Upper bound on list-decoding radius of binary codes

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Abstract—Consider the problem of packing Hamming balls of a given relative radius subject to the constraint that they cover any point of the ambient Hamming space with multiplicity at most $L$. For odd $L \geq 3$ an asymptotic upper bound on the rate of any such packing is proven. The resulting bound improves the best known bound (due to Blinovsky’1986) for rates below a certain threshold. The method is a superposition of the linear-programming idea of Ashikhmin, Barg and Litsyn (that was used previously to improve the estimates of Blinovsky for $L = 2$) and a Ramsey-theoretic technique of Blinovsky. As an application it is shown that for all odd $L$ the slope of the rate-radius tradeoff is zero at zero rate.

Index Terms—Combinatorial coding theory, list-decoding, converse bounds

I. MAIN RESULT AND DISCUSSION

One of the most well-studied problems in information theory asks to find the maximal rate at which codewords can be packed in binary space with a given minimum distance between codewords. Operationally, this (still unknown) rate gives the capacity of the binary input-output channel subject to adversarial noise of a given level. A natural generalization was considered by Elias and Wozencraft [1, 2], who allowed the decoder to output a list of size $L$. In this paper we provide improved upper bounds on the latter question.

Our interest in bounding the asymptotic tradeoff for the list-decoding problem is motivated by our study of fundamental limits of joint source-channel communication [3]. Namely, in [4, Theorem 6] we proposed an extension of the previous result in [3, Theorem 7] that required bounding rate for the list-decoding problem.

We proceed to formal definitions and brief overview of known results. For a binary code $C \subset \mathbb{F}_2^n$ we define its list-size $L$ decoding radius as

$$\tau_L(C) \triangleq \frac{1}{n} \max \{ r : \forall x \in \mathbb{F}_2^n \, |C \cap \{ x + B^r_x \}| \leq L \},$$

where Hamming ball $B^r_x$ and Hamming sphere $S^r_x$ are defined as

$$B^r_x \triangleq \{ x \in \mathbb{F}_2^n : |x| \leq r \}, \quad (1)$$
$$S^r_x \triangleq \{ x \in \mathbb{F}_2^n : |x| = r \}, \quad (2)$$

with $|x| = |\{ i : x_i = 1 \}|$ denoting the Hamming weight of $x$. Alternatively, we may define $\tau_L$ as follows:  

$$\tau_L(C) = \frac{1}{n} \left( \min \left\{ \text{rad}(S) : S \in \binom{C}{L+1} \right\} - 1 \right),$$

where $\text{rad}(S)$ denotes radius of the smallest ball containing $S$ (known as Chebyshev radius):

$$\text{rad}(S) \triangleq \min_{y \in \mathbb{F}_2^n} \max_{x \in S} |y - x|.$$

The asymptotic tradeoff between rate and list-decoding radius $\tau_L$ is defined as usual:

$$\tau_L^*(R) \triangleq \limsup_{n \to \infty} \frac{1}{n} \log |C| \geq 2^n h,$$
$$R^*_L(\tau) \triangleq \limsup_{n \to \infty} \frac{1}{n} \log |C| \geq \tau n$$

The best known upper (converse) bounds on this tradeoff are as follows:

- List size $L = 1$: The best bound to date was found by McEliece, Rodemich, Rumsey and Welch [5]:

$$R^*_1(\tau) \leq R_{LP2}(2\tau), \quad (5)$$
$$R_{LP2}(\delta) \triangleq \min \log 2 - h(\alpha) + h(\beta), \quad (6)$$

where $h(x) = -x \log x - (1-x) \log(1-x)$ and minimum is taken over all $0 \leq \beta \leq \alpha \leq 1/2$ satisfying

$$2^{\alpha(1-\alpha) - \beta(1-\beta)} \leq \delta$$

For rates $R < 0.305$ this bound coincides with the simpler bound:

$$\tau^*_1(R) \leq \frac{1}{2} \delta_{LP1}(R), \quad (7)$$
$$\delta_{LP1}(R) \triangleq \frac{1}{2} - \sqrt{\beta(1-\beta)}, \quad R = \log 2 - h(\beta), \quad (8)$$

where $\beta \in [0, \frac{1}{2}]$.

- List size $L = 2$: The bound found by Ashikhmin, Barg and Litsyn [6] is given as

$$R^*_2(\tau) \leq \log 2 - h(2\tau) + R_{up}(6, \tau), \quad (9)$$

where $R_{up}(6, \alpha)$ is the best known upper bound on rate of codes with minimal distance $\delta n$ constrained to live on Hamming spheres $S^n_{\delta n}$. The expression for $R_{up}(6, \alpha)$ can be obtained by using the linear programming bound from [5] and applying Levenshtein’s monotonicity, cf. [7, Lemma 4.2(6)]. The resulting expression is

$$R^*_2(\tau) \leq \left\{ \begin{array}{ll}
R_{LP2}(2\tau), & \tau \leq \tau_0 \\
\log 2 - h(2\tau) + h(u(\tau)), & \tau > \tau_0,
\end{array} \right.$$
For list sizes $L \geq 3$: The original bound of Blinovsky [8] appears to be the best (before this work):
\[
\tau^*_L(R) \leq \sum_{i=1}^{[L/2]} \frac{(2i-2)!}{(i+1)!} (\lambda(1-\lambda))^i, \quad R = 1 - h(\lambda),
\]
where $\lambda \in [0, \frac{1}{2}]$. Note that [8] also gives a non-constructive lower bound on $\tau^*_L(R)$. Results on list-decoding over non-binary alphabets are also known, see [9], [10].

In this paper we improve the bound of Blinovsky for lists of odd size and rates below a certain threshold. To that end we will mix the ideas of Ashikhmin, Barg and Litsyn (namely, extraction of a large spectrum component from the code) and those of Blinovsky (namely, a Ramsey-theoretic reduction to study of symmetric subcodes).

To present our main result, we need to define exponent of Krawtchouk polynomial $K_{\lambda n}(\xi n) = \exp\{n E_{\beta}(\xi) + o(n)\}$. For $\xi \in [0, \frac{1}{2} - \sqrt{\beta(1-\beta)}]$ the value of $E_{\beta}(\xi)$ was found in [11]. Here we give it in the following parametric form, cf. [12] or [13, Lemma 4]:
\[
E_{\beta}(\xi) = \xi \log(1-\omega) + (1-\xi) \log(1+\omega) - \beta \log \omega \quad (11)
\]
where
\[
\omega \in \left[\frac{\beta}{1-\beta} \sqrt{\frac{1}{\beta(1-\beta)}}\right].
\]
Our main result is the following:

**Theorem 1.** Fix list size $L \geq 2$, rate $R$ and an arbitrary $\beta \in [0, 1/2]$ with $h(\beta) \leq R$. Then any sequence of codes $C_n \subset \{0, 1\}^n$ of rate $R$ satisfies
\[
\limsup_{n \to \infty} \tau_L(C_n) \leq \max_{j, \xi_0} \xi_0 g_j \left(1 - \frac{\xi_1}{2\xi_0}\right) + \left(1 - \xi_0\right) g_j \left(\frac{\xi_1}{2(1-\xi_0)}\right),
\]
where maximization is over $\xi_0$ satisfying
\[
0 \leq \xi_0 \leq 1 - \sqrt{\frac{1}{1-\beta}}
\]
and $j$ ranging over $\{0, 1, 2, \ldots, 2k+1, \ldots\}$ if $L$ is odd and over $\{0, 2, 2k, \ldots\}$ if $L$ is even. Quantity $\xi_1 = \xi_1(\xi_0, \delta, R)$ is a unique solution of
\[
R + h(\beta) - 2E_{\beta}(\xi_0) = h(\xi_0) - \xi_0 h \left(\frac{\xi_1}{2\xi_0}\right) - (1-\xi_0) h \left(\frac{\xi_1}{2(1-\xi_0)}\right),
\]
on the interval $[0, 2\xi_0(1-\xi_0)]$ and functions $g_j(\nu)$ are defined as
\[
g_j(\nu) = \frac{1}{L + j} \left(L\nu - E[|2W - L - j^+]|\right), W \sim \text{Bin}(L, \nu)
\]
As usual with bounds of this type, cf. [14], it appears that taking $h(\beta) = R$ can be done without loss. Under such choice, our bound outperforms Blinovsky’s for all odd $L$ and all rates small enough (see Corollary 3 below). The bound for $L = 3$ is compared in Fig. 1 with the result of Blinovsky numerically. For larger odd $L$ the comparison is similar, but the range of rates where our bound outperforms Blinovsky’s becomes smaller, see Table I.

**Evaluation of Theorem 1** is computationally possible, but is somewhat tedious. Fortunately, for small $L$ the maximum over $\xi_0$ and $j$ is attained at $\xi_0 = \frac{1}{2} - \sqrt{\beta(1-\beta)}$ and $j = 1$. We rigorously prove this for $L = 3$.

**Corollary 2.** For list-size $L = 3$ we have
\[
\tau^*_L(R) \leq \frac{3}{4} \delta - \frac{1}{16} \left(\frac{(2\delta - \xi_1)^3}{\delta^2} + \frac{\xi_1^3}{(1-\delta)^2}\right),
\]
where $\delta \in (0, 1/2]$ and $\xi_1 \in [0, 2\delta(1-\delta)]$ are functions of $R$ determined from
\[
R = h \left(\frac{1}{2} - \sqrt{\delta(1-\delta)}\right),
\]
\[
R = \log 2 - \delta h \left(\frac{\xi_1}{2\delta}\right) - (1-\delta) h \left(\frac{\xi_1}{2(1-\delta)}\right)
\]

Another interesting implication of Theorem 1 is that it allows us to settle the question of slope of the curve $R^*_L(\tau)$ at zero rate. Notice that Blinovsky’s converse bound (10) has a negative slope, while his achievability bound has a zero slope. Our bound always has a zero slope for odd $L$ (but not for even $L$, see Remark 2 in Section II-C).

\^ Notice that proofs of each of the two Corollaries below contain different relaxations of the bound (13), e.g. (22), which are easier to evaluate. Notice also that in Table I for the last two entries ($L = 9, 11$) at the high endpoint of rate the maximum over $\xi_0$ is attained not at $\frac{1}{2} - \sqrt{\beta(1-\beta)}$. 

![Fig. 1. Comparison of bounds on $R^*_L(\tau)$ for list size $L = 3$](image)
Corollary 3. Fix arbitrary odd $L \geq 3$. There exists $R_0 = R_0(L) > 0$ such that for all rates $R < R_0$ we have
\[ \tau^*_L(R) \leq g_1(\delta_{LP_1}(R)), \]
where $g_1(\cdot)$ is a degree-$L$ polynomial defined in (16). In particular,
\[ \frac{d}{dt} \bigg|_{t = \tau^*_L(0)} R^*_L(t) = 0, \]
where the zero-rate radius is $\tau^*_L(0) = \frac{1}{2} - 2^{-L-1}(\frac{L}{n}).$

Before closing our discussion we make some additional remarks:

1) The bound in Theorem 1 can be slightly improved by replacing $\delta_{LP_1}(R)$, that appears in the right-hand side of (14), with a better bound, a so-called second linear-programming bound $\delta_{LP_2}(R)$ from [5]. This would enforce the usage of the more advanced estimate of Litsyn [15, Theorem 5] and complicate analysis significantly. Notice that $\delta_{LP_2}(R) \neq \delta_{LP_1}(R)$ only for rates $R \geq 0.305$. If we focus attention only on rates where new bound is better than Blinovsky’s, such a strengthening only affects the case of $L = 3$ and results in a rather minuscule improvement (for example, for rate $R = 0.33$ the improvement is $\approx 3 \cdot 10^{-5}$).

2) For even $L$ it appears that $h(\beta) = R$ is no longer optimal. However, the resulting bound does not appear to improve upon Blinovsky’s.

3) When $L$ is large (e.g. 35) the maximum in (13) is not always attained by either $j = 1$ or $\xi_0 = \delta_{LP_1}(R)$. It is not clear whether such anomalies only happen in the region of rates where our bound is inferior to Blinovsky’s.

4) The result of Corollary 3 follows by weakening (13) (via concavity of $g_j$, Lemma 8) to
\[ \lim_{n \to \infty} \sup_{C_n} \tau_L(C_n) \leq \max_{j \neq 0} j g_j(\delta_{LP_1}(R)). \]
This is the main intuitive reason why our bound succeeds in improving Blinovsky’s, but only for odd $L$.

Notice that the sphere returned by Kalai-Linial is bigger than that of Elias-Bassalygo (which is the reason our bound deteriorates at large rates), but the good thing is that the subcode $C''$ has another codeword $c_0$ at the center of the Hamming sphere. Now, intuitively $\tau^*_L$ is roughly equivalent to $\tau_{L-1}$. The zero-rate (Plotkin) radius for a list-$L$ decoding of binary codes on Hamming sphere $S^*_L$ is given by
\[ p_L(\xi) = \frac{E[\min(W_{\xi}, L + 1 - W_{\xi})]}{L + 1}, \quad W_{\xi} \sim \text{Bino}(L + 1, \xi). \]
So intuitively, we expect that Blinovsky’s bound should give
\[ \tau^*_L(R) \lesssim p_L(\delta_{GV}(R)) \]
while our bound should give
\[ \tau^*_L(R) \lesssim p_{L-1}(\delta_{LP_1}(R)) \]
Finally, it is easy to check that for even $L$ we have $p_L = p_{L-1}$, while for odd $L$, $p_L > p_{L-1}$. This is the main intuitive reason why our bound succeeds in improving Blinovsky’s, but only for odd $L$.

II. Proofs

A. Proof of Theorem 1

Consider an arbitrary sequence of codes $C_n$ of rate $R$. As in [6] we start by using Delsarte’s linear programming to select a large component of the distance distribution of the code. Namely, we apply result of Kalai and Linial [11, Proposition 3.2]: For every $\beta$ with $h(\beta) \leq R$ there exists a sequence $\epsilon_n \to 0$ such that for every code $C$ of rate $R$ there is a $\xi_0$ satisfying (14) such that
\[ A_{\xi_0,n}(C) \geq \frac{1}{|C|} \sum_{x,x' \in C} 1\{|x-x'| = \xi_0 n\} \]
\[ \geq \exp(n(R + h(\beta) - 2E_\beta(\xi_0) + \epsilon_n)). \]
Without loss of generality (by compactness of the interval $[0, 1/2 - \sqrt{3}/3]$ and passing to a proper subsequence of codes $C'_{n_k}$) we may assume that $\xi_0$ selected in (23) is the same for all blocklengths $n$. Then there is a sequence of subcodes $C'_n$ of asymptotic rate
\[ R' \geq R + h(\beta) - 2E_\beta(\xi_0) \]
such that each $C'_n$ is situated on a sphere $c_0 + S_{\xi_0}$ surrounding another codeword $c_0 \in C$. Our key geometric result is: If there are too many codewords on a sphere $c_0 + S_{\xi_0}$ then it is possible to find $L$ of them that are includable in a small ball that also contains $c_0$. Precisely, we have:
Lemma 4. Fix $\xi_0 \in (0,1)$ and positive integer $L$. There exist a sequence $\epsilon_n \to 0$ such that for any code $C_n \subset S_{\xi_0 n}$ of rate $R' > 0$ there exist $L$ codewords $c_1, \ldots, c_L \in C_n'$ such that
\[
\frac{1}{n} \text{rad}(0, c_1, \ldots, c_L) \leq \theta(\xi_0, R', L) + \epsilon_n,
\]
where
\[
\theta(\xi_0, R', L) \triangleq \max_j \theta_j(\xi_0, R', L)
\]
(24) with $\xi_1 = \xi_1(\xi_0)$ found as unique solution on interval $[0, 2\xi_0(1-\xi_0)]$ of
\[
R' = h(\xi_0) - \xi_0 h\left(\frac{\xi_1}{2\xi_0}\right) - (1-\xi_0)h\left(\frac{\xi_1}{2(1-\xi_0)\xi_0}\right),
\]
(25) and $\theta_j$ and $\xi_0 g_j$ are defined in (16) and $j$ in maximization (25) ranging over the same set as in Theorem 1.

Equipped with Lemma 4 we immediately conclude that
\[
\limsup_{n \to \infty} \tau_L(C_n) \leq \max_{\xi_0 \in [0,\delta]} \theta(\xi_0, R+h(\beta) - 2E_\beta(\xi_0), L).
\]

B. Proof of Lemma 4

Let $T_L$ be the $(2^L-1)$-dimensional space of probability distributions on $\mathbb{F}_2^L$. If $T \in T_L$ then we have
\[
T = (t_v, v \in \mathbb{F}_2^L) \quad \text{such that } \sum_v t_v = 1.
\]
We define distance on $T_L$ to be the $L_\infty$ one:
\[
\|T - T'\| \triangleq \max_{v \in \mathbb{F}_2^L} |t_v - t'_v|.
\]
Permutation group $S_L$ acts naturally on $\mathbb{F}_2^L$ and this action descends to probability distributions $T_L$. We will say that $T$ is symmetric if
\[
T = \sigma(T) \iff t_v = t_{\sigma(v)} \quad \forall v \in \mathbb{F}_2^L
\]
for any permutation $\sigma : [L] \to [L]$. Note that symmetric $T$ is completely specified by $L+1$ numbers (weights of Hamming spheres in $\mathbb{F}_2^L$):
\[
\sum_{v : |v| = j} t_v, \quad j = 0, \ldots, L.
\]
Next, fix some total ordering of $\mathbb{F}_2^n$ (for example, lexicographic). Given a subset $S \subset \mathbb{F}_2^n$ we will say that $S$ is given in ordered form if $S = \{x_1, \ldots, x_{|S|}\}$ and $x_1 < x_2 < \cdots < x_{|S|}$ under the fixed ordering on $\mathbb{F}_2^n$. For any subset of codewords $S = \{x_1, \ldots, x_L\}$ given in ordered form we define its joint type $T(S)$ as an element of $T_L$ with
\[
t_v \triangleq \frac{1}{n} |\{j : x_1(j) = v, \ldots, x_L(j) = v\}|,
\]
where $h(\xi_0, R', L)$ has the property that the number of columns equal to $[1, 0, \ldots, 0]^T$ is the same as the number of columns $[0, 1, \ldots, 0]^T$ etc. For any code $C \subset \mathbb{F}_2^n$ we define its average joint type:
\[
\bar{T}_L(C) = \frac{1}{L!} \left(\sum_{S \subset (\mathbb{F}_2^n)} \sum_{\sigma \in S_{|S|}} \sigma(T(S))\right).
\]
Evidently, $\bar{T}_L(C)$ is symmetric.

Our proof crucially depends on a (slight extension of the) brilliant idea of Blinovsky [8]:

Lemma 5. For every $L \geq 1$, $K \geq L$ and $\delta > 0$ there exist a constant $K_1 = K_1(L, K, \delta)$ such that for all $n \geq 1$ and all codes $C \subset \mathbb{F}_2^n$ of size $|C| \geq K_1$ there exists a subcode $C' \subset C$ of size at least $L$ such that for any $S \subset (\mathbb{F}_2^n)$ we have
\[
\|T(S) - \bar{T}_L(C')\| \leq \delta.
\]
(29)

Remark 1. Note that if $S' \subset S$ then every element of $T(S')$ is a sum of $\leq 2^L$ elements of $T(S)$. Hence, joint types $T(S')$ are approximately symmetric also for smaller subsets $|S'| < L$.

Proof. We first will show that for any $\delta > 0$ and sufficiently large $|C|$ we may select a subcode $C'$ so that the following holds: For any pair of subsets $S, S' \subset C'$ s.t. $|S| = |S'| \leq L$ we have:
\[
\|T(S) - T(S')\| \leq \delta_1
\]
(30)

Consider any code $C_1 \subset \mathbb{F}_2^n$ and define a hypergraph with vertices indexed by elements of $C$ and hyper-edges corresponding to each of the subsets of size $L$. Now define a $\delta_1/2$-net on the space $T_L$ and label each edge according to the closest element of the $\delta_1/2$-net. By a theorem of Ramsey there exists $K_1$ such that if $|C_1| \geq K_1$, then there is a subcode $C'_1 \subset C$ such that $|C'_1| \geq K_1$ and each of the internal edges, indexed by $(C'_1)_L$, is assigned the same label. Thus, by triangle inequality (30) follows for all $S, S' \subset (C'_1)_L$.

Next, apply the previous argument to show that there is a constant $K_{L-1}$ such that for any $C_2 \subset \mathbb{F}_2^n$ of size $|C_2| \geq K_{L-1}$ there exists a subcode $C'_2$ of size $|C'_2| \geq K_{L-1}$ satisfying (30) for all $S, S' \subset (C'_2)_L$. Since $C'_2$ satisfies the size assumption on $C_1$ made in previous paragraph, we can select a further subcode $C'_3 \subset C'_2$ of size $\geq K_L$, so that for $C'_3$ property (30) holds for all $S, S' \subset |L| = L$.

Continuing similarly, we may select a subcode $C'$ of arbitrary $C$ such that (30) holds for all $|S| = |S'| \leq L$ provided that $|C| \geq K_1$.

Next, we show that (30) implies
\[
\|T(S_0) - \sigma(T(S_0))\| \leq C\delta_1,
\]
(31)
where $S_0 \in (C'_L)_L$ is arbitrary and $C = C(L)$ is a constant depending on $L$ only.

Now to prove (31) let $T(S_0) = \{t_v, v \in \mathbb{F}_2^L\}$ and consider an arbitrary transposition $\sigma : [L] \to [L]$. It will be clear that our proof does not depend on what transposition is chosen, so
for simplicity we take $\sigma = \{(L - 1) \leftrightarrow L\}$. We want to show that (30) implies
\[ |t_v - t_{\sigma(v)}| \leq \delta_1. \quad \forall v \in \mathbb{F}_2^L \] (32)
Since transpositions generate permutation group $S_L$, (31) then follows. Notice that (32) is only informative for $v$ whose last two digits are not equal, say $v = [v_0, 0, 1]$. Suppose that $S_0 = \{c_1, \ldots, c_L\}$ given in the ordered form. Let
\[ S = \{c_1, \ldots, c_{L-1}\}, \quad S' = \{c_1, \ldots, c_{L-2}, c_L\} \] (33)
(34)
Joint types $T(S)$ and $T(S')$ are expressible as functions of $T(S_0)$ in particular, the number of occurrences of element $[v_0, 0]$ in $S$ is $t_{[v_0,0]} + t_{[v_0,0,0]}$ and in $S'$ is $t_{[v_0,0,0]} + t_{[v_0,1,0]}$. Thus, from (30) we obtain:
\[ |(t_{[v_0,0]} + t_{[v_0,0,0]}) - (t_{[v_0,0,0]} + t_{[v_0,1,0]})| \leq \delta \]
implying (32) and thus (31).

Finally, we show that (31) implies (29). Indeed, consider the chain
\[ \|T(S) - \hat{T}_L(C')\| = \left\| T(S) - \frac{1}{L!} \sum_{\sigma \in \mathcal{S}_L} \sum_{s' \in \mathcal{C}'_L} \sigma(T(S')) \right\| \]
\[ \leq \frac{1}{L!} \left( \sum_{\sigma \in \mathcal{S}_L} \sum_{s' \in \mathcal{C}'_L} \|T(S) - \sigma(T(S'))\| \right) \]
\[ \leq \frac{1}{L!} \left( \sum_{\sigma \in \mathcal{S}_L} \sum_{s' \in \mathcal{C}'_L} \|T(S) - T(S')\| \right) \]
\[ + \|T(S') - \sigma(T(S'))\| \]
\[ \leq (1 + C)\delta_1. \] (38)
where (36) is by convexity of the norm, (37) is by triangle inequality and (38) is by (30) and (31). Consequently, setting $\delta_1 = \frac{\delta}{1+C}$ we have shown (29).

Before proceeding further we need to define the concept of an average radius (or a moment of inertia):
\[ \overline{\text{rad}}(x_1, \ldots, x_m) \triangleq \min_y \frac{1}{m} \sum_{i=1}^{m} |x_i - y|. \]
Note that the minimizing $y$ can be computed via a per-coordinate majority vote (with arbitrary tie-breaking for even $m$). Consider now an arbitrary subset $S = \{c_1, \ldots, c_L\}$ and define for each $j \geq 0$ the following functions
\[ h_j(S) \triangleq \frac{1}{n} \overline{\text{rad}}(0, \ldots, 0, c_1, \ldots, c_L) \] times
It is easy to find an expression for $h_j(S)$ in terms of the joint-type of $S$:
\[ h_j(S) = \frac{1}{L + j} \left( E[W] - E[|2W - L - j|]\right) \] (39)
\[ \mathbb{P}[W = w] = \sum_{v:|v|=w} t_v, \] (40)
where $t_v$ are components of the joint-type $T(S) = \{t_v, v \in \mathbb{F}_2^L\}$. To check (39) simply observe that if one arranges $L$ codewords of $S$ in an $L \times n$ matrix and also adds $j$ rows of zeros, then computation of $h_j(S)$ can be done per-column: each column of weight $w$ contributes
\[ \min(w, L + j - w) = w - |2w - L - j| \]
to the sum. In view of expression (39) we will abuse notation and write
\[ h_j(T(S)) \overset{\Delta}{=} h_j(S). \]
We now observe that for symmetric codes satisfying (29) average-radii $h_j(S)$ in fact determine the regular radius:

**Lemma 6.** Consider an arbitrary code $C$ satisfying conclusion (29) of Lemma 5. Then for any subset $S = \{c_1, \ldots, c_L\} \subset C$ we have
\[ \left| \text{rad}(0, c_1, \ldots, c_L) - n \cdot \max_j h_j(T_L(C)) \right| \leq 2^L(1 + \delta n), \] (41)
where $j$ in maximization (41) ranges over $\{0, 1, 3, \ldots, 2k + 1, \ldots, L\}$ if $L$ is odd and over $\{0, 2, \ldots, 2k, \ldots, L\}$ if $L$ is even.

**Proof.** For joint-types of size $L$ and all $j \geq 0$ we clearly have (cf. expression (39))
\[ |h_j(T_1) - h_j(T_2)| \leq 2^{L-1}||T_1 - T_2||, \quad \forall T_1, T_2 \in \mathcal{T}_L. \] (42)
We also trivially have
\[ \frac{1}{n} \text{rad}(0, c_1, \ldots, c_L) \geq h_j(S) \quad \forall j \geq 0. \] (43)
Thus from (29) and (42) we already get
\[ \frac{1}{n} \text{rad}(0, c_1, \ldots, c_L) \geq \max_j h_j(T_L(C)) - 2^{L-1}\delta. \]
It remains to show
\[ \frac{1}{n} \text{rad}(0, c_1, \ldots, c_L) \leq \max_j h_j(T_L(C)) + \delta + \frac{2^L}{n}. \] (44)
This evidently requires constructing a good center $y$ for the set $\{0, c_1, \ldots, c_L\}$. To that end fix arbitrary numbers $q = (q_0, \ldots, q_L) \in [0, 1]^L$. Next, for each $v \in \mathbb{F}_2^L$ let $E_v \subset [n]$ be all coordinates on which restriction of $\{c_1, \ldots, c_L\}$ equals $v$. On $E_v$ put $y$ to have a fraction $q_v$ of ones and remaining set to zeros (rounding to integers arbitrarily). Proceed for all $v \in \mathbb{F}_2^L$. Call resulting vector $y(q) \in \mathbb{F}_n^L$.
Denote for convenience $c_0 = 0$. We clearly have
\[ \text{rad}(c_0, c_1, \ldots, c_L) \leq \min_q \max_{p} \sum_{i=0}^{L} p_i |c_i - y(q)|, \] (45)
where $p = (p_0, \ldots, p_L)$ is a probability distribution.
Denote
\[ T(S) = \{t_v, v \in \mathbb{F}_2^L\} \]
\[ \hat{T}_L(C) = \{\bar{t}_v, v \in \mathbb{F}_2^L\} \] (47)
We proceed to computing $|c_i - y(q)|$.

$$|c_i - y(q)| \leq n \sum_{v \in \mathbb{F}_2^n} t_v(q_{w_i}) \{v(i) = 0\}$$

$$+ (1 - q_{w_i}) \{v(i) = 1\} + 2^L,$$

(48)

where $2^L$ comes upper-bounding the integer rounding issues and we abuse notation slightly by setting $v(0) = 0$ for all $v$ (recall that $v(i)$ is the $i$-th coordinate of $v \in \mathbb{F}_2^n$).

By (29) we may replace $t_v$ with $\tilde{t}_v$ at the expense of introducing $2^L \delta n$ error, so we have:

$$|c_i - y(q)| \leq n \sum_{v \in \mathbb{F}_2^n} \tilde{t}_v(q_{w_i}) \{v(i) = 0\}$$

$$+ (1 - q_{w_i}) \{v(i) = 1\} + 2^L(1 + \delta n).$$

(49)

Next notice that the sum over $v$ only depends on whether $i = 0$ or $i \neq 0$ (by symmetry of $\tilde{t}_v$). Furthermore, for any given weight $w$ and $i \neq 0$ we have

$$\sum_{v : |v| = w} 1 \{v(i) = 1\} = \frac{(L^w)_w}{L}.$$ 

Thus, introducing the random variable $\bar{W}$, cf. (39),

$$P[\bar{W} = w] \triangleq \sum_{v : |v| = w} \tilde{t}_v,$$

we can rewrite:

$$\sum_{v \in \mathbb{F}_2^n} \tilde{t}_v(q_{w_i}) \{v(i) = 0\} + (1 - q_{w_i}) \{v(i) = 1\}$$

$$= \frac{1}{L} \mathbb{E}[\bar{W} + (L - 2\bar{W})q_{\bar{W}}].$$

(50)

For $i = 0$ the expression is even simpler:

$$\sum_{v \in \mathbb{F}_2^n} \tilde{t}_v(q_{w_i}) \{v(0) = 0\} + (1 - q_{w_i}) \{v(0) = 1\} = \mathbb{E}[q_{\bar{W}}].$$

Substituting derived upper bound on $|c_i - y(q)|$ into (45) we can see that without loss of generality we may assume $p_1 = \cdots = p_L$, so our upper bound (modulo $O(\delta)$ terms) becomes:

$$\min_q \max_{p \in [0, L-1]} (1 - Lp_1) \mathbb{E}[q_{\bar{W}}] + p_1 \mathbb{E}[\bar{W} + (L - 2\bar{W})q_{\bar{W}}]$$

$$= \min_q \max_{p \in [0, L-1]} p_1 \mathbb{E}[\bar{W}] + \mathbb{E}[q_{\bar{W}}(1 - 2\bar{W}p_1)].$$

(51)

By von Neumann’s minimax theorem we may interchange min and max, thus continuing as follows:

$$= \max_{p_1 \in [0, L-1]} \min_q p_1 \mathbb{E}[\bar{W}] + \mathbb{E}[q_{\bar{W}}(1 - 2\bar{W}p_1)]$$

$$= \max_{p_1 \in [0, L-1]} p_1 \mathbb{E}[\bar{W}] - \mathbb{E}[\bar{W} - (1 - 2\bar{W}p_1)].$$

(52)

The optimized function of $p_1$ is piecewise-linear, so optimization can be reduced to comparing values at slope-discontinuities and boundaries. The point $p_1 = 0$ is easily excluded, while the rest of the points are given by $p_1 = \frac{1}{L}$. with $j$ ranging over the set specified in the statement of Lemma 4. So we continue (52) getting

$$= \max_j \frac{1}{L + j} \left( \mathbb{E}[\bar{W}] - \mathbb{E}[\bar{W} - (1 - 2\bar{W})j] \right)$$

(53)

We can see that expression under maximization is exactly $h_j(\bar{T}_L(C))$ and hence (44) is proved.

Lemma 7. There exist constants $C_1, C_2$ depending only on $L$ such that for any $C \subset \mathbb{F}_2^n$ the joint-type $\bar{T}_L(C)$ is approximately a mixture of product Bernoulli distributions\footnotemark, namely:

$$\left| \bar{T}_L(C) - \frac{1}{n} \sum_{i = 1}^n \text{Bern}^\otimes L(\lambda_i) \right| \leq \frac{C_1}{|C|},$$

(54)

where $\lambda_i = \frac{1}{|C|} \sum_{c \in C} \{c(i) = 1\}$ is the density of ones in the $i$-th column of a $|C| \times n$ matrix representing the code. In particular,

$$h_j(\bar{T}_L(C)) = \frac{1}{n} \sum_j g_j(\lambda_j) \leq \frac{C_2}{|C|},$$

(55)

where functions $g_j$ were defined in (16).

Proof. Second statement (55) follows from the first via (42) and linearity of $h_j(T)$ in the type $T$, cf. (39). To show the first statement, let $M = |C|$, $M_i = \lambda_i M$ and $p_w$ total probability assigned to vectors $v$ of weight $w$ by $\bar{T}_L(C)$. Then by computing $p_w$ over columns of $M \times n$ matrix we obtain

$$p_w = \frac{1}{n} \sum_{i = 1}^n \binom{M_i}{w} \frac{(M - M_i)}{L - w}.$$ 

By a standard estimate we have for all $w = \{0, \ldots, L\}$:

$$\binom{M_i}{w} \frac{(M - M_i)}{L - w} = \binom{L}{w} \lambda^w (1 - \lambda)^{L-w} + O\left(\frac{1}{M}\right),$$

with $O(\cdot)$ term uniform in $w$ and $\lambda_i$. By symmetry of the type $\bar{T}_L(C)$ the result (54) follows.

Lemma 8. Functions $g_j$ defined in (16) are concave on $[0, 1]$.

Proof. Let $W_\lambda \sim \text{Bino}(L, \lambda)$ and $V_\lambda \sim \text{Bino}(L - 1, \lambda)$. Denote for convenience $\bar{\lambda} = 1 - \lambda$ and take $j_0$ to be an integer

\footnotetext{The difference between odd and even $L$ occurs due to the boundary point $p_1 = \frac{1}{L}$, not being a slope-discontinuity when $L$ is odd, so we needed to add it separately.}

\footnotetext{Distribution Bern$^\otimes L(\lambda)$ assigns probability $\lambda^{|v|}(1 - \lambda)^{L - |v|}$ to element $v \in \mathbb{F}_2^n$.}
between 0 and $L$. We have then
\[
\frac{\partial}{\partial \lambda} \mathbb{E} \left[ (W_\lambda - j_0)^+ \right] = \sum_{w=j_0+1}^L \left( \frac{L}{w} \right) (w-j_0) \lambda^w \bar{\lambda}^{L-w} \left\{ w \bar{\lambda}^{-1} - (L-w) \bar{\lambda}^{-1} \right\}
\]
\[= \left( \frac{L}{j_0+1} \right) (j_0+1) \lambda^{j_0} \bar{\lambda}^{L-j_0-1} + \sum_{w=j_0+1}^{L-1} \left( \frac{L}{w+1} \right) (w+1-j_0) (w+1) \lambda^w \bar{\lambda}^{L-w-1} - \left( \frac{L}{w} \right) (w-j_0) (L-w) \lambda^w \bar{\lambda}^{L-w-1}
\]
\[= L \left( \frac{L-j_0}{j_0} \right) \lambda^{j_0} \bar{\lambda}^{L-j_0-1} + \sum_{w=j_0+1}^{L-1} \left( \frac{L-1}{w} \right) \lambda^w \bar{\lambda}^{L-w-1} - \sum_{w=j_0+1}^{L-1} \left( \frac{L-1}{w} \right) \lambda^w \bar{\lambda}^{L-w-1}
\]
\[= L \mathbb{P}[V_\lambda \geq j_0],
\]
where in (57) we shifted the summation by one for the first term under the sum in (56), and in (58) applied identities $(L+1) = \left( \frac{L}{w+1} \right) \frac{w+1}{w+T} = \left( \frac{w}{w+1} \right) \frac{w}{w+T}$. Similarly, if $\theta \in [0,1)$ we have
\[
\frac{\partial}{\partial \lambda} \mathbb{E} \left[ (W_\lambda - j_0 - \theta)^+ \right] = L \mathbb{P}[V_\lambda \geq j_0+1] + (1-\theta) L \mathbb{P}[V_\lambda = j_0].
\]
Similarly, one shows we will need it later in Lemma 9:
\[
\frac{\partial}{\partial \lambda} \mathbb{P}[V_\lambda \geq j_0] = L \mathbb{P}[V_\lambda = j_0 - 1].
\]
Since clearly the function in (60) is strictly increasing in $\lambda$ for any $j_0$ and $\theta$ we conclude that
\[\lambda \mapsto \mathbb{E} \left[ (W_\lambda - j_0 - \theta)^+ \right]
\]
is convex. This concludes the proof of concavity of $g_j$.

**Proof of Lemma 4.** Our plan is the following:

1. Apply Elias-Bassalygo reduction to pass from $C_0^n$ to a subcode $C_0^n$ on an intersection of two spheres $S_{\varepsilon_0,n}$ and $y + S_{\varepsilon_0,n}$.
2. Use Lemma 5 to pass to a symmetric subcode $C''_n \subset C''_n$.
3. Use Lemmas 7-8 to estimate maxima of average radii $h_j$ over $C''_n$.
4. Use Lemma 6 to transport statement about $h_j$ to a statement on $\tau_L(C''_n)$.

We proceed to details. It is sufficient to show that for some constant $C = C(L)$ and arbitrary $\delta > 0$ estimate (24) holds with $\varepsilon_n = C\delta$ whenever $n \geq n_0(\delta)$. So we fix $\delta > 0$ and consider a code $C' \subset S_{\varepsilon_0,n} \subset F_q^n$ with $|C'| \geq \exp(nR + o(n))$. Note that for any $r$, even $m$ with $m/2 \leq \min(r, n-r)$ and arbitrary $y \in S_{r,n}^m$ intersection $\{y + S_{r,n}^m\} \cap S_{r,n}^m$ is isometric to the product of two lower-dimensional spheres:
\[
\{y + S_{r,n}^m\} \cap S_{r,n}^m \cong S_{r-m/2}^r \times S_{m/2}^m.
\]
Therefore, we have for $r = \varepsilon_0 n$ and valid $m$:
\[
\sum_{y \in S_{r,n}^m} |\{y + S_{r,n}^m\} \cap C'| = |C'| \left( \frac{\varepsilon_0 n}{\varepsilon_0 n - m/2} \right) \left( \frac{n(1 - \varepsilon_0)}{m/2} \right).
\]
Consequently, we can select $m = \xi_1 n - o(n)$, where $\xi_1$ defined in (27), so that for some $y \in S_{\varepsilon_0,n}$:
\[
\{|y + S_{\varepsilon_0,n}^m\} \cap C'| > n.
\]
Note that we focus on solution of (27) satisfying $\xi_1 < 2\xi_0(1 - \xi_0)$. For some choices of $R$, $\delta$ and $\varepsilon_0$ choosing $\xi_1 > 2\xi_0(1 - \xi_0)$ is also possible, but such a choice appears to result in a weaker bound.

Next, we let $C'' = \{y + S_{\varepsilon_0,n}^m\} \cap C'$. For sufficiently large $n$ the code $C''$ will satisfy assumptions of Lemma 5 with $K \geq \frac{1}{2}$. Denote the resulting large symmetric subcode $C''$.

Note that because of (62) column-densities $\lambda_i$’s of $C''$, defined in Lemma 7, satisfy (after possibly reordering coordinates):
\[
\sum_{i=1}^{\xi_0 n} \lambda_i = \xi_1 n/2 + o(n), \quad \sum_{i > \xi_0 n} \lambda_i = \xi_1 n/2 + o(n).
\]
Therefore, from Lemmas 7-8 we have
\[
h_j(T_L(C'')) \leq \xi_0 g_j \left( \frac{1 - \xi_1}{2\xi_0} \right) + (1 - \xi_0) g_j \left( \frac{\xi_1}{2(1 - \xi_0)} \right) + \epsilon_n + \frac{C_1}{|C''|}.
\]
where $\epsilon_n \to 0$. Note that by construction the last term in (63) is $O(\delta)$. Also note that the first two terms in (63) equal $\theta_j$ defined in (25).

Finally, by Lemma 6 we get that for any codewords $c_1, \ldots, c_L \in C''$, some constant $C$ and some sequence $\epsilon''_n \to 0$ the following holds:
\[
\frac{1}{n} \text{rad}(0, c_1, \ldots, c_L) \leq \theta(\varepsilon_0, R', L) + \epsilon''_n + C\delta.
\]
By the initial remark, this concludes the proof of Lemma 4.

**C. Proof of Corollary 3**

**Lemma 9.** For any odd $L = 2a + 1$ there exists a neighborhood of $x = \frac{1}{2}$ such that
\[
\max_j g_j(x) = g_1(x),
\]
maximum taken over $j$ equal all the odd numbers not exceeding $L$ and $j = 0$. We also have for some $c > 0$
\[
g_1(x) = \frac{1}{2} \cdot 2^{-L-1} \left( \frac{L}{L-1} \right) + cx + O((2x-1)^2), \quad x \to \frac{1}{2}.
\]
**Proof.** First, the value $g_1(1/2)$ is computed trivially. Then from (60) we have
\[
\frac{d}{dx} g_j(x) = \frac{L}{L+j} \left( 1 - 2P \left( V_x \geq \frac{L+j}{2} \right) \right),
\]
where $j \geq 1$ and $V_x \sim \text{Bino}(x, L-1)$. This implies (65). For future reference we note that (69) (below) and (61) imply
\[
\frac{d}{dx} g_0(x) = 1 - 2P[V_x \geq \frac{L+1}{2}] - P[V_x = \frac{L-1}{2}],
\]
$V_x \sim \text{Bino}(x, L-1)$. (67)
By continuity, (64) follows from showing
\[ g_1(1/2) > \max_{j \in \{0, 3, 5, \ldots, L\}} g_j(1/2). \] (68)

Next, consider \( W_x \sim \text{Bino}(x, L) \) and notice the upper-bound
\[ g_j(x) \leq \frac{1}{L+j} \mathbb{E}[W_x 1\{W_x \leq a\} + (L+j-W_x) 1\{W_x > a+1\}] \]

Then, substituting expression for \( g_1(x) \) we get
\[ g_1(x) - g_0(x) = \frac{1}{L} (\mathbb{P}[W_x > a] - g_1(x)) \] (69)
\[ g_1(x) - g_j(x) \geq \left(\frac{j-1}{L+j}\right) (g_1(x) - \mathbb{P}[W_x > a+1]) \] (70)

Thus, to show (68) it is sufficient to prove that for \( x = 1/2 \) we have
\[ \mathbb{P}[W_x > a+1] < g_1(1/2) < \mathbb{P}[W_x > a+1]. \] (71)

The right-hand inequality is trivial since \( \mathbb{P}[W_{1/2} > a+1] = 1/2 \) while from (65) we know \( g_1(1/2) < 1/2 \). The left-hand inequality, after simple algebra, reduces to showing
\[ \sum_{u=0}^{a-1} (2a + 1 - 2u) \binom{2a + 1}{u} < (2a + 1) \binom{2a + 1}{a}. \]

Notice, that
\[ (n-2u) \binom{n}{u} = n \left( \binom{n-1}{u} - \binom{n-1}{u-1} \right) \forall u \geq 0 \]
and therefore
\[ \sum_{u \leq \ell} (n-2u) \binom{n}{u} = n \binom{n-1}{\ell}. \]

Plugging this identity into the right-hand side of (72) we get
\[ \sum_{u=0}^{a-1} (2a + 1 - 2u) \binom{2a + 1}{u} = (2a + 1) \binom{2a}{a-1} < (2a + 1) \binom{2a + 1}{a} \] (73)
completing the proof of (72).

**Proof of Corollary 3.** We first show that (20) implies (21). To that end, fix a small \( \epsilon > 0 \) so that \( 1/2 - \epsilon \) belongs to the neighborhood existence of which is claimed in Lemma 9. Choose rate so that \( \delta_{L,P_1}(R) = 1/2 - \epsilon \) and notice that this implies \( R = h(\epsilon^2 + o(\epsilon^2)) \),
\[ g_1(\nu) = \nu(1-\nu), \] (78)
\[ g_1(\nu) = \frac{3}{4} \nu - \frac{1}{2} \nu^3, \] (79)
\[ g_3(\nu) = \frac{1}{2} \nu. \] (80)

Note that the right-hand side of (17) is precisely equal to
\[ \delta g_1 \left( \frac{1-\xi_1}{2\delta} \right) + (1-\delta) g_1 \left( \frac{\xi_1}{2(1-\delta)} \right). \]
So this corollary simply states that for \( L = 3 \) the maximum in (13) is achieved at \( j = 1, \xi_0 = \delta \). Let us restate this last statement rigorously: The maximum

\[
\max_{j \in \{0,1,2\}} \max_{\xi_0 \in \delta} g_j \left( 1 - \frac{x}{2(1 - \xi_0)} \right) + (1 - \xi_0) g_j \left( \frac{x}{2(1 - \xi_0)} \right)
\]

(81)
is achieved at \( j = 1, \xi_0 = \delta \). Here \( x = x(\xi_0, \beta) \) is a solution of

\[
2(h(\beta) - E_\beta(\xi_0)) = h(\xi_0) - \xi_0 h \left( \frac{x}{2(1 - \xi_0)} \right) - (1 - \xi_0) h \left( \frac{x}{2(1 - \xi_0)} \right)
\]

(82)

For notational convenience we will denote the function under maximization in (81) by \( g_j(\xi_0, x) \).

We proceed in two steps:

- First, we estimate the maximum over \( \xi_0 \) for \( j = 0 \) as follows:
  
  \[
  \max_{\xi_0} g_0(\xi_0, x) \leq \frac{\log 2 - R}{4 \log 2} \cdot \left( 1 - \frac{1 - \delta}{a_{\max}(1 - a_{\max})} \right) + (1 - \xi_0) g_0(a_{\min}),
  \]

  (83)

  where \( a_{\max}, a_{\min} \leq \frac{1}{2} \alpha \) are given by

  \[
  a_{\max} = h^{-1}(\log 2 - R),
  \]

  (84)

  and

  \[
  a_{\min} = h^{-1} \left( \log 2 - \frac{R}{1 - \delta} \right).
  \]

  (85)

- Second, we prove that for \( j = 1 \) function

  \[
  \xi_0 \mapsto g_j(\xi_0, x(\xi_0))
  \]

  is monotonically increasing.

Once these two steps are shown, it is easy to verify (for example, numerically) that \( g_1(\delta, x(\delta)) \) exceeds both \( \frac{1}{2} \delta \) (term corresponding to \( j = 3 \) in (81)) and the right-hand side of (83) (term corresponding to \( j = 0 \)). Notice that this relation holds for all rates. Therefore, maximum in (81) is indeed attained at \( j = 1, \xi_0 = \delta \).

One trick that will be common to both steps is the following. From the proof of Lemma 4 it is clear that the estimate (24) is monotonic in \( R' \). Therefore, in equation (82) we may replace \( E_\beta(\xi) \) with any upper-bound of it. We will use the well-known upper-bound, which leads to binomial estimates of spectrum components [15, (46)]:

\[
E_\beta(\xi_0) \leq \frac{1}{2} \left( \log 2 + h(\beta) - h(\xi_0) \right).
\]

(86)

Furthermore, it can also be argued that maximum cannot be attained by \( \xi_0 \) so small that

\[
h(\beta) - \frac{1}{2} \left( \log 2 + h(\beta) - h(\xi_0) \right) < 0.
\]

So from now on, we assume that

\[
h^{-1}(\log 2 - h(\beta)) \leq \xi_0 \leq \delta,
\]

and that \( x = x(\xi_0) \leq 2\xi_0(1 - \xi_0) \) is determined from the equation:

\[
\log 2 - R = \xi_0 h \left( \frac{x}{2\xi_0} \right) + (1 - \xi_0) h \left( \frac{x}{2(1 - \xi_0)} \right)
\]

(87)

(we remind \( R = h(\beta) \)).

We proceed to demonstrating (83). For convenience, we introduce

\[
a_1 \triangleq 1 - \frac{x}{2\xi_0}, \quad a_2 \triangleq \frac{x}{2 - 2\xi_0}.
\]

(88)

(89)

By constraints on \( x \) it is easy to see that

\[
0 \leq a_2 \leq \min(a_1, 1 - a_1).
\]

Therefore, we have

\[
\log 2 - R = \xi_0 h(a_1) + (1 - \xi_0) h(a_2) \geq h(a_2)
\]

and thus \( a_2 \leq a_{\max} \) defined in (84). Similarly, we have

\[
\log 2 - R = \xi_0 h(a_1) + (1 - \xi_0) h(a_2) \leq \xi_0 \log 2 + (1 - \xi_0) h(a_2),
\]

and since \( \xi_0 \leq \delta \) we get that \( a_2 \geq a_{\min} \) defined in (85).

Next, notice that \( \frac{h(x)}{x(1-x)} \) is decreasing on \( (0,1/2] \). Thus, we have

\[
h(a_1) \geq g_0(a_1)4 \log 2
\]

(90)

\[
h(a_2) \geq h(a_{\max}) \frac{g_0(a_{\max})}{g_0(a_{\min})}
\]

\[
= \log 2 - R
\]

(91)

\[
\frac{a_{\max}}{a_{\min}} g_0(a_{\min}) \geq c \cdot g_0(a_{\min}).
\]

(92)

Rearranging terms yield (83).

We proceed to proving monotonicity of (82). The technique we will use is general (can be applied to \( L > 3 \) and \( j > 1 \)), so we will avoid particulars of \( L = 3, j = 1 \) case until the final step.

Notice that regardless of the function \( g(\nu) \) we have the equivalence:

\[
\frac{d}{d\xi_0} \xi_0 g(a_1) + (1 - \xi_0) g(a_2) \geq 0 \iff \frac{1}{2} \frac{dx}{d\xi_0} \left( g'(a_2) - g'(a_1) \right) \geq \int_{a_1}^{a_2} (1 - x)(-g''(x))dx - g'(a_2),
\]

(96)

where we recall definition of \( a_1, a_2 \) in (88)-(89). Differentiating (87) in \( \xi_0 \) (and recalling that \( R \) is fixed, while \( x = x(\xi_0) \) is an implicit function of \( \xi_0 \)) we find

\[
\frac{dx}{d\xi_0} = -2 \frac{\log \frac{a_2}{a_1}}{\log \frac{a_2}{a_1} - \frac{a_1}{a_2}} < 0.
\]

Next, one can notice that the map \((\xi_0, x, R) \mapsto (a_1, a_2)\) is a bijection onto the region

\[
\{(a_1, a_2) : 0 \leq a_1 \leq 1, 0 \leq a_2 \leq a_1(1 - a_1)\}.
\]

(97)
With the inverse map given by

\[
\xi_0 = \frac{a_2}{1 - a_1 + a_2}, \quad x = \frac{2a_2^2}{1 - a_1 + a_2}, \quad R = \log 2 - \xi_0 h(a_1) - (1 - \xi_0) h(a_2).
\]

Thus, verifying (96) can as well be done for all \(a_1, a_2\) inside the region (97). Substituting \(g = g_1\) into (96) we get that monotonicity in (82) is equivalent to a two-dimensional inequality:

\[
-2 \log \frac{1 - a_2}{a_1} \cdot (a_1^2 - a_2^2) \geq (2a_1^2 - \frac{4}{3}(a_1^3 - a_2^3) - 1) \log \frac{1 - a_2}{a_1 - 1} - 1. \tag{98}
\]

It is possible to verify numerically that indeed (98) holds on the set (97). For example, one may first demonstrate that it is sufficient to restrict to \(a_2 = 0\) and then verify a corresponding inequality in \(a_1\) only. We omit mechanical details. ☐

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