Universal four-component Fermi gas in one dimension

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Universal four-component Fermi gas in one dimension

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I. INTRODUCTION

Experiments using ultracold atomic gases have achieved striking progress in realizing and studying various many-body systems previously regarded as purely theoretical models. One example is the Tonks-Girardeau gas in one dimension, proposed 50 years ago [1] and realized experimentally in 2004 [2,3]. Another example is the BCS-BEC crossover in a two-component Fermi gas with a short-range two-body interaction [4–6]. It was predicted about 40 years ago [7–9] and hypothesized that these two limits are smoothly connected without phase transitions just as in three dimensions. In the unitarity limit, a one-dimensional Bertsch parameter and its connection to the Tomonaga-Luttinger parameter are introduced in Sec. III C, whose value can be estimated in principle by using $\epsilon$ expansions. Exact relationships involving a contact density are derived in Sec. III D and the contact density is determined from the ground-state energy density in the BCS and BEC limits. Finally, Sec. IV is devoted to the summary of this paper.

II. FEW-BODY PROBLEMS

A. Lattice model

We start with a system of fermions with four components labeled by $\sigma = a,b,c,d$ living on a one-dimensional lattice. We assume that each lattice site can accommodate one, two, or three particles with no change in energy, but an introduction of a fourth particle into a site with three particles releases a finite amount of energy. The lattice Hamiltonian for such a system is

$$H = -t \sum_{\langle xy \rangle, \sigma} c_{x \sigma}^\dagger c_{y \sigma} - U \sum_x c_{1a}^\dagger c_{1b}^\dagger c_{1c} c_{1d} + c_{1b}^\dagger c_{1c} c_{1b} c_{1d}. \quad (1)$$

We will be interested in the dilute limit where the average number of particles per site is small. To find the universal regime, we consider the scattering among all different components of fermions. Such a four-body problem is described by the Schrödinger equation

$$-t \sum_{\sigma} \Delta_{\sigma} + V(x) \Psi(x) = E \Psi(x), \quad (2)$$

where $x = (x_a,x_b,x_c,x_d)$ is a set of coordinates of four particles and $\Delta_{\sigma}$ is the discrete Laplacian with respect to $x_{\sigma}$; $\Delta_{\sigma} \Psi(x_{\sigma}) \equiv \Psi(x_{\sigma} + l) + \Psi(x_{\sigma} - l) - 2\Psi(x_{\sigma})$ with $l$ being
the lattice spacing. The four-body interaction potential is given by $V(x) = -U$ when all $x_i$ are equal and $V(x) = 0$ otherwise.

Since $V(x)$ is translationally invariant, it is convenient to introduce new coordinates $X = (x_1 + x_2 + x_3 + x_4)/4$, $r_1 = (x_1 + x_2 - x_3 - x_4)/2$, $r_2 = (x_1 - x_2 + x_3 - x_4)/2$, and $r_3 = (x_1 - x_2 - x_3 + x_4)/2$ and assume $\Psi(x)$ to be independent of the center-of-mass coordinate $X$. The Schrödinger equation (2) in terms of the remaining three relative coordinates $r = (r_1, r_2, r_3)$ becomes

$$
-\frac{i}{\hbar} \sum_{i=1}^{4} \Delta_i - \delta_{\alpha} g U \right] \Psi(r) = E \Psi(r),
$$

where $\Delta_i \Psi(r) \equiv \Psi(r + e_i) + \Psi(r - e_i) - 2 \Psi(r)$ with $e_1 = \frac{1}{2}(1,1,1)$, $e_2 = \frac{1}{2}(1, -1, -1)$, $e_3 = \frac{1}{2}(-1,1, -1)$, and $e_4 = \frac{1}{2}(-1, -1, 1)$. Equation (3) is equivalent to the Schrödinger equation describing one particle moving in a body-centered cubic lattice with an attractive potential of the magnitude $U$ concentrated at one lattice site.

One can see from Eq. (3) that the zero-energy wave function at a long distance has the form

$$
\Psi(|r| \to \infty)_{E=0} \propto \frac{1}{|r|} - \frac{1}{a},
$$

where $|r|^2 = \sum_{i=1}^{3} (x_i - X)^2 = \frac{1}{4} \sum_{i<j} (x_i - x_j)^2$ is the hyperradius of four particles in one dimension. The form (4) is familiar in two-body scattering problems in three dimensions, which can be understood from the fact that the continuum limit of Eq. (3) with $\frac{\hbar^2}{2m} \equiv \hbar^2/(2m)$ is exactly the Schrödinger equation in three dimensions. Here $a$ is an arbitrary real parameter characterizing the long-distance physics and referred to as the scattering length. By matching the solution of Eq. (3) [18]:

$$
\frac{\Psi(r)}{\Psi(0)} \bigg|_{E=0} = 1 - \frac{\Gamma(\frac{1}{4})^4 U}{32\pi^3} \frac{|U|}{t}
+ \frac{1}{8\pi} \int_{-\pi}^{\pi} \frac{dk}{(2\pi)^3} I - \cos k_1 \cos k_2 \cos k_3
\to 1 - \frac{\Gamma(\frac{1}{4})^4 U}{32\pi^3} \frac{|U|}{t}
+ \frac{1}{8\pi t} \frac{|r|}{|E|} \quad (|r| \to \infty)
$$

with the asymptotic form (4), we find $a$ in units of the lattice spacing $l$ to be

$$
\frac{l}{a} = \frac{\Gamma(\frac{1}{4})^4 U}{4\pi^2} = 8\pi t.
$$

The scattering length $a$ can be fine-tuned to infinite corresponding to the four-body resonance by choosing

$$
\frac{U}{t} = \frac{32\pi^3}{\Gamma(\frac{1}{4})^4} \approx 5.742.
$$

This value of $U/t$ separates the weak coupling regime ($a < 0$) with no bound state from the strong coupling regime ($a > 0$) in which there exists a four-body bound state (tetramer). The wave function and binding energy of the tetramer for $a \gg l$ are given by the universal formulas independent of the lattice parameters:

$$
\Psi(|r| \to \infty) \propto \frac{e^{-|r|/a}}{|r|} \quad \text{and} \quad E_0 = -\frac{\hbar^2}{2ma^2}.
$$

The long-distance physics near the critical value of $U/t$ should be universal and, in particular, scale and conformal invariance are achieved in the unitarity limit $a \to \infty$ [14].

### B. Field-theoretical formulation

The physics in the universal regime can be described by the following continuum-limit Hamiltonian density (hereafter $\hbar = 1$):

$$
\mathcal{H} = -\sum_{\sigma} \left[ \frac{\psi^\dagger \nabla^2 \psi}{2m} - c_0 \bar{\psi}_a \psi_b \bar{\psi}_c \psi_d \psi_e \psi_f \right]\quad (9)
$$

Throughout this paper, we neglect two-body and three-body interactions and interactions involving the same components of fermions. In addition to the translational and Galilean symmetries, the Hamiltonian density has global $U(1)$ and SU(4) symmetries,

$$
\psi_\sigma \to e^{i\theta} \psi_\sigma \quad \text{and} \quad \psi_\sigma \to U_{\sigma\sigma'} \psi_{\sigma'},
$$

corresponding to the conservation charges and SU(4) spins, respectively.

The second term in Eq. (9) describes the four-body contact interaction among different components of the fermionic field $\psi_\sigma$. $c_0$ is a cutoff-dependent coupling constant and can be related to the above-introduced scattering length $a$ by matching in the four-body problem. The four-body scattering amplitude $A(E, p)$ is obtained by summing the Feynman diagrams in Fig. 1 into a geometric series:

$$
[iA(E, p)]^{-1} = \frac{1}{i c_0} + \int_{-\infty}^{\infty} \frac{dk_1 dk_2 dk_3}{(2\pi)^3} \frac{2m}{k_1^2 + k_2^2 + k_3^2 + (k_1 + k_2 + k_3)^2 + p^2/4 - 2mE - i0^+}.
$$

Here the integrations over momenta $k_1, k_2,$ and $k_3$ are linearly divergent. Introducing a momentum cutoff $\sqrt{k_1^2 + k_2^2 + k_3^2} < \Lambda$ and choosing the cutoff dependence of $c_0$ as

$$
\frac{1}{c_0} = \frac{m \Lambda}{3\sqrt{3}\pi} - \frac{m}{4\pi a},
$$

where the dot represents the bare vertex $i\Delta_0$ and the dashed line represents the scattering amplitude $iA(E, p)$. 

![FIG. 1. Feynman diagrams describing the four-body scattering in vacuum.](image-url)
we obtain the following cutoff-independent scattering amplitude in the limit \( \Lambda \to \infty \):

\[
\mathcal{A}(E, p) = \frac{4\pi}{m} \frac{1}{-1/a + \sqrt{p^2 + 2mE - i0^+}}. \tag{13}
\]

In particular, when \( a \) is a real and negative \( E \), indicating the existence of the four-body bound state. Because its binding energy is given by \( E_0 = -1/(2ma^2) \), we can identify \( a \) in Eq. (12) with the scattering length introduced in Eq. (4).

C. Five-body problem and Efimov effect

The above arguments equally apply to four-component bosons in one dimension. However, many-body systems of attractive bosons tend to be unstable to collapse, in contrast to the case of fermions where the Pauli exclusion principle acts against such a collapse. We now show how such a difference already appears in a five-body problem: five bosons develop deep bound states while five fermions do not. This can be seen by studying a scaling dimension of five-body composite operator \( \phi \psi_a \) in the unitarity limit \( a \to \infty \), where \( \phi = c_0 \psi_a \psi_b \psi_c \psi_d \) is a tetramer field. If this operator has a real scaling dimension, the corresponding five-body system is scale invariant and thus does not support bound states. However, if the scaling dimension is complex, the full scale invariance is broken down to a discrete one [19], which indicates the formation of an infinite tower of bound states. Such a connection between the complex scaling dimension and the infinite tower of bound states has been observed in resonantly interacting three bosons or mass-imbalanced fermions in three dimensions [19], two particles interacting with a \( 1/r^2 \) potential [20], and the nonrelativistic anti-de Sitter spacetime and conformal field theory (AdS-CFT) correspondence [21].

Feynman diagrams to renormalize \( \phi \psi_a \) are depicted in Fig. 2. The vertex function \( z(p) \) satisfies the following integral equation:

\[
z(p) = 1 + \lambda \int_{-\infty}^{\infty} \frac{dq}{2\pi} z(q) \frac{8\pi}{\sqrt{5|q|}} \times \int_{-\infty}^{\infty} \frac{dk_1 dk_2}{(2\pi)^2} \frac{1}{2p^2 + 2q^2 + p^2 + k_1^2 + k_2^2 + k_1 k_2}, \tag{14}
\]

where \( \lambda = \pm 1 \) for four-component bosons or fermions. Because of the scale invariance, we can assume \( z(p) \propto (|p|/\Lambda)^\gamma \). Performing the integrations in Eq. (14), we find that the anomalous dimension \( \gamma \) satisfies [22]

\[
1 = -\frac{4\lambda}{\sqrt{15}} \frac{\cos (\gamma \arctan 1/15)}{\gamma \sin (\gamma \arctan 1/15)}. \tag{15}
\]

The anomalous dimensions of even-parity operators satisfy the same equation (15) and their scaling dimensions are given by

\[
\Delta_\phi + \Delta_\psi + \gamma = \frac{5}{2} + \gamma. \tag{16}
\]

For fermions (\( \lambda = -1 \)), we can find a series of real solutions; \( \gamma = 1.59, 4.08, 5.99, \ldots \). According to the operator-state correspondence [23–25], each solution corresponds to the energy of resonantly interacting five fermions in a one-dimensional harmonic potential by

\[
E = \left( \frac{5}{4} + \gamma \right) \omega. \tag{17}
\]

On the other hand, for bosons (\( \lambda = +1 \)), in addition to real solutions \( \gamma = 2.25, 3.91, 6.01, \ldots \), we can find a pair of complex solutions \( \gamma = \pm 0.735i \). This is a signal of the formation of an infinite tower of five-body bound states (pentamers) whose spectrum exhibits the discrete scaling symmetry [26]:

\[
\frac{E_n}{E_{n+1}} = e^{2\gamma/|\Im \gamma|} = (71.8)^2. \tag{18}
\]

For identical bosons with the four-body resonant interaction, the anomalous dimension of the five-body composite operator \( \phi \phi \) satisfies Eq. (15) with \( \lambda = 4 \). In addition to real solutions \( \gamma = 3.43, 6.02, 8.15, \ldots \), it has complex solutions \( \gamma = \pm 1.25i \), and therefore, the spectrum of pentamers is much denser [26]:

\[
\frac{E_n}{E_{n+1}} = e^{2\gamma/|\Im \gamma|} = (12.4)^2. \tag{19}
\]

This is an analog of the Efimov effect for three identical bosons in three dimensions [27]. Note that the ordinary Efimov effect can occur only in a spatial dimension \( 2.30 < d < 3.76 \) and not in one dimension [28]. For comparison, the scaling factor for Efimov trimers in three dimensions is known to be \( e^{2\gamma/|\Im \gamma|} = (22.7)^2 \).

Similarly, the anomalous dimensions of odd-parity operators [e.g., \( 4\phi(\nabla \psi_a) - (\nabla \psi) \phi_a \)] are found to satisfy [22]

\[
1 = -\frac{4\lambda}{\sqrt{15}} \frac{\sin (\gamma \arctan 1/15)}{\gamma \cos (\gamma \arctan 1/15)}. \tag{20}
\]

with the scaling dimensions given by Eq. (16). In this channel, both fermions and bosons have real solutions only; \( \gamma = 0.833, 3.15, 4.87, \ldots \) for \( \lambda = -1 \), \( \gamma = 1.17, 2.85, 5.12, \ldots \) for \( \lambda = +1 \), and \( \gamma = 2.04, 5.53, 6.57, \ldots \) for \( \lambda = 4 \). Therefore, the corresponding states in a harmonic potential are universal and their energies are given by Eq. (17) [23–25].

III. MANY-BODY PROBLEMS

Since bosons with the four-body resonant interaction in one dimension develop deep five-body bound states, the corresponding many-body system cannot be stable to collapse. Therefore, we will study the many-body physics of four-component Fermi gas in one dimension as a function of
the dimensionless parameter \(k_Fa\) characterizing the short-range four-body interaction. Here \(k_F \equiv \pi n/4\) is the Fermi momentum defined by the total number density \(n\). In analogy with the BCS-BEC crossover in three dimensions, we will refer to the weak (strong) coupling limit \(k_Fa \to -(+)0\) as the “BCS” (“BEC”) limit. We caution that this terminology should not be taken literally, since we do not have the spontaneous symmetry breaking in one dimension. We shall see below that the properties of our system in these two limits are consistent with the crossover hypothesis that they are smoothly connected without phase transitions.

### A. BCS limit

The many-body physics is conveniently described by introducing the chemical potential term \(-\mu \psi_j^2 \psi^\dagger_j\) to the Hamiltonian density in Eq. (9). In the weak coupling (BCS) limit \(k_Fa \to -0\), the system develops two Fermi points \(k = \pm k_F\) and low-energy degrees of freedom are excitations around them. Therefore, assuming \(|k| > k_F|\ll k_F\), we can linearize the dispersion relation and express the fermionic field in terms of two slowly varying fields describing excitations around the Fermi points:

\[
\psi_\sigma(x) \simeq e^{ik_Fx} \psi_\sigma^R(x) + e^{-ik_Fx} \psi_\sigma^L(x).
\]

The low-energy effective theory consistent with the original symmetries (10) can be written as

\[
\mathcal{H}_{\text{BCS}} = -i v_F \psi_\sigma^R \nabla \psi_\sigma^R + i v_F \psi_\sigma^L \nabla \psi_\sigma^L
+ g_1 \psi_\sigma^L \psi_\sigma^L \psi_\sigma^R + g_2 \psi_\sigma^R \psi_\sigma^R \psi_\sigma^L + \frac{\Delta_0}{2} (\psi_\sigma^R \psi_\sigma^L \psi_\sigma^R \psi_\sigma^L + \psi_\sigma^L \psi_\sigma^L \psi_\sigma^L \psi_\sigma^R),
\]

where \(v_F \equiv k_F/m\) is the Fermi velocity and summations over \(\sigma(\tau) = a, b, c, d\) are implicitly understood. The \(g_1\) term describes the backward scattering and the \(g_2\) and \(g_4\) terms describe the forward scatterings. The low-energy parameters \(g_1, g_2,\) and \(g_4\) are determined by matching two-body scattering amplitudes at the Fermi points with those from the microscopic theory (9). To leading order in \(k_Fa\), we find

\[
g_1 = g_2 = g_4 = -\frac{4v_F}{\pi} k_F|a| + O((k_Fa)^2).
\]

The spectrum of the low-energy effective theory \(\mathcal{H}_{\text{BCS}}\) can be obtained exactly by bosonization. We introduce charge current operators

\[
J_0^{R(L)} \equiv \psi_\sigma^R \psi_\sigma^{L(\dagger)}
\]

and spin current operators

\[
J_0^{R(L)} \equiv \psi_\sigma^R (\tau_\alpha)_\sigma^\dagger \psi_\sigma^{L(\dagger)},
\]

where \(\tau_\alpha\) with \(\alpha = 1, \ldots, 15\) are generators of SU(4) Lie algebra normalized as \(\text{Tr}(\tau_\alpha \tau_\beta) = \delta_{\alpha\beta}/2\). Using these current operators, \(\mathcal{H}_{\text{BCS}}\) can be expressed in terms of bosons living on two mutually commuting parts (spin-charge separation) [29,30]:

\[
\mathcal{H}_{\text{BCS}} = \mathcal{H}_{\text{ch}} + \mathcal{H}_{\text{sp}}
\]

with

\[
\mathcal{H}_{\text{ch}} = \sum_{\alpha=1}^{15} \left[ \frac{2\pi v_F - g_4}{5} (J_0^R J_\alpha^R + J_\alpha^L J_0^L) - 2g_1 J_0^R J_\alpha^L \right].
\]

The charge part \(\mathcal{H}_{\text{ch}}\) is easily diagonalized by the Bogoliubov transformation and equivalent to the Tomonaga-Luttinger liquid. Introducing a bosonic field \(\phi_0(x)\) by

\[
\partial_x \phi_0 = \frac{\pi}{2} (J_0^R + J_0^L)
\]

and its canonical conjugate by

\[
\Pi_0 = -\frac{i}{2} (J_0^R - J_0^L),
\]

the Hamiltonian density can be brought into the standard form

\[
\mathcal{H}_{\text{ch}} = \frac{\pi K v_\phi}{2} \Pi_0^2 + \frac{v_\phi}{2\pi K} (\partial_x \phi_0)^2,
\]

which describes a gapless excitation transporting a particle number with the linear dispersion relation \(E = \pm v_\phi k\). Here the Tomonaga-Luttinger parameter \(K\) and the sound velocity \(v_\phi\) in the BCS limit \(k_Fa \to -0\) are given by

\[
K = \sqrt{\frac{2\pi v_F + 3g_4 - 4g_2 + g_1}{2\pi v_F + 3g_4 + 4g_2 - g_1}} \to 1 + \frac{6k_F|a|}{\pi^2},
\]

and

\[
v_\phi = \sqrt{\left(\frac{v_F + 3g_4}{2\pi}\right)^2 - \left(\frac{4g_2 - g_1}{2\pi}\right)^2} \to v_F \left(1 - \frac{6k_F|a|}{\pi^2}\right).
\]

We note that the relationship \(K v_\phi = v_F\) is guaranteed by Galilean invariance [31]. The Tomonaga-Luttinger parameter also appears in other physical observables [29,32], for example, in the compressibility

\[
\kappa = \frac{\partial n}{\partial \mu} = \frac{4K}{\pi v_\phi},
\]

and the long-distance asymptotics of correlation functions [see Eqs. (44) and (45) below].

On the other hand, the coupling \(g_1 < 0\) in the spin part \(\mathcal{H}_{\text{sp}}\) is marginally relevant and thus opens up gaps in the spectrum. This can be seen by studying the renormalization group flows of the couplings in Eq. (22). The straightforward one-loop calculations result in

\[
\frac{dg_1}{ds} = -\frac{4g_1^2}{2\pi v_F}, \quad \frac{dg_2}{ds} = -\frac{g_1^2}{2\pi v_F}, \quad \frac{dg_4}{ds} = 0,
\]

where \(\Lambda_s = e^{-s} \Lambda_0\) is the momentum scale at which the couplings \(g_i(s)\) are defined, \(g_4\) and \(4g_2 - g_1\) are exactly marginal as is consistent with the fact that Eq. (26) can be diagonalized, while \(g_1\) evolves as

\[
g_1(s) = \frac{1}{g_1(0) + \frac{4\pi}{2\pi} v_F}.
\]

When \(g_1(0) < 0\), \(g_1(s)\) reaches the Landau pole at \(s = -\frac{2\pi v_F}{4g_1(0)} = \frac{\pi^2}{8k_F|a|}\), indicating that the second term in Eq. (27) develops spin gaps whose magnitude is set by

\[
\Delta \sim v_F \Delta s \sim v_F k_F e^{-\pi^2/(8k_F|a|)}.
\]
The exact gap spectrum can be obtained from the Bethe-ansatz solution if we recognize $\mathcal{H}_\text{sp}$ as the non-Abelian part of the SU(4) chiral Gross-Neveu model [33–35]:

$$\Delta_f \propto v_F k_F e^{-\pi^2/(8k_F a_i)} \sin \left(\frac{f}{\pi} \right).$$  \hspace{1cm} (37)

Here $f = 1, 2, 3$ is a number of excited fermions, and accordingly, there are three distinct gaps which are exponentially small in the BCS limit $k_F a \rightarrow 0$. The degeneracy of $\Delta_1$ and $\Delta_3$ can be traced back to an accidental particle-hole symmetry in $\mathcal{H}_\text{BEC}$ under $\psi^{\dagger}_{\sigma} (r) \rightarrow -\psi_{\sigma} (r)$. This symmetry is broken by quadratic derivative terms $\psi^{(\dagger)} (r) \nabla^2 \psi^{(\dagger)} (r)/(2m)$ neglected in $\mathcal{H}_\text{BEC}$. Because the characteristic momentum scale is set by $\Delta_1 \sim k_F e^{-\pi^2/(8k_F a_i)}$, the small splitting of the degeneracy is estimated to be

$$\Delta_3 - \Delta_1 \sim \frac{\Lambda^2}{2m} \sim v_F k_F e^{-\pi^2/(4k_F a_i)}.$$  \hspace{1cm} (38)

**B. BEC limit**

In the strong coupling (BEC) limit $k_F a \rightarrow +0$, four fermions with all different components form a tightly-bound tetramer and thus the many-body system will be a dilute Bose gas of such tetramers. In this limit, fermionic excitations are largely gapped because of the binding energy of the tetramer $E_0 = -1/(2ma^2)$. The gap spectrum with the fermion number $f = 1, 2, 3$ is simply given by

$$\Delta_f \rightarrow \frac{f}{8ma^2}.$$  \hspace{1cm} (39)

Interestingly, we find that the ordering of the three spin gaps is $\Delta_1 < \Delta_2 < \Delta_3$ in the BEC limit while it is $\Delta_1 \simeq \Delta_3 < \Delta_2$ in the BCS limit [see Eqs. (37) and (38)]. Therefore, there has to be a crossing between two gaps $\Delta_2$ and $\Delta_3$ as a function of $-\infty < (k_F a)^{-1} < \infty$ in the BCS-BEC crossover.

The dilute Bose gas of tetramers to leading order in $k_F a$ is described by the Hamiltonian density

$$\mathcal{H}_\text{BEC} = -\frac{\phi^{\dagger} \nabla^2 \phi}{2M} - \frac{1}{M a_t} \phi^{\dagger} \phi \phi \phi,$$  \hspace{1cm} (40)

where $M \equiv 4m$ is the tetramer mass and the tetramer density is $n_t \equiv n/4$. $a_t$ is a tetramer-tetramer scattering length (analogous to the dimer-dimer scattering length in three dimensions [36]) characterizing the scattering of two tetramers in one dimension. Because the scattering length $a_t$ is the only scale of the system in vacuum, $a_t$ has to be proportional to $a$:

$$a_t = -\eta a.$$  \hspace{1cm} (41)

The coefficient $\eta$ is a universal number obtained by solving the eight-body problem of fermions nonperturbatively. $\eta$ is expected to be positive because tetramers should repel each other due to the fermionic statistics of the constituents. If $\eta > 0$, then the many-body system of tetramers is stable. Here we shall assume $\eta > 0$ and leave the determination of the exact value of $\eta$ as a future problem.

The effective theory of tetramers $\mathcal{H}_\text{BEC}$ is nothing but bosons with a $\delta$-function interaction in one dimension. In contrast to the dilute Bose gas in three dimensions, that in one dimension is strongly interacting because the tetramer-tetramer coupling in Eq. (40) is inversely proportional to the scattering length. As a consequence, the tetramers in the limit $k_F a = +0$ behave as noninteracting spinless “fermions” and the thermodynamic properties of our system in the BEC limit are equivalent to those of a noninteracting Fermi gas with the same mass $M$ and density $n_t$ [1]. Beyond such a hard-core limit, the ground-state energy and the excitation spectrum of $\mathcal{H}_\text{BEC}$ have been obtained exactly in Ref. [37]. In particular, its low-energy physics is described by the Tomonaga-Luttinger liquid (30) again [31]. The Tomonaga-Luttinger parameter $K$ and the sound velocity $v_s$ in the BEC limit $k_F a \rightarrow +0$ are given by

$$K \rightarrow 4(1 - 2n_t a_t) = 4 \left(1 + \frac{2k_F a}{\pi} \right)$$  \hspace{1cm} (42)

and

$$v_s \rightarrow \frac{\pi n_t}{M}(1 + 2n_t a_t) = \frac{v_F}{4} \left(1 + \frac{2k_F a}{\pi} \right).$$  \hspace{1cm} (43)

In the expression for $K$, we have taken into account the fact that the particle number of a tetramer is four [38]. The BCS-BEC crossover hypothesis indicates that $K$ in Eqs. (31), (42) and $v_s$ in Eqs. (32), (43) are smoothly connected, and therefore, there has to be a maximum (minimum) in $K (v_s)$ as a function of $-\infty < (k_F a)^{-1} < \infty$.

The Tomonaga-Luttinger parameter determines the long-distance asymptotics of correlation function. Because the spin degrees of freedom are gapped in the BCS-BEC crossover, only SU(4) singlet operators can have quasi-long-range orderings. Two such examples are the density-density correlation function:

$$\langle \delta n(x) \delta n(0) \rangle_{x \rightarrow \infty} \rightarrow -\frac{2K}{\pi^2 x^2} + A \frac{\cos(2k_F x)}{|k_F x|^2} \cdots$$  \hspace{1cm} (44)

and the tetramer-tetramer correlation function:

$$\langle \phi(x) \phi(0) \rangle_{x \rightarrow \infty} \rightarrow B \frac{|k_F x|^2}{x} \cdots,$$  \hspace{1cm} (45)

where $A, B$ are unknown parameters and both $x \gg k_F^{-1}$ and $x \gg v_F / \Delta_f$ are assumed. We can see that the $2k_F$-charge-density wave is the dominant order for $K \lesssim 2$ (BCS side), while the tetramer quasi-condensation is the dominant order for $K \gtrsim 2$ (BEC side), and there is a crossover in between. We note that these correlation functions have been studied in the context of spin-3/2 Fermi gases with two-body interactions [17,39–43].

**C. Unitarity limit**

It would be difficult to compute $K$ and $v_s$ away from the BCS or BEC limit. However, in the unitarity limit $k_F a \rightarrow \infty$, we can derive exact relationships between $K$ and thermodynamic quantities. Because the density $n$ is the only scale of the system, the ground-state energy density of the unitary Fermi gas can be written as

$$\varepsilon_{\text{unitary}}(n) \equiv \xi \varepsilon_{\text{free}}(n),$$  \hspace{1cm} (46)

where the ground-state energy density of a noninteracting Fermi gas is

$$\varepsilon_{\text{free}}(n) = \frac{\pi^2}{9\hbar^2 m} n^3.$$  \hspace{1cm} (47)
Here \( \xi \), which measures how much energy is gained due to the attractive interaction, is a universal number to characterize the strongly interacting unitary Fermi gas and analogous to the Bertsch parameter in three dimensions [13]. From the thermodynamic relationships, we obtain the pressure as \( P(n) = 2\varepsilon(n) \), and thus, the sound velocity is given by

\[
v^2_s = \frac{1}{m} \frac{\partial P}{\partial n} = \xi v^2_f. \tag{48}\]

Because \( K v_0 = v_F \) is guaranteed by Galilean invariance [31], we find that the Tomonaga-Luttinger parameter is related to the one-dimensional Bertsch parameter by

\[
K = \frac{1}{\sqrt{\xi}}. \tag{49}\]

This relationship implies \( K > 1 \) in the unitarity limit because \( 0 < \xi < 1 \) is expected for the attractive interaction. It is a challenging many-body problem to determine the exact value of \( \xi \).

One possible way to estimate the value of \( \xi \) is to use the \( \epsilon \) expansion [19,44]. Considering the same Hamiltonian density (9) in an arbitrary spatial dimension \( d \), we find that the dimension of the coupling constant \( c_0 \) is given by \( v = 2 - 3d \), which also determines the behavior of the four-body wave function at a short distance:

\[
\Psi(|r| \to 0) \to |r|^\nu. \tag{50}\]

In the limit \( d \to 2/3 \), we have \( v \to 0 \) and the singularity in the wave function disappears. This means that the contact interaction among four fermions disappears and thus the unitary Fermi gas reduces to a noninteracting Fermi gas [45]. On the other hand, in the limit \( d \to 4/3 \), we have \( v \to -2 \) so that the normalization integral of the wave function

\[
\int d^3 r |\Psi(r)|^2 \sim \int dr r^{3-3d} \tag{51}\]

diverges at the origin \( |r| \to 0 \). This means that four fermions behave as a pointlike composite boson and thus the unitary Fermi gas reduces to a noninteracting Bose gas of such tetramers [45,46]. Therefore, \( \xi \) defined as in Eq. (46) is found to be

\[
\xi|_{\nu\to-\frac{1}{3}} \to 1 \text{ and } \xi|_{\nu\to-2} \to 0. \tag{52}\]

It is possible to formulate appropriate perturbation theories around these critical dimensions [47]. Interpolations of two systematic expansions in terms of \( \epsilon = d - \frac{5}{3} \) and \( \epsilon = \frac{4}{3} - d \) would provide a reasonable estimate of \( \xi \) in \( d = 1 \) as in three dimensions [19,44].

D. Exact relationships

Another characteristic of our system (9) that resembles the BCS-BEC crossover in three dimensions is the large-momentum tail of the momentum distribution of fermions and its relationships to other properties of the system [48]. In order to see this, we consider the following operator product expansion (no sum over \( \sigma = a,b,c,d \)):

\[
\psi^\dagger_\sigma \left( x - \frac{y}{2} \right) \psi_\sigma \left( x + \frac{y}{2} \right) = \psi^\dagger_\sigma (x) + \frac{y}{2} \bar{\psi}^\dagger_\sigma \nabla \psi_\sigma (x) - \frac{\sqrt{3}}{8\pi} (mc_0)^2 \psi^\dagger_\sigma \psi^\dagger_b \psi^\dagger_c \psi^\dagger_d \psi_b \psi_c \psi_d \psi_\sigma (x) + O(y^2). \tag{53}\]

This can be confirmed by evaluating expectation values of the both sides for a state consisting of four fermions with all different components [49]. The nonanalytic term \( \sim |y| \) indicates that the momentum distribution of fermions

\[
\rho_\sigma (k) = \int dye^{-iky} \left( \psi^\dagger_\sigma \left( x - \frac{y}{2} \right) \psi_\sigma \left( x + \frac{y}{2} \right) \right) \tag{54}\]

falls off by a power of \( |k| \to \infty \) as

\[
\rho_\sigma (k) \to \sqrt{3} \frac{C}{4\pi k^2}. \tag{55}\]

The coefficient is given by the so-called contact density:

\[
C \equiv \left \langle (mc_0)^2 \psi^\dagger_\sigma \psi^\dagger_b \psi^\dagger_c \psi^\dagger_d \psi_b \psi_c \psi_d \psi_\sigma \right \rangle. \tag{56}\]

The same result can be obtained by the method used in Ref. [50] (see Fig. 3).

From Eqs. (9), (12), and (56), we find that the energy density of the system \( E \equiv \langle \mathcal{H} \rangle \) can be expressed by

\[
E = \sum_\sigma \int_0^\infty \frac{dk}{2\pi m} \left[ \rho_\sigma (k) - \frac{\sqrt{3} C}{4\pi k^2} \right] + \frac{C}{4\pi ma}. \tag{57}\]

This relationship is valid for any state of the system and for any value of the scattering length \( a \). Derivations of the pressure

\[
P = 2E + \frac{C}{4\pi ma}, \tag{58}\]

the adiabatic relationship

\[
\frac{dE}{da} = \frac{C}{4\pi ma^2}, \tag{59}\]

and the generalized virial theorem in the presence of a harmonic potential \( V_\omega = \int dx \frac{1}{2} m\omega^2 x^2 \sum_\sigma \psi^\dagger_\sigma (x) \psi_\sigma (x) \):

\[
E = 2\langle V_\omega \rangle - \int dx \frac{C(x)}{8\pi ma}. \tag{60}\]

are straightforward from the above results by using the methods in Refs. [48,49].

![FIG. 3. Contribution of the contact density C to the fermion propagator iG_\sigma(k_0,k). The integration over k_0 leads to the momentum distribution in Eq. (55).](image-url)
From the adiabatic relationship (59) together with the ground-state energy density in the BCS limit (up to the mean-field correction):
\[ E_{\text{BCS}} = E_{\text{free}} + \frac{4\pi a}{m} \left( \frac{n}{4} \right)^4 + O\left( k_F a^2 \right) \]
and in the BEC limit [37]:
\[ E_{\text{BEC}} = E_0 n_\uparrow + \frac{\pi^2}{6M} n_\uparrow^3 + \frac{\pi^2}{3M} n_\uparrow^4 a_\uparrow + O\left( k_F a^2 \right), \]
we find that the contact density \( C \) is given by
\[ C \rightarrow \frac{16k_F^2 a^2}{\pi^2} \quad (k_F a \rightarrow -0) \]
and
\[ C \rightarrow \frac{4}{k_F a} - \frac{\eta k_F^2 a^2}{3\pi} \quad (k_F a \rightarrow +0). \]
The BCS-BEC crossover hypothesis indicates that both \( E \) and \( C \) smoothly evolve as functions of \( -\infty < (k_F a)^{-1} < \infty \).

References [48,49] have also shown that the contact density in three dimensions is related to the local pair density, which is the number of pairs of spin-up and -down fermions with small separations. Similarly, the contact density in our one-dimensional system is related to the local quadruplet density \( \mathcal{N}_4(R) \), which is the number of sets of four-component fermions with small hyperradii. This can be seen from the following operator product expansion:
\[ \psi_a^\dagger(x_a) \psi_b^\dagger(x_b) \psi_b(x_b) \psi_a(x_a) \psi_c^\dagger(x_c) \psi_d^\dagger(x_d) = \frac{(mc)^2}{16\pi^2} \left( \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} \right)^2 + O(|r|^{-1}). \]
The integral of the left-hand side over the three relative coordinates \( |r| < R \) counts the number of sets of four-component fermions at the fixed center-of-mass coordinate \( X \) but with the hyperradius smaller than \( R \). Therefore, we find that the short-distance asymptotics of the local quadruplet density is related to the contact density by
\[ \mathcal{N}_4(R \rightarrow 0) \rightarrow \frac{C}{4\pi} R. \]

IV. CONCLUSIONS

In summary, we have demonstrated that the four-component Fermi gas in one dimension exhibits the one-dimensional analog of the BCS-BEC crossover as a function of the scattering length characterizing the short-range four-body interaction. We investigated the ground-state energy, the sound velocity, the gap spectrum, and the exact relationships in the BCS-BEC crossover and found that the gap spectrum has the rich structure because of the existence of three distinct gaps. We also showed that the one-dimensional analog of the Efimov effect occurs for five bosons while it is absent for fermions. This work extends our perspectives on the universal few-body and many-body physics to one dimension and possibly opens up a very rich research area. Finally, we note that the system considered in this paper is highly fine-tuned: not only the four-body interaction is tuned to the resonance, two-body and three-body interactions have to be tuned to vanish. Its experimental realization would be challenging.

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[22] The integer solutions \(\gamma = 1\) for \(\lambda = -1\) and \(\gamma = 3\) for \(\lambda = 4\) in Eq. (15) and \(\gamma = 2\) for \(\lambda = 4\) in Eq. (20) have to be eliminated because they correspond to vanishing wave functions [23].
[26] This spectrum in the unitarity limit has been confirmed by numerically solving the integral equation to determine the petrander’s binding energy \(E = -\frac{\mu^2}{2m}\):

\[
Z_\pm(p) = \lambda \int_{-\Lambda}^{\Lambda} \frac{dq}{2\pi} \frac{Z_\pm(q)}{-\frac{1}{2} + \sqrt{\frac{\mu^2}{4} + 2m|E|}}
\times \int_{-\Lambda}^{\Lambda} \frac{dk_1 dk_2}{(2\pi)^2} \frac{1}{2p^2 + 2p^2 - m^2} + k_1^2 + k_2^2 + k_1 k_2 + m|E|,
\]

where \(Z_\pm(-p) = \pm Z_\pm(p)\) correspond to the even- and odd-parity channels, respectively.
[38] Note that \(\varphi_0(x)\) in Eq. (30) is normalized as \(\partial_{n}\varphi_0 = \frac{\xi}{\delta}\delta n(x)\), where \(n + \delta n(x) = \sum_{\sigma} \psi^\dagger_\sigma \psi_\sigma = 4\phi^\dagger \phi\) is the number density operator.
[45] This observation can be supported by extending the four-body scattering amplitude in Eq. (11) to an arbitrary spatial dimension:

\[
A(E, p)|_{\alpha=\infty} = -\left(\frac{2\pi}{m}\right)^{\frac{2d}{4}} \frac{2^d}{\Gamma(1 - \frac{d}{2})} \left[ \frac{\xi^2}{d} - E - i0^+ \right]^{\frac{2d-1}{2}}.
\]

When \(d \to 2/3\) or \(d \to 4/3\), we find \(A \to 0\) corresponding to the noninteracting limit.
[47] We have carried out the expansion in terms of \(\xi = d - \frac{2}{3}\) up to its next-to-leading order and found

\[
(\xi)|_{d=\frac{2}{3}} = 1 - \frac{81\bar{\xi}}{216} \Gamma\left(\frac{1}{3}\right)^3 + O(\bar{\xi}^2).
\]

Due to the large \(O(\bar{\xi})\) correction, the naive extrapolation to \(d \to 1\) does not work (\(\xi \to -0.115\)) and needs to be combined with the expansion in terms of \(\epsilon = \frac{2}{3} - d\).