Principal Stress Line Computation for Discrete Topology Design

by

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ABSTRACT

Principal stress-lines are pairs of orthogonal curves that indicate trajectories of internal forces. Subsequently, these curves idealize paths of material continuity, and naturally encode the optimal topology for any structure for a given set of boundary conditions. Stress-line analysis has the potential to offer a direct and geometrically-provocative approach to optimization that can synthesize both design and structural objectives. However, its application in design has generally been limited due to a lack of standardization and parameterization of the process for generating and interpreting stress lines. Addressing these barriers, this thesis proposes a new implementation framework that enables designers to take advantage of stress-line analysis to inform conceptual structural design. Central to the premise of this research is a new conception of structurally-inspired design exploration that does not impose a singular solution, but instead allows for the exploration of a diverse high-performance design space in order to balance the combination of structural and architectural design objectives. Specifically, the thesis has immediate application for the topological design of both regular and irregular thin shell structures predominately subjected to in-plane and compressive structural actions.

Thesis Supervisor: Caitlin T. Mueller
Title: Assistant Professor of Architecture

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1 Introduction

Geometry exerts a dramatic influence on structural performance. Nature, history and theory each demonstrate that the overall form of a structure tends to affect structural performance more than material, member sizing, or internal topology (Larsen and Tyas 2003; Allen and Zalewski 2010). Although this principle has been studied and demonstrated, building construction practice today typically enforces a dichotomous separation between architecture and engineering. In the conceptual design phase (Wang et al, 2002), an architectural team often defines initial decisions regarding a building's geometry, massing and form. The delayed guidance of engineers generally results in periodic consultation that remains subservient to architectural goals. Consequently, the design process tends to be rigid, linear, and unidirectional, and leads to designs that typically fail to substantially benefit from structural input. The absence of the structural advice constitutes a forgone opportunity that can be measured both practically, and aesthetically. Certainly, structurally efficient designs can reduce material usage and lifecycle costs, and they are generally safer, more durable, and easier to build (Mueller and Ochsendorf 2013). Less known, however, is the opportunity for structural-led exploration to satisfy an aesthetic constraint of the design problem: structural inputs can contribute both to architectural elegance and richness, and lead to the discovery of new and exciting architectural shapes.

1.1 Motivating Example

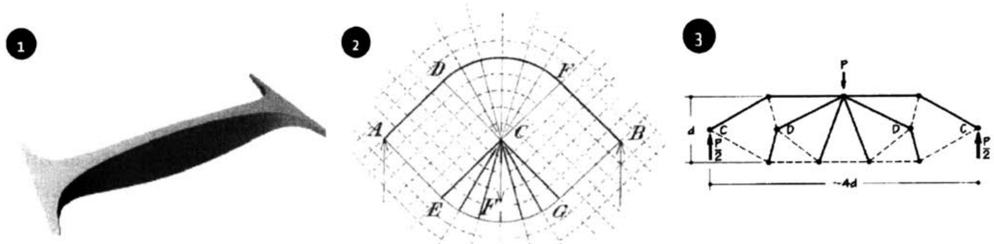


Figure 1.1 Creative designs for the simply-supported beam problem

To motivate the research for structurally-inspired design, presented above are three structural systems for the common boundary condition that prescribe the simply-supported beam. Each example respectively highlight an aspect that is fundamental in the field of structural design: material, flow and geometry.

Figure 1.1.1 shows a beam modified to exhibit varying cross section in both width and height throughout its span. Since the shear and moment capacity of a beam with rectangular cross section is respectively controlled by the width and the depth of the beam, the example redistributes material to where they are needed most in order to achieve constant bending and shear stress throughout the beam.

Instead of proposing a continuum structure, the same design domain can also be satisfied using a truss structure, as in the case in of 1.1.2. A truss is efficient because it carries only axial force, and any planar, shell or solid structure devised to solve a 2-, 2.5- or 3-dimensional boundary condition may be approximated through trusses. The continuum structure can be conceived as an infinitely dense truss - a collection of infinitely small nodes with bars connected to it that are variously oriented. Details of which will be explained in Section 2, the particular orientation exhibited by Figure 1.1.2 corresponds to the flow of forces; famously referred to as the Michell truss, the example represents the optimal configuration of the truss for a simply supported beam.

With no flexural capacity, the orientation of the individual bars directly coincide with the line of forces it carry. It is for this reason that the ideal arch and the hanging cables carrying self-weight are shaped the way they are. In fact, the Michell truss may further be simplified into a series of arches and cables, as demonstrated in Figure 1.1.3. How these examples relate through principal stress lines, and can be used to design effective topology for planar and shell structures constitute the subject of the thesis.

1.2 Geometry and Form

1.2.1 Structural Exploration in Architectural History

The examples above illustrated the objective to improve structural performances is not adverse to the design objectives of geometric complexity and architectural aesthetics. And designers are certainly known to have exploited this potential: structural-led abound in architectural history. Such efforts, while varied in specific methodologies, attempt to capitalize on the critical relationship between architectural geometry and structural behavior, and can lead to innovation in both: efficient structures often entail complex geometric solutions that can be formally compelling.

Historically, the synthesis is achieved through the related practice of physical modelling, which is used as a problem solving tool that addressed both spatial and structural concerns. The alignment of force and geometry, as Example 1c captures, has fascinated designers like Antoni Gaudi, Heinz Iser, and Frei Otto - designers who exploited physical and gravity loaded inverted hanging models as form-finding tools in the design of shell structures. In other structural types, famous examples of integrated architectural and structural form-finding have also included Buckminster Fuller's tensegrity sphere and Jørn Utzon's wooden shell model. These precedents, which are shown Figure 1.2, illustrate that in many cases, geometrically optimized structures can be visually pleasing, and result in a harmonious integration of power and grace that imparts design value and vigor.

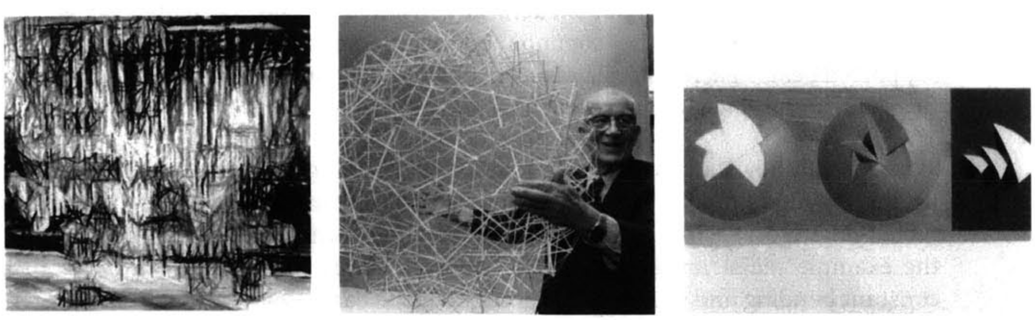


Figure 1.2 Architectural and structural synthesis through model making (Astbury 2014)
From left to right: Antoni Gaudi's catenary arch model, Buckminster Fuller's Tensegrity Sphere and Jørn Utzon's Sydney Opera House.

1.2.2 Natural Optimizers

The intimate relationship between structural performance and geometries is also abundantly visible in nature, where evolution readily rewards 'designs' that achieve the greatest stiffness for the least amount of material. For instance, the living tree rearranges its material in a mechanism similar to Figure 1.1.1: trees that are loaded predominately by one direction can begin to develop elliptical shapes, such that more material is deposited in the portion of the tree that can achieve the greatest bending capacity against the respective load. Matheck noted that the tree might "even assume nearly the shape of an I-beam" oriented against the prevailing load such that little wood is found in the zone of neutral bending.

The underlying properties of principal stress directories that produce the complex lattice pattern demonstrated in Figure 1.3, is also found to be the motivator of bone growth. Initially theorized in the late 19th-century by Wolff, it is now widely supported by studies that the trabecular patterns of the calcaneus are mainly shaped by the principal stress flow that are "characteristics of the standing posture."

13.2 Wolff's Law and Roux's Functional Adaptation Concept 501

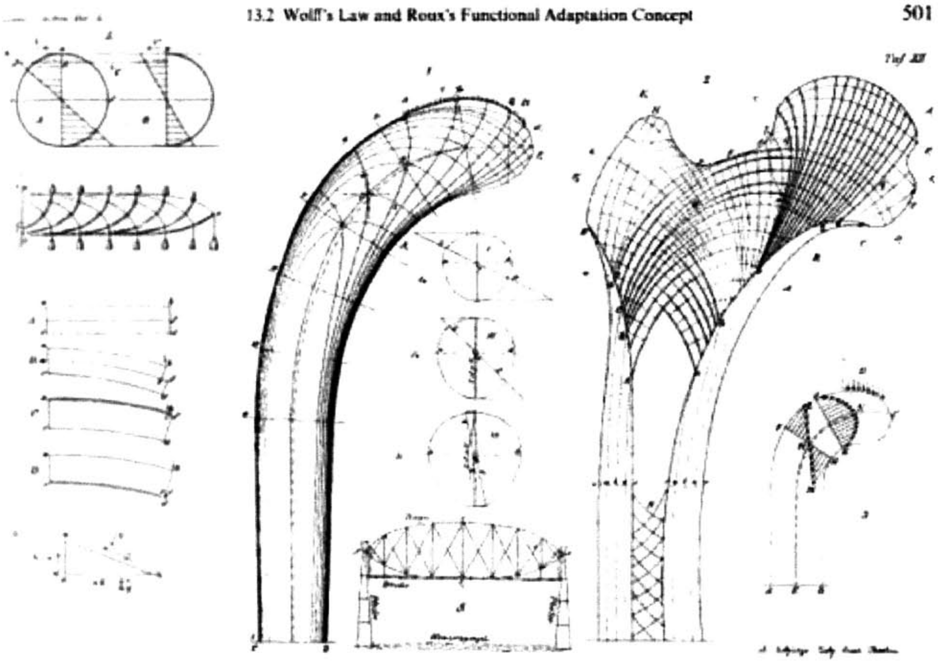


FIGURE 13.2:1 Culmann's crane presented by Wolff in his 1870 paper in Virchow's Archiv.

Figure 1.3 Wolff's law and the Culmann's crane

1.3 Design Tools and Framework

The motivating examples in the preceding section serves as an instructive reminder that design is inherently a structural act. In architectural design, structural performance, efficiency, and expressiveness are often goals that can best be achieved when integrated into generative processes in conceptual design. While recent advances in the field of Computer Aided Design (CAD) has vastly expanded the designer's capacity to address interdisciplinary concerns in design through computation, the potential for increasing synthesis of architecture and structural performance in design has generally been unmet due to a lack of advanced design tools that can effectively synthesize architectural and structural considerations (Mueller and Ochsendorf 2013).

One prominent exception is in the field of surface structures, which have received significant attention from researchers in recent years. The advent of interactive, dynamic, and computational methods for solving three-dimensional equilibrium structures, such as particle-spring systems (Kilian 2005) implemented in parametric tools like Kangaroo (Piker 2013), has combined with advances in digital fabrication to enable designers to generate structurally-inspired forms that are highly curvilinear and complex. However, the focus has centered on primarily global geometries. In comparison, little design-oriented research has been conducted to create high-performance topologies for these complex surfaces.

Since the final structural performance of any structure is influenced considerably by its topology, which also has major for visual appearance and constructability, there is a need for more knowledge and tools that enable designers to create efficient and elegant topologies for surface structures. This thesis proposes such a method, based on the theory of principal stress lines, an increasingly popular concept that nevertheless lacks sufficient investigation in the architectural design realm.

As a final motivating example, a simple shell geometry discretized with a stress-line-based topology is presented here in Figure 1.4 as a motivating example. Analysis has found the stress-line-inspired topology to be more efficient than a grid-based topology.

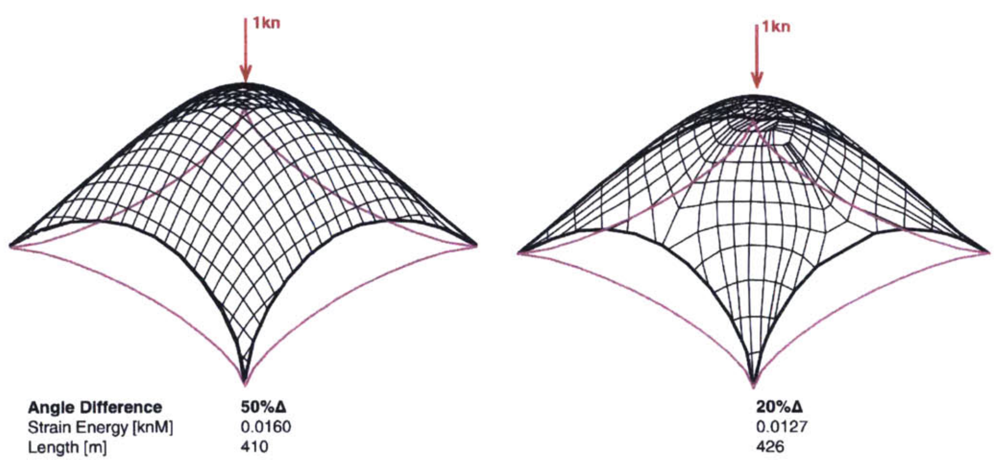


Figure 1.4 Motivating example for stress-line-inspired grid shell topology

In this example shell case under a single point load, the stress-line-based topology performs 20% better than the grid-based topology.

2 Background

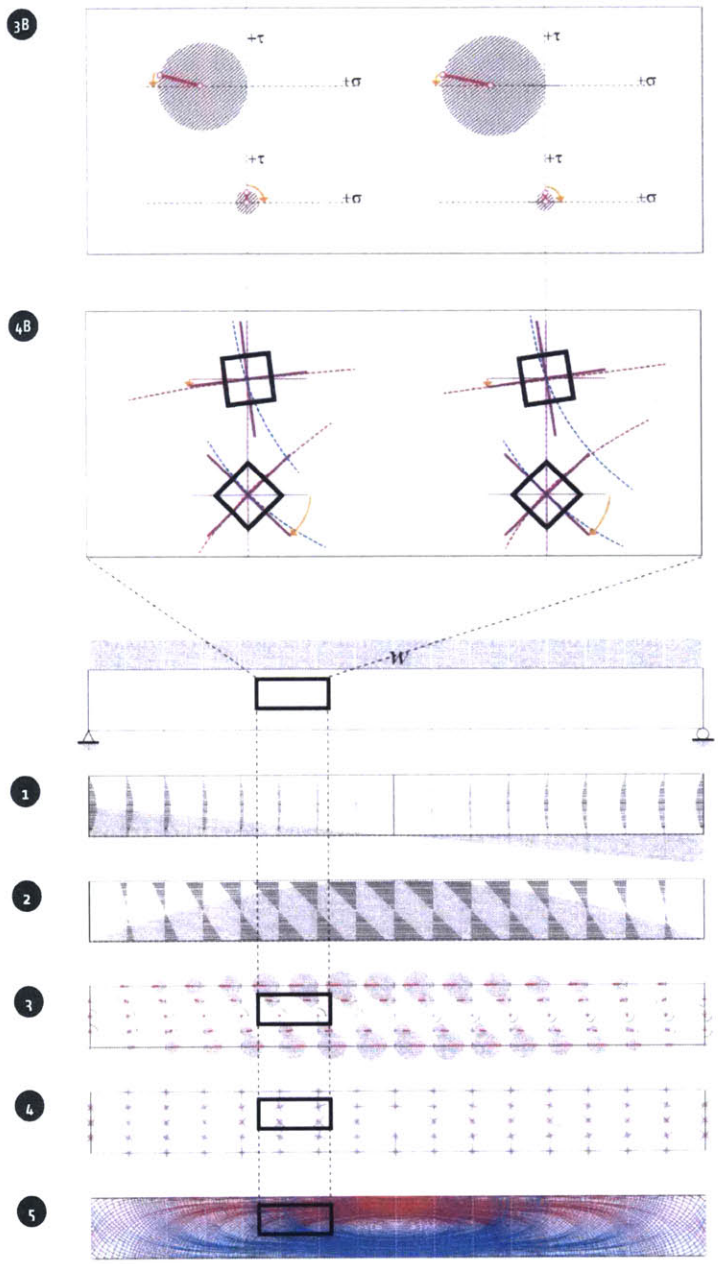


Figure 2.1 Simply-supported beam under uniform loading : from analysis to stress lines

Simply-supported beam under uniformly-distributed load
1) Shear stress distribution over cross Section, and shear diagram
2) Bending stress distribution over cross section, and moment diagram
3) Mohr's circle construction for selected points across beam's span
3B) Detailed Mohr's circle construction for selected beam area
4) Reoriented plane with principal stress directions
4B) Detailed reorientation of plane element and initial stress line projections for selected beam area
5) Principal stress line field for simply-supported beam

2.1 Review of Principal Stress Directions

To understand how principal stress lines are constructed, a review of principal stress directions is needed. As suggested by the motivational example, any structural continuum can be decomposed infinitely into infinitesimal cubical elements to describe the state of stress for each point. If only an individual plane is considered, as is commonly in the case for most structural engineering problem, the state of stress can uniquely be represented by two normal stress and one shear stress component.

Rotating the element, the state of stress will remain unchanged, but the stress components will correspond to the new orientation. Stress transformation allows the orientation of planes with special stress properties to be obtained. Specifically, the normal stress components corresponding to the planar orientation wherein the shear stress is zero and normal stresses are maximum is known as principal stress directions (Hibbeler 2011). Figure 2.1 shows the principal stress orientations at various points across the span of the beam. Of note, Figure 2.1.3, and with greater detail in Figure 2.1.3B, illustrate the common graphical technique used to determine principal stresses known as Mohr's circle.

2.1.1 Principal Stress Lines and Properties

When a sufficient number of principal stress planes are determined for a collection of points across the structural body, the principal stress line field is constructed by connecting these projections. Structural designers are interested with principal stress lines because they provide a visualization of the natural load path of an applied force in the system, which shows the line of desirable material continuity for a given design domain.

As Figure 2.1.5 illustrates, there is a striking regularity and order to the pattern created by principal stress lines. Mathematically, the required transformation orientation is expressed in the following equation:

$$\tan 2\theta = \frac{2\,\tau_{xy}}{(\sigma_x - \sigma_y)}$$

Therefore, the solution has two roots that are set at 90° apart, which establishes the visually distinctive quality of stress lines as two orthogonally intersecting families of curvature.

Given a design domain for a finite system of isotropic material operating within the elastic range, the following properties must also hold, as documented by Chen and Li (2010):

1) Principal stress directions are neither affected by changes to material stiffness, nor the rescaling of the applied forces,

2) Principal stress field are affected by attributes of the design domain, such as the location and degree of fixities of the loading and support conditions respectively, and the geometry of the continuum structure, and

3) The optimal shape for a design domain is contained in the principal stress field.

The consistent regularity of principal stress fields, with their potent capacity to suggest optimal topology as a function of geometry regardless of materiality, make the principal stress line method a compelling methodology for constructing an all-encompassing framework for topological-finding.

2.1.2 Potential Design Application

Although the numerical production of stress-line-based topology constitutes a relatively new inquiry, the research traces its roots to classical studies on structural optimization by Michell. Michell formulated the analytical derivation for several well-known optimal truss structures (1904). While Michell was not explicitly considering principal stress trajectories, his results closely resemble the principal stress lines for the design domain he examined, as Figure 2.2 shows.

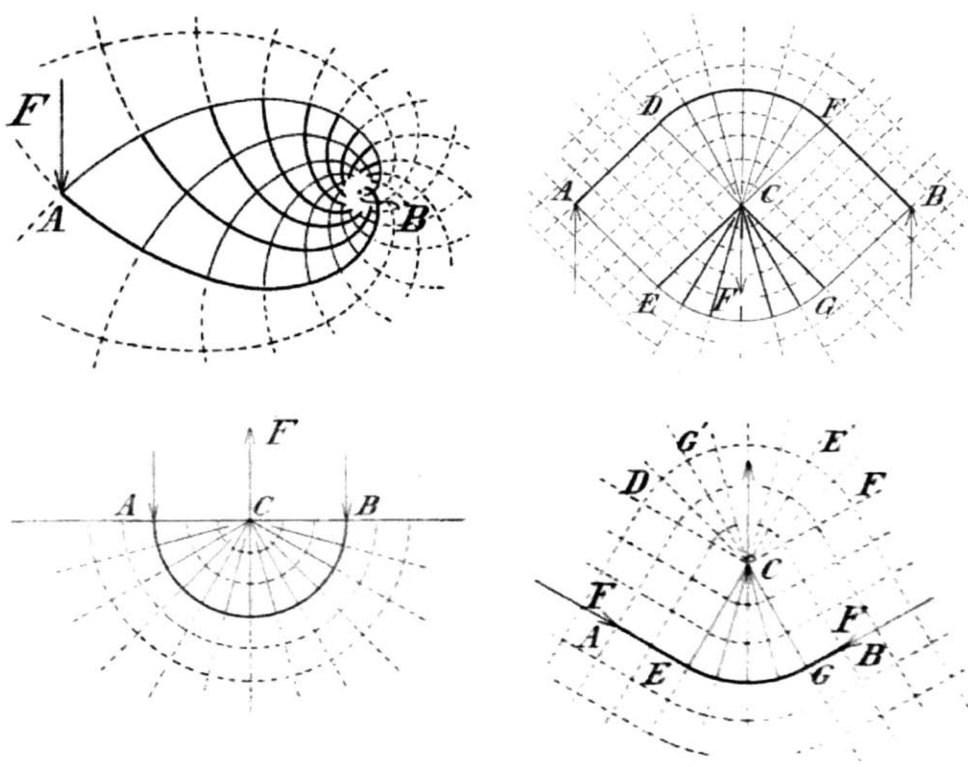


Figure 2.2 Michell's analytical formulation of optimum structure for various cases

The knowledge of Michell's theory, however, does not easily deliver hints on the optimal geometric layout for any design domain (Sokół 2011). Solutions are often infeasible to obtain, such that the considerable research that has since been developed relates largely to numerical methods that seek to reveal the optimum structure through highly computationally intensive procedures. These methods include ground structure and homogeneous methods, which are commonly afflicted with issues like disconnected structures and gray areas that render results unusable (Lu and Kota 2006).

When meaningful results are produced, a convergence is typically evident between the suggested optimal and the principal stress lines of the given design domain, as illustrated in Figure 2.3. These recurring resemblances suggests the potential value of a more direct approach for obtaining optimal structural topologies that uses principal stress analysis.

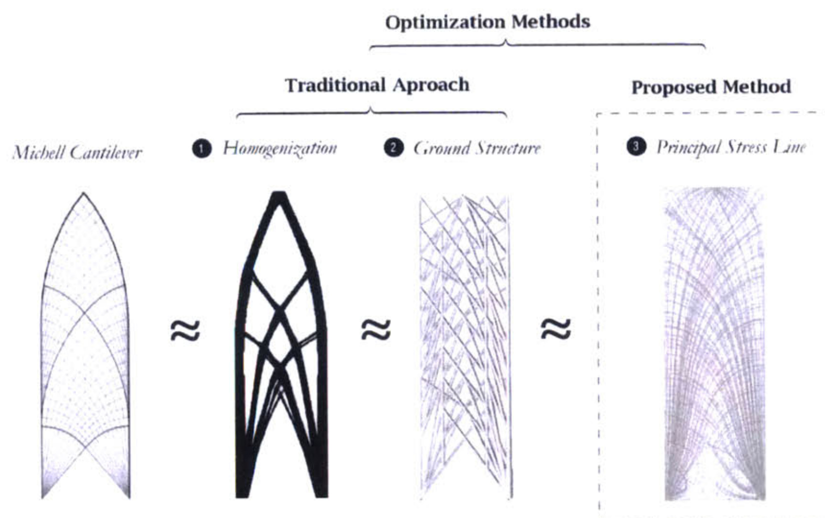


Figure 2.3 Convergence of optimization results

Figure 2.4 illustrates the potential efficacy of a stress-line-based method of topological-finding: the performance of a simply supported truss improves as it becomes more similar to the principal stress lines of its design domain.

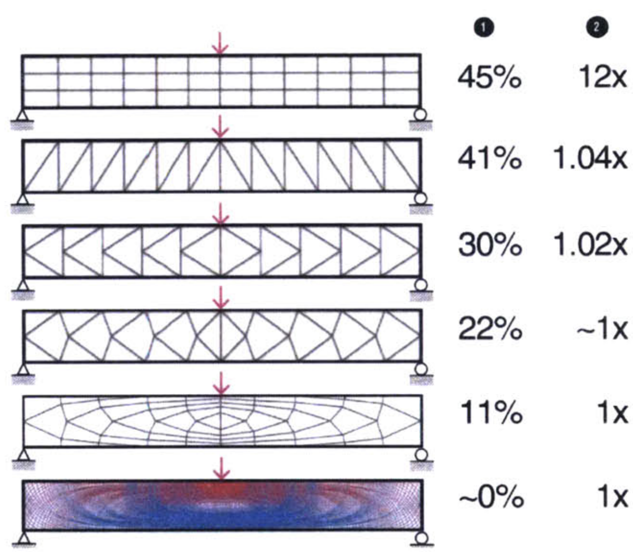


Figure 2.4 Various topologies for the simply-supported trusses
1) Average angle deviation 2) Material quantity

2.2 Potential Design Application

2.2.1 Engineering and architectural precedents

In spite of the research gap in stress line techniques, a number of important practitioners have adapted the theory of stress lines to various creative applications. Techniques related to stress fields, which are related to stress lines, are commonly used engineering to investigate the load-carrying mechanism of complex structures. Schlaich et al. (1987) are generally credited for popularizing its application in engineering practice: their seminal paper presented a method for developing truss models called strut-and-tie models to inform the layout of steel reinforcement in structural concrete design. Figure 2.5 shows a re-implementation of the technique. However, only a few studies have attempted to develop systematic procedures for the development of stress fields since its initial development in the 1980s. Consequently, its application is largely based on intuition and experience (Ruiz and Muttoni 2007).

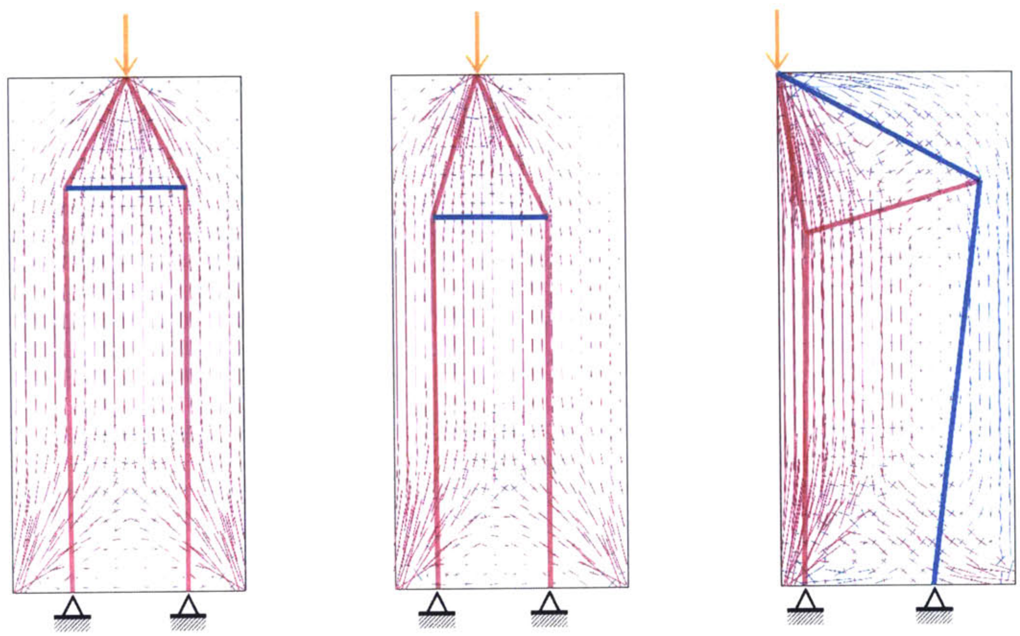


Figure 2.5 Implementation example of strut-and-tie method

Whereas the development of principal stress research in engineering has generally been focused at applying the techniques to the analysis of structures, architecturally-oriented designers have occasionally attempted to materialize stress lines directly. It is widely documented that Nervi's structural designs are frequently influenced by the idea of 'force flow,' with Gatti Wol Mill being the most notable implementation (Halpern, Billington, and Adriaenssens 2013). Similarly, Catalano has capitalized the concept in many of his design proposals (Allen 2010). Since advances in digital fabrication technologies have enabled the rapid production of complex curvilinear forms in recent years (Block and Rippmann 2013), interests in stress lines have reemerged. These examples (see Figure 2.6), however, are exceptional cases led by expert practitioners. To the rest of the engineering and design communities, the valuable guidance potential of stress lines remains unmet due to the lack of parameterization, standardization, and evaluation on the process for generating, interpreting, and analyzing stress lines.

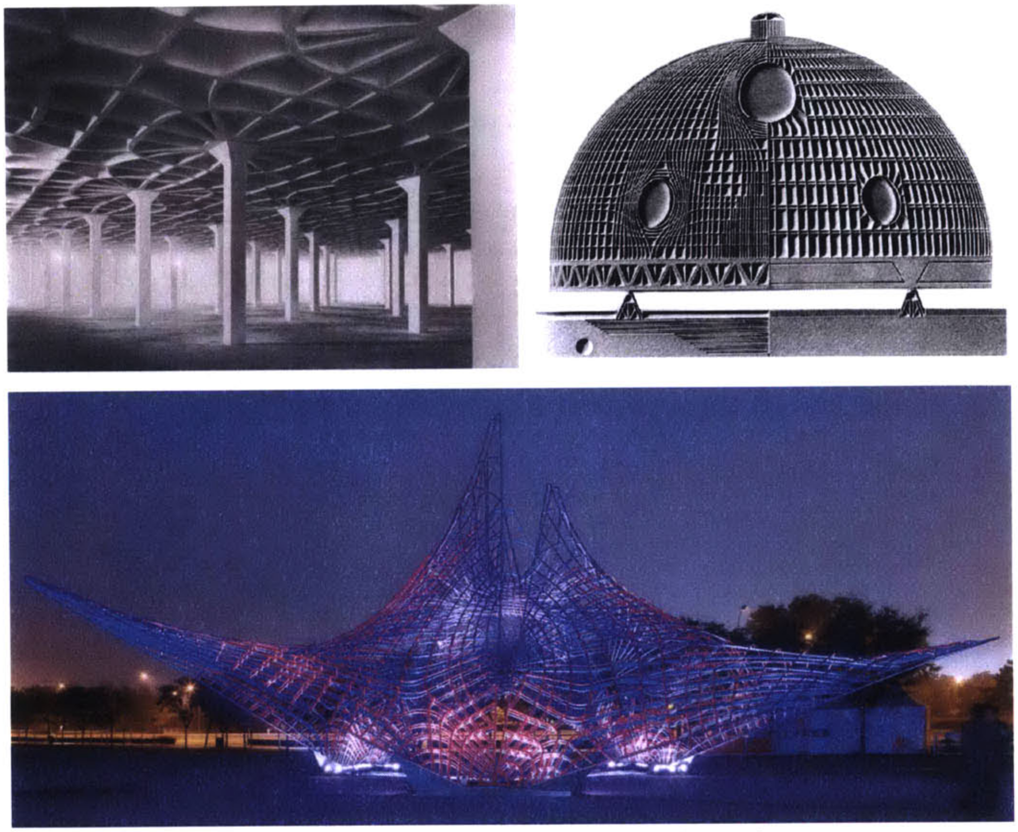


Figure 2.6 Stress-line-inspired architectural precedents
Top left: Gatti Wool Factory floor system (Halpern, Billington, and Adriaenssens 2013)
Top right: Spherical design proposal (Allen 2010)
Bottom: Zaha Hadid Architects London's China International Architectural Biennale (Shi 2014)

2.2.2 Problem and objective

The research is important because conventional tools available to designers for generating stress lines are generally concerned with the visualization of stress flow, as opposed to the manifestation of force flow under pragmatic design constraints intended for materialization.

In tools integrated with parametrized design interfaces, such as Millipede (Panagiotis 2014) and Karamba (Preisinger 2015), few functionalities are available to incorporate designer inputs. Even when adjustments informed by structural considerations are possible, there is a lack of documentation on the structural implication of these settings. More problematic still is the lack of discretization of the various stages of stress line interpolation, as these tools merely deliver an end result, thereby depriving the designer the capacity to alter geometric characteristics of the stress lines that are intrinsic to the method at the various stages of generation. Furthermore, there is no guarantee that the produced stress lines will lead to usable structural patterns, as Figure 2.7 illustrates.

Acknowledging that few applications of stress line methods in design have emerged due to the problems identified, this thesis proposes a consolidated framework for understanding how stress lines can be adapted in design. Particularly, the research is motivated to make stress lines generation a transparent process that is highly configurable by designers. The design application of stress lines is strongest when designers can actively participate in its generation.

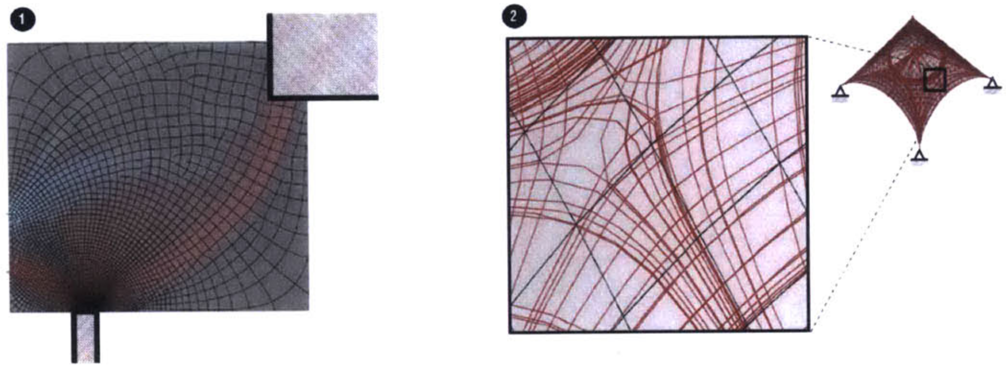


Figure 2.7 Problems with stress lines results in commercial structural analysis tool
1) Discontinuous lines, and 2) Poor resolution, overlaps, and undesirable intersections

3 Methodology

3.1.1 Direct Stress Line Construction

Structural patterns indicative of the internal stress trajectories of forces can be obtained in several approaches. These methods fall under two categories: 1) Direct, or 2) Iterative.

Direct method can be analytical or graphical: the optimum layout is determined either mathematically by satisfying theoretical constraints, or from geometric descriptions that seek to characterize the analytical formulations. Consequently, direct methods are prescribed, and do not require the use finite element analysis (FEA). An example of direct analytical approach is Michell's optimal truss derivation (1904), whereas Mazurek's graphical characterization of the same cantilever is an example of a direct geometrical approach (2011), as shown in Figure 3.1.

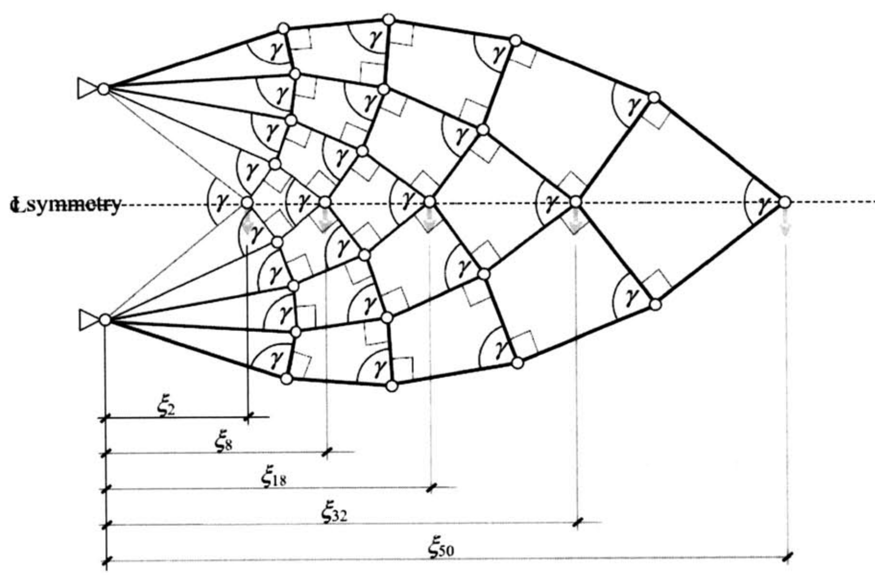


Figure 3.1 Geometric characterization of the Michell cantilever (Mazurek, Baker and Tort 2011)

3.1.2 Iterative Stress Line Interpolation

The problem with the direct approach is that existing formulations are derived only for limited cases. Thus, most stress line construction methods are instead numerical and iterative. The overall process is as followed: given an initial point, or seed, the principal stress directions for that point are found. A line is drawn along these directions, and its end point becomes the starting point for the subsequent iteration. This process repeats until the stress line reaches the design boundaries. With the conclusion of the stress line for one seed, the drawing of a new stress line corresponding to the next seed in the sequence begins (Panagiotis and Kaijima 2014). The number of seeding points are calibrated to create a uniformly dense base stress line field.

Not surprisingly, the quality of stress lines from iterative approaches varies widely: the results depend on the parameters in the different stages of production, and on the methods used to calculate the stress direction. Conventionally, stress directions in the iterative approach are calculated numerically with FEA. Directions, however, can also be calculated exactly by transforming analytically-determined stress components.

While analytical calculations offer precision, in cases involving complex structural designs, it may not be convenient or possible to derive the theoretical derivations. To maximize application potential, this thesis develops a method for constructing stress lines using a common FEA tool, while addressing the problems that are inherent to the iterative numerical approach, which include 1) low stress line resolution 2) poor stress direction interpolation, and 3) stress line discontinuities (Halpern, Billington, and Adriaenssens 2013).

3.2 Post-processing

3.2.1 Density and architectural implication

Although a Michell structure can be constructed for a variety of boundary conditions with varying numbers of members, the theoretically-optimized Michell solution has an infinite number of infinitely small bars with infinitely low stress. It is also known that additional gains in structural efficiency achieved by increasing the number of members in the Michell structure will plateau in as the Michell structure densifies (Mazurek, Baker and Tort 2011).

These conclusions also applies to the stress line-based structure: results for the simple grid-shell case with 4 supports are shown in Figure 3.2. The ideally optimized stress-line structure, which has an infinite number of members, is neither usable in a practical design context nor significantly improved compared to a lower density. Hence the targeted density of the stress-line base structure is ultimately a decision based on constructability and design concerns. This thesis proposes a method that seeks to select the stress lines that most contribute to structural performance.

3.2.2 Post-processing review

As mentioned previously, procedures are applied to process and select stress lines for materialization following the initial construction of a base stress line field. Some recent research related to the post-processing of stress lines are presented here. These methods fall under two categories: 1) Global, and 2) Iterative.

Global post-processing methods refers to defined procedures that seek to systematically adjust the base stress line data in its entirety according to geometric or performative objectives that are supplied externally. One example includes the work of Michlatos, and Kaijima (2014), who implemented a periodic global reparameterziation algorithm that enables the scaling of the principal stress vector field according to an input scalar field, such as a projected image, which may encode performance values. The approach, which is shown in Figure 3.3, allows the density of the resultant field to vary roughly according to stress magnitude.

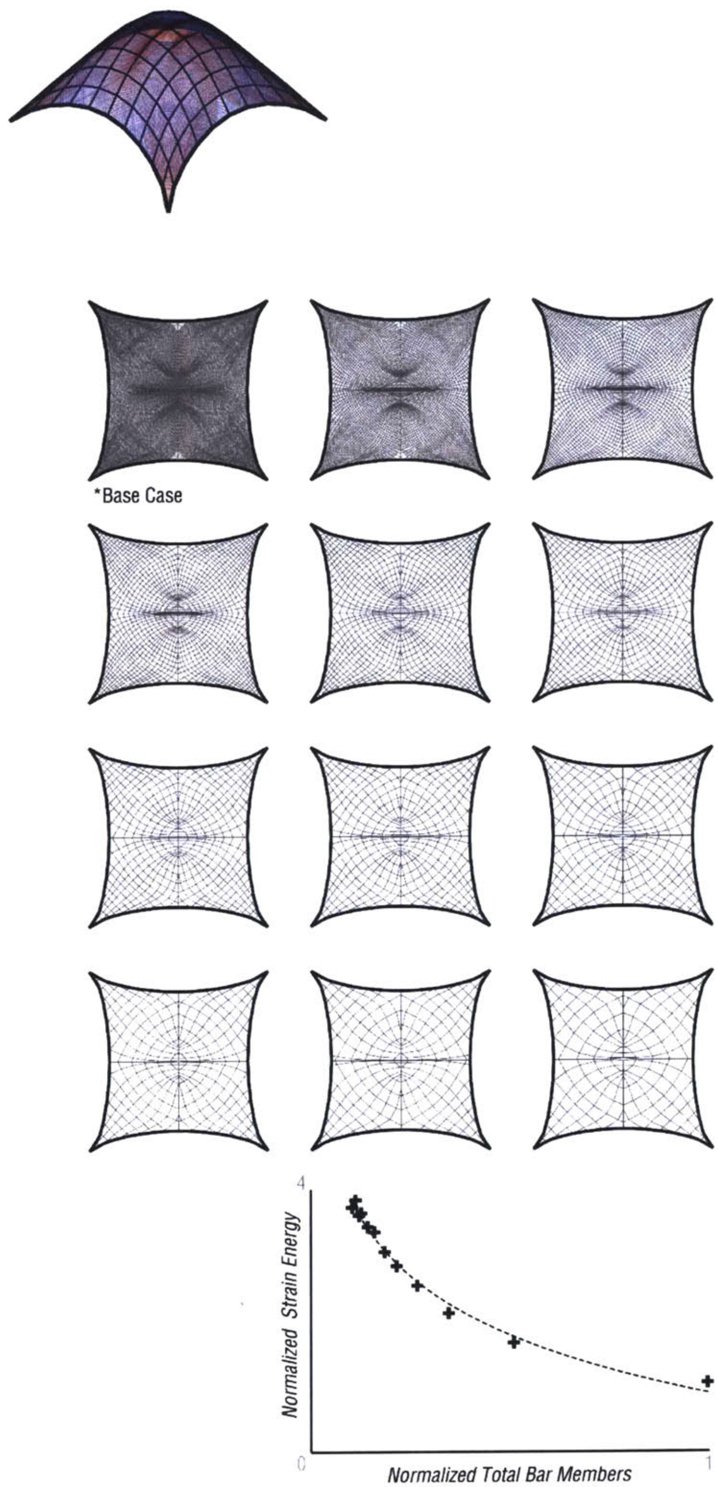


Figure 3.2 Density and performance in stress-line-based structure

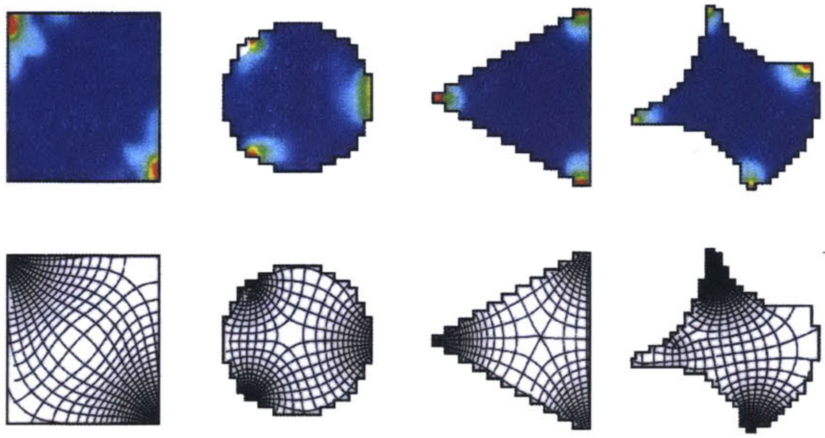


Figure 3.3 Michalato and Kaijima's method to scale stress line by patterns (2014)

The alternative to the global approach is an incremental, or iterative approach, which directly selects stress lines from the base stress line field by maximizing an objective value that is measured at, and is specific to the outcomes in each iteration. For instance, Chen and Li (2006) devised a simple algorithm that incrementally builds a stress line-based structure by co-opting a new set of stress line curvatures to reduce the approximation error in each iteration, thus giving the designer some control over the subdivision of the stress line-based structure (see Figure 3.4).

Generally, these approaches allude to the participation of the designers, who are empowered to affect the initial stress line field according to multiple considerations. More importantly, they do not directly materialize the initially constructed stress line field, but instead use it as a reference on which potential discretized elements are based

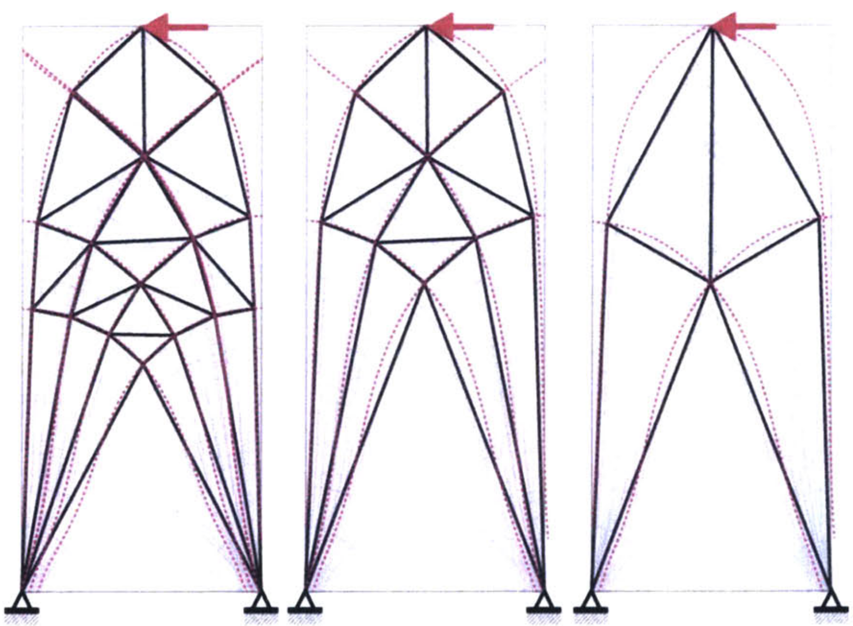


Figure 3.4 Modified implementation of Chen and Li's method (2010)

3.3 Proposed Process Outline

Although a number of interesting research results related to stress lines have emerged recently, the fundamental procedure dictating the drawing of stress lines has remained largely unchanged, even when there is significant room for further development. Research on stress line methods is commonly compartmentalized into two procedures: a brief procedure on seeding and drawing, followed by a disproportionately elaborate post-processing procedure. In addition, there is little quantitative documentation on the variation to the process that can influence structural performance.

This thesis develops a stress line computation and materialization framework that expands the fundamental procedure, and develops the possibilities that are intrinsic to each stage of the process. Specifically, the production of stress-line-inspired topology is divided into three stages: initialization, generation, and selection. These stages are not separate, and feedback is always possible between different stages to further enhance results. Figure 3.5 illustrates the reinterpreted outline for stress-line-inspired design.

The purpose of initialization is to construct an appropriate mesh topology to characterize the design domain investigated, to conduct the initial analysis from which structural data are obtained to form the basis of the stress line construction, and to create an appropriate seeding plan.

In generation, this thesis explores different methods for interpolating stress trajectories, and significantly expands on the general tracing algorithm to include rule-based corrections that can help reduce the numerical noise that is often present in stress line interpolation.

Finally, the implementation concludes with strategies for processing and selecting the stress lines based on performance criteria.

As an overall objective, the proposed framework seeks to minimize the reliance on the need for FEA, by adopting a number of geometric criteria.

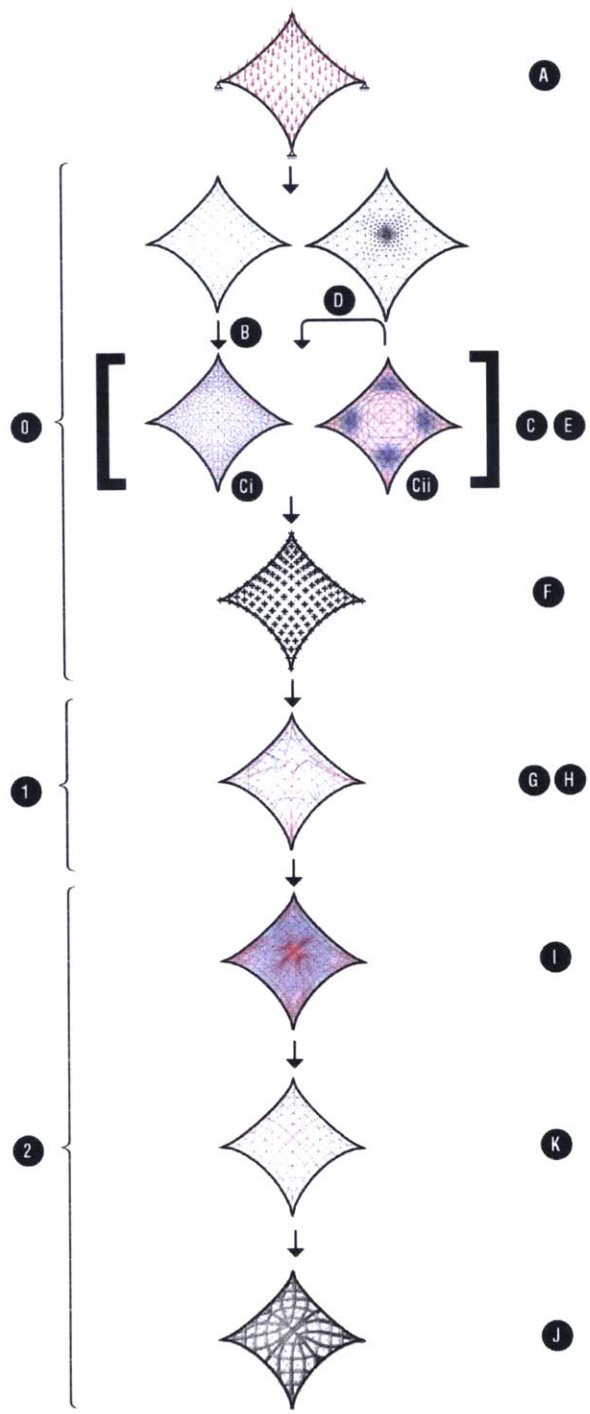


Figure 3.5 Proposed process breakdown of stress-line-based design

Proposed stress line process: 0) Preparation, 1) Generation, and 2) Post-processing; Subroutines: A) Parallel shell domain specification, B) Initial surface meshing, C) FEA structural analysis, which obtains Ci) Principal stress directions, and Cii) Various utilization metrics, D) Remeshing, E) FEA reanalysis, F) Seeding, G) Interpolation, H) Tracing, I) Uniformly-spaced base stress line field, J) Selection, K) Materialization

3.4 Scope and Case Study

Although stress-line-based solutions have theoretical application potential in all structural systems, the implementation demonstrated in this thesis focuses on both planar and form-found 2.5D membrane structures. Since members in these systems are subjected only to in-plane stresses, their normal stresses would be constant across their cross section depth contributed primarily by axial forces with negligible bending. Particularly, the proposed framework is implemented on five main structural types: 1-2) planar cantilever and simply supported beams under a point load, and 3-5) regular form-found grid shells with 3-, 4-, and 5- supports (see Figure 3.6).

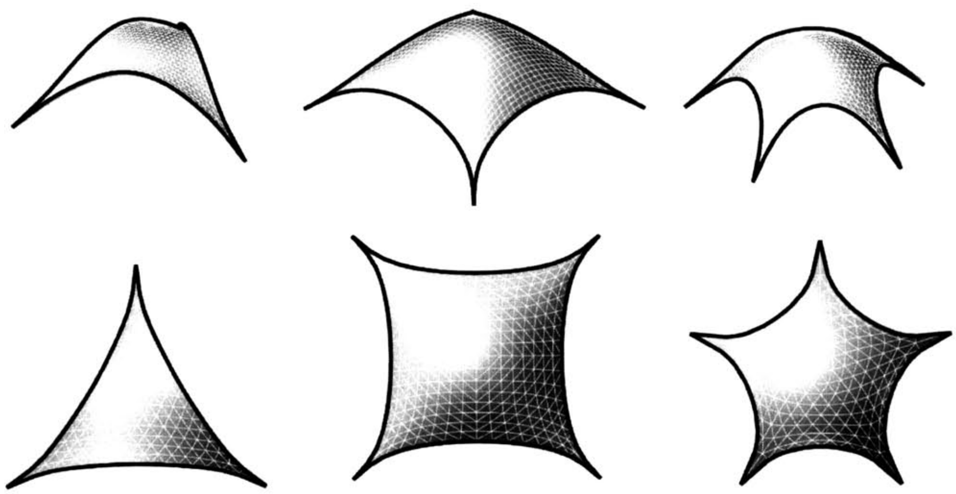


Figure 3.6 Grid shells explored in this thesis

3.5 Commercial Computational Tools

To maximize the design potential of the proposed framework, the research presented on this thesis is developed using popular 3D modeling tool Rhinoceros 3D (McNeel 2015), within the parametric visual programming language environment of Grasshopper 3D (Rutten 2013). Structural analyses were conducted using the plug-in Karamba, whereas the various surfaces used to implement the proposed stress line based framework were initially form-found using Kangaroo Physics. Description of these tools are provided below:

Rhinoceros 3D: Rhinoceros (or Rhino3D) is a commercial 3D computer graphics and computer-aided design (CAD) application software developed by Robert McNeel & Associates. The software is based on NURBS mathematical model, which focuses on producing mathematically precise representation of curves and freeform surfaces in computer graphics.

Grasshopper 3D: Released first in 2007, Grasshopper is a visual programming language (VPL) developed by David Rutten at Robert McNeel & Associates that runs within the Rhinoceros 3D CAD application. Programs (or definitions), are created by dragging components onto a 'canvas.' The components may require inputs and outputs, which are provided by other components connected in a sequence.

Kangaroo Physics and Plankton: Developed by Daniel Piker, Kangaroo is a Live Physics engine for interactive simulation, form-finding, optimization and constraint solving, which consists of a solver library and a set of Grasshopper components. A complimentary library also released by Piker is Plankton, which is a framework for handling n-gonal meshes in Grasshopper.

Karamba Parametric Engineering: Karamba, which is a plug-in fully embedded in the environment of Grasshopper 3D, is used to produce finite element calculations for the parameterized geometric models. The tool was developed by developed by Clemens Preisinger in cooperation with Bollinger-Grohmann-Schneider ZT GmbH Vienna.

4 Implementation

4.1 Principal stress direction conformity

A central metric that is developed on this thesis, which is used both to assess, and steer the development of stress-line-based results, is angle conformity. Since the research is based on the concept that materialization of stress lines can lead to efficient structures, it is important that there is a consistent method for measuring the closeness of a given topology to the base stress line field of the given design domain. Essentially, the orientation of each member is compared against its closest set of principal stress directions in a parallel shell analysis using a highly refined mesh to obtain an average value that suggests the angle approximation error, as shown in Figure 4.1.

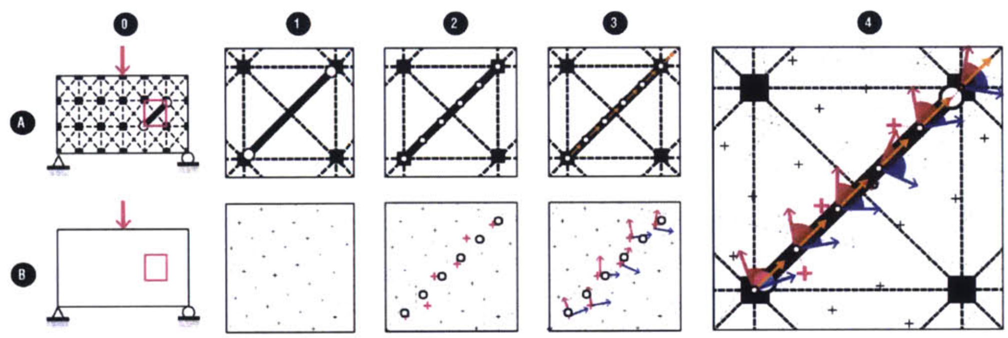


Figure 4.1 Measuring approximation error by angle deviation

0B) Conduct parallel shell analysis for 0A's design domain; 1A) Identify discretized member to be analyzed; 2A) Divide member according to predetermined resolution; 2B) Identify closest finite elements; 3A) Obtain member's tangent vector; 3B) Obtain principal stress directions; 4) Compare angle deviation to both stress directions, obtain average for each directions, and choose the minimum set.

4.2 Preparation

4.2.1 Loading Cases

The analytical information required for the construction of the base stress line field is provided by the parallel shell analysis that is conducted initially. Generally, both the support conditions and the mesh geometry of the shell should mirror the actual constraints as closely as possible. For 2.5D shell cases, the most geometrically usable, and practically meaningful results are obtained when the structure is analyzed with the loading condition used to form-find the shell structure (see Figure 4.2). If the initial form-finding loading condition is not known, gravity loading are used.

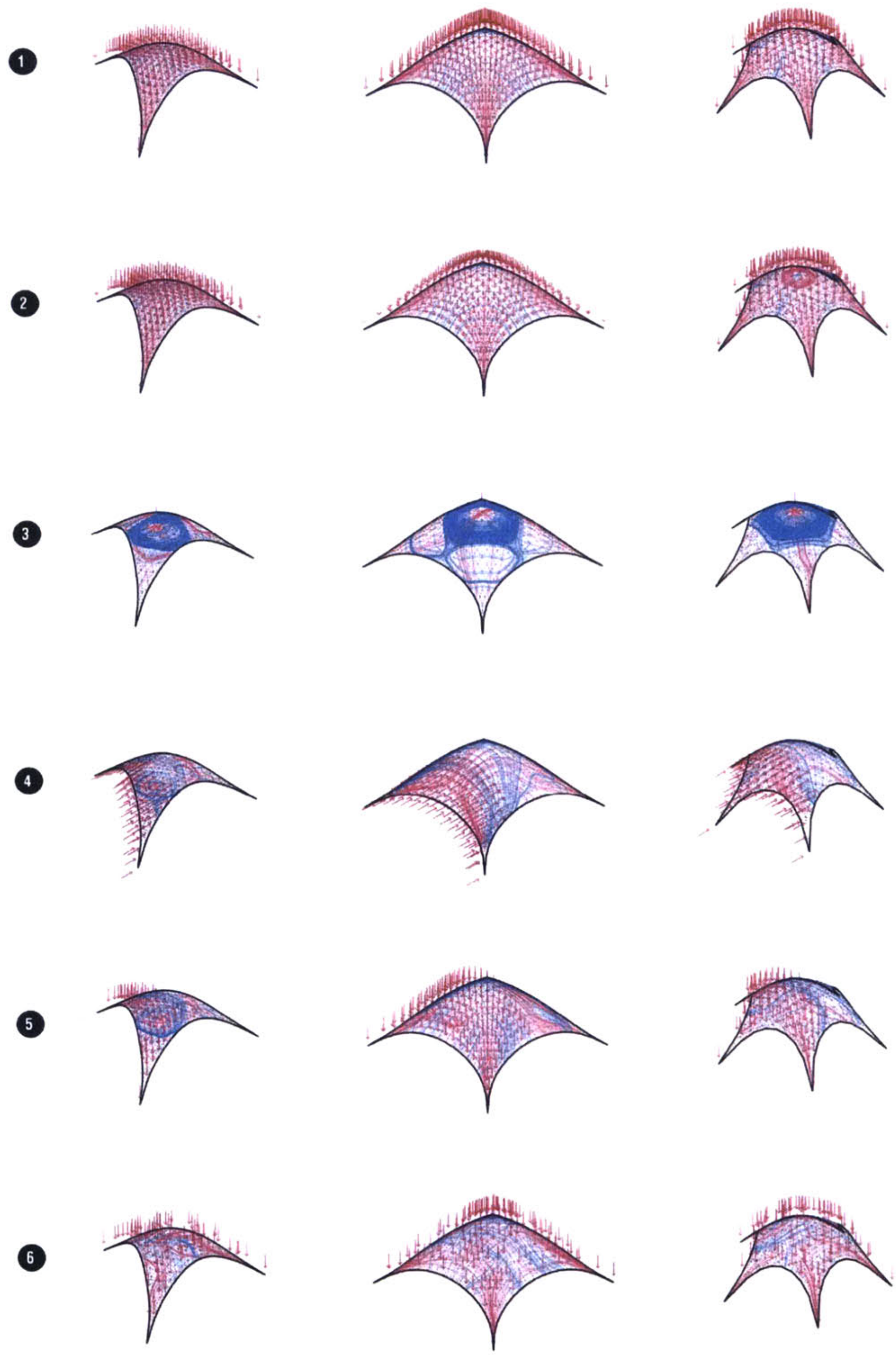


Figure 4.2 Variation in stress lines due to loading conditions

1) Equal point load on all nodes 2) Self-weight 3) Central point load 4) Lateral load 5) Asymmetrically vertical loads 6) Vertical loads on random nodes.

4.2.2 Structural analysis data

The proposed methods are mostly based on three types of analytical data: principal stress directions, principal stress magnitudes and overall member element utilization. Some data processing might be required since interpolation produced by different FEA software might corresponds to different topological features of the input mesh (e.g. Vertices, face centers, etc).

4.2.3 Seeding

Seeding presents an opportunity for the designer to incorporate both spatial and structural objectives that are determined by the nodal configurations. In the absence of particular constraints, the objective of seeding is to determine a collection of starting points from which a uniformly spaced principal stress field can be constructed for later processing and selection.

Conventionally, the designer would sample curvatures that are characteristics of the design domain, such as the design boundaries, to generate the initial seeding points (Chen and Li 2008). While the approach can feasibly generate a uniformly distributed grid for most 2-D planar cases with simple loading conditions, the simplistic approach risks omission of large swaths of the principal stress lines at the minimum direction in more complex 2.5-D and 3-D shell cases where circumferential stresses are present. Thus, the entire area contained within the design domain should be considered as potential starting points.

This thesis identifies two seeding strategies: Guided and Arbitrary. Guided strategies relate the seeding plan to analytical values, such as the magnitude of stresses. Conversely, arbitrary methods may be random, or based on other regular patterns associated with the input mesh that do not necessarily correspond to performance. The results from these approaches are compared in Figure 4.3.

Known nodal locational constraints, which might correspond pragmatically to internal programmatic, light and mechanical constraints, can be incorporated into the seeding plan. However, there is a structural performance tradeoff is when stress lines are skewed from their analytical optimal positions.

The seeding strategy can also be used to articulate known structural performative quality. Figure shows a seeding strategy weighted by shell utilization can achieve a variation of stress field density that closely mirror the utilization values. In this case, n_i random seeding points are distributed into each mesh element Q_i, where n_i is the division of the cell's utilization value by the sum of all utilization values, and multiplied by the desirable total seeding points N_s and an arbitrary scale factor α. The scale factor is adjusted in order to account for the discrepancies between the actual and desirable total number of seeding points due to rounding.

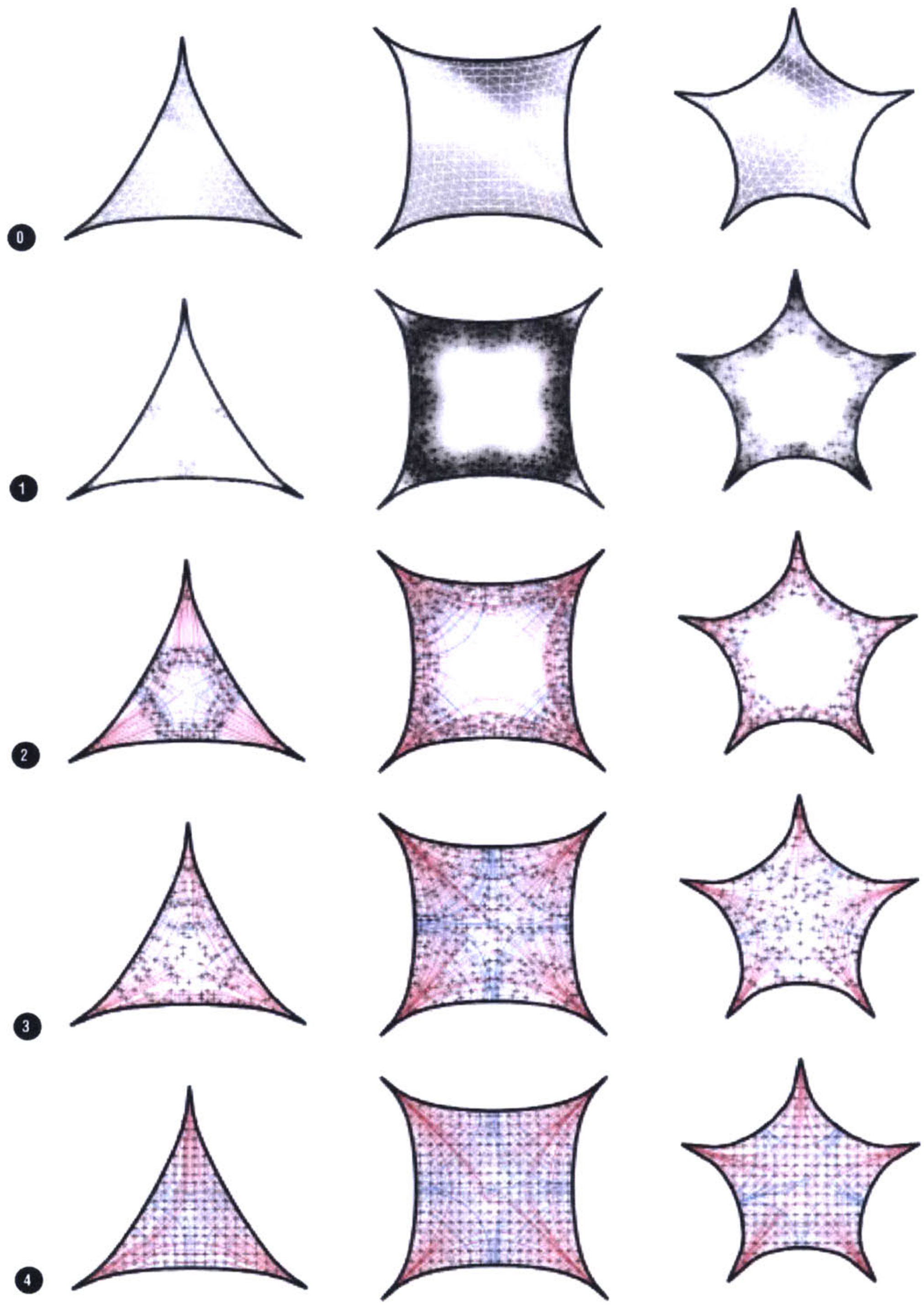


Figure 4.3 Seeding plan variations and stress line results

1) shell element utilization, 2) Seeding by Utilization, 3) random seeding, 4) uniform plan-based seeding

4.2.4 Mesh subdivision by stress direction

Since the quality of the principal stress direction results are related to the overall density of the shell mesh used in the initial analysis, further improvements to the results will require additional mesh subdivision. This thesis proposes the uses of strategic mesh subdivision to reveal greater details where the stress lines are changing in direction the most, such that improvement to the accuracy of the stress lines can be realized without excessive increases to computational demand.

Figure 4.4 illustrates the basic process: principal stress directions for each mesh element are compared to those of its adjacent cells within a pre-determined radius to produce a neighborhood angle deviation value for each cell that is then normalized and projected to a targeted subdivision range. Each cell is then subdivided accordingly to obtain a new set of vertices that are then triangulated with a Delaunay algorithm and relaxed using particle spring simulation to create the refined mesh (1934). Analyses have confirmed that the angle conformity for stress lines are higher in the strategically densified mesh than in the original mesh.

4.3 Generation

4.3.1 Interpolation method

Once a seeding plan is determined, two additional inputs are required to draw the principal stress lines: principal stress directions, and segment length. Figure 4.6 diagrammatically compares the interpolation methods that this section discusses. In a conventional tracing algorithm, the principal stress directions are obtained by finding the mesh element that is closest to the starting point in the current iteration (Michalatos and Kaijima 2014). Referred to here as a first-order approach, this method can generate biased line results because the principal stress direction information is generally a value averaged for the center of the respective finite element.

The most common method devised to address this bias is by linear interpolation. In a third-order approximation method, the three data points forming a triangle element that contain the targeted point are identified; and the stress direction data for the data point is then calculated by interpolation (Chen and Li 2008). Although the third-order interpolation leads to approximation that is more accurate than that of first-order, the criteria for selecting the reference points are not always consistent, especially in irregular meshes.

To achieve consistency and accuracy of respectively the interpolation and approximation methods, a new (1+n)-order approximation method is proposed: for every starting point, the nearest finite element is identified, in addition to n finite elements with centers that are located within a per-determined radius from the starting point. The data corresponding these points are extracted, and weighted according to proximity. Figure 4.6 compares the results using the various approaches, and confirms the improved variation in the (1+n)-order approach.

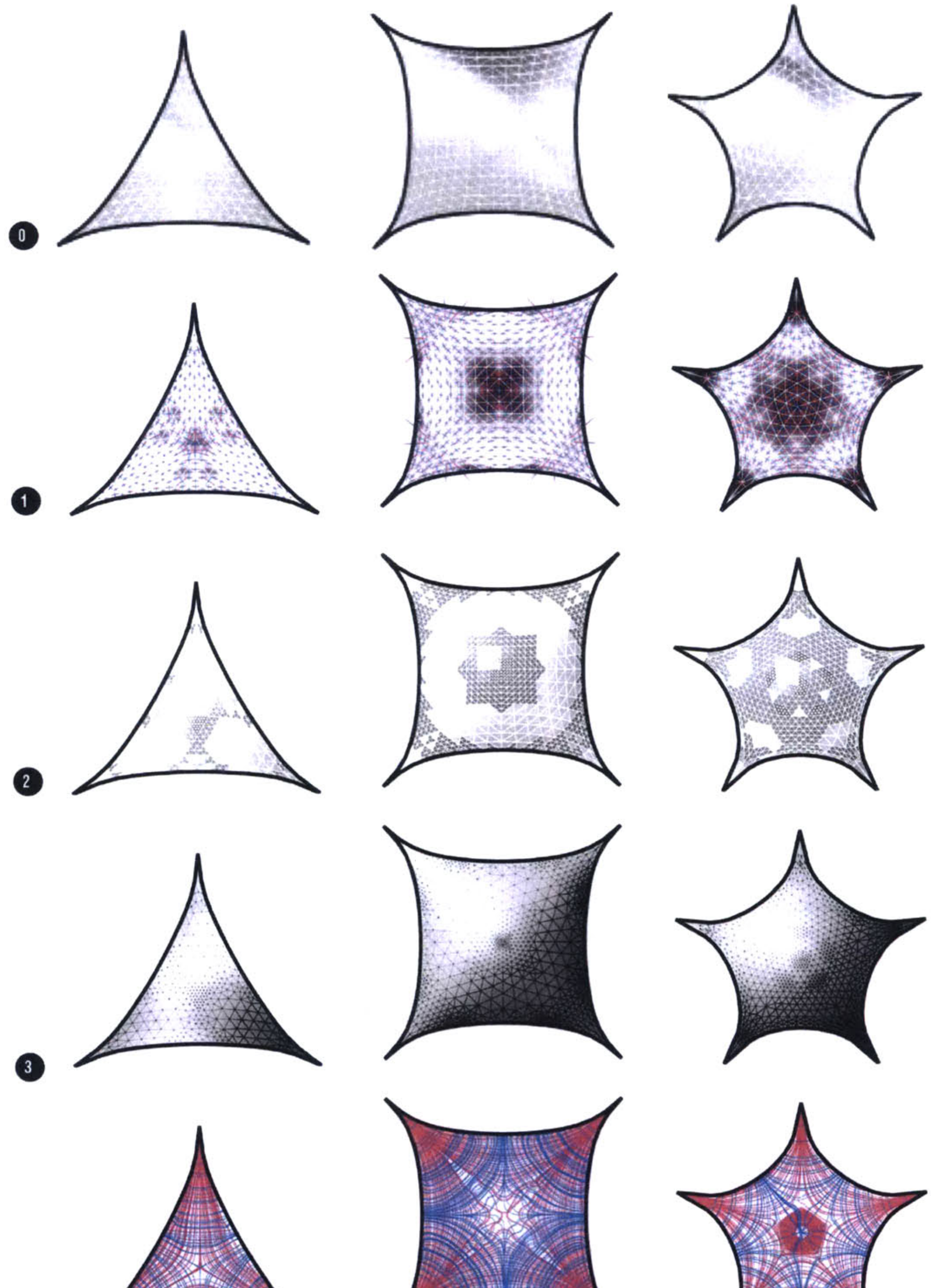


Figure 4.4 Remeshing implementation and corresponding stress line results
1) variation in principal stress directions 2) Subdivision 3) Relaxed Delaunay mesh 4) Refined stress lines

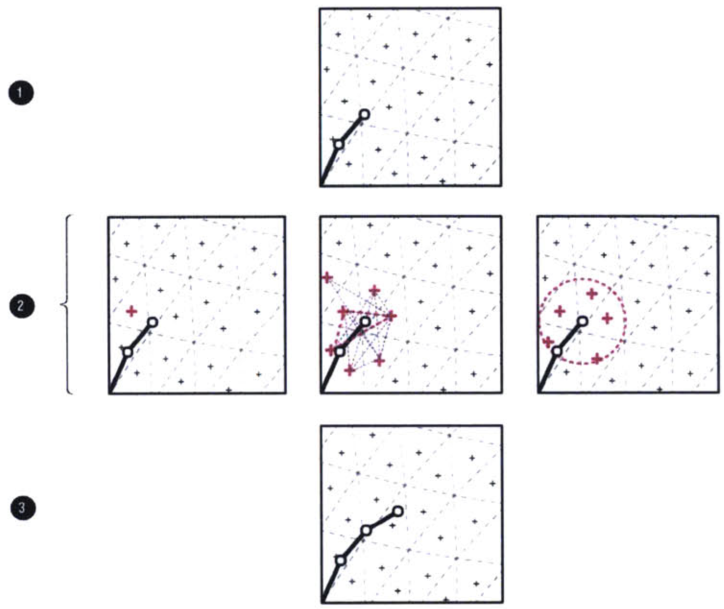


Figure 4.5 Various approximation order for principal stress directions interpolation
1) Input starting point of length segment in current iteration; Interpolation by approximation order of A) 1, B) 3, and C) 1+N; 3) Calculate end point of traced line segment

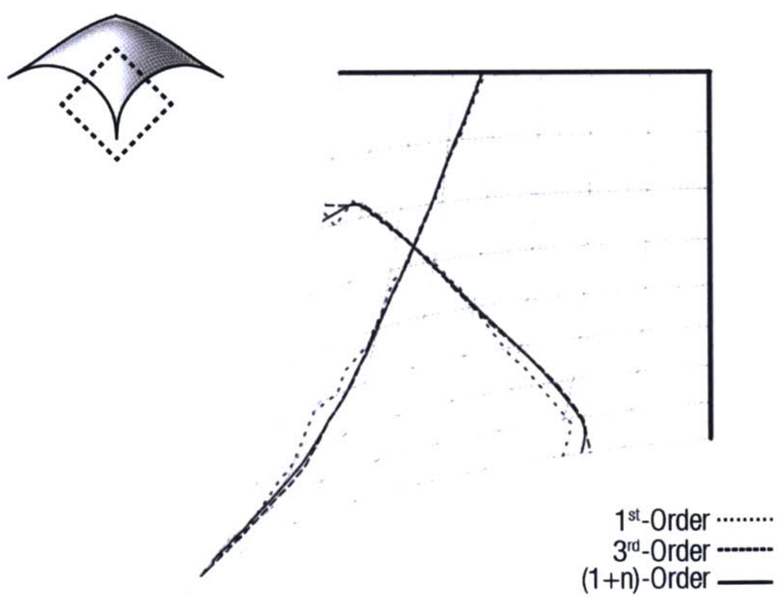


Figure 4.6 Variation to stress line results by approximation order

4.3.2 Segment Length

Generally, bias in stress line results are reduced by decreasing the segment length (Halpern, Billington, and Adriaenssens 2013). The lower bound of the segment length, however, is limited by the interpolation method chosen. For the first-order approach, the limit is set at the average mesh edge length, since a lower length segment would result in the same finite element data being used in numerous iterations. With higher-order approaches, shorter segment lengths can be used to produce smoother, and more accurate stress lines, as Figure 4.7 shows.

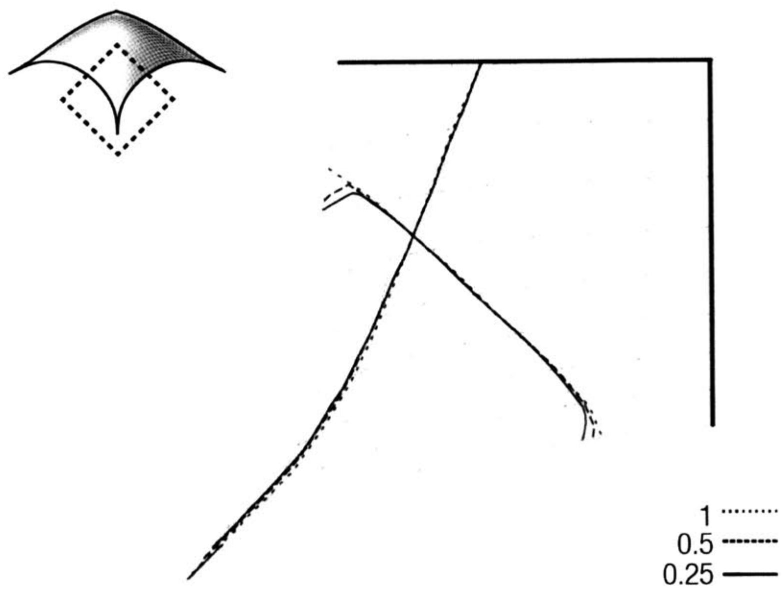


Figure 4.7 Variation to stress line results by segment length in (1+N)-order approach

4.3.3 Sequential iterative tracing

Conventional tracing algorithms prescribe only the termination condition of the tracing procedure, which is met when a stress line reaches the design boundaries. Thus, a considerable number of decisions in the structure of the algorithm are left to the discretion of the designer: points may be traced sequentially or concurrently, and the two principal stress directions may also be traced independently, or simultaneously in the same iteration.

To addresses problems with this approach, this thesis proposes a sequential tracing procedure subjected to rule-based corrections: each seeding point is traced independently from the other seeding points in a sequence, and a pair of stress line curvatures are produced from each seed. The entire algorithm is structured as a two-layer loop. Figure 4.11 illustrates the proposed method. Whereas the outer loop is responsible for supervising and adjusting the seeding plan when necessary, and tracking the information associated to each stress line (such as direction: maximum or minimum), the inner layer loop is tasked with the tracing of the stress lines and the application of rule-based corrections. The proposed approach allows stress lines undergoing tracing to respond to both existing and emergent conditions.

The general algorithm for each iteration j of a given seeding point P_i is as followed: receive the starting point for each of the four length segments of the current iteration, interpolate the stress directions associated with the principal direction for each of the four points, detect location-based geometric conditions for the starting point that warrant corrections, make relevant adjustments to the stress directions, and project the four points according to their directional vector to create the four new segments. Prior to the acceptance of the new length segments, the new lines are examined against the previously constructed stress lines, to determine whether the emergent stress line qualify for several termination rules, which are explained in the following section. The arrival of any four points to the design boundary will trigger a termination tracker that removes the respective point from future iterations. For points eligible for continuation, the projected point becomes the starting point of the subsequent iteration.

4.3.4 Rule-based corrections

Iterative stress line tracing is prone to numerical issues that lead to biased, and even unusable results. By correcting numerical noise as it emerges in the tracing process, a sequenced iterative approach ensures that the stress lines drawn will explicitly meet constructability and spatial requirements. Preliminary implementation of the thesis's rule-based tracing approach are identified here:

Detection of Circumferential Stress: As shown in Figure 4.8, conventional tracers generally loop indefinitely when a stress line reaches an area with circumferential stresses. This rule, which is described in Figure 4.9.1, detects the presence of circumferential stresses and closes the stress lines accordingly.

Enforce Offset: When the starting point of currently traced stress line approaches a defined distance from an existing stress lines, the rules, which is depicted in Figure 4.9.2, ensure that the stress lines for each direction will maintain a sufficient offset, and will not intersect with each other due to approximation biases.

Bypass Seeding: To eliminate the production of redundant and overlapping stress lines, which increases post-processing burden, the detection of a prior principal stress line nearby at the initial tracing iteration for a seeding point will lead to termination for the respective principal stress direction, as Figure 4.9.3shows.

In Figure 4.10, stress lines results from the preliminary implementation of the proposed tracing algorithm are compared with the results generated using Karamba's built-in stress line feature.

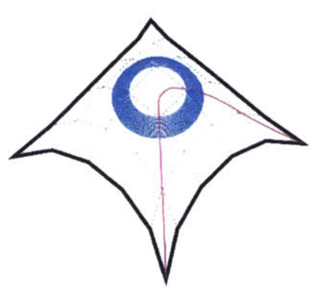


Figure 4.8 Circumferential stress error in commercial tool

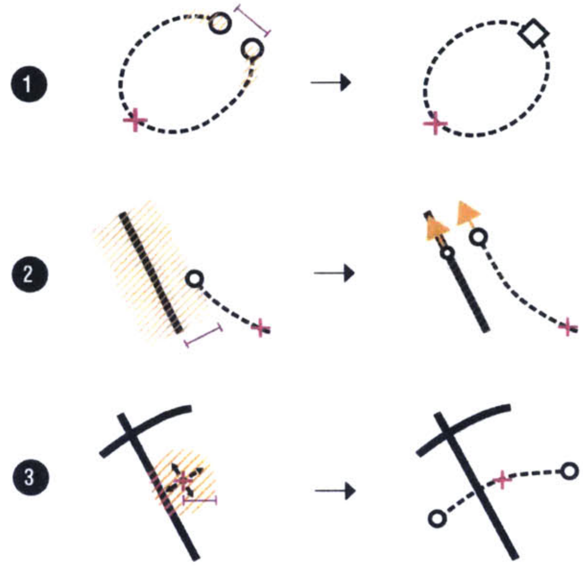


Figure 4.9 Diagrammatic representation of correction rules' parameters

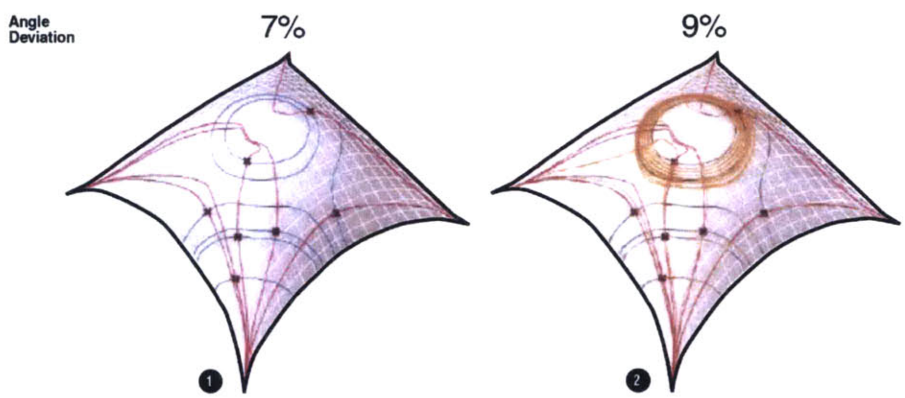


Figure 4.10 Stress line results and angle conformity comparison

1) Proposed algorithm 2) Karamba

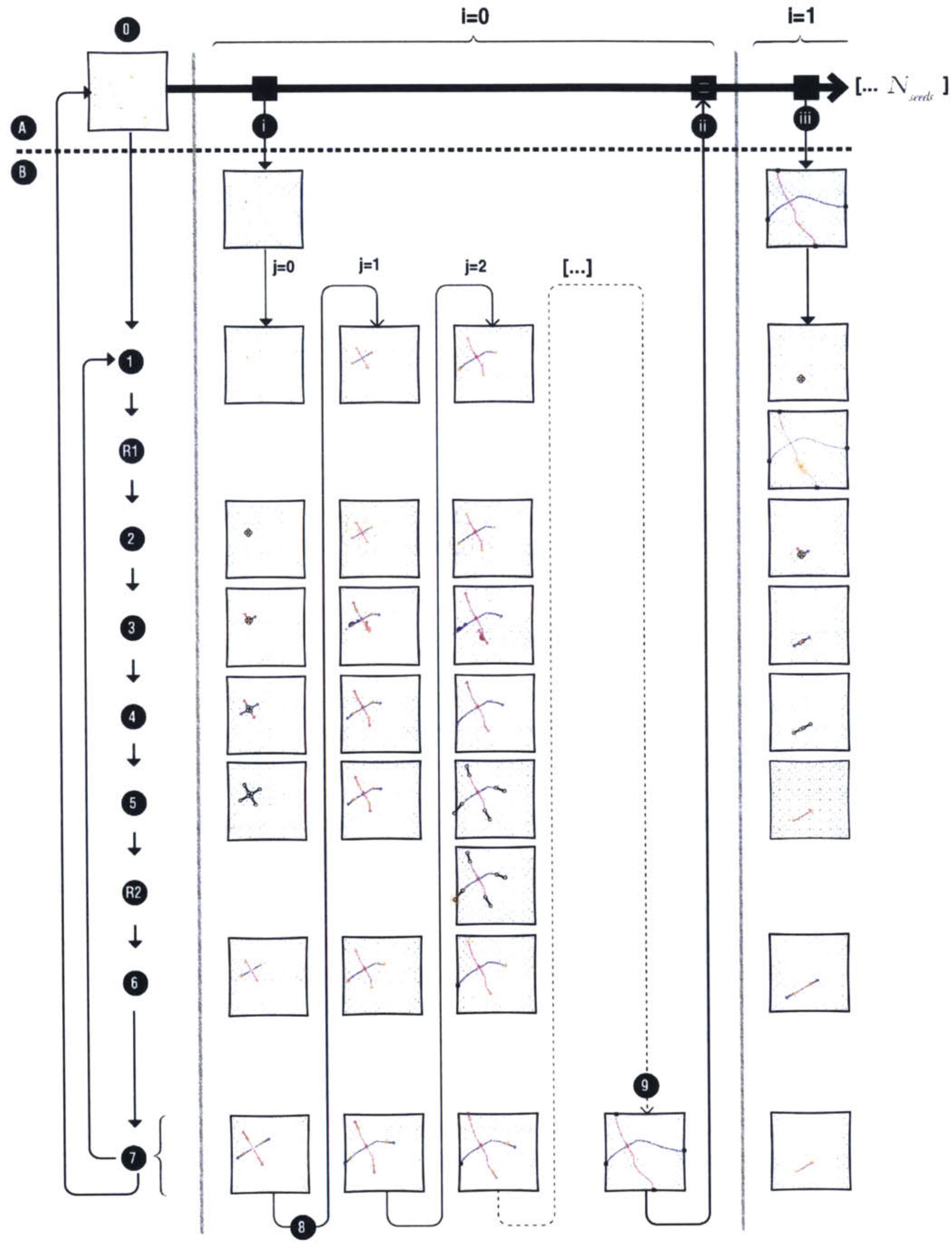


Figure 4.11 Illustrated flowchart of proposed tracing algorithm

The proposed algorithm features an A) Outer loop, and an B) Inner loop, which begins with 0) the input of a seeding plan. The engine i) sequentially inputs a seeding point into the inner loop, followed by ii) the return of fully-traced stress lines, and finally, c) the input of additional seed - a process that repeats until all seeds are traced. The inner loop's subroutines are as follows: 0) Receive starting point; R1) Recognize shape for drawing rules 1; 2) Interpolate; 3) Calculate Vector; 4) Correct flipped vectors ; 5) Draw segments; R2) Recognize shapes for termination rules; 6) Append curves; 7) Output results; 8) Reiterate, and 9) Terminate stress lines when they meet the design boundaries.

4.4 Post-processing

4.4.1 Problem formulation

In structural optimization, efficiency is often defined in terms of structural volume or stiffness. Commonly, the sum of force multiplied by the length for each member of the system is used as an indicator for structural volume (Mueller, and Ochsendorf 2013). However, the method's application is limited to axial-only solutions, in which members are experiencing equal stress across their cross section. Since both the curvature of the stress lines, and the mesh geometry from which the shell analysis is derived, are both approximations of an idealized funicular experience, the discretized structural systems derived from stress lines will inevitably be subjected to bending stresses.

To measure structural efficiency, this thesis uses the minimization of strain energy as the objective function, which is sometimes referred to as the minimization of compliance, or the maximization of stiffness (Rozvany, Bendsøe and Kirsch 1995; Achtziger 1997). The strain energy of a system, which is calculated by multiplying stiffness by deformation squared, can account for structural efficiency when the members of all compared cases are sized to achieve constant total volume.

4.4.2 Comparative analysis procedure

Following any modifications to the stress line-based structure, the members' section areas are rescaled according to a base case prior to ensure that the total structural volume remains constant for all cases. The magnitude of external loads applied to each node are also redistributed according to the percentage of the shell's surface geometry that each node is sup-porting for the updated topology - an estimate obtained through the tributary area method. The load values are normalized such that the total applied force remains constant in all selection cases. With the load values adjusted and the structural members globally resized, an initial analysis is conducted to obtain the stress values required to determine the stress ratio for each member, which is calculated by dividing the maximum stress of each member by the maximum stress of any member in the entire system. Individual members' cross section areas are then rescaled according to their stress ratio, in order to ensure that the structural material within a system is distributed according to stress requirement. A final FEA is then conducted to obtain the strain energy data used for comparison. Similar to the concept of "fully stressed design," the two-iteration approach ensures that each topology is compared using their reasonable optimal sizing. Figure 4.12 illustrates the proposed comparative procedure.

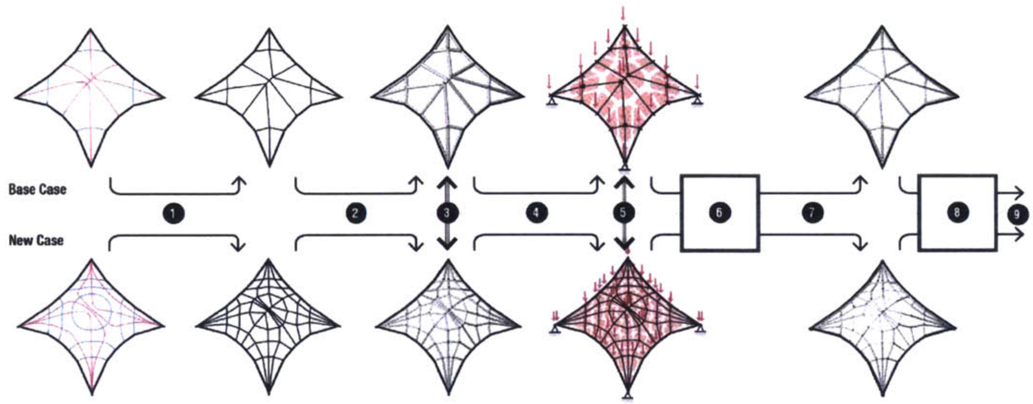


Figure 4.12 Illustrated outline of comparative analysis procedure

1) Assemble linearized frame structure; 2) Adjust constant diameter to ensure 3) equal total volume for all cases; 4) Calculate tributary area and assign loads; 5) Normalize load values to ensure that the total applied forces are equal for all cases; 6) Analyze; 7) Resize members according to stress ratio; 8) Final analysis; 9) Obtain strain energy value for comparison.

4.4.3 Selection by objective function

As illustrated in Figure 4.13, the proposed method is an iterative algorithm that incrementally selects *n* number of stress lines in each iteration for materialization. Selection is heuristic, and based on fitness objectives that can act directly, or indirectly as a proxy for structural performance. This thesis presents in detail the results that were obtained from selection by angle conformity, and selection by strain energy.

As highly complex geometries that are merely visualizations of the force flow for a given de-sign domain, the stress lines are not directly materializable. The process proposed here extracts only the coordinate position of the intersection point of the selected set of stress lines, while maintaining the same topological connectivity. The geometric simplification maximizes the constructability of the materialized stress lines, and ensures that the resultant structure can be modelled as 3D frame structures with mostly axial behavior.

The process begins with an initial starting point input from the user. In each iteration, a random subset of n- stress lines intersecting the input points, or lines are identified and evaluated according to the approximation error of the resultant combined discretized and linearized structure. The combination minimizing the objective is selected, and used as the input lines for the subsequent iteration, which in turn searches for the n stress lines intersecting the new input lines. The approach alternates between the minimum and maximum principal stress directions in each iteration, thus ensuring uniform growth to the stress line density. The subsequent chapter confirms the approaches ability to generate considerable improvement to structural performance within only a few iterations.

The conceptual benefit of an additive process that constructs a stress line-based topology from a blank state is that it gives the designer the opportunity to monitor the trade-off between structural performance and the incremental increase in the density of the stress-line-based structure, such that the designer may precisely identify the optimum density particular to the design constraints. An additive approach is also the least exhaustive computation-ally, as the geometry only gradually increases complexity in each iteration.

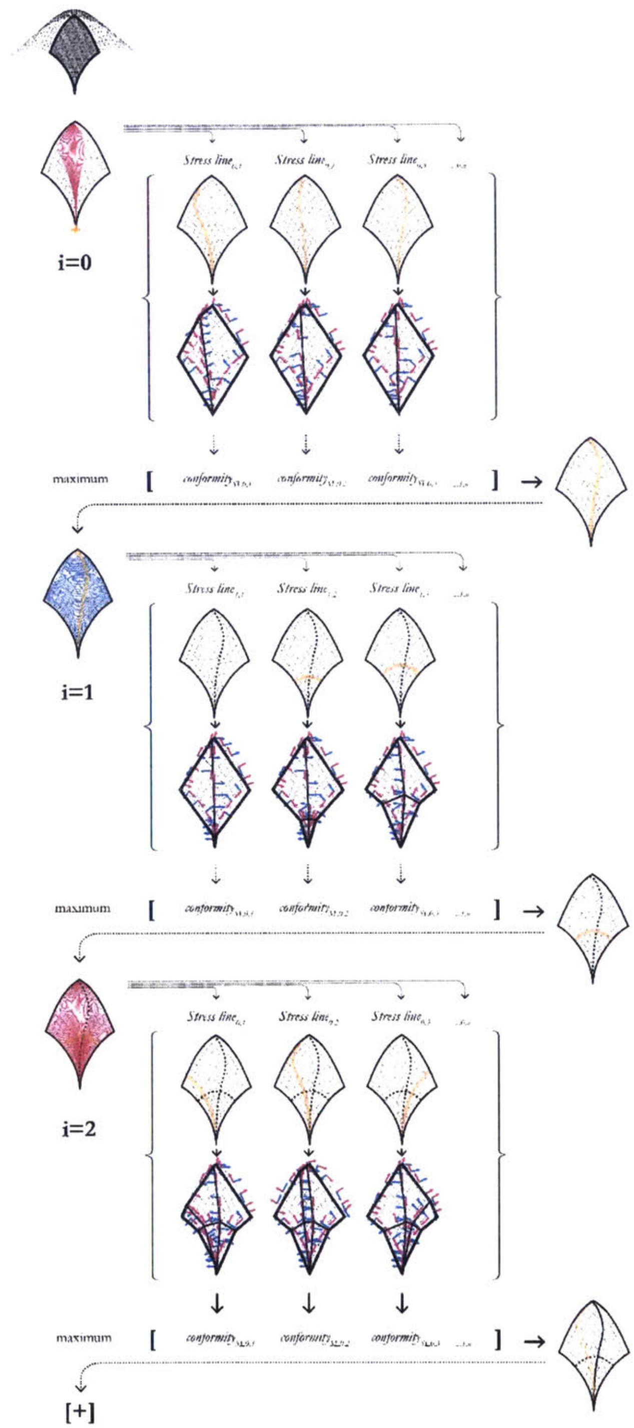


Figure 4.13 Illustrated flowchart of selection algorithm

4.4.4 Subtractive method

The alternative to incrementally adding the stress-line-based topology is to remove stress lines that are regarded as contributing little to the overall performance of the structure. Subtraction may either be predetermined, which refers to analytical values obtained from the parallel shell analysis, or iterative, which is based on ongoing structural analyses conducted in each iteration.

Preliminary implementations corresponding to each approach are described here. One predetermined approach is to remove stress lines based on utilization values: adjacent stress lines are clustered into groups that span similar surface areas, and stress lines are removed from each group according to the average stress magnitudes spanning the same surface area in the parallel shell analysis. The algorithm presents an opportunity for the removal of stress lines that are not expected to be heavily utilized. In the iterative case that was implemented (see Figure 4.14), the initial iteration began with the input of the entire base stress line field, which was linearized and discretized into members that were analyzed under self-weight. The analysis obtained the members' force values, and produced a single force value per stress line. Each iteration proceeded to removes *n* stress lines that are least stressed. Since each stress line is regarded as a design variable, and each stress line may correspond to a sequence of connected members, both approaches differ from traditional topological optimization by reducing the variable space required for optimization.

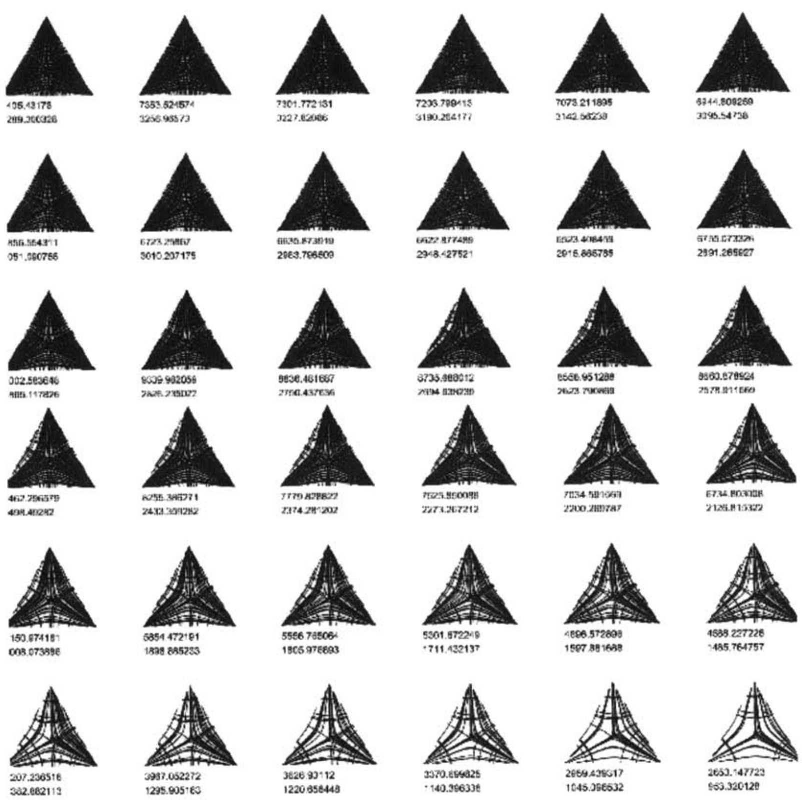
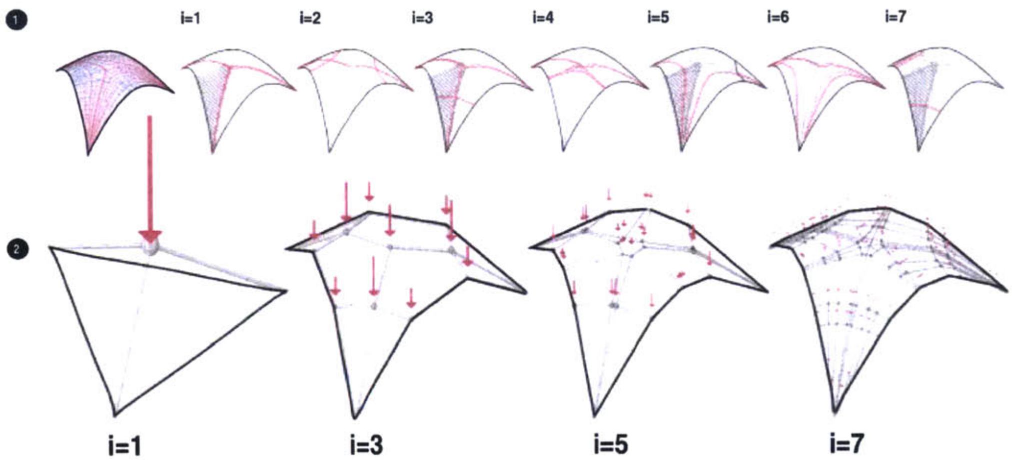


Figure 4.14 Implementation of iterative subtractive approach by removing least-stressed members

5 Case studies and results

5.1 Selection by Angle Conformity

In the first implementation of the incremental method, which was based on a third-order growth model, three initial starting points were supplied and three stress lines were incrementally added to the structure in each iteration. To encourage the development of continuous load paths that cover the largest extent of surface area possible between the shell's centroid and boundaries, the shell's support nodes were input as the starting point to 'grow' the stress-line-based structure by selection.



Angle Deviation (%)	22	11	10	8.9
		Base Case		
Total Length	40	127.5	198	389.40
%Length of Base Case	0.333	1	2	4
Force x Length	5686.4	4500	4100	3738
e = Moment/Force	0.0037	0.10	0.07	0.05
Strain Energy	0.021	0.019	0.017	0.014
Total Force	200			

Figure 5.1 Selected results in initial selection implementation for 3-support shell case

Presented in Figure 5.1 are the edited results obtained from the initial implementation. The initial raw results were manually edited in order to achieve further noise-reduction, and to regularize the topology. To ensure radial symmetry, a representative one-sixth area is identified, mirrored along its local line of symmetry and reproduced in a radial pattern around the shell's radial symmetry centroid.

The results that emerged from the initial implementation largely corroborated with theoretical expectation: as the structure incrementally densifies, which in the process improves its angle conformity to the principal stress directions, various metrics indicative of material efficiency are also decreasing. These measures include force-length and strain energy. Furthermore, the proportion of normal stresses contributed by bending has also decreased in relation to normal stresses due to axial forces, as indicated by the eccentricity calculations.

5.1.1 Spatial objectives: Roof height difference

These early results were encouraging. However, the process was not entirely efficient because it relied on a modest amount of manual post-processing of the stress line results in order to improve their usability as stable, and analyzable topologies. As such, there is no guarantee that the strategy can be easily reproduced in other design cases. To be precise, editing operations include the manual creation of symmetry, and the removal of localized stress line density. Since these problems are physical, and therefore measurable, there is an opportunity to incorporate additional objective values that can characterize these geometric conditions. Thus, the next set of iterations reformulated the objective function as a linear composite function in the following formulation to account for both angle, and geometric conformity:

$J = w_1J_1 + w_2J_2$;

In this formulation, the subscript of 1, and 2 respectively refers to angle conformity, and geometrical conformity. While W accounts for the individual weight, J refers to the individual objective values. In the initial implementation, the measure to account for geometric conformity measured the difference in height between the roof structure framed by the topological results produced in each iteration, and the height in the original shell's design domain. Specifically, the distance between the midpoint of each of the stress-line-based members and their respective vertical projection to the original shell's boundaries are measured, normalized to the original height, and averaged according to the total number of members in the system to obtain a percentage score that would roughly equal in magnitude to the percentage measuring angle conformity. For expediency, the procedure was automated within the Grasshopper environment using a brute-force method of iteration to solve for the combination of stress lines that best minimize this composite measure. Results of the methods are shown in Figure.

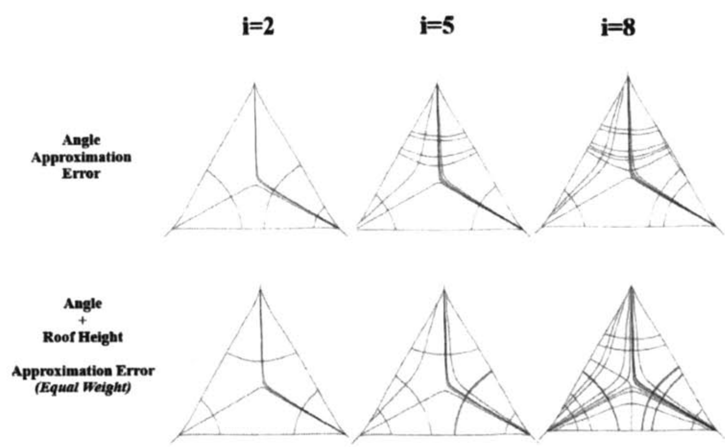


Figure 5.2 Selected results in initial multi-objective selection implementation

Although the results have demonstrated that the linear composite function accounting for both geometric and angle conformity can encourage incremental growth to occur in a more regularized fashion that more closely approximate the referenced shell geometry, the effectiveness of the approach is limited by the quality of the base stress line field, which is asymmetrical due to underlying numerical issues inherent in the interpolation, tracing, and meshing processes. Consequently, manual post-processing is still required.

In the absence of the idealized and perfectly-interpolated stress line results, regular and symmetrical outcomes can only be achieved through explicit means: the stress line selection search space is reduced to a representative portion, which is then reproduced in a circular pattern along the shell's radial symmetry centroid to create the entire grid shell. The remaining studies have adopted this approach.

An unintended computational benefit produced by the new spatial characterization of the selection process is the reduction of stress-line-based design variables in the optimization problem. The reduction of a regular 4-support grid shell into a 1/8 module will, for instance, reduce approximately three-quarter of the stress-line variables. Thus, the new approach produce both desirable spatial outcomes, and computational efficiency.

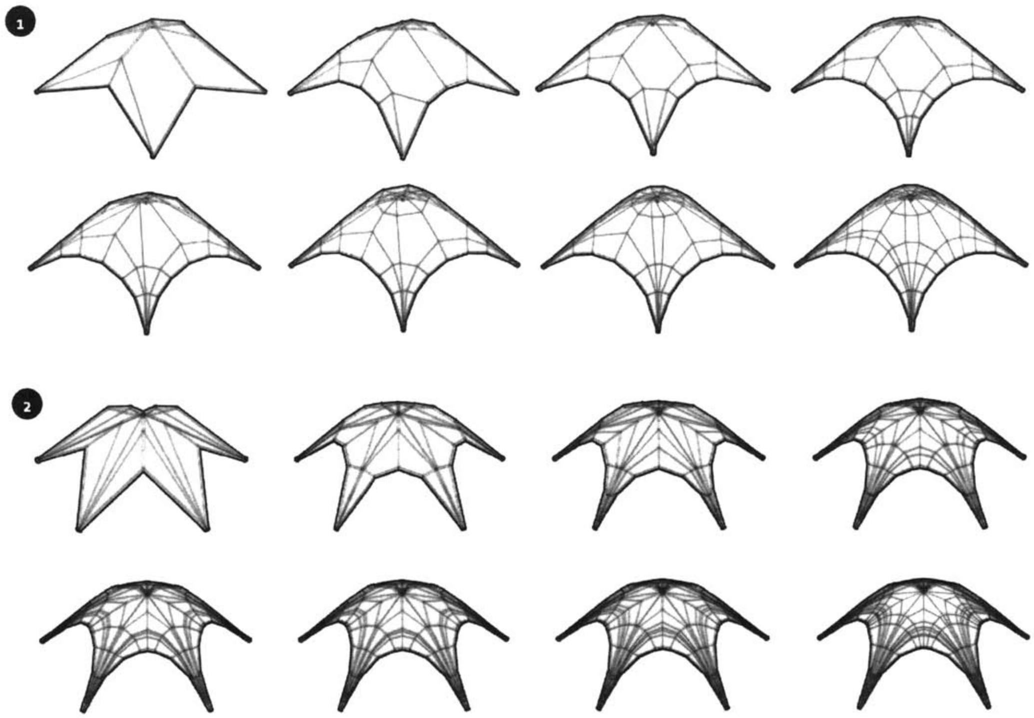


Figure 5.3 Results in multi-objective selection by angle conformity and nodal distribution 1) 4-support case, and 2) 5-support cases

Alternatively, metrics that provides some indication of the relative evenness in the spatial distribution of stress line can also be used as a measure for geometric approximation error. In the figure above, the selection process was guided by a composite objective function that included both angle conformity, and a geometric conformity that considers the distance between adjacent nodes. For each node within the system, the distance between the respective node and its adjacent nodes are measured, and averaged to produce a single value per node. The variance is then calculated from these values, and normalized to the average inter-nodal distance of the entire system to produce a percentage value that approximately indicate the level of unevenness in the spatial distribution of nodes. As is the case with the preceding implementations, the objective is to minimize the composite objective value.

5.2 Revised implementation

The new implementation also inserts frame members corresponding to the boundary of the symmetrical module, so that each linearized and selected stress-line-based member would at minimum be connected at both ends. This eliminates the possible emergence of isolated, and unbraced stress-line-based members, and guarantees the provision of continuous load paths connecting the shell's centroid, the supports, and the selected stress-line-based members. The incorporation of initial frame structures is a strategy commonly used in discrete topological optimization problems to avoid the emergence of kinematically unstable structure (Richardson et al. 2014), which is commonly produced by stochastically-based optimization approaches.

Topologies generated by the revised implementation of the additive selection algorithm by conformity are presented in Figure 5.4 and numerical analytical results in Figure 5.5. Evidently, the effectiveness of the conformity-based metric in generating increasingly optimized structures is not consistent. In both the 4- and 5-support selection cases presented here, the relationship between structural performance and incremental selection appears to be divided into two characteristics phases: typically, continuous incremental improvement is achievable in the initial iterations. However, the consistent improvement in the results of newer iterations, as measured in strain energy, is followed by a recurring performance pattern featuring significant upswing in strain energy followed by their eventual reduction in subsequent iterations. The eccentricity measurements furthermore suggest that bending in the system have increased in these iterations where deteriorating performances are noted. Results from the 3-support selection case, which is not presented here, has also corroborated this pattern.

Considering that improvement in structural performances are consistently and measurably produced in the early iterations for all design cases investigated, the deterioration of structural efficiency documented in the later iterations do not necessarily invalidate the value of the iterative-based stress line selection method. However, these discrepancies indicate that there are limits to the application of this approach, and suggest the need for further investigations to assess the efficacy of the conformity-based metric as an objective value to steer the growth of the stress-line based structure.

Since the results are to be assessed according to their strain energy performance, the optimal selection path that is particular to a given base stress line field can be identified precisely by re-implementing the selection algorithm using strain energy as the objective value. The resultant selection path can then be compared against the conformity-based selection path to generate additional insights.

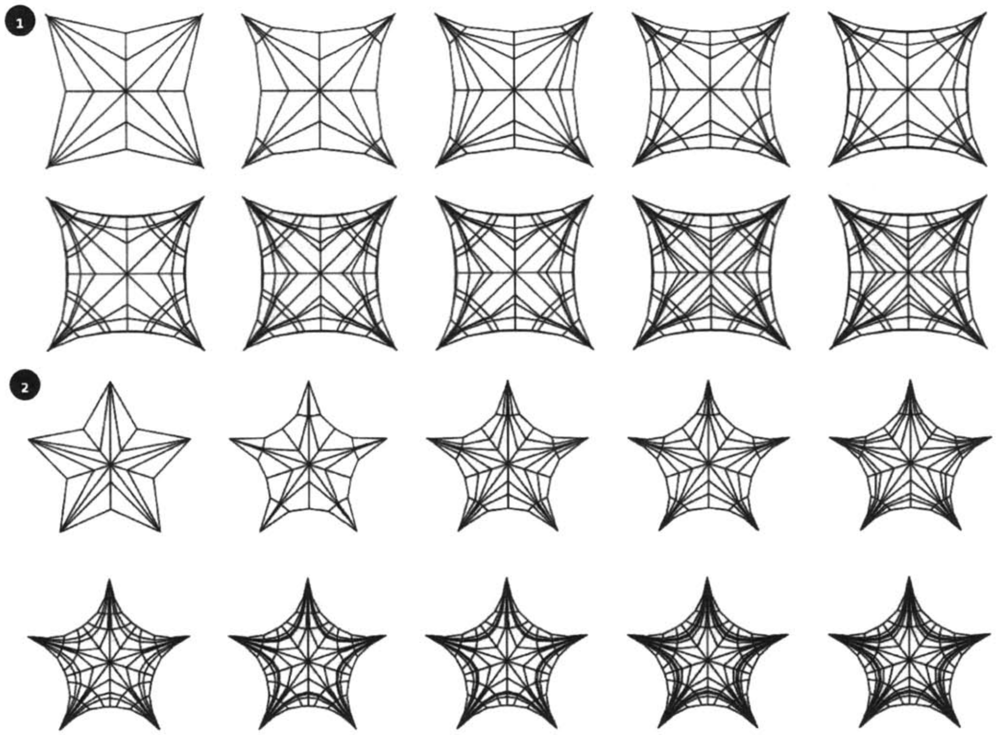


Figure 5.4 Stress line Results from selection with enforced symmetry
1) 4-support case, and 2) 5-support cases

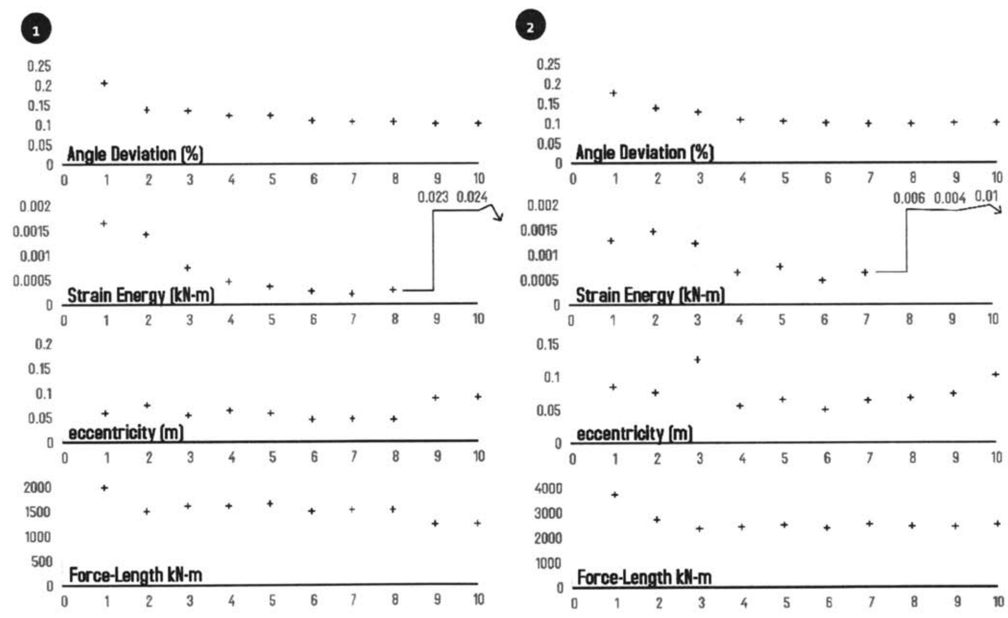


Figure 5.5 Numerical results from selection with enforced symmetry
1) 4-support case, and 2) 5-support cases

5.3 Selection by Strain Energy

As Figure 5.6 shows, the structural performance of topologies generated using strain energy as the objective function also display the similar pattern described earlier. In contrast to the numerical results from the conformity-based case, however, the periodic reversal in structural performances is noticeably less dramatic. Significantly, the capacity for the selection procedure to improve the structural performance is greatest in the first three iterations, when improvement to the conformity of the structure is also relatively the strongest. The strain-energy-based optimal selection path, however, does not necessarily correspond to the conformity-based selection path. Whereas selection by conformity has resulted in stress-line-based structures that are spatially more even and regularly-distributed, the strain-energy-based selection results are characterized by the systematic clustering of previously-selected stress lines. Although further studies will be required to identify the underlying structural properties governing the optimal strain-energy-based selection path, these selection characteristics suggest that the optimal selection path - in stress-line-based topologies generated by the incrementally-additive method - may in fact relate to performative attributes, such as utilization corresponding to the emergent load paths, that is specific to the sequencing of selection in each implementation. More importantly still is the confirmation that structural efficiency, as measured in strain energy, will experience cyclical fluctuation throughout the course of the selection process, while approaching in the long run to the theoretical optimum performance.

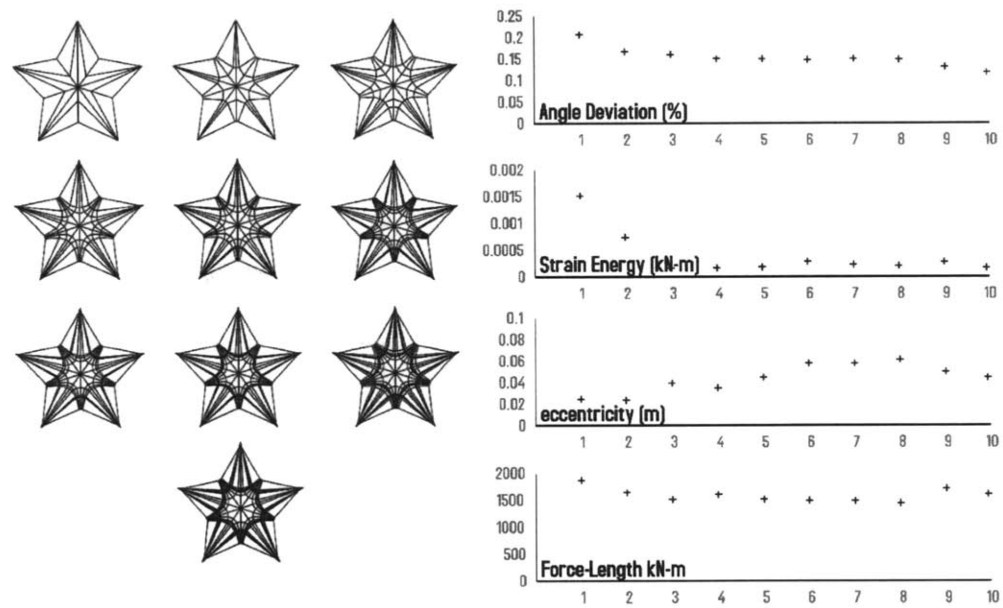


Figure 5.6 Results from strain-energy-based selection for 5-support cases

5.4 Discussion

The various exercises conducted in this thesis has introduced some helpful prescription on the application potential of the selection-based post-processing methods. Although improvements in structural efficiency is generally noticeable and achievable in the early iterations, the gains in structural performance will decrease as the iterations progresses. Significantly, it is identified in these numerical analyses that temporary reversal in structural performance is possible. As such, the effectiveness of the algorithm is dependent on the continuous assessment of all generated topologies from previous iterations, such that the topology selected will perform comparably to other topologies situated along the optimal performance plane revealed through the accumulative performance history in each iteration series. The following section seeks to propose strategies to improve the efficacy of the selection-based process in future developments, and provide some explanations to account for the results identified here.

5.4.1 Theoretical explanation

Although the strategy identified in this thesis has been referred to as a 'densification procedure,' there is a noticeable difference between the numerical densification approach outlined here, and the analytical densification of the Michell structure. Geometrically, each incremental increase in density of the Michell-based structure corresponds to a global readjustment of the density of all stress-line-like members in the system - an adjustment resulting in a topology that is still characterized by a single angle (Mazurek et al. 2011). Conversely, the densification procedure proposed here is a local densification procedure, which relies on a series of iterations to generate a topology that is globally-densified than one from a previous density-order. Therefore, it is likely that the cyclical performance pattern identified earlier is related to the transition of the topology in between different global densities. This is a relevant point to consider since the member resizing procedure, which is subjected to a constant total volumetric constraint and is systematically applied for all analyses in this thesis, ensures that each additional member introduced in the system will have a minimal diameter, thereby penalizing the performance of the entire topology whenever additional members introduced into the system are not meaningfully contributing to the stiffness of overall structure.

The fact that local densification produces negligible contribution to the structural stiffness of an optimum-like topology also corroborates with the known properties of the analytically-derived Michell structure, which has complete reciprocity between its form and force diagram (Baker et al 2013). Thus, the stress-line-based structure under the generation and analytical procedures established in this thesis will only achieve reasonably optimal conditions when selected stress lines geometrically align to the optimal stress lines for a specific order of density. Figure 5.7 illustrate these differences: Figure 5.7.2, which displays a discretized topology derived from stress lines that were selected arbitrarily to achieve a regular and even distribution, is in fact more efficient than Figure 5.7.1, which corresponds to a topology generated in the later iterations of the conformity-based selection approach.

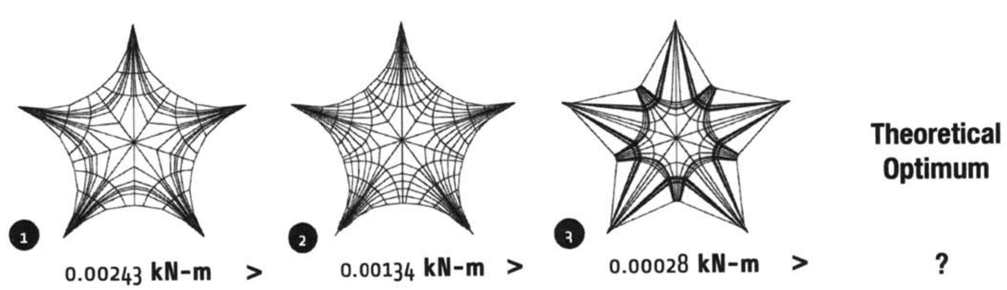


Figure 5.7 Strain-energy comparison for various topologies comprised of 10 stress lines
1) Selection by conformity, 2) Manual regular-based selection, 3) Selection by strain-energy

5.4.2 Measurement of conformity

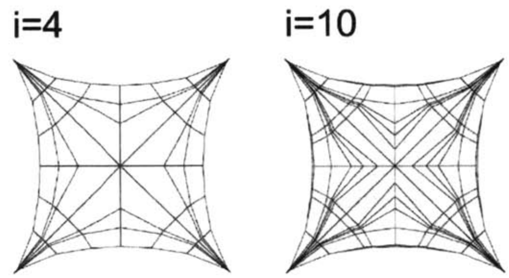


Figure 5.8 Selection biases: localized density clustering

The current formulation of angle conformity is subjected to a number of biases that may lead to selection strategy that deviate from the optimal selection path. Since the angle deviation of each member is weighted according to the ratio of the individual members' length to total member length, the measure does not account for the spatial regularity of selected stress lines. Thus, when the measure is used to steer the growth of the stress-line based structure, local densification of stress lines frequently emerge in later iterations, as the selection process increasingly favors the arbitrary bundling of stress lines in close proximity to areas already hosting previously-selected stress lines, such that these dense stress line aggregation create a localized area of high principal stress direction conformity which induce an upward bias in the conformity measure by virtue of the total member length accumulated in the area (see Figure 5.8).

Future formulation for stress line conformity should incorporate spatial metrics that accounts for the distribution of the stress lines within a design boundary. For instance, the angle measure of each individual member may be weighted by measures that are indicative of densities, such as the average distance of a member to its adjacent members.

5.4.3 Analysis and sizing procedure

Additional areas of considerations include revision to the resizing optimization, which uses stress ratio as a measured. The current analytical process relies on a number of conservative assumptions that effectively exaggerate the sizing of the individual members according to the maximum bending stress of each member. Other measures that can be used in place of stress ratio includes the utilization ratio, or the strain energy density ratio as documented in Chen and Li (2010). Since the reliability of the strain energy value as an indicator of structural efficiency directly hinges on the reliability of the ratio used to determine sizing, it might be possible that the discrepancies noticed in this thesis can partially be attributed to the effectiveness of the resizing optimization used.

5.4.4 Numerical Noises

Other areas that might have contributed to the discrepancies in the results include the enforced radial symmetry, which necessitated the removal of a modest amount of base stress lines and the re-interpolation of new stress lines between the geometrically-defined boundary of the symmetrical module and the remaining stress lines. These newly-interpolated lines may not necessarily conform to actual principal stress directions.

5.5 Summary

In implementing the conformity-based selection strategy for a greater number of iterations and design cases, and developing additional selection approaches based on alternative objective functions, this section has identified a number of issues that were previously unknown in other related research on selection strategies. Generally, the findings from this section suggest the need for a more robust consideration of the relationship between the stress-line-based topologies' geometrical and spatial characteristics, and their structural performances.

6 Conclusion

This thesis aims to bring clarity to a developing field within conceptual structural design that is often misunderstood and applied arbitrarily. Specifically, this is achieved by focusing on the fundamental areas in stress line generation that remain open in related research, such as methods to improve interpolation and the iterative algorithm tracing the stress lines. The basic outline of the stress line method has been reinterpreted, and the many sub-processes involved in stress line generation have been rigorously codified, so that several previously unknown opportunities and issues in the different stages of stress line generation have been discovered.

6.1 Summary of contributions

6.1.1 Stress line methodologies

The specific technical findings outlined in this thesis contributed to the development of a new methodology that is specific to the analysis, and design generation of stress lines. This framework consolidate and incorporate both existing and new research, and seeks to provide a common benchmark to facilitate future researches in the area. Central methods developed in this thesis include:

A) ***Principal stress direction conformity:*** Although the conversion of stress lines to a frame topology constitutes the fundamental intent of stress line materialization, few existing research investigations had developed a metric to quantify the topological conformity of a discretized structure to the underlying principal stress lines of its parallel shell domain. Addressing this research gap, this thesis proposes a geometrically-based angle conformity criteria for a number of stress-line-related operations. In addition to providing an indication on topological conformity, the metric can be used to assess the fidelity of the base stress line field to the principal stress vector field from which it is derived. In design, the metric can steer the selection of stress line for materialization. Finally, the research provides numerical substantiation that angle conformity is correlated to the maximum efficiency achievable for a given discretized topology.

B) ***Comparative analysis method:*** The thesis presents a comparative numerical procedure to address stress-line-based operation ranging from evaluation to optimization. Specifically, the method expedites stress-line-based design exploration by developing automated processes that convert raw selected stress line data into usable geometry with reasonably-optimized sizing, so that normalized numerical analytical results can be quickly obtained for comparative purposes. The framework allows structural performance of different stress-line-inspired topologies for the same design domain to be compared, and the effectiveness of various stress-line-based optimization procedures to be evaluated. Thus, it sets a benchmark that can facilitate the development and communication of future stress-line related research

C) ***Selection and optimization strategies:*** In post-processing, the thesis offers designers several strategies to select stress line results for materialization, each of which entail distinctive spatial and architectural implications. Particularly, the thesis proposed the use of stress lines as design variables that can be optimized to satisfy spatial and structural objectives. The variety of possibilities presented here demonstrates the diversity and versatility of stress-line-inspired design space exploration. The documentation of structural performances furthermore ensures that stress-line-inspired design will not only visualize force flow, but also begin to realize the theoretical potential for stress lines to inspire high-performance topology. In fact, designers and researchers may adopt the comparative framework developed here to propose their own methods of stress-line-based optimization.

6.1.2 Stress line techniques and properties

In codifying and reinterpreting the entire stress-line based design process, the thesis has identified, discretized and expanded a number of overlooked processes related to the generation of stress lines. The techniques developed here seek not only to improve the quality of stress lines, but to render stress line generation a transparent process that welcome designers' participation. Specifically, the thesis has classified these processes into three primary stages: preparation, generation and post-processing. The specific lessons learnt are as follow:

A) ***Preparation:*** In standardizing the parallel shell design domain analysis, the research identified a number of key factors that can influence the stress line outcome with respect to loading conditions, the seeding plan and meshing. First, the best stress line results are obtained using the form-found loading, or gravity loading if the form-finding conditions are not possible. It was identified here that the mesh topology can have a significant impact in the precision of the stress line curvatures. Because global densification can lead to unnecessary increases to computational requirement for stress line computation, the thesis proposed a method to localize mesh densification strategically, such that the resolution of stress line can be improved intelligently at where it is needed most. Finally, the researches on seeding demonstrates that a guided approach to seeding can reduce post-processing burden by focusing the generation of stress line in accordance to performance criteria that may be structural, or spatial.

B) ***Generation:*** Innovations were brought to the core processes of stress line generation, which include interpolation and tracing - procedures which have generally received little attention in spite of their fundamental role in stress line generation. Conventional interpolation methods of principal stress trajectories were reproduced and their effectiveness evaluated. Addressing the common problems associated with these approaches, which include low resolution, and search space inconsistency, a new as the (1+N)-order approximation method was proposed. Comparative results have demonstrated their effectiveness in generating stress lines with higher conformity and resolution.

The most significant change proposed by this research, however, was made to the tracing step. The conventional algorithm in stress line tracing is prone to generating unusable results since it does not seek to addresses biases inherent in interpolation and meshing processes. This thesis acknowledges the numerical basis of stress line generation and proposes a sequential iterative algorithm, which traces the stress line for each seed sequentially, and geometrically detects and responds to the accumulating base stress line field, such that numerical noises are pre-emptively eliminated, and the stress lines are drawn to meet explicit constructability and spatial requirement to reduce post-processing burden. Particularly, the thesis identified and characterized a number of common numerical problems that can be targeted by the integration of corrective geometric rules in the tracing algorithms.

C) ***Post-processing:*** With the base stress line field constructed, stress lines are selected strategically in order to balance both architectural and structural objectives. Here the thesis proposes an additive or subtractive approach. Particularly, structural results were obtained to verify the effectiveness of a number of incrementally-additive methods. This has led both to practical and theoretical findings.

   Conceptually, the relationship between stress line density and performance was clarified. The analytical optimum structure, as referred to Michell, is a regular geometry that can be characterized globally. Similarly, the optimized structure for 2.5D shells is also a regular geometry with a global characterization, whereas local densification will not contribute to further enhancement of structural stiffness. In fact, the optimal selection of stress-line-based structure in an incremental-additive approach is entirely specific to the sequencing of stress line selection, and cannot be directly predicted from the initial outcomes obtained from the parallel shell analysis, as demonstrated in the selection implementation that uses strain energy as the objective function. This finding questions the efficacy of graphical techniques that are popularly adapted to the processing of stress lines, such as the scaling of stress lines to stress magnitudes.

   In practical application, the design contribution of this research is the development of a selection strategy by angle conformity, or strain energy, where an accumulative 60-80% strain energy reduction is consistently achievable in the early course of selection iterations. While these results may not necessarily correspond to the performance of the analytical optimum discrete topology for the given design domain, they nonetheless verify the potential of stress-line-based design approaches to generate a diverse range of similar high-performance topologies.

6.2 Potential Impact

The presented research has numerous applications that directly expand conceptual structural design for architecture at multiple scales. The most immediate design application is the generation of high-performance discretized topology for a variety of 2.5-D geometries, both regular or irregular, in which structural action is predominately in-plane, and compressive. The framework offers a compelling alternative approach to topological finding that differs from conventional strategies that modifies regular grid-based topology.

In other design scales, the research can be extended to digital fabrication techniques that help make the prototyping of geometrically complex and performative stress-line-inspired designs achievable. Future development will enable stress line-inspired topology to be implemented for more complex shell and solid geometries. These methods can also be incorporated into broader implementation of multi-objective design space exploration. Preliminary developments of some of these possibilities are enclosed in the appendix.

6.2.1 Rule-based design space exploration

A rule-based or grammar-based design space exploration offers advantages that are highly appropriate for conceptual structural design. In a conventional parameter-based approach, designs found in these approaches are generally parametric variations of each other with little genuine typological diversity. Furthermore, variation in parameter does not necessarily guarantee structurally sound results.

In contrast to parameter-based approaches, a rule-based system can incorporate structural information to guarantee structural stability. These rules can then be combined to generate an infinite number of new structural typologies. With geometric explorations discretized into specific rules designed to meet structural objectives, design exploration is transformed into an interactive and instructional experience where designers can constantly gain new insights in the relationship between form and forces. More powerful still is the possibility for a rule-based system to infer grammar for existing structural precedents, to accumulate and compile structural grammatical intelligence from designers, and to evolve rules over time, such that the guidance system in perpetual self-refinement.

Although obvious benefits exist in rule-based design space, few implementations of rule-based design guidance tools currently exist. Approaches such as Shea and Cagan's *Shape Annealing* methods (1999), and Geyer's *Multidisciplinary Optimization* (2008) do not take full advantage of the power of grammar over parametric design approaches, since they are limited to pre-determined global structural typology. Compounding the difficulty in achieving a rule-based system in conceptual structural design tools are the typical analytical methods, which produce scalar quantities that do not directly provide geometric recommendations, such that there is a proliferation of non-visual data that must be incorporated into the structural shape. The absence of geometric analytical information suggests that rules are not necessarily aware of the structural implications of their geometric operations. Consequently, optimization is dependent on randomness to a higher degree so an extension of computational time could occur without necessarily guaranteeing improvement. Since stress lines are essentially analytical information that are encoded in curved geometries, it can better capitalize the geometrical potential of rule-based design systems.

Particular to the stress-line-based design framework developed in this thesis, rule-based methods can be applied during the post-processing stage in order 1) to adjust the quality of the stress lines following the construction of the base stress line field, and 2) to transform the discretized structure that is derived from the selection process. While the rule-based methods described in the preceding section assumes a predominately corrective role that targets the biases inherent in the numerical process, the objective for later rule-based transformations is both corrective and generative. Such a system can target generalized geometric conditions independent of the generations of stress lines that may produce structural inefficiencies, and enable the emergence of new geometric characteristics that simultaneously satisfy multi-disciplinary objectives. In any case, further improvement to structural performance is possible, as illustrated in Figure 6.2

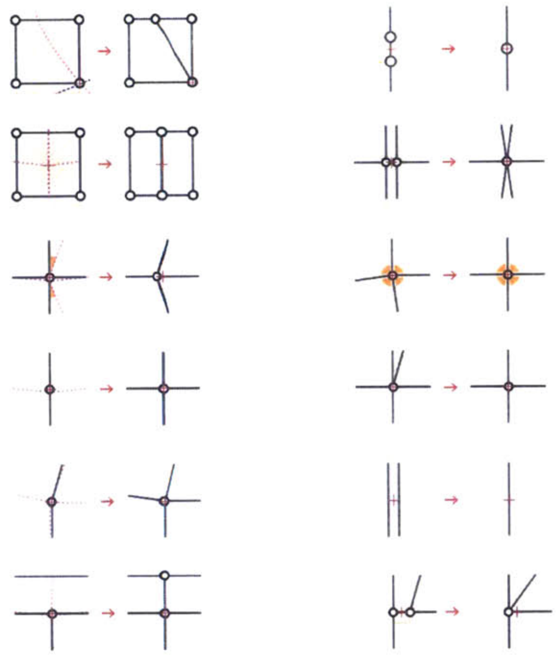


Figure 6.1 Proposed rules for post-processing

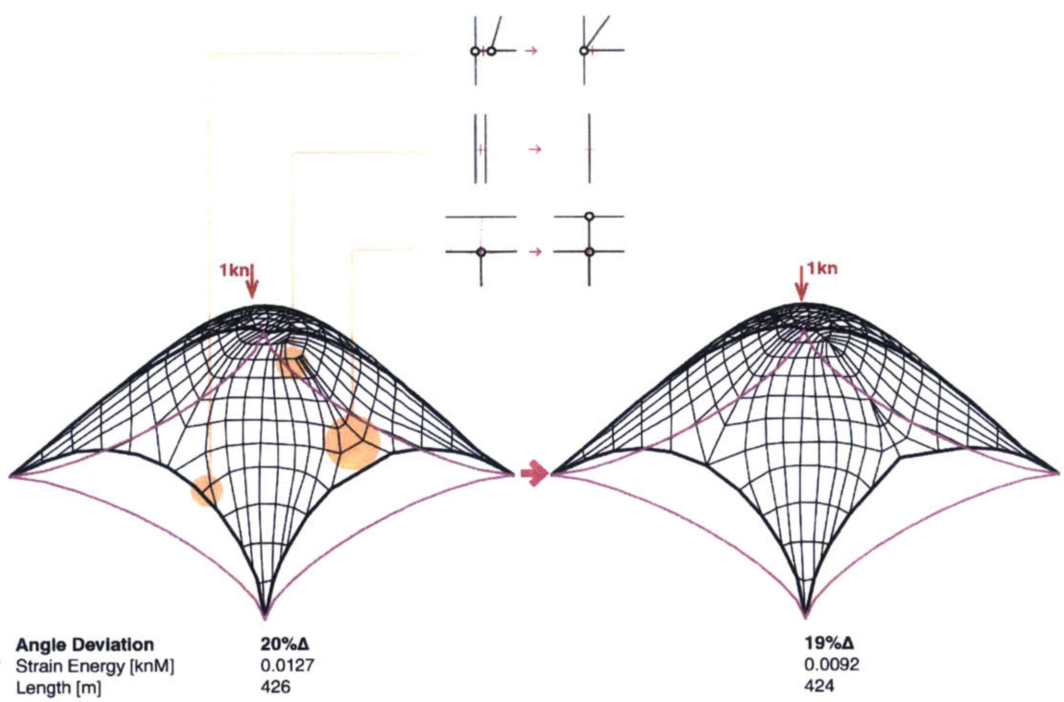


Figure 6.2 Example showing improvements from rule-based post-processing

6.2.2 Stress line additive manufacturing

One specific application currently under investigation by the author and collaborators is the use of stress line research to address limitations of current additive manufacturing (or 3D printing) technologies.

While there are reports of these fabrication technologies being integrated to the design processes of a number of disciplines, from medical to aerospace industries, the application of AM in the field of architecture remains largely as a tool that represents geometry only, whereas its potential to convey both physical performance and geometric form remains generally unmet. In fact, physical models have played a decreasing role in the evaluation of structural behavior, as structural analyses have become increasingly reliant on the use of computational tools, and become detached from the design conceptual design stages. This phenomenon constitutes a departure from the strong tradition of integrated architectural and structural design development in architectural history, whereby physical models have been used to develop, evaluate, and compare design concepts, often in parallel with calculations (Mueller, Irani and Jenett, 2014).

The current capability of using AM has a number of limitations: since common additive manufacturing methods, especially thermoplastic-based fused deposition modelling (FDM), deposit material in a crisscross manner in layers parallel to the horizontal printing bed, the fabrication process leads to anisotropic structural elements with strength and stiffness that varies significantly depending on the orientation of the applied forces. This problem can potentially be solved by adding materials along three-dimensional paths based on stress lines, such that the structural performance of the printed specimen is optimized to reflect the mechanical behavior of the systems that it represents. Such developments will transform rapid prototyping technologies into highly-useful tools for conceptual structural design that are analogous to the conventional use of wind tunnel model to test a building's internal resistance to wind loading. With application both at prototype and end-use scales, stress-line additive manufacturing can lead to high-performing and geometrically novel structures.

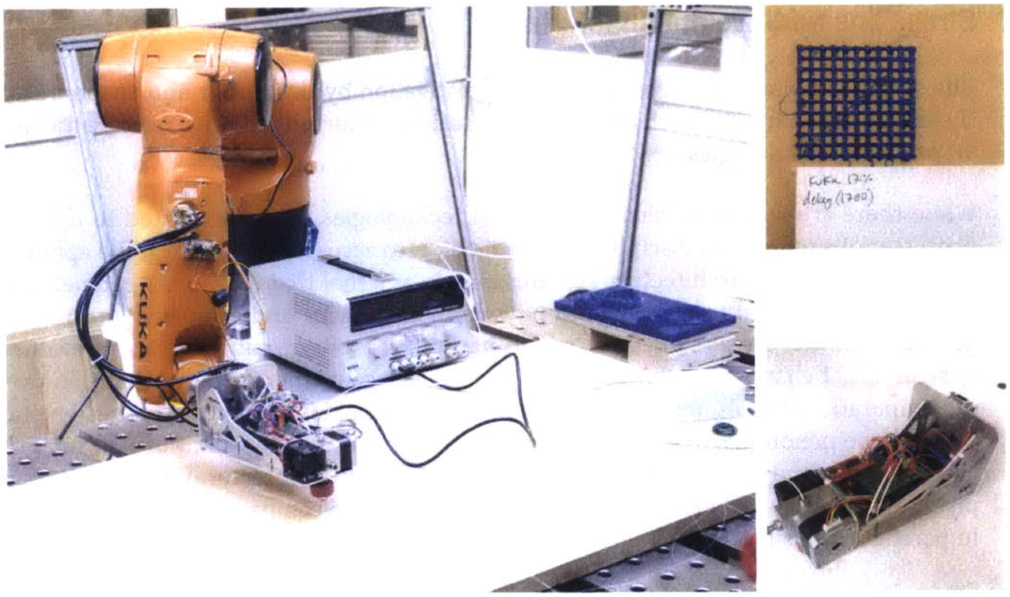


Figure 6.3 Equipment for stress line additive manufacturing

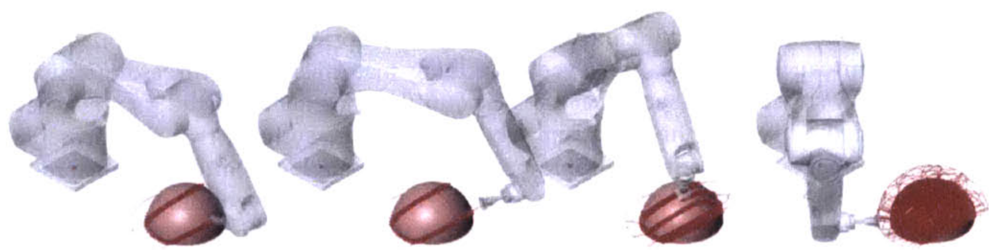


Figure 6.4 Diagram depicting computational path generation

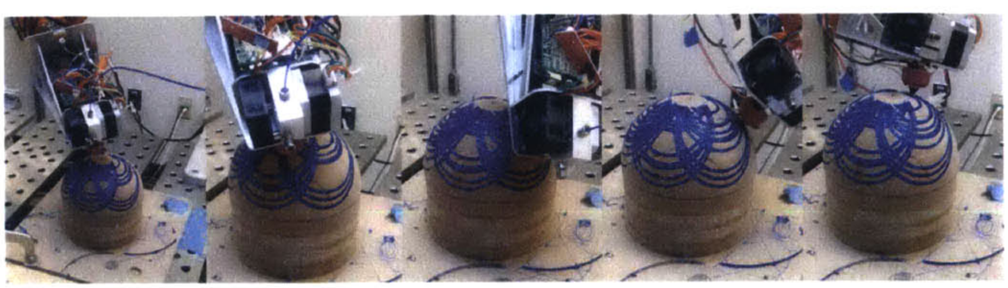


Figure 6.5 Documentation of printing process

6.3 Epilogue

The history of structural design includes a notable collection of projects in which architects and engineers used structural pattern to inform topological design. Whereas the capacity for structural patterns to inspire design is evident (Allen and Zalewski 2010), the processes by which these patterns are generated may be shrouded in ambiguities and inconsistencies, such that the use of structural pattern remains a craft that is highly specialized to the individual practitioners, and employed with great artistic liberty that is not always substantiated by structural logic. In the absence of a well-developed framework that standardizes and parameterizes the process of stress line generation, the potential of stress line-based design has remained inaccessible to the general design community, while the actual effectiveness of stress-line-inspired techniques for producing high-performance design remains a topic that is relatively unstudied.

Recognizing that the materialization of structural pattern has often been relegated to an inspirational or symbolic role, this thesis has sought to develop a feasible method for the design of stress-line-inspired topology that achieves both force-flow visualization and actual structural performance. Since the strongest benefit of stress-line-based techniques is the potential to include the participation of designers, it remains important to achieve a balance between standardization and flexibility. In the process, the research has given weight to a host of geometrically-significant procedures that are often overlooked, and developed a knowledge and implementation framework that will allow designers to make informed decisions both about the generation of stress lines, and the structural assessment of stress-line-inspired topologies. The thesis empowers designers to produce high-performance and visually-compelling results; moreover, it provides a new methodology to facilitate future development of stress-line-related research.

References

Achtziger, W., "Topology optimization of discrete structures: An introduction in view of computational and nonsmooth aspects." In *Topology Optimization in Structural Mechanics*, edited by G.I.N. Rozvany, 57-100. Vienna, AT: Springer, 1997.

Adriaenssens, S., J. N. Richardson, R. F. Coelbo, and P. Bouillard. "Discrete topology optimization." In *Shell Structures for Architecture: Form Finding and Optimization*, edited by S. Adriaenssens, P. Block, D. Veenendaal, and C. Williams, 171-180. New York City: Routledge, 2014.

Allen, E., and W. Zalewski. *Form and Forces: Designing Efficient, Expressive Structures.* Hoboken, N.J.: John Wiley & Sons, 2010.

Astbury, Jon. "Architects Do It with Models: The History of Architecture in 16 Models." *Architectural Review.* February 25, 2014. Accessed May 20, 2015. http://www.architectural-review.com/opinion/architects-do-it-with-models-the-history-of-architecture-in-16-models/8658964.article.

Block P., and M. Rippmann. "Funicular Shell Design Exploration." In *Proceedings of the 33rd Annual Conference of the Association for Computer Aided Design in Architecture*, October 24-26, 2013. Cambridge, ON: University of Waterloo School of Architecture.

Chen, Y., and Y. Li. "Beam Structure Optimization for Additive Manufacturing based on Principal Stress Lines." In *Solid Freeform Fabrication Proceedings*, 2010, ed. D.L Bourell, et al., 666-78. Austin, T.X: SFF, 2010.

Delaunay, Boris: "Sur la sphère vide. A la mémoire de Georges Voronoï." *Bulletin de l'Académie des Sciences de l'URSS, Classe des sciences mathématiques et naturelles*, no. 6 (1934): 793–800.

Geyer, P. "Multidisciplinary Grammars Supporting Design Optimization of Buildings." *Research in Engineering Design* 18, no. 4 (2008): 197-216.

Halpern, A.B., D. P. Billington, and S. Adriaenssens. "The Ribbed Floor Slab Systems of Pier Luigi Nervi." In *Proceedings of the International Association of Shell and Spatial Structures (IASS) Symposium 2013 "Beyond the Limits of Man"*, September 23-27, 2013. Wroclaw, PL: Wroclaw University of Technology.

Hibbeler, R. C. *Statics and Mechanics of Materials.* 3nd ed. Upper Saddle River, N.J.: Pearson/Prentice Hall, 2011.

Kilian, A., and J. Ochsendorf. "Particle-Spring Systems for Structural Form Finding." *Journal of the International Association For Shell And Spatial Structures* 46, no. 2 (2005): 77-85.

Larsen, O. P., and A. Tyas. *Conceptual Structural Design: Bridging the Gap between Architects and Engineers.* London: Thomas Telford, 2003.

Lu, K. and S. Kota. "Topology and Dimensional Synthesis of Compliant Mechanism Using Discrete Optimization." *ASME Journal of Mechanical Design* 128 (2006): 1080-91.

Mattheck, C. *Design in Nature: Learning from Trees.* Berlin: Springer-Verlag, 1998.

Mazurek, A., W. Baker, and C. Tort. "Geometrical Aspects of Optimum Truss Like Structures." *Structural and Multidisciplinary Optimization* 45, no. 2 (2011): 231-42.

Michalatos, P. "Millipede." Grasshopper. March 1, 2014. Accessed May 1, 2015. http://www.grasshopper3d.com/group/millipede.

Michalatos, P., and S. Kaijima. "Eigenshells: Structural patterns on modal forms." In *Shell Structures for Architecture: Form Finding and Optimization*, edited by S. Adriaenssens, P. Block, D. Veenendaal, and C. Williams, 195-210. New York City: Routledge, 2014.

Michell, A.G.M. "The Limits of Economy of Material in Frame-Structures." *Philosophical Magazine* 8, no. 47 (1904): 589-97.

Mueller, C., and J. Ochsendorf. "An Integrated Computational Approach for Creative Conceptual Structural Design." In *Proceedings of the International Association of Shell and Spatial Structures (IASS) Symposium 2013 "Beyond the Limits of Man"*, September 23-27, 2013. Wroclaw, PL: Wroclaw University of Technology.

Mueller, C., A. Irani, and B.E. Jenett. "Additive Manufacturing of Structural Prototypes for Conceptual Design." In *Proceedings of the International Association of Shell and Spatial Structures (IASS) Symposium 2014 "Shells, Membranes and Spatial Structures"*, September 15-19, 2014. Brasilia, BRA: Polytechnic School of the University of São Paulo.

Piker, D., "Kangaroo: Form Finding with Computational Physics." *Architectural Design* 83, no. 2 (2013): 136-37.

Preisinger, C. "Karamba." Grasshopper. February 13, 2015. Accessed May 1, 2015. http://www.grasshopper3d.com/group/karamba.

Rozvany, G.I.N., M.P. Bendsøe, and U. Kirsch, "Layout Optimization of Structures." *Applied Mechanics Reviews* 48, no. 2 (1995): 41-119.

Robert McNeel & Associates. "Rhinoceros." Rhinoceros. Accessed May 11, 2015. http://www.rhino3d.com/.

Ruiz, F. M., and A. Muttoni. "On Development of Suitable Stress Fields for Structural Concrete." *ACI, Structural Journal* 104, no. 4 (2007): 495-502.

Rutten, David. "Back Home." I Eat Bugs For Breakfast. November 10, 2013. Accessed May 11, 2015. https://ieatbugsforbreakfast.wordpress.com/2013/11/10/back-home/.

Schlaich, J., K. Schäfer, and M. Jennewein. "Toward a Consistent Design of Structural Concrete." *Prestressed Concrete Institute Journal* 32, no. 3 (1987): 75-150.

Shea, K., and J. Cagan. "The Design of Novel Roof Trusses with Shape Annealing: Assessing the Ability of a Computational Method in Aiding Structural Designers with Varying Design Intent." *Design Studies* 20, no. 1 (1999): 3-23.

Shi, Xia. "CIAB PAVILLON: Candela Revisited." Karamba: Parametric Engineering. January 17, 2014. Accessed May 1, 2015. http://www.karamba3.com/ciab-pavillon/.

Sokół, T. "A 99 line code for discretized Michell truss optimization written in Mathematica." *Structural and Multidisciplinary Optimization* 43, no. 2 (2011): 181-90.

Wang, L., W. Shen, H. Xie, J. Neelamkavil, and A. Pardasani. "Collaborative conceptual design -- state of the art and future trends." *Computer-Aided Design* 34 (2002): 981-996.