d_0} \sum_{v_1=0}^{v_{\text{max}}} r(d_1, v_1, d_2, v_2, t|d_0)$$
We then make four different approximations to compute the gap distribution. The first one (Approximation 1) is to assume that the distribution of gaps is equal to the simulated headway distribution:

\[ Q_0(d) = Q^s(d) \]  

(3.41)

The second one (Approximation 2) is equivalent to the two body interaction or \( \alpha = 0 \). It considers that the simulated headway distribution is equal to the headway distribution of the first follower. To write it in a matrix form, if

\[ D_{d|d_0} = q(d, d_0) \; ; \; V_{v|d_0} = P(v|d_0) \]  

(3.42)

we have

\[ Q = Q^s = DQ_0 \]  

(3.43)

The second approximation thus reads, with \( q = q_1 \):

\[ Q = Q^s = DQ_0 \]  

(3.44)

which implies

\[ P = VQ_0 = VD^{-1}Q^s \]  

(3.45)

The third approximation (Approximation 3) corresponds to taking \( \alpha = 0.5 \) in Eq. 3.38. It writes:

\[ P = VQ_0 = VD^{-1}Q^s \text{ with } q = \frac{q_1 + q_2}{2} \]  

(3.46)

For the last one (Approximation 4) we chose to take \( \alpha = 0.95 \) in Eq. 3.38. It thus implies:

\[ P = VQ_0 = VD^{-1}Q^s \text{ with } q = (0.05q_1 + 0.95q_2) \]  

(3.47)

The results of these approximation to estimate the transition zone are given in Fig. 3-21, where the transition zone has been isolated and normalized. The last approximation (3.47) is in very good agreement with the simulation result for the transitioning zone. It corresponds to the assumption that the congested system is a series of jams, spaced by a gap distribution, and that the jams are made of many successively stopped vehicles. This provides us with insight into the structure of the traffic, determined by the distribution of
gaps between successively stopped vehicles.

This assumption, however, needs an additional parameter $\alpha$, which corresponds to a correction that takes into account the proportion of leading vehicles to the number of followers. The reason for this parameter is that the time evolution of headway distribution is very different for leaders and followers, unlike the velocity distribution which is quite similar: the leader (which we called follower 1) initially starts at a distance $d_0$ from its leader (which is the stopped vehicle), while follower 2 starts with a headway equal to zero, since it is immediately behind follower 1. For this reason, the headway distribution of follower 1 will always have non zero values for large distances, whereas follower 2 will not. It is important to note that this assumption linking the headway and velocity distributions only works when the system is fully congested (very high densities). At lower densities, when both phases coexist, the headway distribution corresponds to a superposition of distributions for each phase, and it is practically not possible to isolate the jammed one, since its shape depends on density (see Section 3.3.2). We note that the velocity distribution in the three body model, $P(v, d_0)$, was here always assumed to be the one of the first follower, and not averaged between the two vehicles (it is thus different from Eq. 3.35).

### 3.4 Extended Nagel-Schreckenberg Model

Now that we studied the original Nagel-Schreckenberg model from a velocity point of view, we extend the model in order to obtain a finer description of velocity probability distribution functions. This extension follows the discretization described in Ref. [36], it is in some way similar to the Kerner-Klenov-Wolf (KKW) model [32] [33] [34], although it remains simpler as it does not attempt to model the still debated three-phase description of traffic (Ref. [57]).

#### 3.4.1 Extension Description

The extension herein proposed is a finer discretization of the NS model, but we use a different formalism to describe it than the one employed in Ref. [36]. Let us start by separating the model into two descriptions: the continuum version of the model and the discrete version.
Figure 3-21: Normalized transitioning zone of the velocity distribution, found by simulation for $v_{max} = 10$, $p = 0.5$, and the "ES" initial conditions (red), and the four different approximations. The fourth approximation, corresponding to $\alpha = 0.95$ in Eq. (3.38) is in very good agreement with the simulation result. This suggests that a jammed system can be modelled as a succession of many successive stopped vehicles, with a certain distribution of gaps in front of the leader.

The continuum scale is characterized by the following parameters:

\[
\begin{align*}
L & : \text{length of the road in m;} \\
N & : \text{number of vehicles;} \\
\rho & = \frac{N}{L} : \text{density in vehicles/m;} \\
\Lambda & : \text{length of a vehicle in m;} \\
v_{max} & : \text{speed limit in m/s;} \\
D & = \left(\frac{L}{N} - \Lambda\right) : \text{average distance between vehicles in m.}
\end{align*}
\]  

(3.48)

We then obtain a discrete description, starting with the number of levels, $v_{max}$, we chose to describe the velocity. In the discrete description, all the quantities are integers and dimensionless. All the continuum variables are written in capital letters and the discrete
variables are denoted by small letters. We define a velocity increment \( \Delta v = \frac{v_{\text{max}}}{v_{\text{max}}} \) and a time constant \( \tau \), corresponding to the update time, that enables us to go from one description to the other, using the spatial step \( \Delta x = \tau \Delta v \). The variables in discrete space are therefore given by:

\[
\begin{align*}
  l &= \frac{L}{\Delta z} \quad \text{: number of cells on the road;} \\
  \lambda &= \frac{A}{\Delta z} \quad \text{: number of cells occupied by a vehicle;} \\
  v_{\text{max}} &= \frac{v_{\text{max}}}{\Delta v} \quad \text{: speed limit in cells per } \tau; \\
  d &= \left( \frac{1}{N} - \lambda \right) \quad \text{: mean number of cells between two vehicles.}
\end{align*}
\]

In this framework, a single vehicle can occupy several road cells, but every cell cannot contain more than a fraction of a vehicle, as shown in Fig. 3-22. We introduce an additional dimensionless parameter, the occupancy, which corresponds to the fraction of road occupied by vehicles:

\[
O = N \left( \frac{A}{L} \right) = N \left( \frac{\lambda}{L} \right) \in [0; 1]
\]

We will use the occupancy instead of the density from now on as the tunable parameter in the model. Since the vehicles occupy more than one cell, the update rules have to be slightly modified in order to prevent collision. This is achieved by changing the safety distance in the deceleration rule to the number of cells separating the center of gravity of two vehicles minus the number of cells occupied by the vehicles within that space. We also need to change the values of the acceleration and deceleration constants, which are equal to 1 in the original NS model. We therefore introduce two additional parameters:

\[
\begin{align*}
  \alpha &\quad \text{: acceleration constant} \\
  \sigma &\quad \text{: deceleration constant}
\end{align*}
\]

The update rules become:

\[
\begin{align*}
  1) \text{ Acceleration: } v_n &= \min(v_n + \alpha, v_{\text{max}}) \\
  2) \text{ Deceleration: } v_n &= \min(v_n, d_n - (\lambda - 1)) \\
  3) \text{ Randomization: } v_n &= \max(v_n - \eta[0, \sigma], 0) \\
  4) \text{ Movement: } x_n &= x_n + v_n
\end{align*}
\]

where the stochasticity parameter \( p \) is replaced by a distribution function between 0 and \( \sigma \), \( \eta[0, \sigma] \).
Figure 3-22: Sketch of the road discretization used in the extended Nagel-Schreckenberg model. Every cell can be occupied by only a fraction of vehicle but every vehicle occupies several cells.

Numerical Values Chosen for the Simulations

In order to perform numerical simulations, we make the following choices concerning the values of parameters. To have an accurate description of the velocity we fix $v_{\text{max}} = 100$. We keep the original time constant $\tau = 1$ s. Studying the traffic on a highway we consider $V_{\text{max}} = 35$ m/s $\approx 120$ km/h, and a car length of $A = 3.85$ m. This leads to:

$$\begin{align*}
\Delta v &= \frac{V_{\text{max}}}{v_{\text{max}}} = 0.35 \text{ m/s} \\
\Delta x &= \tau \Delta v = 0.35 \text{ m} \\
\lambda &= \frac{A}{\Delta x} = 11
\end{align*}$$

As a side note, it is useful to have an odd number for the size of a vehicle expressed in number of cells, as the position of the vehicle can be taken as the position of the middle cell. In order to keep the same acceleration and deceleration ratio as in the classic NS model, we set them both equal to the car size $\alpha = \sigma = \lambda$, and consider a uniform deceleration distribution. Concerning the choice of the system size and the simulation time, there is a compromise to be made between accuracy and computation time. In fact, the system has to be large enough in order to avoid finite size effects and the simulation time has to be sufficiently long to let the system explore the whole phase space and obtain representative averages (all the averaged values are computed through summation over all vehicles and the complete simulation time), but not too large so the simulations are technically possible. One key rule that has to be verified at all times is

$$\forall \Omega; \quad n = \frac{\Omega l}{\alpha} >> 1$$

So even at very low occupancies, the number of vehicles has to be large. For most of our simulations we choose a system of a hundred kilometers $L = 10^5$ m corresponding to $l = 2.85714 \times 10^5$ cells for a simulation time of $T = 10^6$ s. A major difference with the
original NS model is that the phase space to explored is significantly larger, since it is on the order of magnitude of \( l \times v_{\text{max}} \). Compared to the values of the original NS model, the discrete maximum velocity and road length are each ten times greater, and the phase space is consequently about 100 times larger. It is necessary to keep this in mind, when realizing that the simulation time required to have the same accuracy is practically not attainable. For this reason the initial conditions have a persisting influence on the final outcome close to the critical occupancy. We can however convince ourselves that for occupancy sufficiently far from the critical one, initial conditions will not affect the outcome.

### 3.4.2 Phase Transition and Velocity Distribution

Similarly to the original NS model, its extended version shows a phase transition at critical occupancy between a free flow state and a congested state. This phase transition is easily observable on the fundamental diagram plotted in Fig. 3-23. The peak at the critical
occupancy comes from the mentioned property that the simulation time required to converge
to the most "stable" state tends to infinity around the critical occupancy, and is therefore a
finite size effect as mentioned in Ref. [22]. The change in convexity at high occupancies has
been investigated in Ref. [36]. It is an effect of the higher number of possible configurations
in very jammed scenarios. From a velocity distribution point of view, the difference between
the two states is the non vanishing value of the probability to find a stopped vehicle. Velocity
distributions in both states and a highly congested state where there are no free flow zone,
are plotted in Fig. 3-24. The velocity distribution can be separated into three parts. The
first one corresponds to the fraction of stopped vehicles, which is only non vanishing in
the congested state, and it is therefore a natural candidate as an order parameter. The
second one is the transitioning zone, and the last one is the free flow phase. The free flow
part is known exactly and only depends on the value of the deceleration constant $\sigma$
and the deceleration distribution $\eta_{[0,\sigma]}$. The uniform distribution we choose for the simulation
clearly appears in Fig. 3-24. It is then easy to imagine that, by choosing another form of
distribution, one could recover realistic free flow distributions (probably a Gaussian), and
that it could be calibrated with field data. There is, however, no guarantee that this choice
would conserve the other properties of the model. This is left for further investigation in
the future. The choice of a uniform distribution, however, gives similar result as the original
NS model, and has the advantage that it allows us to analytically solve the two and three
body interactions, as we show next.

3.4.3 Two- and Three-Body Interactions

The choice we made for the stochasticity function $\eta_{[0,1]}$ has the advantage that we can extend
the two- and three-body interaction analyses conducted in Section 3.3.2 to the extended NS
model. This would not be as straightforward had we chosen another form of distribution.
Let us have a look at the solutions.

Two Body Interactions Solution

Considering a vehicle initially stopped and spaced from another stopped vehicle (that will
remain still) by a distance $d_0$, the master equation describing the time evolution of the
crossed distribution of distance and velocity $w(d, v, t|d_0)$ when the vehicle is out out of the
The interaction zone is:

\[
w(d, v, t + 1|d_0) = \frac{1}{\sigma + 1} \sum_{\nu=0}^{\sigma} w(d + v - \alpha + \nu, v - \alpha + \nu, t|d_0)
\]  

(3.55)

It simply considers all the velocities at time \( t \) that can lead to velocity \( v \) at time \( t + 1 \), which is the ensemble \([v - \alpha; \min(v - \alpha + \sigma, v_{\text{max}})]\), sums their probabilities and multiplies the sum by the probability to go from each velocity to \( v \) during the deceleration step, which is given by \( \eta_{[0,1]} \) and is here equal to \( \frac{1}{\sigma+1} \). Now, taking into account all the possible cases and

Figure 3-24: Velocity distribution function for low, medium and high occupancies for \( v_{\text{max}} = 100 \), \( \alpha = \sigma = \lambda = 11 \), "ES" initial conditions, and a uniform deceleration distribution. The simulations with high and medium occupancies are beyond the critical occupancy. The distribution can be separated in three parts, the fraction of stopped vehicles (a), the transitioning zone (b) and free flow fraction (c).
writing \( w(d + v, v, t) = w^d \), the full master equation follows the rules:

\[
W(d, v, t+1|d_0) = \frac{1}{s} W \text{ with } \\
\begin{cases}
W = \left( \min(v-\alpha+\sigma, v_{\text{max}}) \sum_{v=\max(0,v-\alpha)}^{v_{\text{max}}} w^d_v \right) + \Theta(v > v^* - \sigma) \left( \sum_{v=v-\alpha+\sigma+1}^{v_{\text{max}}} w^d_v \right) \\
v^* = \min(d, v_{\text{max}}) \\
s = \frac{\sigma+1-d}{\sigma+1} \text{ if } (d < \alpha \text{ and } v = 0); \quad s = \frac{1}{\sigma+1} \text{ otherwise}
\end{cases}
\]

(3.56)

where we used the definition \( \Theta(A) = 1 \) if \( A \) is true, \( \Theta(A) = 0 \) otherwise. The part \( \left( \min(v-\alpha+\sigma, v_{\text{max}}) \sum_{v=\max(0,v-\alpha)}^{v_{\text{max}}} w^d_v \right) \) corresponds to the case without the interactions (3.55). The term \( v^* \) is used to distinguish whether the maximum accessible velocity is limited by the headway \( d \) or not. The sum \( \left( \sum_{v=v-\alpha+\sigma+1}^{v_{\text{max}}} w^d_v \right) \) represents the principle that when the velocity at time \( t+1 \) is close enough to the limiting speed, hence in the interval \( [v^* - \sigma ; v^*] \), any velocity higher than the lower bound of the interval at time \( t \) can lead to \( v \), because of the deceleration rule. Finally, in the particular case when the velocity at time \( t+1 \) is equal to zero and the distance smaller than the acceleration value \( \alpha \), then the probability that a velocity at time \( t \) maps to zero is equal to \( \frac{\sigma+1-d}{\sigma+1} \). As a side note, we always have the probability to have a velocity higher than the headway equal to zero \( w^d_v \), but it does not need to be stated in the master equation, because the iteration process keeps these values constantly zero. Plotting the transitioning velocity distribution for a large initial distance \( d_0 \) in Fig. 3-25, we realize that we recover the bump at small velocities present in Fig. 3-24, implying that this feature is linked to the model properties (the width of this bump is on the order of \( \alpha \)), and not to the structure of the traffic. If we look at the time evolution of the velocity distribution in Fig. 3-26, its shape is very similar than the one for the follower 1 in the three-body solution of the original NS model (see Fig. 3-18). There are also three phases; namely an accelerating phase, a free flow phase where all the velocities in the range \( [v_{\text{max}} - \sigma ; v_{\text{max}}] \) have the same probability to be encountered, and a deceleration phase. The bump at low velocities in the decelerating phase, which subsists for a long time, is the reason for the bump observed in the velocity distribution (see Fig. 3-25), it is caused by the fact that there are many small distances which can be occupied when the vehicles are very close to each other. This property has already been discussed in Ref. [36]. It is the reason for the change in convexity in the fundamental diagram at very high densities.
Similarly to the classic NS model, the transitioning zone is very flat, except for a small bump very close to zero. This bump recovers the one observed with simulation distribution, but the rest of the transitioning zone is very flat, which is not consistent with numerical observations in Fig. 3-24.

Three-Body Interactions

The three-body interaction uses the same principle, but is more tedious. There is no particular interest of exposing the calculations here, since it is a direct extension of Eq. (3.56), using the four interaction cases described in Section 3.3.2 they are presented in Appendix E. Despite the knowledge of an analytically numerically computable solution for the three-body interaction, we are practically limited by the size of the solution space. Noting $r(d_1, v_1, d_2, v_2, t | d_0)$ the probability to find follower 1 at a distance $d_1$ from the still vehicle at velocity $v_1$, and the follower 2 at distance $d_2$ from follower 1 at a velocity $v_2$ at time $t$ (see notations in Appendix E), for every time step the number of unknowns is of the order of $(d_0,v_{\text{max}},d_0,v_{\text{max}}) = (d_0)^2(v_{\text{max}})^2$. Compared to the original NS model exposed in Section 3.3.2, both velocity and space variables have discretization that is 10 times greater, leading
Figure 3-26: Time evolution of the probability of every velocity for the two-body interaction, for a large initial distance $d_0$. Similarly to the classic NS model, we can observe three phases. An accelerating phase (a) where the vehicle freely accelerates, until it reaches the free flow phase (b) where all the velocities in the interval $[v_{\text{max}} - \sigma; v_{\text{max}}]$ are equiprobable, and eventually decelerates (c) when it comes close enough to the still vehicle until it reaches complete rest with probability 1. The blue curve corresponds to $P(v = 0, t)$

to a number of unknowns that is $10^4$ higher. Calculating the solutions of the three-body interactions reveals itself to be too lengthy and numerically too costly to be included in this study.

**Link between Headway and Velocity Distributions**

Despite the practical impossibility for us to conduct the same study as for the NS original model in Section 3.3.2, the two-body interaction for the extended model is sufficiently computationally cheap for us to use it to link the headway and the velocity distributions. Referring to Section 3.3.2, this is equivalent to the second approximation exposed by Eq. (3.44), when equating the simulated headway distribution, $Q^s$, with the one from the two-body interaction, $Q = D.Q_0$. We thus obtain a velocity distribution $P = V.Q_0 = V.D^{-1}.Q^s$. 97
which is plotted in Fig. 3-27 with the velocity distribution obtained from the simulation. The comparison between the two is not convincing, which suggests that we are missing something, probably the headway distribution of another follower as done in the three-body interaction in order to better capture the internal structure of the system.

### 3.4.4 Continuum Limit

It is possible to take the continuum limit of the NS model, as shown in Ref. [36]. It is then sufficient to consider the positions and velocities to be continuous, and apply the update rules (3.52), with a deceleration distribution $\eta_{[0,1]}$ giving randomly any real number in the segment $[0;1]$ with a chosen distribution. However, the high number of possible velocity we chose in the extension of the NS model, $v_{\text{max}} = 100$, gives us results that are almost exactly similar than using the continuous extension and then separating the velocity distribution in
100 values to visualize the velocity distribution. For this reason we do not investigate further a continuous version of the NS model, but refer the reader to Ref. [36] for this purpose, where interesting considerations about the choice of the acceleration $\alpha$ and deceleration $\sigma$ are discussed.

3.5 Chapter Summary

This Chapter focused on the study of the classical Nagel-Schreckenberg model from a velocity distributions point of view. We showed that this cellular automata model has several important advantages over force-based models. Despite the small number of parameters, it captures global and local traffic behavior, and its simplicity allows us to run simulations on a system sufficiently large over a long enough time to be statistically meaningful and to avoid finite size effects, even fairly close to the critical point. We showed that this out-of-equilibrium system exhibits a phase transition which presents some similarities with other physical systems. It possesses two metastable steady states, a free flow and a congested state, and the transition from one to the other can be compared with activated processes such as nucleation of droplets. The study of the velocity distribution function provides us with a natural order parameter to characterize the system’s state: the probability to find a stopped vehicle. We also showed that the headway and velocity distributions could not be separated into a sum of weighted characteristic distributions, but that their shapes change with the density due to the internal structure of traffic. An analytical model of three-body interactions enables us to make the link between headway and velocity distributions, by assuming the system can be approximated as a succession of jams separated by a gap distribution. In order to obtain more accurate velocity probability distribution functions, we extended the NS model to a close to continuous approximation, and we showed that this model recovers all the properties of the original NS model, with only a few additional parameters. An important feature is that the velocity distribution in the free flow case is exactly known and given by the model properties. It is therefore a good candidate for calibrating the model, provided other parameters are adjusted to recover a realistic transition occupancy and realistic congested distribution.
Chapter 4

Link with Energy Dissipation

When a vehicle travels on a road it encounters a combination of resistive forces due to air drag, internal resistance of the vehicle (engine friction) and the interaction with the pavement (Pavement-Vehicle Interaction, or PVI). It is important to note that the proportion of these forces relatively to each other is velocity dependent (see Fig. 4-1), and the one induced by PVI becomes predominant at low velocities. It then becomes obvious why taking account of the PVI is crucial in the carbon management of road networks. The resistive forces encountered with PVI are produced through mainly three parameters: pavement texture, pavement roughness, and pavement deflection. The combination of these PVI induced resistive forces compose what is commonly called the rolling resistance, and contribute to the excess fuel consumption, namely the quantity of gas a vehicle has to spend to counter balance the rolling resistance. Current estimates of the excess fuel consumption are based on a single speed approximation (SSA), as all vehicles are assumed to travel at the same maximum velocity. As we saw, however, in Chapter 3, the velocity distribution is much more complex than that. Herein, we aim at taking this complexity into account by using velocity probability distribution functions for the estimation of excess fuel consumption, and by incorporating the dissipation estimates into the general framework used to characterize traffic conditions. This approach thus bridges the gap between traffic simulations and mechanistic models. This chapter is devoted to the presentation of the deflection and roughness phenomena and the estimation of their respectively induced PVI, depending on traffic conditions.

Note: The models presented and employed in this chapter were defined in Refs. [38]
[Image -1x0 to 613x792]

Figure 4-1: Part of the different resistive forces acting on a travelling vehicle in the global resistive force, taken from [51] (where it was adapted from [11])

[40], which can be considered as the main references. Additional references will be provided when needed.

4.1 Deflection Model

Pavement deflection is the deformation of the pavement under the load of a vehicle. We focus on the amount of energy that is dissipated that is per traveled length, due to the non linear behaviour of the material constituting the pavement, and its dependence on the external temperature and the vehicle's speed.

4.1.1 Model Adaption and Scaling Relations

In order to describe and model the deflection process, several simplifications and assumptions have to be made. We suppose that the pavement is composed of a visco-elastic beam of thickness $h$, linear density $\rho$, Young’s modulus $E$, relaxation time $\tau$ and viscosity $\eta = \tau E$, on top of an elastic layer of stiffness $k_s$. Concerning the load, we assume that the vehicle is moving under steady-state conditions at a constant speed $v$. We model the interaction between the tire and the pavement as a constant load of intensity $P$ over a rectangular area of width $b$ and length $l$. These considerations are depicted in Fig. 4-2. From these considerations, we can define three essential parameters:
The Winkler length $l_s = \left( \frac{Eh^3}{12k_s} \right)^{\frac{1}{3}}$ which describes the elastic response of the beam;

- The relaxation time $\tau = \frac{\eta}{E}$, that captures the viscous response of the material composing the beam;

- The critical velocity $c_{cr} = l_s \sqrt{\frac{k_s}{m_s}}$, with $m_s$ the surfacic mass of the beam.

Given these parameters, we want to determine the dissipation rate $\mathcal{D}$. It has been shown in [39], using the Vashy-Buckingham theorem, that this system is controlled by four dimensionless numbers

$$\Pi = F(\Pi_1, \Pi_2, \Pi_3)$$  \hspace{1cm} (4.1)

with

$$\begin{align*}
\Pi &= \frac{Dl^2b \kappa}{P^2c_{cr}} \\
\Pi_1 &= \frac{v}{c_{cr}} \\
\Pi_2 &= \frac{\tau_{cr}}{l_s} \\
\Pi_3 &= \frac{l}{l_s}
\end{align*}$$  \hspace{1cm} (4.2)

where $\Pi_2 = \xi = \tau \sqrt{\frac{k_s}{m_s}}$ is the damping ratio, $\Pi_1 = \bar{c} < 1$, and $\Pi_3$ is the ratio between the distance over which the load is applied and the Winkler length. In order to find the value of the dissipation we need to determine the expression of $F$. 

---

Figure 4-2: Uniform load traveling at constant velocity over a visco-elastic beam supported by an elastic layer, adapted from [39]
4.1.2 Exact Solution

As shown in [38], the key to obtaining the solution to this problem is to place ourselves in a moving coordinate system attached to the vehicle. Since this system of coordinates is moving at a constant velocity parallel to the direction of the pavement, it is equivalent to solving the equation of motion either in this system or in the system attached to the pavement. If we assume the beam to be infinite and, in a first approach, perfectly elastic, the equation for the deflection \( w \) of the beam is given, in the moving coordinate \( X = x - vt \), by:

\[
\frac{Eh^3}{12} \frac{\partial^4 w}{\partial X^4} + m_s v^2 \frac{\partial^2 w}{\partial X^2} + k_s w = p
\]  
\((4.3)\)

with \( p = \frac{F}{S_c} \) the load by unit of surface and \( S_c = b.l \) is the contact area of the tire with the pavement. Solving this equation by means of a Fourier transform gives:

\[
\hat{w} = \frac{\hat{p}}{\frac{Eh^3}{12} \lambda^4 - m_s v^2 \lambda^2 + k_s}
\]  
\((4.4)\)

Now that we have the solution for an elastic beam, we consider the material constituting the beam to be a Maxwell visco-elastic material. This material behavior, which can be represented by a spring of stiffness \( E \) being linearly attached to a damper of viscosity \( \eta \), is described by the constitutive equation relating the linear stress and strain rate:

\[
\frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} = \dot{\epsilon}
\]  
\((4.5)\)

or, equivalently

\[
\sigma + \tau \dot{\sigma} = E \tau \dot{\epsilon}
\]  
\((4.6)\)

Now, placing ourselves in the coordinates system moving at constant velocity \( c \) changes the time derivatives in a space derivative \( \frac{d}{dt} = -v \frac{d}{dX} \) and transforms the constitutive law into the equivalent expression

\[
\sigma + \nu \tau \frac{d\sigma}{dX} = E \left( c \tau \frac{d\epsilon}{dX} \right)
\]  
\((4.7)\)

Using once again a Fourier transform in space gives the constitutive equation in terms of \( \hat{\sigma} \) and \( \hat{\epsilon} \), respectively the Fourier transforms of the stress and the strain as

\[
\hat{\sigma} (1 - iv\tau\lambda) = E (-iv\tau\hat{\epsilon})
\]  
\((4.8)\)
This gives us the complex modulus verifying \( \sigma = \hat{E}\xi \) of the form

\[
E^* = \left( -\frac{i\lambda v\tau}{1 - i\lambda v\tau} \right) E
\]  

(4.9)

Then establishing the correspondence principle, we introduce this complex modulus in the Fourier transform of the infinite elastic beam:

\[
\hat{w} = \frac{\hat{p}}{(-\frac{\xi\lambda^2}{1 - \xi^2\lambda^2})} \frac{Eh^4}{12} - m_s v^2 + \lambda_k
\]  

(4.10)

We can simplify this expression by using the dimensionless numbers \( \varepsilon = \frac{v}{c_\varepsilon} \) and \( \xi = \sqrt{\frac{k_s}{m_s}} \), plus the dimensionless Fourier parameters \( \tilde{\lambda} = l_s \lambda, \tilde{\xi} = \frac{\xi}{\rho}, \) and \( \hat{\tilde{w}} = \hat{w} b_{e} \). This leads us to the dimensionless equation:

\[
\hat{\tilde{w}} = \frac{\hat{p}}{(-\frac{\xi\tilde{\lambda}^2}{1 - \xi^2\tilde{\lambda}^2})} \lambda^4 - \varepsilon^2\tilde{\lambda}^2 + 1
\]  

(4.11)

Now that we have an expression for the vertical deflection of the beam, let us find an expression that could link it to the energy dissipation rate. The dissipation rate is written as:

\[
\mathcal{D} = -v \int_{y=-\frac{b}{2}}^{\frac{b}{2}} \int_{x=-\frac{b}{2}}^{\frac{b}{2}} p \frac{dw}{dx} dX dy = -vp\langle \frac{dw}{dX} \rangle b
\]  

(4.12)

with \( \langle \cdot \rangle \) representing the space average over the contact area. From this, and noting \( \mathcal{F}^{-1} \) the inverse Fourier transform, we have:

\[
\mathcal{D} = -v p \langle \mathcal{F}^{-1} (i\lambda \hat{w}) \rangle b
\]  

(4.13)

which can be rewritten in the form:

\[
\mathcal{D} = -v p \left\langle \mathcal{F}^{-1} \left( i\frac{P}{k_s l_s} \hat{\tilde{w}} \right) \right\rangle b
\]  

(4.14)

or, in a dimensionless form, using the linearity of the Fourier Transform

\[
\frac{\mathcal{D} k_s l_s S^2}{vP^2 b} = - \left\langle \mathcal{F}^{-1} (i\tilde{\lambda} \hat{\tilde{w}}) \right\rangle
\]  

(4.15)
or equivalently:

\[ \Pi = -\frac{\Pi_1}{l\Pi_3} \left\langle \mathcal{F}^{-1} \left( i\tilde{\lambda} \hat{w} (\Pi_1, \Pi_2, \Pi_3) \right) \right\rangle \quad (4.16) \]

Hence, the dimensionless function \( F \) is defined as:

\[ F (\Pi_1, \Pi_2, \Pi_3) = -\frac{\Pi_1}{l\Pi_3} \left\langle \mathcal{F}^{-1} \left( i\tilde{\lambda} \hat{w} (\Pi_1, \Pi_2, \Pi_3) \right) \right\rangle \quad (4.17) \]

and the energy dissipation per traveled length reads as:

\[ \mathcal{E} = \frac{D}{v} = \frac{\Pi P^2 c_r}{l^2 k_s b \nu} = -\frac{P^2}{l^2 k_s b l \Pi_3} \left\langle \mathcal{F}^{-1} \left( i\tilde{\lambda} \hat{w} (\Pi_1, \Pi_2, \Pi_3) \right) \right\rangle \quad (4.18) \]

The solution of this expression requires us to define the form of the applied load \( \hat{p} \). We will therefore define it as a Heavyside function of unity amplitude. The Fourier transform for this function is given by (using \( \pi \) as the famous number and not the previous dimensionless number here);

\[ \hat{p}(\lambda) = \int_{-\infty}^{+\infty} \hat{p}(x) e^{-2i\pi x \lambda} dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2i\pi x \lambda} dx \quad (4.19) \]

which transforms as:

\[ \hat{p}(\lambda) = \left( \frac{\sin(\pi \lambda)}{\pi \lambda} \right) \quad (4.20) \]

We assume that this block load is applied between \( x = -\frac{1}{2} \) and \( x = \frac{1}{2} \). It is important to note that the Fourier transform of the dimensionless parameter \( \tilde{p} \) is homogeneous to a length. Using the dimensionless parameter \( \Pi_3 \) this becomes

\[ \hat{p}(\tilde{\lambda}) = \left( \frac{\sin(\theta(\tilde{\lambda}))}{\theta(\tilde{\lambda})} \right) \quad (4.21) \]

and we have \( \theta(\tilde{\lambda}) = \pi \tilde{\lambda} \Pi_3 \). This enables us to write in more details the calculation of \( \Pi \). Integrating over the loading area, it writes:

\[ \left\langle \mathcal{F}^{-1} \left( -i\tilde{\lambda} \hat{w} (\Pi_1, \Pi_2, \Pi_3) \right) \right\rangle = -\Re \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\infty}^{+\infty} i\tilde{\lambda} \frac{\hat{p} e^{2i\pi x \lambda}}{1 - i\tilde{\lambda} \frac{x}{1 - i\tilde{\lambda} x}} \left( \tilde{\lambda}^4 - \tilde{\lambda}^2 + 1 \right) \frac{d\tilde{\lambda}}{l_s} dx \right) \quad (4.22) \]
or equivalently, after the change of variable $\bar{x} = \frac{\bar{t}}{s}$

\[(F^{-1} (-i \hat{\lambda} \hat{w} (\Pi_1, \Pi_2, \Pi_3))) = -\Re \left( \int_{-\Pi_2}^{\Pi_2} \int_{-\infty}^{+\infty} i \bar{x} \left( \frac{l \sin(\theta(\lambda))}{\theta(\lambda)} e^{2i\pi \bar{x} \lambda} \right) d\lambda d\bar{x} \right) \quad (4.23)\]

\[(F^{-1} (-i \bar{\lambda} \bar{w} (\Pi_1, \Pi_2, \Pi_3))) = -\frac{\bar{l}}{\bar{s}} \Re \left( \int_{-\Pi_2}^{\Pi_2} \int_{-\infty}^{+\infty} i \sin(\bar{\pi} \bar{\lambda}) (\cos(2\pi \bar{x} \bar{\lambda}) + i \sin(2\pi \bar{x} \bar{\lambda})) d\lambda d\bar{x} \right) \quad (4.24)\]

Integrating over $\bar{x}$ and taking into account the parity of the trigonometric functions, we obtain, in terms of real numbers:

\[(F^{-1} (-i \hat{\lambda} \hat{w} (\Pi_1, \Pi_2, \Pi_3))) = \int_{-\infty}^{+\infty} \frac{\Pi_1 \Pi_2}{\pi^2} \sin^2(\pi \bar{\lambda} \bar{\Pi}_3) \bar{\lambda}^4 d\bar{\lambda} \quad (4.25)\]

Using the parity of the integrand, this gives

\[(F^{-1} (-i \bar{\lambda} \bar{w} (\Pi_1, \Pi_2, \Pi_3))) = 2\frac{\Pi_1 \Pi_2}{\pi^2} \int_{0}^{+\infty} \frac{\sin^2(\pi \bar{\lambda} \bar{\Pi}_3) \bar{\lambda}^4 d\bar{\lambda}}{(1 - \Pi_1^2 \bar{\lambda}^2)^2 + (\Pi_1 \Pi_2 \bar{\lambda})^2(1 - \Pi_1^2 \bar{\lambda}^2 + \bar{\lambda}^4)^2} \quad (4.26)\]

We thus obtain the energy dissipation per traveled length as:

\[
\mathcal{E} = \frac{2\Pi^2}{l_s^2 k_s b} \frac{\Pi_1 \Pi_2}{(\pi \Pi_3)^2} \int_{0}^{+\infty} \frac{\sin^2(\pi \bar{\lambda} \bar{\Pi}_3) \bar{\lambda}^4 d\bar{\lambda}}{(1 - \Pi_1^2 \bar{\lambda}^2)^2 + (\Pi_1 \Pi_2 \bar{\lambda})^2(1 - \Pi_1^2 \bar{\lambda}^2 + \bar{\lambda}^4)^2} \quad (4.27)\]

or equivalently:

\[
\mathcal{E} = \frac{2\Pi^2 \Pi_1 \Pi_2}{l_s^2 k_s b} \int_{0}^{+\infty} \frac{sinc^2(\pi \bar{\lambda} \bar{\Pi}_3) \bar{\lambda}^6 d\bar{\lambda}}{(1 - \Pi_1^2 \bar{\lambda}^2)^2 + (\Pi_1 \Pi_2 \bar{\lambda})^2(1 - \Pi_1^2 \bar{\lambda}^2 + \bar{\lambda}^4)^2} \geq 0 \quad (4.28)\]

This expression clearly recovers the non-negativity of the energy dissipation, since the integrated function if always positive. Then, writing

\[
\mathcal{E} = \int_{0}^{+\infty} f(\bar{\lambda}, \Pi_1, \Pi_2, \Pi_3) d\bar{\lambda} \quad (4.29)\]

we note several properties on function $f$. First, one readily realizes that $f$ presents no singularities, unless $\Pi_1$ and $\Pi_2$ are simultaneously equal to zero (which never happens in our system according to Table 4.1). Second, it is overall equal to a sinus function times a
Figure 4-3: Dimensionless plot of the deflection, the normalized position is $\frac{Y}{T}$ and the normalized deflection $\frac{u_{kb}}{P}$. The center of the load sees a positive slope, that translates into a wheel being constantly going uphill.

A rational fraction of the order $-6$; and hence vanishes quickly as soon as $\tilde{\lambda}$ becomes large enough. This implies that the integration can be very well numerically calculated over a finite interval $[0 ; \tilde{\lambda}_0]$ with $\tilde{\lambda}_0$ a characteristic value after which the function $f$ is close to zero. We also note that if we have $(\Pi_3 \pi \tilde{\lambda}_0 << 1 \Leftrightarrow l << \frac{L}{\pi \tilde{\lambda}_0})$, we can write sinc$(\Pi_3 \pi \tilde{\lambda}_0) \approx 1$ and the contact length $l$ does not influence the energy dissipation. In Fig. 4-3 we plot the dimensionless solution of the deflection. In this figure we notice that the minimal value of the deflection does not match the center of the load ($X = 0$). As mentioned in Ref. [38] this implies that the tire is not located at the lowest point of the deflection, but ahead of it and everything happens as if the wheel was permanently going uphill. This behavior is currently under investigation experimentally, by the study of a rolling cylinder on a visco-elastic material (see Ref. [45]).
4.1.3 Temperature Dependence of the System

For now, we have presented a model for viscoelastic deflection and its solution, but we have not taken into account possible temperature variations. The effect of temperature change can be effectively implemented in the relaxation time of the visco-elastic beam. Indeed, as mentioned in Ref. [38], if one defines a reference temperature $T_{\text{ref}}$, the relaxation time for a Maxwell material can be rescaled as:

$$\tau(T) = \tau(T_{\text{ref}}) \ a_T(T)$$

with $T$ being the actual temperature and $a_T(T)$ is called the shift factor, which captures the change in relaxation time due to a difference in the temperature from the reference temperature $T_{\text{ref}}$. The expression of the shift factor depends on the nature of the material. For concrete, it has been shown, based on the concept of activation energy that [4]:

$$\log(a_T(T)) = U_c \left( \frac{1}{T} - \frac{1}{T_{\text{ref}}} \right)$$

with the constant $U_c$ being homogeneous to a temperature with the value $U_c = 2,700$ K. For pavement that are made of asphalt mixes, it has been shown in Ref. [51] that the William, Landel and Ferry law holds well. It defines the shift factor as:

$$\log(a_T(T)) = \frac{-C_1(T - T_{\text{ref}})}{C_2 + (T - T_{\text{ref}})}$$

in which $C_1$ and $C_2$ are empirical constants. Typical values are $C_1 = 34$ and $C_2 = 203$ K for a reference temperature $T_{\text{ref}} = 283$ K. From calibrations with respect to finite element simulations [39], the reference relaxation time was determined to be equal to $\tau(T_{\text{ref}}) = 0.0083$ s and it is assumed to be the same for all pavements.

4.1.4 Dimensional Analysis of the Pavements

With the model now in hand, it is instructive to investigate the range of the dimensionless parameters $\Pi_1$, $\Pi_2$ and $\Pi_3$ that are encountered in real road networks. While these three parameters depend on the pavement's mechanical properties, $\Pi_1$ also depends on both the vehicle's velocity, $\Pi_2$ is also a function of of the external temperature through the relaxation time $\tau$ and $\Pi_3$ depends on the contact area between the pavement and the tire. Using
the pavement's classification of Ref. [39] and considering a velocity range encountered on
a highway $v \in [0, 35]$ m.s$^{-1}$, a yearly temperature range typical for New England $T \in
[-30, 40]$ degrees Celsius, and a contact length $l = 0.15$m (used in [39]), we obtain the
values in Table 4.1. The first thing we notice in Table 4.1 is that if the ranges of $\Pi_1$ and $\Pi_3$

<table>
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<tr>
<th>Pavement</th>
<th>Type</th>
<th>$\Pi_1^{min}$</th>
<th>$\Pi_1^{max}$</th>
<th>$\Pi_2^{min}$</th>
<th>$\Pi_2^{max}$</th>
<th>$\Pi_3$</th>
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<td>2.6 $10^4$</td>
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<td>2.2 $10^1$</td>
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<td>PCC</td>
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<td>2.0 $10^0$</td>
<td>2.3 $10^1$</td>
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<td>8.7 $10^{-2}$</td>
<td>2.9 $10^4$</td>
<td>0.38</td>
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<tr>
<td>GPS 7</td>
<td>Composite</td>
<td>0.088</td>
<td>1.2 $10^{-1}$</td>
<td>4.0 $10^4$</td>
<td>0.43</td>
<td></td>
</tr>
</tbody>
</table>

are of the same order of magnitude for all pavements, this is not the case for $\Pi_2$, which spans
six orders of magnitude for asphalt pavements. This comes from the temperature dependence
of the pavements, which follows Eq. (4.31) for concrete pavements and Eq. (4.32) for asphalt
and composite pavements. We also notice that the range of $\Pi_2$ for concrete is included in
the range of $\Pi_2$ for asphalt.

### 4.1.5 Deflection-Induced Dissipation Estimates for Typical Pavements

The deflection-induced dissipation is therefore dependent on the external temperature, the
velocity, the vehicle’s weight on each wheel and the pavement’s properties. The pavement’s
properties non only influence the dissipation because of elastic properties, but the pavement’s
material (concrete or asphalt) also has an effect on the temperature dependence of the
phenomenon. This is illustrated in Fig. 4-4, where the deflection-induced energy dissipation
per traveled length (normalized by its value at temperature $T = 0^\circ$C and velocity $v = 35$
$m/s$) is plotted as a function of the traveling velocity, at different temperatures, for
asphalt and concrete pavements. At a fixed velocity, we notice in Fig. 4-4 that the increase
of temperature has a tendency, both with asphalt and concrete, to increase the energy
dissipation rate (see Fig. 4-4 ). This increase is however much stronger for asphalt than
concrete. If one focuses on high velocity values, the energy dissipation in the asphalt case
spreads over several orders of magnitude between cold and hot temperatures, whereas it
remains at the same magnitude for concrete. This clearly shows the stronger temperature
sensitivity of asphalt compared to concrete. The dissipation dependence on velocity is not as straightforward to evaluate, because it varies with the temperature. The concrete case is simpler and exhibits the same global behavior irrespective of the temperature; that is, an increasing dissipation with velocity from \( v = 0 \) until it reaches a maximum at a certain velocity \( v = v^* \), after which the dissipation is continuously decreasing with velocity.

Increasing the temperature shifts the velocity of maximum dissipation to higher velocities, and the overall maximum dissipation is the same irrespective of temperature. The deflection-induced dissipation on asphalt pavement exhibits a more complex behavior, most likely due to the fact that it presents a much wider range of possible values of \( \Pi_2 \) (see Table 4.1). We observe that for low temperatures, the behavior is similar to the one of concrete, that is, an increasing dissipation with velocity until a maximum is reached, followed by a decrease of dissipation with increasing velocity. For higher values of temperature, however, the dissipation exhibits two maxima and a highly non monotonic behavior of dissipation with velocity. The effect of these dependencies on the dissipation in a road network will be investigated later. The exact dependence on velocity and temperatures certainly depends on the contact length \( l \) but we keep it fixed to a value of \( l = 0.15 \) m, as has been done in
Figure 4-5: Deflection-induced excess fuel consumption per traveled vehicle as a function of the characteristic length $l_0$, for different temperature values, normalized by the deflection-induced excess fuel consumption per traveled meter at $T = 0^\circ C$ and $v = 35$ m/s. The left panel corresponds to a flexible pavement (asphalt) and the right one to concrete.

Ref. [39]. Another interesting observation, complete investigation of which is left for future work, is the following: if one plots the deflection as a function of the characteristic length $l_0 = \tau \times v$, all the deflection curves at different temperatures merge into a master curve for every pavement; as shown in Fig. 4-5. To place ourselves in a dimensionless context, this is equivalent to state that, when one considers a single pavement, the deflection-induced dissipation of this pavement is only a function of the dimensionless parameter $\Pi_4 = \Pi_1 \times \Pi_2$. This has an interesting implication: for every pavement, both temperature and velocity dependence can in fact be captured into a single parameter, which has the potential to considerably simplify deflection-induced dissipation analyses.

In summary, the deflection-induced excess fuel consumption is an increasing function of temperature, and a non-monotonic function of velocity. It increases to reach a maximum followed by a continuous decrease for most of the range of parameters. Concerning the vehicle’s load, we consider throughout the whole study an articulated truck, with the same weight and distribution as given in Ref. [39] and displayed in Fig. 4-6: three axles, one supporting a weight of 3,629 kg and the two others bearing a weight of 14,051 kg. This restriction to one truck, however, has a low influence on our conclusions of deflection-induced
dissipation, because we will most of the time consider normalized quantities, and the presence of the load squared as a multiplicative coefficient in Eq. (4.28) implies that it cancels out in the normalization processes.

4.2 Roughness Model

Another dissipation source that has to be taken into account is the effect of roughness of the road on the excess fuel consumption. The most commonly used model for this phenomena is the HDM-4 model [56], which is based on the measurement of the International Roughness Index (IRI). With a focus on the effect of velocity distribution, we consider a mechanistic model developed in [40], which not only uses the IRI, but also a power spectral density parameter, called the waviness number. This additional parameter adds a dependence on the vehicle’s velocity in the calculation of roughness-induced fuel consumption, which is of high interest to us.

4.2.1 Mechanistic Model

The effect of a road’s roughness on a vehicle’s excess fuel consumption can be computed through its effect on the vehicle’s suspension system. For this purpose we use the two degree of freedom quarter-car model [55], depicted in Fig. 4-7. Going from top to bottom, the suspension system is modelled as a mass $m_s$ (corresponding to the vehicle’s mass), attached to another mass (the wheel and the spinning axis) by a spring of stiffness $k_s$ and a dashpot of viscosity $C_s$ in parallel (the suspension). The tire is modelled as a spring of stiffness $k_t$ linking the mass $m_u$ to the ground. The only dissipative element in this model is the dashpot. The energy dissipation thus results from the vertical motion of the mass $m_s$ relative to the mass $m_u$. If we denote by $z$ this relative displacement, the force that is
applied on the dashpot is:

\[ F = C_s \left( \frac{dz}{dt} \right) = C_s \dot{v} \]  

(4.33)

The energy dissipated for a small vertical displacement \( dz \) is:

\[ \mathcal{E}_d = F.dz \]  

(4.34)

And the energy dissipated per unit of time is:

\[ \delta D = \mathcal{E}_d \left( \frac{dz}{dt} \right) = C_s \dot{z}^2 \]  

(4.35)

Hence, the energy dissipated per traveled length of the vehicle is given by:

\[ \delta \mathcal{E} = \frac{C_s}{v} \dot{z}^2 \]  

(4.36)

with \( v \) the traveling speed of the vehicle, assumed constant (steady-state conditions). In terms of expected values, the expected dissipation is linked to the expected relative displacement by:

\[ E(\delta \mathcal{E}) = \frac{C_s}{v} E(\dot{z}^2) \]  

(4.37)

The vertical displacement \( z \) arises from the roughness of the road \( \xi \), transmitted to the suspension system through the tire of stiffness \( k_t \). The roughness is usually assumed to be distributed as a Gaussian with zero mean value [14], but it can in fact be practically determined from the power spectral density (PSD) \( S_\xi \) of the road profile [53]. This PSD can
be expressed as a power law of the road frequency \( \Omega \)

\[
S_\xi(\Omega) = c|\Omega|^{-w}
\]  

(4.38)

where \( c \) is the unevenness index and \( w \) the waviness number [40]. The expected value of the vertical displacement is given by [30]:

\[
E(z^2) = \int_0^{+\infty} (v(\Omega))^2 |H_s(v\Omega)|^2 S_\xi(\Omega) d\Omega
\]  

(4.39)

with \( H_s \) the transfer function of the response from the road profile in relative vertical displacement. Writing the excitation angular frequency as \( \omega = v\Omega \), we obtain the transfer function from the suspension model shown in Fig. 4-7:

\[
H_s(\omega) = \frac{\omega^2 k_t m_s}{(-m_s\omega^2 + iC_s\omega + k_s)(-m_s\omega^2 + iC_s\omega + k_s) - (iC_s\omega + ks)^2}
\]  

(4.40)

Inserting Eq. (4.38) into Eq. (4.39) gives:

\[
E(\delta E) = cC_v \int_0^{+\infty} (\Omega)^{2-w} |H_s(v\Omega)|^2 d\Omega
\]  

(4.41)

Or, expressing the integrand as the excitation angular frequency,

\[
E(\delta E) = cC_v \omega^{w-2} \int_0^{+\infty} (\omega)^{2-w} |H_s(\omega)|^2 d\omega
\]  

(4.42)

The integrated function being velocity independent, we obtain a power law dependence between the expected dissipated energy per traveled length and the velocity of the car \( E(\delta E) \propto \omega^{w-2} \). Whether the dissipation increases or decreases with the velocity depends on the value of the waviness number \( w \). For \( w > 2 \), the dissipated energy is higher for higher velocities, and the opposite is true for \( w < 2 \). For \( w = 2 \), the dissipation is independent of the traveling velocity. As we will show next, pavements typically exhibit a waviness number of the order of \( w = 2.5 \), thus implying an increase in dissipation with increasing speed.
4.2.2 Link with HDM-4 Model

It is now possible to include the IRI by incorporating the classical HDM-4 model into the mechanistic model previously described. The International Roughness Index is defined, as mentioned in Ref. [40], as "the average rectified velocity of a specific quarter-car (golden car) traveling at \( V_0 = 22 \text{ m.s}^{-1} \)"

\[
IRI = \frac{1000}{V_0L} \int_0^L |\dot{z}_{GC}| dx
\]  

(4.43)

with "GC" referring to the golden car, which properties are given by:

\[
\begin{align*}
\left( \frac{k_a}{m_a} \right)_{GC} &= 653 \text{ s}^{-2} \\
\left( \frac{k_s}{m_s} \right)_{GC} &= 63.3 \text{ s}^{-2} \\
\left( \frac{C_s}{m_s} \right)_{GC} &= 6.0 \text{ s}^{-1} \\
\left( \frac{m_a}{m_s} \right)_{GC} &= 0.15
\end{align*}
\]  

(4.44)

Using the same transfer function as earlier in Eq. (4.40) but with the parameters of the golden car, the expected value of the IRI becomes:

\[
E(\text{IRI}) = \sqrt{\frac{2}{\Pi} \frac{V_0^{w-3} c}{V_0} \int_0^{+\infty} \omega^{2-w} |H_{s,GC}(\omega)|^2 d\omega}
\]  

(4.45)

Finally, by eliminating the unevenness number \( c \) in Eqn. (4.42) and (4.45), we obtain the following equation which closes the loop between the classical HDM-4 model and the mechanistic model:

\[
E(\delta\mathcal{E}) = \frac{\Pi}{2} C_s V_0 \left( \frac{v}{V_0} \right)^{w-2} (E(\text{IRI}))^2 \left( \frac{\int_0^{+\infty} (\omega)^{2-w} |H_s(\omega)|^2 d\omega}{\int_0^{+\infty} \omega^{2-w} |H_{s,GC}(\omega)|^2 d\omega} \right)
\]  

(4.46)

Eq. 4.46 shows that the energy dissipated per unit of traveled length is dependent on the mechanical properties of the vehicle (given from the suspension system), the traveling velocity, the waviness number and the expected value of IRI.
<table>
<thead>
<tr>
<th>Car class Properties</th>
<th>Medium Car</th>
<th>SUV</th>
<th>VAN</th>
<th>Light Truck</th>
<th>Articulated Truck</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{l}$ (ton)</td>
<td>1.46 (4)</td>
<td>2.5 (4)</td>
<td>2.54 (4)</td>
<td>6.5 (4)</td>
<td>34.9 (4)</td>
</tr>
<tr>
<td>$m_{s}$ (kg)</td>
<td>80 (19)</td>
<td>125Φ</td>
<td>134 (20)</td>
<td>395Φ</td>
<td>544 (21)</td>
</tr>
<tr>
<td>$k_{s}$ (KN/m)</td>
<td>29.44 (19)</td>
<td>189Φ</td>
<td>48</td>
<td>337Φ</td>
<td>700 (21)</td>
</tr>
<tr>
<td>$\xi_{b}$</td>
<td>.096 (4)</td>
<td>.072 (4)</td>
<td>.072 (4)</td>
<td>.062 (4)</td>
<td>.059 (4)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>46.98</td>
<td>28.03</td>
<td>31.00</td>
<td>14.90</td>
<td>13.30</td>
</tr>
</tbody>
</table>

Table 4.2: Mechanical properties of different classes of vehicles, taken from [40]

### 4.2.3 Model Calibration

Some of the parameters involved in the computation of the energy dissipation in Eq. (4.46) cannot be accurately determined, and the model therefore needs to be calibrated. This is the case for the stiffness involved in the 2 degrees of freedom model, that are not straightforward to estimate for a vehicle, and of the waviness number. Unlike the derivation of the model's equation, that shows us where the velocity dependence of the dissipation comes, the calibration details are not relevant to our discussion. We simply need to know that the waviness number is obtained from the optimization is $w = 2.4117$, which is close to the $2.5$ value reported in Ref. [37], and the mechanical properties of different vehicles classes are reported in Table 4.2, taken from Ref. [40] where $\xi_{b}$ is the engine efficiency expressed in mL.kW$^{-1}$.s$^{-1}$ and $\beta_{j}$ is a dimensionless parameter defined as

$$\beta_{j} = \sqrt{\frac{k_{l}m_{s}}{k_{s}m_{u}}} \quad (4.47)$$

With these calibration values we have a model for the energy dissipation due to the roughness phenomena that is velocity dependent. This dependency is easier to understand than the one for the deflection model (see Section 4.1) where it is embedded in the transfer function. A main difference here is the independence of the dissipation with respect to the temperature. The roughness-induced dissipation is dependent on the vehicle's velocity, the vehicle properties as well as the waviness and IRI parameters, which are pavement properties. The velocity dependence is plotted in Fig. 4-8 for different pavement types, for an articulated truck (see Table 4.2). The chosen values for IRI and waviness number are taken from Ref. [40] and are shown in Table 4.3. With a waviness number close to $w \approx 2.5$
Figure 4-8: Roughness-induced dissipation normalized by the dissipation at $v = 35$ m/s as a function of velocity for two types of pavements: asphalt (AC) and concrete (PCC).

for both pavement types (see Table 4.3) and the expected dissipation per traveled length proportional to $v^{n-2}$ (see Eq. (4.46)), it is not surprising to see a velocity dependence of the dissipation extremely similar for both pavements (see Fig. 4-8) with a shape similar to a square root and higher dissipations at higher velocities. If the waviness number had been smaller than 2, we would have had a power law with a singularity at zero velocity, and lower dissipation at higher speed.

4.3 Excess Fuel Consumption Estimation

In this Section we bridge the gap between the velocity distributions functions obtained in Chapter 3 and the dissipation models presented in Sections 4.1 and 4.2 to estimate the energy dissipated by PVI on a road.
<table>
<thead>
<tr>
<th>Pavement type</th>
<th>Statistic</th>
<th>IRI [mm/m]</th>
<th>w [1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC</td>
<td>MEAN</td>
<td>1.320</td>
<td>2.475</td>
</tr>
<tr>
<td></td>
<td>STD</td>
<td>0.541</td>
<td>0.417</td>
</tr>
<tr>
<td>PCC</td>
<td>MEAN</td>
<td>1.617</td>
<td>2.479</td>
</tr>
<tr>
<td></td>
<td>STD</td>
<td>0.492</td>
<td>0.314</td>
</tr>
</tbody>
</table>

Table 4.3: Mechanical properties of different classes of pavement: AC for asphalt and PCC for concrete, taken from [40]

4.3.1 Traffic Data and Simulations

To reach our goal we need to link the results from traffic simulations with the data measured on a road network. The precise calibration of the velocity distribution obtained from our models would have to be done using GPS data of velocity and is left for future work. When one sits by the side of the road, the easiest data to measure is the average traffic flow, namely the number of cars passing a certain point per unit of time. The traffic flow averaged over an entire day is usually referred to as AADT, standing for Average Annual Daily Traffic (or AADTT, average annual daily truck traffic, for trucks). However, as has been mentioned in Chapters 3 and 4, the average flow exhibits a non monotonic behavior as a function of the density: it increases linearly with the density until a critical density $\rho_c$, above which the flow decreases when the density increases. Therefore, any value of the measured AADT can be associated with two densities, one below and one above $\rho_c$. The density below the critical point corresponds to a free flow regime, and the density above belongs to a congested regime. In order to know the dissipation in the network, it is important to know both the vehicle density on the road and their velocity distribution. Considering the non-uniqueness of density (and hence distribution) for a measured flow, we need an additional parameter to determine the nature of the flow, or in other words an order parameter. The order parameter we have chosen in Chapter 3 is the probability to find a stopped vehicle. It is a natural parameter for the study from a statistical point of view, but in fact it is not practical: we would need to check on a road segment if any car has actually stopped. Another choice can be made, by having a closer look at the distributions in both the free flow and the congested regimes. We found in Chapter 3 that in the free flow regime, the velocity distribution function is exactly known from our model parameters, and identical for all densities below $\rho_c$. In contrast, at high densities, we found that the probability to find a stopped vehicle is
non zero, and that the distribution is a superposition of the free flow distribution weighted by the fraction of vehicles in the free flow phase, and a jamming distribution (including the transitioning phase) whose precise shape was dependent on the density by the internal structure, that is:

\[ \forall \rho > \rho_c ; \quad P^\rho(v) = \alpha P^j(v) + (1 - \alpha) P_{FF}(v) \quad (4.48) \]

with \( P^\rho \) the jamming distribution at density \( \rho \), \( P_{FF} \) the free flow distribution, and \( \alpha \) the fraction of vehicles that are not in the free flow phase. Since the jamming distribution has higher value at low velocities, the mean velocity defined by:

\[ \langle v \rangle = \sum_{v=0}^{v_{\text{max}}} v P(v) \quad (4.49) \]

will be lower for the congested regime than for the free flow state. This result, in fact very intuitive, implies that the mean velocity can be used as an order parameter. We thus define the free flow mean velocity as:

\[ \langle v \rangle_{FF} = \sum_{v=0}^{v_{\text{max}}} v P_{FF}(v) \quad (4.50) \]

It is exactly known from the model (in the case where there are no interactions). The observed traffic belongs to the congested phase as soon as \( \langle v \rangle < \langle v \rangle_{FF} \). This is illustrated in Fig. 4-9. The mean velocity being a parameter effectively measurable, we can easily conceive that its measurement, combined with the measurement of average flux divided by the number of lanes, will enable us to know the density. Since the velocity distribution function is uniquely defined by the density, AADT and average velocity measurements provide us with sufficient information to estimate the roughness and deflection-induced dissipation on a road section.

### 4.3.2 Single Speed Approximation and Velocity Distribution Function

In the current analyses of energy dissipation in a roadway network, it is assumed that all the vehicles travel at a single speed. Let us call this the single speed approximation (SSA). This speed is generally assumed to be the speed limit of the road but it could also be taken as the measured mean velocity, which is considerably smaller than the speed limit in the presence of congestions. The approach we propose is to use the velocity probability distribution
functions (PDF) obtained in the simulations instead of the SSA to calculate the expected value of dissipation. The energy dissipated using the SSA reads:

$$\mathcal{E}_{SSA} = \mathcal{E}(\langle v \rangle)$$

and the one using the PDF is:

$$\mathcal{E}_{PDF} = \sum_{0}^{u_{max}} \mathcal{E}(v)P(v)$$

In purpose of comparison we use the extended Nagel-Schreckenberg model presented in Section 3.2. The choice of the model will of course have an influence on the precise dissipation values, but this one has the advantage to provide velocity distributions with sufficient accuracy. This is sufficient for our purpose which is to capture the influence on the use of velocity distribution function on the estimation of energy dissipation. For the rest of the chapter,
we compute the energy dissipated per vehicle per traveled lane length using the mechanistic deflection and roughness models previously described for an articulated truck (see Sections 4.1 and 4.2). The asphalt pavement herein considered corresponds to a pavement of type GPS1 and the concrete pavement to a pavement of type GPS3 (see Ref. [39] for details).

The effect of traffic on the dissipation can be calculated from different points of view and we attempt to cover most of them in the coming sections in order to have a better picture of the influence of jams and the differences between both approaches.

### 4.3.3 Dissipation as a Function of Occupancy

The first angle, and the most natural one, is to start directly with the result given by the simulations: looking at the effect of density on the dissipation, since the traffic simulations provide us with a direct relationship between the occupancy and the velocity distribution. We indeed found in Chapter 3 that the function associating every occupancy with the corresponding velocity probability distribution function is well defined: every element from the initial space, the space of occupancies, has at most one image in the space of the velocity distributions. In mathematical terms, calling $\mathcal{F} = [0; 1]$ the ensemble of possible occupancies, $\mathcal{P} = \left\{ (P_0, ..., P_{v_{\text{max}}} \in [0; 1]^{v_{\text{max}}+1} \mid \sum_{i=0}^{v_{\text{max}}} P_i = 1 \right\}$ defines the ensemble of possible velocity distributions and $s_{\text{NSE}}$ the application that associates a given occupancy with the corresponding velocity distribution through the extended NS model:

$$s_{\text{NSE}} : \left\{ \begin{array}{c}
\mathcal{F} \rightarrow \mathcal{P} \\
\mathcal{O} \mapsto s_{\text{NSE}}(\mathcal{O}) = P = (P_0, ..., P_{v_{\text{max}}})
\end{array} \right. \quad (4.53)$$

with:

$$\forall \mathcal{O} \in \mathcal{F} ; \exists P \in \mathcal{P} \text{ so that } s_{\text{NSE}}(\mathcal{O}) = P \quad (4.54)$$

Given this property we can directly plot the dissipation as a function of the occupancy, for both deflection (see Fig. 4-10, left pane) and roughness (See Fig. 4-10, right panel) induced dissipations. We notice that the shape of the roughness-induced dissipation as a function of the occupancy is very similar to the one of the mean velocity depicted in Fig. 4-9. This is easily understandable for the SSA approximation (see Eq. 4.51 ), which gives the dissipation as a power law of the mean velocity. The fact that this is also the case for the PDF approximation (see Eq. (4.52) ) is due to the fact that the increase in occupancy
separates the resulting distribution into a concentration around zero velocity and another one at the maximum velocity, and that these concentrations more or less linearly evolve with the increase of occupancy (see Section 3.2.2). Both dissipations are not very different from each other, implying that roughness-induced dissipation is not very sensitive to this velocity distribution. We observe a similar trend for deflection phenomenon (at \( T = 20^\circ C \)): the SSA and PDF approximations globally exhibit the same occupancy dependence but the difference between both is more accentuated. In fact, both show a strong increase of dissipation with increasing occupancy, until a maximum is reached, after which the dissipation decreases with increasing occupancy. The first increase of dissipation corresponds to the formation of traffic jams, and the latter decrease is explained by the non monotonic behaviour of dissipation with velocity. At higher occupancies, the high concentration of traffic jams implies a low mean velocity, eventually lower than the velocity that induces the maximum dissipation \( v^* \) (see Section 4.1 and Fig. 4-4, left panel). The dissipation then decreases with increasing occupancy. Higher occupancies also imply a high probability to find vehicles at low velocities, and hence eventually lower dissipation. Comparing SSA and PDF approximations, it seems that the SSA tends to overestimate the value of excess fuel consumption, because it does not take into account the lower values of velocities that are present within traffic jams. We should mention, however, that at low velocities the instantaneous fuel consumption is dominated by vehicle idling (see Ref. [9]), and the discussion of the difference between SSA and PDF approximations loses its relevance. The study of asphalt and its non monotonic behavior is left for the study of dissipation as a function of the flux, which gives a better overall picture of the influence of traffic behaviour on energy dissipation.

It is important to note that for both phenomena, the SSA and PDF approximations give the same result at occupancies lower than the critical one \( O_c \), implying that the dissipations estimated from the distribution function in the free flow case is equal to the one calculated from the mean velocity. For this reason all the plotted distributions are normalized by the free flow dissipation, the value of excess fuel consumption per traveled lane mile per vehicle found in the free flow phase.

### 4.3.4 Dissipation as a Function of the Mean Velocity

If we remind ourselves of the plot of mean velocity as a function of the occupancy shown in Fig. 4-9, we observe that the velocity distribution is a well defined function of the mean
velocity, because of the monotonic relation between mean velocity and occupancy. We can then plot the excess fuel consumption as a function of the mean velocity. This enables us to have a better understanding of the effect of taking the velocity distribution into account.

In Fig. 4-11 we show the deflection (left) and roughness (right) induced dissipations per vehicle per traveled lane mile as a function of the mean velocity, normalized by the free flow dissipation, for concrete pavements. The chosen temperature is again 20°C. Low mean velocities correspond to high occupancies (see Fig. 4-9), or, equivalently, to highly congested states with a high probability to find stopped vehicles. It is thus recognized that the deflection-induced dissipation estimation more largely differs between the SSA and the PDF approximations, for the same reason as mentioned previously: the fact that vehicles are stopped and travel at very low velocities is not taken into account in the SSA, and considerably reduces the estimation in the PDF. At high mean velocities, the congestions slowly disappear and the system gets closer to the free flow where both approaches give the same result, which is why they get closer as the mean velocity increases. The roughness-induced dissipation shows few differences between the SSA and the PDF approximations no matter what the mean velocity is. This is consistent with the fact that the velocity
Deflection - GPS3 Roughness - PCC

S1.8 00.9

CL 1.6 - L-o.8-

0.0

0.2 0.4 0.6 0.8 0

Normalized Mean Velocity

Normalized Mean Velocity

Figure 4-11: Normalized dissipation as a function of the mean velocity, for the deflection phenomenon (left) and the roughness phenomenon (right) for an articulated truck on a concrete pavement at a temperature of 20°C. The dissipation is normalized by the dissipation in the free flow phase, for $t < t_c$

dependence of the roughness-induced dissipation is rather weak ($\propto t^{n-2}$).

4.3.5 Dissipation as a Function of Average Flow

Slightly less intuitive, we now compute the dissipation as a function of the average flow. Unlike occupancy and mean velocity, a value of average flow does not uniquely define a distribution. This has been mentioned in Section 4.3.1, and can be understood from the fundamental diagram (see Fig. 3-6): any value of average flow can correspond to two different occupancies, one corresponding to a free flow phase and one to a congested phase. A single average flow can therefore correspond to two different distributions, and therefore two distinct dissipations. We plot the dissipation estimates for both phases in each flow, using the two approaches mentioned earlier. Since the SSA and the PDF approaches give the same dissipation in the free flow phase, we only need three curves: a free flow curve that will be used to normalize the other dissipations; two congested dissipation estimates (one for SSA, one for PDF). They are plotted in Fig. 4-12 for a temperature of 20°C and a concrete pavement. We note that the dissipation in the congested flow is higher for the deflection than the one in the free flow state for most of the flow values, and lower for the roughness.
Figure 4-12: Normalized dissipation as a function of the mean flow, for the deflection phenomenon (left) and the roughness phenomenon (right) for an articulated truck on a concrete pavement at a temperature of 20°C. The dissipation is normalized by the dissipation in the free flow phase, for $\mathcal{O} < \mathcal{O}_c$.

This is consistent with the fact that the congested flow sees a lower mean velocity and lower velocities in its distribution than in free flow. We also recover the large difference between the two approaches in the deflection case at low flow, equivalent to low mean velocity in the congested phase.

To this point, we focussed on the behavior at a single temperature, namely $T = 20^\circ$ C. It is interesting to compare the relative dissipations at different temperature values, for both asphalt and concrete pavement. In the case of deflection-induced dissipation, we have concluded from Fig. 4-4 that an increase in temperature leads to an increase in the dissipation rate. In Fig. 4-13 we compare the deflection-induced excess fuel consumption in the congested and free flow regimes for the SSA and PDF approximations normalized by the free flow dissipation at different temperatures. We note that despite the general increase in dissipation, the ratio between the two regimes decreases with increasing temperature, and even inverts at higher temperatures for the asphalt pavement. In other words, high temperatures show less dissipation in congested flow than in free flow for a certain range of flows. This behavior is inherently linked with the non monotonic relation between deflection-induced dissipation and velocity, shown in Fig. 4-4. This result has a profound meaning and counter
intuitive implications. Discounting idling of vehicles, traveling slower, and even in traffic jams, can reduce the energy dissipated per traveled lane mile by deflection of the pavement, under certain circumstances. On the roughness side, Fig. 4-14 compares the roughness-induced dissipation rate per vehicle in the free flow and congested regimes. The difference between the two pavements is small when normalized, implying that they are both as sensitive to the velocity dependence. This result is consistent with the fact that the power coefficient in the velocity dependence for both pavements are extremely close as shown in Fig. 4.3. The appearance of congestions is then an important factor in the amount of dissipated energy and the effect on the induced dissipation is non trivial. On the one hand it reduces the roughness - induced dissipation per traveled lane mile per vehicle, and increases or decreases (depending on the conditions) the deflection - induced dissipation per vehicle. If ones takes the point of view of the road, overall dissipation on a fixed segment would need to take into account the increase in number of vehicles (which is significant) in the presence of congestions, which would most likely increase the total PVI - induced dissipation. This
Figure 4-14: Normalized roughness-induced dissipation as a function of the mean flow, for an asphalt pavement (left panel) and a concrete pavement (right panel). The scale in every figure is the same and goes from 0 to 1 for the normalized dissipation and 0.3 to 1 for the normalized flow.

observation, together with the precise influence of the contact length $l$ and the comparison of the exact values of deflection and roughness-induced dissipations to see how they scale relatively to one another, is left for future work.

### 4.4 Chapter Summary

In this chapter we described two PVI phenomena, one related to the deflection of the pavement under the vehicle's load, and one related to the roughness of the pavement. We employed two mechanistic models described in Refs. [39] and [40] to obtain an estimate of the energy dissipation per vehicle per traveled lane length and showed that while the deflection-induced dissipation decreases when the vehicle's velocity increases and is an increasing function of the external temperature, the roughness-induced one is independent of the temperature and increases with increasing vehicle's velocity.

Comparing the estimations from the SSA and PDF approximations, we found that due to its smooth velocity dependence, the roughness-induced dissipation is similar for both approximations. The deflection-induced dissipation however shows non-monotonic dependence with the traveling velocity, which translates into greater differences between SSA and PDF,
especially at high occupancies, or equivalently low mean velocities.

Finally, we showed in Section 4.3 that when one only observes the average flow, it is not possible to distinguish whether we are in a free flow regime where all the vehicles travel close to the speed limit or in a congested regime where the vehicle density is much higher and where jams are forming. Considering a single average flow and computing the energy dissipated per traveled lane mile per vehicle for roughness and deflection using two methods (SSA and PDF), we realize that if the traffic is in the free flow state, both approaches give the same value of excess fuel consumption. If the traffic is in the congested state though, the two approaches give different results. This means that the velocity distribution has an influence on the value of the dissipation. However, in both approximations, a vehicle dissipates less energy per traveled lane mile by roughness and can dissipate either less or more energy by deflection, depending on the external temperature and the nature of the pavement. Asphalt pavement and high temperatures tend to see less deflection - induced dissipation in the congested state than in the free flow state, when concrete and lower temperatures tend to exhibit the inverse behaviour. The temperature sensitivity of the deflection phenomena leads to an overall increase in dissipation with increasing temperature, but a decrease in velocity sensitivity: at high temperatures, the estimated dissipations in the free flow and the congested regimes are close to each other. This implies that with increasing temperature, the velocity sensitivity decreases and the appearance of congestions does not lead to major changes in the dissipation compared to the free flow case. We showed that the presence or not of jams can theoretically be determined by the mean velocity, which can serve as an order parameter, when the density is known from the average flow.
Chapter 5

Conclusion

5.1 Summary and Main Findings

We studied in this thesis how existing traffic models can provide relevant information about velocity distribution on a road network, which can effectively be used to estimate energy dissipation due to Pavement-Vehicle Interactions, namely deflection and roughness induced PVIs.

We first focussed on the study of force-base models within a convenient framework to introduce the theory of traffic simulations and its main properties. Within this framework we showed that mechanistic forces can be isolated from behavioral forces. It is this decoupling of forces of different origins which allowed us to focus on velocity distributions separately from the dissipation they provoke in the context of PVIs. We found that the existing force-based models with few fitting parameters were unable to represent the non linear behavior of traffic at both the macroscopic and microscopic scales.

The shortcoming of this class of models brought us to investigate cellular automata models, and specifically the Nagel Schreckenberg model which is capable of reproducing traffic behavior at both micro and macro scales, with just a few parameters. Investigating this model from the point of view of velocity distribution, we identified an order parameter which permits describing a smooth phase transition from a congested flow to a free flow. Obtaining the exact velocity distribution is non trivial, and strongly depends on the density. To address this issue, we proposed an analytical model that takes into account the internal structure of the traffic. We show that this model successfully links the headway distribution to velocity distribution. We then extended the NS model with a finer velocity discretization,
and showed that this refined model exhibits the same characteristic patterns as the original NS model, with only a few more fitting parameters.

The simulation results thus obtained served as input to estimating the excess fuel consumption related to PVI-induced dissipation. After recalling two mechanistic models used to estimate the dissipation due to the deflection of the pavement under a moving load and due to pavement roughness, we investigated how the predictions scale when incorporating traffic conditions. We specifically focused on the difference between the dissipation predicted by assuming a single speed for all the vehicles and by taking into account the complete velocity distribution in both congested and free flow phases. By comparing different pavement types and temperatures, we found that the type of pavement has a low impact on the ratio between free flow and congested dissipations, whereas increasing the temperature tends to decrease this ratio. Thus, realistic traffic flow simulations can be employed to improve predictions of PVI-induced excess fuel consumption.

5.2 Limitations and Possible Future Perspectives

A couple of restrictions of our approach need to be further addressed. We isolated the system to have statistically relevant data for our simulations, placing ourselves in unperturbed traffic conditions and under steady state conditions. However, nothing tells us that the traffic flow on a highway is anything close to a steady state. As far as we know, traffic flow is a very complex system, with no vehicle conservation, filled with perturbations that may place the flow in perpetuate transient conditions. In order to confirm the validity of our approach, a calibration and comparisons with field data from a road section that presents controlled parameters is needed. Another limitation comes from a fundamental assumptions made in the mechanistic models, which is the one of constant vehicle velocity. The change of the reference system would not be possible in the deflection model and the expectation value of energy dissipation would cease to be easily definable without this assumption. But the shape of the velocity distributions which are spread over a large interval of velocity values, implies that the vehicles see a great amount of acceleration and deceleration episodes, which contradicts this assumption. The validity of these models thus requires to compare acceleration and deceleration time scales in real traffic with the characteristic time scale of the mechanistic models.
Finally, there is much room for further investigation on the topics presented in this thesis. When considering the extended NS model, we found that the free flow velocity distribution was exactly given by the deceleration distribution we chose in the model. Specifically, we considered a uniform deceleration distribution as a convenient choice to recover the results of the classical NS model and to extend the analytical solution of the three-body interaction. But there are few chances that this distribution matches field data. A Gaussian deceleration distribution would probably be more realistic for this purpose. Future work thus needs to consider to calibrate the free flow phase with data from real traffic, and then adjust the remaining parameters, such as the acceleration rate, to match velocity distributions of a congested flow with field data. On the dissipation side, the estimates obtained are dependent on the shape of the velocity distribution. This merits further refinement through a detailed study of the impact of distribution parameters, such as the variance (spread around the mean value) and the skewness (symmetry around the mean value) on the calculated PVI-induced dissipations.

Moreover, in this study we discarded agent-based models because of their lack of generality and their complexity. However, with the present development of self-driven cars by companies as Google [44], it is possible that vehicle behavior could become completely deterministic. This would thus exclude the uncertainty that is inherently present with human decisions in the current traffic system. We can then easily imagine that force-based models are up to this task, especially in simple situations such as a highway, where the vehicle's behavior is perfectly known by the constructor and the automated driver. All this hints that coupling traffic flow with mechanistic models can become an integral part of managing our current and future road networks in both operation and maintenance. A very first attempt has been achieved in this thesis.
Appendix A

Nomenclature of Chapter 2

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Units</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_i$</td>
<td>–</td>
<td>Parameter $\alpha$ of vehicle $i$.</td>
</tr>
<tr>
<td>$(x, y)$</td>
<td>[m]</td>
<td>Coordinates of the vehicle in the force based models.</td>
</tr>
<tr>
<td>$A$</td>
<td>[m]</td>
<td>Vehicle length</td>
</tr>
<tr>
<td>$\theta$</td>
<td>–</td>
<td>Angle between vehicle preferential orientation and the Cartesian coordinate system</td>
</tr>
<tr>
<td>$(\xi_r, \xi_\theta)$</td>
<td>–</td>
<td>Coordinate system attached to a vehicle</td>
</tr>
<tr>
<td>$\sum F$</td>
<td>[N]</td>
<td>Sum of the forces acting on a vehicle</td>
</tr>
<tr>
<td>$F^b$</td>
<td>[N]</td>
<td>Total behavioural force acting on a vehicles</td>
</tr>
<tr>
<td>$F^m$</td>
<td>[N]</td>
<td>Total mechanical force acting on a vehicles</td>
</tr>
<tr>
<td>$\rho^a$</td>
<td>[kg],[m]$^{-3}$</td>
<td>Air density</td>
</tr>
<tr>
<td>$C$</td>
<td>–</td>
<td>Drag coefficient</td>
</tr>
<tr>
<td>$S$</td>
<td>[m]$^{-2}$</td>
<td>Typical vehicle front surface</td>
</tr>
<tr>
<td>$f^d$</td>
<td>[N]</td>
<td>Driving force acting on a stopped vehicle</td>
</tr>
<tr>
<td>$v$</td>
<td>[m],[s]$^{-1}$</td>
<td>Vehicle velocity</td>
</tr>
<tr>
<td>$V_{lim}$</td>
<td>[m],[s]$^{-1}$</td>
<td>Limit velocity for the force based models</td>
</tr>
<tr>
<td>$m$</td>
<td>[kg]</td>
<td>Vehicle total mass</td>
</tr>
<tr>
<td>$F^v_j, F^v_{i\rightarrow j}$</td>
<td>[N]</td>
<td>Interaction force from vehicle $j$ acting on the vehicle</td>
</tr>
<tr>
<td>Symbol</td>
<td>Unit</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
<td>------------</td>
</tr>
<tr>
<td>( F^d )</td>
<td>([N])</td>
<td>Driving force</td>
</tr>
<tr>
<td>( \tau )</td>
<td>([s])</td>
<td>Relaxation time for the driving force</td>
</tr>
<tr>
<td>( a )</td>
<td>([m][s]^{-2})</td>
<td>Total acceleration of a vehicle</td>
</tr>
<tr>
<td>( U^{i,j} )</td>
<td>([N].[m])</td>
<td>Interaction potential between vehicles (i) and (j)</td>
</tr>
<tr>
<td>( N )</td>
<td>-</td>
<td>Total number of vehicles</td>
</tr>
<tr>
<td>( t )</td>
<td>([s])</td>
<td>Time</td>
</tr>
<tr>
<td>( \delta t )</td>
<td>([s])</td>
<td>Time increment</td>
</tr>
<tr>
<td>( E_c )</td>
<td>([J])</td>
<td>Total kinetic energy</td>
</tr>
<tr>
<td>( E_p )</td>
<td>([J])</td>
<td>Total Potential energy</td>
</tr>
<tr>
<td>( \delta E )</td>
<td>([J])</td>
<td>Energy increment</td>
</tr>
<tr>
<td>( \Delta x_{i,j} )</td>
<td>([m])</td>
<td>Distance between vehicle (i) and (j)</td>
</tr>
<tr>
<td>( \Delta v_{i,j} )</td>
<td>([m][s]^{-1})</td>
<td>Velocity difference between vehicle (i) and (j)</td>
</tr>
<tr>
<td>( M )</td>
<td>([m].[N])</td>
<td>Total moment acting on the vehicle</td>
</tr>
<tr>
<td>( M^w )</td>
<td>([m].[N])</td>
<td>Moment resulting from the action of the wall</td>
</tr>
<tr>
<td>( M^d )</td>
<td>([m].[N])</td>
<td>Damping moment</td>
</tr>
<tr>
<td>( \delta_{\text{front}} )</td>
<td>([m])</td>
<td>Distance in front of the vehicle center of gravity where the moments are acting</td>
</tr>
<tr>
<td>( \delta_{\text{rear}} )</td>
<td>([m])</td>
<td>Distance behind the vehicle center of gravity where the moments are acting</td>
</tr>
<tr>
<td>( f^w )</td>
<td>([N])</td>
<td>Force acting from the wall on the vehicle</td>
</tr>
<tr>
<td>( U^w )</td>
<td>([N].[m])</td>
<td>Interaction potential between the wall and the vehicles</td>
</tr>
<tr>
<td>( \rho )</td>
<td>([m]^{-1})</td>
<td>Average vehicle density</td>
</tr>
<tr>
<td>( \rho_c )</td>
<td>([m]^{-1})</td>
<td>Critical average vehicle density</td>
</tr>
<tr>
<td>( \rho_{\text{max}} )</td>
<td>([m]^{-1})</td>
<td>Maximum average vehicle density</td>
</tr>
<tr>
<td>( \Phi )</td>
<td>([s]^{-1})</td>
<td>Average vehicle flux</td>
</tr>
<tr>
<td>( \Phi_{\text{max}} )</td>
<td>([s]^{-1})</td>
<td>Maximum average vehicle flux</td>
</tr>
<tr>
<td>( L )</td>
<td>([m])</td>
<td>Road length</td>
</tr>
<tr>
<td>( v_s )</td>
<td>([m].[s]^{-1})</td>
<td>Stationary state velocity</td>
</tr>
<tr>
<td>( \Delta x_s )</td>
<td>([m])</td>
<td>Stationary state vehicle spacing</td>
</tr>
<tr>
<td>Symbol</td>
<td>Unit</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
<td>-------------</td>
</tr>
<tr>
<td>$\mathcal{S}$</td>
<td>-</td>
<td>Space of solutions of the stationary state equation</td>
</tr>
<tr>
<td>$\mathcal{E}$</td>
<td>-</td>
<td>Space of solutions of the locally stable stationary state equation</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>-</td>
<td>Stationary state equation for vehicle spacing</td>
</tr>
<tr>
<td>$\mathcal{G}$</td>
<td>-</td>
<td>Stationary state equation for average density</td>
</tr>
<tr>
<td>$f$</td>
<td>$[m]^{-1}$</td>
<td>Stationary state solution for velocity</td>
</tr>
<tr>
<td>$g$</td>
<td>$[m].[s]^{-1}$</td>
<td>Stationary state solution for density</td>
</tr>
<tr>
<td>$O$</td>
<td>-</td>
<td>Fraction of occupied space by the vehicles</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$[m].[s]^{-1}$</td>
<td>Velocity Increment</td>
</tr>
<tr>
<td>$\lambda \pm$</td>
<td>$[s]^{-1}$</td>
<td>Frequencies of the solution of the stability analysis</td>
</tr>
<tr>
<td>$k$</td>
<td>-</td>
<td>Wave number</td>
</tr>
<tr>
<td>$V_{opt}$</td>
<td>$[m].[s]^{-1}$</td>
<td>Optimal velocity</td>
</tr>
</tbody>
</table>
Appendix B

Nomenclature of Chapter 3

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Units</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_n$</td>
<td></td>
<td>Velocity of vehicle $n$</td>
</tr>
<tr>
<td>$d_n$</td>
<td></td>
<td>Headway of vehicle $n$</td>
</tr>
<tr>
<td>$v_{max}$</td>
<td></td>
<td>Maximum velocity</td>
</tr>
<tr>
<td>$p$</td>
<td></td>
<td>Stochasticity parameter</td>
</tr>
<tr>
<td>$q = 1 - p$</td>
<td></td>
<td>Stochasticity parameter</td>
</tr>
<tr>
<td>$L$</td>
<td>$m$</td>
<td>Total road length</td>
</tr>
<tr>
<td>$T$</td>
<td>$s$</td>
<td>Total simulation time</td>
</tr>
<tr>
<td>$\rho$</td>
<td></td>
<td>Vehicle density - in vehicles per cell</td>
</tr>
<tr>
<td>$A$</td>
<td></td>
<td>Phase Space</td>
</tr>
<tr>
<td>$P$</td>
<td></td>
<td>Velocity probability distribution function</td>
</tr>
<tr>
<td>$Q$</td>
<td></td>
<td>Headway probability distribution function</td>
</tr>
<tr>
<td>$V_{lim}$</td>
<td>$[m].[s]^{-1}$</td>
<td>Speed limit</td>
</tr>
<tr>
<td>$\Delta x$</td>
<td>$[m]$</td>
<td>Cell size</td>
</tr>
<tr>
<td>$\Delta t, \tau$</td>
<td>$[s]$</td>
<td>Iteration time</td>
</tr>
<tr>
<td>$D$</td>
<td>$[m]$</td>
<td>Average distance between two vehicles</td>
</tr>
<tr>
<td>$G_v$</td>
<td></td>
<td>Velocity-velocity correlation function</td>
</tr>
<tr>
<td>$w$</td>
<td></td>
<td>Two body joint distribution function</td>
</tr>
<tr>
<td>$r$</td>
<td></td>
<td>Three body joint distribution function</td>
</tr>
<tr>
<td>$d_0$</td>
<td></td>
<td>Jam spacing</td>
</tr>
<tr>
<td>$Q_0$</td>
<td></td>
<td>Jam spacing distribution function</td>
</tr>
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</table>
\( \alpha \)  
Fitting parameter

\( V_{\text{max}} \)  
Maximum velocity

\( \Lambda \)  
Vehicle length

\( \lambda \)  
Number of cells occupied by a vehicle

\( l \)  
Total number of cells on the road

\( d \)  
Average headway

\( \mathcal{O} \)  
Occupancy

\( \alpha \)  
Total number of cells a vehicle can accelerate

\( \sigma \)  
Total number of cells a vehicle can decelerate
Appendix C

Nomenclature of Chapter 4

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Units</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h )</td>
<td>([m])</td>
<td>Pavement thickness</td>
</tr>
<tr>
<td>( \rho )</td>
<td>([kg],[m]^{-3})</td>
<td>Pavement density</td>
</tr>
<tr>
<td>( E )</td>
<td>([Pa])</td>
<td>Pavement Young’s Modulus</td>
</tr>
<tr>
<td>( \tau )</td>
<td>([s])</td>
<td>Relaxation time</td>
</tr>
<tr>
<td>( \eta )</td>
<td>([Pa],[s]^{-1})</td>
<td>Viscosity</td>
</tr>
<tr>
<td>( k_s )</td>
<td>([Pa],[m]^{-1})</td>
<td>Subgrade modulus of pavement</td>
</tr>
<tr>
<td>( v )</td>
<td>([m],[s]^{-1})</td>
<td>Vehicle velocity</td>
</tr>
<tr>
<td>( P )</td>
<td>([N])</td>
<td>Vehicle loading</td>
</tr>
<tr>
<td>( b )</td>
<td>([m])</td>
<td>Loading width</td>
</tr>
<tr>
<td>( l )</td>
<td>([m])</td>
<td>Loading length</td>
</tr>
<tr>
<td>( S )</td>
<td>([m]^2)</td>
<td>Contact area</td>
</tr>
<tr>
<td>( l_s )</td>
<td>([m])</td>
<td>Winkler length</td>
</tr>
<tr>
<td>( c_{cr} )</td>
<td>([m],[s]^{-1})</td>
<td>Critical velocity</td>
</tr>
<tr>
<td>( D )</td>
<td>([J],[s]^{-1})</td>
<td>Dissipation rate</td>
</tr>
<tr>
<td>( \mathcal{E} )</td>
<td>([J],[m]^{-1})</td>
<td>Dissipated energy per travelled length</td>
</tr>
<tr>
<td>( \Pi_1 = \bar{c} )</td>
<td>–</td>
<td>First dimensionless parameter</td>
</tr>
<tr>
<td>( \Pi_2 = \xi )</td>
<td>–</td>
<td>Second dimensionless parameter</td>
</tr>
<tr>
<td>( \Pi_3 )</td>
<td>–</td>
<td>Third dimensionless parameter</td>
</tr>
<tr>
<td>( \Pi )</td>
<td>–</td>
<td>Fourth dimensionless parameter</td>
</tr>
<tr>
<td>( x )</td>
<td>([m])</td>
<td>Fixed coordinate</td>
</tr>
<tr>
<td>Symbol</td>
<td>Unit</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>----------------</td>
<td>----------------------------------</td>
</tr>
<tr>
<td>$t$</td>
<td>[s]</td>
<td>Time</td>
</tr>
<tr>
<td>$X$</td>
<td>[m]</td>
<td>Moving coordinate</td>
</tr>
<tr>
<td>$w$</td>
<td>[m]</td>
<td>Deflection</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>[m]$^{-1}$</td>
<td>Wave number</td>
</tr>
<tr>
<td>$T$</td>
<td>[K]</td>
<td>External temperature</td>
</tr>
<tr>
<td>$k_s$</td>
<td>[Pa]</td>
<td>Suspension spring stiffness</td>
</tr>
<tr>
<td>$k_t$</td>
<td>[Pa]</td>
<td>Tire stiffness</td>
</tr>
<tr>
<td>$C_s$</td>
<td>[Pa][s]$^{-1}$</td>
<td>Suspension dashpot viscosity</td>
</tr>
<tr>
<td>$m_z$</td>
<td>[kg]</td>
<td>Vehicle mass</td>
</tr>
<tr>
<td>$m_u$</td>
<td>[kg]</td>
<td>Tire mass</td>
</tr>
<tr>
<td>$z$</td>
<td>[m]</td>
<td>Relative vertical displacement</td>
</tr>
<tr>
<td>$S_\xi$</td>
<td>–</td>
<td>Power spectral density</td>
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<td>$w$</td>
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<td>Waviness number</td>
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<td>$c$</td>
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<tr>
<td>$\Omega$</td>
<td>–</td>
<td>Angular wavenumber</td>
</tr>
<tr>
<td>$IRI$</td>
<td>–</td>
<td>International Roughness Index</td>
</tr>
<tr>
<td>$\omega$</td>
<td>[m].[s]$^{-1}$</td>
<td>Excitation angular frequency</td>
</tr>
</tbody>
</table>
Appendix D

Stability Calculations

Since the stability calculations are rather involving and would not facilitate the comprehension of the arguments, they are derived in this appendix.

D.1 String Stability

We perform here the linear expansion of the solution of Eq. (2.71), as presented by Eq. (2.67). We will start by expanding:

\[ q(k)p(k)^{-2} = \frac{q_1}{p_0^2} k \left( 1 + \frac{q_2}{q_1} \right) \left( 1 + \frac{p_1}{p_0} k \right)^{-2} + \mathcal{O}(k^3) \quad (D.1) \]

into

\[ q(k)p(k)^{-2} = \frac{q_1}{p_0^2} k \left( 1 + \left( \frac{q_2}{q_1} - \frac{2p_1}{p_0} \right) k \right) + \mathcal{O}(k^3) \quad (D.2) \]

This leads us to

\[ \left( 1 - 4 \frac{q(k)}{p(k)^{-2}} \right)^{\frac{1}{2}} = 1 - 2 \frac{q_1}{p_0^2} k \left( 1 + \left( \frac{q_2}{q_1} - \frac{2p_1}{p_0} \right) k \right) - \frac{1}{8} \left( \frac{4q_1}{p_0^2} k \right)^2 + \mathcal{O}(k^3) \quad (D.3) \]

Incorporating Eq. (D.3) in Eq. (2.71), we have:

\[ \lambda^+ = -2 \frac{q_1}{p_0^2} \left( \frac{p_0 + p_1 k}{2} \right) \left( k + \left( \frac{q_2}{q_1} - \frac{2p_1}{p_0} + \frac{q_1}{p_0^2} \right) k^2 \right) + \mathcal{O}(k^3) \quad (D.4) \]
which is simplified as follows:

$$\lambda^+ = -\frac{q_1}{p_0}k + \left( \frac{q_1 p_1}{p_0^2} - \frac{q_2}{p_0} - \frac{q_1^2}{p_0^3} \right) k^2 + \mathcal{O}(k^3) \quad (D.5)$$

Using Eq. (2.69),

$$\lambda^+ = i \frac{a_{Ax}}{a_v} k + \left( -\frac{a_{Ax} a_{Av}}{a_v^2} + \frac{a_{Ax}}{2a_v} - \frac{a_{Ax}^2}{a_v^3} \right) k^2 + \mathcal{O}(k^3) \quad (D.6)$$

and factorising the real part by \( \left( \frac{a_{Ax}}{a_v} \right)^2 \) gives Eq. (2.72).

### D.2 Convective Stability

In the convective stability, we start from Eq. (2.67). Using equation Eq. (2.68) and writing \( \cos(k) = X \), one obtains

$$\begin{align*}
\mathbb{R}(p) &= p_r = -a_v + a_{Av}(1 - \cos(k)) = -a_v + a_{Av}(1 - X) \\
\mathbb{S}(p) &= p_i = a_{Av} \sin(k) = a_{Av} \sqrt{1 - X^2} \\
\mathbb{R}(q) &= q_r = a_{Ax}(1 - \cos(k)) = a_{Ax}(1 - X) \\
\mathbb{S}(q) &= q_i = a_{Ax} \sin(k) = a_{Ax} \sqrt{1 - X^2}
\end{align*} \quad (D.7)$$

Writing the determinant of Eq. (2.67) in the exponential form

$$\Delta = p^2(k) - 4q(k) = \rho(k) e^{i\Phi(k)} \quad (D.8)$$

the highest solution of Eq. (2.67) is of the form

$$\lambda^+ = \sigma + i\omega = \frac{1}{2} \left( -p(k) + \sqrt{\rho e^{i\Phi}} \right) \quad (D.9)$$

It's real part is

$$\mathbb{R}(\lambda) = \sigma = \frac{1}{2} \left( -p_r(k) + \sqrt{\rho(\cos(\frac{\Phi}{2}))} \right) \quad (D.10)$$

Now, using trigonometric identities

$$\cos(\frac{\Phi}{2}) = \sqrt{\frac{1 + \cos(\Phi)}{2}} = \sqrt{\frac{1}{2} + \frac{1}{2 \tan^2(\frac{\Phi}{2})}} \quad (D.11)$$
decomposing $\Delta$

$$\Delta = \Delta_r + i\Delta_i$$  \hspace{1cm} (D.12)

and using the definitions of $\Phi$ and $\rho$,

$$\begin{cases} 
\rho = \sqrt{\Delta_r^2 + \Delta_i^2} \\
\Phi = \tan^{-1}(\frac{\Delta_i}{\Delta_r})
\end{cases}$$  \hspace{1cm} (D.13)

we obtain:

$$\cos\left(\frac{\Phi}{2}\right) = \sqrt{\frac{1 + \sqrt{1 + \left(\frac{\Delta_i}{\Delta_r}\right)^2}}{2}} = \sqrt{\frac{1 + |\Delta_r|}{\rho}} = \sqrt{\frac{\rho + |\Delta_r|}{2\rho}}$$  \hspace{1cm} (D.14)

Implementing this expression into Eq. (D.10) gives

$$\sigma = \frac{1}{2} \left( -p_r(k) + \sqrt{\frac{\rho + |\Delta_r|}{2}} \right)$$  \hspace{1cm} (D.15)

Now, recalling the instability condition:

$$\text{Instability} \iff \sigma > 0$$  \hspace{1cm} (D.16)

One has

$$\sigma > 0 \iff \sqrt{\frac{\rho + |\Delta_r|}{2}} > p_r$$  \hspace{1cm} (D.17)

This gives us the first instability condition:

$$\sigma > 0 \iff p_r < 0 \iff -a_v + a_{\Delta v}(1 - X) < 0$$  \hspace{1cm} (D.18)

However, physical considerations of our behavioral forces gives us:

$$\begin{cases} 
a_{\Delta x} \geq 0 \\
a_v \leq 0 \\
a_{\Delta v} \geq 0
\end{cases}$$  \hspace{1cm} (D.19)

which ensures us the positivity of $p_r$. Now, the instability condition can be recast in the form:

$$\sigma > 0 \iff \rho + |\Delta_r| > 2p_r^2 \iff \rho > 2p_r^2 - |\Delta_r|$$  \hspace{1cm} (D.20)
For future use we will recall here the useful relations:

\[
\begin{align*}
\Delta_r &= r^2 - p_r^2 - 4q_r \\
\Delta_l &= 2p_r p_i - 4q_i
\end{align*}
\] (D.21)

and

\[
\rho = \sqrt{(p_r^2 - p_i^2 - 4q_r)^2 + (2p_r p_i - 4q_i)^2} \quad \text{(D.22)}
\]

Now, the instability condition Eq. (D.20) can be separated into two cases, depending on the sign of \( \Delta_r \).

**D.2.1 Case 1: \( \Delta_r > 0 \)**

We will first consider the case,

\[
\Delta_r > 0 \iff p_r^2 - p_i^2 - 4q_r > 0
\] (D.23)

In this case, Eq. (D.20) becomes

\[
\rho > 2p_r^2 - (p_r^2 - p_i^2 - 4q_r) \iff \rho > p_r^2 + p_i^2 + 4q_r
\] (D.24)

Using Eqn. (D.7) and (D.19), we obtain \( q_r > 0 \), and therefore:

\[
p_r^2 + p_i^2 + 4q_r > 0
\] (D.25)

giving

\[
(i) \iff \sigma > 0 \iff \rho^2 > (p_r^2 + p_i^2 + 4q_r)^2
\] (D.26)

with (i) denoting the instability condition for \( \Delta_r > 0 \). We then have:

\[
\begin{align*}
(i) &\iff \Delta_r^2 + \Delta_l^2 > (2p_r^2 - \Delta_r)^2 = 4p_r^4 - 4p_r^2 \Delta_r + \Delta_r^2 \\
(i) &\iff \Delta_r^2 > 4p_r^4 - 4p_r^2 (p_r^2 - p_i^2 - 4q_r) \\
(i) &\iff (2p_r p_i - 4q_i)^2 > 4p_r^2 (p_i^2 + 4q_r) \\
(i) &\iff 4p_r^2 p_i^2 - 16p_r p_i q_i + 16q_i^2 > 4p_r^2 p_i^2 + 16p_r^2 q_r \\
(i) &\iff -16p_r^2 q_r - 16p_r p_i q_i + 16q_i^2 > 0 \\
(i) &\iff q_i^2 - p_r^2 q_r - p_r p_i q_i > 0
\end{align*}
\] (D.27)
Now, using Eq. (D.7), we obtain

\[ (i) \iff a_{\Delta x}^2 (1 - X^2) - p_r^2 a_{\Delta x} (1 - X) - p_r a_{\Delta x} a_{\Delta v} (1 - X^2) > 0 \]
\[ (i) \iff (1 - X) (a_{\Delta x}^2 (1 + X) - p_r^2 a_{\Delta x} - p_r a_{\Delta x} a_{\Delta v} (1 + X)) > 0 \]
\[ (i) \iff (1 - X)((1 + X)(a_{\Delta x}^2 - p_r a_{\Delta x} a_{\Delta v}) - p_r^2 a_{\Delta x}) > 0 \]

Writing \( Y = 1 - X = 1 - \cos(k) \), we obtain the instability condition:

\[ (i) \iff Y ((2 - Y)(a_{\Delta x}^2 - p_r(Y)a_{\Delta x} a_{\Delta v}) - p_r(Y)^2 a_{\Delta x}) > 0 \]

with

\[ p_r(Y) = -a_v + a_{\Delta v} Y \]

Considering that \( Y \in [0 ; 2] \), this reduces to:

\[ (i) \iff (2 - Y)(a_{\Delta x}^2 - p_r(Y)a_{\Delta x} a_{\Delta v}) - p_r(Y)^2 a_{\Delta x} > 0 \]

**D.2.2 Case 2: \( \Delta_r < 0 \)**

Consider now the case \( \Delta_r < 0 \). The instability condition Eq. (D.20) becomes

\[ (ii) \iff \rho > 2p_r^2 + \Delta_r \]

with \( (ii) \) defined as the instability condition for the case \( \Delta_r < 0 \). There are two subcases to consider, depending on the sign of \( 2p_r^2 + \Delta_r \):

\[ 2p_r^2 + \Delta_r > 0 \iff 3p_r^2 - p_r^2 - 4q_r > 0 \]
\[ 2p_r^2 + \Delta_r < 0 \iff 3p_r^2 - p_r^2 - 4q_r < 0 \]
In the case \(3p_r^2 - p_i^2 - 4q_r < 0\), the instability condition is verified. Supposing that \(3p_r^2 - p_i^2 - 4q_r > 0\), we now have:

\[
(ii) \quad \Delta_i^2 + \Delta_r^2 > (2p_r^2 + \Delta_r)^2 = 4p_i^4 + 4p_r^2 \Delta_r + \Delta_r^2
\]

\[
(ii) \quad \Delta_i^2 > 4p_i^4 + 4p_r^2(p_r^2 - p_i^2 - 4q_r)
\]

\[
(ii) \quad (2p_r^2 - 4q_r)^2 > 4p_i^2(2p_r^2 - p_i^2 - 4q_r)
\]

\[
(ii) \quad 4p_r^2 p_i^2 - 16p_r p_i q_i + 16q_i^2 > 8p_i^4 - 4p_r^2 p_r^2 - 16p_r^2 q_r
\]

\[
(ii) \quad 8p_r^2 p_i^2 - 8p_i^4 + 16p_r^2 q_r - 16p_r p_i q_i + 16q_i^2 > 0
\]

\[
(ii) \quad p_r^2 p_i^2 - p_r^4 + 2p_r^2 q_r - 2p_r p_i q_i + 2q_i^2 > 0
\]

\[
(ii) \quad p_r^2 p_i^2 + 2p_r^2 q_r + 2q_i^2 - p_r^4 - 2p_r p_i q_i > 0
\]

\[
(ii) \quad p_r^2(p_r^2 + 2q_r - p_i^2) + 2(q_i^2 - p_r p_i q_i) > 0
\]

using Eq. (D.7) we have:

\[
(p_r^2 + 2q_r - p_i^2) = a_{\Delta v}^2(1 - X^2) + 2a_{\Delta v}(1 - X) - (-a_v + a_{\Delta v}(1 - X))^2
\]

\[
(p_r^2 + 2q_r - p_i^2) = a_{\Delta v}^2(1 - X^2) + 2a_{\Delta v}(1 - X) - a_v^2 + 2a_v a_{\Delta v}(1 - X) - a_{\Delta v}^2(1 - X)^2
\]

\[
(1 - X^2)(1 - X)^2 = 1 - X^2 - 1 + 2X - X^2 = 2X(1 - X)
\]

\[
(p_r^2 + 2q_r - p_i^2) = 2a_{\Delta v}^2 X(1 - X) + 2a_{\Delta x}(1 - X) - a_v^2 + 2a_v a_{\Delta v}(1 - X)
\]

\[
(p_r^2 + 2q_r - p_i^2) = 2(a_{\Delta v}^2(2 - Y) + a_{\Delta x} + a_v a_{\Delta v})Y - a_v^2 ; \text{ with } Y = 1 - X
\]

and

\[
(q_i^2 - p_r p_i q_i) = a_{\Delta x}^2(1 - X^2) - (-a_v + a_{\Delta v}(1 - X))a_{\Delta x} a_{\Delta v}(1 - X)^2
\]

\[
(q_i^2 - p_r p_i q_i) = (1 - X^2)(a_{\Delta x}^2 + a_v a_{\Delta x} a_{\Delta v} - a_{\Delta x} a_{\Delta v}^2(1 - X))
\]

\[
(q_i^2 - p_r p_i q_i) = a_{\Delta x}(1 - X^2)(a_{\Delta x} + a_{\Delta v}(a_v - a_{\Delta v}(1 - X)))
\]

\[
(q_i^2 - p_r p_i q_i) = a_{\Delta x} Y(2 - Y)(a_{\Delta x} + a_{\Delta v}(a_v - a_{\Delta v} Y)) ; \text{ with } Y = (1 - X)
\]

This leads to the instability condition:

\[
(ii) \quad \left( (-a_v + a_{\Delta v} Y)^2 \{ 2(a_{\Delta v}^2(2 - Y) + a_{\Delta x} + a_v a_{\Delta v})Y - a_v^2 \} + 2a_{\Delta x} Y(2 - Y) \{ a_{\Delta x} + a_{\Delta v}(a_v - a_{\Delta v} Y) \} \right) > 0 \quad (D.37)
\]

This gives us the instability conditions for the convective stability.
D.3 Case $a_{\Delta v} = 0$

We now develop the calculations for the specific case $a_{\Delta v} = 0$, which corresponds to the optimal velocity models. Since the local and string stability cases are already presented in Section 2.3, we here show the convective stability conditions only.

This changes the equations Eqn. (D.7) into

$$\begin{cases} 
    p_r = -a_v \\
    p_i = 0 \\
    q_r = a_{\Delta x}(1 - \cos(k)) \\
    q_i = a_{\Delta x} \sin(k)
\end{cases} \quad (D.38)$$

We can express the condition $\Delta_r > 0$ as:

$$\Delta_r(Y) > 0 \iff 4a_{\Delta x} \left( \frac{a_v^2}{4a_{\Delta x}} - Y \right) > 0 \quad (D.39)$$

with $Y = 1 - X = 1 - \cos(k) \in [0 ; 2]$. Using,

$$\begin{cases} 
    \Delta_r(0) = a_v^2 > 0 \\
    \Delta_r(2) = 4a_{\Delta x} \left( \frac{a_v^2}{4a_{\Delta x}} - 2 \right)
\end{cases} \quad (D.40)$$

we have, recalling Eq. (D.19):

$$\Delta_r(Y) > 0 \iff Y \in [0 ; \frac{a_v^2}{4a_{\Delta x}}] \quad (D.41)$$

Consequently,

$$\frac{a_v^2}{a_{\Delta x}} > 8 \iff \forall Y \in [0 ; 2] ; \Delta_r(y) > 0 \quad (D.42)$$

In a more compact form,

$$\exists Y \in [0 ; 2] ; \Delta_r(Y) < 0 \iff a_v^2 < 8a_{\Delta x} \quad (D.43)$$
D.3.1 Case $\Delta_r > 0$

Considering the case $Y \in [0; \frac{a_r^2}{4a_{\Delta x}}]$, the instability condition Eq. (D.31) now reads

\[(i) \iff (2 - Y)a_{\Delta x}^2 > a_r^2a_{\Delta x}\]
\[(i) \iff a_{\Delta x}^2 \left(2 - \frac{a_r^2}{a_{\Delta x}} - Y\right) > 0\]
\[(i) \iff Y < 2 - \frac{a_r^2}{a_{\Delta x}}\]

so that:

\[(i) \iff \frac{a_r^2}{a_{\Delta x}} < 2\]

D.3.2 Case $\Delta_r < 0$

Now, assuming that $\frac{a_r^2}{a_{\Delta x}} < 8$, assume $Y \in [\frac{a_r^2}{4a_{\Delta x}}; 2]$. We first need to check the sign of $2p_r^2 + \Delta_r$ as in Eq. (D.33). This gives

\[p_r^2 + \Delta_r > 0 \iff 3a_r^2 - 4a_{\Delta x}Y > 0p_r^2 + \Delta_r > 0 \iff Y < \frac{3a_r^2}{4a_{\Delta x}}\]

Hence, if $Y \in [\frac{3a_r^2}{4a_{\Delta x}}; 2]$, the system is unstable, whereas if $Y \in [\frac{a_r^2}{4a_{\Delta x}}; \frac{3a_r^2}{4a_{\Delta x}}]$ the instability condition Eq. (D.37) becomes

\[(ii) \iff a_r^2 \{2a_{\Delta x}Y - a_r^2\} + 2a_{\Delta x}^3Y(2 - Y) > 0\]
\[(ii) \iff 2a_{\Delta x}^2Y^2 - 2a_{\Delta x}(2a_{\Delta x} + a_r^2)Y + a_r^4 < 0\]  
\[(ii) \iff H(Y) < 0\]

with

\[H(Y) = 2a_{\Delta x}^2Y^2 - 2a_{\Delta x}(2a_{\Delta x} + a_r^2)Y + a_r^4 < 0\]

Using the determinant of this second order polynomial

\[\Delta_H = 4a_{\Delta x}^2(a_r^2 + 2a_{\Delta x})^2 - 8a_{\Delta x}^2a_r^4\]
\[\Delta_H = 4a_{\Delta x}^2 \left((a_r^2 + 2a_{\Delta x})^2 - 2a_r^4\right)\]
\[\Delta_H = 4a_{\Delta x}^2 \left(a_r^2 + 2a_{\Delta x} + \sqrt{2}a_r^2\right)(a_r^2 + 2a_{\Delta x} - \sqrt{2}a_r^2)\]
\[\Delta_H = 4a_{\Delta x}^2 \left(2a_{\Delta x} + (\sqrt{2} + 1)a_r^2\right) (2a_{\Delta x} - (\sqrt{2} - 1)a_r^2)\]
In order to have $\exists Y ; H(Y) < 0$, one needs to verify $\Delta_H > 0$. We have

\[
\begin{cases}
4a_{\Delta x}^2 > 0 \\
2a_{\Delta x} + (\sqrt{2} + 1)a_v^2 > 0 \\
2a_{\Delta x} - (\sqrt{2} - 1)a_v^2 > 0 \iff a_v^2 < \frac{2}{\sqrt{2} - 1}a_{\Delta x} \iff \frac{a_v^2}{a_{\Delta x}} < \frac{2}{\sqrt{2} - 1} = 4.828
\end{cases}
\] (D.50)

Now, assuming that $\frac{a_v^2}{a_{\Delta x}} < \frac{2}{\sqrt{2} - 1}$, if we notice that

\[
\begin{cases}
H(0) = a_v^4 > 0 \\
H'(0) = -2a_{\Delta x}(a_v^2 + 2a_{\Delta x}) < 0
\end{cases}
\] (D.51)

The instability condition becomes

\[(\exists Y \in [0; 2]; H(Y) < 0) \iff Y_0 - < 2 \] (D.52)

with $Y_0 -$ the smaller root of $H(Y)$. Eqn. (D.51) indeed show that both roots are greater than 0, and we only need to investigate the smallest of them. This gives

\[
\begin{align*}
Y_0 - < 2 & \iff \frac{2a_{\Delta x}(a_v^2 + 2a_{\Delta x}) - \sqrt{\Delta_H}}{4a_{\Delta x}^2} < 2 \\
Y_0 - < 2 & \iff 2a_{\Delta x}a_v^2 + 4a_{\Delta x}^2 - \sqrt{\Delta_H} < 8a_{\Delta x}^2 \\
Y_0 - < 2 & \iff 2a_{\Delta x}a_v^2 - 4a_{\Delta x}^2 < \sqrt{\Delta_H} \\
Y_0 - < 2 & \iff 2a_{\Delta x}(a_v^2 - 2a_{\Delta x}) < \sqrt{\Delta_H} \\
Y_0 - < 2 & \iff a_v^2 < 2a_{\Delta x} \text{ or } 4a_{\Delta x}^2(a_v^2 - 2a_{\Delta x})^2 < \Delta_H
\end{align*}
\] (D.53)

and

\[
\begin{align*}
a_v^2 < 2a_{\Delta x} & \iff \frac{a_v^2}{a_{\Delta x}} < 2
\end{align*}
\] (D.54)

\[
\begin{align*}
4a_{\Delta x}^2(a_v^2 - 2a_{\Delta x})^2 & < \Delta_H \iff (a_v^2 - 2a_{\Delta x})^2 < (a_v^2 + 2a_{\Delta x})^2 - 2a_v^4 \\
4a_{\Delta x}^2(a_v^2 - 2a_{\Delta x})^2 & < \Delta_H \iff 0 < (a_v^2 + 2a_{\Delta x})^2 - (a_v^2 - 2a_{\Delta x})^2 - 2a_v^4 \\
4a_{\Delta x}^2(a_v^2 - 2a_{\Delta x})^2 & < \Delta_H \iff 0 < (2a_v^2)(4a_{\Delta x} - 2a_v^2) \\
4a_{\Delta x}^2(a_v^2 - 2a_{\Delta x})^2 & < \Delta_H \iff 0 < (2a_v^2)(4a_{\Delta x} - a_v^2) \\
4a_{\Delta x}^2(a_v^2 - 2a_{\Delta x})^2 & < \Delta_H \iff \frac{a_v^2}{a_{\Delta x}} < 4
\end{align*}
\] (D.55)
D.3.3 Summary

To conclude this part, the key parameter when $a_{Δv} = 0$ is $\frac{a_r^2}{a_{Δx}}$. For simplicity we will write $\frac{a_r^2}{a_{Δx}} = \xi$. We have several cases:

- 1) $\xi < 2$
- 2) $\xi \in [2 ; 4]$
- 3) $\xi \in [4 ; 8]$
- 4) $\xi > 8$

and we will look at them one by one:

Subcase 1: $\xi < 2$

Under this condition, we know that

$$\Delta_r(Y) > 0 \iff Y \in [0 ; \frac{\xi}{4}]$$

(D.56)

and there is a first instability occurring

$$(i) \iff Y \in [0 ; \min(\frac{\xi}{4}, 2 - \xi)]$$

(D.57)

Concerning the case $(ii)$,

$$\Delta_r(Y) < 0 \iff Y \in [\frac{\xi}{4} ; 2]$$

(D.58)

We have two ranges of instability

$$(ii) \iff Y \in \left[ \min \left( \frac{3\xi}{4}, Y_{0-} \right) ; 2 \right]$$

(D.59)

with

$$Y_{0-} = \frac{(\xi + 2) - \sqrt{(2 + (\sqrt{2} + 1)\xi)(2 - (\sqrt{2} - 1)\xi)}}{2}$$

(D.60)
Subcase 2: $\xi \in [2; 4[$

There is no instability for the range $Y \in [0, \frac{\xi}{4}]$ and we only have

$$(ii) \iff Y \in \left[ \min \left( \frac{3\xi}{4}, Y_{0-} \right), 2 \right]$$

(D.61)

Subcase 3: $\xi \in [4; 8]$

The only range of instability is

$$(ii) \iff Y \in \left[ \frac{3\xi}{4}, 2 \right]$$

(D.62)

Subcase 4: $\xi > 8$

The system is then always stable.
Appendix E

Three-Body Interaction Master Equation for the Extended NS Model

The three-body interaction in the extended NS model using a uniform deceleration distribution can be computed in the following way. We consider three vehicles that are initially stopped, one that will stay still and two moving ones which we will call followers 1 and 2 (follower 1 will sometimes be called the leader of follower 2). Follower 1 is initially spaced from the still vehicle by a distance \(d_1 = d_0\), and the distance between followers 1 and 2 is \(d_2 = 0\). Noting \(v_1\) and \(v_2\) the respective velocities of follower 1 and 2, we want to describe the time evolution of the probability to find our system in the configuration \((d_1, v_1, d_2, v_2)\) at a time \(t\) knowing \(d_0\): \(r(d_1, v_1, d_2, v_2, t|d_0)\). Similarly to Section 3.3, we denote by \(v^*_1 = \min(v_{\text{max}}, d_1)\) and \(v^*_2 = \min(v_{\text{max}}, d_2)\) the respective maximum velocities attainable by followers 1 and 2 with respective headways of \(d_1\) and \(d_2\). In Section 3.3 we showed the particular case when the velocity at time \(t + 1\) was in the interval \([v^* - \sigma, v^*]\), which transforms here into four particular cases, crossing the possibilities for \(v_1\) and \(v_2\) to be in their respective intervals. We can therefore write, using the notation \(r_{d_1, v_1}\):

\[
r(d_1, v_1, d_2, v_2, t + 1|d_0) = \frac{1}{8_1 8_2} (R_1 + R_2 + R_3 + R_4)
\]

(E.1)

The term corresponding to \(v_1 \notin [v^*_1 - \sigma, v^*_1]\) and \(v_2 \notin [v^*_2 - \sigma, v^*_2]\) writes:

\[
R_1 = \left( \sum_{\nu_1 = \max(0, v_1 - \alpha)} \sum_{\nu_2 = \max(0, v_2 - \alpha)} r_{d_1, v_1}^{d_1, v_1} \right)
\]

(E.2)
The one corresponding to \( v_1 \in [v_1^* - \sigma, v_1^*] \) and \( v_2 \notin [v_2^* - \sigma, v_2^*] \) is:

\[
R_2 = \Theta(v_1 > v_1^* - \sigma) \left( \sum_{v_1 = v_1 - \alpha + 1}^{v_{\text{max}}} \sum_{v_2 = \max(0, v_2 - \alpha)}^{v_{\text{max}}} r_{d_1, v_1} \right)
\]

(E.3)

Its symmetric, \( v_1 \notin [v_1^* - \sigma, v_1^*] \) and \( v_2 \in [v_2^* - \sigma, v_2^*] \) reads:

\[
R_3 = \Theta(v_2 > v_2^* - \sigma) \left( \sum_{v_2 = v_2 - \alpha + 1}^{v_{\text{max}}} \sum_{v_1 = \max(0, v_1 - \alpha)}^{v_{\text{max}}} r_{d_1, v_1} \right)
\]

(E.4)

Finally, the case corresponding to \( v_1 \in [v_1^* - \sigma, v_1^*] \) and \( v_2 \in [v_2^* - \sigma, v_2^*] \) is given by:

\[
R_4 = \Theta(v_1 > v_1^* - \sigma) \Theta(v_2 > v_2^* - \sigma) \left( \sum_{v_1 = v_1 - \alpha + 1}^{v_{\text{max}}} \sum_{v_2 = v_2 - \alpha + 1}^{v_{\text{max}}} r_{d_1, v_1} \right)
\]

(E.5)

The \( \Theta \) function is defined as \( \Theta(A) = 1 \) if \( A \) is true, \( \Theta(A) = 0 \) otherwise. The weight terms \( s_1 \) and \( s_2 \) are defined as

\[
s_i = \frac{\sigma + 1 - d_i}{\sigma + 1} \text{ if } (d_i < \alpha \text{ and } v_i = 0) ; \quad s_i = \frac{1}{\sigma + 1} \text{ otherwise}
\]

(E.6)

and take into account the higher probability to find a vehicle stopped if it is sufficiently close to another stopped vehicle ahead.
Bibliography


