

**A Precise Calculus of Paired Lagrangian
Distributions**

by

Mark S. Joshi

B.A.(hons), Oxford University July 1990

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

at the

Massachusetts Institute of Technology

May 1994

© Massachusetts Institute of Technology 1994, All Rights Reserved.

Signature of Author
Department of Mathematics
April 27, 1994

Certified by
Richard B. Melrose
Professor of Mathematics
Thesis Supervisor

Accepted by
David Vogan
Chairman, Departmental Graduate Committee
Department of Mathematics
Science

MASSACHUSETTS INSTITUTE
OF TECHNOLOGY

AUG 11 1994

LIBRARIES

A Precise Calculus of Paired Lagrangian Distributions

by

Mark S. Joshi

Abstract

The topic of this thesis is the construction of a class of distributions associated to a pair of Lagrangian submanifolds which intersect cleanly with codimension one. These distributions are polyhomogeneous Lagrangian distributions away from the intersection and retain the property of polyhomogeneity at the intersection. The concept of a radial operator is introduced and used to give a direct characterization of polyhomogeneous Lagrangian distributions and to give an intrinsic definition of polyhomogeneous, paired Lagrangian distributions.

A symbol map is constructed which allows the construction of a distribution with given homogeneous principal symbol on each Lagrangian submanifold subject to the compatibility condition that the intrinsic Fourier transform at the intersection maps the lead singularity of one symbol into the lead singularity of the other.

Pseudo-differential operators with singular symbols are shown to lie within the class and a composition formula is established. An ellipticity condition is defined which allows the inversion of such operators up to smooth terms. This gives a new version of the construction of parametrices of operators of real principal type. Complex powers of the wave operator are constructed and shown to lie within this class.

Thesis Supervisor: Dr. Richard B. Melrose

Title: Professor of Mathematics

Acknowledgements

I would like to express my deepest gratitude to my supervisor, Richard B. Melrose, for his patience and understanding as well as innumerable helpful discussions.

My parents, Kamalakant and Catherine Joshi, although far away, have been a source of constant faith, love and support throughout my time at M.I.T..

I would also like to thank Victor Guillemin for introducing me to symplectic geometry through his very lucid course and for encouraging me to study microlocal analysis.

My undergraduate tutor Brian Steer inspired me to come to M.I.T. and I am grateful to him for his guidance and teaching throughout my time at Oxford.

My fellow analysis students John Toth, Tanya Christiansen and Andrew Hassell have provided many insights and much encouragement.

I wish also to express my gratitude to all the friends I have made at M.I.T. for making my experience here enjoyable. Special thanks to Alan Blair, Nozomi Minowa and Veronika Requat for their support during difficult times.

CONTENTS

1. Introduction	9
2. Model Forms For Lagrangian Submanifolds	12
3. Marked Lagrangian Distributions	13
4. Paired Lagrangian Distributions	28
5. Polyhomogeneous Lagrangian Distributions	32
6. Polyhomogeneous Paired Lagrangian Distributions	38
7. A Wider Class of Polyhomogeneous Distributions	65
8. The Symbol Map	68
9. Pseudo-differential Operators with Singular Symbols	74
10. Complex Powers of the Constant Coefficient Wave Operator	93
11. Complex Powers of the Wave Operator	100
References	112

1. INTRODUCTION

In [2], Duistermaat and Hörmander constructed the parametrix of an operator of real principal type using Fourier integral operators, although the parametrix itself is not a Fourier integral operator. The wavefront set of its kernel lies on two Lagrangian submanifolds, intersecting cleanly in a submanifold of codimension one, the conormal bundle to the diagonal Λ_0 and the flow out Λ_1 of the characteristic variety intersected with Λ_0 by the bicharacteristic flow. This led Melrose and Uhlmann to develop a symbolic calculus for Lagrangian distributions associated to two Lagrangian submanifolds intersecting cleanly in a submanifold of codimension one in [10], and they give a symbolic construction for the parametrix of a principal type operator. This was later extended, by Guillemin and Uhlmann in [4], to the case where the intersection is of higher codimension and they allowed a wider class of distributions, which includes the parametrices of integral powers of real principal type operators. Such parametrices can be regarded as being pseudo-differential operators with singular symbols as they are associated with the conormal bundle to the diagonal and the flow out of the characteristic variety. Antoniano and Uhlmann, in [1], showed that this calculus of pseudo-differential operators with singular symbols is closed under composition, and discussed microlocal complex powers and pseudo-differential powers of principal type operators in this context.

Melrose adopts a new point of view in [9], where he introduces the concept of a marked Lagrangian distribution. This is a distribution whose wavefront set is contained in a Lagrangian submanifold and which is Lagrangian off a submanifold, the marking, where the singularity is worse. He has shown that the paired Lagrangian distributions of [4] can be decomposed into a sum of marked Lagrangian distributions, associated to Λ_0 and Λ_1 marked by their intersection. We can therefore study paired Lagrangian distributions in terms of this decomposition. The principal problem with this approach is that the decomposition is not unique. One therefore has to cope with

an isotropic distribution supported on the intersection which is independent of the behaviour off the intersection. The idea behind my work is to use polyhomogeneity to remove this uncertainty.

We define a class of paired Lagrangian distributions associated to a pair of Lagrangian submanifolds which intersect cleanly, with codimension one. We do this by introducing the concept of a radial operator for a conic Lagrangian. This is a generalization of the concept of the radial vector field for a submanifold. We show that the polyhomogeneous Lagrangian distributions, $I_{p\Lambda_g}^m(\Lambda)$, associated to a Lagrangian submanifold Λ are precisely the distributions u such that for some fixed radial operator R ,

$$\left(\prod_{j=0}^{N-1} R + m - j \right) u \in I^{m-N}(\Lambda), \text{ for all } N. \quad (1.1)$$

We define the polyhomogeneous Lagrangian distributions analogously: they are the distributions u such that

$$\left(\prod_{j=0}^{N-1} (R_0 + m - j) \prod_{k=0}^{M-1} (R_1 + m - k) \right) u \in I^{m-N,p-M}(\Lambda_0, \Lambda_1), \text{ for all } N, M \quad (1.2)$$

where we define $I^{m,p}(\Lambda_0, \Lambda_1)$ using marked Lagrangian distributions and (R_0, R_1) is a pair of radial operators. We establish an equivalent representation in terms of oscillatory integrals of singular symbols and use this to establish a symbol map. It should be noted that this intrinsic definition does not require any constraints on the dimension of the intersection.

Away from the intersection, our principal symbols are just the usual homogeneous sections of the Maslov bundle tensored with the half-density bundle. These sections may become singular as they approach the intersection. After placing conditions on the nature of these singularities and the relationship between them, we define a symbol map. This allows us to construct a distribution, with a given pair of principal symbols, which is determined up to terms of one lower order everywhere

and in particular of lower order on the intersection. It is this property which makes our class an improvement on that given in [4].

A class of pseudo-differential operators with singular symbols lie within this calculus and we show that this class is closed under composition. We define such an operator to be elliptic if its principal symbol is non-zero away from the singularity and its leading singularity has no zeroes. This in particular allows us to regard many non elliptic pseudo-differential operators as being elliptic in this class. We show that ellipticity is equivalent to the existence of a parametrix which is a pseudo-differential operator with a singular symbol.

Seeley showed in [12] that complex powers of elliptic pseudo-differential are pseudo-differential operators. We look for complex powers of the wave operator in our calculus: Seeley's results are microlocal and suggest the symbol of a complex power of the wave operator will be the complex power of the symbol which will become singular on approach to the characteristic variety. The wave operator on the cartesian product of a Riemannian manifold and \mathbb{R} is an operator of real principal type and so its forward fundamental solution is a polyhomogeneous, paired Lagrangian distribution. Riesz, in [11], constructed a holomorphic family of kernels supported in the forward light cone which obey the group law and form complex powers of the wave operator. We give a new construction of such a family and show that they are also polyhomogeneous paired Lagrangian distributions and, using this, we calculate their symbols. The construction uses the method of descent. Letting K be the kernel of the forward fundamental solution of the wave equation in one extra variable, we define

$$K_s(t, x, x') = 2(\pi_r)_*(\chi_+^{2(-1-s)}(r)K(t, x, x', r)). \quad (1.3)$$

The product is well defined for $\Re s$ large negative and the pushforward is well defined because K is supported in the forward light cone, which implies that the support of K is proper for the projection. These kernels form a holomorphic family obeying the

group law and such that K_1 is the wave operator and K_{-1} is the forward fundamental solution.

To simplify computations, the wave equation is reduced microlocally, by Fourier integral operators, to the wave equation on Euclidean space and this has the effect of mapping K_s to the associated kernels in the flat case, up to smoothing. In the flat case, we establish the alternative representation:

$$K_s = \frac{1}{(2\pi)^{n+1}} \int e^{i(x \cdot \xi + t \cdot \tau)} ((\tau - i0)^2 - \xi^2)^s d\xi d\tau. \quad (1.4)$$

This allows us to calculate the principal symbol on each of the Lagrangian submanifolds and establish the polyhomogeneity of the kernels.

2. MODEL FORMS FOR LAGRANGIAN SUBMANIFOLDS

In this section, we review some theorems about the existence of models for cleanly intersecting conic Lagrangian submanifolds of the cotangent bundle of some smooth manifold X , of dimension greater than or equal to 2. This will enable us to define distributions associated to such Lagrangian submanifolds by doing so for the model and then using Fourier integral operators to reduce the general case to that for the model.

We recall from [10]:

Definition 2.1. *A pair (Λ_0, Λ_1) , where $\Lambda_0 \subset T^*(X) - 0$ is a conic Lagrangian submanifold and $\Lambda_1 \subset T^*(X) - 0$ is a conic Lagrangian submanifold with boundary, is said to be an intersecting pair of Lagrangian submanifolds if $\Lambda_0 \cap \Lambda_1 = \partial\Lambda_1$ and the intersection is clean:*

$$T_\lambda(\Lambda_0) \cap T_\lambda(\Lambda_1) = T_\lambda(\partial\Lambda_1) \text{ for all } \lambda \in \partial\Lambda_1 \quad (2.1)$$

Example 2.1. $\tilde{\Lambda}_0 = N^*\{x = 0\} \subset T^*(\mathbb{R}^n)$, $\tilde{\Lambda}_1^\varepsilon = \{(x, \xi) \in T^*(\mathbb{R}^n) : x'' = 0, \xi_1 = 0, x_1 \geq 0\}$ where $x = (x_1, x'')$ are the standard coordinates on \mathbb{R}^n .

Definition 2.2. *Two intersecting pairs (Λ_0, Λ_1) and (Λ'_0, Λ'_1) , $\Lambda'_i \subset T^*X' - 0$, with given base points $\lambda \in \partial\Lambda_1, \lambda'_1 \in \partial\Lambda'_1$ are said to be locally equivalent if there is a conic neighbourhood V of λ in $T^*(X) - 0$ and a homogeneous symplectic transformation $f : V \rightarrow T^*X' - 0$ such that $f(\lambda) = \lambda', f(\Lambda_0 \cap V) \subset \Lambda'_0$ and $f(\Lambda_1 \cap V) \subset \Lambda_1$.*

Proposition 2.1. *All pointed intersecting pairs of Lagrangian submanifolds in manifolds of a fixed dimension are locally equivalent.*

So, all intersecting pairs are locally equivalent to our example pair. We will henceforth refer to this pair as the model pair.

We will also need a model form for a conic isotropic submanifold of a conic Lagrangian submanifold.

Example 2.2. $\tilde{\Lambda}_0 = N^*\{x = 0\} \subset T^*(\mathbb{R}^n), K_{0,k} = \{(0, \xi) \in \tilde{\Lambda}_0 : \xi' = 0\}$ where $\xi' = (\xi_1, \dots, \xi_k)$.

We recall from [9]:

Proposition 2.2. *If (Λ, K) is a conic Lagrangian submanifold of $T^*(X)$ together with a conic submanifold of codimension k and $p \in K$ then there exists a homogeneous symplectomorphism f from a conic neighbourhood V of p to a conic neighbourhood in $T^*(\mathbb{R}^n)$ such that $f(\Lambda \cap V) \subseteq \tilde{\Lambda}_0$ and $f(K \cap V) \subseteq K_{0,k}$.*

3. MARKED LAGRANGIAN DISTRIBUTIONS

In this section, we review some of the basic facts about marked Lagrangian distributions (from [9]) and establish the equivalence of two new models which we will use later. This will allow us in the next section to define paired Lagrangian distributions as sums of marked Lagrangian distributions.

A Lagrangian distribution is a distribution, associated to a fixed conic, closed, Lagrangian submanifold of the cotangent bundle, whose Sobolev order is stable under the repeated application of first order pseudo-differential operators, which are characteristic on the submanifold. To define a marked Lagrangian distribution, we need a Lagrangian submanifold Λ and a marking of it. Our markings will be single conic submanifolds of Λ . (Much more general markings are possible, see [9]). A marked Lagrangian distribution is a distribution of which the Sobolev order is stable under the repeated application of first order pseudo-differential operators, which are characteristic on Λ and whose bicharacteristic flows are tangent to the marking. Note that any pseudo-differential operator which is characteristic on $\Lambda_0 \cup \Lambda_1$ has bicharacteristic flow tangent to $\Lambda_0 \cap \Lambda_1$, and we can therefore think of marked Lagrangian distributions as paired Lagrangian distributions which are microsupported on Λ_0 (more precisely, in a parabolic neighbourhood of Λ_0 .) Now, as the marked spaces are clearly Fourier integral operator invariant, a filtration can be defined in terms of a canonical model. We recall from [9]:

Let $\langle \xi \rangle_{(1)} = (1 + |\xi'|^4 + |\xi''|^2)^{1/4}$ where $x = (x', x'')$ is a splitting of the x coordinates and (ξ', ξ'') is the associated splitting of the dual coordinates on $T^*(\mathbb{R}^n)$.

Definition 3.1.

$$S_{m_a}^{m,p}(\tilde{\Lambda}_0, K_0) = \left\{ a \in C^\infty(\mathbb{R}^n) : |D_\xi^\alpha a| \leq C_\alpha \langle \xi \rangle^{m-|\alpha''|} \left(\frac{\langle \xi \rangle}{\langle \xi \rangle_{(1)}} \right)^{2p} \langle \xi \rangle_{(1)}^{-|\alpha'|} \right\}$$

Definition 3.2.

$$I^{m,p}(\tilde{\Lambda}_0, K_0) = \{ u = u_1 + u_2 : u_1 \in C^\infty(\mathbb{R}^n), u_2 \in \mathcal{S}'(\mathbb{R}^n), \hat{u}_2 \in S_{m_a}^{m-\frac{n}{4},p}(\Lambda_0, K_0) \}$$

We have thus defined the order of our distribution to be m off the marking and $m + p$ on the marking.

Proposition 3.1. $S_{m_a}^{m,p}(\tilde{\Lambda}_0, K_0)$ is asymptotically complete: if $a_j \in S_{m_a}^{m_j,p_j}(\tilde{\Lambda}_0, K_0)$ and $m_j \rightarrow -\infty, m_j + p_j \rightarrow -\infty$ then putting $m = \max\{m_j\}, p = \max\{m_j + p_j\} - m,$

there exists $a \in S_{ma}^{m,p}(\Lambda_0, K_0)$ such that

$$a - \sum_{j < N} a_j \in S^{\tilde{m}_N, \tilde{p}_N}(\Lambda_0, K_0) \text{ where } \tilde{m}_N = \max_{j \geq N} \{m_j\}, \tilde{p}_N = \max_{j \geq N} \{m_j + p_j\} - \tilde{m}_N$$

The statement of this proposition would be simplified if we replaced p by $m + p$ as this would then give the absolute order on the marking as opposed to the relative one but it is more convenient in general to work with relative orders.

Theorem 3.1. *If F is a properly supported Fourier integral operator of order 0, associated to a homogeneous symplectomorphism preserving $(\tilde{\Lambda}_0, K_0)$ then F preserves $I^{m,p}(\tilde{\Lambda}_0, K_0)$.*

Putting all this together, it is now possible to define a filtration for any (Λ, K) .

Definition 3.3. *Let Λ be a conic embedded Lagrangian submanifold of $T^*(X) - 0$ and let K be a conic embedded submanifold of Λ then $I^{m,p}(\Lambda, K) \subset C^{-\infty}(X)$ consists of those distributions u with $\text{WF}(u)$ contained in Λ and such that for each $p \in \Lambda$ there is a properly supported FIO, F of order 0, elliptic at p , associated to a symplectomorphism taking (Λ, K) to $(\tilde{\Lambda}_0, K_0)$ such that*

$$Fu \in I^{m,p}(\tilde{\Lambda}_0, K_0).$$

The model $(\tilde{\Lambda}_0, K_0)$ is not always convenient so we introduce two alternative models in $T^*(\mathbb{R}^n)$. Our next model will be convenient for considering distributions associated to the bicharacteristic flow out of the characteristic variety, of a differential operator, intersected with the conormal bundle of the diagonal.

Splitting the x coordinates: $x' = (x_1, \dots, x_l); x'' = (x_{l+1}, \dots, x_k)$ and $x''' = (x_{k+1}, \dots, x_n)$, we put:

$$\tilde{\Lambda}_1 = N^*\{x''' = 0\}, K_1 = \{(x, \xi) \in \tilde{\Lambda}_1 | x' = 0\} \quad (3.1)$$

The model cases arise directly when considering parabolic separation of the fundamental solution of $\frac{\partial}{\partial x_1}$. Recall that Duistermaat and Hörmander constructed parametrices for operators of real principal type by reduction to $\frac{\partial}{\partial x_1}$ so we can expect that the reduction will reduce the Lagrangian submanifolds involved to the model ones. Here, the x'' coordinates play the role of type 0 parameters; that is the symbol estimates will be uniform for x'' in a compact set and taking x'' derivatives will preserve the symbol estimates. We will in fact only use the case where $x' = x_1$ but considering the more general case requires no more work. Now, $x' = 0$ will define the marking and so the symbol will gain a half order on the marking when differentiated with respect to $x' = 0$. The space of symbols is that obtained from $S_{ma}^{m,p}(\tilde{\Lambda}_0, K_0)$ by rehomogenization.

Definition 3.4.

$$S_{ma}^{m,p}(\tilde{\Lambda}_1, K_1) = \left\{ a \in C^\infty(\mathbb{R}_{x',x''}^k \times \mathbb{R}_{\xi'''}^{n-k}) : |D_{x'}^{\alpha'} D_{x''}^{\alpha''} D_{\xi'''}^{\alpha'''} a(x', x'', \xi''')| \leq C_{\alpha,K} \langle \xi''' \rangle^{m-|\alpha'''}| \left(\frac{1 + |x'| \langle \xi''' \rangle^{1/2}}{\langle \xi''' \rangle^{1/2}} \right)^{-2p-|\alpha'|} \forall \alpha, (x', x'') \in K, K \text{ compact} \right\}$$

Definition 3.5.

$$J^{m,p}(\tilde{\Lambda}_1, K_1) = \left\{ \int e^{i\langle x''', \xi''' \rangle} a(x', x'', \xi''') d\xi''' : a \in S_{ma}^{m-n/4+k/2,p}(\tilde{\Lambda}_1, K_1) \right\} + C^\infty(\mathbb{R}^n)$$

We could now modify 3.3 to give a filtration of $I(\Lambda, K)$ using $(\tilde{\Lambda}_1, K_1)$ but we will show instead that the two models are equivalent.

Theorem 3.2. *The classes of distributions $I^{m,p}(\tilde{\Lambda}_1, K_1)$ and $J^{m,p}(\tilde{\Lambda}_1, K_1)$ are equal.*

Proof. Away from K_1 , there is nothing to prove as we are then considering ordinary Lagrangian distributions and so the result follows directly from the calculus of FIOs. Hence, we need only consider distributions supported in a small conic neighbourhood of K_1 .

For most of the proof, the x' and x'' behave identically so we group them together as \hat{x} . As we already know that $I^{m,p}(\tilde{\Lambda}_0, K_0)$ is FIO invariant, we need only consider one symplectomorphism which maps $(\tilde{\Lambda}_1, K_1)$ to $(\tilde{\Lambda}_0, K_0)$. The phase function

$$\psi(y, x, \hat{\xi}, \hat{\eta}, \eta) = \langle \hat{x}, \hat{\xi} \rangle + \langle \hat{y}, \hat{\eta} \rangle + \frac{\langle \hat{\xi}, \hat{\eta} \rangle}{|\eta''''|} + \langle y''' - x''', \eta''' \rangle$$

parametrises such a symplectomorphism away from $|\eta''''| = 0$.

$$\begin{aligned} \psi'_{\hat{\xi}, \hat{\eta}} &= \left(\hat{x} + \frac{\hat{\eta}}{|\eta''''|}, \hat{y} + \frac{\hat{\xi}}{|\eta''''|}, y''' - x''' - \langle \hat{\xi}, \hat{\eta} \rangle \frac{\eta''''}{|\eta''''|^3} \right) \\ \psi'_x &= (\hat{\xi}, -\eta'''') \\ \psi'_y &= (\hat{\eta}, \eta'''') \end{aligned}$$

The non-degeneracy of ψ is clear and

$$\Lambda'_{\psi} = \left\{ \left(\frac{-\hat{\eta}}{|\eta''''|}, x''', \hat{\xi}, -\eta'''' , \frac{-\hat{\xi}}{|\eta''''|}, x''' - \langle \hat{\xi}, \hat{\eta} \rangle \frac{\eta''''}{|\eta''''|^3}, -\eta \right) \right\}.$$

So putting, $\hat{x} = \frac{-\hat{\eta}}{\eta''''}, \xi'''' = -\eta''''$, we have that Λ'_{ψ} is the graph of

$$f : (x, \xi) \rightarrow \left(\frac{-\hat{\xi}}{|\xi''''|}, x''' + \frac{\langle \hat{x}, \hat{\xi} \rangle}{|\xi''''|^2} \xi'''' , -\hat{x} |\xi''''|, \xi'''' \right).$$

This is clearly a map from $(\tilde{\Lambda}_0, K_0)$ to $(\tilde{\Lambda}_1, K_1)$ and its Jacobian is invertible, on K_1 :

$$\frac{\partial(\hat{y}, y''', \hat{\eta}, \eta'''')}{\partial(\hat{x}, x''', \hat{\xi}, \xi'''')} = \begin{pmatrix} 0 & 0 & \frac{1}{|\xi''''|} & 0 \\ 0 & \text{Id} & 0 & 0 \\ -|\xi''''| & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{Id} \end{pmatrix}. \quad (3.2)$$

So, f is a homogeneous symplectomorphism in a neighbourhood of K_1 .

Now, let P be a properly supported zeroth order FIO, with phase function ψ and symbol $b(y, \xi, \eta)$ elliptic near K_1 and smoothing outside a small conic neighbourhood

of K_1 and let $u \in I^{m,p}(\Lambda_1, K_1)$, then up to smooth terms we have

$$Pu = \int e^{i\psi(y,x,\hat{\xi},\eta)} b(y, \hat{\xi}, \eta) \left(\int e^{i\langle x''', \mu''' \rangle} a(\hat{x}, \mu''') d\mu''' \right) d\hat{\xi} d\eta dx \quad (3.3)$$

where

$$\begin{aligned} b &\in S_{1,0}^{-\frac{k}{2}}(\mathbb{R}_y^n; \mathbb{R}_{\hat{\xi}}^k \times \mathbb{R}_\eta^n), \\ a &\in S_{ma}^{m+\frac{k}{2}-\frac{n}{4},p}(\tilde{\Lambda}_1, K_1) \end{aligned}$$

and we will take a to be compactly supported in \hat{x} , as we are only interested in behaviour near $\hat{x} = 0$. Thus,

$$Pu = \int e^{i\langle y, \eta \rangle} \left[\int e^{i\left(\langle \hat{x}, \hat{\xi} \rangle + \frac{\langle \hat{x}, \hat{\eta} \rangle}{|\eta'''}|}\right)} b(y, \hat{\xi}, \eta) a(\hat{x}, \eta''') d\hat{x} d\hat{\xi} \right] d\eta \quad (3.4)$$

$$= \int e^{i\langle y, \eta \rangle} \left[\int e^{i\langle \tilde{x}, \hat{\xi} \rangle} b(y, \hat{\xi}, \eta) a\left(\tilde{x} - \frac{\hat{\eta}}{|\eta'''}|, \eta'''\right) d\tilde{x} d\hat{\xi} \right] d\eta. \quad (3.5)$$

Taking a Taylor expansion of b about $\hat{\xi} = 0$, this becomes

$$= \int e^{i\langle y, \eta \rangle} \left[\sum_{|\hat{\alpha}| \leq N-1} \frac{\partial_{\hat{\xi}}^{\hat{\alpha}} b(y, 0, \eta)}{\hat{\alpha}!} \int e^{i\langle \tilde{x}, \hat{\xi} \rangle} (\hat{\xi})^{\hat{\alpha}} a\left(\tilde{x} - \frac{\hat{\eta}}{|\eta'''}|, \eta'''\right) d\tilde{x} d\hat{\xi} \right] d\eta + R_N \quad (3.6)$$

$$= \int e^{i\langle y, \eta \rangle} \sum_{|\hat{\alpha}| \leq N-1} \frac{\partial_{\hat{\xi}}^{\hat{\alpha}} b(y, 0, \eta)}{\hat{\alpha}!} D_{\tilde{x}}^{\hat{\alpha}} a\left(-\frac{\hat{\eta}}{|\eta'''}|, \eta'''\right) d\eta + R_N. \quad (3.7)$$

It is easily checked that $c(\eta) = a(-\eta'/|\eta'''|, -\eta''/|\eta'''|, \eta''') \in S_{ma}^{m+k/2-n/4,p}(\Lambda_0, K_0)$ and of course, $\partial_{\hat{\xi}}^{\hat{\alpha}} b(y, 0, \eta) \in S_{1,0}^{-k/2-|\hat{\alpha}|}$ and so we have

$$\partial_{\hat{\xi}}^{\hat{\alpha}} b(y, 0, \eta) c(\eta) \in S_{ma}^{m-|\hat{\alpha}|-n/4,p}(\Lambda_0 \times \mathbb{R}_y^n, K_0). \quad (3.8)$$

This is the marked symbol space with respect to $(\tilde{\Lambda}_0, K_0)$ with an extra n type 0 parameters. We now want to show that our integral has an asymptotic expansion in this symbol space. To do this, we need to show that the remainder terms, R_N , are harmless. Now,

$$R_N = \int e^{i\langle y, \eta \rangle} \sum_{|\hat{\alpha}|=N} \int e^{i\langle \hat{x}, \hat{\xi} \rangle} (\hat{\xi})^{\hat{\alpha}} b_{\hat{\alpha}}(y, \hat{\xi}, \eta) a\left(\tilde{x} - \frac{\hat{\eta}}{|\eta''''|}, \eta''''\right) d\tilde{x} d\hat{\xi} d\eta \quad (3.9)$$

where $b_{\hat{\alpha}} \in S_{1,0}^{-k/2-|\hat{\alpha}|}$. If we pick $N = K + n + 1 + |p|$ we obtain,

$$\left| \sum_{|\hat{\alpha}|=N} \int e^{i\langle \hat{x}, \hat{\xi} \rangle} (\hat{\xi})^{\hat{\alpha}} b_{\hat{\alpha}}(y, \hat{\xi}, \eta) a\left(\tilde{x} - \frac{\hat{\eta}}{|\eta''''|}, \eta''''\right) d\tilde{x} d\hat{\xi} \right| \leq C \langle \eta \rangle^{-K}. \quad (3.10)$$

as a is compactly supported in \hat{x} . This is clearly symbolic of type $(1/2, 0)$. This establishes our expansion.

So, we have that

$$Pu = \int e^{i\langle y, \eta \rangle} d(y, \eta) d\eta \quad (3.11)$$

where $d(y, \eta) \in S_{ma}^{m-n/4, p}(\Lambda_0 \times \mathbb{R}_y^n, K_0)$. Taking a Taylor expansion in y about $y = 0$ and integrating by parts we obtain an asymptotic expansion in $S^{m-n/4, p}(\tilde{\Lambda}_0, K_0)$ and so $Pu \in I^{m, p}(\tilde{\Lambda}_0, K_0)$.

To complete the proof we now need to show that if Q is a zeroth order FIO associated to f^{-1} then it induces a map from $I^{m, p}(\Lambda_0, K_0)$ to $I^{m, p}(\Lambda_1, K_1)$ and then picking P and Q to be microlocal inverses in a small conic neighbourhood of K_1 , our result will follow.

The phase will be

$$\psi'(x, y, \hat{\xi}, \eta) = -(\langle \hat{y}, \hat{\eta} \rangle + \langle \hat{x}, \hat{\xi} \rangle + \frac{\langle \hat{\xi}, \hat{\eta} \rangle}{|\eta''''|}) + \langle x'''' - y'''' , \eta'''' \rangle .$$

Thus we get

$$Qu = \int e^{i\langle x'''' , \eta'''' \rangle} \left[\int e^{-i\langle \hat{x} + \frac{\hat{\eta}}{|\eta''''|}, \hat{\xi} \rangle} b(x, \hat{\xi}, \eta) \hat{u}(\eta) d\hat{\xi} d\hat{\eta} \right] d\eta'''' \quad (3.12)$$

with $b \in S_{1,0}^{-\frac{k}{2}}$. Now,

$$\begin{aligned} & \int e^{-i\langle \hat{x} + \frac{\hat{\eta}}{|\eta|}, \hat{\xi} \rangle} b(x, \hat{\xi}, \eta) \hat{u}(\eta) d\hat{\xi} d\hat{\eta} \\ &= \int e^{-i\langle \hat{x} - \tilde{\eta}, \hat{\xi} \rangle} b(x, \hat{\xi}, \tilde{\eta}|\eta''', \eta''') \hat{u}(\tilde{\eta}|\eta''', \eta''') d\hat{\xi} d\tilde{\eta} |\eta'''|^k. \end{aligned} \quad (3.13)$$

As $|\eta'''|$ is elliptic on the support of b we have, where the integrand is supported,

$$\begin{aligned} \tilde{b}(x, \hat{\xi}, \tilde{\eta}, \eta''') &= b(x, \hat{\xi}, \tilde{\eta}|\eta''', \eta''') |\eta'''|^k \in S_{1,0}^{k/2}(\mathbb{R}_x^n \times \mathbb{R}_{\tilde{\eta}}^k; \mathbb{R}_{\hat{\xi}}^k \times \mathbb{R}_{\eta'''}^{n-k}) \\ c(\tilde{\eta}, \eta''') &= \hat{u}(\tilde{\eta}|\eta''', \eta''') \in S^{m-\frac{n}{4}, p}(\tilde{\Lambda}_1, K_1) \end{aligned}$$

and thus our integral is equal to

$$e^{-i\langle D_{\tilde{\eta}}, D_{\hat{\xi}} \rangle} \tilde{b}(x, \hat{\xi}, \tilde{\eta}, \eta''') c(\tilde{\eta}, \eta''') |_{\hat{\xi}=0, \tilde{\eta}=\hat{x}} \in S^{m+\frac{k}{2}-\frac{n}{4}, p}(\Lambda_1 \times \mathbb{R}_{\hat{x}}^k, K_1). \quad (3.14)$$

We have an extra $n - k$ type 0 parameters, but these can be removed as above by taking a Taylor expansion in x''' and integrating by parts. \square

Corollary 3.1. *If F is a properly supported Fourier integral operator of order 0, associated to a homogeneous symplectomorphism preserving $(\tilde{\Lambda}_1, K_1)$ then F preserves $I^{m,p}(\tilde{\Lambda}_1, K_1)$.*

The same proof also establishes the equivalence of another model:

$$\tilde{\Lambda}_2 = N^*((x'', x''') = 0), K_2 = \{(x, \xi) \in \tilde{\Lambda}_2 : \xi'' = 0\} \quad (3.15)$$

and, putting $\langle (\xi'', \xi''') \rangle_{(1)} = \langle \xi'' \rangle + \langle \xi''' \rangle^{\frac{1}{2}}$ we use the symbol space:

$$\begin{aligned} S^{m,p}(\tilde{\Lambda}_2, K_2) &= \left\{ a \in C^\infty(\mathbb{R}_{x'}^l \times \mathbb{R}_{\xi''}^{k-l} \times \mathbb{R}_{\xi'''}^{n-k}) : |D_{x'}^{\alpha'} D_{\xi''}^{\alpha''} D_{\xi'''}^{\alpha'''} a(x', \xi'', \xi''')| \leq C_{\alpha, K} \right. \\ & \quad \left. \langle (\xi'', \xi''') \rangle^{m-|\alpha''|} \left(\frac{\langle (\xi'', \xi''') \rangle}{\langle (\xi'', \xi''') \rangle_{(1)}} \right)^{2p} \langle (\xi'', \xi''') \rangle_{(1)}^{-|\alpha''|}, \right. \\ & \quad \left. x' \in K, K \text{ compact} \right\}. \end{aligned} \quad (3.16)$$

We then conclude

Proposition 3.2.

$$I^{m,p}(\tilde{\Lambda}_2, K_2) = \left\{ \int e^{i\langle x'', \xi'' \rangle + i\langle x''', \xi''' \rangle} a(x', \xi'', \xi''') d\xi'' d\xi''' : a \in S^{m-\frac{1}{2}+\frac{n}{4},p}(\tilde{\Lambda}_2, K_2) \right\} \\ + C^\infty(\mathbb{R}^n)$$

This model will be convenient for studying marked pseudo-differential operators, that is operators whose kernel is a marked Lagrangian distribution for which the Lagrangian is the conormal bundle of the diagonal. We have been studying Lagrangian distributions with a marking which is an isotropic submanifold. In the case where this submanifold is of codimension one, it decomposes the Lagrangian into two pieces and since we allow a different order on the marking from the rest of the submanifold, it is natural in this case to allow differing orders on each component of the marking's complement. In particular, we will want to consider distributions micro-supported on one side of the marking. We now take $x = (x_1, x'')$, $\tilde{\Lambda}_0 = N^*\{x = 0\}$, $\tilde{\Lambda}_0^\pm = \{(0, \xi) : \pm \xi_1 > 0\}$. We denote the spline function $H(t)t$ by $S(t)$.

Definition 3.6.

$$S_{ma}^{m,p,r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0^+, \tilde{\Lambda}_0^-) = \left\{ a \in C^\infty(\mathbb{R}_\xi^n) : \right. \\ \left. |D_\xi^\alpha a(\xi)| \leq C \langle \xi \rangle^{m-|\alpha''|} \left(\frac{S(\xi_1) + \langle \xi \rangle^{\frac{1}{2}}}{\langle \xi \rangle^{\frac{1}{2}}} \right)^{2p} \left(\frac{S(-\xi_1) + \langle \xi \rangle^{\frac{1}{2}}}{\langle \xi \rangle^{\frac{1}{2}}} \right)^{2r} \langle \xi \rangle^{\frac{-\alpha_1}{(1)}} \right\}$$

Definition 3.7.

$$I^{m,p,r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0^+, \tilde{\Lambda}_0^-) = \{u = u_1 + u_2 : \\ u_1 \in C^\infty, u_2 \in \mathcal{S}'(\mathbb{R}^n), \hat{u}_2 \in S^{m-\frac{n}{4},p,r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0^+, \tilde{\Lambda}_0^-)\}$$

So our distribution is of order m on the marking, order $m + p$ on $\tilde{\Lambda}_0^+$ and order $m + r$ on $\tilde{\Lambda}_0^-$. As usual, we must check that the space is Fourier integral operator invariant. We show:

Theorem 3.3. *The class $I^{m,p,r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0^+, \tilde{\Lambda}_0^-)$ is invariant under application of properly supported zeroth order Fourier integral operators associated to symplectomorphisms which preserve $(\tilde{\Lambda}_0, \tilde{\Lambda}_0^+, \tilde{\Lambda}_0^-)$.*

Proof. Most of this proof is already contained in the proof of the invariance of $I^{m,p}(\tilde{\Lambda}_0, K_0)$ from [9]. The difference here being that the weight function has an extra factor whilst the metric is the same. Note that we can rewrite our weight function as

$$\begin{aligned} \langle \xi \rangle^m \left(\frac{|\xi_1| + \langle \xi \rangle^{\frac{1}{2}}}{\langle \xi \rangle^{\frac{1}{2}}} \right)^{2p} \left(\frac{S(-\xi_1) + \langle \xi \rangle^{\frac{1}{2}}}{\langle \xi \rangle^{\frac{1}{2}}} \right)^{2(r-p)} \\ = \langle \xi \rangle^m \left(\frac{|\xi_1| + \langle \xi \rangle^{\frac{1}{2}}}{\langle \xi \rangle^{\frac{1}{2}}} \right)^{2r} \left(\frac{S(\xi_1) + \langle \xi \rangle^{\frac{1}{2}}}{\langle \xi \rangle^{\frac{1}{2}}} \right)^{2(p-r)} \end{aligned} \quad (3.17)$$

It will be enough to prove Fourier integral operator invariance for $p - r > 0$ by symmetry. In this case, we have an extra factor of $\left(\frac{S(\xi_1) + \langle \xi \rangle^{\frac{1}{2}}}{\langle \xi \rangle^{\frac{1}{2}}} \right)^{2(p-r)}$. Invariance under coordinate changes in ξ which preserve the model is clear, so we need only consider Fourier integral operators associated to symplectomorphisms which are equal to the identity on the conormal bundle to the origin. Melrose shows in [9] that such operators can be written as

$$\widehat{F}u(\eta) = \int e^{iy \cdot \eta} b(y, \eta) \hat{u}(\Psi(y, \eta)) d\eta \quad (3.18)$$

$$= e^{iD_y D_\eta} (b(y, \eta) \hat{u}(\Psi(y, \eta))) \text{ evaluated at } y = 0. \quad (3.19)$$

where Ψ is homogeneous of degree one, $\Psi(0, \eta) = \eta$ and b is a zeroth order symbol supported near $y = 0$.

We wish to apply Hörmander's results on Gauss transforms (see [7]) to see that $\widehat{F}u$ is in $S_{ma}^{m,p,r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0^+, \tilde{\Lambda}_0^-)$. Hörmander has shown that the class of symbols associated to a slowly varying metric g with g -continuous weight, m , is invariant under the Gauss transform,

$$u(z) \mapsto e^{i\sigma(D_z)} u(z)$$

provided g, m are σ -temperate.

Thus to show $\widehat{F}u$ is in the correct space (as in [9]) we show that $b(y, \eta)\hat{u}(\Psi(y, \eta))$ is in a wider symbol space defined by a slowly varying metric, G , and a G -continuous weight function which when restricted to $y = 0$ yields $S_{m_a}^{m, p, r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0^+, \tilde{\Lambda}_0^-)$.

Let ρ denote a smooth, monotone function on \mathbb{R} which is equal to the identity for $t < \frac{1}{2}$ and 1 for $t > 1$ and put

$$q_{(1)}(y, \eta) = \left(1 + \frac{|\eta_1|^4}{1 + \rho(|y|)^2|\eta|^2} + |\eta''|^2\right)^{\frac{1}{4}} \quad (3.20)$$

$$q_{(2)}(y, \eta) = \left(1 + \frac{|\eta|^4}{1 + \rho(|y|)^2|\eta|^2}\right)^{\frac{1}{4}} \quad (3.21)$$

$$Q_{(1)}(y, \eta) = (1 + |\eta_1|^4 + \rho(|x|)^4|\eta|^4 + |\eta''|^2)^{\frac{1}{4}} \quad (3.22)$$

then we have from [9], $b(y, \eta)\hat{u}(\Psi(y, \eta))$ is in a symbol space with slowly varying metric

$$G = \left(\frac{\langle \eta \rangle}{q_1(y, \eta)}\right)^2 dy^2 + \frac{d\xi_1^2}{q_1(y, \eta)} + \frac{(d\xi'')^2}{q_2(y, \eta)} \quad (3.23)$$

and when $r = p$ with G -continuous weight

$$M(y, \eta) = \begin{cases} \langle \eta \rangle^m \left(\frac{Q_{(1)}(y, \eta)}{\langle \eta \rangle^{\frac{1}{2}}}\right)^{2r} & \text{if } r \geq 0 \\ \langle \eta \rangle^m \left(\frac{q_{(1)}(y, \eta)}{\langle \eta \rangle^{\frac{1}{2}}}\right)^{2r} & \text{if } r < 0 \end{cases} \quad (3.24)$$

We want to consider the quadratic form $\sigma = \tilde{y} \cdot \tilde{\eta}$ where $(\tilde{y}, \tilde{\eta})$ are the dual variables to (y, η) . Our dual metric is then

$$G^\sigma = q_1(y, \eta)^2 dy_1^2 + q_2(y, \eta)^2 (dy'')^2 + \left(\frac{q_1(y, \eta)}{\langle \eta \rangle}\right)^2 d\eta^2. \quad (3.25)$$

We have from [9] that G, m are σ -temperate.

For our class we have an additional weight factor

$$\left(\frac{1 + S(\xi_1 + \rho(|x|) \cdot \langle \xi \rangle) + \langle \xi \rangle^{\frac{1}{2}}}{\langle \xi \rangle^{\frac{1}{2}}}\right)^{2(p-r)} \quad (3.26)$$

so to complete our proof we need to check the G -continuity and σ temperateness of $1 + S(\xi_1 + \rho(|x|) \cdot |\xi|) + \langle \xi \rangle^{\frac{1}{2}}$.

We already know that $\langle \xi \rangle^s$ is G -continuous so there exists c such that

$$G_{x,\xi}(y, \eta) < c \text{ implies } \langle \xi \rangle^s < A_s \langle \xi + \eta \rangle^s.$$

So for $G_{x,\xi}(y, \eta) < c$, we have

$$\begin{aligned} 1 + S(\xi_1 + \eta_1 + \rho(x+y) \langle \xi + \eta \rangle) + \langle \xi + \eta \rangle^{\frac{1}{2}} \\ \geq 1 + S(\xi_1 + \eta_1 + A\rho(x+y) \langle \xi \rangle) + A \langle \xi \rangle^{\frac{1}{2}}. \end{aligned} \quad (3.27)$$

We also have

$$|\eta_1| \leq A(\langle \xi \rangle^{\frac{1}{2}} + |\xi_1|)$$

and

$$|y| \leq A(\langle \xi \rangle^{-\frac{1}{2}} + |\xi_1| \langle \xi \rangle^{-1} |)$$

which imply that

$$\begin{aligned} 1 + S(\xi_1 + \eta_1 + \rho(x+y) \langle \xi + \eta \rangle) + \langle \xi + \eta \rangle^{\frac{1}{2}} \\ \geq 1 + S\left((1 - c(1 + A))\xi_1 + A\rho(|x|) \langle \xi \rangle - c(1 + A) \langle \xi \rangle^{\frac{1}{2}}\right) + A \langle \xi \rangle^{\frac{1}{2}}. \end{aligned} \quad (3.28)$$

So picking c sufficiently small, and noting that $S(x - y) \geq S(x) - |y|$ we have for some C ,

$$\begin{aligned} 1 + S(\xi_1 + \eta_1 + \rho(x+y) \langle \xi + \eta \rangle) + \langle \xi + \eta \rangle^{\frac{1}{2}} \geq \\ C \left(1 + S(\xi_1 + \rho(x) \langle \xi \rangle) + \langle \xi \rangle^{\frac{1}{2}}\right). \end{aligned} \quad (3.29)$$

The reverse equality can be proved similarly and thus we have proven G -continuity.

We use the fact that $\langle \xi \rangle^s$ is σ -temperate to show that

$$1 + S(\xi_1 + \rho(x) \langle \xi \rangle) + \langle \xi \rangle^{\frac{1}{2}}$$

is also. We argue

$$1 + S(\xi_1 + \rho(x) \langle \xi \rangle) + \langle \xi \rangle^{\frac{1}{2}} = \langle \xi \rangle^{-\frac{1}{2}} (1 + S(\xi_1 \langle \xi \rangle^{-\frac{1}{2}} + \rho(x) \langle \xi \rangle^{\frac{1}{2}})) \quad (3.30)$$

$$\leq C \langle \eta \rangle^{\frac{1}{2}} (1 + S((\xi_1 - \eta_1) \langle \xi \rangle^{-\frac{1}{2}} + \rho(x - y) \langle \xi \rangle^{\frac{1}{2}}) + S(\eta_1 \langle \eta \rangle^{-\frac{1}{2}} + \rho(y) \langle \eta \rangle^{-\frac{1}{2}}))(1 + G_{x,\xi}^\sigma(x - y, \xi - \eta))^N \quad (3.31)$$

$$\leq C(\langle \eta \rangle^{\frac{1}{2}} + S(\eta_1 + \rho(y) \langle \eta \rangle^{\frac{1}{2}}))(1 + |\xi_1 - \eta_1| \langle \xi \rangle^{-\frac{1}{2}} + \rho(x - y) \langle \xi \rangle^{\frac{1}{2}})(1 + G_{x,\xi}^\sigma(x - y, \xi - \eta))^N \quad (3.32)$$

$$\leq C(\langle \eta \rangle^{\frac{1}{2}} + S(\eta_1 + \rho(y) \langle \eta \rangle^{\frac{1}{2}}))(1 + G_{x,\xi}^\sigma(x - y, \xi - \eta))^N \quad (3.33)$$

Thus we have σ -temperateness and the theorem follows. \square

We can now define a new filtration for any Lagrangian submanifold which is decomposed by a hypersurface into two pieces.

Definition 3.8. *Let Λ be a conic embedded Lagrangian submanifold of $T^*(X) - 0$ and let K be a conic embedded hypersurface in Λ which decomposes Λ into Λ^+, Λ^- then $I^{m,p}(\Lambda, \Lambda^+, \Lambda^-) \subset C^{-\infty}(X)$ consists of those distributions u with $\text{WF}(u)$ contained in Λ and such that for each $p \in \Lambda$ there is a properly supported FIO, F of order 0, elliptic at p , associated to a symplectomorphism taking $(\Lambda, \Lambda^+, \Lambda^-)$ to $(\tilde{\Lambda}_0, \tilde{\Lambda}_0^+, \tilde{\Lambda}_0^-)$ such that*

$$Fu \in I^{m,p,r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0^+, \tilde{\Lambda}_0^-).$$

We will later want to consider marked Lagrangian distributions which are supported on a Lagrangian submanifold with boundary, with the boundary being the marking, this new filtration gives a way to do this: pick an extension of the Lagrangian submanifold past the boundary and define the class to be those distributions which are of order $-\infty$ on the extended part. This class will be smaller than

that of marked Lagrangian distributions which are microsupported off the extension. To see this, consider the Fourier transform of a function which is supported within a curve which is between a parabola and a cone. We need to check that the class of one-sided distributions so obtained is independent of the choice of extension.

Proposition 3.3. *The class of distributions $I^{m,p,-\infty}(\tilde{\Lambda}_0, \tilde{\Lambda}_0^+, \tilde{\Lambda}_0^-)$ is invariant under zeroth order Fourier integral operators associated to homogeneous symplectomorphisms which preserve $(\tilde{\Lambda}_0^+, \partial\tilde{\Lambda}_0^+)$.*

Proof. We decompose our symplectomorphism into three pieces. The first piece is a change of ξ coordinates which will have no effect on the symbol classes as above. The second will be a symplectomorphism which is equal to the identity on the conormal bundle of the origin and so the results above establish invariance for it. The third will move the model choice of extension to another choice. We can take it to have phase function of the form $\phi = \langle x - y, \xi \rangle + \rho(\xi_1)\psi(x, y, \xi)$ where ρ is smooth and zero on $\xi_1 > 0$.

Now let g be a smooth function on \mathbb{R} which is identically 1 for $t < \frac{1}{2}$ and identically 0 for $t > 1$. We decompose $u \in I^{m,p,-\infty}(\tilde{\Lambda}_0, \tilde{\Lambda}_0^+, \tilde{\Lambda}_0^-)$ using g to define a one sided parabolic cut off.

$$\hat{u}(\xi) = (1 - g)\left(\xi_1|\xi|^{-1/2}\right)\hat{u}(\xi) + g\left(\xi_1|\xi|^{-1/2}\right)\hat{u}(\xi) \quad (3.34)$$

The second piece u_2 is now an isotropic distribution, associated to $\partial\tilde{\Lambda}_0^+$, of order m and so after application of a zero order Fourier integral operator associated to a symplectomorphism preserving $\partial\tilde{\Lambda}_0^+$ will lie in the same class. It is clear that the isotropic distributions of order m are contained in $I^{m,-\infty,-\infty}(\tilde{\Lambda}_0, \tilde{\Lambda}_0^+, \tilde{\Lambda}_0^-)$ and so the invariance of u_2 follows.

We are left with the first piece, $u_1 \in I^{m,r,-\infty}(\tilde{\Lambda}_0, \tilde{\Lambda}_0^+, \tilde{\Lambda}_0^-)$, this has the property

that $\hat{u}_1(\xi)$ is identically zero for $\xi_1 < 0$. Thus

$$\int e^{i\langle x-y, \xi \rangle + \rho(\xi_1)\psi(x, y, \xi)} a(x, \xi) \hat{u}_1(\xi) d\xi = \int e^{i\langle x-y, \xi \rangle} a(x, \xi) \hat{u}_1(\xi) d\xi \quad (3.35)$$

and we know invariance under pseudo-differential operators. \square

This theorem means that the following definition makes sense.

Definition 3.9. *Let Λ^e be a conic embedded Lagrangian submanifold of $T^*(X) - 0$ with boundary $\partial\Lambda^e$ then $I^{m,p}(\Lambda^e, \partial\Lambda^e) = I^{m+p, -p, -\infty}(\Lambda, \Lambda^e, \Lambda^-)$ where Λ is some conic embedded Lagrangian submanifold of $T^*(X) - 0$ without boundary containing Λ^e and Λ^- is the complement of Λ^e .*

An element of $I^{m,p}(\Lambda^e, \partial\Lambda^e)$ will therefore be of order m off the marking and order $m + p$ on the marking. The following is implicit in invariance of the definition and the proofs above.

Corollary 3.2. *Let f be a homogeneous symplectomorphism taking $(\Lambda_1^e, \partial\Lambda_1^e)$ to $(\Lambda_2, \partial\Lambda_2)$. If F is a proper Fourier integral operator, associated to f , of order k then F induces a map*

$$F : I^{m,p}(\Lambda_1^e, \partial\Lambda_1^e) \rightarrow I^{m+k,p}(\Lambda_2, \partial\Lambda_2). \quad (3.36)$$

Just as we needed more than one model for two sided marked Lagrangians, it is convenient to have another model for one sided ones. We obtain such a model by modifying our second model (Definition 3.5). Let $x'' = (x_2, \dots, x_l)$, $x''' = (x_{l+1}, \dots, x_n)$ and then put $\tilde{\Lambda}_1 = N^*(x''' = 0)$, $\tilde{\Lambda}_1^+ = \{(x, \xi) \in \tilde{\Lambda}_1 : x_1 > 0\}$, $\tilde{\Lambda}_1^- = \{(x, \xi) \in \tilde{\Lambda}_1 : x_1 < 0\}$.

Definition 3.10.

$$\begin{aligned} S_{ma}^{m,p,r}(\tilde{\Lambda}_1, \tilde{\Lambda}_1^+, \tilde{\Lambda}_1^-) &= \left\{ a(x_1, x'', \xi''') \in C^\infty(\mathbb{R}_{x', x''}^k \times \mathbb{R}_{\xi'''}^{n-k}) : |D_{x_1}^{\alpha_1} D_{x''}^{\alpha''} D_{\xi'''}^{\alpha'''} a| \right. \\ &\leq C_{\alpha, K} \langle \xi''' \rangle^{m - |\alpha''| + \frac{\alpha_1}{2}} (1 + S(x_1) \langle \xi''' \rangle^{\frac{1}{2}})^{2p - \alpha_1} (1 + S(-x_1) \langle \xi''' \rangle^{\frac{1}{2}})^{2r - \alpha_1} \left. \right\} \end{aligned}$$

Proposition 3.4.

$$I^{m,p,r}(\tilde{\Lambda}_1, \tilde{\Lambda}_1^+, \tilde{\Lambda}_1^-) = \left\{ \int e^{i\langle x''', \xi''' \rangle} a(x_1, x'', \xi''') d\xi''' : a \in S_{ma}^{m-\frac{n}{4}+\frac{k}{2}, p, r}(\tilde{\Lambda}_1, \tilde{\Lambda}_1^+, \tilde{\Lambda}_1^-) \right\} \\ + C^\infty(\mathbb{R}^n)$$

Proof. This is identical to the proof for 3.2. \square

Thus writing $\tilde{\Lambda}_1^e = \tilde{\Lambda}_1^+$ we have:

Corollary 3.3.

$$I^{m,p}(\tilde{\Lambda}_1^e, \partial\tilde{\Lambda}_1^e) = \left\{ \int e^{i\langle x''', \xi''' \rangle} a(x_1, x'', \xi''') d\xi''' : a \in S_{ma}^{m+p-\frac{n}{4}+\frac{k}{2}, -p, -\infty}(\tilde{\Lambda}_1, \tilde{\Lambda}_1^+, \tilde{\Lambda}_1^-) \right\} \\ + C^\infty(\mathbb{R}^n)$$

4. PAIRED LAGRANGIAN DISTRIBUTIONS

Having proven these things about marked Lagrangian distributions, we are able to define some classes of paired Lagrangian distributions.

Definition 4.1. *Let Λ_0 and Λ_1 be cleanly intersecting conic Lagrangian submanifolds of $T^*(X) \setminus 0$ with intersection of codimension k then*

$$I^{m,p}(\Lambda_0, \Lambda_1) = I^{m, \frac{p-m+\frac{k}{2}}{2}}(\Lambda_0, \Lambda_0 \cap \Lambda_1) + I^{p, \frac{m-p+\frac{k}{2}}{2}}(\Lambda_1, \Lambda_0 \cap \Lambda_1)$$

So, the class $I^{m,p}$ consists of distributions which have wavefront sets contained in $\Lambda_0 \cup \Lambda_1$ and have order m on $\Lambda_0 - \Lambda_0 \cap \Lambda_1$, $\frac{m+p+\frac{k}{2}}{2}$ on $\Lambda_0 \cap \Lambda_1$ and p on $\Lambda_1 - \Lambda_0 \cap \Lambda_1$. More, generally we could define filtrations with the order on the intersection independent of m and p but for the distributions we shall encounter this filtration is natural, as it reflects the fact that nothing special happens at the intersection. The extra $\frac{k}{2}$ reflects the fact that in the model case we get the average of the symbolic orders as opposed to the average of Hörmander's Lagrangian orders. We denote by $I(\Lambda_0, \Lambda_1)$ the union

of $I^{m,p}$, over all m and p . The equivalence of this definition to that given by testing by pseudo-differential operators is due to Melrose:

Theorem 4.1. *The class $I(\Lambda_0, \Lambda_1)$ is equal to the class of distributions which have stable local Sobolev order under repeated application of first order, classical, proper, pseudo-differential operators which are characteristic on $\Lambda_0 \cup \Lambda_1$.*

Proof. Let us denote the first order, classical, proper pseudo-differential operators which are characteristic on $\Lambda_0 \cup \Lambda_1$ by $\mathcal{M}(\Lambda_0, \Lambda_1)$ and let the class of distributions which have stable, local, Sobolev order under repeated application of elements of $\mathcal{M}(\Lambda_0, \Lambda_1)$ be denoted by $J(\Lambda_0, \Lambda_1)$.

A pseudo-differential operator which is characteristic on $\Lambda_0 \cup \Lambda_1$ will necessarily have bicharacteristic flow tangent to Λ_0 and Λ_1 and therefore to $\Lambda_0 \cap \Lambda_1$. Thus, elements of $I(\Lambda_0, \Lambda_0 \cap \Lambda_1)$ and $I(\Lambda_1, \Lambda_0 \cap \Lambda_1)$ are stable under repeated application of such operators. Thus $I(\Lambda_0, \Lambda_1) \subseteq J(\Lambda_0, \Lambda_1)$.

Now, the definitions of $J(\Lambda_0, \Lambda_1)$ and $I(\Lambda_0, \Lambda_1)$ are FIO invariant so picking a symplectomorphism f , which maps Λ_0 to $N^*(x = 0)$ and Λ_1 to $N^*(x'' = 0)$ in \mathbb{R}^n (see [4]) and applying a zeroth order elliptic FIO associated to f , we are reduced to the model case $\Lambda_0 = \tilde{\Lambda}_0$ and $\Lambda_1 = \tilde{\Lambda}_1$.

Now let $u \in J(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$, then we want to decompose u into $u_0 + u_1$ with $u_i \in I(\tilde{\Lambda}_i, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1)$. We do this by taking a parabolic cut-off about $\Lambda_0 \cap \tilde{\Lambda}_1$. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \notin \text{supp}(1 - \chi)$ and we put

$$\hat{u}_0 = (1 - \chi) \left(\frac{|\xi'|}{|\xi''|^{1/2}} \right) \hat{u} \quad (4.1)$$

$$\hat{u}_1 = \chi \left(\frac{|\xi'|}{|\xi''|^{1/2}} \right) \hat{u}. \quad (4.2)$$

It is clear that $u = u_0 + u_1$. We must show $u_i \in I(\tilde{\Lambda}_i, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1)$. We do this by showing that the Sobolev order of u_i is stable under first order pseudo-differential operators in

$\mathcal{M}(\tilde{\Lambda}_i, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1)$ the operators which are characteristic on $\tilde{\Lambda}_i$ and have bicharacteristic flow tangent to $\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1$.

Now, if $P \in \mathcal{M}(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$ then letting p be the total left symbol we have from Taylor's theorem

$$p(x, \xi) = \sum_{j=k+1}^n x_j q_j(x, \xi) + \sum_{j=1}^k \sum_{l=1}^k x_j \xi_l r_{jl}(x, \xi) \quad (4.3)$$

which means that $\mathcal{M}(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$ is generated over $\Psi^0(\mathbb{R}^n)$ by operators with symbols

$$x_j \xi_l \quad j = k+1, \dots, n \quad l = 1, \dots, n$$

$$x_j \xi_l \quad j = 1, \dots, k \quad l = 1, \dots, k.$$

And so, \hat{u} is in a fixed weighted L^2 space under application of the vector fields

$$\xi_l \frac{\partial}{\partial \xi_j} \quad j = k+1, \dots, n \quad l = 1, \dots, n \quad (4.4)$$

$$\xi_l \frac{\partial}{\partial \xi_j} \quad j = 1, \dots, k \quad l = 1, \dots, k. \quad (4.5)$$

Whereas, $\mathcal{M}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1)$ is generated by operators with symbols

$$x_j \xi_l \quad j = 1, \dots, n \quad l = 1, \dots, k$$

$$x_j x_l \xi_m \quad j, l = 1, \dots, k \quad m = k+1, \dots, n$$

and so, it is enough show that \hat{u}_0 is in a fixed weighted L^2 space under repeated application of

$$\xi_l \frac{\partial}{\partial \xi_j} \quad j = 1, \dots, n \quad l = 1, \dots, k$$

$$\xi_m \frac{\partial}{\partial \xi_j} \frac{\partial}{\partial \xi_l} \quad j, l = 1, \dots, k \quad m = k+1, \dots, n$$

and this follows from stability under the vector fields

$$|\xi'| \frac{\partial}{\partial \xi_j} \quad j = 1, \dots, n$$

$$|\xi''|^{1/2} \frac{\partial}{\partial \xi_j} \quad j = 1, \dots, k.$$

Now, it is easily checked that $(1 - \chi)\left(\frac{|\xi'|}{|\xi''|^{1/2}}\right) \in S_{ma}^{0,0}(\Lambda_0, K_0)$ and so, using Leibniz rule, it is enough to show that the order of \hat{u} is stable under these vector fields on the support of $(1 - \chi)\left(\frac{|\xi'|}{|\xi''|^{1/2}}\right)$ that is where $|\xi'| \geq C|\xi''|^{1/2}$. But this then follows immediately from (4.4).

The case of \hat{u}_1 is similar. The additional operators generating $\mathcal{M}(\tilde{\Lambda}_1, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1)$ have symbols of the form

$$\frac{\xi_i \xi_j}{|\xi|} \quad i, j = 1, \dots, k$$

but these are bounded on $\text{supp}(\chi\left(\frac{|\xi'|}{|\xi''|^{1/2}}\right))$ and so the result follows. \square

However, the classes of distributions we will encounter are supported in half of one of the Lagrangian submanifolds and we therefore define

Definition 4.2. *Let $(\Lambda_0, \Lambda_1^\varepsilon)$ be an intersecting Lagrangian pair then*

$$I^{m,p}(\Lambda_0, \Lambda_1^\varepsilon) = I^{m, \frac{p-m+\frac{1}{2}}{2}}(\Lambda_0, \partial\Lambda_1^\varepsilon) + I^{p, \frac{m-p+\frac{1}{2}}{2}}(\Lambda_1^\varepsilon, \partial\Lambda_1^\varepsilon). \quad (4.6)$$

It remains to discuss the symbols of paired Lagrangian distributions. The difficulties are that there is not a canonical decomposition of a paired Lagrangian distribution into marked Lagrangian distributions and that the symbol of a marked Lagrangian distribution is a complicated object because of the type $\frac{1}{2}$ behaviour on the marking. Melrose (in [9]) defines the symbol of a marked Lagrangian distribution by picking an infinitesimal Lagrangian extension of the marking and the principal symbol is then invariant under FIOs which fix the Lagrangian submanifold and the infinitesimal extension.

In the case of the decomposition of a paired Lagrangian, there is always a natural choice for the extension of the marking: the other Lagrangian. So, the principal symbol can be defined to be a pair of sections of the Maslov bundle modulo the equivalence relation given by choosing different decompositions. However, for the

distributions we are interested in a different approach can be taken which avoids the necessity for a choice of decomposition. We define a class of polyhomogeneous paired Lagrangian distributions, associated to a pair of cleanly intersecting conic Lagrangian submanifolds with codimension one intersection, such that the symbols' singularities as they approach the intersection determine the behaviour of the distribution at the intersection. To illustrate the ideas we first of all establish a new characterization of polyhomogeneous Lagrangian distributions.

5. POLYHOMOGENEOUS LAGRANGIAN DISTRIBUTIONS

For notational convenience throughout this section, μ will denote a complex number which is the top order of homogeneity of the polyhomogeneous distributions being studied. We will denote the real part of μ by m . The class of polyhomogeneous Lagrangian distributions, $I_{phg}^{\mu+\frac{n}{4}}(\Lambda)$ associated to the conic Lagrangian submanifold Λ are Lagrangian distributions associated to Λ which can locally be written

$$\int e^{i\phi(x,\theta)} a(x,\theta) d\theta \quad (5.1)$$

with ϕ a homogeneous degree one non-degenerate phase function parameterizing Λ and a a classical symbol, that is there exists a sequence of smooth functions on $\mathbb{R}_x^n \times (\mathbb{R}_\theta^N - \{0\})$, $a_{\mu-j}$, such that $a_{\mu-j}$ is homogeneous of degree $\mu - j$ in θ , and

$$\left| D_x^\beta D_\theta^\alpha \left(a(x,\theta) - \sum_{j=0}^{N-1} a_{\mu-j}(x,\theta) \right) \right| \leq C_{\alpha,\beta,N} \langle \theta \rangle^{m-|\alpha|} \text{ for } |\theta| > 1. \quad (5.2)$$

We give an intrinsic characterization of these distributions by introducing the concept of a radial operator for a conic Lagrangian submanifold.

Definition 5.1. *A radial operator for a conic Lagrangian submanifold $\Lambda \subset T^*(M)$ is a properly supported, first order, classical pseudo-differential operator such that for*

all $\mu \in \mathbb{C}$

$$P + \mu : I_{phg}^\mu(\Lambda, \Omega^{\frac{1}{2}}) \rightarrow I_{phg}^{\mu-1}(\Lambda, \Omega^{\frac{1}{2}}). \quad (5.3)$$

We use the subprincipal symbol of a pseudo-differential operator to give a symbolic characterization of radial operators. For a discussion of subprincipal symbols see [2]. It is proven there that if the principal symbol of a pseudo-differential operator, P , vanishes on a Lagrangian submanifold, Λ , then the principal symbol of Pu for $u \in I^m(\Lambda, \Omega^{\frac{1}{2}})$ is

$$\frac{1}{i} \mathcal{L}_{H_P} \sigma_m(u) + \sigma_{sub}(P) \sigma_m(u). \quad (5.4)$$

Theorem 5.1. *Given a conic Lagrangian submanifold Λ there always exists a radial operator R . R is determined up to first order classical operators which map $I^m(\Lambda, \Omega^{\frac{1}{2}}) \rightarrow I^{m-1}(\Lambda, \Omega^{\frac{1}{2}})$, for all m , and is characterized by*

- (1) *the principal symbol of R vanishes on Λ ,*
- (2) *the subprincipal symbol of R equals $n/4$ on Λ ,*
- (3) *the bicharacteristic field of R , H_R , on Λ is equal to the radial vector field multiplied by i^{-1} .*

Note that since the cotangent bundle has a natural homogeneous action by scalars there is a natural radial vector field at every point.

Proof. Note that the difference of any two operators satisfying (1) - (3) has principal symbol vanishing to second order on Λ and subprincipal symbol vanishing on Λ . The vanishing of the principal symbol implies that $(R_1 - R_2)u$ is in $I^m(\Lambda, \Omega^{\frac{1}{2}})$ with principal symbol

$$\frac{1}{i} \mathcal{L}_{H_{R_1 - R_2}} \sigma_m(u) + (\sigma_{sub}(R_1 - R_2)) \sigma_m(u) \quad (5.5)$$

but $H_{R_1 - R_2}$ will be zero on Λ and so will $\sigma_{sub}(R_1 - R_2)$, thus $(R_1 - R_2)u$ is in $I^{m-1}(\Lambda, \Omega^{\frac{1}{2}})$. Thus $R_1 - R_2$ maps $I^m(\Lambda, \Omega^{\frac{1}{2}})$ to $I^{m-1}(\Lambda, \Omega^{\frac{1}{2}})$ and we have proven the uniqueness part of the theorem.

That the principal symbol of Ru will have to vanish on Λ for any $u \in I_{phg}^m(\Lambda, \Omega^{\frac{1}{2}})$ is clear and this implies that the principal symbol of R vanishes on Λ .

The principal symbol of $(R + \mu)u$ will now be

$$\frac{1}{i} \mathcal{L}_{H_R} \sigma_m(u) + (\sigma_{sub}(R) + \mu) \sigma_m(u) \quad (5.6)$$

where \mathcal{L}_{H_R} denotes the Lie derivative acting on 1/2-densities. Thus we need (5.6) to vanish for all m . Since this is true for any polyhomogeneous u , this tells us that if $\sigma_m(u)$ vanishes along a ray so does $\mathcal{L}_{H_R} \sigma_m(u)$ and hence that H_R is tangent to rays which means that it is a multiple of the radial vector field ρ .

Now we can pick local coordinates x on the manifold such that ξ the dual coordinates parametrise Λ and then $|d\xi|^{\frac{1}{2}}$ is a non-vanishing 1/2-density on Λ . In these coordinates the radial vector field is $\xi \frac{\partial}{\partial \xi}$. Writing $H_R = \phi \rho$ we obtain, using Euler's relation, that

$$\left(\frac{1}{i} \phi(\xi) \left(\mu - \frac{n}{4} \right) a_{\mu - \frac{n}{4}} + \frac{1}{i} \phi(\xi) \frac{n}{2} a_{\mu - \frac{n}{4}} + \frac{1}{2i} \sum \xi_j \frac{\partial \phi}{\partial \xi_j} + \mu a_{\mu - \frac{n}{4}} + \sigma_{sub}(R) a_{\mu - \frac{n}{4}} \right) = 0 \quad (5.7)$$

where $a_{\mu - \frac{n}{4}}(\xi) |d\xi|^{\frac{1}{2}}$ is the principal symbol of u . From this, we deduce

$$\mu \left(\frac{\phi(\xi)}{i} + 1 \right) + \left(-\frac{1}{i} \frac{n}{4} \phi(\xi) + \frac{n}{2i} \phi \xi + \frac{1}{2i} \sum \xi_j \frac{\partial \phi}{\partial \xi_j} + \sigma_{sub}(R) \right) = 0. \quad (5.8)$$

This is true for all μ , so $\phi(\xi) = -i$ and $\sigma_{sub}(R) = \frac{n}{4}$, which establishes the properties of R and it is clear from the calculation that any operator with these properties will be a radial operator. \square

We can use the radial operator to characterize the polyhomogeneous distributions associated to a Lagrangian submanifold.

Theorem 5.2. *An element u of $I^m(\Lambda, \Omega^{\frac{1}{2}})$ is polyhomogeneous of order μ if and only if for some (and hence any) radial operator R of Λ*

$$(R + (\mu - N + 1)) \dots (R + \mu - 1)(R + \mu)u \in I^{m-N}(\Lambda, \Omega^{\frac{1}{2}}) \quad \forall N. \quad (5.9)$$

We prove a lemma which allows us to reduce to a particular radial operator in the model case.

Lemma 5.1. *Let $V = x \frac{\partial}{\partial x}$ and let Q be a classical pseudo-differential operator which maps $I^m(\tilde{\Lambda}_0, \Omega^{\frac{1}{2}})$ to $I^{m-k}(\tilde{\Lambda}_0, \Omega^{\frac{1}{2}})$ then $[V, Q] - kQ$ maps $I^m(\tilde{\Lambda}_0, \Omega^{\frac{1}{2}})$ to $I^{m-k-1}(\tilde{\Lambda}_0, \Omega^{\frac{1}{2}})$.*

Proof. If Q is of order l and has symbol q with asymptotic expansion $\sum q_{l-j}$ then the mapping property is equivalent to saying that q_{l-j} vanishes to order $l-j+k$ at $x=0$ for $l-j > -k$. Now we can write

$$q_{l-j} = q'_{l-j} + r_{l-j} \quad (5.10)$$

where q'_{l-j} is a homogeneous polynomial of degree $k+l-j$ in x and r_{l-j} vanishes to order $k+l-j+1$ and then the total symbol of $[V, Q]$ has expansion

$$\sum (x \frac{\partial}{\partial x} - \xi \frac{\partial}{\partial \xi}) q'_{l-j} + \sum (x \frac{\partial}{\partial x} - \xi \frac{\partial}{\partial \xi}) r'_{l-j}. \quad (5.11)$$

The double homogeneity of the q'_{l-j} yields kq'_{l-j} and the r' terms have the correct mapping properties and thus $[V, Q] - kQ$ maps $I^m(\tilde{\Lambda}_0, \Omega^{\frac{1}{2}})$ to $I^{m-k-1}(\tilde{\Lambda}_0, \Omega^{\frac{1}{2}})$. \square

Proof of Theorem 5.2. Both properties here are clearly classical Fourier integral operator invariant so it is enough to consider the model case $\Lambda = \tilde{\Lambda}_0$.

In this case, the radial operators are $x \frac{\partial}{\partial x} + \frac{3n}{4} + P$ where P is a first order, classical pseudo-differential operator which induces a map from $I^m(\tilde{\Lambda}_0, \Omega^{\frac{1}{2}})$ to $I^{m-1}(\tilde{\Lambda}_0, \Omega^{\frac{1}{2}})$. First of all we show that if it is true for any one P then it is true for $P=0$. Let V denote the operator given by $P=0$.

First of all, it is clear that $(V + \mu + P)u \in I^{m-1}(\tilde{\Lambda}_0, \Omega^{\frac{1}{2}})$ implies that $(V + \mu)u$ is also. So now suppose that

$$(V + (\mu - k + 1)) \dots (V + \mu - 1)(V + \mu)u \in I^{m-N}(\tilde{\Lambda}_0, \Omega^{\frac{1}{2}}), k \leq N - 1 \quad (5.12)$$

and our hypothesis says that

$$(V + (\mu - N) + P) \dots (V + \mu - 1 + P)(V + \mu + P)u \in I^{m-N-1}(\tilde{\Lambda}_0, \Omega^{\frac{1}{2}}). \quad (5.13)$$

We commute the P s through to the left to obtain something to which we can apply the inductive hypothesis. Applying lemma 5.1 repeatedly we obtain

$$\sum_{k=0}^N T_k \prod_{j=0}^{N-k} (V + \mu - j)u \in I^{m-N-1}(\tilde{\Lambda}_0, \Omega^{\frac{1}{2}}) \quad (5.14)$$

where $T_0 = \text{Id}$ and $T_k : I^m(\tilde{\Lambda}_0, \Omega^{\frac{1}{2}}) \rightarrow I^{m-k}(\tilde{\Lambda}_0, \Omega^{\frac{1}{2}})$. Hence it follows from the inductive hypothesis that $\prod_{j=0}^N (V + \mu - j)u \in I^{m-N-1}(\tilde{\Lambda}_0, \Omega^{\frac{1}{2}})$

So we have reduced to the case where $P = 0$. Now taking the Fourier transform we obtain

$$\left(\prod_{j=0}^{N-1} \rho - \left(\mu - j - \frac{n}{4} \right) \right) a(\xi) \in S^{m-\frac{n}{4}}(\mathbb{R}_\xi^n) \quad (5.15)$$

where $a|dx|^{\frac{1}{2}}$ is the Fourier transform of ψu , $\psi \equiv 1$ near 0 and of compact support and $\rho = \xi \frac{\partial}{\partial \xi}$.

We need to show that a has an asymptotic expansion in terms of homogeneous factors. That is we want to show there exist $\{a_{\mu-\frac{n}{4}-j}\}$ homogeneous of degree $\mu-\frac{n}{4}-j$ such that

$$a(\xi) - \sum_{j=0}^{N-1} a_{\mu-\frac{n}{4}-j}(\xi) \in S^{m-\frac{n}{4}-N}(\mathbb{R}_\xi^n) \text{ for } |\xi| > 1. \quad (5.16)$$

Let $a_{\mu-\frac{n}{4}}(\xi) = \lim_{\lambda \rightarrow \infty} \lambda^{-(\mu-\frac{n}{4})} a(\lambda^{\mu-\frac{n}{4}} \xi)$. Why does this limit exist? Well,

$$\frac{\partial}{\partial \lambda} \left(\lambda^{-(\mu-\frac{n}{4})} a(\lambda^{\mu-\frac{n}{4}} \xi) \right) = -(\mu - \frac{n}{4}) \lambda^{-1-(\mu-\frac{n}{4})} a(\lambda \xi) + \lambda^{-(\mu-\frac{n}{4})-1} (\lambda \xi) \cdot \frac{\partial a}{\partial \xi}(\lambda \xi) \quad (5.17)$$

and since $(\xi \frac{\partial}{\partial \xi} - (\mu - \frac{n}{4}))a = c \in S^{m-\frac{n}{4}-1}$ this becomes

$$\frac{\partial}{\partial \lambda} \left(\lambda^{-(\mu-\frac{n}{4})} a(\lambda^{\mu-\frac{n}{4}} \xi) \right) = \lambda^{\mu-\frac{n}{4}-1} c(\lambda \xi) = O(\lambda^{-2}). \quad (5.18)$$

So integrating out to infinity the limit exists. It follows immediately from the definition of $a_{\mu-\frac{n}{4}}$ that it is homogeneous of degree $\mu - \frac{n}{4}$. As $\frac{\partial}{\partial \lambda} \left(\lambda^{-(\mu-\frac{n}{4})} a_{\mu-\frac{n}{4}}(\lambda \xi) \right) = 0$, applying the argument above shows that $|a(\xi) - a_{\mu-\frac{n}{4}}(\xi)| = O(|\xi|^{m-\frac{n}{4}-1})$. Changing to polar coordinates (ω, λ) these arguments commute with application of D_ω^α and $\lambda \frac{\partial}{\partial \lambda}$ so we have that

$$a(\xi) - a_{\mu-\frac{n}{4}}(\xi) \in S^{m-\frac{n}{4}-1}(\mathbb{R}_\xi^n) \text{ for } |\xi| > 1. \quad (5.19)$$

To complete our proof by induction we need to show that

$$\left(\prod_{j=0}^{N-1} \rho - (\mu - \frac{n}{4} - 1 - j) \right) (a - a_{\mu-\frac{n}{4}}) \in S^{m-\frac{n}{4}-1-N}(\mathbb{R}_\xi^n). \quad (5.20)$$

This follows from the fact that if $b \in S^{k-1}$ and $(\rho - k - il)b \in S^{k-2}$ then $b \in S^{k-2}$. The point here is that $\rho - k - il$ can only kill terms of order k . A similar computation to above shows that

$$\frac{\partial}{\partial \lambda} (\lambda^{-k-il} b(\lambda \omega)) = O(\lambda^{-3}) \quad (5.21)$$

and integrating out to infinity as before establishes the result. \square

Putting all this together, we have a direct way to define polyhomogeneous Lagrangian distributions. We could define radial operators in terms of the conclusions of Theorem 5.1 and polyhomogeneous distributions in terms of the hypothesis of

Theorem 5.2. Indeed, we could go much further and define for R_Λ a radial operator associated to Λ :

$$\mathcal{M}(\Lambda) = \{P \in \Psi_{cl}^1 : \sigma_1(P)|_\Lambda = 0\} \quad (5.22)$$

$$I^{(s)}(\Lambda) = \{u \in H_{loc}^s(M) : P_1 \dots P_k u \in H_{loc}^s(M), \forall k, P_i \in \mathcal{M}(\Lambda)\} \quad (5.23)$$

$$I_{phg}^\mu(\Lambda) = \left\{ u : \prod_{j=0}^{N-1} (R_\Lambda + \mu - j)u \in I^{(s-N)}(\Lambda), \forall N, \text{ for some } s \right\} \quad (5.24)$$

We are using classical pseudo-differential operators here in our definition. We want to identify their classicality in an intrinsic way. Pseudo-differential operators can be viewed as distributions which are conormal to the diagonal. In the case of the conormal bundle to a submanifold, $\mathcal{M}(\Lambda)$, is generated by the vector fields tangent to the submanifold and the radial vector field is a radial operator. Thus we can make the same definitions but using only vector fields. We thus can obtain a completely intrinsic definition of classicality.

6. POLYHOMOGENEOUS PAIRED LAGRANGIAN DISTRIBUTIONS

We commence by defining the distributions as oscillatory integrals of singular symbols for a model and then show that this is equivalent to a definition involving testing by radial operators. Throughout this section we will denote the orders of homogeneity of a distribution by a complex pair (μ, ν) and their real parts by (m, p) .

We wish to model distributions microsupported on one side of a Lagrangian so for our model we take

Definition 6.1.

$$\tilde{\Lambda}_0 = N^*(x = 0), \tilde{\Lambda}_1^\epsilon = N^*(x'' = 0, x_1 \geq 0) \text{ where } x = (x_1, x'').$$

Our total symbols will be asymptotic sums of homogeneous functions which have conormal singularities at $\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e$.

Definition 6.2. $T^{k,r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ equals the space of $a \in C^\infty(\mathbb{R}_\xi^n - \{\xi_1 = 0\})$ such that a is homogeneous of degree k and there exists a sequence of functions $\{b_j\} \in C^\infty(\mathbb{R}^{n-1} - 0)$ which are homogeneous of degree $k - j - r$ such that

$$\left| D_\xi^\alpha \left(a(\xi) - \sum_{j=0}^{N-1} (\xi - i0)^{r+j} b_j(\xi'') \right) \right| \leq C_\alpha |\xi|^{\Re k - |\alpha|} \left(\frac{|\xi_1|}{|\xi''|} \right)^{\Re r + N - \alpha_1}, \xi_1 \neq 0. \quad (6.1)$$

Definition 6.3. $S_{sing}^{\mu,r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ is the collection of formal sums $\sum_{j=0}^{\infty} a_{\mu-j}$ where $a_{\mu-j} \in T^{\mu-j,r-j}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$.

Away from the intersection, standard arguments show that one can find a classical symbol with asymptotic expansion $\{a_{\mu-j}\}$.

We use a similar definition on $\tilde{\Lambda}_1^e$. First of all, recall the definition of the χ_+^α distribution (see [6])

$$\langle \chi_+^\alpha, f \rangle = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty x^\alpha f(x) dx, \Re \alpha > -1 \quad (6.2)$$

and one can extend via analytic continuation to the whole plane by

$$\langle \chi_+^\alpha, f \rangle = (-1)^k \langle \chi^{\alpha+k}, f^{(k)} \rangle, \Re(\alpha + k) > -1 \quad (6.3)$$

and then χ_+^α restricted to $x > 0$ is $\frac{x^\alpha}{\Gamma(\alpha+1)}$ which vanishes if α is a negative integer as we have $\chi_+^{-k} = \delta^{(k-1)}$.

We require our symbol on $\tilde{\Lambda}_1^e$ to have an expansion in terms of the χ_+ distributions.

Definition 6.4. $T^{k,r}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ equals $a \in C^\infty(\mathbb{R}_{x_1} \times \mathbb{R}_{\xi''}^{n-1}, x_1 > 0)$ such that a is homogeneous of degree k in ξ'' and there exists a sequence of functions b_j homogeneous

of degree k such that

$$\left| D_{x_1}^{\alpha_1} D_{\xi''}^{\alpha''} \left(a(x_1, \xi'') - \sum_{j=0}^{N-1} \chi_+^{j+r}(x_1) b_j(\xi'') \right) \right| \leq C_{\alpha, N} |\xi|^{|\alpha''| - \Re(k)} |x_1|^{\Re(r) + N - \alpha_1}, x_1 > 0. \quad (6.4)$$

Note that in the case where r is a negative integer the fact that the restriction of χ_+^r to $x_1 > 0$ is 0 means that $T^{k, r}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e) = T^{k, 0}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$.

Definition 6.5. $S_{sing}^{\mu, r}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ is the collection of formal sums $\sum_{j=0}^{\infty} a_{\mu-j}$ where $a_{\mu-j} \in T^{\mu-j, r-j}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$.

The relationship between our two classes of symbols lies in the fact that the Fourier transform of χ_+^s is $e^{-i\frac{\pi}{2}(s+1)}(\xi_1 - i0)^{-s-1}$. Our distributions will be asymptotic sums of products $\chi_+^\alpha(x_1)u_j(x'')$ at $\Lambda_0 \cap \tilde{\Lambda}_1^e$.

To quantize our symbols we take a parabolic cut off about $\xi_1 = 0$ to obtain marked symbols, and then show that the singularities of the distribution obtained are independent of the choice of cut off. By a cut off function we shall mean an even function $\phi \in C_0^\infty(\mathbb{R})$ such $\phi \equiv 1$ near 0.

Lemma 6.1. Let ϕ be a cut off function then if $a \in T_{sing}^{\mu, r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ we have

$$(1 - \phi) \left(\frac{\xi_1}{|\xi''|^{1/2}} \right) a(\xi) \in S_{ma}^{m, -\Re(r)/2}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e) \quad (6.5)$$

Proof. We have that

$$|D_\xi^\alpha a(\xi)| \leq C |\xi|^{m-\alpha} \left(\frac{|\xi_1|}{|\xi|} \right)^{\Re(r) - \alpha_1} \quad (6.6)$$

Now on $\text{supp}(1 - \phi) \left(\frac{\xi_1}{|\xi''|^{1/2}} \right)$ we have that

$$|\xi_1|^{-l} \leq \langle \xi \rangle_{(1)}^{-l}$$

so we obtain

$$|(1 - \phi) \left(\frac{\xi_1}{|\xi''|^{1/2}} \right) D_\xi^\alpha a(\xi)| \leq C_\alpha |\xi|^{m-\alpha} \left(\frac{\langle \xi \rangle}{\langle \xi \rangle_{(1)}} \right)^{-\Re(r)} \langle \xi \rangle^{-\alpha_1} \langle \xi \rangle^{m-|\alpha''|}$$

and as $(1 - \phi)\left(\frac{\xi_1}{|\xi''|^{1/2}}\right) \in S_{ma}^{0,0}(\tilde{\Lambda}_0, K_0)$ our result follows. \square

We need a corresponding result on the other side, since the Fourier transform of a product is a convolution this is slightly more complicated.

Lemma 6.2. *Let ϕ be a cut off function then if $b \in T_{sing}^{\mu,r}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ we have that*

$$\int_0^\infty b(y_1, \xi'') |\xi''|^{1/2} \hat{\phi}((x_1 - y_1) |\xi''|^{1/2}) dy_1 \in S_{ma}^{m, -\frac{\Re(r)}{2}}(\Lambda_1, K_1) \quad (6.7)$$

Proof. We have to interpret this convolution as a distribution for $\Re(r) < -1$. We can always decompose our function into sum of products $b_{\mu,j}(\xi'') \chi_+^\alpha(x_1)$ plus an element of $T_{sing}^{\mu,r'}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ with $\Re(r') > -1$.

For $x_1 > \epsilon > 0$,

$$\int_0^\infty b(y_1, \xi'') |\xi''|^{1/2} \hat{\phi}((x_1 - y_1) |\xi''|^{1/2}) dy_1 - b(x_1, \xi'') = O(|\xi''|^{-\infty}) \quad (6.8)$$

Now, we can reduce to the case $\mu = 0$ by premultiplying by $|\xi''|^{-\mu}$ which will not affect the convolution and for $\Re(r') > -1$, we compute using the fact that $\hat{\phi}$ is Schwarz,

$$\begin{aligned}
& \int_0^\infty b(y_1, \xi'') |\xi''|^{1/2} \hat{\phi}((x_1 - y_1) |\xi''|^{1/2}) dy_1 | \\
& \leq C_N \int_{x_1}^\infty |y_1 - x_1|^{\Re(r')} |\xi''|^{1/2} (1 + |y_1| |\xi''|^{1/2})^{-N} dy_1 \\
& = C \int_{|\xi''|^{1/2} x_1}^\infty (|\xi''|^{-1/2} (t - x_1 |\xi''|^{1/2}))^{\Re(r')} (1+t)^{-N} dt \\
& = C |\xi''|^{-\Re(r')/2} \int_{x_1 |\xi''|^{1/2}}^\infty (t - x_1 |\xi''|^{1/2})^{\Re(r')} (1+t)^{-N} dt \\
& \leq C \langle \xi'' \rangle^{-\Re(r')/2} \int_0^\infty t^{\Re(r')} \left(\frac{1}{1+t+t|\xi''|^{1/2}} \right)^N dt
\end{aligned}$$

Applying Cauchy-Schwartz we get

$$\begin{aligned}
& \leq C \langle \xi'' \rangle^{-\Re(r')/2} \left(\int_0^\infty (t^{\Re(r')})^2 \left(\frac{1}{1+t+t|\xi''|^{1/2} x_1} \right)^N dt \right)^{1/2} \\
& \quad \left(\int_0^\infty \left(\frac{1}{1+t+t|\xi''|^{1/2} x_1} \right)^N dt \right)^{1/2} \\
& \leq C \langle \xi'' \rangle^{-\Re(r')/2} (1 + \langle \xi'' \rangle^{1/2} |x_1|)^{1-2N} \\
& \leq C \left(\frac{1 + \langle \xi'' \rangle^{1/2} |x_1|}{\langle \xi'' \rangle^{1/2}} \right)^{\Re(r')}
\end{aligned}$$

picking N correctly. This establishes the top order estimate for $\Re(r') > -1$. To do the general case observe that

$$\langle \chi_+^{l+k}(y_1), |\xi''|^{1/2} \hat{\phi}((x_1 - y_1) |\xi''|^{1/2}) \rangle = |\xi''|^{k/2} \langle \chi_+^l(y_1), |\xi''|^{1/2} \phi^{(k)}((x_1 - y_1) |\xi''|^{1/2}) \rangle \quad (6.9)$$

and then the same arguments as above will work.

Similar arguments establish the estimates for the derivatives also.

□

We can now transform our formal sums into asymptotic sums. Each element of $S^{\mu,r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ is a formal sum $\{\sum a_{\mu-j}\}$ and

$$(1 - \phi) \left(\frac{\xi_1}{|\xi''|^{1/2}} \right) a_{\mu-j} \in S_{ma}^{m-j, \frac{j-\Re(r)}{2}}(\tilde{\Lambda}_0, K_0)$$

and so applying Proposition 3.1 the asymptotic sum exists and thus gives rise to an element of $I_{ma}^{m-\frac{n}{4}, \frac{j-\Re(r)}{2}}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$. This element is independent of the choice of ϕ up to a distribution which is isotropic with respect to $\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e$.

Similarly, each element of $S^{\mu,r}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ is a formal sum $\{\sum b_{\mu-j}\}$ and

$$|\xi''|^{1/2} \phi(x_1 |\xi''|^{1/2}) * b_{\mu-j}(x_1, \xi'') \in S_{ma}^{m-j, \frac{j-\Re(r)}{2}}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$$

and so applying the analogue of 3.1 the asymptotic sum exists and thus gives rise to an element of $I_{ma}^{m-\frac{n}{4}+\frac{1}{2}, \frac{j-\Re(r)}{2}}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$. This element is also independent of the choice of ϕ up to a distribution which is isotropic with respect to $\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e$.

To remove this uncertainty on the isotropic we define a compatibility condition between the symbol on $\tilde{\Lambda}_0$ and the one on $\tilde{\Lambda}_1^e$.

Definition 6.6. *A paired total symbol of order (μ, ν) is a pair of formal sums*

$$\left(\sum a_{\mu-j}, \sum b_{\nu-j} \right)$$

such that

$$\sum a_{\mu-j} \in S_{sing}^{\mu, \mu-\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e), \sum b_{\nu-j} \in S_{sing}^{\nu, \nu-\mu-1}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$$

and such that the Fourier transform in x_1 induces a bijection between their respective expansions at $\xi_1 = 0$ and $x_1 = 0$. That is if

$$a_{\mu-j} \sim \sum_{l=0}^{\infty} (\xi_1 - i0)^{\mu-\nu-j+l} a_{\mu-j,l}(\xi'')$$

and

$$b_{\nu-k} \sim \sum_{r=0}^{\infty} \chi_+^{\nu-\mu-1-k+r}(x_1) b_{\nu-k,r}(\xi'')$$

then

$$(\xi_1 - i0)^{\mu-\nu-j+l} a_{\mu-j,l}(\xi'') = \chi_+^{\nu-\mu-1-l+j}(x_1) \widehat{b_{\nu-l,j}}(\xi''). \quad (6.10)$$

Note that the top order term on each side determines the leading singularity of the top order term on the other side and nothing else about the top order term. So, given a homogeneous function on $\tilde{\Lambda}_0$ and one on $\tilde{\Lambda}_1^\varepsilon$ of which the leading singularities are compatible we can complete to total symbols with these as top order terms.

Theorem 6.1. *There is a quantization map from paired total symbols of order (μ, ν) to $I^{m-n/4, p+\frac{1}{2}-n/4}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^\varepsilon)$ determined up to smooth terms.*

Proof. Picking a cut off function ϕ and applying the arguments above we obtain an element of $I^{m-n/4, p+\frac{1}{2}-n/4}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^\varepsilon)$ which is independent of choice except at $\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^\varepsilon$. Now suppose we pick ϕ_1 and ϕ_2 distinct cut off functions and let I_i denote the distribution obtained by using ϕ_i .

We want to show that $I_1 - I_2 \in C^\infty$. It is enough to show that it is determined up to an arbitrarily low order as a sum of marked Lagrangians. Since, our terms are in $I^{m-j-n/4, \frac{i-m+p}{2}}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^\varepsilon)$ and $I^{p-k-n/4+1/2, \frac{1-p+m+k}{2}}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^\varepsilon)$ all but a finite number will be below a given order.

So fixing some negative order K , we can write

$$\begin{aligned} I_1 - I_2 &= \left(\frac{1}{2\pi}\right)^n \int e^{ix \cdot \xi} \sum_{j=0}^{N-1} (\phi_2 - \phi_1) \left(\frac{\xi_1}{|\xi''|^{\frac{1}{2}}}\right) a_{\mu-j}(\xi) d\xi \\ &\quad + \left(\frac{1}{2\pi}\right)^{n-1} \int e^{ix'' \cdot \xi''} \sum_{k=0}^{M-1} |\xi''|^{\frac{1}{2}} \widehat{\phi_1 - \phi_2}(x_1 |\xi''|^{\frac{1}{2}}) * b_{\nu-k}(x_1, \xi'') d\xi'' + R_K \end{aligned} \quad (6.11)$$

with R_K in $I^{-K, -K}(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$.

Taking the symbols' expansions about $\xi = 0$ and $x_1 = 0$ this becomes

$$\begin{aligned} & \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} \left(\frac{1}{2\pi}\right)^n \int e^{ix \cdot \xi} (\phi_2 - \phi_1) \left(\frac{\xi_1}{|\xi''|^{\frac{1}{2}}}\right) (\xi_1 - i0)^{\mu-\nu-j+l} a_{\mu-j,l}(\xi'') + \\ & \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} \left(\frac{1}{2\pi}\right)^{n-1} \int e^{ix'' \cdot \xi''} |\xi''|^{\frac{1}{2}} (\widehat{\phi_1 - \phi_2})(x_1 |\xi''|^{\frac{1}{2}}) * \chi_+^{\nu-\mu-1-j+l}(x_1) b_{\nu-j,l}(\xi'') + R'_K \end{aligned} \quad (6.12)$$

but our definition of a paired total symbol tells us that these terms cancel and our result follows. \square

We define $J_{phg}^{\mu-n/4, \nu+\frac{1}{2}-n/4}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^c)$ to be the image of this quantization map. In order to reduce our general definition to this case, we must show that this space is invariant under Fourier integral operators associated to model preserving symplectomorphisms. We do this by showing the equivalence of a definition which is *a priori* invariant. A similar approach using radial vector fields has previously been used by Melrose to characterize classes of distributions on manifolds with corners. (see [8])

Definition 6.7. *The pair of properly supported, first order, classical, pseudo differential operators (R_0, R_1) , acting on half densities, is a pair of radial operators for the pair of cleanly intersecting conic Lagrangians (Λ_0, Λ_1) if and only*

- (1) $\sigma_1(R_j)$ vanishes on $\Lambda_0 \cup \Lambda_1$
- (2) $\sigma_{sub}(R_j)$ restricted to Λ_j is $\frac{n}{4}$.
- (3) H_{R_j} restricted to Λ_j is $\frac{1}{i}\rho_{\Lambda_j}$.

This definition is very similar to the alternative definition of the radial operators for a single Lagrangian except that we require the principal symbol of each operator to vanish on both Lagrangians. Note that the results from the previous section show that if we conjugate by an elliptic Fourier integral operator associated to a symplectomorphism f then we obtain a pair of radial operators for $(f(\Lambda_0), f(\Lambda_1))$.

Example 6.1. For $(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$ we can take $R_0 = x \frac{\partial}{\partial x} + \frac{3n}{4}$ and $R_1 = x'' \frac{\partial}{\partial x''} + \frac{3n}{4} - \frac{1}{2}$.

The radial operators in our definition are the operators which one can obtain from these via conjugation by elliptic Fourier integral operators associated to symplectomorphisms mapping $(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$ to (Λ_0, Λ_1) .

Definition 6.8. *Let (Λ_0, Λ_1^e) be an intersecting pair then $u \in I_{phg}^{\mu, \nu}(\Lambda_0, \Lambda_1^e, \Omega^{\frac{1}{2}})$ if and only if for any pair of radial operators for (Λ_0, Λ_1^e)*

$$\left(\prod_{j=0}^{N-1} (R_0 + \mu - j) \prod_{k=0}^{M-1} (R_1 + \nu - k) \right) u \in I^{m-N, p-M}(\Lambda_0, \Lambda_1^e, \Omega^{\frac{1}{2}}) \text{ for all } M, N.$$

We will show that it is enough to check for any one pair of radial operators. Note that the class of distributions $I_{phg}^{\mu, \nu}(\Lambda_0, \Lambda_1^e, \Omega^{\frac{1}{2}})$ is *a priori* invariant under elliptic Fourier integral operators associated to symplectomorphisms which preserve the Lagrangians. We need to show that this definition is equivalent to the already given for the model $(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$. Our results for Lagrangian distributions show that this is true away from $\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e$. In this case multiplication by $|dx|^{\frac{1}{2}}$ gives an identification between half density bundles and functions so we work with functions.

Theorem 6.2. $I_{phg}^{\mu, \nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e) = J_{phg}^{\mu, \nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$

Our proof is similar to that of theorem 5.2 but the use of two Lagrangians makes it more involved. The first thing we need is an analogue of lemma 5.1

Definition 6.9. *If P is classical pseudo-differential operator of order r then P vanishes to order (α, β) on $(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$ if P has total symbol p with asymptotic expansion $\sum p_{r-j}$ and p_{r-j} vanishes to order $r - j + \alpha$ on $\tilde{\Lambda}_0$ and $r - j + \beta$ on $\tilde{\Lambda}_1^e$.*

Lemma 6.3. *If P is of order r and vanishes to order α on $\tilde{\Lambda}_0$ and to order β on $\tilde{\Lambda}_1^e$ then*

- (1) $\left[x \frac{\partial}{\partial x}, P \right] - (r - \alpha)P$ vanishes to order $\alpha + 1$ on $\tilde{\Lambda}_0$ and order β on $\tilde{\Lambda}_1^e$.
- (2) $\left[x'' \frac{\partial}{\partial x''}, P \right] - (r - \beta)P$ vanishes to order α on $\tilde{\Lambda}_0$ and order $\beta + 1$ on $\tilde{\Lambda}_1^e$.

Proof. This is essentially the same as the proof of 5.1 \square

The following properties of vanishing are elementary:

Proposition 6.1. *If P is of order r and P vanishes to order (α, β) on $(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$ then*

$$P : I^{m,p}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e) \rightarrow I^{m+r-\alpha,p+r-\beta}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e). \quad (6.13)$$

Proposition 6.2. *If P_i is of order r_i and P_i vanishes to order (α_i, β_i) on $(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$ then $P_1 P_2$ vanishes to order $(\alpha_1 + \alpha_2, \beta_1 + \beta_2)$ on $(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$.*

Lemma 6.4. *Let $u \in I^{m,p}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$ and suppose that*

$$\left(\prod_{j=0}^{N-1} (R_0 + \mu - j) \prod_{k=0}^{M-1} (R_1 + \nu - k) \right) u \in I^{m-N,p-M}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e) \text{ for all } M, N \quad (6.14)$$

for some pair of radial operators (R_0, R_1) then

$$\left(\prod_{j=0}^{N-1} \left(x \frac{\partial}{\partial x} + \frac{3n}{4} + \mu - j \right) \prod_{k=0}^{M-1} \left(x'' \frac{\partial}{\partial x''} + \frac{3n}{4} - \frac{1}{2} + \nu - k \right) \right) u \in I^{m-N,p-M}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e) \text{ for all } M, N. \quad (6.15)$$

Proof. We have that

$$R_0 = x \frac{\partial}{\partial x} + \frac{3n}{4} + P_0, \quad (6.16)$$

$$R_1 = x'' \frac{\partial}{\partial x''} + \frac{3n}{4} - \frac{1}{2} + P_1 \quad (6.17)$$

where P_0 is first order and vanishes to order $(2, 1)$ on $(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$ and P_1 vanishes to order $(1, 2)$. The result is clearly true for $M = 0$ as it is then just lemma 5.2. So we can prove by induction and analogous arguments to those used in the proof of theorem 5.2 reduce us to showing that

$$\left(\prod_{j=0}^{N-1} \left(x \frac{\partial}{\partial x} + \frac{3n}{4} + \mu - j \right) \sum_{r=0}^{M-1} \sum_{i \leq k} T_{i,i,r+i} \prod_{k=0}^{M-r-1} \left(x'' \frac{\partial}{\partial x''} + \frac{3n}{4} - \frac{1}{2} + \nu - k \right) \right) u \in I^{m-N,p-M}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^\epsilon) \quad (6.18)$$

implies

$$\left(\prod_{j=0}^{N-1} \left(x \frac{\partial}{\partial x} + \frac{3n}{4} + \mu - j \right) \prod_{k=0}^{M-1} \left(x'' \frac{\partial}{\partial x''} + \frac{3n}{4} - \frac{1}{2} + \nu - k \right) \right) u \in I^{m-N,p-M}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^\epsilon) \quad (6.19)$$

where $T_{i,i,r+i}$ is a classical pseudo-differential operator of order i which vanishes to order i on $\tilde{\Lambda}_0$ and to order $r+i$ on $\tilde{\Lambda}_1^\epsilon$ and $T_{0,0,0} = \text{Id}$.

But applying lemma 6.3, we know that

$$\left(x \frac{\partial}{\partial x} + \frac{3n}{4} + \mu - j \right) T_{i,i,r+i} = T_{i,i,r+i} \left(x \frac{\partial}{\partial x} + \frac{3n}{4} + \mu - j \right) + T_{i,i+1,r+i} \quad (6.20)$$

so commuting through we can conclude that

$$\sum_{k=0}^{N-1} \sum_{r=0}^{M-1} \sum_{i \leq k+r} T'_{i,i+k,i+r} \prod_{j=0}^{N-k-1} \left(x \frac{\partial}{\partial x} + \frac{3n}{4} + \mu - j \right) \prod_{l=0}^{M-r-1} \left(x'' \frac{\partial}{\partial x''} + \frac{3n}{4} - \frac{1}{2} + \nu - k \right) u \in I^{m-N,p-M}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^\epsilon) \quad (6.21)$$

where $T'_{0,0,0} = \text{Id}$ and $T'_{i,i,r+i}$ is a classical pseudo-differential operator of order i which vanishes to order i on $\tilde{\Lambda}_0$ and to order $r+i$ on $\tilde{\Lambda}_1^\epsilon$. Thus applying our induction hypothesis and the mapping properties of $T'_{i,i,r+i}$ the result follows. \square

Theorem 6.3. *The space of polyhomogeneous, paired, Lagrangian distributions with respect to a fixed intersecting pair is independent of which radial operators one uses.*

Proof. It is enough to prove this in the model case. Now, Lemma 6.4 reduces an arbitrary pair to a model pair and the proof of 6.4 shows that for a general pair, we

have

$$\begin{aligned}
& \left(\prod_{j=0}^{N-1} (R_0 + \mu - j) \prod_{k=0}^{M-1} (R_1 + \nu - k) \right) u \\
&= \sum_{k=0}^{N-1} \sum_{r=0}^{M-1} \sum_{i \leq k+r} T'_{i,i+k,i+r} \prod_{j=0}^{N-k-1} \left(x \frac{\partial}{\partial x} + \frac{3n}{4} + \mu - j \right) \prod_{l=0}^{M-r-1} \left(x'' \frac{\partial}{\partial x''} + \frac{3n}{4} - \frac{1}{2} + \nu - k \right) u
\end{aligned} \tag{6.22}$$

where $T'_{i,i+k,i+r}$ is as above. So it follows that if the condition holds for the model pair of radial operators then it holds for any pair. \square

Lemma 6.5. *Let (Λ_0, Λ_1) be an intersecting pair then if $u \in I^{m,p}(\Lambda_0, \Lambda_1)$, $u = u_1 + u_2$ with $u_1 \in I(\Lambda_0, \Lambda_0 \cap \Lambda_1)$ and $u_2 \in I(\Lambda_1, \Lambda_0 \cap \Lambda_1)$ and $u_i \in I^{m,p}(\Lambda_0, \Lambda_1)$ then $u_1 \in I^{m, \frac{p-m+\frac{1}{2}+\epsilon}{2}}(\Lambda_0, \Lambda_0 \cap \Lambda_1)$ and $u_2 \in I^{p, \frac{m-p+\frac{1}{2}+\epsilon}{2}}(\Lambda_1, \Lambda_0 \cap \Lambda_1)$ for all $\epsilon > 0$. This is in particular true for any parabolic decomposition of u .*

Proof. It is enough to prove this for the model $\Lambda_0 = N^*(x = 0), \Lambda_1 = N^*(x'' = 0)$. Now, we know that there is such a decomposition $u = v_1 + v_2$ by the definition of $I^{m,p}(\Lambda_0, \Lambda_1)$, with $\epsilon = 0$ and we have

$$u_1 - v_1 = v_2 - u_2 \in I(\Lambda_0, \Lambda_0 \cap \Lambda_1) \cap I(\Lambda_1, \Lambda_0 \cap \Lambda_1). \tag{6.23}$$

So $u_1 - v_1$ is an isotropic distribution with respect to $\Lambda_0 \cap \Lambda_1$ and is contained in $I^{m,p}(\Lambda_0, \Lambda_1)$. Thus we have $u_1 - v_1 = w_1 + w_2$ with $w_1 \in I^{m, \frac{p-m+\frac{1}{2}}{2}}(\Lambda_0, \Lambda_0 \cap \Lambda_1)$ and $w_2 \in I^{p, \frac{m-p+\frac{1}{2}}{2}}(\Lambda_1, \Lambda_0 \cap \Lambda_1)$. Since, $w_1 = u_1 - v_1 - w_2$, it is also isotropic. Letting a be the Fourier transform of w_1 , we have for some k ,

$$|D_\xi^\alpha a(\xi)| \leq C \langle \xi \rangle^{m + \frac{p+\frac{1}{2}}{2} - \frac{n}{4} - |\alpha''|} \langle \xi \rangle_{(1)}^{-\frac{p+\frac{1}{2}}{2} - \alpha_1}, \tag{6.24}$$

$$|D_\xi^\alpha a(\xi)| \leq C \langle \xi \rangle^{k+N-|\alpha''|} \langle \xi \rangle_{(1)}^{-2N-\alpha_1}. \tag{6.25}$$

Raising (6.24) to the power r and (6.25) to the power $s = 1 - r$, and taking their product we obtain that

$$|D_\xi^\alpha a(\xi)| \leq C \langle \xi \rangle^{r(m + \frac{p+\frac{1}{2}}{2} - \frac{n}{4}) + s(k+N) - |\alpha''|} \langle \xi \rangle_{(1)}^{-2Ns - 2pr}. \tag{6.26}$$

Given $\epsilon > 0$ we can pick r such that $r(m + \frac{p+\frac{1}{2}}{2} - \frac{n}{4}) + sk < m + \frac{p+\frac{1}{2}}{2} - \frac{n}{4} + \epsilon$, and then for any N' we can pick N such that $-2pr - 2Ns < -2N'$. Hence,

$$|D_{\xi}^{\alpha} a(\xi)| \leq C \langle \xi \rangle^{m + \frac{p+\frac{1}{2}}{2} - \frac{n}{4} + N' + \epsilon - |\alpha''|} \langle \xi \rangle_{(1)}^{-2N'} \quad (6.27)$$

that is that w_1 is an isotropic distribution of order $m + \frac{p+\frac{1}{2}}{2} + \epsilon$. A similar argument shows that w_2 is isotropic of the same order. Thus $u_1 \in I^{m, \frac{p-m+\frac{1}{2}}{2} + \epsilon}(\Lambda_0, \Lambda_0 \cap \Lambda_1)$ and a similar argument shows the corresponding result for u_2 . \square

Lemma 6.6. *Let $b \in C^{\infty}(\mathbb{R}_{x_1} \times (\mathbb{R}_{\xi''}^{n-1} - \{0\}), x_1 > 0)$ be homogeneous of degree ν in ξ'' and suppose that there exist $\delta > 0$ and q' such that*

$$\left| D_{\xi''}^{\alpha''} D_{x_1}^{\alpha_1} \left(\prod_{j=0}^{N-1} x_1 \frac{\partial}{\partial x_1} - (q+j) \right) b(x_1, \xi'') \right| \leq C_{\alpha, N} |\xi''|^{p-|\alpha''|} |x_1|^{q'+N\delta-\alpha_1} \quad (6.28)$$

then there exists functions $b_j(\xi'') \in S^p(\mathbb{R}_{\xi''}^{n-1})$ which are homogeneous of degree ν in ξ'' such that

$$\left| D_{\xi''}^{\alpha''} D_{x_1}^{\alpha_1} \left(b(x_1, \xi'') - \sum_{j=0}^{N-1} x_1^{j+q} b_j(\xi'') \right) \right| \leq C_{\alpha, N} |\xi''|^{p-|\alpha''|} |x_1|^{q+N-\alpha_1}. \quad (6.29)$$

Proof. This proof is analogous to that of Theorem 5.2 but we consider $x_1 \rightarrow 0+$ instead of $\xi'' \rightarrow +\infty$. In fact, by taking polar coordinates and a radial inversion it can be seen that the two results are very similar. We first of all prove the result when $q' = \Re(q)$ and $\delta = 1$ and then show that the general case follows.

We can always premultiply by $|\xi''|^{-\nu}$ without changing anything so it is enough to consider the case where $\nu = 0$. Now,

$$\frac{\partial}{\partial x_1} (x_1^{-q} b(x_1, \xi'')) = x_1^{-q-1} c(x_1, \xi'') \quad (6.30)$$

where $c = \left(\frac{\partial}{\partial x_1} - q \right) b$ and $|c| \leq |x_1|^{q'+1}$. Hence $\frac{\partial}{\partial x_1} (x_1^{-q} b(x_1, \xi''))$ is bounded as $x_1 \rightarrow 0+$ which implies that $x_1^{-q} b(x_1, \xi'')$ is convergent as $x_1 \rightarrow 0+$. Let $b_q(\xi'')$ be the limit. As b is homogeneous b_q will be also.

As $\frac{\partial}{\partial x_1} \left[(x_1^{-q})(x_1^q b_q(\xi'')) \right] = 0$ it follows that

$$|x_1^{-q}(b(x_1, \xi'') - x_1^q b_q(\xi''))| \leq C|x_1| \quad (6.31)$$

and hence we have

$$|b(x_1, \xi'') - x_1^q b_q(\xi'')| \leq C|x_1|^{q'+1}. \quad (6.32)$$

All these arguments will commute with applications of $x_1 D_{x_1}$ and $\xi_j D_{\xi_k}$ and so we have

$$|D_{\xi''}^{\alpha''} D_{x_1}^{\alpha_1} (b - x_1^q b_q)| \leq C|x_1|^{q'+1-\alpha_1} |\xi''|^{p-|\alpha''|}, \quad (6.33)$$

$$\left| D_{\xi''}^{\alpha''} D_{x_1}^{\alpha_1} \left(\prod_{j=0}^{N-1} x_1 \frac{\partial}{\partial x_1} - (q+j) \right) (b(x_1, \xi'') - b_q(x_1, \xi'') x_1^q) \right| \leq C|\xi''|^{p-|\alpha''|} |x_1|^{q'+N-\alpha_1}. \quad (6.34)$$

To complete our proof by induction we need to show

$$\left| D_{\xi''}^{\alpha''} D_{x_1}^{\alpha_1} \left(\prod_{j=1}^{N-1} x_1 \frac{\partial}{\partial x_1} - (q+j) \right) (b(x_1, \xi'') - x_1^q b_q(\xi'')) \right| \leq |\xi''|^{p-|\alpha''|} |x_1|^{q'+N-\alpha_1}. \quad (6.35)$$

To establish this it is enough to show that if $|a| \leq C|x_1|^{q'+1}$ and $|(x_1 \frac{\partial}{\partial x_1} - q)a| \leq C|x_1|^{q'+N}$ then $|a| \leq C|x_1|^{q'+N}$. As above, we have

$$\left| \frac{\partial}{\partial x_1} (x_1^{-q} a) \right| \leq C|x_1|^{N-1} \quad (6.36)$$

but $x_1^{-q} a \rightarrow 0$ as $x_1 \rightarrow 0+$ so integrating we obtain

$$|x_1^{-q} a| \leq |x_1|^N \quad (6.37)$$

which completes the proof when $q' = \Re(q)$ and $\delta = 1$.

Now in the general case, once we have chosen an N then there exists M such that $q' + \delta M < \Re(q) + N$ and so

$$\left| D_{x_1}^{\alpha_1} D_{\xi''}^{\alpha''} \left(\prod_{j=0}^{N-1} x_1 \frac{\partial}{\partial x_1} - (q+j) \right) \left(\prod_{j=N}^{M-1} x_1 \frac{\partial}{\partial x_1} - (q+j) \right) b(x_1, \xi'') \right| \leq C |\xi''|^{p-|\alpha''|} |x_1|^{\Re(q)+N-\alpha_1}. \quad (6.38)$$

This allows us to deduce the existence of an expansion for

$$\left(\prod_{j=N}^{M-1} x_1 \frac{\partial}{\partial x_1} - (q+j) \right) b(x_1, \xi'')$$

up to order $N-1$ that is there exists b_q, \dots, b_{q+N-1} such that

$$\left| D_{x_1}^{\alpha_1} D_{\xi''}^{\alpha''} \left[\left(\prod_{j=N}^{M-1} x_1 \frac{\partial}{\partial x_1} - (q+j) \right) b(x_1, \xi'') - \sum_{j=0}^{N-1} b_{q+j}(\xi'') x_1^{q+j} \right] \right| \leq C |\xi''|^{p-|\alpha''|} |x_1|^{\Re(q)+N-\alpha_1} \quad (6.39)$$

but we can rewrite this, for some non-zero constants γ_j , as

$$\left| D_{x_1}^{\alpha_1} D_{\xi''}^{\alpha''} \left(\left(\prod_{j=N}^{M-1} x_1 \frac{\partial}{\partial x_1} - (q+j) \right) \left(b(x_1, \xi'') - \sum_{j=0}^{N-1} \gamma_j b_{q+j}(\xi'') x_1^{q+j} \right) \right) \right| \leq C |\xi''|^{p-|\alpha''|} |x_1|^{\Re(q)+N-\alpha_1}. \quad (6.40)$$

This is enough to establish the existence of the expansion up to N terms as

$$\left| \left(x_1 \frac{\partial}{\partial x_1} - k \right) b \right| \leq C |x_1|^r \text{ for } r \leq k$$

implies that

$$|b| \leq C |x_1|^r.$$

To prove this, differentiate $x_1^{-k} b$ and integrate from 1 towards 0. \square

We are now in a position to prove the equivalence of our two definitions.

Proof of Theorem 6.2. We commence by showing $J^{\mu,\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e) \subseteq I_{phg}^{\mu,\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$ and we will use this fact in the second part of our proof. If u is a member of $J^{\mu,\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$, we need to show

$$\left(\prod_{j=0}^{N-1} \left(x \frac{\partial}{\partial x} + \frac{3n}{4} + \mu - j \right) \prod_{k=0}^{M-1} \left(x'' \frac{\partial}{\partial x''} + \frac{3n}{4} - \frac{1}{2} + \nu - k \right) \right) u \in I^{m-N,p-M}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e) \text{ for all } M, N \quad (6.41)$$

The point here is that applying $x \frac{\partial}{\partial x} + (\frac{3n}{4} + \mu - j)$ will kill the terms of homogeneity $\mu - \frac{n}{4} - j$ on $\tilde{\Lambda}_0$ and $x'' \frac{\partial}{\partial x''} + (\frac{3n}{4} + \nu - \frac{1}{2} - j)$ will kill the terms of homogeneity $\nu - \frac{n}{4} + \frac{1}{2} - j$ on $\tilde{\Lambda}_1^e$. We need to be slightly more subtle at the intersection though. For notational simplicity, we shall use $\tilde{\cdot}$ to denote symbolic orders that is $\tilde{\mu} = \mu - \frac{n}{4}$, $\tilde{m} = m - \frac{n}{4}$, $\tilde{\nu} = \nu - \frac{n}{4} + \frac{1}{2}$ and $\tilde{p} = p - \frac{n}{4} + \frac{1}{2}$. Fix M and N . Given $u \in J^{\mu,\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$, near the intersection, we can always decompose as

$$\begin{aligned} u = & \sum_{j=0}^R \int e^{ix \cdot \xi} a_{\tilde{\mu}-j}(\xi) (1 - \phi) \left(\frac{\xi_1}{|\xi''|^{\frac{1}{2}}} \right) d\xi \\ & + \sum_{l=0}^R \int e^{ix'' \cdot \xi''} b_{\tilde{\nu}-l}(x_1, \xi'') *_{x_1} (|\xi''|^{1/2} \hat{\phi}(|x_1| |\xi|^{1/2})) d\xi'' \\ & + \sum_{j,l} \int e^{ix \cdot \xi} (\xi_1 - i0)^{\mu-\nu-j+l} c_{j,l}(\xi'') d\xi'' + v \end{aligned} \quad (6.42)$$

where $v \in I^{m-N,p-M}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$, $c_{j,l}$ is homogeneous of degree $\nu - l$ and $a_{\tilde{\mu}-j}(\xi)$, $b_{\tilde{\nu}-l}(x_1, \xi'')$ vanish to high order at $\xi_1 = 0, x_1 = 0$, respectively. We can assume that $b_{\tilde{\nu}-l}(x_1, \xi'')$ is compactly supported in x_1 . The fourth term is already in the correct space and will remain in it under application of the radial operators. The third term will be killed by application of $x \frac{\partial}{\partial x} + (\frac{3n}{4} + \mu - j)$ or by application of $x'' \frac{\partial}{\partial x''} + (\frac{3n}{4} + \nu - \frac{1}{2} - j)$.

Applying $x \frac{\partial}{\partial x} + (\frac{3n}{4} + \mu - j)$ to the first term we obtain,

$$- \int e^{ix \cdot \xi} a_{\tilde{\mu}-j}(\xi) \xi \frac{\partial}{\partial \xi} (1 - \phi) \left(\frac{\xi_1}{|\xi''|^{\frac{1}{2}}} \right) d\xi. \quad (6.43)$$

This is supported between two parabolae about $\xi_1 = 0$ and if $a_{\tilde{\mu}-j}$ vanishes to order K this will be of order $\tilde{m} - j - K/2$. So provided K is sufficiently large this will be in $I^{m-N,p-M}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$. This leaves the second term, after applying $\xi'' \frac{\partial}{\partial \xi''} - (\tilde{\nu} - l)$ the integrand becomes,

$$\int \xi'' \frac{\partial}{\partial \xi''} \left(\hat{\phi}(y_1 |\xi''|^{1/2}) |\xi''|^{1/2} \right) b_{\tilde{\nu}-l}(x_1 - y_1, \xi'') dy_1 \quad (6.44)$$

and this is equal to

$$\frac{1}{2} \int \frac{\partial}{\partial y_1} \left(y_1 \hat{\phi}(y_1 |\xi''|^{1/2}) |\xi''|^{1/2} \right) b_{\tilde{\nu}-l}(x_1 - y_1, \xi'') dy_1, \quad (6.45)$$

or integrating by parts

$$-\frac{1}{2} \int \left(y_1 \hat{\phi}(y_1 |\xi''|^{1/2}) |\xi''|^{1/2} \right) \frac{\partial}{\partial y_1} b_{\tilde{\nu}-l}(x_1 - y_1, \xi'') dy_1. \quad (6.46)$$

Now, if b vanishes to order K at 0 then b will be C^{K-1} and taking a Taylor expansion about x_1 to order $K - 1$, this is equal to

$$-\frac{1}{2} \int \left(y_1^K \hat{\phi}(y_1 |\xi''|^{1/2}) |\xi''|^{1/2} \right) \tilde{b}_{\tilde{\nu}-l}(x_1, y_1, \xi'') dy_1 \quad (6.47)$$

with $\tilde{b}_{\tilde{\nu}-l}$ continuous. The vanishing of the lower terms comes from applying the Fourier Inversion formula and remembering that $\phi \equiv 1$ near 0. Executing a change of variables, $z_1 = y_1 |\xi''|^{1/2}$ the remaining term is of order $|\xi''|^{\tilde{\nu}-l-K/2+1}$ and so, for K sufficiently large, the partial Fourier transform of this will be in $I^{m-N,p-M}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$. We have therefore shown $u \in I^{\mu,\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$ as needed.

Given $u \in I_{phg}^{\mu,\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$, we establish that the symbol on $\tilde{\Lambda}_1^e$ is an asymptotic sum of homogeneous terms with the correct singularities as $x_1 \rightarrow 0 +$. We then pick a paired total symbol on $(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$ with the same expansion on $\tilde{\Lambda}_1^e$ and quantize it to get a distribution v such that $u - v \in I^{\mu,\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$ and such that $u - v$ has symbol with zero asymptotic expansion on $\tilde{\Lambda}_1^e$ and we use this to show that

$$u - v \in I_{phg}^{\mu,\nu,-\infty}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e) \subseteq J^{\mu,\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$$

which will prove the result. A special argument will be required in the case where $\mu - \nu - \frac{1}{2}$ is a negative integer as in this case we obtain ordinary Lagrangian distributions with respect to $\tilde{\Lambda}_0$.

Let \tilde{u} denote the Fourier transform of u in the last $n - 1$ variables then letting ϕ be a one-sided cut off function we have from Lemma 6.5 that

$$\begin{aligned} & \left| (1 - \phi)(x_1|\xi''|^{\frac{1}{2}}) \prod_{j=0}^{N-1} \left(\frac{\partial}{\partial x_1} x_1 - \xi'' \frac{\partial}{\partial \xi''} + \tilde{\mu} - j \right) \tilde{u} \right| \\ & \leq C \langle \xi'' \rangle^{\tilde{p}} \left(\frac{\langle \xi'' \rangle^{\frac{1}{2}}}{1 + |x_1| \langle \xi'' \rangle^{\frac{1}{2}}} \right)^{\tilde{m} - \tilde{p} + 1 + N + \epsilon}, \end{aligned} \quad (6.48)$$

$$\begin{aligned} & \left| (1 - \phi)(x_1|\xi''|^{\frac{1}{2}}) (\xi'' \frac{\partial}{\partial \xi''} - \tilde{p}) \prod_{j=0}^{N-1} \left(\frac{\partial}{\partial x_1} x_1 - \xi'' \frac{\partial}{\partial \xi''} + \tilde{\mu} - j \right) \tilde{u} \right| \\ & \leq C \langle \xi'' \rangle^{\tilde{p}-1} \left(\frac{\langle \xi'' \rangle^{\frac{1}{2}}}{1 + |x_1| \langle \xi'' \rangle^{\frac{1}{2}}} \right)^{\tilde{m} - \tilde{p} + 2 + N + \epsilon}. \end{aligned} \quad (6.49)$$

We are interested in limits as $|\xi''| \rightarrow \infty$ and for $|x_1| |\xi''|^{\frac{1}{2}} \geq C$ we have

$$\frac{\langle \xi'' \rangle^{\frac{1}{2}}}{1 + |x_1| \langle \xi'' \rangle^{\frac{1}{2}}} \leq \frac{C'}{|x_1|}, \quad (6.50)$$

$$\frac{1 + |x_1| \langle \xi'' \rangle^{\frac{1}{2}}}{\langle \xi'' \rangle^{\frac{1}{2}}} \leq C'' |x_1|. \quad (6.51)$$

So for $|\xi''| \geq x_1^2$ we can rewrite these estimates as

$$\left| (1 - \phi)(x_1|\xi''|^{\frac{1}{2}}) \prod_{j=0}^{N-1} \left(\frac{\partial}{\partial x_1} x_1 - \xi'' \frac{\partial}{\partial \xi''} + \tilde{\mu} - j \right) \tilde{u} \right| \leq C \langle \xi'' \rangle^{\tilde{p}} |x_1|^{\tilde{p} - \tilde{m} - 1 + N + \epsilon}, \quad (6.52)$$

$$\begin{aligned} & \left| (1 - \phi)(x_1|\xi''|^{\frac{1}{2}}) (\xi'' \frac{\partial}{\partial \xi''} - \tilde{p}) \prod_{j=0}^{N-1} \left(\frac{\partial}{\partial x_1} x_1 - \xi'' \frac{\partial}{\partial \xi''} + \tilde{\mu} - j \right) \tilde{u} \right| \\ & \leq C \langle \xi'' \rangle^{\tilde{p}-1} |x_1|^{\tilde{p} - \tilde{m} - 2 + N + \epsilon}. \end{aligned} \quad (6.53)$$

Now let

$$b_{\tilde{\nu},n}(\lambda, x_1, \xi'') = \lambda^{-\tilde{\nu}} \prod_{j=0}^{N-1} \left(\frac{\partial}{\partial x_1} x_1 - \xi'' \frac{\partial}{\partial \xi''} + \tilde{\mu} - j \right) \tilde{u}(x_1, \lambda \xi'') \quad (6.54)$$

then for λ large we have

$$\left| \frac{\partial}{\partial \lambda} (b_{\tilde{\nu},N}) \right| \leq C \lambda^{-2} |x_1|^{\tilde{p}-\tilde{m}-2+\epsilon}. \quad (6.55)$$

Hence $b_{\tilde{\nu},n}$ converges as $\lambda \rightarrow +\infty$ to a function $a_{\tilde{\nu},N}$ which is homogeneous of degree $\tilde{\nu}$ in ξ . Repeating these arguments for the derivatives and noting that any derivatives applied to the cut off will disappear at infinity, we obtain

$$|D_{x_1}^{\alpha_1} D_{\xi''}^{\alpha''} a_{\tilde{\nu},N}| \leq C \langle \xi'' \rangle^{\tilde{p}-|\alpha''|} |x_1|^{\tilde{p}-\tilde{m}-1+N+\epsilon-\alpha_1}. \quad (6.56)$$

So putting $a_{\tilde{\nu}} = a_{\tilde{\nu},0}$ we have applying Euler's relation that

$$\begin{aligned} \left| D_{x_1}^{\alpha_1} D_{\xi''}^{\alpha''} \left(\prod_{j=0}^{N-1} x_1 \frac{\partial}{\partial x_1} + (\tilde{\mu} + 1 - \tilde{\nu} - j) \right) a_{\tilde{\nu}}(x_1, \xi'') \right| \\ \leq C \langle \xi'' \rangle^{\tilde{p}-\alpha''} |x_1|^{\tilde{p}-\tilde{m}-1+N+\epsilon-\alpha_1}. \end{aligned} \quad (6.57)$$

We can now apply Lemma 6.6 to obtain $a_{\tilde{\nu},j}(\xi'')$ homogeneous of degree $\tilde{\nu}$ such that

$$\left| D_{x_1}^{\alpha_1} D_{\xi''}^{\alpha''} \left(a_{\tilde{\nu}}(x_1, \xi'') - \sum_{j=0}^{N-1} x_1^{\tilde{\nu}-\tilde{m}-1+j} a_{\tilde{\nu},j}(\xi'') \right) \right| \leq C \langle \xi'' \rangle^{\tilde{p}-\alpha''} |x_1|^{\tilde{p}-\tilde{m}-1+N+\epsilon-\alpha_1}. \quad (6.58)$$

In order to apply an inductive argument, we now consider $(\tilde{u} - a_{\tilde{\nu}})$ for $|\xi''| \geq C|x_1|^{-2}$. We certainly have for $M \geq 1$

$$\begin{aligned} \left| D_{x_1}^{\alpha_1} D_{\xi''}^{\alpha''} \left(\prod_{k=0}^{M-1} \left(\xi'' \frac{\partial}{\partial \xi''} - (\tilde{\nu} - k) \right) \prod_{j=0}^{N-1} \left(\frac{\partial}{\partial x_1} x_1 - \xi'' \frac{\partial}{\partial \xi''} + \tilde{\mu} - j \right) (\tilde{u} - a_{\tilde{\nu}}) \right) \right| \\ \leq C \langle \xi'' \rangle^{\tilde{p}-M-|\alpha''|} |x_1|^{\tilde{p}-\tilde{m}-1+N-M+\epsilon}. \end{aligned} \quad (6.59)$$

Where as what we need is

$$\begin{aligned} \left| D_{x_1}^{\alpha_1} D_{\xi''}^{\alpha''} \left(\prod_{k=1}^{M-1} \left(\xi'' \frac{\partial}{\partial \xi''} - (\tilde{\nu} - k) \right) \prod_{j=0}^{N-1} \left(\frac{\partial}{\partial x_1} x_1 - \xi'' \frac{\partial}{\partial \xi''} + \tilde{\mu} - j \right) (\tilde{u} - a_{\tilde{\nu}}) \right) \right| \\ \leq C \langle \xi'' \rangle^{\tilde{p}-M-|\alpha''|} |x_1|^{\tilde{p}-\tilde{m}-1+N-M}. \end{aligned} \quad (6.60)$$

Claim: If

$$\left| \left(\xi'' \frac{\partial}{\partial \xi''} - q \right) v \right| \leq C \langle \xi'' \rangle^{\Re(q)-k} |x_1|^l$$

and

$$|v| \leq C \langle \xi'' \rangle^{\Re(q)-k+1} |x_1|^l \text{ with } k \geq 2$$

then

$$|v| \leq C \langle \xi'' \rangle^{\Re(q)-k} |x_1|^l. \quad (6.61)$$

Proof of claim. Putting $\xi'' = \lambda\omega$ and $g = \left(\xi'' \frac{\partial}{\partial \xi''} - q \right) v$ we obtain

$$\frac{\partial}{\partial \lambda} (\lambda^{-q} v(x_1, \lambda\omega)) = \lambda^{-q-1} g(x_1, \lambda\omega). \quad (6.62)$$

So we have that

$$\left| \frac{\partial}{\partial \lambda} (\lambda^{-q} v(x_1, \lambda\omega)) \right| \leq C \lambda^{-k-1} |x_1|^l, \quad (6.63)$$

from which we deduce

$$|v(x_1, \lambda\omega)| \leq C \lambda^{\Re(q)-k} |x_1|^l \quad (6.64)$$

which proves the claim. \square

Hence by induction, we conclude the existence of $a_{\tilde{\nu}-k}(x_1, \xi'')$ homogeneous of degree $\tilde{\nu} - k$ in ξ'' with an expansion as $x_1 \rightarrow 0+$:

$$a_{\tilde{\nu}-k} \sim \sum_j a_{\tilde{\nu}-k,j}(\xi'') x_1^{\tilde{\nu}-k-1-\tilde{\mu}+j} \quad (6.65)$$

$$\tilde{u} \sim \sum_k a_{\tilde{\nu}-k} \quad (6.66)$$

in the sense that

$$\begin{aligned} & \left| D_{x_1}^{\alpha_1} D_{\xi''}^{\alpha''} \left((1 - \phi)(x_1 |\xi''|^{\frac{1}{2}}) \left(\tilde{u} - \sum_{j=0}^{N-1} a_{\tilde{\nu}-j} \right) \right) \right| \\ & \leq C \langle \xi'' \rangle^{\tilde{p}-N-|\alpha''|} \left(\frac{\langle \xi'' \rangle^{\frac{1}{2}}}{1 + |x_1| \langle \xi'' \rangle^{\frac{1}{2}}} \right)^{\tilde{m}-\tilde{p}+1+N+\alpha_1}. \end{aligned} \quad (6.67)$$

In the case where $\tilde{\nu} - \tilde{\mu} - 1 - k + j$ is a negative integer, the coefficient $a_{\tilde{\nu}-k,j}$ will in fact vanish corresponding to the fact that $\chi_+^{\tilde{\nu}-\tilde{\mu}-1+j}(x_1)$ vanishes on $x_1 > 0$. We will investigate this case further later on.

In other cases, we can pick a paired total symbol with expansion $a_{\tilde{\nu}-k}(x_1, \xi'')$ on $\tilde{\Lambda}_1^\epsilon$ and quantize it to get a distribution v with the same expansion on $\tilde{\Lambda}_1^\epsilon$. Thus $u - v$ will have zero expansion on $\tilde{\Lambda}_1^\epsilon$ that is that $(1 - \phi)(x_1|\xi''|^{1/2})\widetilde{u - v} \in S^{-\infty}$. It therefore follows from lemma 6.5 that

$$\prod_{j=0}^{N-1} \left(x \frac{\partial}{\partial x} + \frac{3n}{4} + \mu - j \right) (u - v) \in I_{ma}^{m-N, \frac{p-m+N+\frac{1}{2}}{2} + \epsilon}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1) \text{ for all } N. \quad (6.68)$$

We show that this implies that $u - v$ is a classical Lagrangian distribution with respect to Λ_0 which vanishes to infinite order at the intersection with Λ_1 . Putting $a = \widetilde{u - v}$, we have

$$\prod_{j=0}^{N-1} \left(\xi \frac{\partial}{\partial \xi} - (\tilde{\mu} - j) \right) a \in S_{ma}^{\tilde{m}-N, \frac{\tilde{p}-\tilde{m}+N}{2} + \epsilon}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1) \subset S_{\frac{1}{2}}^{K-\frac{N}{2}}(\mathbb{R}_\xi^n) \quad (6.69)$$

Provided K is large enough. Note that establishing an asymptotic expansion $\sum a_{\tilde{\mu}-j}$, with $a_{\tilde{\mu}-j}$ homogeneous of degree $\tilde{\mu} - j$, is enough to prove the result. Taking polar coordinates (ω, λ) , putting $\rho = \lambda \frac{\partial}{\partial \lambda}$ and factoring the left hand side, we obtain

$$\left| \prod_{j=0}^{N-1} (\rho - (\tilde{\mu} - j)) \prod_{j=N}^{M-1} (\rho - (\tilde{\mu} - j)) a(\lambda, \omega) \right| \leq C |\lambda|^{K-\frac{M}{2}}. \quad (6.70)$$

So picking M sufficiently large, we have $K - \frac{M}{2} \leq \tilde{m} - N$ and therefore applying the arguments in the proof of Theorem 5.2, there exist homogeneous functions $a_{\tilde{\mu}-j}$ such that

$$\left| \left(\prod_{j=N}^{M-1} \rho - (\tilde{\mu} - j) \right) a - \sum_{j=0}^{N-1} a_{\tilde{\mu}-j} \right| \leq C |\lambda|^{k-\frac{M}{2}}. \quad (6.71)$$

Using the homogeneity of $a_{\tilde{\mu}-j}$ and Euler's relation we can rewrite this as

$$\left| \left(\prod_{j=N}^{M-1} \rho - (\tilde{\mu} - j) \right) \left(a - \sum_{j=0}^{N-1} \gamma_j a_{\tilde{\mu}-j} \right) \right| \leq C |\lambda|^{k-\frac{M}{2}} \leq C |\lambda|^{\tilde{m}-M}. \quad (6.72)$$

We now wish to show

$$\left| a - \sum_{j=0}^{N-1} a_{\bar{\mu}-j} \right| \leq C|\lambda|^{k-\frac{M}{2}} \leq C|\lambda|^{\bar{m}-M}. \quad (6.73)$$

and this will be enough to establish the asymptotic expansion.

Claim: $|(\rho - k)b| \leq C|\lambda|^q$ implies $|b| \leq C(|\lambda|^k + |\lambda|^q)$

If $k > q$ then as in many previous arguments we can establish the existence of a function b_k homogeneous of degree k such that $|b - b_k| \leq C|\lambda|^q$ and the claim easily follows.

If $k \leq q$ then differentiating $\lambda^{-k}q$ and integrating up proves the claim.

So we have an asymptotic expansion of homogeneous terms at the top order. Note that the arguments above will commute with the application of spherical derivatives so the homogeneous terms are smooth. It follows from Proposition 18.1.4 in [7] that the expansion must in fact be valid in the symbol space $S^{\bar{m}}(\mathbb{R}_\xi^n)$.

We are not yet done as we need to show that the homogeneous terms vanish to infinite order at $\xi_1 = 0$. We use our original hypothesis to establish an expansion in terms of fractional powers of which the coefficients must all vanish as our terms are smooth. Let

$$a^M = \left(\prod_{k=0}^{M-1} \xi'' \frac{\partial}{\partial \xi''} - (\tilde{\nu} - j) \right) a \quad (6.74)$$

then we have

$$|a^M(\xi)| \leq C \langle \xi \rangle^{\bar{m}} \left(\frac{\langle \xi \rangle}{\langle \xi \rangle_{(1)}} \right)^{\bar{p}-M-\bar{m}+\epsilon}. \quad (6.75)$$

Hence it follows that

$$|\lambda^{-\bar{\mu}} a^M(\lambda\xi)| \leq C \langle \xi \rangle^{\bar{m}} \left(\frac{|\xi| + |\lambda|^{-1}}{|\xi_1| + |\lambda|^{-\frac{1}{2}} |\xi''|^{\frac{1}{2}} + |\lambda|^{-1}} \right)^{\bar{p}-M+\bar{m}+\epsilon}. \quad (6.76)$$

Letting $\lambda \rightarrow +\infty$ this becomes

$$\left| \left(\prod_{k=0}^{M-1} \xi'' \frac{\partial}{\partial \xi''} - (\tilde{\nu} - j) \right) a_{\tilde{\mu}}(\xi) \right| \leq C \left(\frac{|\xi_1|}{|\xi|} \right)^{\tilde{m} - \tilde{p} + M + \epsilon} |\xi|^{\tilde{m}} \quad (6.77)$$

and applying Euler's relation we obtain

$$\left| \left(\prod_{k=0}^{M-1} \xi_1 \frac{\partial}{\partial \xi_1} - (\tilde{\mu} - \tilde{\nu} + j) \right) a_{\tilde{\mu}}(\xi) \right| \leq C \left(\frac{|\xi_1|}{|\xi|} \right)^{\tilde{m} - \tilde{p} + M + \epsilon} |\xi|^{\tilde{m}}. \quad (6.78)$$

This implies the existence of an expansion at $\xi_1 = 0$ with terms of order $\tilde{\mu} - \tilde{p} + j$, the coefficients of this expansion will vanish identically (unless $\tilde{\mu} - \tilde{p} + j$ is a positive integer, the case we have set aside) as $a_{\tilde{\mu}}$ is smooth and so the result follows for the top order term. A similar argument will apply for lower terms after subtracting the higher ones.

It remains to do the case where $\tilde{\nu} - \tilde{\mu}$ is a negative integer. We postpone this case to the next section where we obtain it as a corollary to the invariance of a wider calculus. \square

Corollary 6.1. *$J^{\mu,\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^\epsilon)$ is invariant under proper, zeroth order, classical Fourier integral operators associated to symplectomorphisms which preserve $(\tilde{\Lambda}_0, \tilde{\Lambda}_1^\epsilon)$.*

This gives us an alternative definition for $I_{phg}^{\mu,\nu}$ which is very similar to that of $I_{ma}^{m,p}(\Lambda_0, K_0)$.

Theorem 6.4. *Let $(\Lambda_0, \Lambda_1^\epsilon)$ be an intersecting Lagrangian pair then $I_{phg}^{\mu,\nu}(\Lambda_0, \Lambda_1^\epsilon)$ consists of those distributions u with $\text{WF}(u)$ contained in $\Lambda_0 \cup \Lambda_1^\epsilon$ and such that for each $p \in \Lambda_0 \cup \Lambda_1^\epsilon$ there is a properly supported Fourier integral operator, F , of order 0, elliptic at p , associated to a symplectomorphism taking $(\Lambda_0, \Lambda_1^\epsilon)$ to $(\tilde{\Lambda}_0, \tilde{\Lambda}_1^\epsilon)$ such that*

$$Fu \in J^{\mu,\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^\epsilon). \quad (6.79)$$

Corollary 6.2. *Let $u \in I(\Lambda_0, \Lambda_1^e)$ and suppose there exists $\delta > 0, r, s$ such that for some pair of radial operators (R_0, R_1) ,*

$$\left(\prod_{j=0}^{N-1} (R_0 + \mu - j) \right) u \in I^{r-\delta N, s}(\Lambda_0, \Lambda_1^e) \quad \forall N \quad (6.80)$$

and

$$\left(\prod_{k=0}^{M-1} (R_1 + \nu - k) \right) u \in I^{r, s-\delta M}(\Lambda_0, \Lambda_1^e) \quad \forall N \quad (6.81)$$

then $u \in I_{phg}^{\mu, \nu}(\Lambda_0, \Lambda_1^e)$.

Proof. It is enough to prove this in the model case, combining the two statements we have

$$\left(\prod_{j=0}^{N-1} R_0 + \mu - j \right) \left(\prod_{k=0}^{M-1} R_1 + \nu - k \right) u \in I^{r-N, s}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e) \cap I^{r, s-M}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e). \quad (6.82)$$

Now using lemma 6.5 for any parabolic decomposition of this distribution into $u_1 + u_2$ we have

$$\begin{aligned} u_1 &\in I^{r-\delta N, \frac{s-r+\delta N+\frac{1}{2}}{2}+\epsilon}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e) \cap I^{r, \frac{s-r-\delta M+\frac{1}{2}}{2}+\epsilon}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e) \\ &\subset I^{r-\frac{\delta N}{2}, \frac{s-r+\frac{\delta}{2}(N-M)+\frac{1}{2}}{2}+\epsilon} \end{aligned} \quad (6.83)$$

and

$$\begin{aligned} u_2 &\in I^{s-\delta M, \frac{r-s+\delta M+\frac{1}{2}}{2}+\epsilon}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e) \cap I^{s, \frac{r-s-\delta N+\frac{1}{2}}{2}+\epsilon}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e) \\ &\subset I^{s-\frac{\delta M}{2}, \frac{r-s+\frac{\delta}{2}(M-N)+\frac{1}{2}}{2}+\epsilon}. \end{aligned} \quad (6.84)$$

Inspecting the proof of 6.2, this is enough to establish membership of $J^{\mu, \nu}$ and therefore $I_{phg}^{\mu, \nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$. \square

Proposition 6.3. *Let T be a proper, classical pseudo-differential operator of order r then*

$$T : I_{phg}^{\mu, \nu}(\Lambda_0, \Lambda_1^e) \rightarrow I_{phg}^{\mu+r, \nu+r}(\Lambda_0, \Lambda_1^e).$$

Proof. That the range is contained in $I^{m+r,p+r}(\Lambda_0, \Lambda_1^e)$ is clear from the mapping properties of marked Lagrangian distributions under pseudo-differential operators (see [9]). It is enough to prove the result in the model case.

We need to prove that if $u \in I^{\mu,\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$ then

$$\prod_{j=0}^{N-1} V_{\tilde{\mu}+r-j} \prod_{k=0}^{M-1} W_{\tilde{\nu}+r-k} T u \in I^{m+r-N,p+r-M}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e) \quad (6.85)$$

where $V_{\tilde{\mu}-j} = x \frac{\partial}{\partial x} + \frac{3n}{4} + \tilde{\mu} - j$ and $W_{\tilde{\nu}-k} = x'' \frac{\partial}{\partial x''} + \frac{3n}{4} - \frac{1}{2} + \tilde{\nu} - k$ are the radial operators. Now commuting T through the radial operators as in the proof of 6.4 we obtain

$$\sum_{s=0}^{M-1} \sum_{t=0}^{N-1} P_{s,t} \prod_{j=0}^{s-1} \prod_{k=0}^{t-1} V_{\tilde{\mu}-j} W_{\tilde{\nu}-k} u \quad (6.86)$$

where $P_{s,t}$ is a pseudo-differential operator of order r which vanishes to order (s, t) on $(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$ and the result is now immediate. \square

Corollary 6.3. *Let F be a proper, classical Fourier integral operator of order r associated to a symplectomorphism f mapping (Λ_0, Λ_1^e) to (Λ_2, Λ_3^e) then*

$$F : I_{phg}^{\mu,\nu}(\Lambda_0, \Lambda_1^e) \rightarrow I_{phg}^{\mu+r,\nu+r}(\Lambda_2, \Lambda_3^e). \quad (6.87)$$

Proof. Just decompose F into a zeroth order Fourier integral operator and a pseudo-differential operator of order r . \square

Proposition 6.4. *Let $u_k \in I_{phg}^{\mu-k,\nu}(\Lambda_0, \Lambda_1^e)$ for $k = 0, 1, 2, \dots$ then there exists $u \in I_{phg}^{\mu,\nu}(\Lambda_0, \Lambda_1^e)$ such that*

$$u - \sum_{j=0}^{N-1} u_j \in I_{phg}^{\mu-N,\nu}(\Lambda_0, \Lambda_1^e). \quad (6.88)$$

Proof. It is enough to do this for the model. We need only consider symbols on $\tilde{\Lambda}_0$. The total symbol of u_k on $\tilde{\Lambda}_0$ will be an element of $S_{sing}^{\mu-\frac{n}{4}-\frac{1}{2}-k, \mu-k-\nu-1}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$.

That is a formal sum $\sum_j a_{\mu-\frac{\pi}{4}-\frac{1}{2}-j-k}^k$ such that

$$a_{\mu-\frac{\pi}{4}-\frac{1}{2}-j-k}^k \in T^{\mu-\frac{\pi}{4}-\frac{1}{2}-j-k, \mu-j-k-\nu-1}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e).$$

We put,

$$b_{\mu-\frac{\pi}{4}-\frac{1}{2}-l} = \sum_{j+k=l} a_{\mu-\frac{\pi}{4}-\frac{1}{2}-j-k}^k. \quad (6.89)$$

Each sum is finite and we have $b_{\mu-\frac{\pi}{4}-\frac{1}{2}-l} \in T^{\mu-\frac{\pi}{4}-\frac{1}{2}-l, \mu-l-\nu-1}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$. So we have an element of $S_{sing}^{\mu-\frac{\pi}{4}-\frac{1}{2}, \mu-\nu-1}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ and picking a compatible symbol on $\tilde{\Lambda}_1^e$ and quantizing the result follows. \square

Proposition 6.5. *Let $u_k \in I_{phg}^{\mu, \nu-k}(\Lambda_0, \Lambda_1^e)$ for $k = 0, 1, 2, \dots$ then there exists $u \in I_{phg}^{\mu, \nu}(\Lambda_0, \Lambda_1^e)$ such that*

$$u - \sum_{j=0}^{N-1} u_j \in I_{phg}^{\mu, \nu-N}(\Lambda_0, \Lambda_1^e). \quad (6.90)$$

Proof. Essentially the same as above. \square

Having defined a class of paired Lagrangian distributions, it remains to discuss their relationship to ordinary Lagrangian distributions.

Proposition 6.6.

$$I_{phg}^{\mu}(\Lambda_0) \subset I_{phg}^{\mu, \mu-\frac{1}{2}}(\Lambda_0, \Lambda_1^e)$$

Proof. Considering the model case, this follows from the fact that a smooth function will always have a Taylor series expansion at $\xi_1 = 0$ and the first term will be ξ_1^0 . \square

It is important to note that if the symbol vanishes at the intersection then an element of $I^{\mu}(\Lambda_0)$ will have lower order on the second Lagrangian. In particular, once we have defined a concept of ellipticity, we will see that $u \in I^{\mu}(\Lambda_0)$ may not be elliptic in $I^{\mu}(\Lambda_0)$ or in $I_{phg}^{\mu, \mu-\frac{1}{2}}(\Lambda_0, \Lambda_1^e)$ but may be elliptic in $I_{phg}^{\mu, \mu-\frac{1}{2}-k}(\Lambda_0, \Lambda_1^e)$.

We now prove a version of Egorov's theorem in this category.

Proposition 6.7. *Let f be a homogeneous symplectomorphism of $T^*(X) - 0$ to $T^*(Y) - 0$, let F be a properly supported elliptic Fourier integral operator associated to the graph of f , with parametriz G , and suppose (Λ_0, Λ_1^e) is an intersecting pair of Lagrangian submanifolds which defines a class of operator kernels from $C_c^\infty(X)$ to $C^\infty(X)$ and $P \in I_{phg}^{\mu, \nu}(\Lambda_0, \Lambda_1)$ then $GPF \in I_{phg}^{\mu, \nu}((f \times f)^* \Lambda_0, (f \times f)^* \Lambda_1^e)$ and the symbols are the pullbacks by $f \times f$.*

Proof. The operator condition on (Λ_0, Λ_1^e) means that there are no points in $\Lambda_0 \cup \Lambda_1^e$ of the form $(x, \xi, y, 0)$ or $(x, 0, y, \xi)$.

Now, consider left composition with the Fourier integral operator F :

$$F \circ P(x, y) = \int F(x, z)P(z, y)dz \quad (6.91)$$

$$= \int F(x, z)\delta(y - w)P(z, y)dzdw \quad (6.92)$$

So left composition is equivalent to applying the operator with kernel $F(x, z)\delta(y - w)$. Unfortunately, this is not a Fourier integral operator as we gain additional wavefront, from the points (x, z) such that $(x, z) \in \text{supp}(F)$, of the form $(x, 0, y, 0, w, \xi, w, -\xi)$ but these are irrelevant as our assumptions on the wavefront of $\Lambda_0 \cup \Lambda_1^e$ mean that these do not affect the singularities of the composition. The rest of the wavefront set is $\Gamma'_{f \times \text{Id}}$ and so our left composition is equivalent to applying an Fourier integral operator associated to $\Gamma'_{f \times \text{Id}}$ with the symbol lifted in the extra variables.

A similar argument shows that left composition with G is equivalent to applying a Fourier integral operator associated to $\Gamma'_{\text{Id} \times f}$ and so composing these two operators, we are applying a Fourier integral operator associated to $\Gamma_{f \times f}$ with principal symbol 1 and our result follows. \square

Clearly, a microlocal version of this theorem also holds.

7. A WIDER CLASS OF POLYHOMOGENEOUS DISTRIBUTIONS

In this section, we develop a wider class than that in the previous section and show it has similar properties. We obtain the last steps in the proof of theorem 6.2 as a corollary. The essential difference between our two classes is that we allow the expansion of the total symbol on $x_1 > 0$ to contain negative integral powers of x_1 . We do this by replacing the $\chi_+^{-k}(x_1) = \delta^{(k-1)}(x_1)$ distributions by the $x_{1,+}^{-k}$ distributions. (See [6] for a discussion of these distributions.) There are many different equivalent ways to define $x_{1,+}^{-k}$, we take

$$\langle x_{1,+}^{-k}, f(x_1) \rangle = \frac{(-1)^k}{(k-1)!} \int_0^\infty \log(x_1) \left(\frac{\partial}{\partial x_1} \right)^k f(x_1) dx_1 + \frac{f^{(k-1)}(0)}{(k-1)!} \sum_{j=0}^{k-1} \frac{1}{j} \quad (7.1)$$

An alternative way is to define it as the finite part of $x_{1,+}^s$ as $s \rightarrow -k$. For us, its important properties are :

- (1) $\left(\frac{\partial}{\partial x_1} + k \right) (x_{+,1}^{-k}) = C_k \delta^{(k-1)}(x_1), C_k \neq 0$
- (2) $\left(\frac{\partial}{\partial x_1} + k \right)^2 (x_{+,1}^{-k}) = 0$
- (3) $\text{supp}(x_{+,1}^{-k}) = \{x_1 \geq 0\}$
- (4) $x_{+,1}^{-k} = x_1^{-k}$ on $x_1 > 0$
- (5) $\widehat{x_{+,1}^{-k}}(\xi) = C'_k \xi_1^{k-1} \log(\xi_1 - i0) + C''_k \xi_1^{k-1}$

The first two properties express the fact that $x_{1,+}^{-k}$ is polyhomogeneous but not homogeneous which will allow us to distinguish between elements of the new calculus and the old. For s not a negative integer $x_{1,+}^s$ is just equal $\Gamma(s+1)\chi_+^s$ so there is little difference. It is the poles of the Gamma function that cause the differences at negative integers.

We define our symbols similarly to those in in the previous section.

Definition 7.1.

$$\tilde{T}^{k,r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e) = \begin{cases} T^{k,r} & \text{if } r \text{ is non-integral} \\ T^{k,r} + \log(\xi_1 - i0)T^{k,r} & \text{if } r \text{ is a positive integer} \\ T^{k,r} + \log(\xi_1 - i0)T^{k,0} & \text{if } r \text{ is a negative integer} \end{cases}$$

Definition 7.2. $\tilde{S}_{sing}^{\mu,r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ is the collection of formal sums $\sum_{j=0}^{\infty} a_{\mu-j}$ where $a_{\mu-j} \in T^{\mu-j,r-j}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$.

Definition 7.3. $\tilde{T}^{k,r}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ equals the space of $a \in C^\infty(\mathbb{R}_{x_1} \times \mathbb{R}_{\xi''}^{n-1}, x_1 > 0)$ such that a is homogeneous of degree k in ξ'' and there exists a sequence of functions b_j homogeneous of degree k for which

$$\left| D_{x_1}^{\alpha_1} D_{\xi''}^{\alpha''} \left(a(x_1, \xi'') - \sum_{j=0}^{N-1} x_{1,+}^{j+r} b_j(\xi'') \right) \right| \leq C_{\alpha,N} |\xi|^{k-|\alpha''|} |x_1|^{r+N-\alpha_1}. \quad (7.2)$$

Definition 7.4. $\tilde{S}_{sing}^{\mu,r}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ is the collection of formal sums $\sum_{j=0}^{\infty} a_{\mu-j}$ where $a_{\mu-j} \in \tilde{T}^{\mu-j,r-j}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$.

For r non-integral our classes are of course identical to those in the previous section. If we cut them off we obtain similar estimates but gain an arbitrarily small order from the log terms; we let ϵ an arbitrarily small positive quantity.

Lemma 7.1. Let ϕ be a cut off function then if $a \in \tilde{T}_{sing}^{\mu,r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$, we have

$$(1 - \phi) \left(\frac{\xi_1}{|\xi''|^{1/2}} \right) a(\xi) \in S_{ma}^{m+\epsilon, r/2}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e). \quad (7.3)$$

Lemma 7.2. Let ϕ be a cut off function then if $b \in \tilde{T}_{sing}^{\mu,r}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$, we have that

$$\int_0^\infty b(y_1, \xi'') |\xi''|^{1/2} \hat{\phi}((x_1 - y_1) |\xi''|^{1/2}) dy_1 \in S_{ma}^{m+\epsilon, -\frac{r}{2}}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e). \quad (7.4)$$

The proofs are very similar to those in the previous case. As before we can now transmute our formal sums into asymptotic sums and quantize, subject to a compatibility condition.

Definition 7.5. A paired total symbol with log terms of order (μ, ν) is a pair of formal sums

$$\left(\sum a_{\mu-j}, \sum b_{\nu-j} \right)$$

such that

$$\sum a_{\mu-j} \in \tilde{S}_{sing}^{\mu, \mu-\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e), \sum b_{\nu-j} \in \tilde{S}_{sing}^{\nu, \nu-\mu-1}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$$

and such that the Fourier transform in x_1 induces a bijection between their respective expansions at $\xi_1 = 0$ and $x_1 = 0$.

Theorem 7.1. There is a quantization map from paired total symbols with log terms of order (μ, ν) to $I^{m-\frac{n}{4}+, p+\frac{1}{2}-\frac{n}{4}+}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$ determined up to smooth terms.

The proof is as before and we denote the range $\tilde{J}_{phg}^{\mu-\frac{n}{4}+, \nu+\frac{1}{2}-\frac{n}{4}+}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$.

Definition 7.6. Let (Λ_0, Λ_1^e) be a Lagrangian pair then $u \in \tilde{I}_{phg}^{\mu+, \nu+}(\Lambda_0, \Lambda_1^e, \Omega^{\frac{1}{2}})$ if and only if for any pair of radial operators for (Λ_0, Λ_1^e)

$$\left(\prod_{j=0}^{N-1} (R_0 + \mu - j)^2 \prod_{k=0}^{M-1} (R_1 + \nu - k) \right) u \in I^{m-N+, p-M+}(\Lambda_0, \Lambda_1^e, \Omega^{\frac{1}{2}}) \text{ for all } M, N.$$

Theorem 7.2. $\tilde{I}_{phg}^{\mu+, \nu+}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e) = \tilde{J}_{phg}^{\mu+, \nu+}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$.

Proof. For $J \subset I$, we argue the same as in theorem 6.2 the point being that we require two applications of the radial operator on $\tilde{\Lambda}_0$, to kill terms of the form $x_{1,+}^{-k}$ and one application of the radial operator will never kill such terms.

To prove $I \subset J$ the proof is identical to that of theorem 6.2 except that there is now no problem with the negative integral terms. \square

We can now complete the proof of theorem 6.2 for the case $\nu - \mu - \frac{1}{2}$ is a negative integer. If $u \in I_{phg}^{\mu,\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$ then it is clearly in $\tilde{I}_{phg}^{\mu,\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$ which is equal to $\tilde{J}_{phg}^{\mu,\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$. So, all we need do is show that the coefficients of the $x_{1,+}^{-k}$ terms vanish but as noted above, it will require two applications of the radial operator plus the requisite constant to kill such terms so they are not in $I_{phg}^{\mu,\nu}(\tilde{\Lambda}_0, \tilde{\Lambda}_1^e)$ and we are done.

8. THE SYMBOL MAP

We have defined a class of polyhomogeneous, paired Lagrangian distributions associated to any intersecting pair (Λ_0, Λ_1^e) and established their Fourier integral operator invariance and mapping properties under application of Fourier integral operators associated to symplectomorphisms. It remains to define their symbols. Now away from (Λ_0, Λ_1^e) we have classical Lagrangian distributions so it is clear how to proceed there: the symbols are just the ordinary principal symbols which are homogeneous sections of the Maslov bundle tensored with the half-density bundle. Our construction allows us to take this pair of symbols to be the principal symbol of our distribution provided we define a compatibility condition and make sense of them correctly as distributions as we approach the intersection. It is this property which makes our class an improvement on that given in [4]: we can construct a distribution with specified symbol on both Lagrangians, given a compatibility condition, which is uniquely defined up to distributions of one lower order everywhere.

In this section we shall regard Λ_1^e as a subset of a Lagrangian submanifold without boundary Λ_1 . This is true for our model and therefore can be done locally in general. All we really need is the tangent space at the boundary. Now $\partial\Lambda_1^e$ is $\Lambda_0 \cap \Lambda_1$; $\partial\Lambda_1^e$ is an isotropic submanifold of the symplectic manifold $T^*(X) - 0$ and the symplectic form, ω , therefore induces a non-degenerate pairing:

$$\omega : \frac{T_p(\Lambda_0)}{T_p(\partial\Lambda_1^e)} \times \frac{T_p(\Lambda_1)}{T_p(\partial\Lambda_1^e)} \rightarrow \mathbb{R}. \quad (8.1)$$

Thus we have a canonical isomorphism between the conormal bundle of $\partial\Lambda_1^e$ in Λ_0 and the normal bundle of $\partial\Lambda_1^e$ in Λ_1 .

$$L_\omega : N_p^*(\partial\Lambda_1^e, \Lambda_0) \xrightarrow{\sim} N_p(\partial\Lambda_1^e, \Lambda_1) \quad (8.2)$$

We will use this to identify symbols of distributions conormal to $\partial\Lambda_1^e$, on Λ_0 , with the leading singularities of functions on Λ_1^e .

Theorem 8.1. *There is a canonical isomorphism between L_0 , the Maslov bundle over Λ_0 , and L_1 , the Maslov bundle over Λ_1^e , along $\partial\Lambda_1^e$. This isomorphism maps the natural trivialization of L_0 over $\tilde{\Lambda}_0$ to the natural trivialization of L_1 over $\tilde{\Lambda}_1^e$.*

Proof. Melrose and Uhlmann establish this isomorphism, in [10], by considering a path in the space of Lagrangian subspaces of $T_p(T^*(X))$ transversal to the fibre from $T_p(\Lambda_0)$ to $T_p(\Lambda_1^e)$ and picking a subspace μ which is transversal to the entire path and the fibre. Now an element, a , of $L_{0,p}$ is a map from the set of Lagrangian subspaces, transversal to the fibre and to $T^*(\Lambda_0)$, to \mathbb{C} which when specified on one subspace is given on all the others by transition cycles and similarly for L_1 . So defining $b(\mu) = a(\mu)$ we have an element of $L_{1,p}$. Melrose and Uhlmann show that it is choice independent.

We will use a slightly different map. The problem with the one just given is that it does not map the natural trivialization of $N^*(x = 0)$ to the natural trivialization of $N^*(x'' = 0)$. These natural trivializations come from their nature as conormal bundles; it is shown in [5] that the Maslov bundle has a natural trivialization over a submanifold on which the dimension of the tangent space to the fibre intersecting that of the Lagrangian submanifold is constant. For L_0 the functions are, in fact, constant so evaluation yields a constant. For L_1 , after picking a subspace μ , the trivialization is,

$$e^{i\sigma(\lambda_1, \lambda_2; \mu)} f(\mu)$$

where $\lambda_1 = T_p(\tilde{\Lambda}_1^e)$, $\lambda_2 = T_p(\tilde{\Lambda}_0)$, f is the map defining the element of the bundle and $\sigma(\lambda_1, \lambda_2; \mu)$ is the cross ratio from [5]. This value is independent of μ . Putting $\mu' = \langle \{\frac{\partial}{\partial x_j}\}_{j>1} \rangle + \langle \{\frac{\partial}{\partial x_1} - \frac{\partial}{\partial \xi_1}\} \rangle$ we get

$$e^{-i\frac{\pi}{4}} b(\mu')$$

and μ' satisfies the hypotheses used by Melrose and Uhlmann to define the isomorphism. Thus we multiply their isomorphism by $e^{-i\frac{\pi}{4}}$ to obtain an isomorphism which matches the canonical trivializations in this case. \square

The principal symbols of our distributions will be pull backs by symplectomorphisms of principal symbols in the model case so we need to examine the properties of the principal symbols in that case and recast their properties invariantly: we have $a(\xi)|d\xi|^{\frac{1}{2}}$ and $b(x_1, \xi'')|dx_1|^{\frac{1}{2}}|d\xi|^{\frac{1}{2}}$ where $a \in T^{\mu-\frac{n}{4}, \mu-\nu-\frac{1}{2}}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ and $b \in T^{\nu+\frac{1}{2}-\frac{n}{4}, \nu-\mu-\frac{1}{2}}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$. We want to define these spaces for a general intersecting pair. Our definitions are similar to those in [4].

Proposition 8.1. *The space $T^{q,r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ is invariant under homogeneous diffeomorphisms which preserve $\xi_1 = 0$ and $\xi_1 > 0$.*

Proof. It is clear that a function that is homogeneous and smooth off $\xi_1 = 0$ will retain these properties under such diffeomorphisms so we need to check the effects on the expansion at $\xi_1 = 0$. We thus need only consider behaviour in a small conic neighbourhood of $\xi_1 = 0$.

Now, $a(\xi)$ is an element of $T^{q,r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ if and only if it is homogeneous of degree q and near $\xi_0 = 0$ has an expansion for each N of the form

$$a(\xi) = \sum_{j=0}^{N-1} (\xi_1 - i0)^{j+r} b_j(\xi'') + O(\xi_1^{r+N}). \quad (8.3)$$

Homogeneous changes of coordinates in ξ'' will have no affect so we can assume that the set $\xi_1 = 0$ is fixed. So writing $\xi = \xi(\eta)$, we have $\xi(0, \eta'') = (0, \eta'')$. Now

our hypothesis on preserving $\xi_1 > 0$ means that $\frac{\partial \xi_1}{\partial \eta_1}(0, \xi'') > 0$ on $\xi_1 = 0$. So we can write, $\xi_1(\eta) = \eta_1 \tilde{\xi}_1(\eta'') + \eta_1^2 r_1(\eta)$, with $\tilde{\xi}_1(\eta'')$ homogeneous of degree zero and positive. So, if we do the change of coordinates $\xi_1(\eta) = \eta_1 \tilde{\xi}_1(\eta'')$, $\xi'' = \eta''$, which will just multiply the coefficients of our expansion, we are reduced to the case $\xi_1(\eta) = \eta_1(1 + \eta_1 r_1(\eta))$, $\xi''(\eta) = \eta'' + \eta_1 r''(\eta)$.

So our expansion takes on the form

$$a(\eta) = \sum_{j=0}^{N-1} (\eta_1 - i0)^{j+r} (1 + \eta_1 r_1(\eta))^{j+r} b_j(\eta'' + \eta_1 r''(\eta)) + O(\eta_1^{r+N}) \quad (8.4)$$

but $(1 + \eta_1 r_1(\eta))^{j+r}$ is smooth, near $\eta_1 = 0$, so taking a Taylor expansion about $\eta_1 = 0$ it will be subsumed by the higher order terms and we do the same for $b_j(\eta'' + \eta_1 r''(\eta))$. \square

Proposition 8.2. *The space $T^{q,r}(\tilde{\Lambda}_1^e, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$ is invariant under homogeneous diffeomorphisms which preserve $x_1 = 0$ and $x_1 > 0$.*

Proof. This is essentially the same as proposition 8.1. \square

Now, we know from Theorem 2.1 that there always exist homogeneous symplectic coordinates, (x, ξ) , near a point such that $\Lambda_0 = \{x = 0\}$ and $\Lambda_1^e = \{x_1 \geq 0, x'' = 0, \xi_1 = 0\}$. These put coordinates ξ on Λ_0 and (x_1, ξ'') on Λ_1^e . The change of coordinates obtained by differing symplectomorphisms will satisfy the hypotheses of propositions 8.1, 8.2.

Definition 8.1. *Let (Λ_0, Λ_1^e) be an intersecting pair then $T^{q,r}(\Lambda_0, \Lambda_1^e)$ is the set of smooth homogeneous functions, a , on $\Lambda_0 - \Lambda_1^e$ such that in a conic neighbourhood of any point, there exists homogeneous symplectic coordinates on $T^*M - 0$ which reduce (Λ_0, Λ_1^e) to the model such that in the induced coordinates on Λ_0 , $a \in T^{q,r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$. If L is a complex line bundle over Λ_0 then we define $T^{q,r}(\Lambda_0, \Lambda_1^e, L)$ to be the space of*

smooth sections, s , of L over $\Lambda_0 - \partial\Lambda_1^e$ such that for every homogeneous degree zero section, s^* , of the dual bundle, L^* , $\langle s(x), s^*(x) \rangle \in T^{q,r}(\Lambda_0, \Lambda_1^e)$.

Definition 8.2. Let (Λ_0, Λ_1^e) be an intersecting pair then $T^{q,r}(\Lambda_1^e, \Lambda_0)$ is the set of smooth homogeneous functions, b , on $\Lambda_1^e - \Lambda_0$ such that in a conic neighbourhood of any point, there exist homogeneous symplectic coordinates on $T^*M - 0$ which reduce (Λ_0, Λ_1^e) to the model such that in the induced coordinates on Λ_1^e , $a \in T^{q,r}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^e)$. If L is a complex line bundle over Λ_1^e then we define $T^{q,r}(\Lambda_1^e, \Lambda_0, L)$ to be the space of smooth sections, s , of L over $\Lambda_1^e - \partial\Lambda_1^e$ such that for any homogeneous degree zero section, s^* , of the dual bundle, L^* , $\langle s(x), s^*(x) \rangle \in T^{q,r}(\Lambda_1^e, \Lambda_0)$.

We therefore have if $u \in I_{phg}^{\mu,\nu}(\Lambda_0, \Lambda_1^e)$ that

$$a_0 = \sigma(u|_{\Lambda_0}) \in T^{\mu+\frac{n}{4}, \mu-\nu-\frac{1}{2}}(\Lambda_0, \Lambda_1^e, L_0 \otimes \Omega^{\frac{1}{2}}), \quad (8.5)$$

$$a_1 = \sigma(u|_{\Lambda_1^e}) \in T^{\nu+\frac{n}{4}, \nu-\mu-\frac{1}{2}}(\Lambda_1^e, \Lambda_0, L_1 \otimes \Omega^{\frac{1}{2}}). \quad (8.6)$$

We still need to express the compatibility condition between our leading symbols. The conditions given on the space $T^{\mu+\frac{n}{4}, \mu-\nu-\frac{1}{2}}(\Lambda_0, \Lambda_1^e, L_0 \otimes \Omega^{\frac{1}{2}})$ mean that its elements are distributions which are conormal to $\partial\Lambda_1^e$. This means that they have a well-defined symbol on $N^*(\partial\Lambda_1^e, \Lambda_0)$. This symbol is a section of $N^*(\partial\Lambda_1^e, \Lambda_0, \Omega^{\frac{1}{2}})$ of homogeneity $\nu - \mu - \frac{1}{2}$. We established above an isomorphism, L_ω , from $N^*(\partial\Lambda_1^e, \Lambda_0)$ to $N(\partial\Lambda_1^e, \Lambda_1^e)$ which means that we can regard our symbol as a section of $N(\partial\Lambda_1^e, \Lambda_1^e, \Omega^{\frac{1}{2}})$. It is clear from the nature of the singularity that the symbol will be wholly supported on the part of $N(\partial\Lambda_1^e, \Lambda_1^e)$ pointing into Λ_1^e .

Now as in [4], we can define a space $W^{q,r}(\Lambda_1^e, \partial\Lambda_1^e)$ to express the leading singularities of a_1 as it approaches the intersection.

Definition 8.3. Let L be a complex line bundle over $N(\partial\Lambda_1^e, \Lambda_1) - 0$ then if r is not a negative integer $W^{q,r}(\Lambda_1^e, \partial\Lambda_1^e, L)$ is the set of smooth sections of $(N(\partial\Lambda_1^e, \Lambda_1) - 0, L)$ which are zero on the vectors pointing out of Λ_1^e , are homogeneous of degree q with

respect to the radial action of $\partial\Lambda_1^e$ and are homogeneous of degree r on the fibres of $N(\partial\Lambda_1^e, \Lambda_1) - 0$. If r is a negative integer we define $W^{q,r}(\Lambda_1^e, \partial\Lambda_1^e, L)$ to be the zero section of $(N(\partial\Lambda_1^e, \Lambda_1) - 0, L)$.

As in [4], there exists a short exact sequence

$$0 \rightarrow T^{q,r+1}(\Lambda_1^e, \partial\Lambda_1^e, \Omega^{\frac{1}{2}} \otimes L_1) \rightarrow T^{q,r}(\Lambda_1^e, \partial\Lambda_1^e, \Omega^{\frac{1}{2}} \otimes L_1) \xrightarrow{\beta_{q,r}} W^{q,r}(\partial\Lambda_1^e, \Lambda_1^e, \Omega^{\frac{1}{2}} \otimes L_1) \rightarrow 0. \quad (8.7)$$

Remember that if r is a negative integer then $T^{q,r+1} = T^{q,r}$.

Definition 8.4. $S^{\mu,\nu}(\Lambda_0, \Lambda_1^e)$ is the set of pairs $(a_0, a_1) \in T^{\mu,\mu-\nu-\frac{1}{2}}(\Lambda_0, \Lambda_1^e, L_0 \otimes \Omega^{\frac{1}{2}}) \times T^{\nu,\nu-\mu-\frac{1}{2}}(\Lambda_1^e, \Lambda_0, L_1 \otimes \Omega^{\frac{1}{2}})$ such that $L_\omega^* \beta_{\nu,\nu-\mu-\frac{1}{2}}(a_1) = \sigma_{\partial\Lambda_1^e}(a_0)$ with respect to the canonical identification of Maslov bundles.

Theorem 8.2. *There are three short exact sequences*

$$0 \rightarrow I_{phg}^{\mu-1,\nu-1}(\Lambda_0, \Lambda_1^e) \rightarrow I_{phg}^{\mu,\nu}(\Lambda_0, \Lambda_1^e) \xrightarrow{\sigma_{\mu,\nu}} S^{\mu+\frac{n}{4},\nu+\frac{n}{4}}(\Lambda_0, \Lambda_1^e) \rightarrow 0, \quad (8.8)$$

$$0 \rightarrow I_{phg}^{\mu-1,\nu}(\Lambda_0, \Lambda_1^e) \rightarrow I_{phg}^{\mu,\nu}(\Lambda_0, \Lambda_1^e) \xrightarrow{\sigma_{\mu,\nu-\frac{1}{2}}^0} T^{\mu+\frac{n}{4},\mu-\nu-\frac{1}{2}}(\Lambda_0, \Lambda_1^e, \Omega^{\frac{1}{2}} \otimes L_0) \rightarrow 0 \quad (8.9)$$

and

$$0 \rightarrow I_{phg}^{\mu,\nu-1}(\Lambda_0, \Lambda_1^e) \rightarrow I_{phg}^{\mu,\nu}(\Lambda_0, \Lambda_1^e) \xrightarrow{\sigma_{\nu,\nu-\mu-\frac{1}{2}}^1} T^{\nu+\frac{n}{4},\nu-\mu-\frac{1}{2}}(\Lambda_1^e, \Lambda_0, \Omega^{\frac{1}{2}} \otimes L_1) \rightarrow 0. \quad (8.10)$$

Proof. We define $\sigma_{\mu,\nu}$ to be the pair of principal symbols as a Lagrangian distribution on $(\Lambda_0 - \partial\Lambda_1^e, \Lambda_1^e - \partial\Lambda_1^e)$. It is enough to discuss the model case. The only property which is not then wholly obvious is that the kernel of $\sigma_{\mu,\nu}$ is contained in $I_{phg}^{\mu-1,\nu-1}(\Lambda_0, \Lambda_1^e)$. The reason this is true is that if the top order terms of a paired total symbol are identically zero then this forces the leading singularities of the terms of all orders to be zero. \square

Remark 1. If $\mu - \nu - \frac{1}{2}$ is a positive integer then the space $T^{\mu, \mu - \nu - \frac{1}{2}}(\Lambda_0, \Lambda_1^e, L_0 \otimes \Omega^{\frac{1}{2}})$ consists of sections that are smooth right up to the boundary and therefore have zero symbols when regarded as conormal distributions. This is reflected in our definition of $T^{\nu, \nu - \mu - \frac{1}{2}}(\Lambda_1^e, \Lambda_0, L_1 \otimes \Omega^{\frac{1}{2}})$ which requires the leading singularities to be zero when $\nu - \mu - \frac{1}{2}$ is a negative integer.

Remark 2. In the case where $\mu - \nu - \frac{1}{2}$ is not integral, we could alternatively define $T^{\mu, \mu - \nu - \frac{1}{2}}(\Lambda_0, \Lambda_1^e, L_0 \otimes \Omega^{\frac{1}{2}})$ to be the space of sections of $L_0 \otimes \Omega^{\frac{1}{2}}$ which are classical conormal of order $\nu - \mu - \frac{1}{2} + \frac{n-1}{2}$ to $\partial\Lambda_1^e$ and have wavefront set contained in the part of $N(\partial\Lambda_1^e, \Lambda_1^e)$ pointing into Λ_1^e .

9. PSEUDO-DIFFERENTIAL OPERATORS WITH SINGULAR SYMBOLS

The original reason why paired Lagrangian distributions were invented by Melrose and Uhlmann in [10] was to allow a symbolic construction of parametrices for operators of real principal type. These parametrices can be thought of as pseudo-differential operators with singular symbols: the symbol of the parametrix on the conormal bundle to the diagonal blows up on approach to the characteristic variety of the original operator. The second Lagrangian in this case is the flow out of the characteristic variety by the bicharacteristic flow. Antoniano and Uhlmann showed in [1] that the Guillemin-Uhlmann calculus of [4] is closed under composition in this case and calculated the principal symbols. We show that our calculus is also closed in this case and calculate the principal symbols. We also define an ellipticity condition which allows us to construct parametrices up to smooth terms. Later on, we shall show that the complex powers of the wave operator also lie in this class.

We recall from [2]:

Definition 9.1. *Let $P \in \Psi^m(X)$ be a properly supported pseudo-differential operator. We shall say that P is of real principal type in X if P has a real homogeneous principal*

part p of order m and no complete bicharacteristic of P stays over a compact set in X .

Definition 9.2. *If P is of real principal type in X , we shall say that X is pseudo-convex with respect to P if for every compact set $K \subset X$ there is another compact set $K' \subset X$ such that K' contains any interval in the bicharacteristic curve with respect to P having both end points in K .*

Definition 9.3. *If p is a first order symbol of real principal type such that X is pseudo-convex then the flow out, Λ , associated with p is the flow out of $\Delta \cap \{p = 0\}$ by the Hamiltonian flow of p in the first variable in positive time. In this case, we shall call Λ a flow out Lagrangian.*

As in [2], Λ will be a homogeneous canonical relation. Note that $\tilde{\Lambda}$ is the flow out associated to $\frac{\partial}{\partial x_1}$.

If Λ is a flow out, then in terms of canonical relations, we have that $\Delta \circ \Lambda = \Lambda = \Lambda \circ \Delta$ and so no new Lagrangians are generated when composing elements of $I(\Delta, \Lambda)$.

As we are considering pseudo-differential operators, it is more convenient to use a model in which one of the Lagrangians is the conormal bundle to the diagonal. So we take

$$\Delta = N^*(x = y), \tilde{\Lambda} = N^*(x'' = y'', x_1 \geq y_1) \subset T^*(\mathbb{R}^n \times \mathbb{R}^n) - 0. \quad (9.1)$$

The second Lagrangian is the one associated to ξ_1 . In this case our radial operators can be taken to be the vector fields

$$R_\Delta = (x - y) \frac{\partial}{\partial x} + n, R_{\tilde{\Lambda}} = (x'' - y'') \frac{\partial}{\partial x''} + (n - 1) + \frac{1}{2}. \quad (9.2)$$

Pseudo-differential operators are much simpler objects than Lagrangian distributions when it comes to making invariant sense of the principal symbol. The symplectic

form induces a natural half-density on the conormal bundle to the diagonal and the structure of conormality to the diagonal induces a natural trivialization of the Maslov bundle. Thus we can regard the symbol on the diagonal as being a simple function. On the flow out, the special structure can also be used to simplify the half-density and Maslov bundles but in a less direct fashion. The dimension of the intersection of the tangent space to Λ and the tangent space of the fibre will always be $n - 1$. This is true because the Hamiltonian vector field will not be tangent to the fibre, and the dimension of the tangent space of the conormal bundle intersected with the fibre will be n and p will define a hypersurface in the conormal bundle to the diagonal. Thus, as the dimension of the intersection with the fibre is constant, there will be a natural trivialization of the Maslov bundle along $\Delta \cap \Lambda$; this can be extended to the entire flow out by transporting along the bicharacteristic flow. We will always use this trivialization; for $\tilde{\Lambda}$ this will agree with the natural trivialization from its conormality. Another way to arrive at this trivialization is to consider the natural trivialization on the diagonal and use the isomorphism of Maslov bundles to give a trivialization above Λ . We have defined the isomorphism so that these will agree in the model case and hence by functoriality will agree in general.

Suppose Λ is the flow out associated with p , then picking coordinates z on $\Delta \cap \{p = 0\}$, we can write

$$\alpha(z, p)|dz||dp| = |dx||d\xi| \quad (9.3)$$

where $|dx||d\xi|$ is the symplectic density and α is a positive real function. Now, we can define coordinates (z, s) on Λ by

$$(z, s) \mapsto \exp(sH_p)(z) \quad (9.4)$$

and therefore can define a density

$$\mu_p = \alpha(z, 0)|ds||dz|. \quad (9.5)$$

This is not intrinsic as it depends upon the choice of p . It is easily checked to be independent of the choice of z , however. We examine the dependence on p ; suppose $\tilde{p} = \beta p$ for some non zero function β . Then,

$$\mu_{\tilde{p}} = \frac{\alpha(z, 0)}{\beta(z, 0)\beta(z, s)} |dz| |ds| = \frac{1}{\beta(z, 0)\beta(z, s)} \mu_p. \quad (9.6)$$

The $\beta(z, 0)$ factor comes from the scaling of α on the diagonal and the $\beta(z, s)$ factor from the scaling of the Hamiltonian flow. This class of half-densities allows us to define an intrinsic convolution along the bicharacteristics.

Proposition 9.1. *There exists a natural convolution of half-densities on Λ defined by*

$$(a\mu_p^{\frac{1}{2}} * b\mu_p^{\frac{1}{2}})(r_1, r_0) = \int_0^t a(r_1, \gamma(s))b(\gamma(s), r_0)ds\mu_p^{\frac{1}{2}} \quad (9.7)$$

where γ is the bicharacteristic of p such that $\gamma(t) = r_1$ and $\gamma(0) = r_0$.

Proof. We need to check that this is independent of the choice of p . If we take \tilde{p} , as above, instead then

$$a(q_1, q_0)\mu_{\tilde{p}}^{\frac{1}{2}} = a\beta(q_1)^{\frac{1}{2}}\beta(q_0)^{\frac{1}{2}}\mu_p^{\frac{1}{2}}$$

and similarly for b . Letting $\tilde{\gamma}$ be the relevant bicharacteristic of \tilde{p} we get for the convolution with respect to \tilde{p}

$$\int_0^{t'} a(r_1, \tilde{\gamma}(\tilde{s}))b(\tilde{\gamma}(\tilde{s}), r_0)\beta(r_1)^{\frac{1}{2}}\beta(r_0)^{\frac{1}{2}}\beta(\tilde{\gamma}(\tilde{s}))d\tilde{s}\mu_{\tilde{p}}^{\frac{1}{2}} \quad (9.8)$$

which upon executing the change of variables $\tilde{s} = \tilde{s}(s)$ where s is the flow out variable for p becomes the convolution with respect to p . \square

This convolution will yield the value of the principal symbol of a composition on the flow out. This also makes sense for singular symbols provided we interpret their singularities as $\chi_+^\alpha(s - t)$ distributions. Convolutions will always be well defined as the singularities of the two factors will occur at opposite ends of the bicharacteristic.

However we are still several steps away from calculating the principal symbol. We cut our paired Lagrangians up in order to examine the composition:

Proposition 9.2. *If $K_1 \in I^{m_1, p_1}(\Delta, \Delta \cap \tilde{\Lambda})$ and $K_2 \in I^{m_2, p_2}(\Delta, \Delta \cap \tilde{\Lambda})$ and are properly supported then $K_1 \circ K_2 \in I^{m_1+m_2, p_1+p_2}(\Delta, \Delta \cap \tilde{\Lambda})$.*

Proof. Recalling theorem 3.2, we can represent K_j as

$$\int e^{i\langle x-y, \xi \rangle} a_j \left(\frac{x+y}{2}, \xi \right) d\xi + C^\infty \quad (9.9)$$

where $|D_x^\alpha D_\xi^\beta a_j(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-|\alpha|} \left(\frac{\langle \xi \rangle}{\langle \xi \rangle_{(1)}} \right)^{2p} \langle \xi \rangle_{(1)}^{-\alpha_1}$. Now this means that K_j is a type $(\frac{1}{2}, 0)$ pseudo-differential operator of order $M_j = \max\{m_j, m_j + p_j\}$.

We can therefore apply Hörmander's Weyl calculus (see [7]) and conclude that $K_1 \circ K_2$ is a type $(\frac{1}{2}, 0)$ pseudo-differential operator of order $M_1 + M_2$ with total symbol $a_1 \# a_2$ which has an asymptotic expansion in $S_{\frac{1}{2}, 0}$:

$$\sum_j \left(\frac{(i(D_x D_\eta - D_\xi D_y))^j}{j!} \right) a_1(x, \xi) a_2(y, \eta) \text{ evaluated at } (x, \xi) = (y, \eta). \quad (9.10)$$

Now the j^{th} term here is in $S_{ma}^{m_1+m_2-\frac{j}{2}, p_1+p_2}(\Delta, \Delta \cap \tilde{\Lambda})$ and so it follows that $a_1 \# a_2 \in S_{ma}^{m_1+m_2, p_1+p_2}(\Delta, \Delta \cap \tilde{\Lambda})$; to get the derivative estimates we just take sufficiently large number of terms in the expansion that the derivative estimates in $S_{\frac{1}{2}, 0}$ of the remainder are of lower order than the requisite ones. \square

Proposition 9.3. *Let $K \in I^{m, r}(\Delta, \Delta \cap \tilde{\Lambda})$ be properly supported and suppose L is in $I^{p, s}(\tilde{\Lambda}, \Delta \cap \tilde{\Lambda})$ then $K \circ L, L \circ K \in I^{m+r+p, s}(\tilde{\Lambda}, \Delta \cap \tilde{\Lambda})$.*

Proof. It is enough to prove this for $K \circ L$ as, taking adjoints, it will then follow for $L \circ K$.

Recalling theorem 3.5, we can write

$$L(z, y) = \int e^{i\langle z''-x'', \xi'' \rangle} b \left(\frac{z_1+y_1}{2}, \frac{z_1-y_1}{2}, y'', \xi'' \right) d\xi'' \quad (9.11)$$

where $|D_s^k D_t^l D_{y''}^{\alpha''} D_{\xi''}^{\beta''} b(s, t, y'', \xi'')| \leq C_{k,l,\alpha'',\beta''} \langle \xi'' \rangle^{p+\frac{1}{2}-|\beta''|} \left(\frac{\langle \xi'' \rangle^{\frac{1}{2}}}{|t| + \langle \xi'' \rangle^{\frac{1}{2}}} \right)^{2s+l}$; we will denote this space $S^{p+\frac{1}{2},s}(\tilde{\Lambda}, \tilde{\Lambda} \cap \Delta)$. From theorem 3.2

$$K(x, z) = \int e^{i\langle x-z, \xi \rangle} a(x, \xi) d\xi \quad (9.12)$$

with $|D_x^\alpha D_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\alpha''|} \left(\frac{\langle \xi \rangle}{\langle \xi \rangle_{(1)}} \right)^{2r} \langle \xi \rangle_{(1)}^{-\alpha_1}$. Putting these together we obtain,

$$K \circ L = \int e^{i\langle x''-y'', \xi'' \rangle} c(x, y, \xi'') d\xi'' \quad (9.13)$$

where

$$c(x, y, \xi'') = \int e^{i\langle x_1-z_1, \xi_1 \rangle} a(x, \xi) b\left(\frac{z_1+y_1}{2}, \frac{z_1-y_1}{2}, y'', \xi''\right) dz_1 d\xi_1 \quad (9.14)$$

but we want our integral to be of the same form as that expressing L , so letting $c(x, y, \xi'') = \tilde{c}\left(\frac{x_1+y_1}{2}, \frac{x_1-y_1}{2}, x'', y'', \xi''\right)$, we have

$$\begin{aligned} \tilde{c}(s, t, x'', y'', \xi'') &= \int e^{i\langle s+t-z_1, \xi_1 \rangle} a(s+t, x'', \xi) \\ &\quad b\left(\frac{z_1+s-t}{2}, \frac{z_1-s+t}{2}, y'', \xi''\right) d\xi_1 dz_1. \end{aligned} \quad (9.15)$$

Letting $\tilde{z}_1 = z_1 - s$, we have,

$$\tilde{c}(s, t, x'', y'', \xi'') = \int e^{i\langle t-\tilde{z}_1, \xi_1 \rangle} a(s+t, x'', \xi) b\left(\frac{\tilde{z}_1+2s-t}{2}, \frac{\tilde{z}_1+t}{2}, y'', \xi''\right) d\xi_1 dz_1. \quad (9.16)$$

So in the language of Gauss transforms,

$$\tilde{c}(s, t, x'', y'', \xi'') = e^{iD_{\tilde{z}_1} D_{\xi_1}} \left(a(s+t, x'', \xi) b\left(\frac{\tilde{z}_1+2s-t}{2}, \frac{\tilde{z}_1+t}{2}, y'', \xi''\right) \right) \Big|_{\xi_1=0, \tilde{z}_1=t} \quad (9.17)$$

We now want to apply Hörmander's results on Gauss transforms to conclude that \tilde{c} is in $S^{p+m+r+\frac{1}{2},s}(\tilde{\Lambda}, \tilde{\Lambda} \cap \Delta)$ with some extra parameters. (For a discussion of Gauss transforms see [7].) To do so, we must identify the symbol space in which our operand lies in terms of a slowly varying metric, g , and a g -continuous weight function, m ,

and then show that this space restricts to yield $S^{p+m+r+\frac{1}{2},s}(\tilde{\Lambda}, \tilde{\Lambda} \cap \Delta)$ with the extra parameters. We can take

$$g = ds^2 + (dx'')^2 + (dy'')^2 + \frac{(d\xi'')^2}{1 + |\xi|^2} + \frac{d\xi_1^2}{\langle \xi \rangle_{(1)}^2} + \left(\frac{\langle \xi \rangle^{\frac{1}{2}}}{1 + \langle \xi \rangle^{\frac{1}{2}} |z_1 + t|} \right)^2 dt^2 + \left(\frac{\langle \xi \rangle^{\frac{1}{2}}}{1 + \langle \xi \rangle^{\frac{1}{2}} |z_1 + t|} \right)^2 dz_1^2, \quad (9.18)$$

$$m = \langle \xi \rangle^p \left(\frac{\langle \xi \rangle}{\langle \xi \rangle_{(1)}} \right)^{2r} \left(\frac{\langle \xi \rangle^{\frac{1}{2}}}{1 + \langle \xi \rangle^{\frac{1}{2}} |z_1 + t|} \right)^{2s}. \quad (9.19)$$

Note that the restriction of these to $z_1 = 0, t = 0$ yields the requisite space. Since the coefficients of $d\xi_j^2$ are between $\langle \xi \rangle^{-1}$ and $\langle \xi \rangle^{-2}$ and $\langle \xi \rangle^s, \langle \xi \rangle_{(1)}^s$ are continuous with respect to the metrics $\langle \xi \rangle^{-2} d\xi^2, \langle \xi \rangle^{-1} d\xi^2$ they are also g -continuous. Thus to check the slow variation of g and the g -continuity of m , it is sufficient to show that $1 + |t + z_1| \langle \xi'' \rangle^{\frac{1}{2}}$ is g -continuous. This is equivalent to showing that $\langle \xi'' \rangle^{-\frac{1}{2}} + |t + z_1|$ is g -continuous. As $\langle \xi'' \rangle^{-\frac{1}{2}}$ is g -continuous, we have that for $g_{\xi, z_1, t}(\tilde{\xi}, \tilde{z}_1, \tilde{t}) < c$

$$\langle \xi + \tilde{\xi} \rangle^{-\frac{1}{2}} + |z_1 + \tilde{z}_1 + t + \tilde{t}| \geq C \langle \xi \rangle^{-\frac{1}{2}} + |z_1 + \tilde{z}_1 + t + \tilde{t}| \quad (9.20)$$

$$\geq C \langle \xi \rangle^{-\frac{1}{2}} + |z_1 + t| - |\tilde{z}_1| - |\tilde{t}| \quad (9.21)$$

$$\geq C'(\langle \xi \rangle^{-\frac{1}{2}} + |z_1 + t|). \quad (9.22)$$

The same argument shows the estimate from above so we have proven g -continuity.

Our symmetric bilinear form on the dual space is

$$A(z'_1, \xi'_1, z''_1, \xi''_1) = \frac{1}{2}(z'_1 \xi''_1 + z''_1 \xi'_1). \quad (9.23)$$

and thus our dual metric is

$$g^A = \infty(ds^2 + (dx'')^2 + (dy'')^2 + (d\xi'')^2 + dt^2) + (\langle \xi \rangle_{(1)}^2) dz_1^2 + \left(\frac{\langle \xi \rangle^{\frac{1}{2}} + |z_1 + t|}{\langle \xi \rangle^{\frac{1}{2}}} \right)^2 d\xi_1^2. \quad (9.24)$$

The g, A temperateness of $\langle \xi \rangle^s, \langle \xi \rangle_{(1)}$ are clear so we need only check the g, A temperateness of $|t + z_1| + \langle \xi \rangle^{\pm \frac{1}{2}}$. This follows easily from the fact that $\langle \xi \rangle^{-\frac{1}{2}}$ is g, A temperate and $g_\xi^A(z_1) \geq \langle \xi \rangle^{\frac{1}{2}} |z_1|^2$.

Thus applying the Gauss transform we have an element of the correct space with some extra z'' parameters. These can be removed by taking a Taylor expansion about $z'' = 0$ and integrating by parts. \square

Proposition 9.4. *Let $K_i \in I^{p_i, r_i}(\tilde{\Lambda}, \partial\Lambda_1^e)$ be of the form*

$$K_i = \int e^{i\langle x'' - y'', \xi'' \rangle} a \left(\frac{x_1 + y_1}{2}, \frac{x_1 - y_1}{2}, x'', \xi'' \right) d\xi'' + C^\infty \quad (9.25)$$

and with $a_i(s, t, x'', \xi'') \equiv 0$ for $t \leq 0$, be supported on $x_1 \geq y_1 - K, |x'' - y''| \leq K$ for some K then $K_1 \circ K_2 \in I^{p_1 + p_2 + \frac{1}{2} + \max\{r_1, r_2, 0\}, r_1 + r_2 - 1 - \max\{r_1, r_2, 0\}}(\tilde{\Lambda}, \partial\Lambda_1^e)$.

The extra $\max\{r_1, r_2, 0\}$ expresses the fact that the extra degree of singularity at the marking is smeared out along the Lagrangian.

Proof. Our support conditions imply that our compositions are well defined. Discarding smooth terms we can write,

$$K_1 = \int e^{i\langle x'' - y'', \xi'' \rangle} a_1 \left(\frac{x_1 + y_1}{2}, \frac{x_1 - y_1}{2}, x'', \xi'' \right) d\xi'' \quad (9.26)$$

and

$$K_2 = \int e^{i\langle y'' - z'', \xi'' \rangle} a_2 \left(\frac{y_1 + z_1}{2}, \frac{y_1 - z_1}{2}, y'', \xi'' \right) d\xi'' \quad (9.27)$$

with $a_i(s, t, y'', \xi'')$ in the space of symbols $S^{p_i + \frac{1}{2}, r_i}(\tilde{\Lambda}, \tilde{\Lambda} \cap \Delta)$ and zero for $t < 0$. Note that from our hypothesis, we have $a_2 \left(\frac{y_1 + z_1}{2}, \frac{y_1 - z_1}{2}, y'', \xi'' \right)$ for K_2 rather than dependence on z'' but we can easily go from one to the other via the standard process

of going from a general quantization to a right quantization for pseudo-differential operators i.e taking a Taylor expansion about $z'' = y''$ and integrating by parts. Thus

$$K \circ L = \int e^{i\langle x'' - z'', \xi'' \rangle} \left[\int a \left(\frac{x_1 + y_1}{2}, \frac{x_1 - y_1}{2}, x'', \xi'' \right) b \left(\frac{y_1 + z_1}{2}, \frac{y_1 - z_1}{2}, x'', \xi'' \right) dy_1 \right] d\xi'' \quad (9.28)$$

Thus, we have

$$K \circ L = \int e^{i\langle x'' - z'', \xi'' \rangle} c \left(\frac{x_1 + z_1}{2}, \frac{x_1 - z_1}{2}, x'', z'', \xi'' \right) d\xi'' \quad (9.29)$$

where

$$c(s, t, x'', z'', \xi'') = \int a \left(\frac{s + t + y_1}{2}, \frac{s + t - y_1}{2}, x'', \xi'' \right) b \left(\frac{s + t - y_1}{2}, \frac{y_1 - s + t}{2}, z'', \xi'' \right) dy_1. \quad (9.30)$$

Changing variables and invoking the support conditions in our hypothesis, this becomes

$$\frac{1}{2} \int_0^t a(y_1 + s, t - y_1, x'', \xi'') b(y_1 + s - t, y_1, z'', \xi'') dy_1. \quad (9.31)$$

So c is zero for $t \leq 0$ and for $t > 0$ we have

$$|c| \leq C \langle \xi'' \rangle^{p_1 + p_2 + 1} \int_0^t \left(\frac{\langle \xi'' \rangle^{\frac{1}{2}}}{1 + |t - y_1| \langle \xi'' \rangle^{\frac{1}{2}}} \right)^{2r_1} \left(\frac{\langle \xi'' \rangle^{\frac{1}{2}}}{1 + |y_1| \langle \xi'' \rangle^{\frac{1}{2}}} \right)^{2r_2} dy_1 \quad (9.32)$$

$$= C \langle \xi'' \rangle^{p_1 + p_2 + 1} \int_0^t \left(\langle \xi'' \rangle^{-\frac{1}{2}} + |t - y_1| \right)^{-2r_1} \left(\langle \xi'' \rangle^{-\frac{1}{2}} + |y_1| \right)^{-2r_2} dy_1. \quad (9.33)$$

We split the integral into two pieces, $I_1 + I_2$,

$$\begin{aligned}
|c(s, t, x'', z'', \xi'') < \xi'' >^{-(p_1+p_2+1)}| \leq \\
C \int_0^{t/2} \left(\frac{< \xi'' >^{\frac{1}{2}}}{1 + |t - y_1| < \xi'' >^{\frac{1}{2}}} \right)^{2r_1} \left(\frac{< \xi'' >^{\frac{1}{2}}}{1 + |y_1| < \xi'' >^{\frac{1}{2}}} \right)^{2r_2} dy_1 \\
+ C \int_{t/2}^t \left(\frac{< \xi'' >^{\frac{1}{2}}}{1 + |t - y_1| < \xi'' >^{\frac{1}{2}}} \right)^{2r_1} \left(\frac{< \xi'' >^{\frac{1}{2}}}{1 + |y_1| < \xi'' >^{\frac{1}{2}}} \right)^{2r_2} dy_1. \quad (9.34)
\end{aligned}$$

The first piece can be estimated

$$I_1 \leq C (< \xi'' >^{-\frac{1}{2}} + |t|)^{-2r_1} \int_0^{t/2} (< \xi'' >^{-\frac{1}{2}} + y_1) dy_1 \quad (9.35)$$

$$\leq C ((< \xi'' >^{-\frac{1}{2}} + |t|)^{-2r_1} t < \xi'' >^{-\frac{r}{2}}). \quad (9.36)$$

Doing the same for I_2 , we obtain

$$|c| \leq C < \xi'' >^{p_1+p_2+1} (< \xi'' >^{-\frac{1}{2}} + |t|)^{-\min\{2r_1, 2r_2\}t} < \xi'' >^{\frac{\max\{2r_1, 2r_2\}}{2}}. \quad (9.37)$$

When both r_1 and r_2 are negative a better estimate can be obtained by maximizing each of the components of the integrand:

$$|c| \leq C < \xi'' >^{p_1+p_2+1} (< \xi'' >^{-\frac{1}{2}} + |t|)^{-2r_1-2r_2} t. \quad (9.38)$$

Putting all this together and noting $t < (< \xi'' >^{-\frac{1}{2}} + |t|)$ we have

$$|c| \leq C < \xi'' >^{p_1+p_2+1+\max\{0, r_1, r_2\}} (< \xi'' >^{-\frac{1}{2}} + |t|)^{-2(r_1+r_2-1-\max\{0, r_1, r_2\})} \quad (9.39)$$

which is the result. \square

Theorem 9.1. *Let $A \in I(\Delta, \tilde{\Lambda})$, $B \in I(\Delta, \tilde{\Lambda})$ then $A \circ B \in I(\Delta, \tilde{\Lambda})$*

Proof. This follows from taking one sided parabolic cut offs to decompose $A = A_1 + A_2$ and $B = B_1 + B_2$ so that $A_1, B_1 \in I(\Delta, \Delta \cap \tilde{\Lambda})$, $A_2, B_2 \in I(\tilde{\Lambda}, \Delta \cap \tilde{\Lambda})$ and A_2, B_2 satisfy the hypotheses of proposition 9.4. \square

The composition theorem for polyhomogeneous, paired Lagrangian distributions now follows. Note that the orders are those expected from the Duistermaat-Guillemin clean intersection calculus. ([3])

Theorem 9.2. *Let $K \in I_{phg}^{\mu_1, \nu_1}(\Delta, \tilde{\Lambda})$, $B \in I_{phg}^{\mu_2, \nu_2}(\Delta, \tilde{\Lambda})$ then*

$$A \circ B \in I_{phg}^{\mu_1 + \mu_2, \nu_1 + \nu_2 + \frac{1}{2}}(\Delta, \tilde{\Lambda}).$$

Proof. We know from theorem 9.1 that the composition is a paired Lagrangian so we need only check its polyhomogeneity and its order.

We use the definition which involves testing by the radial vector fields. Firstly, we consider doing so on the diagonal: we show that applying the top order operator yields a composition of elements of lower orders.

$$\begin{aligned} & \left((z-x) \frac{\partial}{\partial x} - n - \mu_1 - \mu_2 \right) (K \circ L)(x, z) = \\ & -\mu_2 K \circ L + (\pi_y)_* \left((y-x) \frac{\partial}{\partial x} - \mu_1 - n \right) K(x, y) L(y, z) + (\pi_y)_* \left((z-y) \frac{\partial K}{\partial x} L(y, z) \right) \end{aligned} \quad (9.40)$$

The second term is the composition of something in $I_{phg}^{\mu_1-1, \nu_1}(\Delta, \tilde{\Lambda})$ and L . Putting $P = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) K$, can rewrite the other two terms as

$$(\pi_y)_* \left((z-y) P(x, y) L(y, z) \right) - (\pi_y)_* \left((z-y) \frac{\partial K}{\partial y} (x, y) L(y, z) \right) - \mu_2 K \circ L. \quad (9.41)$$

The first term here is the composition of an element of $I_{phg}^{\mu_1, \nu_1}(\Delta, \tilde{\Lambda})$ with an element of $I_{phg}^{\mu_2-1, \nu_2}(\Delta, \tilde{\Lambda})$ and we can rewrite the rest as

$$(\pi_y)_* \left(\frac{\partial}{\partial y} ((z'' - y'') KL) \right) - \left[(\pi_y)_* \left((z-y) \frac{\partial L}{\partial y} K \right) - n - \mu_2 (K \circ L) \right] \quad (9.42)$$

The second term here is the composition of an element of $I_{phg}^{\mu_1, \nu_1}(\Delta, \tilde{\Lambda})$ with an element of $I_{phg}^{\mu_2-1, \nu_2}(\Delta, \tilde{\Lambda})$ and the first term is equal to zero.

Letting K_i denote an element of $I_{phg}^{\mu_1-i, \nu_1}(\Delta, \tilde{\Lambda})$ and L_i an element of $I_{phg}^{\mu_2-i, \nu_2}(\Delta, \tilde{\Lambda})$ we have

$$\left((z-x) \frac{\partial}{\partial x} - n - \mu_1 - \mu_2 \right) (K \circ L) = K_1 \circ L_0 + K_0 \circ L_1. \quad (9.43)$$

It follows by induction that

$$\left(\prod_{j=0}^{N-1} (z-x) \frac{\partial}{\partial x} - n - \mu_1 - \mu_2 + j \right) (K \circ L) = \sum_{l=0}^N K_l \circ L_{N-l}. \quad (9.44)$$

We argue similarly for the flow out. We consider

$$\begin{aligned} & \left((z''-x'') \frac{\partial}{\partial x''} - (n-1) - \frac{1}{2} - (\nu_1 + \nu_2 + \frac{1}{2}) \right) (K \circ L)(x, z) = \\ & (\pi_y)_* \left[\left((y''-x'') \frac{\partial}{\partial x''} - (n-1 + \frac{1}{2} + \nu_1) \right) K(x, y) L(y, z) \right] + \\ & (\pi_y)_* \left((z''-y'') \frac{\partial K}{\partial x''}(x, y) L(y, z) \right) - (\nu_1 + \nu_2 + \frac{1}{2}) K \circ L(x, z). \end{aligned} \quad (9.45)$$

The first term here is the composition of an element of $I_{phg}^{\mu_1, \nu_1-1}(\Delta, \tilde{\Lambda})$ with an element of $I_{phg}^{\mu_2, \nu_2}(\Delta, \tilde{\Lambda})$. As above, $P = \frac{\partial K}{\partial x''} + \frac{\partial K}{\partial y''} \in I_{phg}^{\mu_1, \nu_1}(\Delta, \tilde{\Lambda})$ and we can rewrite the last two terms as

$$(\pi_y)_* \left((z''-y'') P(x, y) L(y, z) \right) - (\pi_y)_* \left((z''-y'') \frac{\partial K}{\partial y''}(x, y) L(y, z) \right) - (p_2 + \frac{1}{2}) K \circ L. \quad (9.46)$$

The first term is the composition of an element of $I_{phg}^{\mu_1, \nu_1-1}(\Delta, \tilde{\Lambda})$ with an element of $I_{phg}^{\mu_2, \nu_2}(\Delta, \tilde{\Lambda})$. Using the product rule the rest becomes

$$\begin{aligned} & (\pi_y)_* \left[\frac{\partial}{\partial y''} \left((z''-y'') K(x, y) L(y, z) \right) \right] - \\ & (\pi_y)_* \left[K(x, y) \left((z''-y'') \frac{\partial}{\partial y''} - (n-1 + \frac{1}{2} + \nu_2) \right) L(y, z) \right]. \end{aligned} \quad (9.47)$$

The first term disappears and the second term is the composition of an element of $I_{phg}^{\mu_1, \nu_1}(\Delta, \tilde{\Lambda})$ with an element of $I_{phg}^{\mu_2, \nu_2-1}(\Delta, \tilde{\Lambda})$. So, putting all this together and

arguing inductively, we obtain

$$\begin{aligned} & \left(\prod_{j=0}^{M-1} (R_{\Delta} + \mu_1 + \mu_2 - j) \right) \left(\prod_{k=0}^{N-1} \left(R_{\Lambda} + \nu_1 + \nu_2 + \frac{1}{2} - k \right) \right) (K \circ L) \\ & = \sum_{j=0}^N \sum_{k=0}^M K_{j,k} \circ L_{M-j, N-k} \quad (9.48) \end{aligned}$$

with $K_{j,k} \in I_{phg}^{m_1-j, \nu_1-k}(\Delta, \tilde{\Lambda})$ and $L_{j,k} \in I_{phg}^{\mu_2-j, \nu_2-k}(\Delta, \tilde{\Lambda})$.

To simplify computations we check orders only for applications of the radial fields on one Lagrangian submanifold at a time. This is sufficient by corollary 6.2. As in the proof of theorem 9.1 we cut up K_i, L_j into sums of marked Lagrangian distributions. It then follows from our results on their compositions that

$$\begin{aligned} & \left(\prod_{j=0}^{N-1} R_{\Delta} + \mu_1 + \mu_2 - j \right) (K \circ L) \in I^{m_1+m_2-N, p_1+p_2}(\Delta, \Delta \cap \tilde{\Lambda}) \\ & \quad + \sum_{j_1+j_2=N} I^{p_2+\frac{m_1-j_1+p_1+\frac{1}{2}+\epsilon}{2}, \frac{m_2-p_2-j_2+\frac{1}{2}+\epsilon}{2}}(\tilde{\Lambda}, \Delta \cap \tilde{\Lambda}) \\ & \quad + \sum_{j_1+j_2=N} I^{p_1+\frac{m_2-j_2+p_2+\frac{1}{2}+\epsilon}{2}, \frac{m_1-p_1-j_1+\frac{1}{2}+\epsilon}{2}}(\tilde{\Lambda}, \Delta \cap \tilde{\Lambda}) \\ & \quad + \sum_{j_1+j_2=N} I^{p_1+p_2+\frac{1}{2} \max\{m_2-j_2-p_2, m_1-p_1-j_1\}+\frac{1}{2}+\epsilon, \frac{1}{2} \min\{m_2-j_2-p_2, m_1-p_1-j_1\}+\frac{1}{2}+\epsilon-1}(\tilde{\Lambda}, \Delta \cap \tilde{\Lambda}) \end{aligned} \quad (9.49)$$

which means that

$$\begin{aligned} & \left(\prod_{j=0}^{N-1} R_{\Delta} + \mu_1 + \mu_2 - j \right) (K \circ L) \in I^{m_1+m_2-N, p_1+p_2}(\Delta, \tilde{\Lambda}) \\ & \quad + I^{m_1+m_2-N, p_2+\frac{m_2+p_1+\frac{1}{2}+\epsilon}{2}}(\Delta, \tilde{\Lambda}) + I^{m_1+m_2-N, p_1+\frac{m_1+p_2+\frac{1}{2}+\epsilon}{2}}(\Delta, \tilde{\Lambda}) \\ & \quad + I^{m_1+m_2-N+\frac{1}{2} \max\{m_1-p_1, m_2-p_2\}, p_1+p_2+\frac{1}{2} \max\{m_1-p_1, m_2-p_2\}+\frac{1}{2}+\epsilon}(\Delta, \tilde{\Lambda}) \end{aligned} \quad (9.50)$$

which is enough for the conormal bundle to the diagonal.

The argument for the flow out is essentially the same. The important point being

that it is enough to find r, s such that

$$\left(\prod_{j=0}^{N-1} R_{\Lambda} + p_1 + p_2 + \frac{1}{2} - j \right) (K \circ L) \in I^{r, s - \frac{N}{2}}(\Delta, \Lambda). \quad (9.51)$$

The principal symbol on the diagonal is immediate from the calculus of pseudo-differential operators. To obtain the principal symbol on the flow out we decompose our operators into three pieces. The first piece will be a marked pseudo-differential operator of low order at the marking, the second will be a flow out Fourier integral operator marked at the diagonal and the third will be a sum of operators of the form $\chi_{+}^{\alpha}(x_1 - y_1)P(x_1, y_1, x'', y'')$ with P a pseudo-differential operator in x'' . How do we do this? Well by construction in the model case, we can lower the order on the marking by subtracting such products and then parabolically decompose. (This will introduce additional singularities at $\xi'' = 0$ but they are irrelevant on the flow out.)

So after stripping away the product terms, we do a one-sided parabolic cut off and obtain $K = K_1 + K_2$ and $L = L_1 + L_2$ with K_1, L_1 marked pseudo-differential operators with sufficiently low order at the marking that they have no effect on the principal symbol on the flow out (by Prop 9.3) where as K_2, L_2 will be of the form

$$K_2 = \int e^{i\langle x'' - y'', \xi'' \rangle} (1 - \phi)(|x_1 - y_1| |\xi''|^{\frac{1}{2}}) a(x_1, y_1, x'', \xi'') d\xi'', \quad (9.52)$$

$$L_2 = \int e^{i\langle y'' - z'', \xi'' \rangle} (1 - \phi)(|y_1 - z_1| |\xi''|^{\frac{1}{2}}) b(y_1, z_1, z'', \xi'') d\xi''. \quad (9.53)$$

Composing K_2 with L_2 we obtain

$$\int e^{i\langle x'' - y'', \xi'' \rangle} c(x_1, y_1, x'', z'', \xi'') d\xi'' \quad (9.54)$$

and

$$c(x_1, z_1, x'', z'', \xi'') = \int (1 - \phi)(|x_1 - y_1| |\xi''|^{\frac{1}{2}}) a(x_1, y_1, x'', \xi'') \\ (1 - \phi)(|y_1 - z_1| |\xi''|^{\frac{1}{2}}) b(y_1, z_1, z'', \xi'') dy_1. \quad (9.55)$$

To pass from this to principal symbols we need to consider

$$c_{\tilde{p}_1+\tilde{p}_2} = \lim_{\lambda \rightarrow \infty} \lambda^{-\tilde{p}_1-\tilde{p}_2} c(x_1, z_1, x'', z'', \lambda \xi''). \quad (9.56)$$

The corresponding limits of a and b will give their principal parts, $a_{\tilde{p}_1}$, $b_{\tilde{p}_2}$ and these can be taken to be zero to second order at the intersection provided the order of the distribution is sufficiently low there. It then follows easily that

$$c_{\tilde{p}_1+\tilde{p}_2} = a_{\tilde{p}_1} * b_{\tilde{p}_2} \quad (9.57)$$

which is the result. Of course, we still have to consider the product type terms and their compositions with K_2, L_2 . A typical product type term will be

$$\int e^{i\langle x''-y'', \xi'' \rangle} \chi_+^\alpha(x_1 - y_1) a_{\tilde{p}_i}(x_1, y_1, x'', \xi'') d\xi'' \quad (9.58)$$

where $a_{\tilde{p}_i}$ is homogeneous of degree \tilde{p}_i . If $\Re \alpha > 0$ then composing two such terms will be convolution in y_1 and pseudo-differential operator composition in the other variables and so the principal symbol is the convolution of the principal symbols. When $\Re \alpha < 0$, we need to interpret (9.58) distributionally and taking Taylor series around $x_1 = y_1$, we need consider

$$\int \chi_+^\alpha(x_1 - y_1) a'(x_1, x'', \xi'') \chi_+^\beta(y_1 - z_1) b'(z_1, z'', \xi'') dy_1 \quad (9.59)$$

which is just

$$\chi_+^{\alpha+\beta-1}(x_1 - z_1) a'(x_1, x'', \xi'') b'(z_1, z'', \xi'') \quad (9.60)$$

and so this case follows.

It remains to discuss compositions between product type terms and K_2, L_2 . These follow by a combination of the arguments in the two other cases. \square

Corollary 9.1. *If Λ is a flow out Lagrangian and $K_i \in I_{phg}^{m_i, p_i}(\Delta, \Lambda)$ and the projection from $\{(x, y, z) \in \text{supp}(K(x, y)L(y, z))\}$ to (x, z) is proper then*

$$K_1 \circ K_2 \in I_{phg}^{\mu_1+\mu_2, \nu_1+\nu_2+\frac{1}{2}}(\Delta, \Lambda). \quad (9.61)$$

The principal symbol on the diagonal is the product of the principal symbols and on the flow out is the intrinsic convolution of the principal symbols with respect to the natural trivialization.

Proof. Our support condition means that the composition is well-defined. Our method of proof is to reduce to the model case. Let p be a homogeneous degree one function which induces the flow out, Λ . In a conic neighbourhood of a point where p is non-zero, the operators are pseudo-differential and there is nothing to prove.

At a point q on the characteristic variety, we pick homogeneous symplectic coordinates (x, ξ) such that $p = \xi_n$ on a conic neighbourhood $V \subset T^*(X) - 0$. (as in [2]) We extend these coordinates so they include the entire bicharacteristic through q using H_p . Let $W = \bigcup_t \exp(tH_p)V$ and define for $r \in W$,

$$(x, \xi)(\exp(tH_p)(r)) = (x, \xi)(r) + (0, e_n)t \quad (9.62)$$

where $e_n = (0, 0, \dots, 0, 1)$. This is well-defined because the Hamiltonian flow of ξ_n is $\frac{\partial}{\partial x_n}$ and is symplectic because $\exp(tH_p)$ is a symplectomorphism.

Let f denote the coordinate map. Consider a point (r, q) on the bicharacteristic through (q, q) , by definition, $r = \exp(t_0 H_p)(q)$ for some $t_0 > 0$. If we decompose each of K_1, K_2 into two pieces such that the first piece, K'_i is microsupported in W and on the part of Λ with $t < t_0 + 1$ and the rest is not microsupported on a neighbourhood of the bicharacteristic from q to r then the singularity at (r, q) is determined purely by the composition $K'_1 \circ K'_2$. We pick a Fourier integral operator F associated to Γ_f which is elliptic on the bicharacteristic from (q, q) to (r, q) and conjugate. Using Egorov's theorem (Prop. 6.7) for paired Lagrangian distributions, this reduces Λ to $\tilde{\Lambda}$ and the result follows. \square

With a composition calculus defined, it is natural to examine invertibility of operators in $I_{phg}^{\mu, \nu}(\Delta, \Lambda)$. We define an ellipticity condition weaker than that for pseudo-

differential operators which leads to the existence of a parametrix modulo smoothing. The obvious condition is to require that the principal symbol on the diagonal is non-zero off the intersection. We make the additional requirement that the leading singularity at the intersection is non-zero. For $\mu - \nu - \frac{1}{2}$ non-integral, this is equivalent to requiring the principal symbol of the symbol at the intersection to have no zeroes but for integer values this breaks down as the terms in our expansion are smooth. We therefore make a definition analogous to definition 8.3.

Definition 9.4. Let $W^{\mu,q}(\Lambda_0, \Lambda_1^e, L_0)$ be the smooth sections, f , of $(N(\partial\Lambda_1^e, \Lambda_0) - 0, L_0)$ which are homogeneous of degree $\mu - q$ with respect to the conic action on $\partial\Lambda_1^e$, are homogeneous of degree q with respect to the conic action of $N(\partial\Lambda_1^e, \Lambda_0)$ and such that $f(p, -v) = e^{-i\pi q} f(p, v)$ where $\omega(v, w) > 0$ for $w \in T_p(\Lambda_1)$ pointing into Λ_1^e .

There exists a short exact sequence

$$0 \rightarrow T^{\mu,q-1}(\Lambda_0, \Lambda_1^e, L_0) \rightarrow T^{\mu,q}(\Lambda_0, \Lambda_1^e, L_0) \xrightarrow{\alpha^{\mu,q}} W^{\mu,q}(\Lambda_0, \Lambda_1^e, L_0) \rightarrow 0. \quad (9.63)$$

Definition 9.5. We shall say $K \in I_{phg}^{\mu,\nu}(\Delta, \Lambda)$ is elliptic if and only if $\sigma_\mu^0(K)$ is non-zero on $\Delta - \partial\Lambda_1^e$ and $\alpha_{\mu,\mu-\nu-\frac{1}{2}}(K)$ has no zeroes.

Example 9.1. Let Λ be the flow out in $T^*(\mathbb{R}^n) - 0$ induced by $p = \xi_n$ then if $P_k = D_{x_n}^k \delta(x-y)$ it has principal symbol ξ_n^k and is elliptic as an element of $I_{phg}^{k,-\frac{1}{2}}(\Delta, \Lambda)$. It is not elliptic as an element of $I_{phg}^{k,j-\frac{1}{2}}(\Delta, \Lambda)$ for $0 < j \leq k$.

Example 9.2. Let P be a polyhomogeneous operator of real principal type of order m with principal symbol p then let Λ be the flow out associated with qp for any q which is homogeneous of order $1 - m$ and non-zero on the characteristic variety of p then P^k is elliptic as element of $I_{phg}^{km, km-k-\frac{1}{2}}(\Delta, \Lambda)$. Note that Λ will vary according to the sign of q on each component of the characteristic variety of p .

Theorem 9.3. Let $P \in I_{phg}^{\mu,\nu}(\Delta, \Lambda)$ then there exists a two sided parametrix up to smoothing in $I_{phg}^{-\mu,-\nu-1}(\Delta, \Lambda)$ if and only if P is elliptic in $I_{phg}^{\mu,\nu}(\Delta, \Lambda)$.

Proof. That the existence of such a parametrix implies ellipticity is clear from considering the principal symbol of a composition.

Initially, we solve just on the diagonal and use an argument similar to that used for elliptic pseudo-differential operators. The symbol, $\sigma_{\mu, \mu - \nu - \frac{1}{2}}^0(P)$, is an element of $T^{\mu, \mu - \nu - \frac{1}{2}}(\Delta, \Lambda)$. Off $\partial\Lambda_1^e$, the ellipticity condition means that the reciprocal is non singular and homogeneous of degree $-\mu$. We have to check that it has the right form of singularity at $\partial\Lambda_1^e$. Membership of $T^{\mu, \mu - \nu - \frac{1}{2}}(\Delta, \Lambda)$ is equivalent to requiring for some coordinates, x , on the base

$$\sigma_{\mu, \mu - \nu - \frac{1}{2}}^0(P) \sim (\xi_1 - i0)^{\mu - \nu - \frac{1}{2}} a_0(x, \xi'') \left(1 + \sum_{j>0} (\xi_1 - i0)^j a_j(x, \xi'') \right) \text{ as } \xi_1 \rightarrow 0+ \quad (9.64)$$

with a_j homogeneous of degree $\nu + \frac{1}{2} - j$. The second part of the definition of ellipticity means precisely that $a_0(\xi'')$ has no zeroes. This means that $\sigma_{\mu, \mu - \nu - \frac{1}{2}}^0(P)^{-1}$ has an expansion

$$\sigma_{\mu, \mu - \nu - \frac{1}{2}}^0(P)^{-1} \sim (\xi_1 - i0)^{\mu - (-\nu - 1) - \frac{1}{2}} a_0^{-1}(x, \xi'') \left(1 + \sum_{j>0} (\xi_1 - i0)^j a_j(x, \xi'') \right)^{-1}. \quad (9.65)$$

Of course, we have to check that $\left(1 + \sum_{j>0} (\xi_1 - i0)^j a_j(x, \xi'') \right)$ is invertible as an asymptotic expansion. Suppose we have an inverse up to N terms so that

$$\begin{aligned} \left(1 + \sum_{j>0} (\xi_1 - i0)^j a_j(x, \xi'') \right) \left(1 + \sum_{j=0}^{N-1} (\xi_1 - i0)^j b_j(x, \xi'') \right) \\ = 1 + (\xi_1 - i0)^N c_N(x, \xi'') + O(\xi_1^{N+1}) \end{aligned} \quad (9.66)$$

then

$$\begin{aligned} \left(1 + \sum_{j>0} (\xi_1 - i0)^j a_j(x, \xi'') \right) \left(1 + \sum_{j=0}^{N-1} (\xi_1 - i0)^j b_j(x, \xi'') \right) \\ \left(1 - (\xi_1 - i0)^N c_N(x, \xi'') \right) = 1 + O(\xi_1^{N+1}) \end{aligned} \quad (9.67)$$

so expanding the product and proceeding by induction will yield an inverse to arbitrarily high order. Note that the coefficients of the first N terms do not change when finding the term of order $N + k$, so taking the infinite asymptotic expansion given by the limit of the partial expansions we obtain our inverse.

So, $\sigma_{\mu, \mu - \nu - \frac{1}{2}}^0(P)^{-1} \in T^{-\mu, -\mu - (-\nu - 1) - \frac{1}{2}}(\Delta, \Lambda)$ and it is therefore by (8.9) the principal symbol of $Q_1 \in I_{phg}^{-\mu, -\nu - 1}(\Delta, \Lambda)$. Thus $PQ_1 \in I_{phg}^{0, -\frac{1}{2}}(\Delta, \Lambda)$ and $\sigma_{0,0}^0(PQ_1) = 1$. Thus $R = \text{Id} - PQ_1 \in I_{phg}^{-1, -\frac{1}{2}}(\Delta, \Lambda)$.

So $R^k \in I_{phg}^{-k, -\frac{1}{2}}(\Delta, \Lambda)$ and we can use a Neumann series to find the diagonal part of our parametrix. Let $Q_2 \sim \sum_{k \geq 0} R^k$ in $I_{phg}^{0, -\frac{1}{2}}(\Delta, \Lambda)$ using Proposition 6.4. We then have

$$PQ_1Q_2 - \text{Id} = (\text{Id} - R) \left(\sum_{k=0}^{N-1} R^k \right) + (\text{Id} - R) \left(Q_2 - \sum_{k=0}^{N-1} R^k \right) - \text{Id} \quad (9.68)$$

which shows that $PQ_1Q_2 - \text{Id} \in I_{phg}^{-N, -\frac{1}{2}}(\Delta, \Lambda)$ for all N . That is $KQ_1Q_2 - \text{Id} \in I_{phg}^{-\infty, -\frac{1}{2}}(\Delta, \Lambda)$. We have that $Q_1Q_2 \in I_{phg}^{-\mu, -\nu - 1}(\Delta, \Lambda)$.

Now $I_{phg}^{-\infty, -\frac{1}{2}}(\Delta, \Lambda)$ is contained in $I_{phg}^{-\frac{1}{2}}(\Lambda)$ and thus we have to solve the equation

$$(\text{Id} - A)(\text{Id} + B) - \text{Id} \in C^\infty \quad (9.69)$$

with $A \in I_{phg}^{-\frac{1}{2}}(\Lambda)$ for $B \in I_{phg}^{-\frac{1}{2}}(\Lambda)$. We can not take a Neumann series here as $A^k \in I_{phg}^{-\frac{1}{2}}(\Lambda)$ so our terms do not gain regularity. Passing to principal symbols on the flow out we have,

$$\sigma_{-\frac{1}{2}}^1(B) - \sigma_{-\frac{1}{2}}^1(A) - \sigma_{-\frac{1}{2}}^1(A) * \sigma_{-\frac{1}{2}}^1(B) = 0. \quad (9.70)$$

This is a Volterra equation as our symbols are supported in the positive flow out. We can therefore solve for $\sigma_{-\frac{1}{2}}^1(B)$ and then picking B with such a symbol, we have

$$(\text{Id} - A)(\text{Id} + B) - \text{Id} \in I_{phg}^{-\frac{3}{2}}(\Lambda). \quad (9.71)$$

Now, if $C \in I_{phg}^{-\frac{3}{2}}(\Lambda)$ then $C^k \in I_{phg}^{-k-\frac{1}{2}}(\Lambda)$ and so we can take a Neumann series and we have constructed a right parametrix. The same argument constructs a left parametrix and the result follows. \square

10. COMPLEX POWERS OF THE CONSTANT COEFFICIENT WAVE OPERATOR

The constant coefficient wave operator on $\mathbb{R}^n \times \mathbb{R}$, $D_t^2 - \Delta$, where Δ is the Laplacian, has principal symbol $\tau^2 - \xi^2 = (\tau - |\xi|)(\tau + |\xi|)$. Taking the flow out of the characteristic variety, Λ , it is elliptic as an element of $I_{phg}^{2, \frac{1}{2}}(\Delta, \Lambda)$ and therefore the forward fundamental solution is an element of $I_{phg}^{-2, -\frac{3}{2}}(\Delta, \Lambda)$. Complex powers of elliptic pseudo-differential operators are pseudo-differential operators so, by analogy, we look for complex powers of the wave operator in $I(\Delta, \Lambda)$.

Applying the constant coefficient wave operator is equivalent to multiplying the Fourier transform by $\tau^2 - \xi^2$, so naively, we would like to define the s^{th} complex power to be multiplication of the Fourier transform by $(\tau^2 - \xi^2)^s$ but the problem is that the zeroes of $\tau^2 - \xi^2$ introduce singularities. We can surmount this problem by regarding $(\tau^2 - \xi^2)^s$ as the boundary value of a holomorphic function defined in an open cone, as in [6].

As noted above, for the complex powers of the constant coefficient wave operator, we therefore take the representation

$$K_s = \left(\frac{1}{2\pi}\right)^{n+1} \lim_{\epsilon \rightarrow 0^+} \int e^{i(x \cdot \xi + t \cdot \tau)} ((\tau - i\epsilon)^2 - \xi^2)^s d\xi d\tau. \quad (10.1)$$

Provided we can show this makes sense, it is clear that $K_1 = D_t^2 - D_x^2$, $K_0 = \text{Id}$ and $K_\tau \circ K_s = K_{\tau+s}$, when we regard K_s as a convolution operator.

Although, the defining integral is dependent upon ϵ , we can make it independent of ϵ , by letting the exponent vary with ϵ . And, indeed we can also vary the imaginary

part of ξ , provided we remain within an appropriate cone:

$$\Gamma_c \equiv \{(\tau, \xi) \in \mathbb{C}^{n+1} \mid \Im\tau < 0, |\Im\xi| < c|\Im\tau|\} \text{ where } c < 1. \quad (10.2)$$

Of course, we must check that this complex power makes sense and that we have a well defined boundary distribution.

Lemma 10.1. $(\tau^2 - \xi^2)^s$ is well defined in Γ_c and converges to a well defined distribution as $\Im(\tau, \xi) \rightarrow 0$ in Γ_c .

Proof. To define $(\tau^2 - \xi^2)^s$ in Γ_c , it is enough to show that $\tau^2 - \xi^2$ does not take values in a half-line and picking an appropriate branch of log, put

$$(\tau^2 - \xi^2)^s = e^{s \log(\tau^2 - \xi^2)} \quad (10.3)$$

Now, if we put

$$\tau = a + i\epsilon \text{ and } \xi = b + i\nu$$

then we obtain

$$\tau^2 - \xi^2 = (a^2 - b^2 + \nu^2 - \epsilon^2) + 2i(\epsilon a - \nu b) \quad (10.4)$$

This will never be in \mathbb{R}_+ as $\Im(\tau^2 - \xi^2) = 0$ will imply that $|a| > |b|$ so that $\tau^2 - \xi^2 < 0$. Hence, we can pick the branch of log defined in a plane cut along the positive real axis, with $\log(-1) = i\pi$ and this will define $(\tau^2 - \xi^2)^s$ in Γ_c .

To define the boundary distribution (see [6], 3.1), we need to show that there exists k such that

$$|(\tau^2 - \xi^2)^s| \leq C|(\epsilon, \nu)|^{-k}, \text{ for } (\tau, \xi) \in \Gamma_c. \quad (10.5)$$

Now,

$$|\tau^2 - \xi^2|^2 = |a^2 - b^2 + \nu^2 - \epsilon^2|^2 + 4|\epsilon a - \nu b|^2.$$

If $|\epsilon a - \nu b| \leq (1 - c)|\epsilon||a|$ then as $|\epsilon| > |\nu|$ we have $|b| \geq |a|$ which implies that

$$a^2 - b^2 + \nu^2 - \epsilon^2 \leq (c^2 - 1)\epsilon \leq -C'|(\epsilon, \nu)|^2 \text{ some } C' > 0.$$

or

$$|a^2 - b^2 + \nu^2 - \epsilon^2| \geq C'|(\epsilon, \nu)|^2 \text{ some } C' > 0.$$

Otherwise, we have $|\epsilon a - \nu b| \geq (1 - c)|\epsilon||a|$ so $|\epsilon a - \nu b| \geq C''|(\epsilon, \nu)||a|$. Pick K such that $c^2 - (1 - K^{-2}) < 0$ then if $|\tau| \geq |\epsilon|/K$ we have

$$|\epsilon a - \nu b| \geq C|(\epsilon, \nu)|$$

and if $|\tau| \leq |\epsilon|/K$ then

$$\begin{aligned} a^2 - b^2 + \nu^2 - \epsilon^2 &\leq \epsilon^2/K^2 - b^2 + \nu^2 - \epsilon^2 \\ &\leq \nu^2 - (1 - 1/K^2)\epsilon^2 \\ &\leq (c^2 - (1 - 1/K^2))\epsilon^2. \\ &\leq -C'|(\epsilon, \nu)|^2. \end{aligned}$$

So, putting all this together, we have that

$$|\tau^2 - \xi^2| \geq C'|(\epsilon, \nu)|^2 \text{ some } C' > 0$$

which implies that, for $\Re s < 0$,

$$|(\tau^2 - \xi^2)^s| \leq C'_s|(\epsilon, \nu)|^{2\Re(s)} \text{ for some } C'_s > 0.$$

In the case, where $\Re s \geq 0$, $(\tau^2 - \xi^2)^s$ is continuous up to the boundary so there is nothing to prove.

□

Now, let $u_{\epsilon, \nu}$ be the inverse Fourier transform of $((\tau + i\epsilon)^2 - (\xi + i\nu)^2)^s$ where $(\epsilon, \nu) \in \Gamma_c$ then, if $\phi \in C_0^\infty(\mathbb{R}^{n+1})$ we have

Lemma 10.2. $\langle u_{\epsilon, \nu}, \phi e^{-\epsilon t - \nu \cdot x} \rangle$ is independent of $(\epsilon, \nu) \in \Gamma_c$.

Proof. First of all, since ϕ is compactly supported, multiplying it by an exponential leaves us with a compactly supported function, which is therefore Schwartz. So, the pairing makes sense. Now, by definition of the Fourier Transform on tempered distributions, we have $\langle u_{\epsilon, \nu}, \phi e^{-\epsilon t - \nu \cdot x} \rangle = \langle ((\tau + i\epsilon)^2 - (\xi + i\nu)^2)^s, \widehat{\phi e^{-\epsilon t - \nu \cdot x}} \rangle$ and we can rewrite this as

$$\langle u_{\epsilon, \nu}, \phi e^{-\epsilon t - \nu \cdot x} \rangle = \iint ((\tau + i\epsilon)^2 - (\xi + i\nu)^2)^s \hat{\phi}(\psi + i\nu, \tau + i\epsilon) d\tau d\xi$$

and as $\hat{\phi}$ is Schwartz the integral is absolutely convergent and is equal to

$$\iint_{\Im \xi = \nu, \Im \tau = \epsilon} (\tau^2 - \xi^2)^s \hat{\phi}(\xi, \tau) d\tau d\xi.$$

Since our integrand is holomorphic, we can now use Cauchy's theorem to change contours. Suppose (ϵ_1, ν_1) and (ϵ_2, ν_2) are such that $|\epsilon_1| < |\epsilon_2|$. First of all, we move the ϵ contour, keeping $\nu = \nu_1$ fixed. We can always increase the size of ϵ without leaving Γ_c , so we take a rectangular contour with sides of length $2R$ horizontally and going from $i\epsilon_1$ to $i\epsilon_2$ vertically, and let R go to infinity. This will establish equality, provided the vertical integrals go to zero, but this follows immediately from the Paley-Wiener-Schwartz theorem, since ϕ is a compactly supported smooth function.

The same process will also take care of changing ν , provided we do the coordinates which decrease in modulus first (in order to stay within Γ_c .) \square

With this proven, we can now establish the support properties of these complex powers:

Theorem 10.1. K_s is supported within the forward light cone.

Proof. This means precisely that if $\text{supp}(\phi) \subseteq \{(x, t) : |x| > t\}$ then $\langle K_s, \phi \rangle = 0$.

Let (x_0, t_0) be such that $|x_0| > |t_0|$ we will show that if ϕ is supported in a sufficiently small neighbourhood of (x_0, t_0) that this is true, which is sufficient. There exists a small cone Γ in Γ_c and a small open set U containing (x_0, t_0) such that for

$$(x, t) \in U \text{ and } (\epsilon, \nu) \in \Gamma, \epsilon t + \nu \cdot x > 0.$$

When $t_0 \leq 0$ this is clear, and when $t_0 > 0$ take

$$\epsilon_0 = |x_0|/2(-1 - |x_0|/|t_0|), \nu_0 = x_0$$

and then

$$t_0 + \epsilon_0 > \frac{-|x_0|^2}{t} t + |x_0|^2 = 0$$

and this will persist nearby. Hence, we can take Γ and U such that for $(\epsilon, \nu) \in \Gamma$ and $(t, x) \in U$

$$\epsilon t + \nu \cdot x > C|(\epsilon, \nu)|.$$

This implies that the Schwartz norms of $\phi e^{-\epsilon t - \nu \cdot x}$ will be exponentially decreasing as $|(\epsilon, \nu)|$ go to infinity in Γ , if $\text{supp}(\phi) \subset U$. As the Fourier transform is a continuous map of Schwartz space, the same will be true of $\widehat{\phi e^{-\epsilon t - \nu \cdot x}}$. Now, the distributions $u_{\epsilon, \nu}$ are within a bounded subset of the tempered distributions, so we have that

$$|\langle u_{\epsilon, \nu}, \phi e^{-\epsilon t - \nu \cdot x} \rangle| \leq C e^{-C'|(\epsilon, \nu)|} \quad (10.6)$$

but the left hand side of 10.6 is independent of (ϵ, ν) by our lemma and the right hand side goes to 0 as $|(\epsilon, \nu)| \rightarrow \infty$. \square

The next thing to be done is to establish the parabolic decomposition of these kernels and to calculate their symbols. In this case, the Lagrangians are the conormal bundle to the origin, Λ_0 , and the closure of the conormal bundle of the light cone, Λ_1^ϵ . i.e. $\overline{N^*(|x| = t, t > 0)}$

Theorem 10.2. $K_s \in I_{phg}^{2s-\frac{n}{4}, s-\frac{n}{4}-\frac{1}{2}}(\Lambda_0, \Lambda_1)$. On Λ_0 away from Λ_1 , the symbol of K_s is $((\tau - i0)^2 - \xi^2)^s$ and on Λ_1^c away from Λ_0 the symbol has the asymptotic expansion

$$\sum_{j=0}^{\infty} (\pm 1)^j e^{i\frac{\pi}{2}(s+j)} \frac{s(s-1)\dots(s-j+1)}{j!} (2|\xi|)^{s-j} t^{-s-j-1} / \Gamma(-s-j)$$

with respect to the phase functions $x \cdot \xi \pm t|\xi|$.

Proof. We have defined K_s to be the Fourier transform of $((\tau - i0)^2 - \xi^2)^s$, so away from $\tau^2 - \xi^2 = 0$ it is clear that we have an element of $I_{phg}^{2s}(\Lambda_0)$. The characteristic variety has two components $\tau = |\xi|$ and $\tau = -|\xi|$ and thus the flow out will also. We discuss $\tau = |\xi|$, as $\tau = -|\xi|$ will be the same with a few sign changes. We cut off close to $\tau = |\xi|$, letting $\psi(\xi, \tau) = \phi(1 - |\xi|^{-1}|\tau|)$, where ϕ is a cut-off function, we compute

$$K'_s = \int e^{i(x \cdot \xi + t\tau)} \psi(\xi, \tau) ((\tau - i0)^2 - \xi^2)^s d\xi d\tau \quad (10.7)$$

$$= \int e^{i(x \cdot \xi + t\tau)} \psi(\xi, \tau) (\tau - i0 - |\xi|)^s (\tau + |\xi|)^s d\xi d\tau \quad (10.8)$$

$$= \int e^{i(x \cdot \xi + t\tau + t|\xi|)} (\tau - i0)^s \phi(|\xi|^{-1}\tau) (\tau + 2|\xi|)^s d\xi d\tau. \quad (10.9)$$

We want to write K'_s as a Fourier integral operator applied to a distribution associated with the model so we can expand the last integral to be

$$\int e^{i(\langle x-y, \xi \rangle + \langle t-\tau, \tau \rangle + t|\xi|)} a(x, \xi, t, \tau) \int e^{i(y \cdot \eta + r\tau)} (\gamma - i0)^s \phi(|\eta|^{-1}\gamma) (\gamma + 2|\eta|)^s d\eta d\gamma dy dr d\tau d\gamma + C^\infty \quad (10.10)$$

where a is a symbol which is identically one in a conic neighbourhood of $\tau = 0$ and zero outside a conic neighbourhood. Our distribution is in the form of a Fourier integral operator applied to the Fourier transform of an element of $T^{2s, s}(\tilde{\Lambda}_0, \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^c)$. This establishes membership of $I_{phg}^{2s-\frac{n}{4}, s-\frac{n}{4}-\frac{1}{2}}(\Lambda_0, \Lambda_1)$. To find the symbol on the flow out we just take the Fourier transform of the Taylor expansion term by term. \square

Later on we will want to consider conjugation by Fourier operators so we rephrase the value of the symbol in a more invariant way. The forward light cone is generated by the flow out of the characteristic variety at the origin by the Hamiltonians $p_{\pm} = \tau \pm |\xi|$. These therefore yield a natural coordinates on the flow out, (ξ, t) . We can also think of these coordinates as the geodesic flow at time t with initial point $x = 0, t = 0, \tau = \pm|\xi|$.

Corollary 10.1. *The principal symbol on Λ_1^e away from the intersection with Λ_0 is*

$$\frac{t^{-s-1}}{\Gamma(-s)} (2|\xi|)^s e^{i\frac{\pi}{2}s} |dt|^{\frac{1}{2}} |d\xi|^{\frac{1}{2}}$$

in the coordinates induced by the geodesic flow from $(0, 0, \xi, \pm|\xi|)$ with respect to the trivialization of the Maslov bundle given by the canonical trivialization at the origin transported by the geodesic flow.

Proof. We must check that the trivialization of the Maslov bundle given by the phase $\phi = x.\xi \pm t|\xi|$ agrees with the natural trivialization on the flow out. Let $\lambda_1 = T_p(\Lambda)$, λ_2 be the tangent space to the fibre and let $\psi(x, t)$ be a generating function parametrizing a Lagrangian, $\Gamma = (x, t, \psi'_{x,t})$, transversal to the fibre and to Λ . Letting $\mu = T_p(\Gamma)$ we have that the transition function is given by

$$e^{i\frac{\pi}{4}(\sigma(\lambda_1, \lambda_2; \mu) + S(\phi, \psi))}$$

where σ is the cross ratio as defined in [5] and $S(\phi, \psi)$ is the signature of the matrix

$$\begin{pmatrix} \phi''_{\xi\xi} & \phi''_{\xi(x,t)} \\ \phi''_{(x,t)\xi} & \phi''_{(x,t)(x,t)} - \psi''_{(x,t)(x,t)} \end{pmatrix}.$$

It will be enough to do all this at some fixed point. Take $\phi = x.\xi + |\xi|t$. We choose a point, $p = (0, 0, \xi, |\xi|)$, at the intersection. We then have that λ_1 is the span of $\{\xi_j \frac{\partial}{\partial \tau} + |\xi| \frac{\partial}{\partial \xi_j}, \sum_i \frac{\partial}{\partial x_i} - \frac{\xi_j}{|\xi|} \frac{\partial}{\partial \tau}\}$ and λ_2 is the span of $\{\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \tau}\}$. Thus we need ψ giving μ transversal to λ_1 . (Transversality to the fibre is automatic.) Let $\psi(x, t) = x.\xi + |\xi|t + \frac{t^2}{2}$

we have

$$\Gamma = \{(x, t, \xi, |\xi| + t)\}$$

and μ is the span of $\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\}$. Transversality is clear and our matrix is

$$\begin{pmatrix} 0 & \text{Id} & \frac{\xi}{|\xi|} \\ \text{Id} & 0 & 0 \\ \frac{\xi^t}{|\xi|} & 0 & -1 \end{pmatrix}. \quad (10.11)$$

Executing a change of variables, this will have the same signature as

$$\begin{pmatrix} \text{Id} & 0 & \frac{1}{2} \frac{\xi}{|\xi|} \\ 0 & -\text{Id} & -\frac{1}{2} \frac{\xi}{|\xi|} \\ \frac{1}{2} \frac{\xi^t}{|\xi|} & -\frac{1}{2} \frac{\xi^t}{|\xi|} & -1 \end{pmatrix}. \quad (10.12)$$

This has signature -1 as signature is a homotopy type invariant in the space of invertible matrices and the off diagonal elements can be shrunk to zero.

We are left to compute $\sigma(\lambda_1, \lambda_2; \mu)$ this is defined to be equal to $\sigma(\lambda_1^\rho, \lambda_2^\rho; \mu^\rho)$ where the superscript ρ denotes that the subspaces have been intersected by $(\lambda_1 \cap \lambda_2)^\perp$ and reduced by $\lambda_1 \cap \lambda_2$ then $\lambda_1^\rho, \lambda_2^\rho, \mu^\rho$ are all transversal one dimensional Lagrangian subspaces $\frac{(\lambda_1 \cap \lambda_2)^\perp}{\lambda_1 \cap \lambda_2}$. Picking symplectic linear coordinates, (r, η) , such that $\lambda_1 = \{r = 0\}$, $\lambda_2 = \{\eta = 0\}$ and $\mu = \{\xi = Ax\}$, $\sigma(\lambda_1^\rho, \lambda_2^\rho; \mu^\rho)$ is the signature of A . Of course, A is a number here, so we just need to know its sign. It is easily calculated to be 1.

The argument is similar for $\phi = x \cdot \xi - t|\xi|$. \square

11. COMPLEX POWERS OF THE WAVE OPERATOR

In this section, we define the complex powers of the wave operator on the cartesian product of a Riemannian manifold M and \mathbb{R} , show that they form a holomorphic family satisfying the group law and by reducing to the constant case establish that they are polyhomogeneous paired Lagrangian distributions and calculate their symbols.

The forward fundamental solution of the wave operator on $M \times \mathbb{R} \times \mathbb{R}$ can be written as $K(x, x', t, r)$, since the wave operator is constant coefficient in t and r . We define

Definition 11.1. For $\Re s \ll 0$, $L_s = 2e^{i\pi(s+1)}(\pi_r)_*(\chi_+^{2(-s-1)}(r)K(x, x', t, r))$

The problem here, is that we are taking a product not permitted by the calculus of wavefront sets but this is permissible provided one of the factors is sufficiently smooth, which is why we take $\Re s \ll 0$. There is no problem with the push forward, because for (x, x', t) in a compact set, it is compactly supported in r . We will use analytic continuation to extend to all $s \in \mathbb{C}$. The fact that K is supported in the set $\{t > 0, d(x, x')^2 + r^2 \leq t^2\}$ immediately implies that L_s is supported in the set $\{t > 0, d(x, x')^2 \leq t^2\}$ which is the forward light cone.

In the case, where $-s$ is a positive integer, we have the alternative representation

$$L_s = e^{i\pi(s+1)} \frac{(\pi_r)_*(r^{2(-1-s)}K(x, x', t, r))}{\Gamma(-2(s+1)+1)} \quad (11.1)$$

which follows from the fact that K is even in r . This definition will make sense independent of the size of s since in this case $r^{2(-1-s)}$ is smooth, we denote these additional kernels L'_s . Now, let $\phi \in C_0^\infty(\mathbb{R}^{n+1})$ and then in the case of $s = -1$, we have

$$\begin{aligned} \langle (D_t^2 - \Delta)L'_1, \phi \rangle &= \langle K, (D_t^2 - \Delta)^t \pi_r^* \phi \rangle \\ &= \langle K, (D_t^2 - \Delta - D_r^2)^t \pi_r^* \phi \rangle \\ &= \langle \delta(t, x - x', r), \pi_r^* \phi \rangle \\ &= \langle \delta(t, x - x'), \phi \rangle . \end{aligned}$$

This means that $(D_t^2 - \Delta)L'_1 = \text{Id}$ when the kernels are interpreted as operators.

And for $-s - 1$ an integer greater than zero we have

$$\langle (D_t^2 - \Delta)L'_s, \phi \rangle = e^{i\pi(s+1)} \frac{\langle K, (D_t^2 - \Delta)^t r^{2(-s-1)} \pi_r^* \phi \rangle}{\Gamma(2(-s-1) + 1)} \quad (11.2)$$

$$= e^{i\pi(s+1)} \frac{\langle K, (D_t^2 - \Delta - D_r^2)^t r^{2(-s-1)} \pi_r^* \phi \rangle}{\Gamma(2(-s-1) + 1)} \quad (11.3)$$

$$+ \frac{\langle K, D_r^2 r^{2(-s-1)} \pi_r^* \phi \rangle}{\Gamma(2(-s-1) + 1)} \quad (11.4)$$

$$= -e^{i\pi(s+1)} \frac{\langle K, \left(\frac{\partial}{\partial r}\right)^2 r^{2(-s-1)} \pi_r^* \phi \rangle}{\Gamma(2(-s-1) + 1)} \quad (11.5)$$

$$= \langle L'_{s+1}, \phi \rangle. \quad (11.6)$$

The first term in (11.4) vanishes because $r^{2(-s-1)} \delta(t, x - x', r) = 0$.

So using the equality of L_{-k} and L'_k , for k a large positive integer, we have that $(D_t^2 - \Delta)^k L_{-k} = \delta(x - x', t)$. Now, using a similar argument, we can show that $(D_t^2 - \Delta)^k L_s = L_{s+k}$ where both are defined. Putting, these two facts together and regarding the L_s as operators we have that $L_k \circ L_s = L_{s+k}$ for k a large positive integer.

It remains to extend the composition law to all $p \in \mathbb{C}$ with $\Re p \ll 0$. We do this (see [11]) by using

Theorem 11.1. *If f is a holomorphic function on a half plane, dominated by a non-vanishing holomorphic function and vanishes on the integers then f is identically zero.*

Now, $K \in (C^0)'$ so if we let $\phi \in C_0^\infty(\mathbb{R}^{n+1})$ and choose $\psi(r) \in C_0^\infty(\mathbb{R})$ so that $\psi \equiv 1$ on a sufficiently large set then as K is supported inside the light cone, we have

$$|\langle L_s, \phi \rangle| \leq \frac{2}{|\Gamma(-2s-1)|} \|K\|_0 \|\chi_+^{-2s-2}(r) \psi(r) \phi(x, x', t)\|_{C^0} \quad (11.7)$$

$$\leq \frac{C'}{|\Gamma(-2s-1)|} |C^{2s}| |e^{\pi i(s+1)}|. \quad (11.8)$$

Thus, we have that

$$| \langle L_{s+p}, \phi \rangle | \leq \frac{C'}{|\Gamma(-2(s+p)-1)|} |C^{2(s+p)}| |e^{\pi i(s+p+1)}|. \quad (11.9)$$

We need to establish a similar bound for $\langle L_s \circ L_p, \phi \rangle$. Picking $\psi(r) \in C_0^\infty(\mathbb{R})$ identically 1 on a sufficiently large set we obtain

$$\begin{aligned} \langle L_s \circ L_p, \phi \rangle &= 2e^{i\pi(s+p)} \int \dots \int \chi_+^{2(-s-1)}(r-r') \chi_+^{2(-p-1)}(r') \\ &\quad K(x, x', t-t', r-r') K(x', x'', t', r') dr' dt' dx' \psi(r) \phi(x, x', t) dt dx dx'' \end{aligned} \quad (11.10)$$

$$= 2e^{i\pi(s+p)} \iint \chi_+^{2(-s-1)}(r-r') \chi_+^{2(-p-1)}(r') b(r', r-r') \psi(r) dr dr' \quad (11.11)$$

where

$$b(r, s) = \int \dots \int K(x, x', t-t', s) K(x', x'', t', r) \phi(x, x'', t) dt dx dx' dt' dx''. \quad (11.12)$$

The advantage of this representation is that $b(r, s)$ is continuous which allow us to remove it from the integral. Now, $b(r, s)$ is defined by a pushforward of a paired Lagrangian distribution but points near the flow out are wiped out by the pushforward and we only retain points that are conormal to $r = 0$ or to $s = 0$. Now, since on the diagonal K is a conormal distribution of order -2 we obtain

$$b(r, s) = \iint e^{i(r\gamma+s\tau)} c(r, s, \gamma, \tau) d\gamma d\tau + C^\infty \quad (11.13)$$

where c is a product type symbol in (γ, τ) of order $(-2, -2)$. This means that b is paired Lagrangian distribution associated to $N^*(s=0)$, $N^*(t=0)$ and $N^*(s=t=0)$ with symbolic order $-2, -2, -4$. This shows that $b \in H^{\frac{3}{4}}(\mathbb{R}^2)$ and so by the Sobolev embedding theorem b is continuous.

Hence, we have

$$| \langle L_s \circ L_p, \phi \rangle | \leq \sup(|b|) \int |\psi(r)| \int \chi_+^{2(-s-1)}(r-r') \chi_+^{2(-p-1)}(r') dr' dr \quad (11.14)$$

$$= \sup(|b|) \int |\psi(r)| \chi_+^{2(-s-p-1)}(r) dr \quad (11.15)$$

$$\leq \frac{\sup(|b|)}{|\Gamma(2(-s-p-1)+1)|} |C^{2(-s-p-1)}| |e^{i\pi(s+p)}|. \quad (11.16)$$

Thus, if we regard $\langle L_p \circ L_s - L_{p+s}, \phi \rangle$ as a function of p , it satisfies the hypotheses of 11.1 and so is identically zero, which proves the composition law.

We are now in a position to define L_s , for all complex s , by analytic extension.

Definition 11.2. $L_s = (D_t^2 - \Delta)^k L_{s-k}$ where $\Re(s-k) \ll 0$.

We must check that this is independent of the choice of k . But if $k > l$ we have

$$(D_t^2 - \Delta)^k L_{s-k} = (D_t^2 - \Delta)^l (D_t^2 - \Delta)^{k-l} L_{s-k} = (D_t^2 - \Delta)^l L_{s-l} \quad (11.17)$$

so it is well defined.

Using analyticity, we can extend the previously proven relations to the whole complex plane. Thus, we have

Theorem 11.2. L_s is an entire holomorphic family of kernels supported in the forward light cone such that

$$L_p \circ L_s = L_{p+s} \text{ where } s, p \in \mathbb{C} \quad (11.18)$$

$$L_0 = \text{Id} \quad (11.19)$$

$$L_1 = D_t^2 - \Delta \quad (11.20)$$

$$L_{-1} = \text{the forward fundamental solution.} \quad (11.21)$$

Our next task is to calculate the symbols in the variable coefficient case. We do this by reducing to the constant coefficient case using FIOs. Our proof mimics

that of Duistermaat and Hörmander to construct a parametrix for an operator of real principal type. First of all, we must check that our two representations in the constant coefficient case are equal.

Lemma 11.1. *If $\mathcal{M} = \mathbb{R}^n$ then $K_s = L_s$.*

Proof. As we have two holomorphic families it is enough to show equality for $\Re s \ll 0$ and $\Im s > 0$. Letting K be the forward fundamental solution of the wave equation, we compute

$$L_s = e^{i\pi(s+1)} 2(\pi_\tau)_* (\chi_+^{2(-1-s)}(r) K) \quad (11.22)$$

$$= e^{i\pi(s+1)} \frac{2}{(2\pi)^3} \int e^{i(\langle x, \xi \rangle + t\tau)} \left[\lim_{\delta \rightarrow 0^+} \int e^{ir\eta} \chi_+^{2(-1-s)}(r) \frac{1}{(\tau - i0)^2 - \xi^2 - \eta^2} dr d\eta \right] d\xi d\tau \quad (11.23)$$

$$= i e^{i\pi(2s+1)} \frac{2}{(2\pi)^3} \int e^{i(\langle x, \xi \rangle + t\tau)} \left[\lim_{\delta \rightarrow 0^+} \int \frac{(-\eta - i0)^{1+2s}}{a_\delta^2 - \eta^2} d\eta \right] d\xi d\tau \quad (11.24)$$

where $(\tau - i\delta)^2 - \xi^2 = a_\delta^2$ and we can take $\Im a_\delta < 0$.

We want to evaluate the inner integral, this is equal to a contour integral along the real axis with a small semi-circular detour below the axis at the origin, using Cauchy's theorem. Taking $\Im s > 0$ and considering a large semi-circular contour below the axis, we conclude from Cauchy's Residue theorem that

$$\int \frac{(-\eta - i0)^{1+2s}}{a_\delta^2 - \eta^2} = -2\pi i \operatorname{Res} \left(\frac{(-\eta - i0)^{1+2s}}{a_\delta^2 - \eta^2}, a_\delta \right) \quad (11.25)$$

$$= -i\pi e^{-i\pi(2s+1)} (a_\delta)^{2s} \quad (11.26)$$

And hence,

$$L_s = \frac{1}{(2\pi)^2} \int e^{i(\langle x, \xi \rangle + t\tau)} ((\tau - i0)^2 - \xi^2)^s d\xi d\tau \quad (11.27)$$

□

Now, the principal symbol of the variable coefficient wave operator is $\tau^2 - \sum g^{ij}(x)\xi_i\xi_j$ where g is the Riemann metric on M whereas that of the constant coefficient wave operator, $D_t^2 - \Delta_F$, is $\tau^2 - \xi^2$.

Lemma 11.2. *Let $q \in T^*(X \times \mathbb{R}) - 0$ be a point in the characteristic variety of the wave operator then there exists a homogeneous symplectomorphism, f , from a conic neighbourhood U in $T^*(\mathbb{R}^{n+1}) - 0$ to a conic neighbourhood $V \subset T^*(X \times \mathbb{R}) - 0$ containing the bicharacteristic through q such that $f^*(\tau^2 - \sum g^{ij}(x)\xi_i\xi_j) = \tau^2 - \xi^2$.*

Proof. We prove that there exist homogeneous symplectic coordinates, (y, η) , in a conic neighbourhood of the bicharacteristic such that in these coordinates

$$\tau^2 - \sum g^{ij}(x)\xi_i\xi_j = \eta_1\eta_2.$$

This will mean that we can in particular do so for the constant coefficient wave operator and so the result follows.

We can write

$$\tau^2 - \sum g^{ij}(x)\xi_i\xi_j = \left(\tau - \left(\sum g^{ij}(x)\xi_i\xi_j \right)^{\frac{1}{2}} \right) \left(\tau + \left(\sum g^{ij}(x)\xi_i\xi_j \right)^{\frac{1}{2}} \right). \quad (11.28)$$

So we define

$$\eta_1 = \left(\tau - \left(\sum g^{ij}(x)\xi_i\xi_j \right)^{\frac{1}{2}} \right), \quad (11.29)$$

$$\eta_2 = \left(\tau + \left(\sum g^{ij}(x)\xi_i\xi_j \right)^{\frac{1}{2}} \right). \quad (11.30)$$

These functions are smooth in a conic neighbourhood of the characteristic variety and are homogeneous of degree one. We also have

$$\{\eta_1, \eta_2\} = 0 \quad (11.31)$$

and we can therefore extend to a homogeneous, symplectic coordinate system, (y, η) , in a small conic neighbourhood W . (see [7], Chap. 21)

At a point in the characteristic variety either $\eta_1 = 0$ or $\eta_2 = 0$. As the situation is symmetrical, we suppose $\eta_1 = 0$. We use the Hamiltonian flow to extend our coordinates to a neighbourhood of the bicharacteristic through q . On $T^*(\mathbb{R}^n) - 0$, η_1 induces the flow $\frac{\partial}{\partial y_1}$ and on $T^*(M) - 0$ it induces a multiple of the bicharacteristic flow defined by the wave operator. Thus we can define for $p \in V = \bigcup_t \exp(tH_{\eta_1})W$

$$(y, \eta)(\exp(tH_{\eta_1})(p)) = (y, \eta)(p) + (0, e_1)t \quad (11.32)$$

where $e_1 = (1, 0, 0, \dots, 0)$. This has the right properties as η_1, η_2 Poisson commute. \square

Proposition 11.1. *Given $q \in T^*(X \times \mathbb{R}) - 0$, there exist classical Fourier integral operators $A \in I^0(X \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, \Gamma'_f)$ and $B \in I^0(\mathbb{R}^n \times \mathbb{R} \times X \times \mathbb{R}, \Gamma'_{f-1})$ such that $q \notin WF'(AB - I)$ and $(q, f(q)) \notin WF'((D_t^2 - \Delta)A - A(D_t^2 - \Delta_F))$. If q is in the characteristic variety then the result holds for the entire bicharacteristic through q .*

Proof. If q is characteristic, let f be the symplectomorphism from lemma 11.2. Otherwise considering the square root of the modulus of $\tau^2 - \sum g^{ij}\xi_i\xi_j$ as a symplectic coordinate we see that such coordinates exist in a conic neighbourhood, V , of q . To do the two cases at once, let W be the set $\{q\}$, if q is non-characteristic and the set of points in the bicharacteristic through q , if q is characteristic.

Let A_1 be an element of $I^0(X \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, \Gamma'_f)$ which is elliptic on $(W, f(W))$ and have a classical symbol and let $B_1 \in I^0(\mathbb{R}^n \times \mathbb{R} \times X \times \mathbb{R}, \Gamma'_{f-1})$ satisfy $W \cap WF'(A_1B_1 - I) = \emptyset$ then from the calculus of FIOs, we have that the principal symbol of $Q = B_1(D_t^2 - \Delta)A_1$ is $\tau^2 - \xi^2$ and that

$$(W, f(W)) \cap WF'((D_t^2 - \Delta)A - A(D_t^2 - \Delta_F)) = \emptyset.$$

This reduces us to proving that if Q is a first order classical pseudo-differential operator then there exists zeroth order classical pseudo-differential operator A_2 , which

is micro-elliptic on W , such that

$$(D_t^2 - \Delta_F + Q)A_2 - A_2(D_t^2 - \Delta_F) \in \Psi^{-\infty} \quad (11.33)$$

and putting $A = A_1 A_2$ our result follows.

To solve (11.33), we rewrite it in the form

$$[D_t^2 - \Delta_F, A_2] + Q A_2 \in \Psi^{-\infty}. \quad (11.34)$$

If we denote the principal symbol of Q by q_1 and of A_2 by a_0 then the vanishing of the principal symbol requires,

$$\left(2\tau \frac{\partial}{\partial t} - 2\xi \cdot \frac{\partial}{\partial x}\right) a_0 + q_1 a_0 = 0.$$

Dividing this through by $|\tau, \xi|$ gives us

$$\left(2 \frac{\tau}{|\tau, \xi|} \frac{\partial}{\partial t} - 2 \frac{\xi}{|\tau, \xi|} \cdot \frac{\partial}{\partial x}\right) a_0 + \frac{q_1}{|\tau, \xi|} a_0 = 0$$

The coefficients of this equation are homogeneous of degree zero and so if we specify non-zero, degree 0 homogeneous initial data on a non-characteristic, conic hypersurface, we can solve to obtain a non-zero degree 0 homogeneous function. So, letting A_0 have principal symbol a_0 , we have solved our equation up to zeroth order. We now use an iterative process to obtain lower order terms. We now choose A_{-1}, A_{-2}, \dots such that A_j is classical and has homogeneous principal symbol of degree j and such that

$$[D_t^2 - \Delta_F, A_0 + A_{-1} + \dots + A_{-N}] + q(A_0 + A_{-1} + \dots + A_{-N}) \in \Psi^{-N-1}. \quad (11.35)$$

The condition on A_N is therefore

$$[D_t^2 - \Delta_F, A_{-N}] + q A_{-N} = R_{-N} \in \Psi^{-N} \quad (11.36)$$

or, in terms of principal symbols, we have

$$\left(2\tau \frac{\partial}{\partial t} - 2\xi \cdot \frac{\partial}{\partial x}\right) a_{-N} + q_1 a_{-N} = r_{-N}. \quad (11.37)$$

If we divide through as before and pick initial data of homogeneity $-N - 1$ this will have a solution of homogeneity $-N - 1$.

And so having constructed $\{A_j\}$ we pick $A_2 \sim \sum A_j$ and we are done. \square

Theorem 11.3. *If A and B are as in proposition 11.1 then*

$$(q, q) \notin \text{WF}'(AK_{F,s}B - K_s).$$

and if q is characteristic, the bicharacteristic through (q, q) in the first variable does not meet $\text{WF}'(AK_{F,s}B - K_s)$.

Proof. We will first of all, assume that $\Re s \ll 0$ and then use the microlocality of the wave operator to deduce the general case.

In this proof, all our operators are constant coefficient in r and so we will regard our kernels as functions of (x, x', t, t', r) and we let

$$\text{WF}'(L) = \{(x, t, \xi, \tau, x, t, \xi, \tau, r, \eta) : (x, t, \xi, \tau, x, t, -\xi, -\tau, r, \eta) \in \text{WF}(L)\}.$$

We denote by \bar{A} and \bar{B} , the operators with kernels $A(x, t, x', t')\delta(r)$ and $B(x, t, x', t')\delta(r)$.

We can write

$$\begin{aligned} AK_{F,s}B - K_s &= A((\pi_r)_*(\chi_+^s(r)K_F))B - (\pi_r)_*(\chi_+^s(r)K) \\ &= (\pi_r)_*(\chi_+^s(r)(\bar{A}K_F\bar{B} - K)). \end{aligned}$$

So,

$$(x_0, t_0, \xi_0, \tau_0, x_1, t_1, \xi_1, \tau_1) \in \text{WF}'(AK_{F,s}B - K_s)$$

implies

$$\exists r (x_0, t_0, \xi_0, \tau_0, x_1, t_1, \xi_1, \tau_1, r, 0) \in \text{WF}'(\chi_+^s(r)(\bar{A}K_F\bar{B} - K))$$

which means that either

$$\exists r (x_0, t_0, \xi_0, \tau_0, x_1, t_1, \xi_1, \tau_1, r, 0) \in \text{WF}'(\bar{A}K_F\bar{B} - K)$$

or

$$\exists \eta (x_0, t_0, \xi_0, \tau_0, x_1, t_1, \xi_1, \tau_1, 0, \eta) \in \text{WF}'(\bar{A}K_F\bar{B} - K).$$

Thus, in order to show $(q, q') \notin \text{WF}'(AK_{F,s}B - K_s)$, it is certainly sufficient to show that there does not exist (r, η) such that

$$(q, q', r, \eta) \in \text{WF}'(\bar{A}K_F\bar{B} - K). \quad (11.38)$$

We prove this by using the microlocal uniqueness of the forward fundamental solution of the wave equation, which we reprove using propagation of singularities. For simplicity, we write $\square = D_t^2 - \Delta$ and $\square_F = D_t^2 - \Delta_F$. We compute

$$\begin{aligned} (\square - D_r^2)(\bar{A}K_F\bar{B} - K) &= \square\bar{A}K_F\bar{B} - \bar{A}D_r^2\bar{B} - \text{Id} \\ &= \bar{A}(\square_F - D_r^2)K_F\bar{B} + (\square\bar{A} - \bar{A}\square_F)K_F\bar{B} - \text{Id} \\ &= (\bar{A}\bar{B} - \text{Id}) + (\square\bar{A} - \bar{A}\square_F)K_F\bar{B}. \end{aligned}$$

Now, our construction of A and B ensures that $W \cap \text{WF}'(AB - I) = \emptyset$ and so we have that $(q', q, r, \eta) \notin \text{WF}'(\bar{A}\bar{B} - I)$ for all q' in W . The same holds for $(\square\bar{A} - \bar{A}\square_F)K_F\bar{B}$ this is true because $(\square A - A\square_F)$ is smoothing at (q', p) for any p , (remember A is associated to a canonical graph) and so the lifted version will be smooth at (q', p, r, η) . Hence,

$$(q', q, r, \eta) \notin \text{WF}'((\square\bar{A} - \bar{A}\square_F)K_F\bar{B})$$

by the composition law for wavefront sets.

Now, any wavefront set of $\bar{A}K_F\bar{B} - K$ with non-zero (ξ, τ, ξ', τ') component must be supported in the forward flow out as this is true of K and K_F and any additional

singularities introduced by \bar{A} and \bar{B} will have zero (ξ, τ, ξ', τ') component. Thus if $(q', q) \in W$, we have

$$(q', q, r, \eta) \notin \text{WF}'(\bar{A}K_F\bar{B})$$

since otherwise the entire backward bicharacteristic through (q', q, r, η) would be in $\text{WF}'(\bar{A}K_F\bar{B} - K)$.

So, this establishes for $\Re s \ll 0$ that

$$(q', q, r, \eta) \notin \text{WF}'(\bar{A}K_F\bar{B} - K).$$

which implies the result, for $\Re s \ll 0$.

Using the microlocality of \square , we have

$$(q', q, r, \eta) \notin \text{WF}'(\square(AK_F^s B - K^s))$$

and

$$\square(AK_F^s B - K^s) = A \square_F K_F^s B - K^{s+1} + (\square A - A \square_F) K_F^s B.$$

The third term is smoothing at (q', q) and so this implies that

$$(q', q) \notin \text{WF}'(AK_F^{s+1} B - K^{s+1})$$

and the general case now follows by induction. \square

Letting, Λ_0 denote the conormal bundle to the diagonal and Λ_1 the flow-out of the characteristic variety's intersection with the diagonal in positive time, we are now ready to prove

Theorem 11.4. $L_s \in I_{phg}^{2s, s-1/2}(\Lambda_0, \Lambda_1)$. The principal symbol on Λ_0 off Λ_1 is

$$((\tau - i0)^2 - \sum g^{ij}(x)\xi_i\xi_j)^s$$

and on Λ_1 off Λ_0 , the principal symbol is

$$\frac{e^{i\frac{\pi}{2}s}}{\Gamma(-s)} (4 \sum g^{ij}(\xi))^{s/2} t^{-s-1} |d\xi|^{\frac{1}{2}} |dx|^{\frac{1}{2}} |dt|^{\frac{1}{2}}$$

at the point induced by the geodesic flow, in the first variable, at time t from the point $(x, \xi, x, \xi, 0, \pm |\sum g^{ij}(x)\xi_i\xi_j|^{\frac{1}{2}})$ on the diagonal with respect to the trivialization of the Maslov bundle given by transporting the natural trivialization at the diagonal by the geodesic flow.

Note that the density $|d\xi|^{\frac{1}{2}}|dx|^{\frac{1}{2}}$ is intrinsic here as it is the square root of the symplectic density on $T^*(M)$ and t is of course the ordinary coordinate on \mathbb{R} .

Proof. We already know this for K_s in the constant coefficient case and using the equality of K_s and L_s in that case, this is an immediate consequence of Egorov's theorem (Prop 6.7). \square

REFERENCES

1. J. Antoniano and G. Uhlmann, A Functional Calculus for a Class of Pseudodifferential Operators with Singular Symbols, *Proceedings of Symposia in Pure Mathematics*. Vol 43 (1985).
2. J.J. Duistermaat and L. Hörmander, Fourier Integral Operators II. *Acta Mathematicae* 1972.
3. J.J Duistermaat and V.W. Guillemin, The Spectrum of Positive Elliptic Operators with Periodic Bicharacteristics, *Inventiones Math.* 29 (1975)
4. V.W. Guillemin and G. Uhlmann, Oscillatory Integrals with Singular Symbols, *Duke Mathematical Journal*. Vol 48, No 1. (1981)
5. L. Hörmander, Fourier Integral Operators I, *Acta Mathematicae* 1971.
6. L. Hörmander, Analysis of Linear Partial Differential Operators Vol.1, *Springer Verlag, Berlin* 1985
7. L. Hörmander, Analysis of Linear Partial Differential Operators Vol.3, *Springer Verlag, Berlin* 1985
8. R.B. Melrose, Differential Analysis on Manifolds with Corners. *forthcoming*.
9. R.B. Melrose, Marked Lagrangian Distributions. *forthcoming*.
10. R.B. Melrose and G. Uhlmann, Lagrangian Intersection and the Cauchy Problem. *Communications in Pure and Applied Mathematics* 1979.
11. M. Riesz, L'integrale de Riemann Liouville et le Probleme de Cauchy. *Acta Mathematicae* 1948.

12. R.T. Seeley, Complex Powers of an Elliptic Operator. *A.M.S. Proc. Symp. Pure Math 10 (1967)*
Corrections in: The resolvent of an elliptic boundary problem. *Am. J. Math 91 (1969)*