Managing $[\text{subscript } 5]$ in Dimensional Regularization II: the Trace with more $[\text{subscript } 5]$'s

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Managing $\gamma_5$ in Dimensional Regularization II: 
the Trace with more $\gamma_5$’s

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Abstract

In the present paper we evaluate the anomaly for the abelian 
axial current in a non abelian chiral gauge theory, by using di-
mensional regularization. This amount to formulate a procedure 
for managing traces with more than one $\gamma_5$.

The suggested procedure obeys Lorentz covariance and cyclic-
ity, at variance with previous approaches (e.g. the celebrated ’t 
Hooft and Veltman’s where Lorentz is violated)

The result of the present paper is a further step forward in 
the program initiated by a previous work on the traces involving 
a single $\gamma_5$. The final goal is an unconstrained definition of $\gamma_5$ 
in dimensional regularization. Here, in the evaluation of the 
anomaly, we profit of the axial current conservation equation, 
when radiative corrections are neglected. This kind of tool is 
not always exploited in field theories with $\gamma_5$, e.g. in the use of 
dimensional regularization of infrared and collinear divergences.

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1 Introduction

In paper I ([1]) we solved the problem of defining the trace of gamma’s with zero or one $\gamma_5$ in generic $D$ dimensions [2]-[4], by using an integral representation. The $\gamma_5$ problem has been widely discussed in the literature [5]-[29].

The new representation sets the rules for managing the algebra in a Lorentz covariant formalism, consistent with the cyclicity of the trace. The ABJ anomaly [30]-[33] and the LFE (Local Functional Equation) [34]-[37] associated to the abelian local chiral transformations have been verified by explicit calculations.

In the present paper we consider the case of a trace with more than one $\gamma_5$, that frequently occurs in actual Feynman amplitude calculations. There is a further cogent reason to consider such a case, i.e. the need to formulate local chiral non abelian gauge transformations, as in the electroweak model. Were it not possible to do it in a consistent way, then the $\gamma_5$ manipulation in generic dimension would be of limited significance.

In this work we go through the explicit calculation of the divergence of the abelian axial current

$$\partial_\mu J^5_\mu$$

up to one loop correction in a $SU(2)$ nonabelian chiral (massless) theory. We use dimensional regularization and the limit $D = 4$ is taken.

We make some assumptions, hoping that they are mutually consistent:

1. Gamma’s and $\gamma_\chi$ (our $\gamma_5$ in generic $D$) form an associative algebra.

2. We study the generic trace where the Lorentz indices are all contracted with vectors (e.g. momenta and polarization vectors)

$$Tr(p) \equiv Tr\left(\cdots \gamma_\chi \cdots \gamma_{\mu j} \cdots \gamma_\chi \cdots \gamma_{\mu h} \cdots \right) \cdots p_{\mu j} \cdots p_{\mu h} \cdots.$$ (2)

Then our Ansatz is that:

2. In a neighborhood of $D = 4$ the trace admits an expansion

$$Tr(p) = \sum_{h=0} A_h(p)(D - 4)^h,$$ (3)

where $A_h(p)$ are Lorentz invariants in $D = 4$ dimensions (the tensor $\varepsilon_{\mu \nu \rho \sigma}$ might be present).
3. The limit $D = 4$ is smooth. For instance

$$\{\gamma_\chi, \gamma_\mu\} = \mathcal{O}(D - 4), \forall \mu.$$  \hspace{1cm} (4)

To our opinion the integral representation of the trace with zero or one $\gamma_\chi$, thoroughly studied in I, can be extended to the case of multiple $\gamma_\chi$. However we have not been able yet to continue our integral representation for any number of $\gamma_\chi$ to non integer $D$; i.e. the manipulations, requiring an integer $D$, provide little help in order to extend the results to non integer $D$. For these reasons and for sake of brevity and conciseness we do not discuss here the extension to multiple $\gamma_\chi$ of the results in I. Instead we manipulate in a formal way the gamma’s, assuming that they exist somehow.

For instance the trace $Tr(\gamma_\chi\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)$ need not to be given. In the evaluation of the anomaly only the following quantity is required

$$Tr(\{\gamma_\chi, \gamma_\alpha\}\gamma_\beta\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = Tr(\{\gamma_\alpha, \gamma_\beta\}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma).$$ \hspace{1cm} (5)

The strategy for evaluating the trace with many $\gamma_\chi$ turns out to be very simple at the one-loop level.

1. We move around, inside a trace, a $\gamma_\chi$ by introducing the anticommutator. For instance

$$\gamma_\chi\gamma_\mu = -\gamma_\mu\gamma_\chi + \{\gamma_\chi, \gamma_\mu\}$$

$$[\gamma_\chi, \gamma_\chi] = 0.$$ \hspace{1cm} (6)

2. Once the anticommutator $\{\gamma_\chi, \gamma_\mu\}$ is introduced into the trace we get only $\mathcal{O}(D - 4)$ quantities or of higher order in $D - 4$.

3. If we need only terms of first order in $D - 4$ and $\{\gamma_\chi, \gamma_\mu\}$ is present, then we can use the $D = 4$ algebra in the subsequent manipulation (e.g. $\gamma_\chi^2 = 1$ and $\{\gamma_\chi, \gamma_\mu\} = 0$).

4. Eventually the trace contains at most one $\gamma_\chi$, if $\{\gamma_\chi, \gamma_\mu\}$ is present and if only first $D - 4$ order terms are required.

Trace with at most one $\gamma_\chi$ have been dealt in I.

To summarize, the method is very simple and straightforward. Once the $\mathcal{O}(D - 4)$ factor is introduced into the trace via a single anticommutator
\{\gamma_x, \gamma_\mu\}$, the $D = 4$ naive algebra can be used
\[
\begin{align*}
\gamma_x p_1 \ldots p_k \gamma_x & \rightarrow (-)^k p_1 \ldots p_k \\
\gamma_x^2 & = 1 .
\end{align*}
\]
However powers of $\gamma_x$ need some care as it is discussed in Section 2.

In the present paper we apply the above outlined method to the evaluation of the anomaly present in the operator (1). First we organize all contributions to the operator $\partial_\mu \phi^5$ in such a way that they identically vanish if one uses the naive $D = 4$ algebra (i.e. if poles in $D = 4$ are neglected). With this procedure we can factorize $\{\gamma_x, \gamma_\mu\}$ in the trace. Then the evaluation of the anomaly is straightforward.

\section{More Algebraic Properties}

The algebra of $\gamma_x$ with the other gamma’s is not know. Thus the algebraic manipulations go around this difficulty. As an example, used frequently in I, we quote the following identity
\[
\begin{align*}
Tr\left(\{\gamma_\alpha, \gamma_x\}\gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\lambda \gamma_\mu\right) & \delta_{\alpha \lambda} = Tr\left(\gamma_x \{\gamma_\alpha, \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\lambda \gamma_\mu\}\right) \delta_{\alpha \lambda} \\
& = (2 - D)Tr\left(\gamma_x \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\mu\right) + Tr\left(\gamma_x \gamma_\lambda \begin{bmatrix} (6 - D)\gamma_\rho \gamma_\beta \gamma_\sigma - 4(\delta_\rho_\beta \gamma_\sigma - \delta_\rho_\sigma \gamma_\beta + \delta_\sigma_\beta \gamma_\rho) \gamma_\mu \end{bmatrix}\right) \\
& = Tr\left(\gamma_x \begin{bmatrix} (2(D - 4)\gamma_\rho \gamma_\beta \gamma_\sigma - 4(\delta_\rho_\beta \gamma_\sigma - \delta_\rho_\sigma \gamma_\beta + \delta_\sigma_\beta \gamma_\rho) \gamma_\mu \end{bmatrix}\right)
\end{align*}
\]
which is zero both for $D = 4$ and $D = 2$, as it should be.

Here we list some rules and some caveat. It should be reminded that the naive $D = 4$ algebra can be used only under the protection of a $O(D - 4)$ factor in the trace. For instance
\[
\gamma_x^2 = 1
\]
cannot be used under all circumstances. Here is an example of some unpleasant difficulty
\[
\begin{align*}
Tr\left(\gamma_\mu \gamma_x \gamma_\alpha \gamma_\rho \gamma_x \gamma_\beta \gamma_\sigma \gamma_\lambda \gamma_x\right) & = -Tr\left(\gamma_x \gamma_\mu \gamma_\alpha \gamma_\rho \gamma_x \gamma_\beta \gamma_\sigma \gamma_\lambda \gamma_x\right) + Tr\left(\{\gamma_\mu, \gamma_x\}\gamma_\alpha \gamma_\rho \gamma_x \gamma_\beta \gamma_\sigma \gamma_\lambda \gamma_x\right) \\
& = -Tr\left(\gamma_x \gamma_\mu \gamma_\alpha \gamma_\rho \gamma_x \gamma_\beta \gamma_\sigma \gamma_\lambda \gamma_x\right) - Tr\left(\{\gamma_\mu, \gamma_x\}\gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\lambda \gamma_x\right) \\
& = -Tr\left(\gamma_x \gamma_\mu \gamma_\alpha \gamma_\rho \gamma_x \gamma_\beta \gamma_\sigma \gamma_\lambda \gamma_x\right) - Tr\left(\{\gamma_\mu, \gamma_x\}\gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\lambda \gamma_x\right)
\end{align*}
\]
But also

\[
\begin{align*}
    Tr \left( \gamma_\mu \gamma_\chi \gamma_\alpha \gamma_\nu \gamma_\chi \gamma_\beta \gamma_\sigma \gamma_\iota \right) &= Tr \left( \gamma_\chi \gamma_\mu \gamma_\chi \gamma_\alpha \gamma_\nu \gamma_\chi \gamma_\beta \gamma_\sigma \gamma_\iota \right) \\
    &= -Tr \left( \gamma_\mu \gamma_\chi \gamma_\alpha \gamma_\nu \gamma_\chi \gamma_\beta \gamma_\sigma \gamma_\iota \right) + Tr \left( \{\gamma_\chi, \gamma_\mu\} \gamma_\chi \gamma_\alpha \gamma_\nu \gamma_\chi \gamma_\beta \gamma_\sigma \gamma_\iota \right) \\
    &= -Tr \left( \gamma_\mu \gamma_\chi \gamma_\alpha \gamma_\nu \gamma_\chi \gamma_\beta \gamma_\sigma \gamma_\iota \right) + Tr \left( \{\gamma_\chi, \gamma_\mu\} \gamma_\alpha \gamma_\nu \gamma_\beta \gamma_\sigma \gamma_\iota \right) \\
    &= -Tr \left( \gamma_\mu \gamma_\chi \gamma_\alpha \gamma_\nu \gamma_\chi \gamma_\beta \gamma_\sigma \gamma_\iota \right) + Tr \left( \{\gamma_\chi, \gamma_\mu\} \gamma_\alpha \gamma_\nu \gamma_\beta \gamma_\sigma \gamma_\iota \right)
    \end{align*}
\]

(11)

Thus eqs. (10) and (11) are in contradiction if we use \( \gamma_\chi^2 = 1 \). The last identity can be used only inside a trace where a \( \mathcal{O}(D - 4) \) term already is present.

Moreover one can easily derive

\[
Tr \left( \left[ \gamma_\mu, \gamma_\chi \gamma_\chi \right] \gamma_\alpha \gamma_\nu \gamma_\chi \gamma_\beta \gamma_\sigma \gamma_\iota \right) = 2Tr \left( \{\gamma_\chi, \gamma_\mu\} \gamma_\alpha \gamma_\nu \gamma_\beta \gamma_\sigma \gamma_\iota \right)
\]

(12)

which shows once more how \( \gamma_\chi^2 \) is difficult object to deal with.

In some cases we can use \( \gamma_\chi^2 = 1 \) in proximity of \( D = 4 \). In our calculation we encounter two cases of this sort.

\[
\begin{align*}
    Tr \left( \left[ \gamma_\mu, \gamma_\chi^2 \gamma_\alpha \gamma_\nu \gamma_\beta \right] \right) \\
    Tr \left( \left[ \gamma_\mu, \gamma_\chi^2 \right] \gamma_\alpha \gamma_\nu \gamma_\beta \gamma_\sigma \gamma_\iota \right)
    \end{align*}
\]

(13)

We can easily prove that around \( D = 4 \) they can be neglected. For instance

\[
\begin{align*}
    Tr \left( \left[ \gamma_\mu, \gamma_\chi^2 \right] \gamma_\alpha \gamma_\nu \gamma_\beta \right) \delta_{\mu\alpha} &= (D - 4)Tr \left( \gamma_\chi^2 \gamma_\rho \gamma_\beta \right) + 4\delta_{\rho\beta} Tr \left( \gamma_\chi^2 \right) \\
    -DT Tr \left( \gamma_\chi^2 \gamma_\rho \gamma_\beta \right) &= 0
    \end{align*}
\]

(14)

and

\[
\begin{align*}
    Tr \left( \left[ \gamma_\mu, \gamma_\chi^2 \right] \gamma_\alpha \gamma_\nu \gamma_\beta \gamma_\sigma \gamma_\iota \right) \delta_{\mu\alpha} &= (D - 8)Tr \left( \gamma_\chi^2 \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \right) \\
    +4\delta_{\rho\beta} Tr \left( \gamma_\chi^2 \gamma_\sigma \gamma_\iota \right) - 4\delta_{\rho\sigma} Tr \left( \gamma_\chi^2 \gamma_\beta \gamma_\iota \right) + 4\delta_{\rho\iota} Tr \left( \gamma_\chi^2 \gamma_\sigma \gamma_\beta \right) \\
    +4\delta_{\beta\sigma} Tr \left( \gamma_\chi^2 \gamma_\rho \gamma_\iota \right) - 4\delta_{\beta\iota} Tr \left( \gamma_\chi^2 \gamma_\rho \gamma_\sigma \right) + 4\delta_{\sigma\iota} Tr \left( \gamma_\chi^2 \gamma_\rho \gamma_\beta \right) \\
    -DT Tr \left( \gamma_\chi^2 \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \right) &= 0
    \end{align*}
\]

(15)

are compatible with both traces in eq. (13) being zero at \( D \sim 4 \).
3 The Anomaly of isoscalar $J^5_\mu$ in Chiral Nonabelian
Gauge Theories

In chiral theory every vertex carries a factor

$$\frac{1}{2}(1 + \gamma_\chi).$$  (16)

The triangular graph gives the amplitude (a factor $i^3$ from fermion propagators, a factor $i^2$ for interaction vertexes and a factor $-1$ from fermion loop. Total $-i$)

$$T_{\mu\rho\sigma}(k, p) = -i \int \frac{d^Dq}{(2\pi)^D} \frac{1}{(q-k)^2 q^2 (q+p)^2}$$

$$Tr \left( \gamma_\mu \gamma_\chi (q-k)_\alpha \gamma_\alpha \gamma_\rho (1 + \gamma_\chi) q_\beta \gamma_\beta \gamma_\sigma (1 + \gamma_\chi)(q+p) \gamma_\iota \right).$$  (17)

The crossed graph will be added later on. The trace on the internal group indices contributes by a factor

$$Tr(t_a t_b) = \frac{1}{2} \delta_{ab}. \quad (18)$$

Eventually we consider the divergence of the current (eq. (1))

$$i (p + k)_\mu T_{\mu\rho\sigma}(k, p) = \frac{1}{4} (q + p - q + k)_\mu$$

$$\int \frac{d^Dq}{(2\pi)^D} \left\{ \frac{Tr \left( \gamma_\mu \gamma_\chi (q-k)_\alpha \gamma_\alpha \gamma_\rho (1 + \gamma_\chi) q_\beta \gamma_\beta \gamma_\sigma (1 + \gamma_\chi)(q+p) \gamma_\iota \right)}{(q-k)^2 q^2 (q+p)^2} - \frac{Tr \left( (q-k)_\mu \gamma_\rho \gamma_\chi (q-k)_\alpha \gamma_\alpha \right)}{(q-k)^2}$$

$$\frac{\gamma_\rho (1 + \gamma_\chi) q_\beta \gamma_\beta \gamma_\sigma (1 + \gamma_\chi)(q+p) \gamma_\iota}{q^2 (q+p)^2}$$

$$+ \frac{Tr \left( \gamma_\chi (q-k)_\alpha \gamma_\alpha \gamma_\rho (1 + \gamma_\chi) q_\beta \gamma_\beta \gamma_\sigma (1 + \gamma_\chi) \right)}{(q-k)^2 q^2 (q+p)^2} \right\} \} \quad (19)$$

The crossed graph yields

$$i (p + k)_\mu T_{\mu\rho\sigma}(p, k) = \frac{1}{4} (q + k - q + p)_\mu$$

$$\int \frac{d^Dq}{(2\pi)^D} \left\{ \frac{Tr \left( \gamma_\mu \gamma_\chi (q-p)_\alpha \gamma_\alpha \gamma_\sigma (1 + \gamma_\chi) q_\beta \gamma_\beta \gamma_\rho (1 + \gamma_\chi)(q+k) \gamma_\iota \right)}{(q-p)^2 q^2 (q+k)^2} \right\} \} \quad (19)$$
\[
\begin{align*}
&= \frac{1}{4} \int \frac{d^D q}{(2\pi)^D} \left\{ -T r \left( \frac{(q - p)\mu\gamma_\mu\gamma_\chi(q - p)\alpha\gamma_\alpha}{(q - p)^2} \right) \\
&\quad \frac{\gamma_\sigma(1 + \gamma_\chi) q_\beta\gamma_\beta \gamma_\rho (1 + \gamma_\chi)(q + k)\gamma_\iota}{q^2(q + k)^2} \\
&\quad + T r \left( \frac{\gamma_\chi(q - p)\alpha\gamma_\alpha \gamma_\sigma(1 + \gamma_\chi) q_\beta\gamma_\beta \gamma_\rho (1 + \gamma_\chi)}{(q - p)^2q^2} \right) \right\}. \\
\end{align*}
\]

(20)

We shift the variable \( q \to q - k \) in the first integral and \( q \to q + p \) in the second of eq. (20).

\[
\begin{align*}
&i (p + k)\mu T_{\mu\sigma\rho}(p, k) \\
&= \frac{1}{4} \int \frac{d^D q}{(2\pi)^D} \left\{ -T r \left( \frac{(q - k - p)\mu\gamma_\mu\gamma_\chi(q - k - p)\iota\gamma_\iota}{(q - k - p)^2} \right) \\
&\quad \frac{\gamma_\sigma(1 + \gamma_\chi) (q - k)\alpha\gamma_\alpha \gamma_\rho (1 + \gamma_\chi)q_\beta\gamma_\beta}{q^2(q - k)^2} \\
&\quad + T r \left( \frac{\gamma_\chi q_\beta\gamma_\beta \gamma_\sigma(1 + \gamma_\chi) (q + p)\iota\gamma_\iota \gamma_\rho (1 + \gamma_\chi)}{(q + p)^2q^2} \right) \right\}. \\
\end{align*}
\]

(21)

By inspection one sees that the first term in eq. (19) cancels the second term in eq. (21) if one uses naively the algebra in \( D = 4 \). The same happens to the second term in eq. (19) with the first term in eq. (21). Our strategy is to find the anomaly in the lack of these cancellations, when radiative corrections are taken into account. As an example we deal with one of these two cases. Thus we have

\[
\begin{align*}
&\frac{1}{4} \int \frac{d^D q}{(2\pi)^D} \left\{ -T r \left( \frac{(q - k)\mu\gamma_\mu\gamma_\chi(q - k)\alpha\gamma_\alpha}{(q - k)^2} \right) \\
&\quad \frac{\gamma_\rho(1 + \gamma_\chi) q_\beta\gamma_\beta \gamma_\sigma(1 + \gamma_\chi)(q + p)\iota\gamma_\iota}{q^2(q + p)^2} \\
&\quad + T r \left( \frac{\gamma_\chi q_\beta\gamma_\beta \gamma_\sigma(1 + \gamma_\chi) (q + p)\iota\gamma_\iota \gamma_\rho (1 + \gamma_\chi)}{(q + p)^2q^2} \right) \right\} \\
&= \frac{1}{4} \int \frac{d^D q}{(2\pi)^D} \left\{ -T r \left( \frac{(q - k)\mu\gamma_\mu\gamma_\chi(q - k)\alpha\gamma_\alpha}{(q - k)^2} \right) \\
&\quad \frac{\gamma_\rho(1 + \gamma_\chi) q_\beta\gamma_\beta \gamma_\sigma(1 + \gamma_\chi)(q + p)\iota\gamma_\iota}{q^2(q + p)^2} \right\}. \\
\end{align*}
\]

7
\[
\frac{Tr\left(q_\beta \gamma_\beta \gamma_\sigma (1 + \gamma_X)(q + p)\gamma_\iota \left(-\gamma_X \gamma_\rho + \{\gamma_X, \gamma_\rho\}\right)(1 + \gamma_X)\right)}{(q + p)^2 q^2}
\]
\[
= \frac{1}{4} \int \frac{d^Dq}{(2\pi)^D} \left\{ -Tr\left(\{\gamma_\mu, \gamma_X\} \gamma_\alpha \gamma_\rho (1 + \gamma_X) \gamma_\beta \gamma_\sigma (1 + \gamma_X)\right) \right.
\]
\[
+ Tr\left(\gamma_\beta \gamma_\sigma (1 + \gamma_X) \gamma_\iota \{\gamma_X, \gamma_\rho\}(1 + \gamma_X)\right) \frac{q_\beta (q + p)\gamma_\iota}{(q + p)^2 q^2} \left(\{\gamma_\mu, \gamma_X\} \gamma_\alpha \gamma_\rho (1 + \gamma_X) \gamma_\beta \gamma_\sigma (1 + \gamma_X)\right) \right\}
\]

(22)

Eq. (22) gives a contribution to the triangular graph anomaly. The cross term will be added later on. Noticeable is the emerging inside the trace of the factors \{\gamma_\mu, \gamma_X\} and \{\gamma_X, \gamma_\rho\} of order \mathcal{O}(D - 4).

3.1 Reduction of \gamma_X’s

We proceed to remove all \gamma_X’s in eq. (22) where it is possible. The guiding idea is that the presence in the trace of the factors \{\gamma_\mu, \gamma_X\}, which is of order \mathcal{O}(D - 4), allows us the use

\[
\{\gamma_X, \gamma_\nu\} = 0, \forall \nu
\]

\[
\gamma_X^2 = 1
\]

for all the other remaining \gamma_X’s. The generic value \(D\) is kept throughout the computation and the limit \(\gamma_X \to \gamma_5\) is taken as a last step of the algebraic manipulation of \{\gamma_\mu, \gamma_X\}.

Thus we consider the gamma content of the first term in eq. (19)

\[
Tr\left(\{\gamma_\mu, \gamma_X\} \gamma_\alpha \gamma_\rho (1 + \gamma_X) \gamma_\beta \gamma_\sigma (1 + \gamma_X) \gamma_\iota\right)
\]

\[
= Tr\left(\{\gamma_\mu, \gamma_X\} \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota\right)
\]

\[
+ Tr\left(\{\gamma_\mu, \gamma_X\} \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_X \gamma_\iota\right)
\]

\[
+ Tr\left(\{\gamma_\mu, \gamma_X\} \gamma_\alpha \gamma_\rho \gamma_X \gamma_\beta \gamma_\sigma \gamma_\iota\right)
\]

\[
+ Tr\left(\{\gamma_\mu, \gamma_X\} \gamma_\alpha \gamma_\rho \gamma_X \gamma_X \gamma_\beta \gamma_\sigma \gamma_\iota\right)
\]

(24)

The first term in the RHS of eq. (24) gives

\[
Tr\left(\{\gamma_\mu, \gamma_X\} \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota\right) = Tr\left(\gamma_X \{\gamma_\mu, \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota\}\right)
\]

(25)
The fourth term in the RHS of eq. (24) gives
\[
TR \left( \{\gamma_\mu, \gamma_X\} \gamma_\alpha \gamma_\rho \gamma_\chi \gamma_\beta \gamma_\sigma \gamma_\chi \gamma_\iota \right) = TR \left( \{\gamma_\mu, \gamma_X\} \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\chi \gamma_\iota \right)
\]
\[
= TR \left( \{\gamma_\mu, \gamma_X\} \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\chi \gamma_\iota \right) = TR \left( \gamma_X \{\gamma_\mu, \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\chi \gamma_\iota \} \right) \quad (26)
\]
Finally the first and the fourth together yield
\[
TR \left( \{\gamma_\mu, \gamma_X\} \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\chi \gamma_\iota \right) + TR \left( \{\gamma_\mu, \gamma_X\} \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_X \gamma_\iota \right)
\]
\[
= 2TR \left( \gamma_X \{\gamma_\mu, \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\chi \gamma_\iota \} \right).
\]
(27)
Now we consider the second and third terms in eq. (24) i.e. where an even number of $\gamma_X$ is present.
\[
TR \left( \{\gamma_\mu, \gamma_X\} \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\chi \gamma_\iota \right) + TR \left( \{\gamma_\mu, \gamma_X\} \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\chi \gamma_\iota \right)
\]
\[
= TR \left( \{\gamma_\mu, \gamma_X^2\} \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\chi \gamma_\iota \right) = 0 \quad (28)
\]
according to the arguments of Section 2.
The same analysis has to be performed on the gamma content of the second term in eq. (22) which should match the first eq. (22) or of eq. (19)
\[
TR \left( \{\gamma_X, \gamma_\rho\} (1 + \gamma_X) \gamma_\beta \gamma_\sigma \gamma_X (1 + \gamma_X) \gamma_\iota \right)
\]
\[
= TR \left( \{\gamma_X, \gamma_\rho\} \gamma_\beta \gamma_\sigma \gamma_\iota \right)
\]
\[
+ TR \left( \{\gamma_X, \gamma_\rho\} \gamma_X \gamma_\beta \gamma_\sigma \gamma_\iota \right)
\]
\[
+ TR \left( \{\gamma_X, \gamma_\rho\} \gamma_\beta \gamma_\sigma \gamma_X \gamma_\iota \right)
\]
\[
+ TR \left( \{\gamma_X, \gamma_\rho\} \gamma_X \gamma_\beta \gamma_\sigma \gamma_X \gamma_\iota \right) \quad (29)
\]
We elaborate on the single terms as for eq. (24)
\[
= TR \left( \gamma_X \{\gamma_\rho, \gamma_\beta \gamma_\sigma \gamma_\iota \} \right)
\]
\[
+ TR \left( \{\gamma_\rho, \gamma_X^2\} \gamma_\beta \gamma_\sigma \gamma_\iota \right)
\]
\[
+ TR \left( \{\gamma_X, \gamma_\rho\} \gamma_X^2 \gamma_\beta \gamma_\sigma \gamma_\iota \right) \quad (30)
\]
where now all terms are zero for $D \sim 4$.
Finally the only surviving of the gamma’s algebra is the term in the RHS of eq. (27)
\[
i (p + k)_\mu (T_{\mu \rho \sigma}(k, p) + T_{\mu \sigma \rho}(p, k))
\]
\[
\frac{1}{4} \int \frac{d^D q}{(2\pi)^D} \left\{ 2Tr \left( \gamma_\chi \{ \gamma_\mu, \gamma_\alpha \gamma_\rho, \gamma_\gamma, \gamma_\sigma, \gamma_i \} \right) \frac{(q - k)_\mu (q - k)_\alpha q_\beta (q + p)_\iota}{(q - k)^2 q^2 (q + p)^2} \right. \\
+ (k \leftrightarrow p)(\rho \leftrightarrow \sigma) \right\}.
\]

(31)

### 3.2 Symmetric Integration

We use Feynman parameterization in order to perform a symmetric integration over \( q \)

\[
i (p + k)_\mu (T_{\mu\rho\sigma}(k, p) + T_{\mu\sigma\rho}(p, k))
= \frac{1}{4^2} \int_0^1 dx \int_0^x dy \int \frac{d^D q}{(2\pi)^D} \left\{ 2Tr \left( \gamma_\chi \{ \gamma_\mu, \gamma_\alpha \gamma_\rho, \gamma_\gamma, \gamma_\sigma, \gamma_i \} \right) \frac{(q + r - k)_\mu (q + r - k)_\alpha (q + r)_\beta (q + r + p)_\iota}{(q^2 - \Delta)^3} \right. \\
+ (k \leftrightarrow p)(\rho \leftrightarrow \sigma) \right\}
\]

(32)

with

\[
r_\nu \equiv (yk - xp + yp)_\nu.
\]

(33)

We keep only those terms that survive in the limit \( D = 4 \)

\[
i (p + k)_\mu (T_{\mu\rho\sigma}(k, p) + T_{\mu\sigma\rho}(p, k))
= \frac{1}{4D^2} \int_0^1 dx \int_0^x dy \int \frac{d^D q}{(2\pi)^D} \frac{q^2}{(q^2 - \Delta)^3} \left\{ 4Tr \gamma_\chi \left[ \left( \delta_{\mu\alpha} \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_i - \delta_{\mu\rho} \gamma_\alpha \gamma_\beta \gamma_\sigma \gamma_i + \delta_{\mu\beta} \gamma_\alpha \gamma_\rho \gamma_\sigma \gamma_i \right. \\
- \delta_{\mu\sigma} \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_i + \delta_{\mu\alpha} \gamma_\rho \gamma_\beta \gamma_i \right] \left( \delta_{\mu\alpha} r_\beta (r + p)_\iota + \delta_{\mu\beta} (r - k)_\alpha (r + p)_\iota + \delta_{\mu\iota} (r - k)_\alpha r_\beta \right) \right. \\
+ (k \leftrightarrow p)(\rho \leftrightarrow \sigma) \right\}
\]

(34)

\[
= \frac{1}{4D^2} \int_0^1 dx \int_0^x dy \int \frac{d^D q}{(2\pi)^D} \frac{q^2}{(q^2 - \Delta)^3} \left\{ 4Tr \gamma_\chi \left[ \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_i \left( Dr_\beta p_\alpha + (r - k)_\beta (r + p)_\iota + k_\beta r_\iota \right) \right. \\
+ (k \leftrightarrow p)(\rho \leftrightarrow \sigma) \right\}.
\]
\[-\gamma_{\rho} \gamma_{\beta} \gamma_{\varepsilon} \gamma_{i} \left( r_{\beta} p_{i} - (r - k)_{\beta} (r + p)_{i} - k_{\beta} r_{i} \right) \\
+ \gamma_{\rho} \gamma_{\beta} \gamma_{\varepsilon} \gamma_{i} \left( -r_{\beta} p_{i} - D (r - k)_{\beta} (r + p)_{i} + k_{\beta} r_{i} \right) \\
- \gamma_{\rho} \gamma_{\beta} \gamma_{\varepsilon} \gamma_{i} \left( r_{\beta} p_{i} - (r - k)_{\beta} (r + p)_{i} - k_{\beta} r_{i} \right) \\
+ \gamma_{\rho} \gamma_{\beta} \gamma_{\varepsilon} \gamma_{i} \left( -r_{\beta} p_{i} + (r - k)_{\beta} (r + p)_{i} - D k_{\beta} r_{i} \right) \right] + (k \leftrightarrow p)(\rho \leftrightarrow \sigma) \}
= \frac{2}{D} \int_{0}^{1} dx \int_{0}^{x} dy \int \frac{d^{D} q}{(2\pi)^{D}} \frac{q^{2}}{(q^{2} - \Delta)^{3}(D - 4)} \left( \gamma_{\chi} \gamma_{\rho} \gamma_{\beta} \gamma_{\varepsilon} \gamma_{i} \right) \left( r_{\beta} p_{i} - (r - k)_{\beta} (r + p)_{i} - k_{\beta} r_{i} \right) \left( k \leftrightarrow p \right) (\rho \leftrightarrow \sigma)
= \frac{2}{D} \int_{0}^{1} dx \int_{0}^{x} dy \int \frac{d^{D} q}{(2\pi)^{D}} \frac{q^{2}}{(q^{2} - \Delta)^{3}(D - 4)} \left( \gamma_{\chi} \gamma_{\rho} \gamma_{\beta} \gamma_{\varepsilon} \gamma_{i} \right) k_{\beta} p_{i} \left( k \leftrightarrow p \right) (\rho \leftrightarrow \sigma)
= \frac{2}{D} \int \frac{d^{D} q}{(2\pi)^{D}} \frac{q^{2}}{(q^{2} - \Delta)^{3}(D - 4)} \left( D - 4 \right) \text{Tr} \left( \gamma_{\chi} \gamma_{\rho} \gamma_{\beta} \gamma_{\varepsilon} \gamma_{i} \right) k_{\beta} p_{i}.
(34)

where the dependence of \( \Delta \) from \( x, y \) has been neglected due to the vanishing factor \( D - 4 \).

### 3.3 The Triangle Anomaly

The expression in eq. (34) provides the anomaly in presence of two external vector mesons. Only the pole part of the integral provides a non vanishing result

\[
i (p + k)_{\mu} \left( T_{\mu \nu \sigma} (k, p) + T_{\mu \sigma \rho} (p, k) \right)
= \frac{2}{D} \left( -\frac{i}{(4\pi)^{2}} \right) \frac{2}{D - 4} \left( D - 4 \right) \text{Tr} \left( \gamma_{\chi} \gamma_{\rho} \gamma_{\beta} \gamma_{\varepsilon} \gamma_{i} \right) k_{\beta} p_{i}
= -\frac{i}{(4\pi)^{2}} \text{Tr} \left( \gamma_{\chi} \gamma_{\rho} \gamma_{\beta} \gamma_{\varepsilon} \gamma_{i} \right) k_{\beta} p_{i}.
(35)
\]

Finally we add the group factor from eq. (18)

\[
i (p + k)_{\mu} \left( T_{\mu \nu \sigma}^{ab} (k, p) + T_{\mu \sigma \rho}^{ab} (p, k) \right)
= -\frac{1}{2} \delta_{ab} \left( \frac{i}{(4\pi)^{2}} \right) \text{Tr} \left( \gamma_{\chi} \gamma_{\rho} \gamma_{\beta} \gamma_{\varepsilon} \gamma_{i} \right) k_{\beta} p_{i}.
(36)
\]

In terms of fields this is

\[
\partial_{\mu} J_{\mu}^{5} = -\frac{1}{4} \left( \frac{i}{(4\pi)^{2}} \right) \text{Tr} \left( \gamma_{\chi} \gamma_{\rho} \gamma_{\beta} \gamma_{\varepsilon} \gamma_{i} \right) \partial_{\beta} A_{\mu}^{a} \partial_{\epsilon} A_{\mu}^{a}.
\]

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\begin{eqnarray}
&=& -\frac{1}{2} \left(\frac{i}{(4\pi)^2}\right) \text{tr} \left( \gamma_\chi \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_i \right) \text{tr} \left( \partial_\beta A_\nu \partial_\sigma A_\sigma \right) 
\end{eqnarray}

which is in agreement with the result in I.

4 One-loop Box Contribution

The amplitude for the box diagram (by neglecting the group factors) is given by the Feynman rules $-i^4 \frac{1}{2^5}$ (four propagators, three vertices and a $- \epsilon$ due to the fermion loop)

$$T_{\mu\rho\nu\sigma}^{\text{Box}}(k, p, l) = \frac{i}{2^5} \int \frac{d^Dq}{(2\pi)^D} T \left( \gamma_\mu \gamma_\chi q_\alpha \gamma_\alpha (1 + \gamma_\chi)(q + k) \gamma_\beta \gamma_\beta \gamma_\sigma (1 + \gamma_\chi)(q + k + p) \gamma_\rho \gamma_\rho \right)$$

$$\gamma_\sigma (1 + \gamma_\chi)(q + k + p) \gamma_\gamma_\mu (1 + \gamma_\chi)(q + k + p + l) \gamma_\delta \gamma_\delta$$

$$[q^2(q + k)^2(q + k + p)^2(q + k + p + l)^2]^{-1}$$

where incoming momenta and polarizations are $(k, \rho)$, $(p, \sigma)$ and $(l, \nu)$.

4.1 One-loop Box Contribution: the Divergence of the Current

We include also the group factor $\text{tr}(t_a t_b t_c) = \frac{i}{4} \varepsilon_{abc}$. Thus the divergence of the current at one loop is

$$i(p + k + l)_\mu T_{\mu\rho\nu\sigma}^{\text{Box Div}}(k, p, l) = \frac{i}{2^5} \varepsilon_{abc}$$

$$-\frac{i}{2^5} \int \frac{d^Dq}{(2\pi)^D} (p + k + l)_\mu \text{tr} \left( \gamma_\mu \gamma_\chi q_\alpha \gamma_\alpha \right)$$

$$\gamma_\rho (1 + \gamma_\chi)(q + k) \gamma_\beta \gamma_\beta \gamma_\gamma_\sigma (1 + \gamma_\chi)(q + k + p) \gamma_\rho \gamma_\rho$$

$$\gamma_\sigma (1 + \gamma_\chi)(q + k + p + l) \gamma_\delta \gamma_\delta$$

$$[q^2(q + k)^2(q + k + p)^2(q + k + p + l)^2]^{-1}$$

$$= -\frac{i}{2^5} \int \frac{d^Dq}{(2\pi)^D} \left\{ -\text{tr} \left( q_\mu \gamma_\mu \gamma_\chi q_\alpha \gamma_\alpha \gamma_\rho (1 + \gamma_\chi)(q + k) \gamma_\beta \gamma_\beta \gamma_\gamma_\sigma (1 + \gamma_\chi)(q + k + p) \gamma_\rho \gamma_\rho \right.$$$$\left. (1 + \gamma_\chi)(q + k + p + l) \gamma_\delta \gamma_\delta \right) [q^2(q + k)^2(q + k + p)^2(q + k + p + l)^2]^{-1}$$

$$+ \text{tr} \left( \gamma_\gamma_\chi q_\alpha \gamma_\alpha \gamma_\rho (1 + \gamma_\chi)(q + k) \gamma_\beta \gamma_\beta \gamma_\gamma_\sigma (1 + \gamma_\chi)(q + k + p) \gamma_\rho \gamma_\rho \right.$$$$\left. (1 + \gamma_\chi)(q + k + p + l) \gamma_\delta \gamma_\delta \right) [q^2(q + k)^2(q + k + p)^2(q + k + p + l)^2]^{-1} \right\}$$

(39)

The sum over the permutations of $(a, \rho, k)$, $(b, \sigma, p)$ and $(c, \nu, l)$ is understood.
4.2 Identities at $D = 4$

It is convenient to disclose the identities that would be satisfied in a situation where $D$ can be taken equal to 4. Thus we consider the first integral of the RHS where we identify the part responsible for the anomaly (i.e. $\{\gamma_\mu, \gamma_\chi\}$) of eq. (39)

$$-i\frac{\varepsilon_{abc}}{2^5} \int \frac{d^Dq}{(2\pi)^D}$$

$$-Tr \left( \left\{ \{\gamma_\mu, \gamma_\chi\} - \gamma_\chi\gamma_\mu \right\}g_\mu q_\alpha\gamma_\alpha\gamma_\rho(1 + \gamma_\chi)(q + k)_\beta\gamma_\beta$$

$$\gamma_\sigma (1 + \gamma_\chi)(q + k + p)_\gamma\gamma_\nu(1 + \gamma_\chi)(q + k + p + l)_\delta\gamma_\delta$$

$$[q^2(q + k)^2(q + k + p)^2(q + k + p + l)^2]^{-1}$$

(40)

The non anomalous part ($-\gamma_\chi\gamma_\mu$) should contribute to the cancellations in the divergence of the isoscalar axial current. We elaborate this quantity by replacing $q \rightarrow q - k$

$$-i\frac{\varepsilon_{abc}}{2^5} \int \frac{d^Dq}{(2\pi)^D}$$

$$Tr \left( \gamma_\chi\gamma_\rho(1 + \gamma_\chi)q_\beta\gamma_\beta \gamma_\sigma (1 + \gamma_\chi)$$

$$(q + p)_\gamma\gamma_\nu(1 + \gamma_\chi)(q + p + l)_\delta\gamma_\delta \right) \right) [q + p)^2(q + p + l)^2q^2]^{-1}$$

(41)

We add the expression in eq. (41) to the second term in eq. (39) on which we perform the cyclic permutation $(a, \rho, k) \rightarrow (b, \sigma, p) \rightarrow (c, \nu, l) \rightarrow (a, \rho, k)$. The result of this sum is

$$-i\frac{\varepsilon_{abc}}{2^5} \int \frac{d^Dq}{(2\pi)^D} \left\{ Tr \left( \gamma_\chi\gamma_\rho(1 + \gamma_\chi)q_\beta\gamma_\beta \gamma_\sigma (1 + \gamma_\chi)$$

$$(q + p)_\gamma\gamma_\nu(1 + \gamma_\chi)(q + p + l)_\delta\gamma_\delta \right) \right) \right) [q + p)^2(q + p + l)^2q^2]^{-1}$$

$$+ Tr \left( \gamma_\chi q_\beta\gamma_\beta \gamma_\sigma (1 + \gamma_\chi)(q + p)_\gamma\gamma_\nu (1 + \gamma_\chi)$$

$$(q + p + l)_\delta\gamma_\delta\gamma_\rho(1 + \gamma_\chi) \right) \right) [q + p)^2(q + p + l)^2q^2]^{-1}$$

$$= -i\frac{\varepsilon_{abc}}{2^5} \int \frac{d^Dq}{(2\pi)^D} Tr \left( \left\{ \gamma_\chi, \gamma_\rho \right\}(1 + \gamma_\chi)q_\beta\gamma_\beta \gamma_\sigma (1 + \gamma_\chi)$$

$$(q + p)_\gamma\gamma_\nu(1 + \gamma_\chi)(q + p + l)_\delta\gamma_\delta \right) [(q + p)^2(q + p + l)^2q^2]^{-1}.$$
We see that the expression in eq. (42) is vanishing if \( \{ \gamma_{\alpha}, \gamma_{\rho} \} = 0 \).

The same result can be obtained for all terms generated from eq. (39) by using the permutations on the external variables \((a, \rho, k)\), \((b, \sigma, p)\) and \((c, \nu, l)\).

### 4.3 The Box Anomaly

From the previous calculation we get the final result for the anomaly coming from the box. It is given by the sum over all permutations on the external vector mesons of the term proportional to \( \{ \gamma_{\mu}, \gamma_{\chi} \} \) in eq. (40) and of the expression in eq. (42)

\[
-\frac{i\varepsilon_{abc}}{2^5} \int \frac{d^D q}{(2\pi)^D} \left\{ -Tr \left( \{ \gamma_{\mu}, \gamma_{\chi} \} q_{\mu} q_{\alpha} \gamma_{\alpha} \gamma_{\rho}(1 + \gamma_{\chi})(q + k)_{\beta} \gamma_{\beta} \gamma_{\sigma} (1 + \gamma_{\chi})(q + k + p + l)_{\delta} \gamma_{\delta} \right) \right. \\
\left. \gamma_{\chi}(q + k + p), \gamma_{\gamma}(1 + \gamma_{\chi})(q + k + p + l)_{\delta} \gamma_{\delta} \right\} \left[ q^2(q + k)^2(q + k + p)^2(q + k + p + l)^2 \right]^{-1} + Tr \left( \{ \gamma_{\chi}, \gamma_{\rho} \} (1 + \gamma_{\chi}) q_{\beta} \gamma_{\beta} \gamma_{\sigma} (1 + \gamma_{\chi})(q + p)_{\delta} \gamma_{\delta} \right) [(q + p)^2(q + p + l)^2q^2]^{-1} \right. \\
\left. \right\} . \tag{43}
\]

### 4.4 The First Term in Eq. (43)

Let us consider the first term in eq. (43). Since \( \{ \gamma_{\mu}, \gamma_{\chi} \} = \mathcal{O}(D - 4) \) the gamma trace reduces to

\[
Tr \left( \{ \gamma_{\mu}, \gamma_{\chi} \} \gamma_{\alpha} \gamma_{\rho}(1 + \gamma_{\chi}) \gamma_{\beta} \gamma_{\sigma} (1 + \gamma_{\chi}) \gamma_{\gamma}(1 + \gamma_{\chi}) \right) = 4Tr \left( \{ \gamma_{\mu}, \gamma_{\chi} \} \gamma_{\alpha} \gamma_{\rho} \gamma_{\beta} \gamma_{\sigma} \gamma_{\gamma} \gamma_{\delta}(1 + \gamma_{\chi}) \right) . \tag{44}
\]

Let us focus now on the momentum integration. Only the divergent part of the \(q\)-integral can yield a non-zero result; i.e. the 4-th powers of \(q\) in the numerator. After Feynman parameterization, shift by \(q_{\mu} \to q_{\mu} + r_{\mu}\) and symmetric integration we get

\[
q_{\mu} q_{\alpha} q_{\beta} q_{\mu} \to \frac{q^4}{D(D + 2)} \left[ \delta_{\mu\alpha} \delta_{\beta\gamma} + \delta_{\mu\beta} \delta_{\alpha\gamma} + \delta_{\mu\gamma} \delta_{\alpha\beta} \right] . \tag{45}
\]
Thus we can neglect the second $\gamma_x$ at the far right in eq. (44) and the numerator of the first term in eq. (43) after symmetric integration becomes

\[-Tr \left( \{\gamma_{\mu_x}, \gamma_{x}\} q_{\mu} q_{\alpha} \gamma_{\alpha} \gamma_{\rho} (1 + \gamma_{x}) (q + k) \gamma_{\beta} \gamma_{\beta} \gamma_{\sigma} \gamma_{\sigma} (q + k) (q + k + p) \gamma_{\nu} (1 + \gamma_{x}) (q + k + p + l) \delta \gamma_{\delta} \right)\]

\[= - \frac{q^4}{D(D+2)} 4Tr \left( \gamma_{x} \left\{ \gamma_{\mu}, \gamma_{x}\right\} \gamma_{\alpha} \gamma_{\beta} \gamma_{\sigma} \gamma_{\sigma} \gamma_{\nu} \gamma_{\nu} (1 + \gamma_{x}) (q + k + p + l) \delta \gamma_{\delta} \right)\]

\[= - \frac{q^4}{D(D+2)} 4Tr \left( \gamma_{x} \left\{ \gamma_{\mu}, \gamma_{x}\right\} \gamma_{\alpha} \gamma_{\beta} \gamma_{\sigma} \gamma_{\sigma} \gamma_{\nu} \gamma_{\nu} (1 + \gamma_{x}) (q + k + p + l) \delta \gamma_{\delta} \right)\]

\[= - \frac{q^4}{D(D+2)} 4Tr \left( \gamma_{x} \left\{ \gamma_{\mu}, \gamma_{x}\right\} \gamma_{\alpha} \gamma_{\beta} \gamma_{\sigma} \gamma_{\sigma} \gamma_{\nu} \gamma_{\nu} (1 + \gamma_{x}) (q + k + p + l) \delta \gamma_{\delta} \right)\]

(46)

We neglect the last line of eq. (46) since, after the use of the Kronecker delta, too few gammas are left for a non-zero limit of $D = 4$. Thus we have

\[-Tr \left( \{\gamma_{\mu_x}, \gamma_{x}\} q_{\mu} q_{\alpha} \gamma_{\alpha} \gamma_{\rho} (1 + \gamma_{x}) (q + k) \gamma_{\beta} \gamma_{\beta} \gamma_{\sigma} \gamma_{\sigma} (q + k) (q + k + p) \gamma_{\nu} (1 + \gamma_{x}) (q + k + p + l) \delta \gamma_{\delta} \right)\]

\[= - \frac{q^4}{D(D+2)} 4Tr \left( \gamma_{x} \left\{ \gamma_{\mu}, \gamma_{x}\right\} \gamma_{\alpha} \gamma_{\beta} \gamma_{\sigma} \gamma_{\sigma} \gamma_{\nu} \gamma_{\nu} (1 + \gamma_{x}) (q + k + p + l) \delta \gamma_{\delta} \right)\]

\[= - \frac{q^4}{D(D+2)} 4Tr \left( \gamma_{x} \left\{ \gamma_{\mu}, \gamma_{x}\right\} \gamma_{\alpha} \gamma_{\beta} \gamma_{\sigma} \gamma_{\sigma} \gamma_{\nu} \gamma_{\nu} (1 + \gamma_{x}) (q + k + p + l) \delta \gamma_{\delta} \right)\]

(46)
\[= -2(D - 4) \frac{q^4}{D(D + 2)} 4Tr (\gamma \gamma_\rho \gamma_\sigma \gamma_\nu \gamma_i) \]
\[
\left( [2 - D] + (2 - D) - (6 - D) \right) (r + k + p + l)_i \\
+ [-(6 - D) - (2 - D) + (10 - D)] (r + k + p)_i \\
+ [2 - D] + (6 - D) - (10 - D) \right] (r + k)_i \\
- \left[ (2 - D) + (2 - D) - (6 - D) \right] r_i \\
\]
\[= -2(D - 4)(D + 2) \frac{q^4}{D(D + 2)} 4Tr (\gamma \gamma_\rho \gamma_\sigma \gamma_\nu \gamma_i) \]
\[
\left( -(r + k + p + l) + (r + k + p) - (r + k) + r \right)_i \\
\]
\[= 8(D - 4) \frac{q^4}{D} Tr (\gamma \gamma_\rho \gamma_\sigma \gamma_\nu \gamma_i)(k + l)_i \quad (47)\]

### 4.5 The Second Term in Eq. (43)

The second term in eq. (43) has also to be evaluated in the process of symmetric integration over \(q\) after the shift

\[q_\mu \rightarrow q_\mu + r_\mu. \quad (48)\]

Thus we have

\[q_\mu q_\nu \rightarrow \frac{q^2}{D} \delta_{\mu \nu}. \quad (49)\]

We have

\[Tr \left( \{\gamma_\chi, \gamma_\rho \}(1 + \gamma_\chi)q_\beta \gamma_\beta \gamma_\sigma \gamma_\nu (1 + \gamma_\chi) \right) \]
\[\]
\[(q + p)_i \gamma_\nu (1 + \gamma_\chi)(q + p + l) \delta \gamma_\delta \]
\[= 4Tr \left( \{\gamma_\chi, \gamma_\rho \} \gamma_\beta \gamma_\sigma \gamma_\nu (1 + \gamma_\chi) \gamma_\delta \right) \]
\[
\left[ (q + r)_\beta (q + r + p)_\delta (q + r + p + l) \right] \quad (50)\]

After symmetric integration the second \(\gamma_\chi\) in the RHS of eq. (50) can be neglected by following the argument in Section 2

\[Tr \left( \{\gamma_\chi, \gamma_\rho \}(1 + \gamma_\chi)q_\beta \gamma_\beta \gamma_\sigma \gamma_\nu (1 + \gamma_\chi) \right) \]
\[(q + p)_\gamma \gamma_\nu (1 + \gamma_\chi)(q + p + l)\delta_\gamma_\delta \]
\[= 4 \frac{q^2}{D} Tr \left( \gamma_\chi \{ \gamma_\rho, \gamma_\beta \gamma_\sigma \gamma_\iota \gamma_\nu \gamma_\delta \} \right) \]
\[\left[ \delta_\beta_\iota (r + p + l)_\delta + \delta_\iota_\delta r_\beta + \delta_\beta_\delta (r + p)_\iota \right] \]  

(51)

We evaluate the Kronecker delta’s
\[Tr \left( \{ \gamma_\chi, \gamma_\rho \} (1 + \gamma_\chi) q_\beta \gamma_\sigma (1 + \gamma_\chi) \right) \]
\[= 4 \frac{q^2}{D} Tr \left( \gamma_\chi \left\{ \gamma_\rho, \left[ (2 - D) \gamma_\sigma \gamma_\nu \gamma_\delta (r + p + l)_\delta \right. \right. \]
\[\left. \left. + (2 - D) \gamma_\beta \gamma_\sigma \gamma_\iota r_\beta + (6 - D) \gamma_\sigma \gamma_\iota \gamma_\nu (r + p)_\iota \right] \right\} \right) = 0 \]  

(52)

around \( D = 4 \).

5 Anomaly from the Box

By restoring the initial factor of eq. (43) the anomaly in the current conservation is
\[-i \varepsilon_{abc} \int \frac{d^D q}{(2\pi)^D} \left\{ 8(D - 4) \frac{q^4}{D} Tr (\gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota (k + l)_\iota) \right\} (q^2 - \Delta)^{-4} \]
\[= -i \varepsilon_{abc} \frac{1}{2^5 D (4\pi)^2} \frac{2}{(D - 4)} 8(D - 4) Tr (\gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota (k + l)_\iota) \]
\[\frac{\varepsilon_{abc}}{2D} \frac{1}{(4\pi)^2} Tr (\gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota (k + l)_\iota) \]  

(53)

The sum over the permutations at \( (D = 4) \) gives
\[\varepsilon_{abc} \frac{1}{(4\pi)^2} \frac{2}{D} Tr (\gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota (k + p + l)_\iota) \]. \]  

(54)

In terms of fields we have
\[\partial_\mu J^5_\mu = \frac{1}{(4\pi)^2} \frac{1}{D} Tr (\gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota (k + p + l)_\iota) (-4i) tr \left( i \partial_\mu A_\rho A_\sigma A_\iota \right) \]
\[= \frac{1}{(4\pi)^2} Tr (\gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota) tr \left( \partial_\mu A_\rho A_\sigma A_\iota \right) \]  

(55)
where $tr$ is the trace over the $SU(2)$ internal indices.

Together eqs. (37) and (55) give the anomaly in the covariant form

$$\partial_\mu J^5_\mu = \frac{1}{(4\pi)^2} \frac{i}{8} Tr \left( \gamma_\chi \gamma_\beta \gamma_\rho \gamma_\iota \gamma_\sigma \right) tr \left( G_{\beta\rho} G_{\iota\sigma} \right)$$

(56)

where

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu].$$

(57)

6 Conclusions

The present analytic calculation of the anomaly of the axial isoscalar current in the $SU(2)$ chiral theory indicates that a consistent definition of the trace with $\gamma_5$ in dimensional regularization is at hand.

In this work we used the ingredients expected to be present in a consistent solution of the problem: associative algebra for the gamma’s, Lorentz covariance, cyclicity, smooth limit at $D = 4$

$$Tr(p) = \sum_{h=0} A_h(p)(D-4)^h.$$

(58)

$Tr(p)$ is any trace of gamma’s and $\gamma_\chi$, where the Lorentz indices are all saturated by vectors and tensors (e.g. $\delta_{\mu\nu}$) and $A_h(p)$ are $D = 4$ Lorentz invariants (being $\varepsilon_{\alpha\beta\rho\sigma}$ allowed).

The outlook is the extension of the integral representation of the trace, discussed in a previous paper (I), to the situation where more than one $\gamma_5$ is present.

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References


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