Portfolio Optimization and the Endurance Investors' Case

15.060 Data, Models and Decisions

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Development of Portfolio Model Data

\( P_{A,t} \) = stock price of asset \( A \) at end of month \( t \)

\( D_{A,t} \) = dividends paid during month \( t \)

\( R_{A,t} \) = monthly rate of return of asset \( A \) in month \( t \):

\[
R_{A,t} = \left( \frac{P_{A,t} + D_{A,t} - P_{A,(t-1)}}{P_{A,(t-1)}} \right)
\]

\( R_A \) = *future* monthly rate of return of asset \( A \)

\( R_A \) is a random variable

Estimate of mean and standard deviation of \( R_A \):

\[
\text{MEAN}_{R_A} = \frac{1}{T} \sum_{t=1}^{T} (R_{A,t})
\]

\[
\text{VAR}_A = \frac{1}{T-1} \sum_{t=1}^{T} \left( R_{A,t} - \text{MEAN}_{R_A} \right)^2
\]

\[
\text{SD}_A = \sqrt{\text{VAR}_A}
\]

Estimate of covariance and correlation of \( R_A \) and \( R_B \):

\[
\text{COV}(R_A, R_B) = \frac{1}{T-1} \sum_{t=1}^{T} \left( R_{A,t} - \text{MEAN}_{R_A} \right) \left( R_{B,t} - \text{MEAN}_{R_B} \right)
\]

\[
\text{CORR}(R_A, R_B) = \frac{\text{COV}(R_A, R_B)}{\text{SD}_A \text{SD}_B}
\]
Critique of Methodology

- Uses past performance to estimate future performance
- Some months might be very unusual
- Might instead use “exponential smoothing”
- How many months should we use? 12, 24, 60, 120, …
- This method (based on past data) is a good estimator of future covariance and correlation, but it is not a very good estimator of future expected returns

- There are much better ways to estimate MEANR_A based on the Capital Asset Pricing Model (CAPM):

\[
MEANR_A = R_f + \beta \cdot ( MEANR_{MARKET} - R_f )
\]

\( R_f \) is the risk-free rate of return

\( MARKET \) is the entire stock market/economy

\( \beta = \frac{COV( R_A, R_{MARKET} )}{VAR_{MARKET}} \)

Converting monthly estimates to annual estimates? Rates of return are multiplicative, not additive. Handled using logarithms.
More Advanced Modeling Methods for Portfolio Optimization

Set-Up and Basic Data Requirements

\[ i = 1, \ldots, n \] is the group of assets under consideration

We currently own \( H_i \) shares of asset \( i \)

\( \text{PCURR}_i = \text{yesterday’s closing price of asset } i \)

\( \text{PFUT}_i = \text{future price of asset } i \) (say, one quarter from now)

\( \text{PFUT}_i \) is a random variable

We have estimates of the following data:

\( \text{EPFUT}_i \) (expected future price of asset \( i \))

\( \text{COV}(\text{PFUT}_i, \text{PFUT}_j) \) (covariance of future price of assets \( i \) and \( j \))
Decision Variables:

$Y_i = \text{the amount we will purchase or sell of asset } i$

$Y_i \geq 0 \text{ if we are buying}$

$Y_i \leq 0 \text{ if we are selling}$

Expected Value and Risk of the Portfolio

EXPECTED VALUE =

$$\sum_i \text{EPFUT}_i \cdot (H_i + Y_i)$$

STANDARD DEVIATION =

$$\sqrt{\sum_i \sum_j \text{COV}(\text{PFUT}_i, \text{PFUT}_j)(H_i + Y_i)(H_j + Y_j)}$$

Transaction Fees

Transaction fee of $A_i$, measured in $\text{ per share traded}$

Typically $A_i = 0.03 \text{ (or less) for large investors, depending on the asset}$

Total transaction costs are:

$$\sum_i A_i |Y_i|$$
Price Impact

Suppose average daily volume of Boeing is 200,000 shares

Suppose a large investment firm wants to purchase 100,000 shares

The closing price of Boeing yesterday was $60.00

What will happen?
For example, it might turn out that the firm can purchase 20,000 shares at $61.00
40,000 shares at $62.00
40,000 shares at $63.00

If the transaction fees are $0.03/share, then:

Transaction fees: $3,000 = 0.03 • 100,000

Impact on price: $220,000 = $20,000 • 1.00
+ 40,000 • 2.00
+ 40,000 • 3.00

Cost of price impact is significantly greater than the transaction fees

We model: \( P_{\text{NEW}}_i = P_{\text{CURR}}_i + C_i \cdot Y_i \)

(\( C_i \) is typically determined by regression on past data)

Therefore:

\[
\text{PRICE IMPACT} = ( P_{\text{CURR}}_i + C_i \cdot Y_i ) \cdot Y_i - ( P_{\text{CURR}}_i ) \cdot Y_i \\
= C_i \cdot (Y_i)^2
\]
Liquidity

If a firm owns 50% of all outstanding stock in asset $i$, it will be more difficult to sell the stock.

Let $T_i =$ total number of outstanding shares of stock in asset $i$

Suppose we do not want to own more than 4% of the total outstanding shares in asset $i$

$$H_i + Y_i \leq 0.04 \cdot T_i$$

Turnover

We can impose a constraint on the volume traded of asset $i$:

$$|Y_i| \leq \text{LIMIT}_i$$

and we can impose a bound on the total volume of all trades:

$$\sum_i P_{NEW_i} \cdot |Y_i| \leq 0.05 \cdot \left( \sum_i P_{CURR_i} \cdot H_i \right)$$

Cash Balance

We keep a small amount of the fund in cash to use for transactions, otherwise we must balance out the cash used and created:

$$|\sum_i P_{NEW_i} \cdot Y_i| \leq \text{LIMIT}$$
No Short Selling

\[ H_i + Y_i \geq 0 \]

Closeness to Benchmark

Investment firms often restrict the weights of stocks in their portfolios to be close to some benchmark. Here is one way that this is done if the benchmark is the S&P500:

\[ T_i \text{ is the number of outstanding shares of asset } i \text{ in the S&P500} \]

\[ \text{MCAP}_i = \text{PCURR}_i \cdot T_i \]

\[ \text{WBENCH}_i = \frac{\text{MCAP}_i}{\sum_{k=1}^{500} \text{MCAP}_k} \]

\[ \text{TOTALINVEST} = \sum_i \text{PCURR}_i \cdot (H_i + Y_i) \]

We then use the following constraint:

\[ \sum_i \left| \frac{\text{PCURR}_i \cdot (H_i + Y_i)}{\text{TOTALINVEST}} - \text{WBENCH}_i \right| \leq \text{LIMIT} \]
The Overall Model

Maximize $\sum_i EPFUT_i \cdot (H_i + Y_i) - A_i | Y_i | - C_i \cdot (Y_i)^2$

Subject to:

$\sqrt{\sum_i \sum_j \text{COV}(PFUT_i, PFUT_j)(H_i + Y_i)(H_j + Y_j)} \leq \text{TGTAR Std. Dev.}$

$H_i + Y_i \leq 0.04 \cdot T_i$ (Liquidity)

$| Y_i | \leq \text{LIMIT}_i$ (Turnover)

$\sum_i PNEW_i \cdot | Y_i | \leq 0.05 \cdot (\sum_i \text{PCURR}_i \cdot H_i)$ (Turnover)

$| \sum_i PNEW_i \cdot Y_i | \leq \text{LIMIT}$ (Cash Balance)

$H_i + Y_i \geq 0$ (No short selling)

$\sum_i \frac{\text{PCURR}_i \cdot (H_i + Y_i)}{\text{TOTALINVEST}} - \text{WBENCH}_i \leq \text{LIMIT}$ (Benchmark)

$PNEW_i = \text{PCURR}_i + C_i \cdot Y_i$ (Price Impact)
Multi-Period Portfolio Models

Capture the dynamics that decisions can be delayed in order to take advantage of economies of scale in trading as well as to lessen the price impact of trades for large institutional investors

Based on “dynamic programming” or “dynamic optimization”

Take 15.071 Decision Methodologies for Managers ("The Edge")
Spring semester, Professor Bertsimas
Additional Reading

Basic level:


Advanced level:


Final Note

The Nobel Memorial Prize in Economic Science was awarded in 1990 to Merton Miller, William Sharpe, and Harry Markowitz for their work on portfolio theory and portfolio models (and the implications for asset pricing).