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Reducing Phase Noise in Multi-phase Oscillators

Paolo Maffezzoni, Senior Member, IEEE, Bichoy Bahr, Student Member, IEEE, Zheng Zhang, Student Member, IEEE, and Luca Daniel, Member, IEEE

Abstract—This paper investigates phase noise mechanism in arrays of resonant LC oscillators. Such arrays represent today a promising solution for the generation of multi-phase signals needed in several advanced applications. The analysis presented in this paper relies on consolidated phase-domain macromodels as well as on the original concept of noise transfer function illustrated herein. The proposed analysis sheds new light on noise generation in oscillator arrays and is able to explain certain noise degradation effects observed in nonreciprocal coupling networks. Phase-domain simulation together with noise transfer function concept provide a very efficient computational tool for rapid calculations of phase response and output noise. Thanks to this efficient tool and to the gained qualitative understanding, we are able to propose a chain array configuration enhanced by the injection of a clean, low-noise, signal. In this paper, it is shown how the injected chain array can provide the prescribed phase separation while significantly reducing output phase noise.

Index Terms—Multi-phase oscillators, noise reduction, noise transfer function.

I. INTRODUCTION

Arrays of weakly coupled resonant oscillators can be employed to generate multi-phase harmonic signals, i.e. sinusoids with the same oscillating frequency and with prescribed phase separations. Multi-phase signals are now indispensable in many advanced emerging applications, such as in extremely high-frequency synthesis and multiphase clock distribution [1]–[3], as well as in brain-inspired parallel computing for data analysis [4]. For such applications, stringent phase-noise specifications are required.

Previous studies and experimental evidences have shown how coupled oscillators can exhibit a reduction of their phase noise compared to the free-running case [5], [6]. More specifically, the phase noise spectrum near the carrier is reduced by a factor proportional to the number $N$ of stages in the array, while the noise spectrum far from the carrier remains almost unchanged. Such a noise reduction effect is commonly not enough to meet noise specifications. In fact, the noise spectrum near the carrier continues to be shaped as $1/f^2$ (or $1/f^3$ down to the corner frequency where flicker noise gets dominant) even if reduced compared to the free-running case. In addition, it has been shown that certain array configurations, such as nonreciprocal unilaterally coupled chain arrays, may result in unexpected deterioration of the noise spectrum with the appearance of spurious peaks [5], [6]. This anomalous behavior and performance limitations call for further investigations.

In this paper, we use the phase-domain macromodel presented in [6] and the concept of Noise Transfer Function (NTF) to shed new light on phase-noise mechanism in oscillator arrays. Thanks to this understanding, we demonstrate that the noise performance of the multi-phase chain array can be definitely enhanced, even at low frequencies, by properly injecting an external clean low-noise signal. Such a clean signal is in fact available in the majority of frequency synthesizers and clock generation systems and is obtained by locking one oscillator (within a PLL or with a pulsed injection) to a stable low-frequency reference (i.e., the output of a crystal oscillator). The problem with an external injection is that it may disrupt the correct phase separation if not properly dimensioned. In this paper, we show how injection strength can be set so as to reduce noise while preserving the prescribed phase separation.

The novel contributions of this paper may be summarized as follows:

1) We investigate the form of the NTFs that describe how the noise sources internal to oscillators are transferred to the output phase noise. We derive how such NTFs depend on the specific choices of array topologies.

2) A behavioral method is provided that allows modeling internal noise sources in locked oscillators through a single macro noise source.

3) A chain array topology enhanced by the injection of a clean, low-noise, signal is presented. It is shown that the proposed array provides multi-phase signals with $\pi/N$ phase separation while improving noise spectrum over a wide frequency band. Results are checked, in a few test cases, via comparisons with SpectreRF simulations.

The topics listed above are organized in the paper as follows: Sec. II describes oscillator array structure and review its phase-domain model. In Sec. III, we illustrate phase noise modeling for individual free-running or locked oscillators and then describe array phase noise analysis and NTF concept. In Sec. IV, we focus on the relevant case of a chain array and derive the conditions under which an external injection becomes effective in noise reduction. Finally, Sec. V is devoted to numerical experiments and validation.

II. OSCILLATOR ARRAY

We consider an array composed with $N$ identical LC resonant CMOS oscillators, as shown in Fig. 1(Top). Each oscillator, when free-running, oscillates autonomously at the angular frequency $\omega_0$ and its output voltage, measured at the LC tank nodes, is purely harmonic

$$V_0(t) = V_M \cos(\omega_0 t),$$

(1)
with $V_M$ being the voltage amplitude. To construct the array, oscillators are coupled by a transconductance, implemented by differential-pair transistors. Fig. 1(Top) shows, the coupling circuit between two oscillator stages of the array with index $k$ and $j$. In this example, a differential-pair circuit reads the voltage $V_k(t)$ at the $k$th-stage output and injects a proportional differential current

$$I_j(t) = g_{jk} V_k(t)$$

into the tank nodes of the $j$th oscillator. The module of parameter $g_{jk}$ corresponds to the transconductance of the associated differential-pair transistor while its sign refers to the way differential current $I_j(t)$ is injected into the nodes $n_j^+$ and $n_j^-$. In this paper, it is conventionally assumed that a positive $g_{jk}$ corresponds to $I_j(t)$ exiting node $n_j^+$ and entering $n_j^-$, as it is the case shown in Fig. 1(Top). A negative $g_{jk}$, instead, corresponds to $I_j(t)$ exiting node $n_j^-$ and entering $n_j^+$ and can be implemented by switching the way differential-pair drains are connected to the injected nodes. In the array in Fig. 1(Top) coupling is bilateral since a second differential pair transistor reads the voltage $V_j(t)$ of the $j$th stage and injects a proportional current $I_k(t) = g_{kj} V_j(t)$ into the $k$th stage. The array topology can be schematically represented by the system shown in Fig. 1(Bottom) where couplings are indicated by oriented arches of strength $g_{kj}$. Array topology can be described by the conductance matrix $G = \{g_{kj}\} \in \mathbb{R}^{N \times N}$ that collects all coupling strength coefficients $g_{jk}$.

It is known that for resonant oscillators with well matched free-running frequency, weak coupling (e.g. with the transconductance of coupling transistors one order of magnitude smaller than that of oscillator transistors) is enough to keep oscillators synchronized. Under this hypothesis, the array response can be realistically simulated with a phase-domain macromodel [7]–[12]. According to this method, the output voltage of the $k$th oscillator in the array is written as

$$V_k(t) = V_0 (t + \alpha_k(t)) = V_M \cos(\omega_0 t + \omega_0 \alpha_k(t))$$

where $\alpha_k(t)$ represents the time-dependent time shift of the perturbed response with respect to the free-running one and $\phi_k(t) = \omega_0 \alpha_k(t)$ is the associated excess phase variable. Then, the time-shift variable and injected current $I_k(t)$ are related by the scalar differential equation [7], [8]

$$\dot{\alpha}_k(t) = \Gamma_k(t + \alpha_k(t)) I_k(t)$$

where $\Gamma_k(t)$ is the periodic phase-sensitivity function to current injection at the tank. It has been found that for LC resonant oscillators $\Gamma(t)$ is sinusoidal and delayed by a $\pi/2$ phase angle with respect to the output response [16], i.e.

$$\Gamma_k(t) = \Gamma_M \cos(\omega_0 t - \pi/2).$$

The phase response of the whole chain array is governed by the following set of nonlinear differential-algebraic equations

$$\dot{\phi}_k(t) = \omega_0 \alpha_k(t)$$

$$\dot{\alpha}_k(t) = \Gamma_k(t + \alpha_k(t)) I_k(t)$$

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oscillator, (Dashed line) spectrum
sources internal to the oscillator circuit [17], [18]. The P SD
φ
 S
Density (PSD)
source
in a compact way, by the average stochastic equation
(10) can be extracted by fitting the shape of the available
output spectrum of the locked oscillator results of the type
S
lock
(f) = \left( K_w + \frac{K_f}{f} \right) \frac{1}{|i f + f_B|^2} (12)
as it is qualitatively shown in Fig. 3 under the realistic
hypothesis that \( f_B >> f_c \). In view of (9), we derive that
the macro noise source \( n_{lock}^k(t) \) to be associated to a clean
low-noise oscillator has the PSD
S
lock
(f) = \left( K_w + \frac{K_f}{f} \right) \frac{1}{|f|^2 |f + f_B|^2} (13)
which vanishes for \( f << f_B \), as shown in Fig. 3.

C. Oscillator array

The oscillator array is formed by coupling \( N \) free-running
LC oscillators as shown in Fig. 1. In addition, one low-
noise oscillator (whose noise model has been described in
subsection III-B) may be present in the system. An example
of such an array topology is investigated in the next section
and is shown in Fig. 6. The low-noise oscillator is supposed
to inject unilaterally into the other stages of the array so that
its behavior (and its low noise output) is not affected by array
operation.

To study phase noise in the oscillator array, the macro
noise source \( n_k(t) \) associated to the \( k \)th oscillator is added
to equation (6a), i.e.
\[
\hat{\alpha}_k(t) = \Gamma_k(t + \alpha_k(t)) I_k(t) + n_k(t). \tag{14}
\]
For oscillators that were running in free mode before being
coupled, the noise source \( n_k(t) \) in (14) has PSD of the type
(11), whereas for the low-noise oscillator, the associated noise
source \( n_k(t) \) has PSD of the type (13).

Combining (14) with (6b), we are led to the following set
of stochastic differential equations
\[
\hat{\alpha}_k(t) = \Gamma_k(t + \alpha_k(t)) \sum_{j=1}^{N} g_{kj} V_j(t + \alpha_j(t)) + n_k(t). \tag{15}
\]
The presence of noise sources induces extra random fluctuations $\tau_k(t)$ of variables $\alpha_k(t)$ around their noiseless regime values $\tilde{\alpha}_k(t)$, i.e.

$$\alpha_k(t) = \tilde{\alpha}_k(t) + \tau_k(t).$$

Exploiting the fact that $\tau_k(t) \ll \tilde{\alpha}_k(t)$, in the Appendix it is shown that the PSD of the noise-induced excess phase $\theta_k(t) = \omega_0 \tau_k(t)$ for the $k$th oscillator in the array is given by

$$S_{\theta_k}(f) = \frac{\omega_0^2}{2} \sum_{j=1}^{N} \text{NTF}_{kj}(f),$$

where

$$\text{NTF}_{kj}(f) = |t_{kj}(f)|^2.$$  

In the expression above, the complex coefficient $t_{kj}(f)$ represents the signal transfer function from input $n_j(t)$ of $j$th oscillator to output phase $\theta_k(t)$ of $k$th oscillator, while NTF$_{kj}(f)$ is the associated noise transfer function in terms of noise power.

Furthermore, transfer functions $t_{kj}(f)$ are the entries of the complex matrix

$$T(f) = \{t_{kj}\} = (i2\pi f I_N - A)^{-1},$$

where $I_N$ is the identity matrix of size $N$, “$-1$” denotes inverse operator, while $A = \{a_{kj}\} \in \mathbb{R}^{N \times N}$ is defined as follows

$$a_{kk} = B \sum_{j=1, j \neq k}^{N} g_{kj} \cos(\Phi_{kj})$$

$$a_{kj} = -B g_{kj} \cos(\Phi_{kj}) \quad \text{for } k \neq j,$$

with $B = \omega_0 \Gamma_M V_M/2$, as derived in the Appendix.

In the remainder of this section, we better investigate the form of the signal transfer functions $t_{kj}(f)$. As underlined in the Appendix, for any phase separation $\Phi_{kj}$ that corresponds to a stable solution of (6), the eigenvalues $\lambda_k$ of $A$ are such that: $\lambda_1 = 0$, while remaining ones have negative real part $\Re(\lambda_k) < 0$ for $k = 2, \ldots, N$. For exposition simplicity, in what follows we suppose that such eigenvalues are all distinct (this is in fact the case for the chain array topologies that we will consider later). Under this hypothesis, matrix $A$ has eigenvalue decomposition as follows:

$$A = V \cdot D_A \cdot W^T,$$

where the diagonal matrix $D_A$ collects the $\lambda_k$ eigenvalues, whereas the columns of matrix $V$ are the related eigenvectors $V_k$. In particular, eigenvector $V_1$ associated to $\lambda_1 = 0$ spans the null space of matrix $A$, i.e. $A \cdot V_1 = 0$. The columns $\vec{W}_j$ of matrix $W$ defined as $W^T = V^{-1}$ are the rows of the inverse matrix $V^{-1}$. From (19), it results

$$T(f) = V \cdot D_{\alpha n} \cdot W^T,$$

where the diagonal matrix $D_{\alpha n}$ collects the elements $(i2\pi f - \lambda_k)^{-1}$. Expression (22) can be further expanded into

$$T(f) = \vec{V}_1 \vec{W}_1^T \frac{1}{i2\pi f} + \sum_{k=2}^{N} \vec{V}_k \vec{W}_k^T \frac{1}{i2\pi f - \lambda_k}.$$  

The following qualitative observations are in order.

1) Coupling oscillators to form an array results in filtering the macro noise source $\tau_k(t)$ through transfer functions $t_{kj}(f)$ whose poles are the eigenvalues $\lambda_k$ of matrix $A$.

2) If eigenvalues $\lambda_k$ are complex conjugate (i.e. they have a nonzero imaginary part), these transfer functions may give resonance effects with unwanted spikes in the output phase noise spectrum. This phenomenon, which was observed in previous studies [5] and [6], becomes more pronounced when the number $N$ of stages is increased. Such spikes in the output spectrum are removed if the conductance matrix $G = g_{jk}$ is made symmetric, i.e. if coupling is bilateral with $g_{kj} = g_{jk}$. In this case in fact $A$ is symmetric and thus its eigenvalues $\lambda_k$ are purely real, which eliminates any resonance effect. For these reasons, from now on our analysis will be focused on symmetric arrays.

3) The first term in expansion (23) is the most critical one since its transfer function $\propto 1/f$, which corresponds to integrating noise in time, produces large phase noise components at low frequencies.

4) Nonzero elements of vector $\vec{W}_1$ tell us which noise sources in the array are actually integrated through $1/f$ transfer function, in other words the noise source $n_j(t)$ associated to $j$th oscillator is integrated in time if and only if the $j$th element in vector $\vec{W}_1$ is non zero. As a result, in order to minimize output noise, vector $\vec{W}_1$ should have non zero elements only in correspondence to low-noise oscillators.

We conclude this section by observing that vector $\vec{W}_1$ is the eigenvector of $A^T$ associated to $\lambda_1 = 0$ and thus it spans the null space of matrix $A^T$. This is easily seen by transposing (21) and observing that $W^T = V^{-1}$, which yields

$$A^T = W \cdot D_A \cdot W^{-1}.$$  

IV. CHAIN ARRAYS WITH CLEAN SIGNAL INJECTION

An array topology of particular importance is the chain array where only nearest neighbor oscillators are coupled, i.e. $g_{kj} \neq 0$ only for $j \in (k-1, k+1)$. Chain arrays are interesting for implementation reasons since they require a limited number of coupling stages. More importantly, it has been proved that for a synchronized chain array only one steady-state phase separation $\Phi_{kj}$ corresponds to a stable solution of (6) and thus it is observable in practice [6], [15]. This stable phase separation only depends on array topology and coupling strengths, i.e. on conductance matrix $G$, while it does not depend on initial phase conditions.

In this paper, in particular, we will focus on the chain array topology portrayed in Fig. 4 where neighboring oscillators are connected by a symmetric bilateral coupling of strength $-g$ (with $g$ being the differential-pair transistor transconductance). The chain array is closed at the ends with the first and last stages that are bilaterally coupled with strength $g$. The associated conductance matrix $G$ is reported in Fig. 5(Top).

For this chain array, it has been proved that when synchronization is achieved, steady-state phase variables are such that:

$$\Phi_{k+1, k} = \phi_{k+1}(t) - \phi_k(t) = \pm \pi/N.$$  

(25)
Thus, denoting \( x = g B \cos(\Phi_{k+1,k}) \), the symmetric matrix \( A = A^T \) exhibits the structure shown in Fig. 5(Bottom) and its null space is spanned by the vector \( \vec{W}_1 \) formed by all ones. This implies that all of the noise sources \( n_k(t) \) in the array are transferred to the outputs through the critical network function \( 1/i2\pi f \). We thus expect that for the chain array in Fig. 4 the output power spectra, even if reduced compared to the free-running case, will continue to be shaped as \( 1/f^2 \) (and \( 1/f^4 \) at the very low frequencies down to \( f_C \)).

To improve the noise performance, in what follows we investigate the enhanced array arrangement shown in Fig. 6: in this arrangement the oscillators in the chain array are injected unilaterally with the clean signal provided by a low-noise oscillator labelled \( O_1 \). The macro noise source \( n_1(t) \) associated to \( O_1 \) has the PSD described in (11) which vanishes for \( f < f_B \). Signal injection from \( O_1 \) should reduce the small-signal phase fluctuations induced by noise while minimally affecting the large-signal phases \( \phi_k(t) \) and related phase separations. To achieve this goal, the injection strength is fixed to a value \( g_r << g \). The conductance matrix \( G \) for the injected chain is augmented by an all-zero row (oscillator 1 is not injected by others) and one column collecting injection strength coefficients \( g_r \) as shown in Fig. 7(Top). Supposing that (25) remains unchanged, and adopting the notations \( x = g B \cos(\Phi_{k+1,k}) \), \( x_r = g_r B \cos(\Phi_{k+1,k}) \), and \( y = -2x + x_r \), the structure of the \( A^T \) matrix (i.e. its non zero entries) is shown in Fig. 7(Bottom). The null-space spanning vector \( \vec{W}_1 \) has a one in the first position and all zeros in the others. This means that in the injected array the only noise source which is transferred through the critical network function \( 1/i2\pi f \) is that of the clean source \( n_1(t) \) associated to the locked oscillator. This is expected to improve significantly the array noise performance at the low frequencies.

V. NUMERICAL RESULTS

In this section, we present numerical results for chain arrays formed with \( N = 5 \) identical LC oscillators and for configurations without external injection and with injection. The circuit of each LC oscillator is shown in Fig. 1 with the device parameters reported in Table I. A single oscillator is first simulated with the periodic steady state (pss) analysis of SpectreRF and then the \( \Gamma(t) \) function is extracted with the method described in [16]. The circuit oscillates at \( f_0 = 1.0261 \text{GHz} \) and its output voltage \( V_0(t) \) and \( \Gamma(t) \) function are harmonic as in (1) and (5) with peak values \( V_M = 3.45 \text{V} \), \( \Gamma_M = 151.8 \text{A}^{-1} \), respectively.
Oscillator stages are coupled as in Fig. 4, to form a chain array with no injection. A fixed differential-pair transconductance parameter $g = g^0 = 10^{-5} \Omega^{-1}$ is chosen. The phase response of the chain array is then simulated with the model (6), starting from initial random phase values. Fig. 8 shows the time evolution of the $\phi_k(t)$ phase variables: the phase difference among nearby oscillators converges to the constant value $\Delta \phi = \pi/5 \approx 0.628 \text{rad}$ meaning that the oscillator array is synchronized.

Hence, the array with unilateral external injection as described in Fig. 6 is considered: the low-noise oscillator $O_1$ injects unilaterally in the chain stages, numbered from 2 to $N+1 = 6$, with transconductance strength $g_r = g^0 = 10^{-6} \Omega^{-1}$. Fig. 9 shows the simulated time evolution of the $\phi_k(t)$ phase variables for the array with injection: compared to the case with no injection, array synchronization takes a longer time but eventually phase separations among chain array stages, i.e. for $k = 2, \ldots, N+1$, converge to $\Delta \phi \approx \pi/5$ with a relative error which is smaller than 2%. Besides that, chain array oscillators synchronize with the external signal in the way that oscillator $O_2$ in the center of the chain array, is almost in antiphase with $O_1$, i.e. $\phi_1(t) - \phi_2(t) \approx \pi$. This behavior is fully confirmed by detailed circuit-level simulations with SpectreRF. Fig. 10 reports the oscillators output voltages derived with the phase-domain simulation and model (3) and those obtained through simulation with SpectreRF: the waveforms (after being properly delayed) match with great accuracy. We thus conclude that, for the selected coupling strengths, the prescribed phase separation is preserved in the presence of external injection.

We pass now to analyze noise. To this aim, the output phase noise of the individual free-running oscillator is computed with the nnoise analysis of SpectreRF [19]. For the considered oscillator device, phase noise is dominated by the thermal white noise down to a corner frequency $f_C$ of some hundreds hertz. Over the frequency range of interest, the spectrum of the free-running oscillator is well approximated by

$$S_\phi(f) \approx \frac{100}{f^2} \text{rad}^2/\text{Hz}. \quad (26)$$

which corresponds to (10) with $K_f = 0$. In view of (9) or (11), the associated macro noise source $n(t)$ has a constant PSD $S_n(f) = S_n = 10^{-16} \text{rad}^2/\text{Hz}$. Noise in each oscillator in the chain array is thus modelled with a macro noise source $n(t)$ having constant PSD $S_n(f) = S_n$. For the array configuration with external injection, oscillator $O_1$ is a locked low-noise oscillator whose PSD of the type (12) is well approximated by

$$S_{\phi_1}(f) = \frac{100}{|f + f_B|^2}, \quad (27)$$

with control bandwidth $f_B = 10 \text{MHz}$. From (11), we deduce that macro noise source $n_1(t)$ has PSD

$$S_{n_1}(f) = \frac{100}{f_B^2} \frac{f^2}{|f + f_B|^2}. \quad (28)$$

First, we focus on the array with no injection and, starting from the simulated phase separations $\Phi_{kj}$ shown in Fig. 8 and using (19) and (20), we calculate the NTFs. Fig. 11 shows NTF$_{22}(f)$ describing self-noise transfer from source $n_2(t)$ to output $\phi_2(t)$ for oscillator number 2 (the same curves are found for the other oscillators in the chain). It also shows NTF$_{2j}(f)$, with $j \in \{1, 3, 4, 5\}$, describing noise transfer from other oscillators. We conclude that, even though such NTFs are attenuated compared to NTF($f$) = $1/(|2\pi f|^2$ for

### Table 1: Parameters of the LC Oscillator

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{DD}$</td>
<td>2.5 V</td>
</tr>
<tr>
<td>$I_p$</td>
<td>640 µA</td>
</tr>
<tr>
<td>$C$</td>
<td>0.3 pF</td>
</tr>
<tr>
<td>$L$</td>
<td>40 nH</td>
</tr>
<tr>
<td>$R$</td>
<td>11 kΩ</td>
</tr>
<tr>
<td>$(W/L)$</td>
<td>30</td>
</tr>
</tbody>
</table>
the free-running oscillator, they still vary as $1/f^2$ at the low frequencies.

Second, we calculate the NTFs for the array with injection using the simulated phase separations $\Phi_{kj}$ shown in Fig. 9. In this case, we see from Fig. 12 how self-noise function $\text{NTF}_{22}(f)$ and transfer functions $\text{NTF}_{2j}(f)$ for $j \in (3, 4, 5, 6)$ are significantly reduced down to $\approx 1\, \text{MHz}$ where they tend to constant values. The only NTF that keeps varying as $1/f^2$ is the NTF$_{21}(f)$ that weights clean noise source $n_1(t)$. This should reduce the total output phase noise. Fig. 13 shows the total output phase noise $S_{\phi}(f)$ computed with the phase-domain model for the cases of array without injection and with injection. External injection is seen to yield a remarkable noise reduction at the low frequencies. This result is fully confirmed by circuit-level phase noise simulations with SpectreRF as reported in Fig. 13 for comparison. Detailed SpectreRF simulation is indeed time consuming, (e.g. for the relatively simple case of $N = 5$ with external injection, the single simulation requires about 20 minutes on a quad core) and thus it is used only for verification purpose. By contrast, phase domain analysis requires only a few seconds and thus it allows extensive exploration of array performances as a function of parameters values. In the remainder of this section, we exploit the efficiency of the phase domain model to investigate two issues that are relevant for practical implementations. A first issue is connected to the variability of coupling coefficients due to fabrication uncertainty. To study this effect, we assume that coupling transconductance $g$ and injection strength $g_e$ undergo random variations around their nominal
values $g^0$ and $g^r$, according to $g = g^0 \cdot [1 + U(-a, a)]$ and $g_r = g^r \cdot [1 + U(-a, a)]$, respectively. The symbol $U(-a, a)$ denotes a stochastic variable uniformly distributed over the interval $(-a, a)$. Hence, we perform Monte Carlo simulations where for each randomly generated $g$ and $g_r$ values, we simulated the phase-domain response of the injected array and determine the steady-state phase-difference values. Fig. 14 shows the statistical distribution of the asymptotic phase difference $\Phi_2 = \Phi_2(t) = \Phi_3$ (very similar curves are obtained for the other differences) calculated with 500 Monte Carlo iterations and for $\pm 2\%$ coupling variability, i.e. $a = 0.02$: the resulting phase difference tends to be normally distributed around its mean value 0.625 rad with standard deviation $< 1\%$. Thus, the assumed variability of coupling coefficients does not significantly affect the array phase response. We also verify that the associated phase-noise spectra remain very close to the curve (iii) in Fig. 13 calculated in the absence of variability. It is worth underlining that the whole Monte Carlo simulation with the phase-domain model is accomplished in only 25 minutes while it would require more than one week if it were performed with detailed SpectreRF simulations.

The second issue is related to the importance of tightly matching the frequency $\omega_1$ of the locked low-noise oscillator $O_1$ to the free-running frequency $\omega_0 = 2\pi f_0$ of the oscillators in the array. To investigate this aspect, we assume $\omega_1 \neq \omega_k = \omega_0$ for $k = 2, \ldots, N + 1$, and we study the effect of a small frequency detuning $\omega_0 - \omega_1 = \Delta \omega = 2\pi \times (1\, \text{MHz})$. In the presence of frequency detuning, synchronization of each oscillator in the array with the low-noise one requires that, for $t \to \infty$, the following condition holds [13]

$$\omega_1 t + \omega_1 \phi_1(t) - \omega_k t + \omega_k \phi_k(t) = \Phi_{1k},$$

(29)

with $\Phi_{1k}$ being a constant. This means that, at synchronization, the phase differences

$$\phi_j(t) - \phi_k(t) = \omega_j t - \omega_k t + \Phi_{1k} = \Delta \omega t + \Phi_{1k}$$

(30)

for $k = 2, \ldots, N + 1$ should contain a term $\Delta \omega t$ growing linearly with time that compensates for frequency detuning. For oscillators within the array, instead, mutual synchronization condition remains as in (7) for $k, j \in (2, \ldots, N + 1)$.

Fig. 15 shows the phase response of the injected array in the presence of the assumed frequency detuning and for the coupling parameters $g = 10^{-5} \Omega^{-1}$ and $g_r = 10^{-6} \Omega^{-1}$ considered so far. With these parameters, condition (30) is not met and the oscillators within the arrays do not synchronize with $O_1$. As a result, the phase differences among nearby oscillators, do not reach constant values but exhibit fluctuations ($\pm 10\%$) with period $2\pi / \Delta \omega$, as shown in Fig. 16.

Array synchronization with $O_1$ is completely recovered if coupling coefficients values are increased to $g = 2.5 \cdot 10^{-7} \Omega^{-1}$ and $g_r = 2.5 \cdot 10^{-6} \Omega^{-1}$. With these parameters, the array phase response shown in Fig. 17 satisfies synchronization condition (30) and phase differences among nearby oscillators reach the prescribed constant phase separations. We also verified that, in this condition, the resulting phase-noise spectra are still very similar to the curve (iii) shown in Fig. 13.

![Fig. 14. Statistical distribution of the phase difference $\Phi_2 = \phi_2(t) - \phi_3(t)$ due to coupling coefficients variability.](image)

![Fig. 15. Phase response of the array in the presence of frequency detuning $\Delta \omega$ and for coupling $g = 10^{-5} \Omega^{-1}$ and $g_r = 10^{-6} \Omega^{-1}$.](image)

![Fig. 16. Phase differences $\Delta \phi_{k,k+1}(t) = \phi_k(t) - \phi_{k+1}(t)$ among nearby oscillators in the array when they do not synchronize with $O_1$.](image)
In view of (3) and (5), that are valid for harmonic oscillators, (32) is transformed into
\[
\dot{\tau}_k(t) = \begin{cases} \Gamma_M V_M \omega_0 \cos[\omega_0 (t + \tilde{\alpha}_k)] \\ + \sum_{j=1}^{N} g_{kj} \cos[\omega_0 (t \pm \tilde{\alpha}_j)] \tau_j(t) \\ + \sum_{j=1}^{N} g_{kj} \cos[\omega_0 (t + \tilde{\alpha}_j) + \pi/2] \tau_j(t) + n_{k}(t). \end{cases}
\]

We then use averaging [21], [22] and keep only the low-frequency terms arising from the cosine products in (33), obtaining
\[
\dot{\tau}_k(t) \approx B \left( \sum_{j=1}^{N} g_{kj} \cos(\Phi_{kj}) \right) \cdot \tau_k(t) - B \sum_{j=1}^{N} g_{kj} \cos(\Phi_{kj}) \cdot \tau_j(t) + n_{k_{eq}}(t). \tag{34}\]

where \( B = \omega_0 \Gamma_M V_M / 2 \) and where \( \Phi_{kj} \) are defined in (7).

Denoting \( \bar{\theta}(t) \) the vector that collects all excess phase variables \( \theta_k(t) = \omega_0 \tau_k(t) \) and \( \bar{n}(t) \) the vector of macro noise sources, from (34) we deduce
\[
\frac{d}{dt} \bar{\theta}(t) = A \cdot \bar{\theta}(t) + \omega_0 \bar{n}(t) \tag{35}\]

where the elements of matrix \( A \in \mathbb{R}^{N \times N} \) are given by
\[
a_{kk} = B \sum_{j=1, j \neq k}^{N} g_{kj} \cos(\Phi_{kj}) \quad a_{kj} = -B g_{kj} \cos(\Phi_{kj}) \quad \text{for } k \neq j. \tag{36}\]

Finally, Fourier transforming (35) and passing to power noise, the closed-form expression (17) is obtained with matrix \( T(f) \) defined as in (19).

We also observe how entries of matrix \( A \) are decided by the array topology, which is described by \( g_{kj} \) coupling coefficients, and by the phase-difference values \( \Phi_{kj} \) (7) that are reached at synchronization. Matrix \( A \) is singular with rank \( N - 1 \) and thus has a null eigenvalue, let’s say \( \lambda_1 = 0 \). In addition, from (35) we see that the eigenvalues of \( A \) govern the dynamics induced by any perturbation of the steady-state solution of (6) [20]. As a result, for any phase separation \( \Phi_{kj} \) that corresponds to a stable solution of (6), i.e. that can be obtained by numerically simulating (6) in time, the eigenvalues \( \lambda_k \) for \( k = 2, \ldots, N \) should necessarily have negative real part, i.e. \( \Re(\lambda_k) < 0 \).

\textbf{REFERENCES}


Paolo Maffezzoni (M’08–SM’15) received the Laurea degree (summa cum laude) in Electrical Engineering from Politecnico di Milano, Italy, in 1991 and the PhD degree in Electronic Information Engineering from Universita’ di Brescia, Italy, in 1996. Since 1998, he has been with Politecnico di Milano where he is an Associate Professor of Electrical Engineering. His research interests include analysis, simulation and design of nonlinear dynamical circuits and systems, oscillating devices, synchronization, stochastic simulation. He has over 120 research publications among which 64 papers in international journals. He has served as an Associate Editor for the IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems.

Bichoy Bahr (S’10) received the B.Sc. degree with honors in 2008 and the M.Sc. degree in 2012, both in electrical engineering, from Ain Shams University, Cairo, Egypt. He worked as an Analog/Mixed Signal and MEMS Modeling/Design Engineer at MEMS Vision, Egypt. Mr. Bahr is currently working towards the Ph.D. degree in the Department of Electrical Engineering at Massachusetts Institute of Technology (MIT), Cambridge, MA, USA. He is a research assistant in the HybridMEMS group, MIT. His research interests include the design, fabrication, modeling and optimization of monolithically integrated unrelease MEMS resonators, in standard ICs technology. He is also interested in multi-GHz MEMS-based monolithic oscillators, coupled oscillator-arrays and unconventional signal processing.

Zheng Zhang (S’09-M’15) received his Ph.D degree (2015) in Electrical Engineering and Computer Science from Massachusetts Institute of Technology (MIT), Cambridge, MA, his M.Phil degree (2010) in Electrical and Electronic Engineering from the University of Hong Kong, Hong Kong, and his B. Eng degree (2008) in Electronics from Huazhong University of Science and Technology, Wuhan, China. He is currently a postdoc associate with the Mathematics and Computer Science Division at Argonne National Laboratory, Argonne, IL. His research interests include high-dimensional uncertainty quantification, tensors and data analysis. Applications of interest include nano-scale devices, circuits and systems, energy systems and biomedical computation. Dr. Zhang received the 2015 Doctoral Dissertation Seminar Award (i.e., best dissertation award) from the Microsystems Technology Laboratory of MIT, the 2014 Best Paper Award from IEEE Transactions on CAD of Integrated Circuits and Systems, the 2011 Li Ka Shing Prize (an annual best M.Phil/Ph.D. award) from the University of Hong Kong, the 2010 Mathworks Fellowship from MIT, and three additional best paper nominations in international conferences. His industrial research experiences include Coventor Inc. and Maxim-IC.

Luca Daniel (S’98-M’03) received the Ph.D. degree in Electrical Engineering from the University of California, Berkeley, in 2003. He is currently a Full Professor in the Electrical Engineering and Computer Science Department of the Massachusetts Institute of Technology (MIT). Industry experiences include HP Research Labs, Palo Alto (1998) and Cadence Berkeley Labs (2001). His current research interests include integral equation solvers, uncertainty quantification and parameterized model order reduction, applied to RF circuits, silicon photonics, MEMs, Magnetic Resonance Imaging scanners, and the human cardiovascular system. Prof. Daniel was the recipient of the 1999 IEEE Trans. on Power Electronics best paper award; the 2003 best PhD thesis award from the Electrical Engineering and the Applied Math departments at UC Berkeley; the 2003 ACM Outstanding Ph.D. Dissertation Award in Electronic Design Automation; the 2009 IBM Corporation Faculty Award; the 2010 IEEE Early Career Award in Elctronic Design Automation; the 2014 IEEE Computer Aided Design best paper award; and seven best paper awards in conferences.