Path selection in the growth of rivers

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Path selection in the growth of rivers

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River networks exhibit a complex ramified structure that has inspired decades of studies. However, an understanding of the propagation of a single stream remains elusive. Here we invoke a criterion for path selection from fracture mechanics and apply it to the growth of streams in a diffusion field. We show that, as it cuts through the landscape, a stream maintains a symmetric groundwater flow around its tip. The local flow conditions therefore determine the growth of the drainage network. We use this principle to reconstruct the history of a network and to find a growth law associated with it. Our results show that the deterministic growth of a single channel based on its local environment can be used to characterize the structure of river networks.


The authors declare no conflict of interest.

Significance

The complex patterns of river networks evolve from interactions between growing streams. Here we show that the principle of local symmetry, a concept originating in fracture mechanics, explains the path followed by growing streams fed by groundwater. Although path selection does not by itself imply a rate of growth, we additionally show how local symmetry may be used to infer how rates of growth scale with water flux. Our methods are applicable to other problems of unstable pattern formation, such as the growth of hierarchical crack patterns and geologic fault networks, where dynamics remain poorly understood.

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neglect the Poisson term in Eq. 3, and the field can be approximated as (23, 24)
\[ \nabla^2 \phi = 0. \]  
[4]
For a semiinfinite channel on the negative x axis with boundary conditions
\[ \phi(\theta = \pm \pi) = 0, \]  
[5]
the harmonic field around the tip can be expressed in cylindrical coordinates as (23)
\[ \phi(r, \theta) = a_1 r^{1/2} \cos \left( \frac{\theta}{2} \right) + a_2 r \sin(\theta) + O\left(r^{3/2}\right), \]  
[6]
where, as shown in Fig. 1, \( r \) is the distance from the channel head and the channel is located at \( \theta = \pm \pi \). The coefficients \( a_i, i = 1,2 \) are determined by the shape of the water flux coming from the outer boundary.

Because the leading term in the expansion of Eq. 6 is symmetric with respect to \( \theta \), we do not expect it to influence the direction in which the channel grows. Thus, we must consider the subdominant term that breaks the symmetry and can therefore cause the stream to bend as it grows. Other terms in the expansion are negligible in the vicinity of the channel head.

In fracture mechanics, a crack maintains a symmetric elastic field around its tip to release the maximum stress as it propagates (10, 13). Inspired by this example, we suggest that a channel grows in the direction which maintains a locally symmetric groundwater flow. Accordingly, we formulate an analog of the principle of local symmetry as follows: A channel grows in the direction for which the coefficient \( a_2 \) vanishes. Fig. 2 expresses this notion pictorially.

**Evaluation of the Principle of Local Symmetry**

We now evaluate the principle of local symmetry (PLS) from three points of view. First, we justify its application to channel networks based on simple physical reasoning. We then show that the PLS is mathematically equivalent to assuming that a channel grows along the groundwater flow lines, an assumption that has been shown to be consistent with field observations (24). Finally, we describe a numerical procedure to grow a network according to this principle.

**Negative Feedback Induced by Groundwater.** As a channel grows, it moves the boundary of the network, and this changes the groundwater flow near the tip of the channel. Mathematically, the coefficients \( a_n \) of the expansion of the local flow field near the tip [6] vary during the channel’s growth.

To understand how groundwater controls growth, we study the influence of the coefficient of the first two dominant terms (as \( r \to 0 \)) \( a_1 \) and \( a_2 \) on the distribution of groundwater into the channel. At a distance \( r \) from the channel’s head, the amount of water collected through the right-hand side of the stream reads
\[ Q_{\text{right}} \approx \int_0^r \frac{\partial \phi}{\partial \theta} \frac{dr'}{r'} \approx a_1 \sqrt{r} - a_2 r. \]  
[7]
Similarly, the amount of water collected through the left-hand side reads
\[ Q_{\text{left}} \approx a_1 \sqrt{r} + a_2 r. \]  
[8]
For illustration, let us assume that the second coefficient \( a_2 \) is negative. Eqs. 7 and 8 indicate that more groundwater seeps into the stream through its right-hand bank (\( \theta < 0 \)) than through its left-hand bank (\( \theta > 0 \)). We expect this asymmetric seepage to erode the bank collecting more groundwater faster, thus causing...
Streamline growth: (A) in the physical plane, and (B) in the mathematical plane. The channel (solid blue) growing along a flow line that enters the tip is geodesic (green). Red line: example of nongeodesic growth.

The geometry of a network draining a diffusion field depends on the dynamics of its growth. In particular, when two nascent tips grow off of a parent channel, the angle of this bifurcation depends on the growth rule. For instance, if a channel grows along the stream line that intersects its tip (Fig. 3A), it should bifurcate at $2\pi/5 = 72^\circ$ (24). This value accords with field measurements collected in a small river network near Bristol, Florida. This observation suggests that a channel maintains a locally symmetric groundwater flow in the vicinity of the tip as it grows.

Equivalence with Geodesic Growth. The geometry of a network growing according to the PLS, (i) solve the diffusion field around the network, (ii) evolute the network according to the local properties of the field near its tips, and repeat these operations (23). Whether the diffusion field satisfies the Poisson equation or the Laplace equation does not impact significantly the numerical procedure. However, complex analysis facilitates considerably the derivation of formal results about network growth in Laplacian fields.

In particular, the Laplacian field representing the groundwater flow defines a complex map from the physical plane to the upper half-plane (Fig. 3). This map encapsulates both the geometry of the network at a specific time, and the groundwater flow around it. Translating geodesic growth and local symmetry into this formalism, we find that they define the same growth rule (SI Equivalence Between Geodesic Growth and Local Symmetry).

This equivalence does not necessarily hold when the channel grows in a field which satisfies the Poisson equation. However, because the above demonstration is based on the local properties of the field, the source term of the Poisson equation comes as an external flux only. We therefore expect that the PLS is but a reformulation of geodesic growth, which accords with field observation. An important consequence of this reformulation is the ability to specify the growth direction in a precise, well-controlled manner, to which we now turn.

Numerical Implementation. We design a numerical method to calculate trajectories that explicitly maintain local symmetry, and compare its results to an analytic solution. We consider a simple case in which one channel grows in a confined rectangular geometry $-1 < x < 1, -y < y < 30$ in a Laplacian field. We first calculate the trajectory using the PLS. Our algorithmic implementation of this principle requires that at each step streams grow in the direction for which $\phi_x$ vanishes. (Further details are in SI Propagation of a Channel.) We apply the following boundary conditions: a zero elevation at the bottom ($\phi = 0$ at $y = 0$), which corresponds to a main river or an estuary; no flux at the sides ($\partial \phi / \partial x = 0$ at $x = 1, -1$), which corresponds to a groundwater divide; and a constant flux of water from the top, ($\partial \phi / \partial y = 1$). We then initiate a small slit ($l = 0.01$) perpendicular to the bottom edge, and allow it to grow according to the PLS. Not surprisingly, a stream initiated at the middle of the lower edge ($x = 0; y = 0$) continues straight. However, when we break the symmetry and initiate a slit left of the center ($x = -0.5; y = 0$) the stream bends toward the center of the box.

To validate our numerical implementation of the PLS, we compare our numerical trajectory to the evolution of a path in a Laplacian field according to the deterministic Loewner equation (26–28). In the Loewner model, the properties of analytic functions in the complex plane are used to map the geometry into the complex half-plane or into radial geometry, and to find the solution for the field. Then, at each time step, a slit is added to the tip of the channel based on the gradient of the field entering the tip. Fig. 4 compares results from the two approaches. We find that the two solutions exhibit the same trajectory.

Fig. 4. (Left) Trajectory of a single stream initiated at the bottom, to the left of center. Blue: the analytical solution (28). Red: the numerical trajectory of a stream grown according to the PLS. (Inset) The initial slit. (Right) The average error ($\|\Delta x(y)\|$) between the numerical trajectory and the analytical solution with the decrease of the step size, $ds$.
Growth of a Real Stream Network

The evolution of a channel is defined by the field in the vicinity of the tip. However, this field is nonlocal and highly dependent on the boundary conditions imposed by the ramified network of the streams. In this section, the numerical method developed in the previous section is used to compute trajectories in more general settings where no analytic solutions are possible.

Growth According to Local Symmetry. We seek to determine if growth of a real stream network is consistent with the PLS. We study a network of seepage valleys located near Bristol, FL on the Florida Panhandle (17). The network is presented in Fig. 5. The valley network is obtained from a high-resolution LIDAR (Light Detection and Ranging) map with horizontal resolution of 1.2 m and vertical resolution of less than 5 cm (17). In this network, groundwater flows through unconsolidated sand above the relatively impermeable substratum, and into the streams (17, 29). Previous analyses indicate that the homogeneity of the sand is consistent with the assumption of constant $\kappa$ (18, 19, 24, 30).

The flow is determined by the Poisson equation

$$\partial \phi = \kappa \nabla^2 \phi,$$

which we identify with $\nabla \phi$ in the expansion 6 that corresponds to the water flux entering the tip. Then, we remove a segment, $l_i$, from the tip of the $i$th tributary and propagate it forward to its original length in five small steps (to reduce numerical error). The growth of each stream is characterized by two variables: its growth rate and the direction of its growth. We assume that the velocity of a stream is proportional to the magnitude of the gradient of the field, raised to a power $\eta$:

$$v \sim |\nabla \phi|^\eta \sim a_1^\eta. \quad [9]$$

A similar growth law has been considered in Laplacian path models (25, 28) and related erosion models (31–33). Thus, the length $l_i$ of each segment that we remove from a channel, and later add as it grows forward, is defined according to its relative velocity: $l_i \propto \sqrt{|\nabla^2 \phi_i|}$, where $v_i \propto a_1^\eta$, of the $i$th channel ($\eta = 1$) and $v = 1/\eta \sum_i v_i$ is the mean velocity. We fix the total length removed from the network of $n$ tributaries to be $n$ meters. Each channel then grows in a direction that fulfills local symmetry, i.e., in the direction for which $a_1$ vanishes. After we grow the network back to its original length, we study each of the tributaries separately, and measure the angle, $\beta$, between the real trajectory of the stream and the reconstructed trajectory. We perform this calculation for 255 channel heads in the Florida network. We obtain a mean of $\beta$ around zero with an SD of $\sim 7^\circ$. Different values of $\eta$ give a similar distribution. We find that a mean around 0 of the angle error is consistent with a growth that fulfills local symmetry. However, some of the streams deviate significantly from their real growth direction, which may suggest that other factors account for their growth.

To evaluate the significance of the results, we suggest a null hypothesis in which the streams grow in the direction of the tangent regardless of the value of $a_2$. We obtain the direction of the tangent based on the last two grid points (approximately 2 m) of the channel trajectory after retraction. Then, we calculate the angle, $\phi$, between the tangent direction and the real trajectory, as shown in Fig. 6. We find that a growth according to the PLS reduces on average the error angle by 50% compared with growth in the direction of the tangent, and therefore improves prediction of future growth relative to the null model. An $F$ test (34) shows that the reduction of the variance using PLS is statistically significant, with $P < 10^{-9}$ (assuming each measurement is statistically independent).

Growth Law. To understand the deviations between the real and the calculated path, we hypothesize that the deviant streams grew in a different environment than currently exists, e.g., the neighboring tributaries were relatively undeveloped (or overdeveloped) when the studied stream reached its current location. To illustrate this idea, in Fig. 7 we show the trajectories of two streams with different velocities, and compare their evolution in Poisson field for different growth exponents $\eta$. One notices that for smaller $\eta$ the slower streams are more likely to deviate from their real trajectory, but for higher $\eta$ the faster streams change their course. Only when $\eta = \eta_0$ (the correct value of $\eta$), any errors will remain uncorrelated to the velocity of the streams.

Motivated by this reasoning, we study the correlation between the flux entering the tip, which we identify with $a_1$, and the angle $\beta$ for different values of $\eta$. Retracting the network with different $\eta$ creates different boundary conditions and influences the
The importance of the growth exponent $\eta$ is in the evolution of the network: Negative $\eta$ will generate a stable network in which each perturbation, or small channel, will survive regardless of the water flow entering the tip. A positive $\eta$ results in an unstable structure in which a small difference in the velocity of two competing channels is amplified and may lead to a screening mechanism and the survival only of the faster channel (28). Fig. 9 contains a schematic representation of this concept. The small positive exponent found for the stream network in Florida indicates that this network is unstable. This conclusion is consistent with the prediction of a highly ramified network.

**Summary**

In summary, we offer a criterion for path selection of a stream in a diffusing field. We show that this criterion, which is based on the PLS (10, 13), predicts accurately the evolution of channels fed by groundwater. We suggest a method to infer the history of a real network by reconstructing it according to the PLS and evaluating errors for different growth laws. We parameterize the large-scale relationship between water flux and sediment transport with a single exponent and show that for the Florida network this growth exponent is roughly 0.7. We envision that our methods may also be applied to other problems, such as the growth of hierarchical crack patterns (35–37) and geological fault networks (38), to provide a better understanding of their evolution.

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