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An Open Boundary Condition for Numerical Coastal Circulation Models

PEIFENG MA
Singapore-MIT Alliance for Research and Technology, Singapore, Republic of Singapore

OLE SECHER MADSEN
Massachusetts Institute of Technology, Cambridge, Massachusetts

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ABSTRACT

Open boundaries (OBs) are usually unavoidable in numerical coastal circulation simulations. At OBs, an appropriate open boundary condition (OBC) is required so that outgoing waves freely pass to the exterior without creating reflections back into the interior of the computational domain. In this paper, the authors derive, based on the shallow-water equations including bottom friction and neglecting Coriolis effect and by means of nonlinear characteristic analysis, an OBC formulation with two predictive parameters, phase speed $c_r$ and decay time $T_f$. Simple idealized tests are performed to demonstrate the proposed OBC’s excellent skills in elimination of unwanted reflections at OBs when the motion is periodic, as assumed in its theoretical derivation. It turns out that the formulas for the two OBC parameters become independent of period in the limit of small friction and/or short period. This feature is used to derive an OBC applicable when information about the typical period of the motion to be simulated is unavailable. Simple, idealized tests of this period independent OBC demonstrate its ability to afford excellent results, even when the limitations inherent in its derivation are exceeded. Finally, the OBC is applied in more realistic simulations, including Coriolis effects of 2D tidal flows, and is shown to yield excellent results, especially for residual flows.

1. Introduction

In most numerical simulations in coastal waters, open boundaries are always present, and therefore an appropriate open boundary condition (OBC) must be applied. This OBC should be able to prevent any reflection of outgoing waves generated at the open boundary. Unless the entire computational domain is located in open waters, a nonglobal simulation must have open boundaries being at least partially placed in shallow waters. Due to shallow water depth, the bottom friction may play an important role in wave propagation in shallow waters; for example, the bottom friction will cause wave attenuation and reduction of phase speed. Thus, the bottom friction influences should be taken into account in the open boundary condition.

Numerous OBC studies have been conducted in the last few decades (e.g., Flather 1976; Orlanski 1976; Chapman 1985; Blumberg and Kantha 1985; Palma and Matano 1998, 2001; Marchesiello et al. 2001; Nycander and Döös 2003; Blayo and Debreu 2005; Marsaleix et al. 2006; Carter and Merrifield 2007; Herzfeld 2009) to develop or test a variety of OBCs: for example, the gravity wave OBC, Orlanski OBC (Orlanski 1976), Flather OBC (Flather 1976), and characteristic wave amplitudes methods (e.g., Röed and Cooper 1987). However, none of these OBCs accounts for the influence of bottom friction and very few of the OBC studies, with the exception of Chapman (1985), have attempted to investigate the influences of bottom friction on OBC performance.

Besides bottom friction, Coriolis force is another effect that has been neglected in most well-known OBCs: for example, those tested by Chapman (1985) and Palma and Matano (1998) and reviewed by Blayo and Debreu (2005). In fact, Coriolis force may also affect wave propagation in coastal waters; for example, the phase speed of long waves in open waters with Coriolis effect becomes

$$c = c_r / \sqrt{1 - (f/\omega)^2}$$

in which $f$ is the Coriolis parameter, $\omega$ the wave frequency, and $c_r$ is the phase speed without Coriolis effect (e.g., Gill 1982). Hence, the neglect of Coriolis effects in an OBC formulation may influence the...
performance of the OBC when these effects are included in the simulation.

The objective of the present study is to propose an open boundary condition with explicit inclusion of bottom friction effect, that is, one of the two aforementioned effects, and to test the proposed OBC in a number of shallow water conditions, from idealized to more realistic flows. The paper is organized as follows. An OBC formulation including bottom friction effects is derived in section 2. Some numerical experiments are conducted to test the OBC for 1D periodic motions in section 3. In section 4, the OBC is applied in simulations of 2D tidal flows that include Coriolis effects. Finally, conclusions are provided in section 5.

2. Formulation

We consider the 1D shallow-water equations expressed in terms of surface elevation \( \eta \) and depth-averaged velocity \( U \) without rotational effects,

\[
\frac{\partial \eta}{\partial t} + \frac{\partial U(h + \eta)}{\partial x} = 0 \tag{1}
\]

and

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = -g \frac{\partial \eta}{\partial x} - \frac{\tau_b}{\rho(h + \eta)}, \tag{2}
\]

where \( h \) is the still water depth and \( \tau_b \) is the bed shear stress which can be related to velocity \( U \) by a quadratic form,

\[
\tau_b/\rho = C_b |U| U, \tag{3}
\]

with \( C_b \) denoting a dimensionless bottom friction factor. This friction factor is related to bottom roughness \( k_{na} \) through \( C_b = [\kappa \ln(11h/k_{na})]^2 \), which is obtained from \( \eta \) or \( U \).

The presence of wind waves can significantly enhance the bottom roughness (e.g., Grant and Madsen 1979) experienced by a current, and the value of \( C_b \) can therefore vary over a wide range in coastal waters. For a water depth of \( h = 20 \) m, typical wind wave conditions with wave period \( T_w = 8 \) s and wave heights \( H = 0.5 \text{ to } 3 \) m, and bottom sediment diameters \( d = 0.2, 0.4, \) and 0.6 mm, the apparent bottom roughness \( k_{na} \), and hence, the value of \( C_b \), for a \( U = 0.5 \text{ m s}^{-1} \) current can be estimated from the wave–current–sediment interaction model presented by Madsen (2002). The results, shown in Fig. 1, reveal that the friction factor is small in weak wave conditions when the bed remains flat and increases significantly when ripples are present due to stronger wind waves. When the wave height increases further, very strong near-bed flows wash out ripples and the bottom friction factor is smaller than for rippled bed conditions: for example, the case with grain size of 0.2 mm for large waves in Fig. 1. For this range of coastal conditions, the estimated bottom friction factor, as shown in Fig. 1, varies from \( C_b = 0.93 \times 10^{-3} \), corresponding to weak flow and fine sediment, to \( C_b = 6.92 \times 10^{-3} \) for strong wave and coarse sediment conditions.

We consider a small-amplitude periodic wave in constant depth \( h \) so that the nonlinear terms in Eqs. (1) and (2) can be neglected. We also assume a linear relation between the bottom shear stress and the volume transport rate, \( U(h + \eta) \):

\[
\tau_b/\rho = RU(h + \eta). \tag{4}
\]

For \( R = R(t) \) the linearized equations can then be written as

\[
\frac{\partial^2 \eta}{\partial t^2} - gh \frac{\partial^2 \eta}{\partial x^2} + R \frac{\partial \eta}{\partial t} = 0. \tag{5}
\]

From (3) and (4) the linear friction factor \( R \) can be related to the nonlinear friction factor \( C_b \) through

\[
R = C_b |U|/(h + \eta). \tag{6}
\]

Obviously, (6) gives a friction factor \( R \) varying both in time and space. For convenience of derivation, we may neglect the term with \( \eta \) and replace \( |U| \) in (6) by a constant, such as the amplitude of \( |U| \) or the period-averaged value of \( |U| \), so that the friction factor \( R \) becomes a constant.
We assume Eq. (5) has a progressive wave solution
\[ \eta = Ae^{i(kx - \omega t)} \]  
(7)
and introduce (7) into (5) to obtain the complex wavenumber
\[ k = k_r + ik_i = \frac{\omega}{\sqrt{gh}} = \frac{\omega}{\sqrt{g(h + \eta)}} \]  
(8)
where the plus (minus) sign represents a wave propagating in the +x (-x) direction. In the present study, we just consider waves propagating in the +x direction, for which the real and imaginary parts of the complex wavenumber are
\[ k_r = \frac{\omega}{\sqrt{gh}} \left( \frac{\sqrt{1 + R^2\omega^2} + 1}{2} \right)^{1/2} \]  
(9a)
and
\[ k_i = \frac{\omega}{\sqrt{gh}} \left( \frac{\sqrt{1 + R^2\omega^2} - 1}{2} \right)^{1/2} \]  
(9b)
Thus, the progressive wave (7) becomes
\[ \eta = a_0 e^{-k_x} e^{i(kx - \omega t)} = a(x) e^{i(kx - \omega t)} \]  
(10)
where \(a(x) = a_0 e^{-k_x}\) is the attenuating wave amplitude due to bottom friction, with \(a_0\) the amplitude at \(x = 0\).

Now, we have the analytical solution for attenuating long waves. To derive the OBC formulation, we introduce the notation \(c^2 = g(h + \eta)\) and, taking advantage of the constancy of \(h\) [e.g., \(\partial \eta / \partial x = \partial (h + \eta) / \partial x\) and analogies], write Eqs. (1) and (2) in characteristic form. Following, for example, Henderson (1966), we rewrite (1) and (2):
\[ \frac{\partial c}{\partial t} + 2U \frac{\partial c}{\partial x} + c \frac{\partial U}{\partial x} = 0 \]  
(11a)
and
\[ \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + 2c \frac{\partial c}{\partial x} = -RU \]  
(11b)
The sum and the difference of (11b) and (11a) read
\[ \frac{\partial (U + 2c)}{\partial t} + (U + c) \frac{\partial (U + 2c)}{\partial x} = -RU \]  
(12a)
and
\[ \frac{\partial (U - 2c)}{\partial t} + (U - c) \frac{\partial (U - 2c)}{\partial x} = -RU \]  
(12b)
These two equations describe wave propagations along the characteristics \(dx/dt = U + c\) and \(dx/dt = U - c\), respectively. Our attention is focused on the characteristic carrying information in the +x direction expressed in (12a), that is, with our sign convention from the interior toward the OB of the computational domain. Introducing the—in most cases excellent—approximation,
\[ c = \sqrt{gh + \eta} = \sqrt{g(h + \eta)} \left[ 1 + \frac{1}{2} \frac{\eta}{h} - \frac{1}{8} \left( \frac{\eta}{h} \right)^2 + \ldots \right] \]
\[ \approx \sqrt{gh} + \frac{1}{2} \sqrt{g/h} \eta, \]  
(13)
and approximating \(U\) by its expression for a linear progressive wave,
\[ U = \frac{\omega}{k h} = \frac{\omega}{\sqrt{h + \eta}} \left( \frac{1}{1 + \sqrt{1 + iR/\omega}} \right) \]  
(14)
where \(k\) given by (8) was used, we obtain for constant \(h\) and \(R\)
\[ \frac{\partial \eta}{\partial t} + (c + U) \frac{\partial \eta}{\partial x} = -\frac{R}{1 + \sqrt{1 + iR/\omega}} \eta \]  
(15)
as an approximate equation for \(\eta\) along positive characteristics. This equation can be written as
\[ \frac{\partial \eta}{\partial t} + c_r \frac{\partial \eta}{\partial x} = -\frac{R}{1 + \sqrt{1 + iR/\omega}} \eta + (c_r - c - U) \frac{\partial \eta}{\partial x} \]  
(16)
in which \(c_r\) is defined by
\[ c_r = \frac{\omega}{k_r} \sqrt{h + \eta}/h + U \]
\[ = \sqrt{gh + \eta} \sqrt{\frac{1 + R^2\omega^2 + 1}{2}} + U. \]  
(17)
With \(c_r\) given by (17) and \(c\) by (13), the last term in (16) becomes
\[ (c_r - c - U) \frac{\partial \eta}{\partial x} = \sqrt{1 + \eta/h} \left( \frac{\omega}{k_r} - \sqrt{gh} \right) \frac{\partial \eta}{\partial x} \]
\[ \approx \left( \frac{\omega}{k_r} - \sqrt{gh} \right) \frac{\partial \eta}{\partial x} \]  
(18)
to \(O(\eta/h)\) accuracy. Using the expression for \(R/\omega = i(1 - ghk^2/\omega^2)\) obtained from (8), the next to last term in (16) may, after some algebra, be written as
\[ -\frac{R}{1 + \sqrt{1 + iR/\omega}} \eta = -ik\left(\frac{\omega}{k} - \sqrt{gh}\right) \eta. \]  

Introducing (18), with \( \partial \eta/\partial x = ik \eta \), and (19) into (16) we have, to \( O(\eta/h) \) accuracy, the rhs of (16):

\[ \frac{R}{1 + \sqrt{1 + iR/\omega}} \eta + (c_r - c - U) \frac{\partial \eta}{\partial x} = -ik\eta\left(\frac{\omega}{k} - \sqrt{gh}\right) + ik\eta\left(\frac{\omega}{k_r} - \sqrt{gh}\right) = -\frac{\omega k}{k_r} \eta. \]

We define the decay time \( T_f \):

\[ T_f = \frac{k_r}{\omega k_i} = \frac{\sqrt{1 + R^2/\omega^2} + 1}{\sqrt{1 + R^2/\omega^2} - 1} \frac{1}{\omega} \]  

In which (9a) and (9b) are used. Hence, (16) becomes

\[ \frac{\partial \eta}{\partial t} + c_r \frac{\partial \eta}{\partial x} = -\frac{\eta}{T_f}. \]  

In this OBC, the phase speed \( c_r \), given by (17), and decay time \( T_f \), given by (21), are both related to the bottom friction condition. Some nonlinear terms are involved in (22) through \( c_r \), which are not taken into account in most commonly used OBCs. If neglecting these nonlinear terms, (22) becomes identical to the OBC derived based on linear characteristic analysis by Ma and Madsen (2010).

The OBC given by (22) has the exact same form as the passive form of the Blumberg and Kuntha (BK) OBC (Blumberg and Kuntha 1985), which was also used in other studies (e.g., Chapman 1985; Palma and Matano 1998, 2001). However, the phase speed in BK OBC is given by \( c_r = \sqrt{gh} \), and the time scale \( T_f \) with the value being selected more or less arbitrarily, was introduced as a restoring time to handle external data: that is, with a time scale \( O(T_f) \) the model-predicted value at the boundary is forced to known external data or mean sea level drifting problems rather than to account for bottom friction effects.

The OBC given by (22) can be reduced to other commonly used OBCs by choosing different combinations of \( c_r \) and \( T_f \). The radiation gravity wave (GW) OBC can be obtained for \( c_r = \sqrt{gh} \) and \( T_f = \infty \), with \( g = 9.81 \text{ m s}^{-2} \), the gravity acceleration. The Orlanski (1976) boundary condition is obtained if \( T_f = \infty \) and the phase speed \( c_r \) is computed from numerical results at interior points in previous time steps as \( c_r = -(\partial \eta/\partial t)/(\partial \eta/\partial x) \). Finally, if \( T_f = 0 \), (22) becomes a clamped condition, \( \eta = 0 \).

It is noted that the above derivation is based on the constant linear friction factor \( R \) in (6). Even though a preliminary estimate of \( U \) along the open boundary may be available, for example, from a large-scale regional model that, in addition to \( \eta \), provides values for the velocity at the OB, we may alternatively calculate \( c_r \) and \( T_f \) in a very convenient way by using the instantaneous velocity and elevation at the closest interior grid points and taking these values to compute \( R \) from (6). This seems to be inconsistent with the derivation but is actually theoretically justifiable by expressing the OBC along \( dx/dt = -\eta \) with \( dx/dt = -\eta T_f \). Numerically, it can be expressed as \( \Delta \eta/\Delta t = -\eta T_f \) in which the elevation is the one at time \( t \) and, therefore, the corresponding \( T_f \) should be related to the elevation and velocity at the same time: that is, \( \eta(t) \) and \( U(t) \).

For some extreme conditions, the expressions for \( c_r \) in (17) and \( T_f \) in (21) can be significantly simplified. Thus, for \( R/\omega \ll 1 \), (17) and (21) can be approximated by

\[ c_r \approx \sqrt{g(h + \eta)(1 - (R/\omega)^2/8 + \cdots) + U} \]

and

\[ T_f \approx 2/(1 + (R/\omega)^2/8 + \cdots) \approx 2/R. \]

For this relatively low friction, or short period wave case, the leading term is independent of \( \omega \) and (22) can be applied without the wave period being known a priori. This may be a very useful feature since the typical wave period is often unknown for many coastal flow conditions.

On the other hand, if \( R/\omega \ll 1 \), (17) and (21) reduce to

\[ c_r \approx \sqrt{g(h + \eta)} \sqrt{2/(R/\omega)} + U \]

and

\[ T_f \approx 1/\omega, \]

which do not simplify applications as all variables remain in the formulations.

Based on (17) without the nonlinear terms and (21), the values of nondimensional speed and decay time, \( c_r/\sqrt{gh} \) and \( T_f/T \) with \( T = 2\pi/\omega \), are computed for \( R/\omega \) ranging from 0.01 to 100 and plotted in Figs. 2a,b. The parameters computed from the simplified formulas, (23a), (23b), and (24a), (24b) without nonlinear terms in (23a) and (24a) are also shown. Evidently, the simplified formulas (23a), (23b) provide quite acceptable approximations to \( c_r/\sqrt{gh} \) and \( T_f/T \) computed from (17) and (21) for \( R/\omega \) up to \( O(1) \).

In coastal waters, the typical order of magnitudes of relevant parameters in the above formulations are \( h \sim O(10 \text{ m}) \), \( U \sim O(1 \text{ m s}^{-1}) \), \( C_b \sim O(10^{-3}) \), and
the order of magnitude of \( R/\omega \) is \( O(1) \). This implies that one needs to be careful when using the approximate formulations of \( c_r \) and \( T_f \) in numerical coastal simulations. For example, for a water depth of 20 m, semidiurnal tide with velocity amplitude of \(|U| \approx 1 \text{ m s}^{-1}\), and friction coefficient of \( C_b = 0.005 \), the value of \( R/\omega \) with \( R \) computed from (6) is about 1.78; therefore \( c_r/\sqrt{gh} \approx 0.8 \) and \( T_f \approx 0.27 \) \( T = 3.3 \text{ h} \). The approximation (23b) gives \( T_f \approx 2/R \approx 2.2 \text{ h} \) and (24b) gives \( T_f = T/2\pi \approx 2.0 \text{ h} \).

### 3. Application to 1D flows

With the OBC, given by (22), we may first test it in 1D flows. Hereafter, we refer to (22) with period-dependent parameters \( c_r \) from (17) and \( T_f \) from (21) as OBCP and with period-independent parameters \( c_r \) and \( T_f \) from (23a), (23b) as OBCA. Two commonly used OBCs are also applied in the present study to compare with OBCP and OBCA. One is the GW OBC, \( \partial \eta / \partial t + \sqrt{gh} \partial \eta / \partial x = 0 \), and the other is the OBC proposed by Flather (1976) (Fla OBC), which (or some of its variants) are recommended by many OBC studies (e.g., Palma and Matano 1998, 2001; Nycander and Doo s 2003; Blayo and Debreu 2005; Carter and Merrifield 2007) for good performance in some realistic simulations. The Fla OBC can be used either as a mixed (combined active and passive) or purely passive OBC. In the present study, we only consider the passive case; that is, \( U = \eta \sqrt{gh} \).

The barotropic (2D depth-averaged) mode of the Princeton Ocean Model (POM) (Blumberg and Mellor 1987) is employed to carry out all tests in the present study. To save computational time and avoid influences of the north–south boundaries, the 2D program is simplified to its 1D form in this section. Two scenarios of tests are performed in this section: one is a single harmonic wave case when both OBCP and OBCA may be implemented and the other is a combined two-wave case for which only OBCA is applicable. The tests are carried out in a 1D open channel of constant width, length \( L = 80 \text{ km} \), and still water depth \( h = 20 \text{ m} \). In this test, a grid size of 2 km and time step of 12 s are used. Three types of bottom friction conditions are considered in the test, low friction condition: \( C_b = 0.001 \); moderate friction condition: \( C_b = 0.0025 \); and high friction condition: \( C_b = 0.008 \).

#### a. Simple periodic waves

In this subsection, we apply and compare the performances of the four OBCs—OBCP, OBCA, GW OBC, and Fla OBC—for two semidiurnal tidal wave cases with period \( T = 12.4 \text{ h} \). One has incident wave amplitude of \( a_0 = 1 \text{ m} \), which gives \( R/\omega < 1 \) for most of the time. In this case, OBCA is not applied owing to its close performance to OBCP. The other has a large incident wave amplitude of \( a_0 = 3 \text{ m} \) in which \( R/\omega \) can be smaller or greater than unity, depending on the friction factor. In both cases, the semidiurnal tidal wave is prescribed at the upstream open boundary \( x = 0 \) as \( \eta = a_0 \sin \omega t \) and OBCs are applied at the downstream open boundary \( x = L = 80 \text{ km} \). In the present study, Fla OBC is implemented in terms of velocity as provided in the original POM program, whereas all other OBCs are applied in terms of surface elevation. A numerical simulation in a very long channel with length \( L = 6000 \text{ km} \) is conducted to provide a no-reflection solution. The model is run for six days, and the time series of elevation over the period from 130 to 142.4 h are Fourier analyzed to yield amplitudes of different harmonics. To check if the periodic steady state is reached, the elevation time series over this period is compared with those over the period taken on the 100th day from a 100-day simulation and the two results are virtually identical.

The first harmonic amplitudes of elevation for long-channel results and those with various OBCs are plotted.
in Fig. 3 for the small amplitude wave case and in Fig. 4a for the large amplitude wave case. It is seen in Fig. 3 that OBCP (and hence OBCA) outperforms the GW OBC and Fla OBC for all three friction conditions. The relative errors at \(x = 80\) km caused by reflections, defined by \((a_{\text{OBC}} - a_{\text{long}})/a_{\text{long}}\), predicted by OBCP are 0.6%, −0.7%, and −0.7% for \(C_b = 0.001, 0.0025,\) and 0.008, respectively, as opposed to 3.7%, 6.9%, and 11.8% by GW OBC and −3.7%, −6.4%, and −10.6% by Fla OBC. The reason for the poor performance of GW OBC is clearly that \(T_f\) has an infinite value in GW OBC, whereas the predicted \(T_f\) are merely several hours for the three friction conditions. The poor agreement predicted by the Fla OBC may be caused by the neglect of the friction-dependent term in \(U\), as shown in (14).

Because of the inclusion of nonlinear terms in the phase speed \(c_r\), given by (17) or (23a), that are not considered in some well-known OBCs (e.g., GW OBC, Fla OBC, and BK OBC), (22) is expected to afford better results in this large-amplitude wave case. Moreover, this large-wave case may provide an opportunity to examine the performance of OBCA since values of \(R/v\) (Fig. 4b) are up to 0.59, 1.02, and 1.81 for \(C_b = 0.001, 0.0025,\) and 0.008, respectively.

It can be observed in Fig. 4a that both OBCP and OBCA perform very well, with relative errors at \(x = L\) of −0.8%, −0.5%, and 0.7% predicted by OBCP and −0.4%, 0.6%, and 2.5% by OBCA for the three friction conditions, respectively. In this case, the results (not shown owing to large differences) predicted by GW OBC and Fla OBC are quite poor, with relative errors at \(x = L\) of 7.1%, 10.3%, and 17.4% from GW OBC and −6.3%, −9.7%, and −16.3% from Fla OBC.

Although OBCA is derived for \(R/v < 1\), it performs quite well for all three friction conditions, even though \(R/v < 1\) is violated. The good results with OBCA for the weak and moderate friction conditions may be explained by expressing the OBC along the linear characteristic. Thus, along \(dx/dt = c_r\), the OBC reads \(d\eta/dt = −\eta/T_f\), which has the solution \(\eta \propto \exp(−t/T_f) = \exp[−x/(c_r T_f)]\). With the aid of (23a), (23b), the \(O(R/v)^2\) terms in the product \(c_r T_f\) vanish, and we have \(c_r T_f = 2 \sqrt{gh/R(1 - O(R^4/v^4))}\). Therefore, the OBCA is accurate to \(O(R/v)^4\) and should afford an excellent approximation, even when \(R/v\) is close to 1. For the large friction condition, this argument is not valid as the value of \(R/v\) is greater than unity for most of the time (Fig. 4b). All we can say is that the good performance of OBCA in this case may be, at least partially, due to a similar cancellation of errors as for \(R/v < 1\), that is, an overprediction of \(c_r\) and underprediction of \(T_f\).
Nonlinear waves are usually accompanied by quite strong mean elevation and mean (or residual) flow, which may be of importance in some practical problems, for example, transport of sediment or pollutants in coastal waters. Consequently, it is worthwhile examining the influence of choice of OBC on the prediction of mean values in this large-wave case.

Before looking at the results, it is appropriate to check if \( L = 80 \) km is sufficiently long for the prescribed linear wave at \( x = 0 \) to develop into the full nonlinear state. To do this, the long-channel solutions are Fourier analyzed. The second harmonic amplitudes, which are taken as indicators of nonlinearity, suggest that its value at \( x = 80 \) km for the low friction condition is about 70% of its maximum value obtained at \( x = 240 \) km, for the moderate friction condition about 95% of the maximum value obtained at \( x = 120 \) km, and for the high friction condition the maximum value is obtained at \( x = 50 \) km. This reveals that \( L = 80 \) km is sufficient for the incident waves to have developed essentially full nonlinear features when they reach the open boundary.

Some mean values at selected locations—mean velocity at \( x = 0, 40, \) and \( 80 \) km, denoted by \( \bar{U}_{x}, \bar{U}_{40}, \bar{U}_{80} \); and mean elevations at \( x = 40 \) and \( 80 \) km, denoted by \( \eta_{40}, \eta_{80} \), for friction factor \( C_b = 0.001, 0.0025, \) and \( 0.008 \)—are listed in Table 1. It can be seen that the mean values from OBCP and OBCA are quite close with a difference that increases with increasing friction factor; for example, relative differences of \( \eta_{80} \) to long-channel solutions from OBCP for the three friction conditions are \( -0.8\%, -6.0\%, \) and \( -17.5\% \), respectively, and those from OBCA are \( -1.5\%, -9.3\%, \) and \( -27.2\% \). The mean values obtained from OBCP and OBCA are much closer to the long-channel solutions than those from the Fla OBC: for example, \( \eta_{80} \) (\( \bar{U}_{80} \)) from OBCP for the three friction conditions show differences of \( -0.8\% \) (16.9\%), \( -6.0\% \) (22.9\%), and \( -17.5\% \) (183.1\%), respectively, as opposed to \( -64.7\% \) (177\%), \( -69.6\% \) (257\%), and \( -75.4\% \) (725\%) from Fla OBC. The results from GW OBC are poorer than those from OBCP and OBCA for most of the cases except for \( \bar{U}_{40} \) and \( \bar{U}_{80} \) with the friction factor of \( C_b = 0.0025 \). Considering the poor performance of GW OBC on first harmonic amplitudes, the good results for these particular cases might be caused by some error cancellations. It can also be observed in Table 1 that the mean velocities from Fla OBC at \( x = 40 \) and \( 80 \) km have positive values for most of the cases, whereas those from the long-channel solution and other OBCs are negative. This could be of critical importance if one is interested in predicting long-term-averaged flows, such as residual tidal flows.

The physical origin of the mean elevations and velocities listed in Table 1 may be elucidated by invoking the period-averaged volume and momentum conservation principles. With an overbar denoting period average, volume conservation requires

\[
\frac{\partial \eta_{\text{net}}}{\partial x} = \frac{\partial}{\partial x} \left[ (h + \eta) \bar{U} \right] = \frac{\partial}{\partial x} \left[ h \bar{U} + \eta \bar{U} \right] = \frac{\partial}{\partial x} (h \bar{U} + \bar{U} \eta) = -\frac{\partial \bar{U}}{\partial t}
\]

(25)
where \( \overline{q}_a \), the wave-associated mean discharge or “mass transport,” is proportional to \( a_0^2(x) \) by virtue of (10) and (14). The conservation of mean momentum can be written (\( h = \text{const} \))

\[
\frac{\partial}{\partial x} \left[ \rho \overline{U}^2 (h + \eta) + \frac{1}{2} \rho g (h + \eta)^2 \right] + \tau_b = \frac{\partial}{\partial x} \left[ \rho h \overline{U}^2 + \frac{1}{2} \rho g (h + \eta)^2 \right] + \rho g \partial \eta \partial x + \tau_b = \frac{\partial S_0}{\partial x} + \rho g \partial \eta \partial x + \rho h\overline{U} = -\rho \frac{\partial \overline{U}}{\partial t}, \tag{26}
\]

where \( S_0 \), the wave-associated momentum thrust or “radiation stress,” is proportional to \( a_0^2(x) \) [cf. (10) and (14)] and the bottom shear stress \( \tau_b \) is expressed in its linear form, with \( R \) given by (6).

As waves propagate into the channel their amplitudes decay due to frictional attenuation. This implies that both \( \partial q_0/\partial x \) and \( \partial S_0/\partial x \) are negative. Assuming steady state in the mean to be reached for \( x < 80 \text{ km} \) at \( t \sim 130 \text{ h} \) (Table 1), that is, \( \partial \eta / \partial t = \partial \overline{U} / \partial t = 0 \), (25) shows that \( \partial \overline{U} / \partial x > 0 \), and, disregarding \( \tau_b \), we obtain \( \partial \eta / \partial x > 0 \) from (26). With the exception of Flä OBC for the large friction case, \( C_b = 0.008 \), it is encouraging to note that all results presented in Table 1 exhibit these features.

It is of some interest to note that, whereas \( \overline{q}_a = 0 \) at \( x = 0 \) [i.e., \( \overline{q}(x) \) can be obtained by integration of \( \partial \eta / \partial x \)], the value of \( \overline{U} \) at \( x = 0 \) is a priori unknown. This is so because (25) for steady-state conditions simply states \( \partial \overline{q}_\text{net} / \partial x = 0 \), or \( \overline{q}_\text{net} = \text{const} \). If the channel were closed, one would have \( \overline{q}_\text{net} = 0 \), but here the channel is infinitely long. Thus, a prediction of \( \overline{q}_\text{net} = \text{const} \) for \( 0 < x < 80 \text{ km} \) just implies that a constant discharge \( \overline{q}_\text{net} \) passes through this channel section. The magnitude of \( \overline{q}_\text{net} \) depends on the volume supply required to allow the flow disturbance to “diffuse” into the quiescent water at \( x \to \infty \). So, a determination of \( \overline{q}_\text{net} \) would involve solution of the unsteady forms of (25) and (26), and this, of course, is exactly what the numerical simulation for the long channel does.

b. Combination of two waves

In the preceding subsection, all tests were conducted for single harmonic wave conditions for which \( \omega \) is known. However, for a motion consisting of the combination of two waves of different frequencies, \( \omega \) is undefined and, although \( R \) can be obtained from (6), \( c \), and \( T \) cannot be evaluated from (17) and (21) because these require \( R/\omega \) to be known. Thus, for a combined wave case, OBCA is the only available choice since it is period independent. Again, GW OBC and Flä OBC are applied in this combined wave case.

In the two-wave test, we consider a rather extreme condition by choosing one wave as a semidiurnal tide with period of \( T_1 = 12.4 \text{ h} \) and amplitude \( a_{01} = 1 \text{ m} \) and the other as a slowly varying motion with period of \( T_2 = 5T_1 = 62 \text{ h} \) and a large amplitude \( a_{02} = 3 \text{ m} \), which could be thought of as simulating the slow motion associated with the response to a wind stress. The time series of linearly combined elevations are specified at the upstream boundary at \( x = 0 \) as \( \eta = a_{01} \sin \omega_1 t + a_{02} \sin \omega_2 t \), in which \( \omega_1 \) and \( \omega_2 \) are angular frequencies corresponding to the two waves, respectively. At the downstream open boundary at \( x = L = 80 \text{ km} \), we apply OBCA, GW OBC, and Flä OBC. As for the tests with single waves, a numerical simulation in a very long channel provides a no-reflection solution, and three friction conditions, \( C_b = 0.001, 0.0025, \) and \( 0.008 \), are again considered.

To present the results, \( T_2 = 62\text{-h-long time series of elevations at different locations along the channel, obtained after establishing periodicity, are Fourier analyzed. From this analysis, the wave amplitudes along the channel, \( a_1(x) \) and \( a_2(x) \), corresponding to the two wave components, \( \omega_1 \) and \( \omega_2 \), can be obtained, and those for friction condition of \( C_b = 0.0025 \) are shown in Figs. 5a and 5b, respectively. The time series of \( R/\omega = 5R/\omega_1 \), which provides a measure of OBCA’s expected validity, are shown in Fig. 6.

It can be observed in Fig. 6 that the ratio \( R/\omega_2 \) is larger than unity for most of the time for all three friction conditions; for example, \( R/\omega_2 \) is up to 5 for friction condition \( C_b = 0.0025 \), which is in serious violation of the assumption underlying OBCA. Nevertheless, the results for wave amplitudes in Figs. 5a and 5b show good agreement, with relative errors from OBCA at \( x = L = 80 \text{ km} \) being \(-4.6\% \) for the \( \omega_1 \) component and \(-0.4\% \) for the \( \omega_2 \) component, as opposed to \( 17.9\% \) and \( 25.6\% \) from GW OBC and \(-21.7\% \) and \(-26\% \) from Flä OBC. Additional simulations suggest that the relative differences among the OBCs for the other two friction conditions are similar to the moderate friction case shown in Figs. 5a,b, with relative errors at \( x = L = 80 \text{ km} \) for \( C_b = 0.008 \) from OBCA, with \( R/\omega_2 \) up to 10, being \(-8.6\% \) and \(-10.0\% \) for the two harmonic components, respectively, as opposed to \( 26\% \) and \( 43\% \) from GW OBC and \(-32\% \) and \(-42\% \) from Flä OBC.

For \( C_b = 0.001 \), the errors are \(-2.8\% \) and \( 1.1\% \) from OBCA, \( 12.2\% \) and \( 15.4\% \) from GW OBC, and \(-13.3\% \) and \(-14.8\% \) from Flä OBC. Obviously, OBCA’s predictions are superior to those of GW OBC and Flä OBC, even though the assumption underlying OBCA, \( R/\omega \ll 1 \), is seriously violated. This implies that the application of OBCA for many common coastal flow conditions should yield acceptable results.
It may again be worthwhile to examine the OBCA ability to predict mean water level and velocity in this combined wave case. To do this, a period average over \( T_2 = 62 \) h (from 280 to 342 h after startup) is obtained from the time series of elevation and velocity. The resulting mean values for the long-channel solution and with OBCA, GW OBC, and Fla OBC are listed in Table 2. It is seen that the Fla OBC provides the poorest agreement among the three OBCs, especially for mean velocities, which have the wrong direction in most of the cases. OBCA has the best performance in predicting mean elevation for all the cases and mean velocity for weak and moderate friction conditions; for example, the relative errors of \( \eta_{80} \) and \( \bar{U}_{80} \) for \( C_b = 0.0025 \) are 2.7% and 6.9% from OBCA as opposed to –43% and 66% from GW OBC and –55% and –209% from Fla OBC.

One weakness of OBCA for the high friction condition shown in Table 2 is that the mean velocity at \( x = 80 \) km is positive, whereas the long-channel solution gives a negative value. This is, however, an extreme case since both the amplitude of the slowly varying wave and the friction factor are assumed extremely large. Far better results can be expected in simulations of less extreme coastal flows. As an example in support of this expectation, another combined wave test was performed with equal amplitudes of the two wave components; that is, \( a_{01} = a_{02} = 1 \) m. For this reduction in amplitude of the slowly varying wave motion, which reduces \( R/v^2 \) by roughly a factor of 3, the mean velocities predicted by OBCA for \( C_b = 0.008 \) are –1.10, –0.60, and –0.27 cm s\(^{-1}\) at \( x = 0, 40, \) and 80 km, respectively, in reasonable agreement with the long-channel solution 1.48, –0.93, and –0.59 cm s\(^{-1}\).

In general, the tests in this section suggest that one may be quite confident in applying OBCP and/or OBCA in coastal flow conditions when bottom friction effects are dominant.

4. Application to 2D flows, including Coriolis effects

In the preceding section, numerical experiments performed for idealized 1D flows suggest that our proposed OBCP and OBCA both are able to yield excellent results. It is noted, however, that outgoing waves in 1D cases are exactly normal to the OBs, and this is not true for most realistic flows. To examine the OBC capability in more realistic flows, we apply it to two 2D tidal flow cases in this section. One is the tidal flow over a circular bump in a straight channel with or without Coriolis...
effects included in the simulations. The other is a tidal flow on a continental shelf with Coriolis effects included.

a. Tidal flow over a circular bump

This test case is similar to the one presented by Park and Wang (1994) for studying tidal vorticity over isolated topographic features. Two scenarios are considered: one excludes the Coriolis force to be consistent with the OBC derivation and the other includes the Coriolis force to test the OBC in a more realistic flow simulation. The simulations are conducted in a straight open channel 60 km long, 30 km wide, and 20 m deep outside the bump area (see sketch in Fig. 7a). The x axis is oriented in the along-channel direction, and the y axis is in the cross-channel direction. The bump, centered at (40 km, 15 km) with (0, 0) referring to the left-hand bottom corner, has a circular Gaussian shape with radius 10 km and maximum height 10 m at the center.

We specify an incident semidiurnal tidal wave of amplitude $a_0 = 1$ m at $x = 0$ and apply OBCs at $x = L = 60$ km in the simulations. OBCs used in the tests are GW OBC, Fla OBC, and OBCA. The following parameters are used: constant Coriolis parameter of $f = 10^{-4}$ s$^{-1}$ for the case with Coriolis effects; horizontal eddy viscosity $A_M = 50$ m$^2$ s$^{-1}$; bottom friction factors $C_b = 0.001, 0.0025, and 0.008$; grid size $\Delta x = \Delta y = 1$ km; and time step $\Delta t = 12$ s. Solutions without reflections from the downstream open boundary are obtained from the simulations in a long channel with length 3000 km. The model is run for four days. Nearly identical results at the 4th and 11th days from a 12-day simulation suggest that a periodicity has been established in 4 days.

The tidal elevations at four selected locations, A, B, C, and D (Fig. 7a), are examined. Time series of tidal elevation at point B for $C_b = 0.0025$ and with Coriolis effects are Fourier analyzed and the percentage of relative errors for the first-harmonic amplitudes are listed in Table 3 and demonstrate that OBCA, both with and without Coriolis effects included in the simulations, outperforms GW and Fla for almost all cases, especially for the large friction case in which the errors from OBCA are less than 0.3% and 0.5% for the simulations with and without Coriolis effects, respectively, as opposed to about 2%–10% from GW and Fla OBCs. It can be also seen in Table 3 that the agreement predicted by OBCA becomes better when the friction factor is larger, whereas those predicted by GW and Fla are poorer for stronger friction conditions.

Topographically induced residual tidal flows are very common and could be very important to some physical processes in coastal waters: for example, the evolution of large-scale bed forms (Zimmerman 1981). Hence, it is worthwhile to look at the residual flow induced by the bump. The residual velocities, $\vec{U}$ and $\vec{V}$, are computed by averaging over a wave period after steady state has been achieved. The presence of the bump disturbs the incoming tidal flow and leads to three tilted circulation cells of residual tidal flow over the bump as shown in Fig. 8 for friction condition $C_b = 0.0025$ and including Coriolis effects. The pattern of this residual flow is very similar to that presented by Park and Wang (1994). The asymmetry

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>$\eta_{50}$ (cm)</th>
<th>$\eta_{80}$ (cm)</th>
<th>$U_0$ (cm s$^{-1}$)</th>
<th>$U_{40}$ (cm s$^{-1}$)</th>
<th>$U_{80}$ (cm s$^{-1}$)</th>
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<tbody>
<tr>
<td>Long channel</td>
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<td>-5.49</td>
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<td>11.00</td>
<td>-8.99</td>
<td>-7.54</td>
<td>-6.37</td>
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<tr>
<td>GW OBC</td>
<td>4.20</td>
<td>6.74</td>
<td>-3.02</td>
<td>-1.60</td>
<td>-0.63</td>
</tr>
<tr>
<td>Fla OBC</td>
<td>3.43</td>
<td>4.82</td>
<td>-0.28</td>
<td>1.57</td>
<td>3.36</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>$\eta_{50}$ (cm)</th>
<th>$\eta_{80}$ (cm)</th>
<th>$U_0$ (cm s$^{-1}$)</th>
<th>$U_{40}$ (cm s$^{-1}$)</th>
<th>$U_{80}$ (cm s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long channel</td>
<td>7.62</td>
<td>12.03</td>
<td>-6.43</td>
<td>-4.73</td>
<td>-3.50</td>
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<td>OBCA</td>
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<td>12.36</td>
<td>-6.44</td>
<td>-4.90</td>
<td>-3.74</td>
</tr>
<tr>
<td>GW OBC</td>
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<td>6.85</td>
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<td>-2.03</td>
<td>-1.20</td>
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<tr>
<td>Fla OBC</td>
<td>4.87</td>
<td>5.41</td>
<td>-0.59</td>
<td>1.68</td>
<td>3.81</td>
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<th>Boundary condition</th>
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<th>$\eta_{80}$ (cm)</th>
<th>$U_0$ (cm s$^{-1}$)</th>
<th>$U_{40}$ (cm s$^{-1}$)</th>
<th>$U_{80}$ (cm s$^{-1}$)</th>
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<tbody>
<tr>
<td>Long channel</td>
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<td>13.78</td>
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<td>-2.69</td>
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<tr>
<td>OBCA</td>
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<td>11.20</td>
<td>-2.61</td>
<td>-0.98</td>
<td>0.17</td>
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<tr>
<td>GW OBC</td>
<td>5.59</td>
<td>7.10</td>
<td>-3.08</td>
<td>-1.52</td>
<td>-0.89</td>
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<tr>
<td>Fla OBC</td>
<td>6.49</td>
<td>4.78</td>
<td>-0.83</td>
<td>1.42</td>
<td>3.44</td>
</tr>
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</table>
in Fig. 8 is caused by the Coriolis force as the residual flow without Coriolis effects show perfect symmetry about the center line of the channel, \( y = 15 \text{ km} \).

The contour lines of the magnitude of residual currents in the case with Coriolis effects, \( U_{res} = (U^2 + V^2)^{1/2} \), for the long-channel solution and the differences, \( U_{res, OBC} - U_{res, long} \), from OBCA, GW, and Fla for \( C_b = 0.0025 \), are shown in Figs. 9a–d, respectively. The residual flow magnitudes (Fig. 9a) are from about 0.4 to 42.5 mm s\(^{-1}\) with the maximum about 3 km away from the bump center. The maximum difference from the long-channel solution obtained from OBCA (Fig. 9b) is about 4.8 mm s\(^{-1}\), which is much smaller than the maximum difference 16.9 mm s\(^{-1}\) obtained from GW (Fig. 9c) and 10.8 mm s\(^{-1}\) from Fla (Fig. 9d).

For the simulation without Coriolis effects, the performance of OBCA is even better. For example, the maximum difference of residual velocity predicted by
OBCA is 4.8 mm s\(^{-1}\) as opposed to 16 and \(-13\) mm s\(^{-1}\) predicted by GW and Fla, respectively. The results of residual currents for the other friction cases with \(C_b = 0.001\) and 0.008, which are not presented in detail, suggest that the comparisons among the different OBCs show similar behaviors as those for \(C_b = 0.0025\).

### b. Tidal flow on a continental shelf

As far as the OBC test is concerned, the tidal flow case presented in section 4a is similar to a 1D case since outgoing waves at the boundary only deviate slightly from the normal direction. In this subsection, we perform

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
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<td>0.9</td>
<td>0.6</td>
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<td>Fla</td>
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</table>

\(f = 0\) s\(^{-1}\) \n
<table>
<thead>
<tr>
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<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
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<tbody>
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<td>-1.2</td>
<td>-0.9</td>
<td>-0.6</td>
</tr>
<tr>
<td>GW</td>
<td>1.1</td>
<td>1.6</td>
<td>0.9</td>
<td>0.6</td>
</tr>
<tr>
<td>Fla</td>
<td>-1.8</td>
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<td>-1.7</td>
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</table>

\(f = 10^{-4}\) s\(^{-1}\) \n
<table>
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<tr>
<th>Boundary condition</th>
<th>A</th>
<th>B</th>
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</thead>
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<tr>
<td>OBCA</td>
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<td>-0.4</td>
<td>-0.5</td>
<td>-0.4</td>
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<tr>
<td>GW</td>
<td>4.3</td>
<td>9.5</td>
<td>5.1</td>
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<tr>
<td>Fla</td>
<td>-4.0</td>
<td>-7.8</td>
<td>-4.3</td>
<td>-1.9</td>
</tr>
</tbody>
</table>

**TABLE 3. Relative errors of first harmonic amplitudes, \((a_{OBC} - a_{long})/a_{long} \times 100\), at the four locations, A–D, identified in Fig. 7a.**

\[ f = 0 \text{ (s}^{-1}) \]

\[ f = 10^{-4} \text{ (s}^{-1}) \]

**FIG. 8. Vector plot of residual flow over the bump from large domain solution for \(a_0 = 1\) m, \(C_b = 0.0025\), and Coriolis parameter \(f = 10^{-4}\) s\(^{-1}\) (the circle indicates the bump region).**
a simulation on a continental shelf including the Coriolis effects, where outgoing waves may pass across the open boundary at any angle of incidence. The computational domain is rectangular with the same alongshore and cross-shore dimensions of $L = W = 100$ km. The $x$ axis is along the southern coastline, oriented to the east, and the $y$ axis originates at the coast and is oriented to the north. The water depth is unchanged alongshore and exponentially increases offshore as 

$$h(y) = 8 + 2 \exp(4y/W),$$  \hspace{1cm} (27)$$ which gives a depth of 10 m at the shoreline and about 117 m at the offshore open boundary.

An incident semidiurnal tidal wave, $\eta = a_0(y) \sin \omega t$, is prescribed at the upstream open boundary, $x = 0$. Based on analytical solutions of coastal trapped waves, for example, Mysak (1968), we assume a wave amplitude $a_0(y)$ at $x = 0$.

**Fig. 9.** Contours (mm s$^{-1}$) of (a) residual flow magnitude for $C_b = 0.0025$ and $f = 10^{-4}$ s$^{-1}$ from the long-channel solution and differences in residual flow magnitude predicted by (b) OBCA, (c) GW OBC, and (d) Fla OBC.
where $\beta = 2$, based on $\beta \sim kW$ (e.g., Mysak 1968) with $k$ the wavenumber. This amplitude variation gives a wave amplitude decreasing from $a_0 = 2$ m at the coast, $y = 0$, to $a_0 = 0$ at the offshore boundary, $y = W$. It also leads to a decaying surface slope from a maximum at the coast to zero at the offshore boundary, which avoids unrealistically large currents near the offshore boundary. A clamped condition, $\eta = 0$, is used at the offshore open boundary, $y = W$. At the downstream open boundary, $x = L = 100$ km, OBCA or Fla OBC is applied.

The following parameters are used in the simulation: constant Coriolis parameter $f = 10^{-4}$ s$^{-1}$, horizontal eddy viscosity $A_M = 50$ m$^2$ s$^{-1}$, bottom friction factor $C_b = 0.0025$, grid size $\Delta x = \Delta y = 2$ km, and time step $\Delta t = 24$ s. Solutions with negligible reflections from the downstream open boundary are obtained from simulations for a long shelf with length 5000 km. The simulation period is 6 days. The comparisons between the results at the 6th and 20th days from a 20-day simulation indicate that periodic steady state is achieved after 6 days.

The elevation and velocity time series over the last wave cycle of the simulation when the periodic steady state has been achieved are Fourier analyzed to yield the mean values, $\bar{\eta}$, $\bar{U}$, and $\bar{V}$, and the first-harmonic amplitudes, $a_1$, $U_1$, and $V_1$. Due to the decrease of $a_0(y)$ in the offshore direction, waves rapidly spread out toward the offshore region when leaving the upstream boundary. As a result, the decay in the alongshore direction is much more significant than in the 1D cases discussed in section 3. For example, the first-harmonic amplitude along the coast in this case is reduced from 2 m at $x = 0$ to only 0.15 m at $x = 100$ km, whereas the amplitude in the 1D case decreases from 1 to 0.8 m. The resulting weak flows at the downstream boundary, $x = L = 100$ km, leads to less significant bottom friction effects due to a small friction factor $R$ computed by (6) and therefore implies that our OBCA may not outperform Fla OBC.

The results of first harmonic amplitudes $a_1$ suggest that both OBCA and Fla OBC predict substantial errors at the southeastern corner with similar maximum differences from the long-shelf solution, 9.8 cm for OBCA.
and 9.2 cm for Fla OBC. This means that the two OBCs predict similar primary flows.

The contour lines of mean surface elevations for the long-shelf solution $\eta_{\text{long}}$ shown in Fig. 10 reveal that the largest mean elevation, 5.4 cm, is obtained near the coast about 25 km east of the upstream open boundary. It can also be seen that the contour lines at the northeastern corner nearly parallel to the coastline. Because of deep waters and weak flow in this area, the bottom friction force is relatively unimportant and the gradient of mean surface elevation is balanced by the Coriolis force. This geostrophic balance will give rise to an essentially alongshore mean flow at the eastern boundary, $x = 100$ km, with a magnitude that can be estimated from $f\bar{U} = \frac{g\partial \eta}{\partial y}$. For example, the mean velocity at $x = 98$ km, $y = 60$ km can be computed from the two uppermost contour lines in Fig. 10 as $\bar{U} \approx 9.81 \times 10^{-4} \times 1\text{ cm/22 km} \approx 4.5 \text{ cm s}^{-1}$, which is very close to the simulated result of 4.3 cm s$^{-1}$.

The differences of mean surface elevations, defined by $\eta_{\text{diff}} = \eta_{\text{OBC}} - \eta_{\text{long}}$, predicted by the two OBCs are shown in Fig. 11 for the eastern half domain since those in the western half domain are insignificant (less than $\sim 1$ mm). The maximum differences predicted by the two OBCs are quite close, 4.8 mm predicted by OBCA and 6.7 mm predicted by Fla OBC, but the pattern of the contour lines of the differences are dramatically different. As shown in Fig. 11a, the differences predicted by OBCA spread out from the southeastern corner, whereas those predicted by Fla OBC are concentrated in the vicinity of the northeast downstream open boundary, leading to strong surface gradients in the alongshore direction. These strong gradients indicate that there are large mean velocity differences, which may be estimated in the same manner as for the long-shelf solution: that is, assuming a geostrophic balance. For example, the mean velocity difference based on the contour lines of 4 and 6 mm at $y = 90$ km is

**Fig. 11.** Differences of mean surface elevations (mm) for $C_s = 0.0025$ and $f = 10^{-4} \text{ s}^{-1}$ from (a) OBCA and (b) Fla OBC.
This is quite close to the simulated differences of cross-shore velocity, 121 mm s$^{-1}$, at the same location as shown in Fig. 12b. The contour lines in Fig. 12a show that OBCA predicts much smaller difference of $V$, up to 13 mm s$^{-1}$. One can also observe in Fig. 12b that the large errors predicted by Fla OBC remain in the vicinity of the downstream open boundary. This feature does not change with time, as nearly identical results are obtained from a 20-day simulation. In contrast to large differences of mean cross-shore flow, the two OBCs predict similar magnitudes of differences of mean alongshore velocity $U$ with the largest difference of 1.5 cm s$^{-1}$ from OBCA and 1.3 cm s$^{-1}$ from Fla OBC. In general, the results from this test suggest that the two OBCs predict quite similar primary flow and OBCA predicts better residual or mean flows. The results of tests for other friction cases with $C_b = 0.001$ and $C_b = 0.008$, without details presented, suggest that the comparisons between the OBCs show similar behavior as those for $C_b = 0.0025$. With the Fla OBC implemented in terms of surface elevation, a test for $C_b = 0.0025$ suggests the overall agreement, for both primary and residual flows, is slightly poorer than those obtained when Fla OBC is implemented in terms of velocity. The results in these more realistic experiments suggests that OBCA is capable of predicting good results for both actual and residual flows for more realistic coastal flow conditions, when the Coriolis force is included in the simulation.

5. Conclusions

Based on the nonlinear 1D shallow-water equations along the positive characteristic and including bottom friction, we have derived an approximate open boundary condition, (22), along with expressions for the two key parameters, phase speed $c_r$, given by (17), and decay time $T_f$, given by (21). This open boundary condition,
referred to as OBCP, was derived for a periodic motion, and the two parameters were found to depend on the ratio of a linear friction factor $R$ given by (6) and the radian frequency of the wave $R/\omega$. Tests with both small and large amplitude waves show that this open boundary condition (OBCP) is superior to other commonly used OBCs in its ability to eliminate reflections at the open boundary. In addition to virtual elimination of reflections, tests also suggest that OBCP outperforms other OBCs in the prediction of period-averaged mean water levels and mean velocities.

However, OBCP suffers a serious limitation: the wave period must be known a priori to predict $c_r$ and $T_f$ from (17) and (21). To remove this restriction, we obtained approximate expressions for the two parameters, $c_r$ and $T_f$, given by (23a), (23b), that are independent of wave period and, strictly speaking, valid only for $R/\omega \ll 1$. Through test applications with these period-independent parameters, referred to as OBCA, it was found that OBCA performs as well as OBPC for periodic motions, so long as $R/\omega < 1$, nearly as well as OBCP when $R/\omega$ exceeds unity by a moderate amount ($R/\omega$ up to $\sim 2$) and may still provide acceptable results even when $R/\omega$ reaches values as large as 10. Thus, when the typical period of the motion to be simulated is not known, OBCA may be applied with some confidence if order of magnitude estimates suggest a moderate value of $R/\omega$, say $\lesssim 2$, and with caution if $R/\omega$ is estimated to be much larger than unity.

With the confidence in OBCA established from some idealized cases, it is applied to simulate 2D tidal flows with Coriolis effects included in the simulation: one over a circular bump in a straight channel and the other on a continental shelf. The results of the former case suggest that OBCA is able to produce good results for both actual and residual flows over the bump and again outperforms commonly used OBCs like the GW and Fla OBCs, even for simulations with Coriolis force included. For the more realistic shelf case, OBCA performs as well as Fla OBC in predicting the primary flow and far better than Fla OBC in predicting residual flows. These results indicate that OBCA can be applied with some confidence in numerical simulations of realistic flows in coastal waters.

Despite the excellent performance of OBCP for periodic motions and the capabilities of OBCA for more realistic coastal flows, it should be recognized that our OBC is derived based on 1D equations. Extension of our OBCs to more realistic scenarios, for example, obliquely incident waves, is in principle straightforward but may not be simple to implement. Also, taking Coriolis effects explicitly into account in our OBC formulation should enhance its capability for large-scale simulations of flows on continental shelves. Moreover, changes in bathymetry in the vicinity of an open boundary, that is, sloping rather than horizontal bottom, and divergence of flow as the open boundary is approached should be explicitly accounted for in formulating a general OBC. We are currently pursuing extensions of our OBC along these lines as well as examining its performance for more realistic scenarios, such as wind-induced circulation and setup.

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