Combinatorial Joint Source-Channel Coding

by

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Submitted to the Department of Electrical Engineering and Computer Science
in partial fulfillment of the requirements for the degree of

Master of Science

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2015

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May 18, 2015

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Abstract

Traditional error correction and source coding has focused on the stochastic setting where separation based schemes are optimal, and current solutions for applications requiring both lossy compression and noise resilience reflect this approach. However, in the adversarial setting, with worst case errors, separation based schemes are far from being even asymptotically optimal. This work investigates fundamental limits, achievability and converse bounds, practical codes, and algorithms for joint source channel coding (JSCC) in the adversarial setting. Particular attention is paid to the cases of flip and erasure errors.

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Chapter 1

Background

Coding theory, more specifically, source and channel coding, has a long and fruitful history paved with many seminal papers and culminating in comprehensive textbooks. Traditionally, these coding problems have been divided into two categories, stochastic and adversarial/combinatorial, based on the error model, average for stochastic and worst case for combinatorial. The focus of this work is combinatorial joint source-channel coding (CJSCC). In particular, CJSCC in Hamming space with flip or erasure errors where the distortion measure of interest is the Hamming distance.

1.1 Combinatorial Coding

In Hamming space, the combinatorial coding problem is a packing problem and is addressed extensively in [1] and [2]. The packing problem seeks the maximum number of points with the distance between any two points greater than a given minimal distance, or, equivalently, the maximum number of disjoint balls of a given radius that can be packed into binary Hamming space of a given dimension. An exact asymptotic solution is open, the best known lower bound is the Gilbert-Varshamov bound and the best known upper bound is the MRRW bound [3]. The multiple packing problem [4] is an extension of the packing problem wherein any Hamming ball of a given radius cannot contain more than a given number of points.

1.2 Combinatorial Compression

The corresponding compression or source coding problem is a covering problem, and a near comprehensive treatment is given in [5]. Historically, the covering problem has proven to be much more attainable than the packing problem and an exact asymptotic characterization has been found. Moreover, it is shown in [6] that the rate distortion function in the stochastic and adversarial settings are equal. The combinatorial covering problem is further subdivided into linear and nonlinear, i.e. whether the collection of points form a subspace. The asymptotic rate of the optimal linear covering is given in [7] for binary Hamming space and [8] for non binary Hamming space. The techniques used in [8] also demonstrate that more general tilings, beyond Hamming spheres, can be used as efficient covers.

1.3 Stochastic Joint Source-Channel Coding

As mentioned, the primary focus of this work is CJSCC and the literature on this topic appears to be lacking. In the stochastic setting the separation principle [9], [10], asymptotically there is no loss in separate source and channel coding, has supported individual study for compression and coding. As such, research in stochastic JSCC has been minimal, but in the interest of completeness we mention a few such endeavors. In [11] and [12] joint coding techniques are used in estimation theory to derive new lower bounds for signal parameter estimation. More recently, the nonasymptotic performance of JSCC has been sharpened with the introduction of a second order term called the JSCC dispersion [13], and the exponent of decay for probability of success is given in [14].

1.4 Combinatorial Joint Source-Channel Coding

The adversarial joint source-channel problem and a framework for analysis were introduced in [15] and expanded in [16]. An adversarial joint source-channel problem is specified by a source and an adversarial channel. A source consists of a source alphabet with a given probability distribution, a reconstruction alphabet and a distortion metric between source and reconstructed symbols. An adversarial channel consists of an input alphabet, an output alphabet and a conditional distribution. The adversary is restricted to outputs that are strongly typical given the input with respect to the conditional distribution. Two error models are addressed: flip errors, analogous to the BSC in stochastic JSCC, and erasure errors, analogous to the BEC, and minimization is over the worst-case adversarial action.

For flip errors, the alphabets are all binary Hamming space, the distortion metric is the Hamming distance and the adversary is restricted to outputs whose Hamming distance to the input are bounded according to the channel parameter. The CJSCC problem is characterized by choice of an optimal encoder-decoder pair. Error-reducing codes, introduced in [17], are very similar to the CJSCC problem with flip errors and used to construct traditional error-correcting codes. More specifically, an encoder-decoder pair is an error-reducing code if it is a CJSCC over a window of values. Jointly tailoring matched encoder-decoder pairs has been previously investigated, but, to the author's knowledge, not in the general framework presented in [15]. In particular, in [18] a cryptographic based encoder-decoder pair is used to improve known results for the adversarial channel with computationally bounded noise.

For flip and erasure errors tradeoff between optimal distortion and bandwidth expansion factor is sought. In [15] it is shown that the optimal CJSCC with flip errors and unit bandwidth expansion factor is the identity map and this is strictly better than any separated scheme. For higher order bandwidth expansion factors an analog of the identity scheme is the repetition code, and, unlike the traditional stochastic setting, the performance of the repetition code is nontrivial. This observation instigated an investigation into the repetition of other small dimension codes. In [19], the asymptotic performance of repeating a small dimensional code and a more detailed analysis for repetition of the perfect seven four Hamming code for flip errors is given. Repetition of small order codes is much more straightforward for erasure errors and the corresponding asymptotics will be analyzed.

Given an encoder one can calculate the performance of and give an explicit representation for the optimal decoder, and similarly for a given decoder. In particular, for an encoder, the optimal decoder is the Chebyshev center of the preimage of a Hamming ball dependent on the input. Unlike its Euclidean analog the Chebyshev center is not unique, and determining the Chebyshev center and the corresponding Chebyshev radius of a set in Hamming space is computationally intensive. Therefore, practical implementations of CJSCC will require efficient algorithms for finding Chebyshev radii. These and similar questions are addressed in [20], where, among other things, an efficient approximation algorithm for calculating Chebyshev radii using a linear programming relaxation is given.

Chapter 2

Preliminaries

The notation for the n fold product of the field of two elements $\mathbb{F}_2^n$ is used for n dimensional binary Hamming space, $d(\cdot, \cdot)$ is the Hamming distance and $w(\cdot)$ is the Hamming weight. Given a set $S \subset \mathbb{F}_2^n$ its Chebyshev radius is the radius of the smallest Hamming ball containing all of its points

$$\text{rad}(S) = \min_{y \in \mathbb{F}_2^n} \max_{x \in S} d(x, y),$$

a point y_0 achieving this minimum is called a Chebyshev center, and its covering radius is the radius of the smallest covering by points in S

$$r_{\text{cov}}(S) = \max_{y \in \mathbb{F}_2^n} \min_{x \in S} d(x, y).$$

These two quantities satisfy an important relation

$$\text{rad}(S) = n - r_{\text{cov}}(S).$$

Proof. Let c be a Chebyshev center of S and choose $s \in S$ such that $d(s, \mathbf{1} + c) \leq \text{cov}(S)$, then

$$n = d(c, \mathbf{1} + c) \leq d(c, s) + d(s, \mathbf{1} + c) \leq \text{rad}(S) + r_{\text{cov}}(S).$$

Suppose $\text{rad}(S) + r_{\text{cov}}(S) > n$, then there exists $x_0 \in \mathbb{F}_2^n$ such that

$$n - \text{rad}(S) < r_{\text{cov}}(S) = \max_{x \in \mathbb{F}_2^n} \min_{s \in S} \text{wt}(x+s) = \min_{s \in S} \text{wt}(x_0 + s).$$

Thus

$$\text{rad}(S) > \max_{s \in S}\{n - \text{wt}(x_0 + s)\} = \max_{s \in S} \text{wt}((\mathbf{1} + x_0) + s),$$

a contradiction. □

There are also some combinatorial quantities of interest:

- $K(n, r)$ – minimal number of points covering $\mathbb{F}_2^n$ with radius r balls;
- $A(n, d)$ – maximal number of points in $\mathbb{F}_2^n$ with distance between any two points at least d;
- $A_L(n, r)$ – the maximal number of points in $\mathbb{F}_2^n$ such that any ball of radius r contains at most L points.

The two packing numbers are related $A_1(n, r) = A(n, 2r + 1)$.

Lemma 1. *For all* $x \in \mathbb{F}_2^n$

i) $B_r(x) = B_r(0) + x$;

ii) $B_r^C(x) = B_{n-r-1}(x + 1)$.

Proof.

i) Let $f(y) = y + x$. Claim: $f : B_r(0) \to B_r(x)$ is an isomorphism. Suppose $f(y) = f(z)$, then $y+x = z+x$ and $y = z$. Let $z \in B_r(x)$. Then $z = (z-x)+x = f(z-x)$, where $d(z - x, x) = w(z - x + x) = w(z) \leq r$ implies $z - x \in B_r(x)$. Hence $B_r(x) = f(B_r(0)) = B_r(0) + x$.

ii) Similar to part i, $B_r^C(x) = B_r^C(0) + x$. Let $g(y) = y + 1$. Claim: $g : B_{n-r-1}(0) \to B_r^C(0)$ is an isomorphism. From part i, it is in injective. Let $z \in B_r^C(0)$, then $z = f(z - 1)$. For all $y \in \mathbb{F}_2^n$, $w(y - 1) = w(y + 1) = n - w(y)$. Therefore, if $w(z) \geq r+1$, then $w(z-1) \leq n-r-1$ and $z-1 \in B_{n-r-1}(0)$. Thus $B_r^C(0) = f(B_{n-r-1}(0)) = B_{n-r-1}(0) + 1$.

□

Lemma 2. *Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$.*

i) $\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$, *whenever* $\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$ *exists.*

ii) For $a_n, b_n \geq 0$, $\liminf_{n\to\infty} a_n b_n \geq (\liminf_{n\to\infty} a_n)(\liminf_{n\to\infty} b_n)$.

iii) For $0 \leq \lambda \leq 1$, $\liminf_{n\to\infty} a_{\lfloor \lambda n \rfloor} = \liminf_{n\to\infty} a_n$.

Proof.

i) As $\{(n,n) \mid n \geq k\} \subset \{(i,j) \mid i,j \geq n\}$, for all n

$$\inf_{k\geq n}(a_n + b_n) \geq \inf_{(i,j)|i,j\geq n}(a_i + b_j) = \inf_{i\geq n} a_i + \inf_{j\geq n} b_j.$$

If $\liminf_n a_n$ and $\liminf_n b_n$ are both finite, then a limit will distribute over the sum. By assumption $\liminf_n a_n + \liminf_n b_n$ is well defined, i.e. infinite of the same sign. It suffices to show that $\liminf_{n\to\infty}(a_n + b_n)$ is infinite when either $\liminf_{n\to\infty} a_n$ or $\liminf_{n\to\infty} b_n$ is infinite. This follows from the above equation.

ii) Similarly, by nonnegativity,

$$\inf_{k\geq n}(a_n b_n) \geq \inf_{(i,j):i,j\geq n}(a_i b_j) = \inf_{i\geq n} a_i \inf_{j\geq n} b_j.$$

iii) $\{a_{\lfloor \lambda n \rfloor}\} = \{a_n\}$.

□

The floor function is superadditive.

Lemma 3. *For $x, y \in \mathbb{R}$*

$$\lfloor x+y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor.$$

Proof. Let $x, y \in \mathbb{R}$, with $x = n_x + r_x$ and $y = n_y + r_y$ where $n_x, n_y \in \mathbb{Z}$ and $0 \le r_x, r_y < 1$. Then

$$\lfloor x + y \rfloor = n_x + n_y + \lfloor r_x + r_y \rfloor \ge n_x + n_y = \lfloor x \rfloor + \lfloor y \rfloor.$$

□

The Chebyshev radius in Hamming space is additive.

Lemma 4. *For all $m, n \in \mathbb{N}$, $A \subset \mathbb{F}_2^m$ and $B \subset \mathbb{F}_2^n$,* $\mathrm{rad}(A \oplus B) = \mathrm{rad}(A) + \mathrm{rad}(B)$.

Proof. Let x_A be a Chebyshev center for A and x_B a Chebyshev center for B. Let $x \in A$ and $y \in B$, then

$$d([x\, y], [x_A\, y_B]) = d(x, x_A) + d(y, y_B) \le \mathrm{rad}(A) + \mathrm{rad}(B)$$

Hence $\mathrm{rad}(A \oplus B) \le \mathrm{rad}(A) + \mathrm{rad}(B)$. Similarly, let $[x_{A\oplus B}\, y_{A\oplus B}]$ be a Chebyshev center of $A \oplus B$. For all $x \in A$ and $y \in B$

$$d(x, x_{A\oplus B}) + d(y, y_{A\oplus B}) = d([x\, y], [x_{A\oplus B}\, y_{A\oplus B}]) \le \mathrm{rad}(A \oplus B).$$

As this holds for all x, y

$$\begin{aligned}\mathrm{rad}(A) + \mathrm{rad}(B) &\le \max_x d(x, x_{A\oplus B}) + \max_y d(y, y_{A\oplus B}) \\ &= \max_{x,y} [d(x, x_{A\oplus B}) + d(y, y_{A\oplus B})] \\ &\le \mathrm{rad}(A \oplus B).\end{aligned}$$

Hence $\mathrm{rad}(A \oplus B) = \mathrm{rad}(A) + \mathrm{rad}(B)$. □

Chapter 3

Flip Errors

3.1 CJSCC with Flip Errors

Definition 1. *Let* $k, n \in \mathbb{N}$, $E \in \{0, \ldots, k\}$ *and* $\Delta \in \{0, \ldots, n\}$. *A pair of maps* $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$ *and* $g : \mathbb{F}_2^n \to \mathbb{F}_2^k$ *is a* $(k, n\,; E, \Delta)$ *CJSCC if, for all* $(x, y) \in \mathbb{F}_2^k \times \mathbb{F}_2^n$,

$$d(f(x), y) \leq \Delta \quad \Longrightarrow \quad d(x, g(y)) \leq E,$$

or, equivalently, $E(\Delta\,; k, n, f, g) \leq E$, *where*

$$E(\Delta\,; k, n, f, g) := \max_{(x,y):d(f(x),y)\leq\Delta} d(x, g(y)).$$

In the sequel the k and n may be dropped when understood from the context. Moreover, the notation $E(\Delta\,; h)$ is used when h is either an *encoder* or *decoder*, an encoder being a map from the source space to the channel space and a decoder being a map from the channel space to the source space. In the interest of notational consistency, typically, an encoder is denoted with an f, a decoder with a g, the source dimension is k and the channel dimension is n.

Definition 2. *The optimal distortion for a* $(k, n\,; E, \Delta)$ *CJSCC is*

$$E^*(\Delta\,; k, n) := \min_{f,g} E(\Delta\,; f, g),$$

with minimization over $f:\mathbb{F}_2^k \to \mathbb{F}_2^n$ *and* $g:\mathbb{F}_2^n \to \mathbb{F}_2^k$.

The following is a simplified characterization of the CJSCC performance of encoders and decoders.

i) For all $f:\mathbb{F}_2^k \to \mathbb{F}_2^n$,

$$E(\Delta\,;f) := \min_{g} E(\Delta\,;f,g) = \max_{y\in\mathbb{F}_2^n} \operatorname{rad}\left(f^{-1}B_\Delta(y)\right),$$

the optimal decoder is $g^*(y\,;\Delta,f) \in \operatorname{cen}(f^{-1}B_\Delta(y))$.

ii) For all $g:\mathbb{F}_2^n \to \mathbb{F}_2^k$,

$$E(\Delta\,;g) := \min_{f} E(\Delta\,;f,g) = \max_{x\in\mathbb{F}_2^k}\min_{z\in\mathbb{F}_2^n}\max_{y\in B_\Delta(z)} d(g(y),x),$$

the optimal encoder is $f^*(x\,;\Delta,g) \in \arg\min_{z\in\mathbb{F}_2^n}\max_{y\in B_\Delta(z)} d(g(y),x)$.

Proof.

i)

$$\begin{aligned}\max_{(x,y):d(f(x),y)\le\Delta} d(x,g(y)) &= \max_{y\in\mathbb{F}_2^n}\max_{x\in f^{-1}B_\Delta(y)} d(x,g(y))\\ &\ge \max_{y\in\mathbb{F}_2^n}\min_{z\in\mathbb{F}_2^k}\max_{x\in f^{-1}B_\Delta(y)} d(x,z)\\ &= \max_{y\in\mathbb{F}_2^n}\operatorname{rad}(f^{-1}B_\Delta(y)),\end{aligned}$$

and $g(y;\Delta,f) \in \operatorname{cen}(f^{-1}B_\Delta(y))$ achieves the bound.

ii)

$$\begin{aligned}\max_{(x,y):d(f(x),y)\le\Delta} d(x,g(y)) &= \max_{x\in\mathbb{F}_2^k}\max_{y\in B_{\delta n}(f(x))} d(x,g(y))\\ &\ge \max_{x\in\mathbb{F}_2^k}\min_{z\in\mathbb{F}_2^n}\max_{y\in B_{\delta n}(z)} d(x,g(y)),\end{aligned}$$

and $f(x;\Delta,g) \in \arg\min_{z\in\mathbb{F}_2^n}\max_{y\in B_{\delta n}(z)} d(x,g(y))$ achieves the bound.

□

In the sequel, an encoder $f:\mathbb{F}_2^k \to \mathbb{F}_2^n$ (resp. decoder $g:\mathbb{F}_2^n \to \mathbb{F}_2^k$) may be called a $(k,n\,;E,\Delta)$ CJSCC if $E(\Delta\,;f)\le E$ (resp. $E(\Delta\,;g)\le E$).

i) A function $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$ is a $(k, n; E, \Delta)$ CJSCC if and only if, for all $S \subset \mathbb{F}_2^n$, $\text{rad}(S) \leq \Delta$ implies $\text{rad}(f^{-1}(S)) \leq E$.

ii) Let $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$ be a $(k, n; E, \Delta)$ CJSCC

a)

$$\text{rad}(f(S)) \leq \Delta \qquad \Longrightarrow \qquad \text{rad}(S) \leq E.$$

b)

$$d(x, y) \geq 2E + 1 \qquad \Longrightarrow \qquad d(f(x), f(y)) \geq 2\Delta + 1.$$

c) If $f(0) = 0$, e.g. f is linear,

$$w(x) \geq 2E + 1 \qquad \Longrightarrow \qquad w(f(x)) \geq 2\Delta + 1.$$

The later implications are very weak necessity conditions for any CJSCC f and comprise the basis for a series of converse bounds.

3.2 Converse Bounds

Two important converse bounds arise by studying the behavior of intrinsic combinatorial objects, i.e. coverings and packings, under the action of a CJSCC.

3.2.1 Covering Converse

Theorem 1. *(Covering Converse) If a $(k, n; E, \Delta)$ CJSCC exists, then*

i) $K(k, E) \leq K(n, \Delta)$;

ii) $K(k, k - E - 1) \geq K(n, n - \Delta - 1)$.

Proof.

i) Let $C \subset \mathbb{F}_2^n$ be a minimal $K(n, \Delta)$ covering. Partition $\mathbb{F}_2^n$ into $\{U_c : c \in C\}$ with $\text{rad}(U_c) \leq \Delta$ for all c. By the CJSCC condition, $\{f^{-1}U_c\}$ is a partition of $\mathbb{F}_2^k$

with $\mathrm{rad}(f^{-1}U_c) \leq E$. For each c choose a Chebyshev center c' of $f^{-1}U_c$. Let $C' = \{c'\}$, then $r_{\mathrm{cov}}(C') \leq E$ and thusly $K(k, E) \leq |C'| = |C| = K(n, \Delta)$.

ii) Suppose $K(k, k-E-1) < K(n, n-\Delta+1)$. Let $S \subset \mathbb{F}_2^k$ be a minimal $K(k, k-E-1)$ covering. Then $|f(S)| \leq K(k, k-E-1) < K(n, n-\Delta-1)$. Thus $\mathrm{rad}(f(S)) \leq \Delta$ and $\mathrm{rad}(f^{-1}f(S)) \geq \mathrm{rad}(S) = E+1 > E$, a contradiction.

The statements are equivalent by Theorem 4. □

3.2.2 Packing Converse

Theorem 2. *Let f be a $(k, n; E, \Delta)$ CJSCC. If an L-multiple packing of radius E exists in $\mathbb{F}_2^k$, then its image under f is an L-multiple packing of radius Δ and*

$$A_L(k, E) \leq L A_L(n, \Delta).$$

Proof. Let C be an L-multiple packing of radius E. Suppose $f(C)$ is not an L-multiple packing of radius Δ. Then there exists $y_0 \in \mathbb{F}_2^n$ such that $|f(C) \cap B_\Delta(y_0)| > L$. By construction $\mathrm{rad}(f(C) \cap B_\Delta(y_0)) \leq \Delta$. Thus there exists x_0 such that $f^{-1}(f(C) \cap B_\Delta(y_0)) \subset B_E(x_0)$. For all $c_0 \in C$,

$$f(c_0) \in f(C) \cap B_\Delta(y_0) \quad \Longrightarrow \quad c_0 \in C \cap f^{-1}(f(C) \cap B_\Delta(y_0)).$$

Hence $|C \cap B_E(x_0)| \geq |C \cap f^{-1}(f(C) \cap B_\Delta(y_0))| \geq |f(C) \cap B_\Delta(y_0)| > L$, a contradiction. The bound follows from $|f^{-1}(f(c_0))| \leq L$. □

With $L = 1$, this yields the coding converse of [15],

$$A(k, 2E+1) = A_1(k, E) \leq A_1(n, \Delta) = A(n, 2\Delta+1).$$

3.3 Basic CJSCCs

3.3.1 (Pseudo)-Identity Code

The (pseudo)-identity code $I_{k,n}$ maps by identity the first $\min\{k,n\}$ bits and zero pads the remaining $n - \min\{k,n\}$ bits.

Lemma 5. *Let $k,n \in \mathbb{N}$ and $\Delta \in \{0,\ldots,n\}$. The distortion of the (pseudo)-identity map $I_{k,n} : \mathbb{F}_2^k \to \mathbb{F}_2^n$ is*

$$E(\Delta\,; I_{k,n}) = \min\{k, \Delta + \max\{0, k-n\}\}.$$

Proof. Let $f = I_{k,n}$. Suppose $k < n$. By construction, for all $y \in \mathbb{F}_2^n$, $f^{-1}B_\Delta(y\,;n) = f^{-1}B_\Delta([y_1^k\ \underline{0}]\,;n) = B_{\min\{k,\Delta\}}(y_1^k\,;k)$. Thus $E(\Delta\,;f) = \min\{k,\Delta\}$. Suppose $k \geq n$. By construction, for all $y \in \mathbb{F}_2^n$, $f^{-1}B_\Delta(y\,;n) = B_\Delta(y\,;k) \oplus \mathbb{F}_2^{k-n}$. Thus $E(\Delta\,;f) = \Delta + (k-n)$. □

3.3.2 Repetition Code

Let $k,n \in \mathbb{N}$. By the division algorithm there exists unique q,r such that $n = qk + r$ with $0 \leq r < k$. The repetition codes repeats the i-th bit $q+1$ times if $i \leq r$ and q times if $i > r$, i.e. $x_i \mapsto \begin{bmatrix} x_i^{(1)} & \ldots & x_i^{(m)} \end{bmatrix}$ where $m \in \{q, q+1\}$. Therefore, the first r bits will be repeated $q+1$ times and the last $k-r$ bits will be repeated q times. As such, the adversary will erase the later bits first and the corresponding distortion is

$$E(\Delta\,;R_{k,n}) = \begin{cases} \frac{\Delta}{\lceil q/2 \rceil} & 0 \leq \Delta \leq (k-r)\lceil q/2 \rceil \\ \frac{\Delta - (k-r)\lceil q/2 \rceil}{\lceil (q+1)/2 \rceil} + (k-r) & (k-r)\lceil q/2 \rceil < \Delta \leq n \end{cases}.$$

The case $n = \rho k$, $\rho \in \mathbb{N}$ is called the ρ-repetition code and is denoted $R_\rho : \mathbb{F}_2^k \to \mathbb{F}_2^{\rho k}$.

3.3.3 Separated Code

For $2 \leq M \leq \max\{A(n, 2\Delta+1), 2\}$, the M-separated code $S_{M,k,n}$ sets a correspondence between a radius E cardinality M covering in $\mathbb{F}_2^k$ and a radius Δ cardinality M packing in $\mathbb{F}_2^n$ and maps points according to their respective approximation point. The resulting distortion is the largest E such that

$$K(k, E) \leq A(n, 2\Delta + 1).$$

3.3.4 Composition of Encoders

Lemma 6. *(Composition Lemma) Let $k, m, n \in \mathbb{N}$ and $\Delta \in \{0, \ldots, n\}$. For all $f_1 : \mathbb{F}_2^k \to \mathbb{F}_2^m$ and $f_2 : \mathbb{F}_2^m \to \mathbb{F}_2^n$*

$$E(\Delta\,; f_2 \circ f_1) \leq E(E(\Delta\,; f_2)\,; f_1)$$

and

$$E^*(\Delta\,; k, n) \leq E^*(E^*(\Delta\,; m, n)\,; k, m).$$

Proof. Let g_1 and g_2 be the optimal Chebyshev decoders. Then $d((f_2 \circ f_1)(x), y) \leq \Delta$ implies $d(f_1(x), g_2(y)) \leq E(\Delta\,; f_2)$ implies $d(x, (g_1 \circ g_2)(y)) \leq E(E(\Delta\,; f_2)\,; f_1)$. The second statement follows immediatley from the first using the optimal encoders. □

Combined with the pseudo-identity code this establishes a weak continuity result.

Lemma 7. *Let $k, n, a, b \in \mathbb{N}$.*

i) For all $\Delta \leq n$,

$$E^*(\Delta\,; k + a, n + b) \leq E^*(\Delta\,; k, n) + a.$$

ii) For all $\Delta \geq b$,

$$E^*(\Delta\,; k + a, n + b) \geq E^*(\Delta - b\,; k, n).$$

Proof.

i) Let f be a $(k,n\,;E^*(\Delta\,;k,n),\Delta)$ CJSCC. By Lemmas 6 and 5

$$E(\Delta\,;I_{n,n+b}\circ f\circ I_{k+a,k})\leq E(E(E(\Delta\,;I_{n,n+b})\,;f)\,;I_{k+a,k})=E(\Delta\,;f)+a.$$

ii) Similarly, let f be a $(k+a,n+b\,;E^*(\Delta\,;k+a,n+b),\Delta)$ CJSCC and $\Gamma=\Delta-b$. By Lemmas 6 and 5

$$E(\Gamma\,;I_{n+b,n}\circ f\circ I_{k,k+a})\leq E(E(E(\Gamma\,;I_{n+b,n})\,;f)\,;I_{k,k+a})=E(\Gamma+b\,;f)=E(\Delta\,;f).$$

□

3.4 Linear Encoders

A linear $(k,n\,;E,\Delta)$ CJSCC is an $n\times k$ matrix $A\in\mathbb{F}_2^{n\times k}$ and satisfies, as $A0=0$, for all $x\in\mathbb{F}_2^k$,

$$w(x)\geq 2E+1 \qquad\Longrightarrow\qquad w(Ax)\geq 2\Delta+1. \tag{3.1}$$

The structural conditions imposed by a linear encoder induce an equivalence relation on the preimage of Hamming balls thereby reducing the number of points that need to be evaluated. In particular when evaluating the distortion of A,

$$E(\Delta\,;A)=\max_{y\in\mathbb{F}_2^n}\ \mathrm{rad}(A^{-1}B_\Delta(y)),$$

one can restrict to a subset of $\mathbb{F}_2^n$ consisting of the coset leaders of $A(\mathbb{F}_2^k)$ with weight less than or equal to Δ.

The performance of the repetition code, in particular, for odd ρ, has proven to be nontrivial. In the class of linear codes where all of the rows have unit weight, the repetition code assigns equal favor to all coordinates. Moreover, one can show that the repetition code or some permutation thereof is the unique linear code such that $\mathrm{wt}(Ax)=\rho\,\mathrm{wt}(x)$ for all ρ. The following technical Lemma and its practical extension establish that the repetition code is asymptotically optimal in the class of linear codes with unit weight rows for odd ρ.

Lemma 8. *Let $\rho > 0$, $\{c_i \in \mathbb{N} : 1 \leq i \leq k\}$ with $c_1 \leq c_2 \leq \ldots \leq c_k$ and $\sum_{i=1}^{k} c_i = \lfloor \rho k \rfloor$. For all $0 \leq m \leq k$*

$$\sum_{k=1}^{m} \lceil c_i/2 \rceil \leq \frac{\frac{1}{k}\lfloor \rho k \rfloor + 1}{2} m.$$

Proof. Let $\tau = \lfloor \rho k \rfloor / k$. As the c_i are integers,

$$\sum_{i=1}^{k} \left\lceil \frac{c_i}{2} \right\rceil \leq \sum_{i=1}^{k} \frac{c_i + 1}{2} = \frac{\tau + 1}{2} k.$$

Suppose there exists m, such that

$$\sum_{i=1}^{m} \lceil c_i/2 \rceil > \frac{\tau + 1}{2} m.$$

By monotonicity of the c_i, for $j \geq m$

$$\lceil c_j/2 \rceil m \geq \lceil c_m/2 \rceil m \geq \sum_{i=1}^{m} \lceil c_i/2 \rceil > \frac{\tau + 1}{2} m.$$

Thus

$$\begin{aligned} \sum_{i=1}^{k} \lceil c_i/2 \rceil &= \sum_{i=1}^{m} \lceil c_i/2 \rceil + \sum_{i=m+1}^{k} \lceil c_i/2 \rceil \\ &> \frac{\tau + 1}{2} m + \frac{\tau + 1}{2} (k - m) = \frac{\tau + 1}{2} k, \end{aligned}$$

a contradiction. □

Proposition 1. *Let $A : \mathbb{F}_2^k \to \mathbb{F}_2^{\rho k}$. If $wt(a_i) = 1$, for all i, then for all $\Delta \in \{0, \ldots, n\}$*

$$E(\Delta; A) \geq \min \left\{ \left\lceil \frac{2}{\rho + 1} \Delta - 1 \right\rceil, k \right\}.$$

Proof. Let $\{c_i : 1 \leq i \leq k\}$ be the sums of the k columns of A. By construction $\sum_{i=1}^{k} c_i = \rho k$. WLOG assume $c_1 \leq \ldots \leq c_n$, CJSCC performance is unaffected by permutation of rows and columns. If all of the rows have weight one, then the adversary's only option is to distort individual bits in $\mathbb{F}_2^k$, i.e. flip $\lceil c_i/2 \rceil$ bits in $\mathbb{F}_2^{\rho k}$ wheres c_i is the number of bits in column i thereby distorting bit i in $\mathbb{F}_2^k$. Thus, the adversary will flip bits in order of the smallest column weights according to the following maximization

$$M = \max \left\{ m : \sum_{i=1}^{m} \lceil c_i/2 \rceil \leq \Delta \right\}.$$

The corresponding distortion is

$$E(\Delta\,; A) = \min\{M, k\}.$$

By Lemma 8,

$$\Delta < \sum_{i=1}^{M+1} \lceil c_i/2 \rceil \leq \frac{\rho+1}{2}(M+1),$$

and thusly $M \geq \left\lceil \frac{2}{\rho+1}\Delta - 1 \right\rceil$. □

Lemma 9. *Let $A \in \mathbb{F}_2^{n \times k}$ and $\underline{X}$ a k-dimensional vector of i.i.d. Bernoulli(q) random variables. Then*

$$E\left[w(A\underline{X})\right] = \frac{n}{2} - \frac{1}{2}V_A(1-2p),$$

where V_A is a generating function for the weight of the rows of A.

Proof. Let a_i be an enumeration of the rows of A and V_A the corresponding generating function.

$$\begin{aligned} E\left[w(A\underline{X})\right] &= \sum_{i=1}^{n} E\left[\langle a_i,\, \underline{X}\rangle \mod 2\right] \\ &=^{(a)} \sum_{i=1}^{n} \frac{1}{2}\left(1-(1-2q)^{w(a_i)}\right) \\ &= \frac{n}{2} - \frac{1}{2}V_A(1-2q), \end{aligned}$$

where (a) follows because, for all $j \geq 0$,

$$E\left[\sum_{i=1}^{j} X_j \bmod 2\right] = \frac{1}{2}(1-(1-2q)^j).$$

□

The following simple Proposition follows immediately.

Proposition 2. *(Conservation of weight) For all $A \in \mathbb{F}_2^{n \times k}$*

$$\sum_{x \in \mathbf{F}_2^k} w(Ax) = \frac{n - zr(A)}{k} \sum_{x \in \mathbf{F}_2^k} w(x),$$

where $zr(A)$ is the number of identically zero rows of A.

Proof. Let $\{X_i \mid i = 1, \ldots, k\}$ be a sequence of i.i.d. Bernoulli 1/2 random variables, $\underline{X} = (X_1, \ldots, X_k)$ a vectorization and $A \in \mathbb{F}_2^{n \times k}$. By Lemma 9,

$$\begin{aligned} E[w(A\underline{X})] &= \frac{n}{2} - \frac{1}{2}V_A(0) \\ &= \frac{n - zr(A)}{2} \\ &= \frac{n - zr(A)}{k}\frac{k}{2} = \frac{n - zr(A)}{k} E[w(\underline{X})]. \end{aligned}$$

As all sequences are equiprobable, multiplying both sides by 2^k gives the result. □

If $n = \rho k$, then the previous theorem is more succinctly $\sum_{x \in \mathbf{F}_2^k} w(Ax) \leq \rho \sum_{x \in \mathbf{F}_2^k} w(x)$. Thus there does not exist a linear encoder A such that $w(Ax) > \rho w(x)$ for all x.

3.5 Duality

3.5.1 Dual Problem

The adversarial CJSCC problem is

$$\begin{aligned}
E^*(\Delta\,; k,n) &= \min_{f,g} \max_{(x,y)\in\mathbb{F}_2^k\times\mathbb{F}_2^n \,:\, d(f(x),y)\le\Delta} d(x,g(y)) \\
&= \min_{f} \max_{y\in\mathbb{F}_2^n} \min_{z\in\mathbb{F}_2^k} \max_{x\in f^{-1}B_\Delta(y)} d(x,z) \\
&= \min_{f} \max_{y\in\mathbb{F}_2^n} \mathrm{rad}(f^{-1}B_\Delta(y)) \\
&= \min_{g} \max_{x\in\mathbb{F}_2^k} \min_{y\in\mathbb{F}_2^n} \max_{z\in gB_\Delta(y)} d(z,x),
\end{aligned}$$

where $f:\mathbb{F}_2^k\to\mathbb{F}_2^n$ and $g:\mathbb{F}_2^n\to\mathbb{F}_2^k$. The corresponding dual problem is

$$\begin{aligned}
\Delta^*(E\,; k,n)+1 &= \max_{f,g} \min_{(x,y)\in\mathbb{F}_2^k\times\mathbb{F}_2^n \,:\, d(x,g(y))\ge E+1} d(f(x),y) \\
&= \max_{g} \min_{x\in\mathbb{F}_2^k} \max_{z\in\mathbb{F}_2^n} \min_{y\in g^{-1}B_E^C(x)} d(y,z) \\
&= \max_{g} \min_{x\in\mathbb{F}_2^k} r_{\mathrm{cov}}(g^{-1}B_E^C(x)) \\
&= \max_{f} \min_{y\in\mathbb{F}_2^n} \max_{x\in\mathbb{F}_2^k} \min_{z\in fB_E^C(x)} d(z,y),
\end{aligned}$$

where $f:\mathbb{F}_2^k\to\mathbb{F}_2^n$ and $g:\mathbb{F}_2^n\to\mathbb{F}_2^k$. A few basic properties of binary Hamming space provide an immediate relationship between the primal and dual problem.

Theorem 3. *(Duality)*

$$\Delta^*(E\,; k,n) = n - E^*(k-E-1\,; n,k) - 1.$$

Proof. For $S \subset \mathbb{F}_2^n$, $r_{\text{cov}}(S) = n - \text{rad}(S)$. Combined with Lemma 1

$$\begin{aligned}\Delta^*(E\,;k,n)+1 &= \max_g \min_{x\in\mathbb{F}_2^k} r_{\text{cov}}(g^{-1}B_E^C(x))\\ &= n - \min_g \max_{x\in\mathbb{F}_2^k} \text{rad}(g^{-1}B_{k-E-1}(x+1))\\ &=^{(a)} n - \min_g \max_{x\in\mathbb{F}_2^k} \text{rad}(g^{-1}B_{k-E-1}(x))\\ &=^{(b)} n - E^*(k-E-1\,;n,k),\end{aligned}$$

where (a) follows because $\mathbb{F}_2^n + 1 \cong \mathbb{F}_2^n$ and (b) follows because the minimization is over $g : \mathbb{F}_2^n \to \mathbb{F}_2^k$. □

We collect some equivalent conditions and a few more implications and relationships between the primal and dual problems.

Lemma 10.

i) $E^*(\Gamma\,;k,n) \le F$ *if and only if* $\Delta^*(F\,;k,n) \ge \Gamma$.

ii) If $E^*(\Gamma+1\,;k,n) \ge F+1$, *then* $\Delta^*(F\,;k,n) \le \Gamma$.

iii) If $\Delta^*(F-1\,;k,n) \le \Gamma - 1$, *then* $E^*(\Gamma\,;k,n) \ge F$.

Proof.

i) Suppose $E^*(\Gamma\,;k,n) \le F$. Let f^* and g^* be the maps achieving $E^*(\Gamma\,;k,n)$. Then, by the contrapositive to the JSCC condition, $d(x, g^*(y)) \ge E^*(\Gamma\,;k,n)+1$ implies $d(f^*(x), y) \ge \Gamma + 1$. Thus, as $F \ge E^*(\Gamma\,;k,n)$,

$$\begin{aligned}\Delta^*(F\,;k,n)+1 &= \max_{f,g} \min_{(x,y)\in\mathbb{F}_2^k\times\mathbb{F}_2^n\,:\,d(x,g(y))\ge F+1} d(f(x),y)\\ &\ge \min_{(x,y)\in\mathbb{F}_2^k\times\mathbb{F}_2^n\,:\,d(x,g^*(y))\ge F+1} d(f^*(x),y)\\ &\ge \Gamma+1.\end{aligned}$$

Suppose $\Delta^*(F\,;k,n) \ge \Gamma$. Let f^* and g^* be the maps achieving $\Delta^*(F\,;k,n)$. Then $d(x, g^*(y)) \ge F+1$ implies $d(f^*(x), y) \ge \Delta^*(F\,;k,n)+1$.

ii) Let $\Delta_0 = \Delta^*(F\,; k, n)$. Suppose $\Delta_0 \geq \Gamma + 1$. Let f^* and g^* be the maps achieving Δ_0. Then $d(x, g^*(y)) \geq F + 1$ implies $d(f^*, y) \geq \Delta_0 + 1$. Thus $E^*(\Gamma + 1\,; k, n) \leq E^*(\Delta_0\,; k, n) \leq F$, a contradiction.

iii) Let $E_0 = E^*(\Gamma\,; k, n)$. Suppose $E_0 \leq E - 1$. Let f^* and g^* be the maps achieving E_0. Then $d(f^*(x), y) \leq \Gamma$ implies $d(x, g^*(y)) \leq E_0$. Thus $\Delta^*(F - 1\,; k, n) \geq \Delta^*(E_0\,; k, n) \geq \Gamma$, a contradiction.

□

Corollary 1.

i)

$$E^*(\Delta\,; k, n) = E_\Delta,$$

where E_Δ is the smallest F such that $\Delta^*(F\,; k, n) \geq \Delta$.

ii)

$$\Delta^*(E\,; k, n) = \Delta_E,$$

where Δ_E is the largest Γ such that $E^*(\Gamma\,; k, n) \leq E$.

Proof. Let $E^*(\cdot) := E^*(\cdot\,; k, n)$ and $\Delta^*(\cdot) := \Delta^*(\cdot\,; k, n)$.

i) $\Delta^*(E_\Delta) \geq \Delta \Longrightarrow E^*(\Delta) \leq E_\Delta$ and $\Delta^*(E_\Delta - 1) \leq \Delta - 1 \Longrightarrow E^*(\Delta) \geq E_\Delta$.

ii) $E^*(\Delta_E) \leq E \Longrightarrow \Delta^*(E) \geq \Delta_E$ and $E^*(\Delta_E + 1) \geq E + 1 \Longrightarrow \Delta^*(E) \leq \Delta_E$.

□

Corollary 2.

$$\Delta^*(k - E - 1\,; k, n) = n - \Delta_E - 1,$$

where Δ_E is the smallest Γ such that $E^*(n - \Gamma - 1\,; k, n) \leq k - E - 1$.

Corollary 3. *If $E^*(\Delta + 1\,; k, n) > E^*(\Delta\,; k, n)$, then*

$$\Delta^*(E^*(\Delta\,; k, n)\,; k, n) = \Delta.$$

Proposition 3.

$$E^*(E^*(\Delta\,; k, n)\,; n, k) = \Delta_E,$$

where Δ_E *is the smallest* Γ *such that* $E^*(n - \Gamma - 1\,; k, n) \le k - E^*(\Delta\,; k, n) - 1$.

Proof. Combining the results of Corollary 2 and Theorem 3

$$\begin{aligned} E^*(E^*(\Delta\,; k, n)\,; n, k) &= n - \Delta^*(k - E^*(\Delta\,; k, n) - 1\,; k, n) - 1 \\ &= n - (n - \Delta_E - 1) - 1 = \Delta_E. \end{aligned}$$

□

3.5.2 Operational Duality

The following Theorem provides a correspondence between achievable distortion points at source-channel dimensions (k, n) and (n, k).

Theorem 4. *(Operational Duality) A* $(k, n\,; E, \Delta)$ *CJSCC exists if and only if an* $(n, k\,; n - \Delta - 1, k - E - 1)$ *CJSCC exists.*

Proof. If the pair (f, g) is a $(k, n\,; E, \Delta)$ CJSCC then the pair $(g + \mathbf{1}_k, f + \mathbf{1}_n)$ is an $(n, k\,; n - \Delta - 1, k - E - 1)$ CJSCC. More specifically, for all $(x, y) \in \mathbb{F}_2^n \times \mathbb{F}_2^k$,

$$\begin{aligned} & d(g(x) + \mathbf{1}_k, y) \le k - E - 1 \\ \implies\ & d(g(x), y) \ge E + 1 \qquad (w(z + \mathbf{1}_k) = k - w(z)) \\ \implies\ & d(x, f(y)) \ge \Delta + 1 \qquad \text{(CJSCC contrapositive)} \\ \implies\ & d(x, f(y) + \mathbf{1}_n) \le n - \Delta - 1. \end{aligned}$$

The reverse implication follows by symmetry. □

Chapter 4

Asymptotic Flip Errors

Asymptotically we allow the user to choose the optimal sequence of source and channel dimensions for a given bandwidth expansion factor $\rho > 0$.

Definition 3. *For bandwidth expansion factor $\rho > 0$, the asymptotically optimal CJSCC is*

$$D^*(\delta\,;\rho) := \inf_{\{\Delta_m\},\{k_m\},\{n_m\}} \liminf_{m\to\infty} \frac{1}{k_m} E^*(\Delta_m\,; k_m, n_m),$$

where the infimum is over unbounded sequences of natural numbers $\{k_m\}$, $\{n_m\}$ and $\{\Delta_m \in \{0,\dots,n_m\}\}$ such that

$$\lim_{m\to\infty} \frac{1}{n_m}\Delta_m = \delta \qquad \lim_{m\to\infty} \frac{n_m}{k_m} = \rho.$$

A triplet of sequences $(\{\Delta_m\},\{k_m\},\{n_m\})$ satisfying the conditions of Definition 3 is said to be a admissible (δ,ρ) source-channel sequence. The point (D,δ) is asymptotically achievable if there exists a sequence of $(k_m, n_m\,; E_m, \Delta_m)$ CJSCCs (f_m, g_m) such that $\frac{1}{k_m}E_m \to D$ and $\frac{1}{n_m}\Delta_m \to \delta$. The region of asymptotically achievable (D,δ) with bandwidth expansion factor ρ is lower bounded by the curve $(D^*(\delta\,;\rho),\delta)$. Moreover, the region is completely characterized by this lower boundary, i.e. a $(k,n\,;E,\Delta)$ CJSCC is a $(k,n\,;F,\Gamma)$ CJSCC for all $F \geq E$ and $\Gamma \leq \Delta$.

The following Lemma provides an approximation for the limit of a sequence of functions acting on a sequence. This result is used to extend nonasymptotic results, in particular converse bounds, into an asymptotic setting.

Lemma 11. *Let* $\{f_n : \mathbb{R} \to \mathbb{R}\}$ *be a sequence of functions,* $\{x_n \in \mathbb{R}\}$ *a sequence,* $\{n_k \in \mathbb{N}\}$ *a subsequence and* $\underline{x} = \liminf_{k\to\infty} x_{n_k}$*. Suppose there is an interval* (a, b) *where the* f_n *are nonincreasing and* $\liminf_{n\to\infty} f_n$ *is bounded below by a real valued right continuous function* g*. If* $a < \underline{x} < b$*, then*

$$\limsup_{k\to\infty} f_{n_k}(x_{n_k}) \geq g(\underline{x}).$$

Proof. Let $\varepsilon > 0$ and $K_0 \in \mathbb{N}$. Let $h_k = \inf_{m\geq k} f_{n_m}$ and $h = \lim_{k\to\infty} h_k$. By right continuity of g on (a, b), there exists $\delta_\varepsilon > 0$ such that, for all $\delta < \delta_\varepsilon$, $|g(\underline{x}+\delta)-g(\underline{x})| < \varepsilon$. Fix a $\delta_0 < \min\{\delta_\varepsilon, \underline{x} - a, b - \underline{x}\}$. Since $h \geq g > -\infty$, h_k converges pointwise on (a, b). Thus, there exists $K_1 \geq K_0$ such that, for all $k \geq K_1$, $|h_k(\underline{x}+\delta_0)-h(\underline{x}+\delta_0)| < \varepsilon$. Thus, for all $k \geq K_1$,

$$h_k(\underline{x} + \delta_0) > h(\underline{x} + \delta_0) - \varepsilon \geq g(\underline{x} + \delta_0) - \varepsilon > g(\underline{x}) - 2\varepsilon.$$

There exists $K_2 \geq K_1$ such that, for all $k \geq K_2$, $x_{n_k} > \underline{x}-\delta_0$ and there exists $K_3 \geq K_2$ such that $x_{n_{K_3}} < \underline{x} + \delta_0$. Hence, as the f_n are nonincreasing on (a, b),

$$f_{n_{K_3}}(x_{n_{K_3}}) \geq f_{n_{K_3}}(\underline{x} + \delta_0) \geq h_{K_3}(\underline{x} + \delta_0) > g(\underline{x}) - 2\varepsilon.$$

□

Corollary 4. *Let* $\{f_n : \mathbb{R} \to \mathbb{R}\}$ *be a sequence of functions,* $\{x_n \in \mathbb{R}\}$ *a sequence,* $\{n_k \in \mathbb{N}\}$ *a subsequence and* $\underline{x} = \liminf_{k\to\infty} x_{n_k}$*. Suppose there is an interval* (a, b) *where the* f_n *are nondecreasing and* $\limsup_{n\to\infty} f_n$ *is bounded above by a real valued right continuous function* g*. If* $a < \underline{x} < b$*, then*

$$\liminf_{k\to\infty} f_{n_k}(x_{n_k}) \leq g(\underline{x}).$$

Corollary 5. *Let* $\{a_n\}$ *be a sequence,* $\{n_k\}$ *a subsequence and* $\{X_n\}$ *an i.i.d. sequence* *of random variables. If* $\liminf_{k\to\infty} a_{n_k} < E[X]$*, then*

$$\limsup_{k\to\infty} \Pr\left(\frac{1}{n_k}\sum_{i=1}^{n_k} X_i \geq a_{n_k}\right) = 1.$$

Proof. By the law of large numbers

$$\lim_{n\to\infty} \Pr\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq a\right) = \mathbb{1}\{a \leq E[X]\}.$$

□

4.1 Asymptotic Converse Bounds

This section serves primarily to extend the known converse bounds of [15] and any converses explicitly named reference converses given therein. The nonasymptotic converse bounds of the preceding chapter are extended by analyzing the limit of their normalized rate.

The information theoretic converse (IT) and asymptotic coding converse (CC) are

$$D_{\mathrm{IT}}(\delta\,;\rho) := \begin{cases} h^{-1}(|1-\rho(1-h(\delta))|^{+}) & 0 \leq \delta \leq \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} < \delta \leq 1 \end{cases}$$

and

$$D_{\mathrm{CC}}(\delta\,;\rho) := \begin{cases} \frac{1}{2}h^{-1}(1-\rho R_{\mathrm{MRRW}}(2\delta)) & 0 \leq \delta \leq \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} < \delta \leq \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} < \delta \leq 1 \end{cases}, \tag{4.1}$$

where $h(x) := -x\log x - (1-x)\log(1-x)$ with base 2 logarithms and

$$R_{\mathrm{MRRW}}(\delta) := \min_{0\leq u\leq 1-2\delta} 1 + \hat{h}(u^2) - \hat{h}(u^2 + 2(1+u)\delta),$$

where $\hat{h}(u) := h((1-\sqrt{1-u})/2)$. The maximum of these two lower bounds represents the current state of the art, and our contribution is an improvement for all δ and ρ, excluding the combination of $\delta \leq 1/2$ and $\rho \leq 1$, where the IT converse is optimal.

4.1.1 Asymptotic Covering Converse

Asymptotically Theorem 1 and Lemma 11 yield a lower-bound on $D^*(\delta\,;\rho)$ given by the following function:

$$D_{\text{cov}}(\delta;\rho) = \begin{cases} h^{-1}(|1-\rho(1-h(\delta))|^+) & \delta \leq \frac{1}{2} \\ 1-h^{-1}(|1-\rho(1-h(1-\delta))|^+) & \delta > \frac{1}{2} \end{cases}. \tag{4.2}$$

Corollary 6. *For all* $0 \leq \delta \leq 1$ *and* $\rho > 0$,

$$D^*(\delta\,;\rho) \geq D_{cov}(\delta\,;\rho),$$

where, $D_{cov}(\delta\,;\rho) : [0,1] \to [0,1]$, $D_{cov}(\delta\,;\rho) :=$

$$\begin{cases} 0 & 0 \leq \delta < 1-\theta_\rho \\ h^{-1}(1-\rho(1-h(\delta)) & 1-\theta_\rho \leq \delta < 1/2 \\ 1-h^{-1}(1-\rho(1-h(1-\delta))) & 1/2 \leq \delta < \theta_\rho \\ 1 & \theta_\rho \leq \delta \leq 1 \end{cases}.$$

and θ_ρ *is the largest* γ *such that* $h(1-\gamma) \geq 1-1/\rho$.

Proof. Let $(\{\Delta_m\},\{k_m\},\{n_m\})$ be an admissible (δ,ρ) source-channel sequence, $\delta_m = \frac{1}{n_m}\Delta_m$, $D_m = \frac{1}{k_m}E^*(\Delta_m\,;k_m,n_m)$, $D = \liminf_{m\to\infty} D_m$ and $f_m : [0,1] \to [0,1]$, $f_m(x) := \frac{1}{m}\log K(m,\lfloor xm \rfloor)$. Then $\lim_{m\to\infty} f_m$ is the limit of the normalized rate for the asymptotically optimal covering and, see e.g. [5] ch. 12, this limit exists

$$f(x) := \lim_{m\to\infty} f_m(x) = \begin{cases} 1-h(x) & 0 \leq x \leq 1/2 \\ 0 & 1/2 < x \leq 1 \end{cases}.$$

Thus, as f_m is nonincreasing, $f_m(x)$ satisfies the conditions of Lemma 11 and $f_m(1-x)$ satisfies the conditions of Corollary 4. Combined with the result of Theorem 1

$$f(D) \leq \limsup_{m\to\infty} f_{k_m}(D_m) \leq \limsup_{m\to\infty} \frac{n_m}{k_m} f_{n_m}(\delta_m) = \rho f(\delta),$$

where the equality follows because f is continuous and δ_m converges. Similarly

$$f(1-D) \geq \liminf_{m\to\infty} f_{k_m}(1-D_m-1/k_m) \geq \liminf_{m\to\infty} \frac{n_m}{k_m} f_{n_m}(1-\delta_m-1/n_m) = \rho f(1-\delta).$$

□

It should be noted that, for $1/2 < \delta \leq 1$, $D_{\text{cov}}(\delta\,;\rho)$ is monotonically *increasing* in ρ with $D_{\text{cov}}(\delta\,;\rho) > \delta$ for $\rho > 1$ and

$$\lim_{\rho\to 0} D_{\text{cov}}(\delta\,;\rho) = \tfrac{1}{2} \qquad \lim_{\rho\to\infty} D_{\text{cov}}(\delta\,;\rho) = \begin{cases} 0 & 0 \leq \delta < \frac{1}{2} \\ \frac{1}{2} & \delta = \frac{1}{2} \\ 1 & \frac{1}{2} < \delta \leq 1 \end{cases}.$$

Thus, for all $\delta > 1/2$,

$$\lim_{\rho\to\infty} D^*(\delta\,;\rho) = 1$$

and combined with (4.6), for all $\delta < 1/2$,

$$\lim_{\rho\to 0} D^*(\delta\,;\rho) = \tfrac{1}{2}.$$

4.1.2 Asymptotic L-Multiple Packing Converse

With $L = 1$, Theorem 2 is asymptotically equivalent to (4.1), and the novelty here is using it for $\delta > 1/4$ or $L > 1$. As per the numerical evaluations given in Section 4.5, using $L = 2$ gives the best asymptotic converse bound in $0 \leq \delta \leq 1/4$. Lemma 14 gives an explicit characterization of the bound.

The following Theorem shows that coding with non-unit bandwidth expansion factor $\rho \neq 1$ probably yields no gain in the region $1/4 < \delta < 1/2$.

Theorem 5. *Let $\rho > 0$.*

i) (Plotkin-Levenshtein) For all $m \in \mathbb{N}$,

$$D^*_{HAD}\left(\left(\frac{1}{2}\frac{m}{2m-1}\right)+;\rho\right) \geq \frac{1}{2}\frac{m}{2m-1},$$

where the minimization is over admissible (δ, ρ) source channel sequences such that $\{k_m\}$, $\{2E^*(\Delta_m;\, k_m, n_m) + 1\}$ satisfy the conditions of Theorem 9.

ii) (Blinovsky) For all $\ell \in \mathbb{N}$,

$$D^*\left(\frac{1}{2}\left(1-\binom{2\ell}{\ell}2^{-2\ell}\right)-;\rho\right) \geq \frac{1}{2}\left(1-\binom{2\ell}{\ell}2^{-2\ell}\right).$$

Proof. (Sketch)

i) Evaluate the coding converse using the Plotkin-Levenshtein solution to $A(n, d)$, [1] ch. 7.3.

ii) Evaluate Blinovsky's upper and lower bounds at the endpoint of the upper bound for ranging values of L.

□

The full proof is given below.

Theorem 6. *[4] (Blinovsky Achievability) Let*

$$\begin{aligned}
f_{e,\ell}(s) &:= -\log\sum_{i=0}^{\ell}\binom{2\ell+1}{i}2^{-is/(2\ell+1)};\\
F_{e,\ell}(s) &:= 1 + \frac{1}{2\ell}(f_{e,\ell}(s) - sf'_{e,\ell}(s));\\
f_{o,\ell}(s) &:= -\log\left(\sum_{i=0}^{\ell-1}\binom{2\ell}{i}2^{-is/(2\ell)} + \binom{2\ell}{\ell}2^{-s/2-1}\right);\\
F_{o,\ell}(s) &:= 1 + \frac{1}{2\ell-1}(f_{o,\ell}(s) - sf'_{o,\ell}(s)).
\end{aligned}$$

For all $\ell \in \mathbb{N}$ *and* $s \in [0, \infty)$,

$$\log A_{2\ell}(n, f'_{e,\ell}(s)n) \geq nF_{e,\ell}(s) + o(n);$$
$$\log A_{2\ell-1}(n, f'_{o,\ell}(s)n) \geq nF_{o,\ell}(s) + o(n).$$

Theorem 7. *[4] (Blinovsky Converse) Let*

$$g_\ell(\lambda) := \sum_{i=1}^{\ell-1} \binom{2i}{i} \frac{1}{i+1} (\lambda(1-\lambda))^{i+1}.$$

For all $\ell \in \mathbb{N}$ *and* $\lambda \in [0, 1/2]$,

$$\log A_L(n, g_\ell(\lambda)n) \leq n(1 - h(\lambda)) + o(n),$$

where L *is* either 2ℓ *or* $2\ell - 1$.

Lemma 12. *Let* $\alpha_k, c_k > 0$ *for* $1 \leq k \leq n$ *and* $g(s) = 1 + \sum_{k=1}^{n} c_k e^{-\alpha_k s}$. *Then* $-\log g(s)$ *is strictly increasing with a continuous strictly decreasing derivative.*

Proof. Let $f(s) = -\log g(s)$. Then f is real analytic on $[0, \infty)$, as g is real analytic on $\mathbb{R}$, log is real analytic on $[1, \infty)$ and $g(s) \in [1, \infty)$. Therefore $f'(s)$, $f''(s)$ are continuous on $[0, \infty)$, and it suffices to show that $f'(s) > 0$ and $f''(s) < 0$ on $(0, \infty)$. By construction $g(s), g''(s) > 0$ on $(0, \infty)$ and $g'(s) < 0$ on $(0, \infty)$. Thus the first derivative $f'(s) = -g'(s)/g(s)$ is strictly positive on $(0, \infty)$. The second derivative is

$$f''(s) = \frac{-g(s)g''(s) + g'(s)^2}{g(s)^2}.$$

By Cauchy-Schwarz

$$\begin{aligned}
g'(s)^2 &= \left(\sum_{k=1}^{n} \alpha_k c_k e^{-\alpha_k s}\right)^2 \\
&= \left(\sum_{k=1}^{n} \left|\sqrt{c_k e^{-\alpha_k s}}\right| \left|\alpha_k \sqrt{c_k e^{-\alpha_k s}}\right|\right)^2 \\
&\leq \left(\sum_{k=1}^{n} c_k e^{-\alpha_k s}\right)\left(\sum_{k=1}^{n} \alpha_k^2 c_k e^{-\alpha_k s}\right) \\
&= (g(s) - 1)g''(s).
\end{aligned}$$

Therefore, $-g(s)g''(s) + g'(s)^2 = -g''(s)$ and thusly $f''(s) < 0$ on $(0, \infty)$. □

Lemma 13. *For all* $\ell \geq 1$

$$\frac{\sum_{i=1}^{\ell} \binom{2\ell}{i-1}}{\sum_{i=0}^{\ell} \binom{2\ell+1}{i}} = \frac{\sum_{i=1}^{\ell-1} \binom{2\ell-1}{i-1} + \frac{1}{4}\binom{2\ell}{\ell}}{\sum_{i=0}^{\ell-1} \binom{2\ell}{i} + \frac{1}{2}\binom{2\ell}{\ell}} = \sum_{i=0}^{\ell-1} \binom{2i}{i} \frac{1}{i+1} 2^{-2(i+1)}.$$

Proof. Expanding the numerator and denominator of the first term

$$\begin{aligned}
\sum_{i=1}^{\ell} \binom{2\ell}{i-1} &= \sum_{i=0}^{\ell-1} \binom{2\ell}{i} = \frac{1}{2}\left(2^{2\ell} - \binom{2\ell}{\ell}\right) \\
\sum_{i=0}^{\ell} \binom{2\ell+1}{i} &= \frac{1}{2} 2^{2\ell+1},
\end{aligned}$$

with ratio $\frac{1}{2}\left(1 - \binom{2\ell}{\ell} 2^{-2\ell}\right)$. Similarly for the second term

$$\begin{aligned}
\sum_{i=1}^{\ell-1} \binom{2\ell-1}{i-1} + \frac{1}{4}\binom{2\ell}{\ell} &= \sum_{i=0}^{\ell-2} \binom{2\ell-1}{i} + \frac{1}{4}\binom{2\ell}{\ell} = \frac{1}{2} 2^{2\ell-1} + \frac{1}{4}\binom{2\ell}{\ell} \\
\sum_{i=0}^{\ell-1} \binom{2\ell}{i} + \frac{1}{2}\binom{2\ell}{\ell} &= \frac{1}{2}\left(2^{2\ell} - \binom{2\ell}{\ell}\right) + \frac{1}{2}\binom{2\ell}{\ell} = \frac{1}{2} 2^{2\ell},
\end{aligned}$$

with ratio $\frac{1}{2}\left(1 - \binom{2\ell}{\ell} 2^{-2\ell}\right)$. The Catalan series is

$$\sum_{i=0}^{\infty} \binom{2i}{i} \frac{1}{i+1} z^i = \frac{1 - \sqrt{1 - 4z}}{2z},$$

for $|z| \leq 1/4$. Thus the final term is

$$\sum_{i=0}^{\ell-1} \binom{2i}{i} \frac{1}{i+1} 2^{-2(i+1)} = \frac{1}{4}\left(2 - \sum_{i=\ell}^{\infty} \binom{2i}{i} \frac{1}{i+1} 2^{-2i}\right),$$

and it suffices to show that

$$\sum_{i=\ell}^{\infty} \binom{2i}{i} \frac{1}{i+1} 2^{-2i-1} = \binom{2\ell}{\ell} 2^{-2\ell}.$$

This follows from

$$\begin{aligned}
\binom{2\ell}{\ell} 2^{-2\ell} - \binom{2(\ell+1)}{(\ell+1)} 2^{-2(\ell+1)} &= \binom{2\ell}{\ell} 2^{-2\ell-1}\left(2 - \frac{1}{2}\frac{\ell!\ell!}{(2\ell)!}\frac{(2\ell+2)!}{(\ell+1)!(\ell+1)!}\right) \\
&= \binom{2\ell}{\ell} 2^{-2\ell}\left(2 - \frac{1}{2}\frac{(2\ell+2)(2\ell+1)}{(\ell+1)(\ell+1)}\right) \\
&= \binom{2\ell}{\ell} 2^{-2\ell}\left(2 - \frac{2\ell+1}{\ell+1}\right) \\
&= \binom{2\ell}{\ell} \frac{1}{\ell+1} 2^{-2\ell}.
\end{aligned}$$

□

Corollary 7. *For all $\ell \in \mathbb{N}$*

i) $f'_{e,\ell}$ *and* $f'_{o,\ell}$ *are continuous and strictly decreasing on* $[0, \infty)$.

ii) $F_{e,\ell}$ *and* $F_{o,\ell}$ *are continuous and strictly increasing on* $[0, \infty)$.

iii) g_ℓ *is continuous and strictly decreasing on* $[0, 1/2]$.

iv) $f'_{e,\ell}(0) = f'_{o,\ell}(0) = g_\ell(1/2) = \frac{1}{2}\left(1 - \binom{2\ell}{\ell} 2^{-2\ell}\right)$

v) $F_{e,\ell}(0) = F_{o,\ell}(0) = 0$.

Proof. Parts i and ii follow from Lemma 12 and parts iv and v follow from Lemma 13. Part iii follows because $\lambda(1-\lambda)$ is continuous and strictly increasing on $[0, 1/2]$. □

Lemma 14. *Let $\ell \in \mathbb{N}$ and $\delta \in g_\ell([0, 1/2])$. If there exists an admissible (δ, ρ) source-channel sequence $(\{\Delta_m\}, \{k_m\}, \{n_m\})$ such that* $\liminf_{m\to\infty} \frac{1}{k_m} E(\Delta_m\, ; k_m, n_m) < g_\ell(1/2)$, *then*

$$D^*(\delta\,;\rho) \geq (f'_{e,\ell} \circ F_{e,\ell}^{-1})(\rho(1-(h\circ g_\ell^{-1})(\delta)))$$

and

$$D^*(\delta\,;\rho) \geq (f'_{o,\ell} \circ F_{o,\ell}^{-1})(\rho(1-(h\circ g_\ell^{-1})(\delta))).$$

Proof. Let $(\{\Delta_m\},\{k_m\},\{n_m\})$ be any such admissible (δ,ρ) source-channel sequence, $\delta_m = \frac{1}{n_m}\Delta_m$, $D_m = \frac{1}{k_m}E^*(\Delta_m\,;k_m,n_m)$, $D = \liminf_{m\to\infty} D_m$, $f_m(x) = \frac{1}{m}\log A_{2\ell}(m,\lfloor xm\rfloor)$ and $I_e = (0, g_\ell(1/2))$. By Corollary 7, $f_{e,\ell}(s) \leq f_{e,\ell}(0) = g_\ell(1/2)$ and $f_{e,\ell}(s)$ is invertible. Moreover, $\lim_{s\to\infty} f_{e,\ell}(s) = 0$. Thus $F_{e,\ell}\circ(f'_{e,\ell})^{-1}$ is well defined on I_e and, by Theorem 6, for all $x\in I_e$, $\liminf_{k\to\infty} f_m(x) \geq (F_{e,\ell}\circ(f'_{e,\ell})^{-1})(x)$. Therefore, as the f_m are nonincreasing, combining the results of Lemma 11, Theorem 2 and Theorem 7

$$\begin{aligned}
(F_{e,\ell}\circ(f'_{e,\ell})^{-1})(D) &\leq \limsup_{k\to\infty} f_{k_m}(D_m)\\
&\leq \limsup_{k\to\infty}\left(\frac{n_m}{k_m}f_{n_m}(\delta_m) + \frac{1}{k_m}\log 2\ell\right)\\
&\leq \limsup_{k\to\infty}\left(\frac{n_m}{k_m}(1-h(g_\ell^{-1}(\delta_m))) + o(1)\right)\\
&= \rho(1-h(g_\ell^{-1}(\delta))).
\end{aligned}$$

The given expression follows from Corollary 7 part *i*, *ii* and extension to $f'_{o,\ell}$ and $F_{o,\ell}$ follows by symmetry. □

Theorem 8. *(Plotkin) For* $2d > n$,

$$A(n,d) \leq 2d/(2d-n).$$

Theorem 9. *(Levenshtein) Provided enough Hadamard matrices exist (if* $k := \lfloor d/(2d-n)\rfloor$, *then Hadamard matrices of order* $2k$ *(k even),* $2k+2$ *(k odd),* $4k$ *and* $4k+4$ *are sufficient)* *i) If d is even and $2d > n \geq d$*

$$A(n,d) = 2\left\lfloor \frac{d}{2d-n} \right\rfloor .$$

ii) If d is odd and $2d+1 > n \geq d$

$$A(n,d) = 2\left\lfloor \frac{d+1}{2d+1-n} \right\rfloor .$$

Proof of Theorem 5

i) Let $(\{\Delta_m\}, \{k_m\}, \{n_m\})$ be any such admissible (δ, ρ) source channel sequence, $D_m = \frac{1}{k_m} E^*(\Delta_m\,; k_m, n_m)$, $D = \liminf_{m\to\infty} D_m$,

$$f_m(x) = 2\min\left\{ \left\lfloor \frac{2x+1/m}{4x-1+2/m} \right\rfloor, \left\lfloor \frac{2x+2/m}{4x-1+3/m} \right\rfloor \right\}$$

and $\mathcal{A} = \{\frac{1}{2}\frac{n}{2n-1} : n \in \mathbb{N}\}$. Then, for $1/4 < x < 1/2$,

$$f(x) := \lim_{m\to\infty} f_m(x) = 2\begin{cases} \lfloor \frac{2x}{4x-1} \rfloor & x \notin \mathcal{A} \\ \lfloor \frac{2x}{4x-1} \rfloor - 1 & x \in \mathcal{A} \end{cases},$$

as $2x/(4x-1)$ is strictly decreasing. By Theorem 2, $A(k_m, 2D_m k_m + 1) \leq A(n_m, 2\Delta_m + 1)$. Thus

$$\begin{aligned} f(D) &\leq^{(a)} \limsup_{m\to\infty} f_{k_m}(D_m) \\ &\leq^{(b)} \limsup_{m\to\infty} A(k_m, 2D_m k_m + 1) \\ &\leq \limsup_{m\to\infty} A(n_m, 2\Delta_m + 1) \\ &\leq^{(c)} \limsup_{m\to\infty} 2\frac{2\Delta_m + 1}{4\Delta_m + 2 - n_m} \\ &= 2\frac{2\delta}{4\delta - 1}, \end{aligned}$$

where (a) is Lemma 11, (b) is Theorem 9 and (c) is Theorem 8. Therefore, for all $m \in \mathbb{N}$, $\delta > \frac{1}{2}\frac{m}{2m-1}$ implies $D > \frac{1}{2}\frac{m}{2m-1}$ if $D \notin \mathcal{A}$ and $D > \frac{1}{2}\frac{m+1}{2(m+1)-1}$ if $D \in \mathcal{A}$. Hence, for all $m \in \mathbb{N}$, $\delta > \frac{1}{2}\frac{m}{2m-1}$ implies $D \geq \frac{1}{2}\frac{m}{2m-1}$.

ii) By Lemma 14, for all $\varepsilon > 0$,

$$D^*(g_\ell(\delta) - \varepsilon\,; \rho) \geq (f_{e,\ell} \circ F_{e,\ell}^{-1})(\rho(1 - h(g_\ell^{-1}(g_\ell(\delta) - \varepsilon))))$$

and by Corollary 7 all functions on the RHS are continuous.

4.1.3 Repetition Scheme Converse

An L-repetition scheme $f^{\oplus L} : \mathbb{F}_2^{Lk} \to \mathbb{F}_2^{Ln}$ is the L times concatenation of a based code $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$. Previous results [15] have demonstrated that repeating a small base code may yield good CJSCC. The asymptotic performance of L-repetition schemes is characterized in [19, Thm. 2] where it is shown that, for all $\rho > 0$, $k \in \mathbb{N}$ and $f : \mathbb{F}_2^k \to \mathbb{F}_2^{\lfloor \rho k \rfloor}$, the limit function

$$D(\delta\,; f^\infty) := \lim_{L \to \infty} \frac{1}{Lk} E(\lfloor \delta L \lfloor \rho k \rfloor \rfloor\,; f^{\oplus L})$$

exists and is concave in δ. The concavity in δ and the covering converse, $D_{\text{cov}}(\delta\,; \rho)$, yield the following lower bound on the asymptotic performance of any repetition scheme:

Lemma 15. *Let $0 \leq \delta \leq 1$, $\rho > 0$. For all $f : \mathbb{F}_2^k \to \mathbb{F}_2^{\lfloor \rho k \rfloor}$,*

$$D(\delta\,; f^{\oplus\infty}) \geq \begin{cases} \frac{D_{cov}(\delta_0\,;\rho)}{\delta_0}\delta & 0 \leq \delta \leq \delta_0 \\ D_{cov}(\delta\,;\rho) & \delta_0 < \delta \leq 1 \end{cases}, \tag{4.3}$$

where δ_0 is the unique solution in $1/2 < \delta < \theta_\rho$ to

$$\delta D'_{cov}(\delta\,;\rho) - D_{cov}(\delta\,;\rho) = 0$$

and θ_ρ is the largest γ such that $h(1-\gamma) \geq 1 - 1/\rho$.

Proof. Let $h_\rho(\delta) := D_{\text{cov}}(\delta\,;\rho)$. By the covering converse and [19, Thm. 2], $D(\delta\,; f^{\oplus\infty})$ is lower bounded by the least concave function upper bounding $h_\rho(\delta)$ and thus it suffices to show that this is the given function. Let $I_\rho = (1/2, \theta_\rho) \subset (0,1)$. A necessity of concavity on $[0,1]$ is $h'_\rho(\delta)\delta \leq h_\rho(\delta)$. The binary entropy function h is strictly increasing and strictly concave on $(0, 1/2)$ and thusly its inverse h^{-1} is strictly increasing and strictly convex on $(0,1)$. Therefore, $h_\rho : I_\rho \to (1/2, 1)$ is strictly increasing and strictly concave. Let $g(\delta) = \delta h'_\rho(\delta) - h_\rho(\delta)$, then

$$g'(\delta) = h'_\rho(\delta) + \delta h''_\rho(\delta) - h'_\rho(\delta) = \delta h''_\rho(\delta) < 0,$$

since h''_ρ is strictly concave and $\delta \in I_\rho$. Thus $g(\delta)$ is strictly decreasing. Using the inverse function theorem

$$h'_\rho(\delta) = \frac{\rho h'(1-\delta)}{h'(1-h_\rho(\delta))},$$

where $h'(\delta) = \log \frac{1-\delta}{\delta}$. Furthermore, h' is strictly decreasing on $(0, 1/2)$ with $\lim_{\delta\searrow 0} h(\delta) = \infty$ and $h'(1/2) = 0$. Therefore as $h'(1-\theta_\rho)$ is bounded and $h_\rho(\theta_p) = 1/2$

$$\lim_{\delta \nearrow \theta_\rho} h'_\rho(\delta) = 0,$$

and, by L'Hôpital's rule,

$$\begin{aligned}
\lim_{\delta \searrow 1/2} h'_\rho(\delta) &= \lim_{\delta \searrow 1/2} \frac{-\rho h''(1-\delta)}{-h''(1-h_\rho(\delta)) h'_\rho(\delta)} \\
&= \lim_{\delta \searrow 1/2} \frac{-\rho h''(1-\delta)}{-h''(1-h_\rho(\delta))} \lim_{\delta \searrow 1/2} \frac{1}{h'_\rho(\delta)}.
\end{aligned}$$

Thus $\lim_{\delta \searrow 1/2} h'_\rho(\delta) = \sqrt{\rho}$, and

$$\lim_{\delta \searrow 1/2} g(\delta) = 1/2(\sqrt{\rho} - 1) > 0, \qquad \lim_{\delta \nearrow \theta_\rho} g(\delta) = -1.$$

Hence by continuity of g there exists a unique $\delta_0 \in I_\rho$ such that $g(\delta_0) = 0$. Linear interpolation up to δ_0 yields a concave function. □

This bound is increasing in ρ, strictly greater than δ for all $\rho > 1$ and shows that repetition schemes are suboptimal for low distortion and large bandwidth expansion factor, e.g. when compared to separated schemes.

4.2 Asymptotic Linear Encoders

For linear encoders we can sharpen the double staircase result given by the multiple packing bound and established a restriction on the weights of rows for a sequence of linear encoders achieving zero distortion. We begin with a technical Lemma.

Lemma 16. *Let A be a linear $(k,n;E,\Delta)$ CJSCC. For all $0 \leq q \leq 1$,*

$$\Pr(w(A\underline{X}) \geq 2E+1) \leq \frac{1}{2\Delta+1} \begin{cases} \frac{1}{2}\left(1-(1-2q)^{\frac{\omega(A)}{k}}\right)n & q < \frac{1}{2} \\ qn & q \geq \frac{1}{2} \end{cases},$$

where $\underline{X}$ is a k-dimensional i.i.d. Bernoulli(q) vector, $\omega(A) := \sum_{i=1}^{n} w(a_i)$ and $\{a_i\}$ is an enumeration of the rows of A.

Proof. Using (3.1),

$$\begin{aligned} E[w(A\underline{X})] &= E[w(A\underline{X}),\, w(\underline{X}) \leq 2E] + E[w(A\underline{X}),\, w(\underline{X}) > 2E] \\ &\geq E[w(A\underline{X})\colon w(\underline{X}) > 2E] \Pr(w(\underline{X}) > 2E) \\ &\geq (2\Delta+1)\Pr(w(\underline{X}) > 2E). \end{aligned}$$

By Lemma 9

$$E\left[\mathrm{wt}(A\underline{X})\right] = \frac{n}{2} - \frac{1}{2}V_A(1-2p),$$

where V_A is a generating function for the weight of the rows of A. Let $\alpha = 1-2p$. Then

$$V_A(1-2p) = V_A(\alpha) = \sum_{i=1}^{n} \alpha^{w(a_i)}.$$

If $q < 1/2$, then α is positive, $V_A(\alpha)$ is convex in $w(a_i)$ and

$$V_A(\alpha) = n\frac{1}{n}\sum_{i=1}^{n} \alpha^{w(a_i)} \geq n\alpha^{\omega(A)/n}.$$

If $q \geq 1/2$, then α is nonpositive and, as $w(a_i) \in \{0, \ldots, n\}$, the least positive term is α. □

Theorem 10. *For all $\rho > 0$ and $1/4 < \delta < 1/2$, the asymptotically optimal distortion for linear encoders is bounded*

$$D^*_{lin}(\delta\,; \rho) \geq \delta.$$

Proof. Let $(\{\Delta_m\}, \{k_m\}, \{n_m\})$ be an admissible (δ, ρ) source-channel sequence, $E_m = E^*_{\text{lin}}(\Delta_m\,; k_m, n_m)$, A_m a linear $(k_m, n_m\,; E_m, \Delta_m)$ CJSCC, $D = \liminf_{m\to\infty} \frac{1}{k_m} E_m$ and $\underline{X}_m$ a k_m dimensional i.i.d. Bernoulli(q) vector. Choose $q > \max\{2D, 1/2\}$. By Lemma 16,

$$\Pr\left(w(\underline{X}_m) \geq 2E_m + 1\right) \leq \frac{qn_m}{2\Delta_m + 1}.$$

Taking limits

$$1 = \limsup_{m\to\infty} \Pr\left(w(\underline{X}_m) \geq 2E_m + 1\right) \leq \frac{q}{2\delta},$$

where the equality is Lemma 11 and the law of large numbers. Hence $q \geq 2\delta$, and thusly $D \geq \delta$. □

Proposition 4. *Let $(\{\Delta_m\}, \{k_m\}, \{n_m\})$ be an admissible (δ, ρ) source-channel sequence for $\rho > 0$ and $0 < \delta < 1/4$ and $\{A_m\}$ a sequence of $(k_m, n_m\,; E(\Delta_m\,; A_m), \Delta_m)$ CJSCCs. If $\limsup_{m\to\infty} \frac{1}{k_m}\omega(A_m) < \infty$, then*

$$\liminf_{m\to\infty} \frac{1}{k_m} E(\Delta_m\,; A_m) > 0.$$

Proof. Let $E_m = E(\Delta_m\,; A_m)$, $D = \liminf_{m\to\infty} \frac{1}{k_m} E_m$ and $M = \limsup_{m\to\infty} \frac{1}{k_m}\omega(A_m)$. If $D \geq 1/4$ the assertion follows. Suppose $D < 1/4$. Choose $q < 1/2$ such that

$q > 2D$. Let $\underline{X}_m$ be a k_m dimensional i.i.d. Bernoulli(q) vector. By Lemma 16

$$\Pr\left(w(\underline{X}_m) \geq 2E_m + 1\right) \leq \frac{1}{2}\frac{n_m(1-(1-2q)^{\omega(A_m)/n_m})}{2\Delta_m + 1}.$$

Therefore, by Corollary 5,

$$\begin{aligned} 1 &\leq \limsup_{m\to\infty} \frac{1}{2}\frac{n_m(1-(1-2q)^{\omega(A_m)/n_m})}{2\Delta_m + 1} \\ &= \frac{1 - \liminf_{m\to\infty}(1-2q)^{\omega(A_m)/n_m}}{4\delta} \\ &\leq^{(a)} \frac{1-(1-2q)^{M/\rho}}{4\delta}, \end{aligned}$$

where (a) follows because $(1-2q)^x$ is decreasing and continuous. Rearranging provides

$$q \geq \frac{1}{2}\left[1 - \exp\left(\rho \log(1-4\delta)/M\right)\right] > 0,$$

as $0 < \delta < 1/4$. Hence $D > 0$. □

In particular, a linear encoder with vanishing density of ones in every row cannot achieve zero distortion for a positive channel parameter.

4.3 $D - \delta$ Trade-Off as a Function of ρ

In the information theoretic setting there is both monotonicity and continuity in ρ. This section partially extends these properties to the combinatorial setup. A basis for this analysis is the performance of CJSCCs combined by composition.

Of particular interest is the canonical admissible (δ, ρ) source-channel sequence $(\lfloor\delta\lfloor\rho k\rfloor\rfloor, k, \lfloor\rho k\rfloor)$. To facilitate in the analysis of such sequences we define upper and lower limits

$$\overline{e}(\delta\,;\rho) := \limsup_{k\to\infty} \frac{1}{k}E^*(\lfloor\delta\lfloor\rho k\rfloor\rfloor\,; k, \lfloor\rho k\rfloor)$$

and

$$\underline{e}(\delta\,;\rho) := \liminf_{k\to\infty} \frac{1}{k}E^*(\lfloor\delta\lfloor\rho k\rfloor\rfloor\,; k, \lfloor\rho k\rfloor).$$

The notation $e(\delta\,;\rho)$ is used in statements that apply to both.

An immediate application of the composition Lemma shows that $e(\delta\,;\rho)$ is more or less impervious to finite deviations and provides a limited monotonicity result.

Proposition 5. *For all $\rho > 0$, $a, b \in \mathbb{N}$ and $0 \le \delta \le 1$*

$$\underline{e}(\delta-\,;\rho) \le \liminf_{k\to\infty} \frac{1}{k+a} E^*(\lfloor \delta(\lfloor \rho k \rfloor + b)\rfloor\,; k{+}a, \lfloor \rho k \rfloor{+}b) \le \underline{e}(\delta+\,;\rho),$$

and

$$\overline{e}(\delta-\,;\rho) \le \limsup_{k\to\infty} \frac{1}{k+a} E^*(\lfloor \delta(\lfloor \rho k \rfloor + b)\rfloor\,; k{+}a, \lfloor \rho k \rfloor{+}b) \le \overline{e}(\delta+\,;\rho).$$

Proof. If $\delta = 0$ or $\delta = 1$, then the statement is vacuous. Suppose $0 < \delta < 1$. Let $\Delta_k = \lfloor \delta(\lfloor \rho k \rfloor + b)\rfloor$ and $0 < \varepsilon < \min\{\delta, 1-\delta\}$. There exists K_1 such that, for all $k \ge K_1$, $\lfloor (\delta-\varepsilon)\lfloor \rho k \rfloor \rfloor \le \Delta_k - b$ and there exists K_2, such that, for all $k \ge K_2$, $\Delta_k \le \lfloor (\delta+\varepsilon)\lfloor \rho k \rfloor \rfloor$. There exists $K_3 \in \mathbb{N}$ such that, for all $k \ge K_3$, $b \le \Delta_k \le \lfloor \rho k \rfloor$. Let $K = \max\{K_1, K_2, K_3\}$. By Lemma 7, for all $k \ge K$,

$$\begin{aligned} E^*(\lfloor (\delta-\varepsilon)\lfloor \rho k \rfloor \rfloor\,; k, \lfloor \rho k \rfloor) &\le E^*(\Delta_k - b\,; k, \lfloor \rho k \rfloor) \\ &\le E^*(\Delta_k\,; k+a, \lfloor \rho k \rfloor + b) \\ &\le E^*(\Delta_k\,; k, \lfloor \rho k \rfloor) + a \\ &\le E^*(\lfloor (\delta+\varepsilon)\lfloor \rho k \rfloor \rfloor\,; k, \lfloor \rho k \rfloor) + a. \end{aligned}$$

Taking limits gives the result. □

Proposition 6. *Let $\rho, \tau > 0$ and $0 \le \delta \le 1$. If $\limsup_{k\to\infty} \frac{1}{\lfloor \tau k \rfloor} E^*(\lfloor \delta \lfloor \rho k \rfloor \rfloor\,; \lfloor \tau k \rfloor, \lfloor \rho k \lfloor) \le \delta$, then $\overline{e}(\delta\,;\rho) \le \overline{e}(\delta+\,;\tau)$.*

Proof. Let $\varepsilon > 0$. Let $\Delta_k = \lfloor (\delta{+}\varepsilon)\lfloor \tau k \rfloor \rfloor$, $E_k = E^*(\Delta_k\,; k, \lfloor \tau k \rfloor)$, $\overline{E} = \limsup_{k\to\infty} \frac{1}{k} E_k$, $\Gamma_k = \lfloor \delta \lfloor \rho k \rfloor \rfloor$, $F_k = E^*(\Gamma_k\,; \lfloor \tau k \rfloor, \lfloor \rho k \rfloor)$, $\overline{F} = \limsup_{k\to\infty} \frac{1}{\lfloor \tau k \rfloor} F_k$. There exists $K_1 \in \mathbb{N}$ such that for all $k \ge K_1$, $E_k < (\overline{E} + \varepsilon)k$. By assumption, $\overline{F} \le \delta$. Thus there exists K_2 such that, for all $k \ge K_2$, $F_k < \Delta_k$. Let $K_3 = \max\{K_1, K_2\}$. Using Lemma 6 with $m = \lfloor \tau k \rfloor$ and $n = \lfloor \rho k \rfloor$, for all $k \geq K_3$,

$$E^*(\Gamma_k\,; k, \lfloor \rho k \rfloor) \leq E^*(F_k\,; k, \lfloor \tau k \rfloor) \leq E^*(\Delta_k\,; k, \lfloor \tau k \rfloor) \leq (\overline{e}(\delta + \varepsilon\,; \tau) + \varepsilon)k.$$

□

Lemma 17. *For all $\rho, \tau \in \mathbb{Q}$*

$$\bigcup_{k \in \mathbb{N}} \{(\lfloor \tau k \rfloor, \lfloor \rho k \rfloor)\} \subset \bigcup_{b=0}^{qs-1} \bigcup_{n \in \mathbb{N}} \{(n + \lfloor \tau b \rfloor, \lfloor (\rho/\tau) n \rfloor + \lfloor \rho b \rfloor)\},$$

where $\rho = p/q$ and $\tau = r/s$ are the unique representation of ρ and τ as the quotient of relatively prime integers.

Proof. Any $k \in \mathbb{N}$ can be expressed as $k = aqs + b$ where $0 \leq b < qs$. Therefore,

$$\lfloor \rho k \rfloor = \lfloor p/q(aqs + b) \rfloor = \lfloor pas + pb/q \rfloor = pas + \lfloor pb/q \rfloor = \rho aqs + \lfloor \rho b \rfloor,$$

$$\lfloor \tau k \rfloor = \tau aqs + \lfloor \tau b \rfloor$$

and

$$\{(\lfloor \tau k \rfloor, \lfloor \rho k \rfloor)\} = \{(raq + \lfloor \tau b \rfloor, pas + \lfloor \rho b \rfloor)\}.$$

Let $n \in \{aqr : a \in \mathbb{N}\}$

$$\{(n, \lfloor (\rho/\tau) n \rfloor)\} = \{(aqr, \lfloor ((ps)/(qr))(aqr) \rfloor)\} = \{(aqr, psa)\}.$$

□

Lemma 18. *Let $\{c_n(\lambda) : n \in \mathbb{N},\ \lambda \in \Lambda\}$ where for all λ*

$$\limsup_{n \to \infty} c_n(\lambda) \leq a.$$

Let $a_k = c_{n_k}(\lambda_{n_k})$ where $\{n_k\}$ is any subsequence. If Λ is finite then

$$\limsup_{k \to \infty} a_k \leq a.$$

Proof. Let $\varepsilon > 0$. For each $\lambda \in \Lambda$, there exists N_λ such that, for all $n \geq N_\lambda$, $c_n(\lambda) < a + \varepsilon$. Let $N = \max_{\lambda\in\Lambda} N_\lambda$. As $n_k \geq k$ and $\{a_k\} \subset \bigcup_{\lambda\in\Lambda}\{c_n(\lambda)\}$, for all $k \geq N$, $a_k < a + \varepsilon$. □

Lemma 19. *If $\rho, \tau \in \mathbb{Q}$, then*

$$\limsup_{k\to\infty} \frac{1}{\lfloor \tau k \rfloor} E^*(\lfloor \delta \lfloor \rho k \rfloor \rfloor \,;\, \lfloor \tau k \rfloor, \lfloor \rho k \rfloor) \leq \overline{e}(\delta+ \,;\, \rho/\tau).$$

Proof. Combine the results of Lemmas 17, 18 and Proposition 5. □

4.4 Achievable Asymptotic Curves

The operational duality established in Section 3.5.2 yields an inverse function relation between the asymptotically achievable (D, δ) at bandwidth expansion factors ρ and $1/\rho$. By Theorem 4, the point (D, δ) is asymptotically achievable with bandwidth expansion factor ρ if and only if the point $(1 - \delta, 1 - D)$ is asymptotically achievable with bandwidth expansion factor $1/\rho$.

Lemma 20. *If the curve $(D(\delta), \delta)$ is asymptotically achievable with bandwidth expansion factor ρ and $D(\delta)$ is left continuous at $D^{-1}(1-\delta)$, then the point $(1-D^{-1}(1-\delta), \delta)$ is asymptotically achievable with bandwidth expansion factor $1/\rho$, where $D^{-1}(\delta)$ is a generalized inverse*

$$D^{-1}(\delta) := \sup\{\gamma : D(\gamma) \leq \delta\}.$$

Proof. By Theorem 4, for $0 \leq t \leq 1$, the curve $(1 - t, 1 - D(t))$ is asymptotically achievable at $1/\rho$. Parametering $t(\delta) = D^{-1}(1 - \delta)$, the curve

$$(1 - D^{-1}(1 - \delta), 1 - D(D^{-1}(1 - \delta)))$$

is asymptotically achievable at $1/\rho$. Choose an increasing sequence $\{\gamma_n\} \in \{\gamma : D(\gamma) \leq 1 - \delta\}$ such that $\gamma_n \nearrow D^{-1}(1 - \delta)$. By left continuity of $D(\delta)$ at $D^{-1}(1 - \delta)$

$$D(D^{-1}(1 - \delta)) = \lim_{n\to\infty} D(\gamma_n) \leq 1 - \delta,$$

where the inequality follows by construction of $\{\gamma_n\}$. Hence $1 - D(D^{-1}(1-\delta)) \geq \delta$ and thusly $(1 - D^{-1}(1-\delta), \delta)$ is asymptotically achievable. $\square$

Lemma 21. *If the point (D, δ) is asymptotically achievable, then, for all $0 < \varepsilon \leq \delta$, the point $(D, \delta - \varepsilon)$ is asymptotically achievable using the canonical $(\delta - \varepsilon, \rho)$ source-channel sequence $(\lfloor(\delta-\varepsilon)\lfloor \rho k\rfloor\rfloor, k, \lfloor \rho k \rfloor)$.*

Proof. Let $0 < \varepsilon \leq \delta$. Let (f_m, g_m) be a sequence of $(k_m, n_m\,; E_m, \Delta_m)$ CJSCCs such that

$$\frac{n_m}{k_m} \to \rho \qquad \frac{\Delta_m}{n_m} \to \delta \qquad \frac{E_m}{k_m} \to D.$$

Thus $n_m = \lfloor \rho k_m \rfloor + a_m$ and $\Delta_m = \lfloor \delta \lfloor \rho k_m \rfloor \rfloor + b_m$ where a_m and b_m are $o(k_m)$. There exists $M \in \mathbb{N}$ such that, for all $m \geq M$,

$$\lfloor(\delta-\varepsilon)\lfloor \rho k_m\rfloor\rfloor \leq \lfloor \delta \lfloor \rho k_m \rfloor\rfloor + b_m - \max\{0, a_m\} = \Delta_m - \max\{0, a_m\}.$$

Using a (psuedo)-identity code and the composition Lemma, for all $m \geq M$,

$$\begin{aligned} E^*(\lfloor(\delta-\varepsilon)\lfloor \rho k_m\rfloor\rfloor\,; k_m, \lfloor \rho k_m\rfloor) &\leq E^*(\Delta_m - \max\{0, a_m\}\,; k_m, \lfloor \rho k_m \rfloor) \\ &\leq E(\Delta_m - \max\{0, a_m\}\,; I_{\lfloor \rho k_m\rfloor + a_m, \lfloor \rho k_m \rfloor} \circ f_m) \\ &\leq E(E(\Delta_m - \max\{0, a_m\}\,; I_{\lfloor \rho k_m\rfloor + a_m, \lfloor \rho k_m \rfloor})\,; f_m) \\ &= E(\Delta_m\,; f_m) = E_m. \end{aligned}$$

For any sequence $\{k_m \in \mathbb{N}\}$, $\{(\lfloor \delta \lfloor \rho k_m \rfloor\rfloor, k_m, \lfloor \rho k_m \rfloor)\} \subset \{(\lfloor \delta \lfloor \rho k\rfloor\rfloor, k, \lfloor \rho k \rfloor)\}$. Thus

$$\underline{e}(\delta - \varepsilon\,; \rho) \leq \liminf_{m\to\infty} \frac{1}{k_m} E^*(\lfloor(\delta-\varepsilon)\lfloor \rho k_m\rfloor\rfloor\,; k_m, \lfloor \rho k_m\rfloor) \leq \liminf_{m\to\infty} \frac{1}{k_m} E_m = D.$$

$\square$

Corollary 8. *For all $0 < \varepsilon \leq \delta$,*

$$\underline{e}(\delta - \varepsilon\,; \rho) \leq D^*(\delta\,; \rho) \leq \underline{e}(\delta\,; \rho).$$

A continuity relation between ρ and δ follows.

Proposition 7. *Let* $0 \le \delta \le 1$ *and* $\rho > 0$. *For all* $0 \le \varepsilon \le \delta(1-\delta)$,

$$D^*\left(\delta\,;\left(1-\frac{\varepsilon}{\delta}\right)\rho\right) \le D^*\left(\delta+\frac{\varepsilon}{\delta}\,;\rho\right).$$

Proof. By the composition Lemma, Lemma 6,

$$\begin{aligned} &E^*\left(\lfloor(\delta-\varepsilon)\lfloor\rho k\rfloor\rfloor\,;k,\left\lfloor\left(1-\frac{\varepsilon}{\delta}\right)\lfloor\rho k\rfloor\right\rfloor\right) \\ &\qquad\le E^*\left(\lfloor(\delta-\varepsilon)\lfloor\rho k\rfloor\rfloor+\lfloor\rho k\rfloor-\left\lfloor\left(1-\frac{\varepsilon}{\delta}\right)\lfloor\rho k\rfloor\right\rfloor\,;k,\lfloor\rho k\rfloor\right). \end{aligned}$$

For all $\varepsilon' < \varepsilon$, there exists K such that for all $k \ge K$

$$\lfloor(\delta-\varepsilon)\lfloor\rho k\rfloor\rfloor+\lfloor\rho k\rfloor-\left\lfloor\left(1-\frac{\varepsilon}{\delta}\right)\lfloor\rho k\rfloor\right\rfloor \le \left\lfloor\left(\delta-\varepsilon'+\frac{\varepsilon}{\delta}\right)\lfloor\rho k\rfloor\right\rfloor.$$

Thus

$$D^*\left(\delta\,;\left(1-\frac{\varepsilon}{\delta}\right)\rho\right) \le \underline{e}\left(\delta-\varepsilon'+\frac{\varepsilon}{\delta}\,;\rho\right)$$

and the result follows by Corollary 8. □

4.5 Numerical Evaluations

The section collects numerical evaluations comparing simple achievable schemes with the best known converse bounds.

4.5.1 Asymptotic Performance of Basic CJSCCs

- The distortion of the asymptotic (pseudo)-identity map I_ρ is

$$D(\delta\,;I_\rho) = \min\{1, \rho\delta + \max\{0, 1-\rho\}\}. \tag{4.4}$$

- The distortion of the asymptotic ρ-repetition code R_ρ is

$$D(\delta\,;R_\rho) = \min\left\{1, \frac{\rho}{\lceil\rho/2\rceil}\delta\right\}. \tag{4.5}$$

- The distortion of the asymptotic separated ρ-code S_ρ is [15, Sec. III-C] $D(\delta\,; S_\rho) =$

$$\begin{cases} h^{-1}(|1-\rho(1-h(2\delta))|^{+}) & 0 \le \delta < \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \le \delta < \frac{1}{2} \end{cases}. \tag{4.6}$$

The correspondence between achievable (D, δ) with bandwidth expansion factors ρ and $1/\rho$ of Lemma 20 yields dual versions of the preceding CJSCCs. The dual separated ρ-code is of particular interest.

- The distortion of the asymptotic dual (pseudo)-identity code $I_\rho^\perp$ is

$$D(\delta\,; I_\rho^\perp) = \min\{1, \rho\delta\}.$$

- For $\rho \in 1/\mathbb{N}$, the distortion of the asymptotic dual ρ-repetition code $R_\rho^\perp$ is

$$D(\delta\,; R_\rho^\perp) = \max\left\{0, \rho\lceil 1/(2\rho)\rceil\delta + (1-\rho\lceil 1/(2\rho)\rceil)\right\}.$$

- The distortion of the asymptotic dual separated ρ-code $S_\rho^\perp$ is, for $1/2 \le \delta \le 1$,

$$D(\delta\,; S_\rho^\perp) = 1 - \tfrac{1}{2}h^{-1}(|1-\rho(1-h(1-\delta))|^{+}). \tag{4.7}$$

Together the dual separated code and the covering converse, 4.7 and 4.2, establish the maximal δ asymptotically achieving nontrivial distortion $D < 1$ for bandwidth expansion factor ρ

$$\delta^*(1-; \rho) = 1 - h^{-1}\left(\left|1 - \frac{1}{\rho}\right|^{+}\right). \tag{4.8}$$

4.5.2 Comparison for $\rho = 3$

Figure 4-1 gives the best known converse and achievability bounds for bandwidth expansion factor $\rho = 3$. The dotted black line represents the uncoded or $\rho = 1$ case where the identity scheme is optimal. Deviation from this line is of interest.

The achievability bound is given as follows:

- $0 \leq \delta \lessapprox 0.185$ – the separated 3-code (4.6);
- $0.185 \lessapprox \delta < 1/3$ – the 3-repetition code (4.5);
- $1/3 \leq \delta < 1/2$ – the separated 3-code (4.6);
- $1/2 \leq \delta \leq 1$ – the dual separated 3-code (4.7).

The converse bound is given as follows:

- $0 \leq \delta \leq 1/4$ – the asymptotic L multiple packing converse Theorem 2 using $L = 2$ and the upper bound from [21];
- $1/4 < \delta < 1/2$ – the interlacing of the bounds in Theorem 5;
- $1/2 \leq \delta \leq 1$ – the asymptotic covering converse Theorem 1.

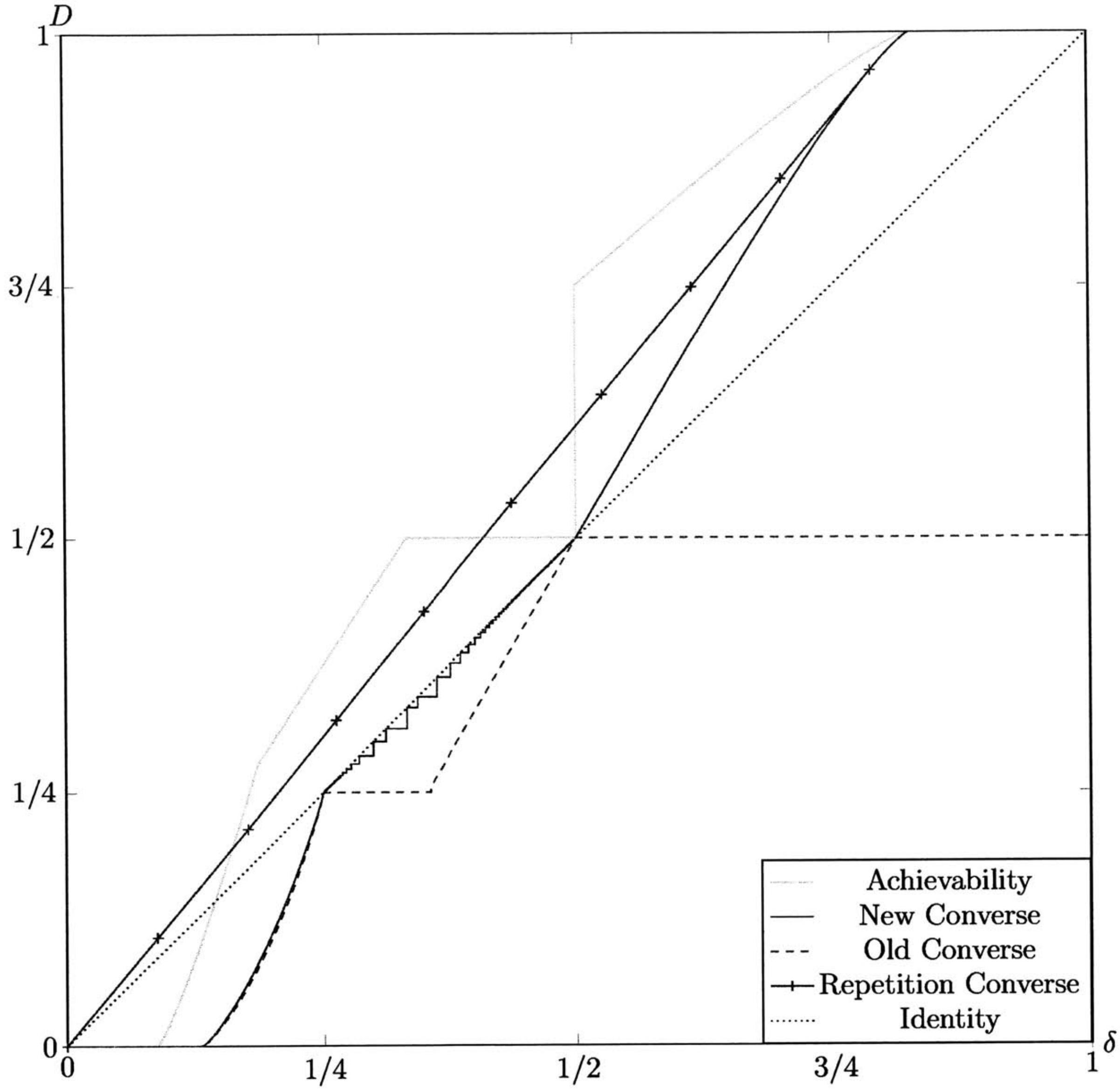


Figure 4-1: Best known achievability and converse bounds for $\rho = 3$.

Chapter 5

Erasure Errors

5.1 CJSCC with Erasure Errors

Definition 4. *Elements $x, y \in \{0,1,?\}^n$ are said to be equivalent modulo an index set $I \subset [n] = \{1,2,\ldots,n\}$, denoted $x_I = y_I$, if $x_i = y_i$ for all $i \in I$.*

Definition 5. *The set of achievable erasure points given an element $y \in \mathbb{F}_2^n$ and an index set $I \subset [n]$ is*

$$\mathcal{E}_I(y) := \{z \in \{0,1,?\}^n \mid z_I = y_I,\ z_{[n]\setminus I} \equiv ?\}.$$

Definition 6. *Let $k, n \in \mathbb{N}$, $E \in \{0,\ldots,k\}$ and $\mathcal{E} \in \{0,\ldots,n\}$. A pair of maps $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$ and $g : \{0,1,?\}^n \to \mathbb{F}_2^k$ is a $(k,n\,;E,\mathcal{E})$ CJSCC if, for all $x \in \mathbb{F}_2^k$, $I \subset [n]$ with $|I| \geq n - \mathcal{E}$ and $y \in \{0,1,?\}$ such that $y \in \mathcal{E}_I(f(x))$, $d(x, g(y)) \leq E$. Equivalently*

$$E(\mathcal{E}\,; k, n, f, g) := \max_{\{x\in\mathbb{F}_2^k\}} \max_{\{I:|I|\geq n-\mathcal{E}\}} \max_{\{y\in\mathcal{E}_I(f(x))\}} d(x, g(y)) \leq E.$$

In the sequel the k and n may be dropped when understood from the context.

Definition 7. *The I plane around an element $y \in \mathbb{F}_2^n$ is*

$$C_I(y\,; n) := \{z \in \mathbb{F}_2^n \mid z_I = y_I\}.$$

Theorem 11. *For all $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$ and $\mathcal{E} \in \{0, \dots, n\}$*

$$D(\mathcal{E}\,; f) := \inf_g D(\mathcal{E}\,; f, g) = \max_{x \in \mathbb{F}_2^k} \max_{I:|I|=n-\mathcal{E}} \operatorname{rad}(f^{-1}C_I(f(x))),$$

where the minimization is over $g : \{0, 1, ?\}^n \to \mathbb{F}_2^k$.

Proof. For all $I \subset [n]$, the following two sets are equal

$$\{(x, y) \mid x \in \mathbb{F}_2^k,\, y \in \mathcal{E}_I(f(x))\} = \{(x, y) \mid y_I \in \{0, 1\}^{|I|},\, y_{[n]\setminus I} \equiv ?,\, f(x)_I = y_I\}.$$

Therefore, an equivalent maximization is the following

$$\begin{aligned}
\max_{\{x \in \{0,1\}^k\}} \max_{\{y \in \mathcal{E}_I(f(x))\}} d(x, g(y)) &= \max_{\{y: y_I \in \{0,1\}^{|I|}, y_{[n]\setminus I} \equiv ?\}} \max_{\{x: f(x)_I = y_I\}} d(x, g(y)) \\
&\geq \max_{\{y: y_I \in \{0,1\}^{|I|}, y_{[n]\setminus I} \equiv ?\}} \min_{\{z \in \mathbb{F}_2^k\}} \max_{\{x: f(x)_I = y_I\}} d(x, z) \\
&=^{(a)} \max_{\{y: y_I \in \{0,1\}^{|I|}, y_{[n]\setminus I} \equiv ?\}} \operatorname{rad}(\{x : f(x)_I = y_I\}) \\
&= \max_{\{y: y_I \in \{0,1\}^{|I|}, y_{[n]\setminus I} \equiv ?\}} \operatorname{rad}(f^{-1}C_I(y))
\end{aligned}$$

where (a) follows by definition of the Chebyshev radius and $g(y) \in \operatorname{cen}(f^{-1}(B_I(y)))$ achieves this bound independent of I because I is known. Moreover, for all y such that $y_I \in \{0, 1\}^{|I|}$ and $y_{[n]\setminus I} \equiv ?$

$$f^{-1}C_I(y) = \begin{cases} f^{-1}C_I(f(x_0)) & \exists\, x_0 \text{ s.t. } y_I = f(x_0)_I \\ \emptyset & \text{else} \end{cases}.$$

Thus it suffices to maximize over $f(\mathbb{F}_2^k)$. Combining provides

$$D(\mathcal{E}\,; f) = \max_{I:|I| \geq n-\mathcal{E}} \max_{x \in \mathbb{F}_2^k} \operatorname{rad}(f^{-1}C_I(f(x))).$$

If $J \supset I$, then for all $y \in \mathbb{F}_2^n$ $C_J(y) \subset C_I(y)$. Thus maximizing over $\{I : |I| \geq n - \lfloor \varepsilon n \rfloor\}$ is equivalent to maximizing over $\{I : |I| = n - \lfloor \varepsilon n \rfloor\}$. ☐

In the sequel, we focus attention on the case $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$ and g the Chebyshev decoder. Moreover, the characterization of the previous theorem is taken to be the definition of a $(k,n\,;E,\mathcal{E})$ CJSCC and the optimal CJSCC is

$$E^*(\mathcal{E}\,;k,n) := \min_f E(\mathcal{E}\,;k,n,f).$$

5.2 Converse Bounds

A few of the flip error converse bounds carry over with appropriate factors of 1/2.

Lemma 22. *(2-point converse) Let f be a $(k,n\,;E,\mathcal{E})$ CJSCC. Then*

$$d(x,y) \geq 2E+1 \qquad \Longrightarrow \qquad d(f(x),f(y)) \geq \mathcal{E}+1.$$

Proof. If $d(f(x),f(y)) \leq \mathcal{E}$, then there exists I_0 such that $|I_0| \geq n-\mathcal{E}$ and $f(y) \in C_{I_0}(f(x))$. Moreover, for all $S \subset \mathbb{F}_2^k$, $\operatorname{diam}(S) \leq 2\operatorname{rad}(S)$. □

Lemma 23. *Let $E_k = E^*(\mathcal{E}\,;k,n)$. Then*

$$A(k,2E_k+1) \leq A(n,\mathcal{E}+1).$$

Proof. Let f be a $(k,n\,;E_k,\mathcal{E})$ CJSCC and $C \subset \mathbb{F}_2^k$ achieve $A(k,2E_k+1)$. Then $d(x,z) \geq 2E_k+1$ implies $d(f(x),f(z)) \geq \mathcal{E}+1$, and thusly f is injective when restricted to C. Thus $A(k,2E_k+1) = |C| = |f(C)| \leq A(n,\mathcal{E}+1)$. □

5.3 L-repetition

As a corollary to the repetition converse for flip errors and as the repetition code achieves $E=\mathcal{E}$ for erasure errors, the performance of any L-repetition scheme is no better than the repetition code. In the interest of completeness a characterization of L-repetition schemes is given. In particular, the L-repetition distortion diagonalizes.

Theorem 12. *Let f be a $(k,n\,;E,\mathcal{E})$ CJSCC. The distortion of its L-repetition is*

$$E(\mathcal{E}\,;f^{\oplus L}) = \max_{\mathcal{E}_1+\ldots+\mathcal{E}_L\leq\mathcal{E}} \sum_{i=1}^{L} E(\mathcal{E}_i\,;f).$$

Proof. Let $f_L := f^{\oplus L}$. For all L,

$$E(\mathcal{E}\,;f_L) = \max_{x\in\mathbb{F}_2^{Lk}} \max_{I:|I|\geq Ln-\mathcal{E}} \mathrm{rad}(f_L^{-1}C_I(f_L(x))).$$

Consider the 2-repetition encoding f_2

$$\begin{aligned}
E(\mathcal{E}\,;f_2) &= \max_{[x_1\ x_2]\in\mathbb{F}_2^{2k}} \max_{I:|I|\geq 2n-\mathcal{E}} \mathrm{rad}\left(f_2^{-1}C_I([f(x_1)\ f(x_2)])\right)\\
&= \max_{[x_1\ x_2]\in\mathbb{F}_2^{2k}} \max_{I:|I|\geq 2n-\mathcal{E}} \mathrm{rad}\left(f^{-1}C_{I_1}(f(x_1))\oplus f^{-1}C_{I_2}(f(x_2))\right) \quad (I=I_1\cup I_2)\\
&= \max_{[x_1\ x_2]\in\mathbb{F}_2^{2k}} \max_{I:|I|\geq 2n-\mathcal{E}} \mathrm{rad}\left(f^{-1}C_{I_1}(f(x_1))\right)+\mathrm{rad}\left(f^{-1}C_{I_2}(f(x_2))\right) \quad \text{(Lemma 4)}.
\end{aligned}$$

By induction

$$E(\mathcal{E}\,;f_L) = \max_{x\in\mathbb{F}_2^{Lk}} \max_{I\subset[Ln]:|I|\geq Ln-\mathcal{E}} \sum_{j=1}^{L} \mathrm{rad}\left(f^{-1}C_{I_j}(f(x_j))\right),$$

where $x=[x_1\ \ldots\ x_L]$ and $I=I_1\cup\ldots\cup I_L$. The result follows as

$$\{I\subset[Ln]:|I|\geq Ln-\mathcal{E}\} = \left\{\bigcup_{j=1}^{L} I_j : I_j\subset[n],\ |I_j|\geq n-\mathcal{E}_j, \sum_{j=1}^{L}\mathcal{E}_j\leq\mathcal{E}\right\}.$$

□

5.3.1 Asymptotics

Lemma 24. *For all $f:\mathbb{F}_2^k\to\mathbb{F}_2^n$, as a function of ε,*

$$\limsup_{L\to\infty}\frac{1}{Lk}E(\lfloor\varepsilon Ln\rfloor\,;f^{\oplus L}) \leq \inf\{\phi \mid k\phi(\cdot/n)\geq E(\cdot\,;f),\ \phi-\textit{concave nondecreasing}\}.$$

Proof. Let $k\phi(\cdot/n) \geq E(\cdot\,; f)$ be a concave nondecreasing function. For all L

$$\begin{aligned}
E(\lfloor \varepsilon Ln \rfloor\,; f^{\oplus L}) &= \max_{\mathcal{E}_1+\ldots+\mathcal{E}_L \leq \lfloor \varepsilon Ln \rfloor} \sum_{i=1}^{L} E(\mathcal{E}_i\,; f) \\
&\leq \max_{\mathcal{E}_1+\ldots+\mathcal{E}_L \leq \lfloor \varepsilon Ln \rfloor} k \sum_{i=1}^{L} \phi(\mathcal{E}_i/n) \quad (k\phi(\cdot/n) \geq E(\cdot\,; f)) \\
&= Lk \max_{\mathcal{E}_1+\ldots+\mathcal{E}_L \leq \lfloor \varepsilon Ln \rfloor} \frac{1}{L} \sum_{i=1}^{L} \phi(\mathcal{E}_i/n) \\
&\leq Lk \max_{\mathcal{E}_1+\ldots+\mathcal{E}_L \leq \lfloor \varepsilon Ln \rfloor} \phi\left(\frac{1}{Ln} \sum_{i=1}^{L} \mathcal{E}_i\right) \quad (\phi \text{ concave}) \\
&\leq Lk\phi\left(\frac{\lfloor \varepsilon Ln \rfloor}{Ln}\right) \quad (\phi \text{ nondecreasing}) \\
&\leq Lk\phi\,(\varepsilon)\,.
\end{aligned}$$

and this holds in the limit $\limsup_{L\to\infty} \frac{1}{Lk} E(\lfloor \varepsilon Ln \rfloor; f^{\oplus L}) \leq \phi$. Hence, as ϕ was arbitrary, the inequality holds over the infimum of such functions. □

To better coincide with the limiting case we consider parameterizations.

Lemma 25. *For all* $0 \leq \varepsilon \leq 1$,

$$E(\lfloor \varepsilon Lk \rfloor\,; f^{\oplus L}) = \max_{\varepsilon_1+\ldots+\varepsilon_L \leq \varepsilon L} \sum_{i=1}^{L} E(\lfloor \varepsilon_i k \rfloor\,; f),$$

where $0 \leq \varepsilon_i \leq 1$.

Proof. It suffices to show that

$$\left\{(e_1, \ldots, e_L) \mid e_i \in \mathbb{N}, \sum_{i=1}^{L} e_i \leq \lfloor \varepsilon Ln \rfloor\right\} = \left\{(\lfloor z_1 n \rfloor, \ldots, \lfloor z_L n \rfloor) \mid z_i \in [0,1], \frac{1}{L}\sum_{i=1}^{L} z_i \leq \varepsilon\right\}.$$

For the forward containment, consider a vector $(e_1, \ldots, e_n)$. Let $z_i = \frac{e_i}{n}$, then $\lfloor z_i n \rfloor = e_i$ and

$$\frac{1}{L}\sum_{i=1}^{L} z_i = \frac{1}{L}\sum_{i=1}^{L} \frac{e_i}{n} = \frac{1}{Ln}\sum_{i=1}^{L} e_i \leq \frac{\lfloor \varepsilon Ln \rfloor}{Ln} \leq \varepsilon.$$

For the reverse containment, by Lemma 3 and monotonicity of floor,

$$\sum_{i=1}^{L} \lfloor z_i n \rfloor \le \left\lfloor \sum_{i=1}^{L} z_i n \right\rfloor \le \lfloor \varepsilon L n \rfloor .$$

□

Lemma 26. *For all $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$, $\liminf_{L\to\infty} \frac{1}{Lk} E(\lfloor \varepsilon L n \rfloor ; f^{\oplus L})$ upper bounds the base distortion $\frac{1}{k} E(\lfloor \varepsilon n \rfloor ; f)$ and is concave nondecreasing.*

Proof. Let $D(\varepsilon) := \frac{1}{k} E(\lfloor \varepsilon n \rfloor ; f)$, $D_L(\varepsilon) := \frac{1}{Lk} E(\lfloor \varepsilon L n \rfloor ; f^{\oplus L})$ and $D_\infty(\varepsilon) := \liminf_{L\to\infty} D_L(\varepsilon)$. By Lemma 25, for all L,

$$D_L(\varepsilon) = \max_{\varepsilon_1 + \ldots + \varepsilon_L \le L\varepsilon} \frac{1}{L} \sum_{i=1}^{L} D(\varepsilon_i) \ge \frac{1}{L} \sum_{i=1}^{L} D(\varepsilon) = D(\varepsilon)$$

and thusly, this holds in the limit. Let $X_L(\varepsilon) = \left\{ z \in [0,1]^L : \frac{1}{L} \sum_{i=1}^{L} z_i \le \varepsilon \right\}$ and, $g_L : [0,1]^L \to [0,1]$, $g_L(z) := \frac{1}{Lk} \sum_{i=1}^{L} E(\lfloor z_i n \rfloor ; f)$. Then $D_L(\varepsilon) = \max_{z \in X_L(\varepsilon)} g_L(z)$. Let $0 \le \lambda \le 1$ and $\varepsilon, \delta \in [0,1]$

$$\begin{aligned}
&X_L(\lambda\varepsilon + (1-\lambda)\delta) \\
&\supset \left\{ z : \frac{1}{\lfloor \lambda L \rfloor} \sum_{i=1}^{\lfloor \lambda L \rfloor} z_i \le \varepsilon,\ \frac{1}{\lfloor (1-\lambda) L \rfloor} \sum_{i=\lfloor \lambda L \rfloor + 1}^{\lfloor \lambda L \rfloor + \lfloor (1-\lambda) L \rfloor} z_i \le \delta,\ z_{\lfloor \lambda L \rfloor + \lfloor (1-\lambda) L \rfloor + 1}^{L} = 0 \right\} \\
&\supset \left\{ z : \frac{1}{\lfloor \lambda L \rfloor} \sum_{i=1}^{\lfloor \lambda L \rfloor} z_i \le \varepsilon,\ \frac{1}{\lfloor \bar{\lambda} L \rfloor} \sum_{i=\lfloor \lambda L \rfloor}^{\lfloor \lambda L \rfloor + \lfloor \bar{\lambda} L \rfloor} z_i \le \delta,\ z_{\lfloor \lambda L \rfloor + \lfloor \bar{\lambda} L \rfloor + 1}^{L} = 0 \right\} \\
&= \left\{ z_1^{\lfloor \lambda L \rfloor} \in X_{\lfloor \lambda L \rfloor}(\varepsilon) \right\} \cap \left\{ z_{\lfloor \lambda L \rfloor + 1}^{\lfloor \lambda L \rfloor + \lfloor (1-\lambda) L \rfloor} \in X_{\lfloor (1-\lambda) L \rfloor}(\delta) \right\} \cap \left\{ z_{\lfloor \lambda L \rfloor + \lfloor (1-\lambda) L \rfloor + 1}^{L} = 0 \right\},
\end{aligned}$$

where the first containment follows from $\lfloor x \rfloor \le x$. Similarly, $g_L(z)$ decomposes as

$$\begin{aligned} g_L(z) &= \frac{1}{L}\sum_{i=1}^{L} D(z_i) \\ &= \lambda \frac{\lfloor \lambda L\rfloor}{\lambda L}\frac{1}{\lfloor \lambda L\rfloor}\sum_{i=1}^{\lfloor \lambda L\rfloor} D(z_i) + (1-\lambda)\frac{\lfloor (1-\lambda)L\rfloor}{(1-\lambda)L}\frac{1}{\lfloor (1-\lambda)L\rfloor}\sum_{i=\lfloor \lambda L\rfloor+1}^{\lfloor \lambda L\rfloor+\lfloor (1-\lambda)L\rfloor} D(z_i) \\ &\quad + \frac{1}{L}\sum_{i=\lfloor \lambda L\rfloor+\lfloor (1-\lambda)L\rfloor+1}^{L} D(z_i) \\ &= \lambda \frac{\lfloor \lambda L\rfloor}{\lambda L} g_{\lfloor \lambda L\rfloor}(z_1^{\lfloor \lambda L\rfloor}) + (1-\lambda)\frac{\lfloor (1-\lambda)L\rfloor}{(1-\lambda)L} g_{\lfloor (1-\lambda)L\rfloor}(z_{\lfloor \lambda L\rfloor+1}^{\lfloor \lambda L\rfloor+\lfloor (1-\lambda)L\rfloor}) \\ &\quad + \frac{1}{L}\sum_{i=\lfloor \lambda L\rfloor+\lfloor (1-\lambda)L\rfloor+1}^{L} D(z_i). \end{aligned}$$

Combining these two decomposition and the fact that $D(0)=0$

$$D_L(\lambda\varepsilon + (1-\lambda)\delta) \ge \lambda\frac{\lfloor \lambda L\rfloor}{\lambda L} D_{\lfloor \lambda L\rfloor}(\varepsilon) + (1-\lambda)\frac{\lfloor (1-\lambda)L\rfloor}{(1-\lambda)L} D_{\lfloor (1-\lambda)L\rfloor}(\delta).$$

Using the properties of the limit inferior from Lemma 2

$$\begin{aligned} D_\infty(\lambda\varepsilon + (1-\lambda)\delta) &\ge \liminf_{L\to\infty}\left[\lambda\frac{\lfloor \lambda L\rfloor}{\lambda L} D_{\lfloor \lambda L\rfloor}(\varepsilon) + (1-\lambda)\frac{\lfloor (1-\lambda)L\rfloor}{(1-\lambda)L} D_{\lfloor (1-\lambda)L\rfloor}(\delta)\right] \\ &\ge \lambda\left(\liminf_{L\to\infty}\frac{\lfloor \lambda L\rfloor}{\lambda L}\right)\left(\liminf_{L\to\infty} D_{\lfloor \lambda L\rfloor}(\varepsilon)\right) \\ &\quad + (1-\lambda)\left(\liminf_{L\to\infty}\frac{\lfloor (1-\lambda)L\rfloor}{(1-\lambda)L}\right)\left(\liminf_{L\to\infty} D_{\lfloor (1-\lambda)L\rfloor}(\delta)\right) \\ &= \lambda D_\infty(\varepsilon) + (1-\lambda)D_\infty(\varepsilon). \end{aligned}$$

Hence D_∞ is concave. □

Proposition 8. *For all $f : \mathbb{F}^k \to \mathbb{F}_2^n$, the limit $D(\varepsilon\,; f^{\oplus\infty}) := \lim_{L\to\infty}\frac{1}{Lk}D(\lfloor \varepsilon Ln\rfloor\,; f^{\oplus L})$ exists and is equal to the upper convex envelop of $\frac{1}{k}E(\lfloor \varepsilon n\rfloor\,; f)$,*

$$\lim_{L\to\infty} D(\cdot\,; f^{\oplus L}) = \inf\{\phi \mid k\phi(\cdot/n) \ge E(\cdot\,; f),\ \phi - \textit{concave nondecreasing}\}.$$

Proof. Let $D_L(\varepsilon) := \frac{1}{Lk}E(\lfloor \varepsilon Ln \rfloor\,; f^{\oplus L})$. Combining the results of Lemmas 24 and 26

$$\limsup_{L\to\infty} D_L \le \inf\{\phi \mid k\phi(\cdot/n) \ge E(\cdot\,; f),\ \phi - \text{concave nondecreasing}\} \le \liminf_{L\to\infty} D_L.$$

Since $\liminf \le \limsup$ these inequalities are equalities and the assertion follows. □

5.4 Linear Encoders

Lemma 27. *Let $k, n \in \mathbb{N}$ and $\mathcal{E} \in \{0, \ldots, n\}$. If $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$ is linear, then*

$$E(\mathcal{E}\,; f) = \max_{I:|I|=n-\mathcal{E}} \operatorname{rad}(f^{-1}C_I(0)).$$

Proof. For all $x \in \mathbb{F}_2^k$

$$\begin{aligned} f^{-1}C_I(f(x)) &= \{z \mid f(z)_I = f(x)_I\} \\ &= \{z \mid f(z-x)_I = 0_I\} \\ &= \{z \mid z - x \in f^{-1}C_I(0)\} \\ &= f^{-1}C_I(0) + x, \end{aligned}$$

and $\operatorname{rad}(f^{-1}C_I(0) + x) = \operatorname{rad}(f^{-1}C_I(0))$. □

Proposition 9. *Let $k, n \in \mathbb{N}$ and $\mathcal{E} \in \{0, \ldots, n\}$. The optimal distortion minimizing over linear $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$ CJSCCs is*

$$E^*_{lin}(\mathcal{E}; k, n) = \min_{A\in\mathbb{F}_2^{n\times k}} \max_{I:\,|I|=n-\mathcal{E}} \operatorname{rad}(\ker(A_I)).$$

Proof. Let $A \in \mathbb{F}_2^{n\times k}$ be a linear $(k, n\,; E(\mathcal{E}\,; f), \mathcal{E})$ CJSCC. Then

$$A^{-1}C_I(0) = \{x \in \mathbb{F}_2^k \mid [Ax]_I = 0_I\} = \ker(A_I),$$

where A_I is the $|I| \times k$ matrix consisting of the rows of A with indices in I. The result follows by Lemma 27. □

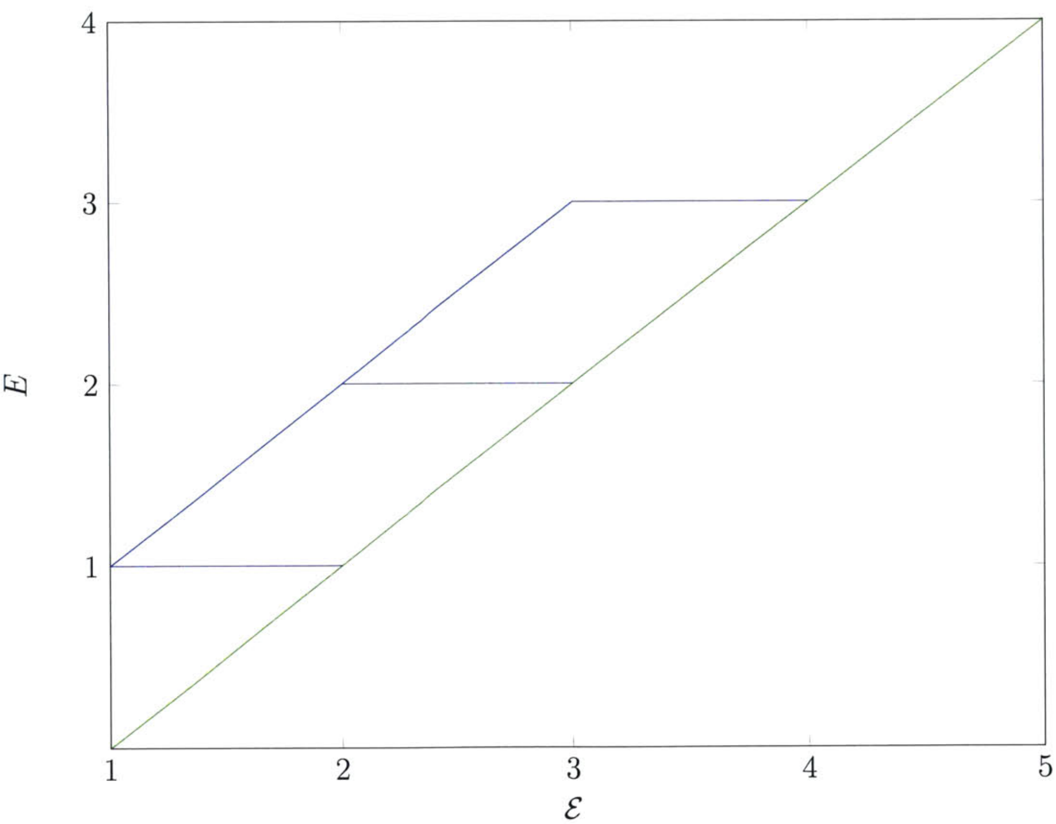
The following sections find the optimal linear encoder for a given (k, n) through exhaustive simulation.

5.4.1 $k = 4$ and $n = 5$

There are 4 achievable distortion patterns

$$\left\{\begin{array}{ccccc} 1 & 2 & 3 & 3 & 4 \\ 1 & 2 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{array}\right\}.$$

Observing the following plot, of the superposition of all achievable distortions, the optimal distortion of Theorem 9 is achievable.



The single parity check uniquely achieves this optimum $\left[\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array}\right]$, the green line,

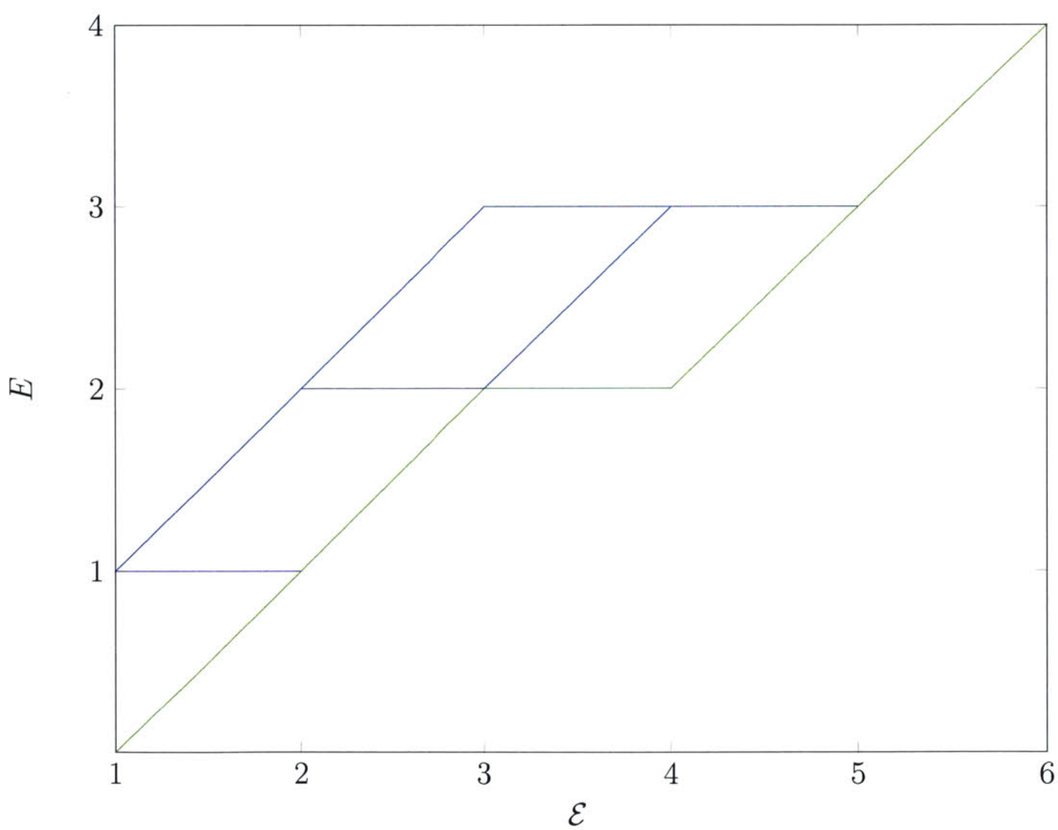
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

5.4.2 $k = 4$ and $n = 6$

There are 6 achievable distortion patterns

$$\left\{\begin{matrix} 1 & 2 & 3 & 3 & 3 & 4 \\ 1 & 2 & 2 & 3 & 3 & 4 \\ 1 & 1 & 2 & 3 & 3 & 4 \\ 0 & 1 & 2 & 3 & 3 & 4 \\ 1 & 1 & 2 & 2 & 3 & 4 \\ 0 & 1 & 2 & 2 & 3 & 4 \end{matrix}\right\}.$$

Observing the following plot, of the superposition of all achievable distortions, the optimal distortion of Theorem 9 is achievable.



Two matrices achieve $\begin{bmatrix} 0 & 1 & 2 & 2 & 3 & 4 \end{bmatrix}$, the green line,

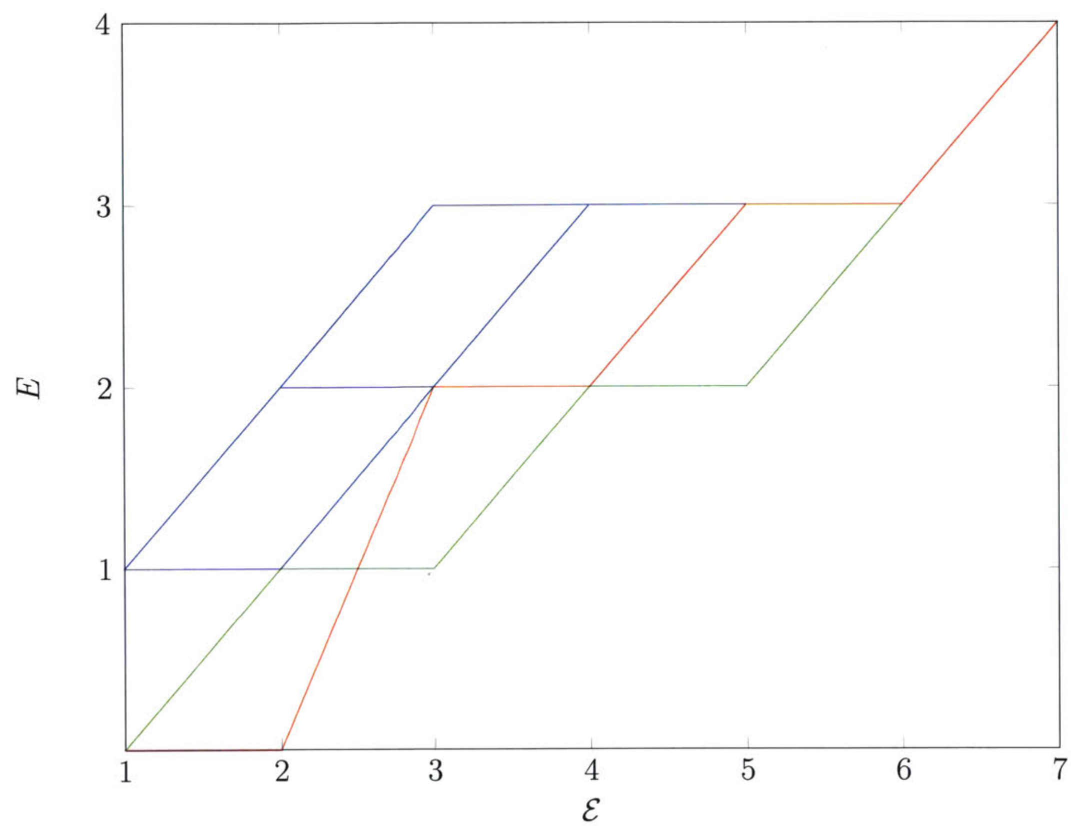
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

5.4.3 $k = 4$ and $n = 7$

There are 11 achievable distortion patterns

$$\left\{\begin{array}{ccccccc} 1 & 2 & 3 & 3 & 3 & 3 & 4 \\ 1 & 2 & 2 & 3 & 3 & 3 & 4 \\ 1 & 1 & 2 & 3 & 3 & 3 & 4 \\ 0 & 1 & 2 & 3 & 3 & 3 & 4 \\ 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 1 & 1 & 2 & 2 & 3 & 3 & 4 \\ 0 & 1 & 2 & 2 & 3 & 3 & 4 \\ 0 & 1 & 1 & 2 & 3 & 3 & 4 \\ 1 & 1 & 1 & 2 & 3 & 3 & 4 \\ 0 & 1 & 1 & 2 & 2 & 3 & 4 \\ 0 & 0 & 2 & 2 & 3 & 3 & 4 \end{array}\right\}.$$

Observing the following plot, of the superposition of all achievable distortions, the optimal distortion of Theorem 9 is not achievable.



The red line $\begin{bmatrix} 0 & 0 & 2 & 2 & 3 & 3 & 4 \end{bmatrix}$ is achieved uniquely by the Hamming code. Two matrices achieve $\begin{bmatrix} 0 & 1 & 1 & 2 & 2 & 3 & 4 \end{bmatrix}$, the green line,

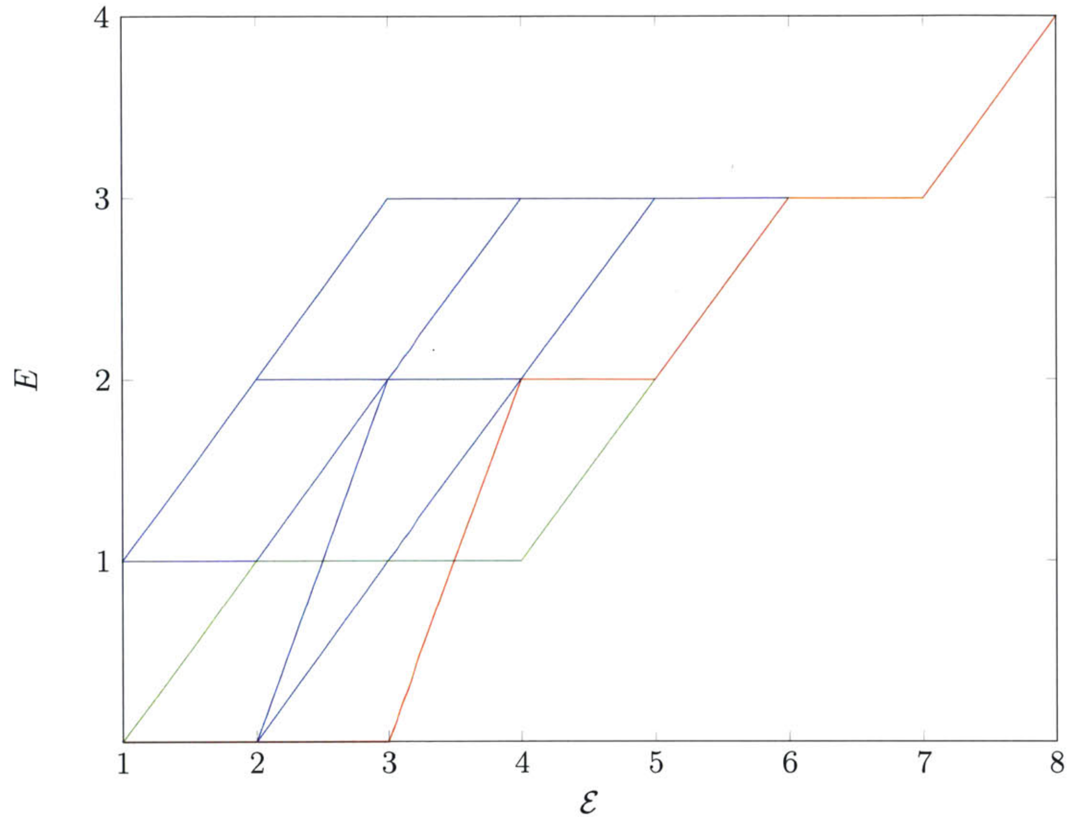
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

5.4.4 $k = 4$ and $n = 8$

There are 19 achievable distortion patterns

$$
\left\{\begin{array}{cccccccc}
1 & 2 & 3 & 3 & 3 & 3 & 3 & 4 \\
1 & 2 & 2 & 3 & 3 & 3 & 3 & 4 \\
1 & 1 & 2 & 3 & 3 & 3 & 3 & 4 \\
0 & 1 & 2 & 3 & 3 & 3 & 3 & 4 \\
1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\
1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 \\
0 & 1 & 2 & 2 & 3 & 3 & 3 & 4 \\
0 & 1 & 1 & 2 & 3 & 3 & 3 & 4 \\
1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\
0 & 1 & 2 & 2 & 2 & 3 & 3 & 4
\end{array}\right\}
\quad
\left\{\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 3 & 3 & 3 & 4 \\
1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\
0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\
0 & 0 & 2 & 2 & 2 & 3 & 3 & 4 \\
0 & 0 & 1 & 2 & 2 & 3 & 3 & 4 \\
1 & 2 & 2 & 2 & 2 & 3 & 3 & 4 \\
0 & 1 & 1 & 1 & 2 & 3 & 3 & 4 \\
0 & 0 & 2 & 2 & 3 & 3 & 3 & 4 \\
0 & 0 & 0 & 2 & 2 & 3 & 3 & 4
\end{array}\right\}.
$$

Observing the following plot, of the superposition of all achievable distortions, the optimal distortion of Theorem 9 is not achievable.



The red line $\begin{bmatrix} 0 & 0 & 0 & 2 & 2 & 3 & 3 & 4 \end{bmatrix}$ is achieved uniquely by the extended Hamming code. Three matrices achieve $\begin{bmatrix} 0 & 1 & 1 & 1 & 2 & 3 & 3 & 4 \end{bmatrix}$, the green line,

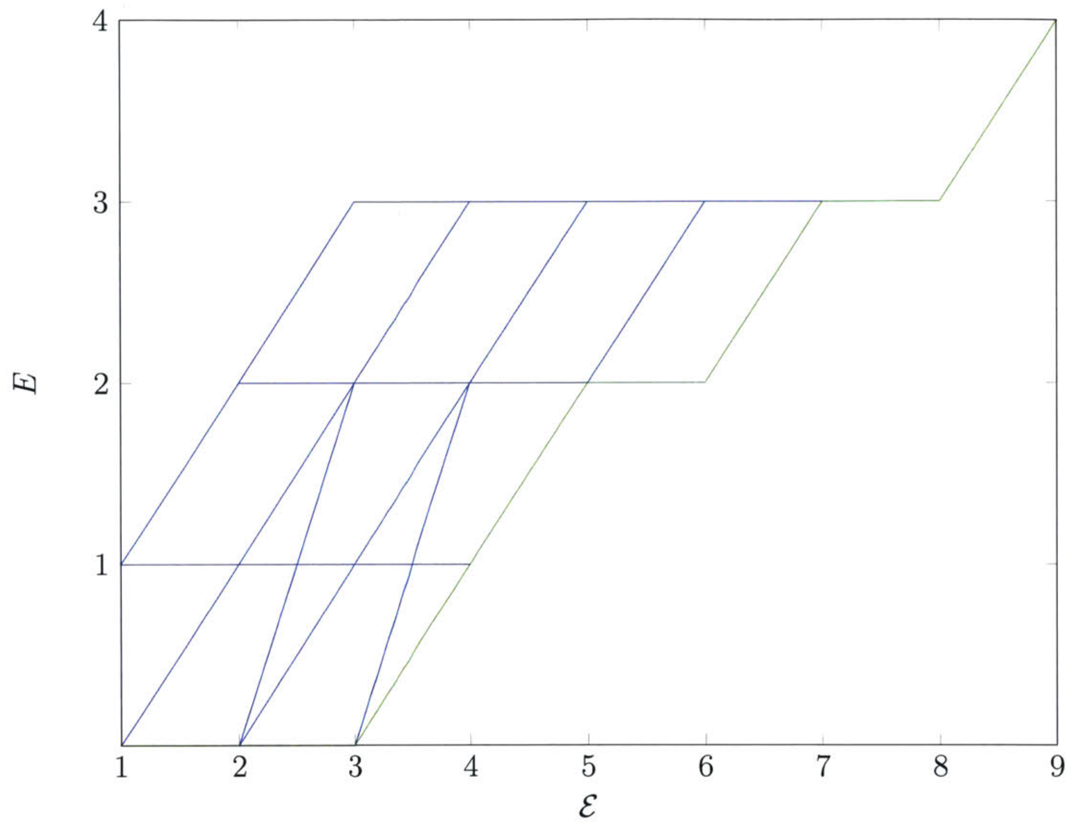
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

5.4.5 $k = 4$ and $n = 9$

There are 31 achievable distortion patterns

$$
\left\{\begin{matrix}
1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 4 \\
1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 \\
1 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 4 \\
0 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 4 \\
1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 \\
1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 4 \\
0 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 4 \\
0 & 1 & 1 & 2 & 3 & 3 & 3 & 3 & 4 \\
1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\
1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\
0 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\
1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 \\
0 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 \\
0 & 0 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\
0 & 0 & 1 & 2 & 2 & 3 & 3 & 3 & 4 \\
1 & 1 & 1 & 2 & 3 & 3 & 3 & 3 & 4
\end{matrix}\right\}
\quad
\left\{\begin{matrix}
1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\
0 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\
1 & 1 & 1 & 1 & 2 & 3 & 3 & 3 & 4 \\
0 & 1 & 1 & 1 & 2 & 3 & 3 & 3 & 4 \\
1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\
0 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\
0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\
0 & 0 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\
0 & 0 & 2 & 2 & 3 & 3 & 3 & 3 & 4 \\
0 & 0 & 0 & 2 & 2 & 2 & 3 & 3 & 4 \\
1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 4 \\
0 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 4 \\
0 & 0 & 0 & 2 & 2 & 3 & 3 & 3 & 4 \\
0 & 0 & 2 & 2 & 2 & 2 & 3 & 3 & 4 \\
0 & 0 & 0 & 1 & 2 & 2 & 3 & 3 & 4
\end{matrix}\right\}.
$$

Observing the following plot, of the superposition of all achievable distortions, the optimal distortion of Theorem 9 is achievable.



Two matrices achieve $\begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 2 & 3 & 3 & 4 \end{bmatrix}$, the green line,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

5.5 Double Identity Code

Consider $\rho = 2$ and let $f_k : \mathbb{F}_2^k \to \mathbb{F}_2^{2k}$ where $f_k(x) = A_k x = \begin{bmatrix} I_K & \overline{I_k} \end{bmatrix}^T x$. For all $x \in \mathbb{F}_2^k$

$$w(f_k(x)) = \begin{cases} 2\,w(x) & w(x) \text{ is even} \\ k & w(x) \text{ is odd} \end{cases}.$$

Follows from $w(f_k(x)) = w(I_k x) + w(\overline{I_k} x)$. Any given x has $w(x)$ nonzero entries and in the multiplication $\overline{I_k} x$ these nonzero entries will coincide with $w(x)$ rows with one zero and $k - w(x)$ rows with no zeros. Thus, if $w(x)$ is even this results in $w(x)$ odd sums, giving one, and $k - w(x)$ even sums, giving zero. Similarly, if $w(x)$ is odd this yields $w(x)$ zeros and $k - w(x)$ ones.

Distortion, for $0 \leq \mathcal{E} < k$

$$D(\mathcal{E}\,; f_k) = \max\left\{ \left\lfloor \frac{\mathcal{E}}{2} \right\rfloor - 1, 0 \right\}.$$

Given $\mathcal{E}$ erasures the adversary can erase at most all of the codewords of even weight supported on any $\lfloor \frac{\mathcal{E}}{2} \rfloor$ positions in $\mathbb{F}_2^k$. If $\lfloor \frac{\mathcal{E}}{2} \rfloor$ is even choose any x_0 with an odd number of ones and, if $\lfloor \frac{\mathcal{E}}{2} \rfloor$ is odd choose any x_0 with an even number of ones. In either case, the antipodal of x_0, on these positions, has odd weight, but $\lfloor \frac{\mathcal{E}}{2} \rfloor - 1$ is achievable by some codeword of even weight.

Parameterizing $\mathcal{E}$ this yields

$$\frac{1}{k}\left(\left\lfloor \frac{\varepsilon 2k}{2} \right\rfloor - 1 \right) = \frac{1}{k}\left(\lfloor \varepsilon k \rfloor - 1 \right).$$

This beats repetition for all k, but asymptotically they coincide.

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