THE BASIS AND APPLICATION OF PERTURBATION THEORY TO NEUTRON DIFFUSION

by

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ABSTRACT

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The role of the adjoint flux in neutron diffusion is clarified and extended to a time dependent concept in a generally non-critical reactor. The fractional importance is defined as the fractional contribution made by one neutron or precursor to the ultimate population of neutrons and precursors. From this physical concept, the mathematical properties of the variational form of calculation are deduced and the equations for the time dependent importance are derived in the diffusion and Boltzmann representations. The equivalence of neutrons is introduced as a time independent concept also of validity in the non-critical reactor, and shown to be related to the fractional importance.

A proof of the general properties of the variational form for the diffusion approximation with multigroups of neutrons and precursors is given and three theorems relating to the stationary property of the bilinear concomitant are derived. A discussion of the completeness of eigenfunction expansions is given in which the concept of joint error is introduced as an extension of the mean square error for non-self adjoint systems. A proof is given that the joint error is minimized (in a certain sense) by the use of the adjoint expansion coefficients. The multigroup equations for the system and the adjoint system are then derived as a consequence of a single variational principle, based on an appropriate Lagrange density.

The application of the adjoint flux to perturbation theory is reviewed and a physical account given of the perturbation form. A detailed commentary is made on the parameters in the light of the physical derivation of the importance. These results are illustrated by a series of experiments in the 1MW enriched uranium-heavy water research reactor of the Massachusetts Institute of Technology. The perturbation formula is generalized to include the time dependent reference reactor, and theorems are proved relating to the arbitrary nature of the kinetics parameters, with particular reference to the reactivity. The generating time is introduced and shown to lead to a significant simplification of the reactor kinetics equations.

The property of equivalence of perturbations and cross-sections is considered and the static reactivity is compared with the kinetic reactivity. A number of new applications of perturbation theory to the calculation of reactivities are introduced. Control rod calculations and Xenon effects in reflected reactors are treated. Surface perturbation methods for reflectors are developed. Experimental results are used to verify theoretical predictions. The enhancement is introduced as a unifying parameter describing the geometrical effects of spatially varying samples. The enhancement is related to the statistical weight and typical values calculated for a number of important reactor processes.

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> But other fell into good ground and brought forth fruit, some an hundredfold, some sixtyfold, some thirtyfold.
>
> Mark, IV, viii.

1.1 Scope of Work

The general field of enquiry concerns the kinetic behaviour of nuclear reactors with spatially varying properties and the basis and application of variational techniques to the problem. On the one hand, the treatment of reactor kinetics can be regarded as a mathematical excercise in eigenvalue problems. The bilinear or variational forms of calculations are well known to have improved accuracy in approximation methods; perturbation theory can be regarded as one application of the general variational approach. We are also concerned to develop a full and consistent physical basis for these mathematical techniques since we believe that such a physical understanding will improve the value of their subsequent application to the particular problems. Thus the exposition, in Section 3, given on physical grounds, itself leads to certain mathematical developments for the basis of the theory, given in Section 4. The second main portion of the thesis concerns applications to what is conveniently divided into Kinetics (Section 5) and Statics (Section 6).

A reactor described by diffusion theory with one group of neutrons and having uniform, constant properties, has a kinetic behaviour well known to be given by the inhour relation. To understand why the more general problem with spatially varying properties and several groups of

neutrons requires the application of variational methods, it is necessary to investigate the nature of the parameters appearing in the inhour relation and the nature of the general as opposed to the particular solution to the kinetics problem. The remainder of this introduction serves to give the model and assumptions used in the subsequent analyses and also to clarify the role played by the kinetic parameters. Here we will develop a matrix notation suitable for the generality in which we wish to work. In Section 2 we give an historical account which relates the considerable body of work on variational methods known to classical physics with the field of reactor physics. On the basis of this discussion we can define the concepts that lead to a generalisation of current concepts in reactor physics and a detailed derivation of the properties of the mathematical techniques. In view of our accent on the physical basis, we are fortunate that the start-up programme of the Research Reactor of the Massachusetts Institute of Technology enables us to offer experimental verification of points of theory and application.

1.2 Neutron Balance Equations

Although the simple diffusion approximation even in the multigroup treatment is not considered as the best representation of the behaviour of neutrons in a reactor, from its simplicity and tractability it is undoubtedly the most widely used. Because of this common use we choose to give our development in the diffusion approximation so that the physical implications can be seen directly. In an Appendix to Section 3 we give a derivation in terms of the Boltzmann integro-differential equation to show that all those corrections customarily made to diffusion

theory on the basis of the Milne problem are applicable to our own development.

We shall not consider the statistical fluctuations occurring in a system nor any problems of quantum as opposed to classical continuum theory. We shall have occasion to refer to the existence of one neutron in a reactor however. Such a statement is intended literally and not as referring to the average of a large number of neutrons, normalised to unity. To reconcile this usage with our assumption of the diffusion equation, we must interpret the results in the sense of the Gibbsian ensemble; that the diffusion equation will approximate the average behaviour of one neutron in a very large number of duplicate or identical reactors.

The commonly used one-group equation with no delayed neutrons has the form

$$D \Delta \phi + k \Sigma \phi - \Sigma \phi = V^{-1} \frac{\partial \phi}{\partial t} \qquad (1.1)$$

where D is the diffusion coefficient, k, the production rate probability and Σ the absorption rate probability. In this form however, the equation is only valid for a reactor in which the diffusion coefficient and thus the scattering properties do not vary. If there is more than one region we must consider the appropriate interface conditions between the regions. It is more convenient in discussing the properties of solutions to eq. (1.1) to utilise the form which allows all parameters to vary over the reactor space and which does not require the separate considerations of interface conditions. Thus the form we shall employ has the single equation valid for every small element of volume within

the reactor:

$$\nabla \cdot D \nabla \phi + k \Sigma \phi - \Sigma \phi = V^{-1} \frac{\partial \phi}{\partial t} \qquad (1.2)$$

Here all parameters are regarded as functions of position, e.g., $D = D(\underline{r})$. Of course in some parts of the reactor these parameters may actually be zero, e.g., $k\Sigma$ vanishes within the reflector. We shall not deal with the more general problems in which the parameters are also functions of time.

We also assume that we are presented with some initial conditions represented by the initial value of the flux, $\phi(o)$. Later, the adjoint flux will be introduced, ϕ^+, and we shall consider this also to be a function of time with its appropriate initial conditions. We might suppose that some arrangement of sources, subsequently removed from the reactor, can induce any specified initial neutron distribution – the subsequent behaviour of the flux is then our fundamental problem.* We remark finally that we consider throughout that the flux, $\phi(\underline{r}, t)$ – and the adjoint flux – is a real physical entity and we shall not treat the transformed flux, $\phi(\underline{r}, s)$ say, in the non-physical Laplace transform space. The transform variable, s, however, does correspond closely to the logarithmic time derivative of the flux, $\frac{\partial \phi}{\partial t}/\phi$:

$$\frac{\partial \phi}{\partial t} = s\,\phi - \phi(\underline{r}, o). \qquad (1.3)$$

*Weinberg and Wigner (1) discuss the property of real dissipative systems, as represented by the diffusion term, that the initial conditions must have occurred at some finite point in time which cannot be infinitely far back. A similar difficulty with the double sided Laplace transform is well known where the initial conditions must be taken to be zero if the system is not to have dissipated infinite energy.

Henceforth we shall employ s for the logarithmic derivative. In general s may be regarded as a function of time and space. It is well known however that the problem of the reactor behaviour can be expressed by considering solutions with constant values of s (the eigenwert).

1.3 Multigroup

The general form of the multigroup equations has the form typified by the two-group example

$$\begin{aligned} \nabla \cdot D_1 \nabla \phi_1 - \Sigma_1 \phi_1 + \frac{k}{p} \Sigma_2 \phi_2 &= v_1^{-1} \frac{\partial \phi_1}{\partial t} \\ p \Sigma_1 \phi_1 + \nabla \cdot D_2 \nabla \phi_2 - \Sigma_2 \phi_2 &= v_2^{-1} \frac{\partial \phi_2}{\partial t} \end{aligned} \qquad (1.4)$$

An important physical point concerns the off-diagonal coefficients, $\frac{k}{p} \Sigma_2 \phi_2$ and $p \Sigma_1 \phi_1$ in equation (1.4). Since there will be some fast leakage to balance in the reactor, the supply of neutrons from fission to the fast group must, on the average over the reactor at any rate, be greater than the return of neutrons to the thermal group by slowing down. Then $\frac{k}{p} \Sigma_2 \phi_2$ must on the average be larger than $p \Sigma_1 \phi_1$. This asymmetry leads to a certain amount of mathematical complexity.

A difficult question arises when the scattering cross section varies so much with energy that the extrapolation distances at the outer boundary are significantly different between groups. Obviously this effect will be most important in small reactors. In general, it is no longer possible to obtain closed solutions to such systems and the problem is difficult. In common with general practice, we will ignore such cases, leaving them to transport theory (see (1) however).

The 2 and 1/2 group example just given in equation (1.4) is probably the only multigroup equation solved by hand, though for some special cases we shall use 3 group calculations. The theoretical development of general results will require large sets of simultaneous equations.

1.4 Precursors and Delayed Neutrons

The simplest representation of delayed neutron effects involves one group of neutrons and one group of precursors with equations similar in appearance to the two group equations previously discussed.

$$\begin{aligned} V^{-1}\frac{\partial \phi}{\partial t} &= \nabla \cdot D\nabla\phi - \Sigma\phi + (1-\beta)F\phi \qquad +\lambda C \\ \frac{\partial C}{\partial t} &= \beta F \phi \qquad -\lambda C \end{aligned} \tag{1.5}$$

The most significant difference from the two-group equations is in the absence of a diffusion term for leakage in the precursor balance. We shall not consider those problems corresponding to circulating fuel reactors where there might be more complicated effects.

Owing to the disappearance of the leakage term, the neutron precursor ratio can be expressed explicitly in terms of the eigenvalues. Thus the second equation of (1.5), replacing $\frac{\partial C}{\partial t}$ with $s\,C$, gives

$$\frac{C}{\phi} = \frac{\beta F}{\lambda + s} \tag{1.6}$$

Such a substitution can only be carried out in the analogous equation with two groups of neutrons when the leakage term is reducable to an algebraic buckling, i.e., in the one region reactor. The absence of the leakage term from the precursor equations leads to a useful simplification of the equation for the eigenvalue, s.

A second difference lies in the nature of the decay probabilities λ_i.

The effect of the precursors is not to take away any neutrons from the system – there is no true absorption – but merely to delay the appearance of a neutron before it arrives in the neutron group.

1.5 Matrix Representation

The scope of this work makes it useful to find a shorthand notation for the simultaneous equations we have discussed. We might want to allow for the difference in energy between prompt and delayed neutrons by writing separate neutron groups for each energy. With, say, six precursors and a few groups of photoneutrons, the number of equations mounts to an unwieldy figure. Fortunately the matrix and vector notation is admirably suited to represent simultaneous equations. The particular scheme we use has the appearance of the simplest one energy representation which should help to keep the physical principles in view.

To avoid writing $(k\Sigma)$, in view of the fact that in the multigroup representation this term corresponds more closely to $\nu\Sigma_f$ and, finally, to avoid confusion with the absorption term, Σ, already appearing in the equation, we define a new term, F, to be the total fission probability per unit flux. Thus $F\phi$ gives the rate of production of prompt neutrons and precursors per unit volume. $F\phi$ replaces $k\Sigma\phi$ in equation (1.2) et. seq. When precursors are considered, F gives the total rate of production of prompt neutrons and precursors.

For the multigroup cases we write matrices for the different processes. For example, to represent the two-group case, equation (1.4), we write:

$$\text{diffusion:} \begin{bmatrix} \nabla \cdot D_1 \nabla & 0 \\ 0 & \nabla \cdot D_2 \nabla \end{bmatrix} = \nabla \cdot D \nabla ; \tag{1.7}$$

$$\text{transfer and absorption;} \begin{bmatrix} -\Sigma_1 & 0 \\ p\Sigma_1 & -\Sigma_2 \end{bmatrix} = -\Sigma ; \tag{1.8}$$

$$\text{production:} \begin{bmatrix} 0 & \frac{k}{p}\Sigma_2 \\ 0 & 0 \end{bmatrix} = F ; \tag{1.9}$$

$$\text{inverse speeds:} \begin{bmatrix} V_1^{-1} & 0 \\ 0 & V_2^{-1} \end{bmatrix} = V^{-1} ; \tag{1.10}$$

Then, if we write a flux vector, ϕ, as

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \phi , \tag{1.11}$$

the usual rules of matrix multiplication make the equation

$$(\nabla \cdot D \nabla - \Sigma + F)\, \phi = V^{-1} \frac{\partial \phi}{\partial t} \tag{1.12}$$

an exact representation of the 2 group equations, (1.4).

Even this expression can be compressed by defining the increase matrix M,

$$M = (\nabla \cdot D \nabla - \Sigma + F). \tag{1.13}$$

Then any number of groups can be represented by the concise equation

$$M\phi = V^{-1} \frac{\partial \phi}{\partial t} \tag{1.14}$$

Corresponding to eq. (1.14) there is an eigenvalue equation which takes the form

$$M\phi = s V^{-1} \phi \tag{1.15}$$

The full representation of the neutron and precursor balances with allowance for the energy differences between prompt and delayed fission neutrons, requires large numbers of simultaneous equations. It is suggestive and economical to express all of these equations in a single matrix equation.

Consider the elements of the neutron and precursor balance, which can be divided into two main terms. First, we have the terms involving the production of prompt neutrons and precursors and the destruction of neutrons (including neutron transfer). All these terms were present in the simplified case where precursor hold-up was neglected. Thus we extend the meaning of our increase matrix, M, to include precursor and prompt neutron production as well as neutron destruction. The remaining terms in the population balance involve the transfer of neutrons from the form of precursors to the form of neutrons. This precursor transfer will be expressed by a second matrix, T. The matrix M will lead to an expression for the reactivity or other measure of criticality. The matrix T will simplify and vanish in our normal treatment owing to the fact that no precursors are really lost to the system, but reappear as neutrons.

For the simple one group example of eq. (1.5), we would define

$$M = \begin{bmatrix} \nabla \cdot D\nabla - \Sigma + (1-\beta) F & 0 \\ \beta F & 0 \end{bmatrix} \tag{1.16}$$

$$T = \begin{bmatrix} 0 & +\lambda \\ 0 & -\lambda \end{bmatrix} \tag{1.17}$$

Then in defining the generalized flux vector, ψ, by

$$\psi = \begin{bmatrix} \phi \\ C \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ C_1 \\ \vdots \\ C_i \end{bmatrix} \quad \text{(in general)} \tag{1.18}$$

the matrix expression $(M+T)\psi$, exactly reproduces the neutron and precursor balance, the right hand side of equation (1.5).

The rate of change of the flux and precursor densities can be expressed in a similar fashion. Define the diagonal matrix A by

$$A = \begin{bmatrix} V^{-1} & 0 \\ 0 & 1 \end{bmatrix} \tag{1.19}$$

Then $A \frac{\partial \psi}{\partial t} = (M+T)\psi$, (1.20)

with the corresponding eigenvalue equation

$$s A\psi = (M+T)\psi. \tag{1.21}$$

1.6 Some Properties of the Matrices

For the most part, the matrices we have defined have the properties normally associated with algebraic matrix and vector representation as given for example by Hildebrand (2). Thus, all the matrices are associative:

$$\nabla \cdot D' \nabla = \begin{bmatrix} \nabla \cdot D_1 \nabla + \nabla \cdot \delta D_1 \nabla & 0 \\ 0 & \nabla \cdot D_2 \nabla + \nabla \cdot \delta D_2 \nabla \end{bmatrix} = \nabla \cdot D \nabla + \nabla \cdot \delta D \nabla \tag{1.22}$$

We now introduce a second vector, ϕ^+, to be identified later with the adjoint flux or the importance of neutrons and precursors. The vector ϕ^+ is always compatible with ϕ, i.e., it has the same number of elements:

$$\phi^+ = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix} \tag{1.23}$$

Then it is possible to write a scalar product such as $\phi^+ F \phi$. Here we do not need to distinguish between row and column vectors. Since we shall only be forming this simple type of scalar product, the positions of the vectors, row to the left and column to the right, serve to identify the nature of the vectors in an equation.

The purely algebraic matrices commute in the sense that

$$\phi^+ F \phi = \phi F^T \phi^+ \tag{1.24}$$

at every point in the reactor for any ϕ and ϕ^+. In equation (1.24), F^T

is the transpose of the production matrix, F, or in terms of the example of equation (1.9):

$$F^T = \begin{bmatrix} 0 & 0 \\ \frac{k}{p}\Sigma_2 & 0 \end{bmatrix} \tag{1.25}$$

The diffusion matrix, $\nabla \cdot D\nabla$, however, is not purely algebraic but contains operators. Thus $\phi^+ \nabla \cdot D\nabla \phi$ does not commute as in equation (1.24). We shall show later that it does commute on the average, when integrated over the whole reactor, under suitable boundary conditions. It follows that the matrix M which contains the diffusion matrix, does not commute in general either, but will do so under the same conditions as the diffusion matrix.

We can however take the transpose of $\nabla \cdot D\nabla$ as well as A and V^{-1}. Since these matrices are purely diagonal, each one is equal to its own transpose.

When we extend the matrix representation to include precursors and delayed neutrons, we can extend our vector ϕ^+ to a vector, ψ^+ by defining:

$$\psi^+ = \begin{bmatrix} \phi_1^+ \\ \phi_2^+ \\ \vdots \\ C_1^+ \\ \vdots \\ C_i^+ \end{bmatrix} \tag{1.26}$$

Again ψ^+ is defined to be compatible with ψ so that we can form the scalar product $\psi^+(M+T)\psi$.

It is convenient to be able to express an actual reactor in terms of the properties of a reference reactor and the difference in these properties or the perturbation. We use a prime to indicate the actual properties while an "unprime" refers to the reference value. The perturbation is then written in terms of a δ, though at this moment, we do not imply that the perturbation is necessarily small. The arbitrary choice of the reference reactor establishes the nature of the perturbation.

Starting with equation (1.2), we consider the corresponding equation when the system is perturbed at a number of points to new properties such that

$$\begin{aligned} D' &= D + \delta D \\ F' &= F + \delta F \\ \text{and} \quad \Sigma' &= \Sigma + \delta\Sigma \\ \phi' &= \phi + \delta\phi \end{aligned} \tag{1.27}$$

Then we have

$$\nabla \cdot D' \nabla \phi' - \Sigma' + F' \phi' = V^{-1} \frac{\partial \phi'}{\partial t} \tag{1.28}$$

We retain the symbol M for the increase matrix in the actual (primed) reactor. We do not prime M however since we use a separate symbol, R, for the value of the increase matrix in the reference or unperturbed reactor. Thus, if the reference reactor is chosen to be critical

$$M\phi' = s V^{-1} \phi', \tag{1.29}$$

$$R\phi = 0. \tag{1.30}$$

The difference in M and R is defined to be P, the perturbation in the properties. We have, in view of the associative property,

$$P = M - R = \nabla \cdot \delta D \nabla - \delta \Sigma + \delta F \tag{1.31}$$

1.7 Counting the Population

To find the total number of neutrons in a reactor in the one-group approximation we need only integrate the neutron density over the total volume. Thus the population, N, would be given by

$$N = \int V^{-1} \phi' \, dv \tag{1.32}$$

All such integrals are understood to be over the reactor volume. In the multigroup treatment, however, we must form a scalar product of the neutron density. To do this we use the 'Ones' vector, I, to count the neutrons where in the 2 group example

$$I = [1, \ 1]$$

Then explicitly

$$I V^{-1} \phi' = [1 \ \ 1] \begin{bmatrix} V_1^{-1} & 0 \\ 0 & V_2^{-1} \end{bmatrix} \begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix}$$

$$= V_1^{-1} \phi_1' + V_2^{-1} \phi_2' \tag{1.33}$$

The population is then given, for any number of groups, by

$$N = \int I V^{-1} \phi' \, dv \, . \tag{1.34}$$

This counting is done by giving equal weight to all neutrons

irrespective of position or energy. The ones vector is obviously just a special value of a general weighting function which counts the neutrons with a different importance of weight depending on energy and position such as the ϕ^+ of equation (1.23).

When we include the representation of precursors and delayed neutrons, the definition of the reactor population is extended to include both neutrons and precursors. Thus if we take I to be a compatible vector, the total population is given by

$$N = \int I A \psi' \, dv = \int I V^{-1} \phi' \, dv + \int I C' \, dv \, . \qquad (1.35)$$

Again, the vector ψ^+ is a generalisation of the ones vector by means of which the population can be counted with a weighting function dependent on energy and position, rather than with the unit weighting.

1.8 Eigenvalues and the Inverse Period

One approach to the solution of the time-dependent equations is based on the following observation. Whatever the initial flux condition in a reactor, if the properties are fixed (independent of time) then the flux finally settles to a distribution where the logarithmic time derivative, $\frac{\partial \phi}{\partial t}/\phi$, is the same at every point in the reactor. This logarithmic time derivative is the inverse asymptotic period, s. Furthermore, the final flux shape is fixed only by the properties and not by the initial flux distribution.

It is important to realize that the period or its inverse is the only observable physical characteristic of the reactor as a whole. It is the sign and magnitude of the asymptotic period that is the only true measure of the criticality of a reactor. Our problem can now be

reduced to finding the solution, not of equation (1.14), but of the corresponding eigenvalue equation

$$M\phi' = (\nabla \cdot D'\nabla - \Sigma' + F')\phi' = s V^{-1}\phi' . \qquad (1.36)$$

Mathematically it is found that a number of solutions of eq. (1.36) can be found, each with a value of s, s_n say, and a flux distribution, ϕ'_n. Each of these fluxes will then change in time exponentially as $e^{s_n t}$. Now, the asymptotic flux must eventually dominate all the other fluxes and it must also, for physical reasons, be positive at every point within the reactor and in each energy group. We wish to find the solution corresponding to the algebraically largest s_n; in general it is found that all of the other mathematical solutions cannot exist physically independently of the asymptotic flux since all other ϕ_n change sign somewhere within the reactor, i.e., they oscillate.

The German word eigenwert is translated as characteristic value and indeed s is the one characteristic of a reactor that has significance and can be measured. It might be called an operational quantity. Other quantities such as the effective multiplication factor and reactivity, which we shall discuss, have a clear physical meaning and yet cannot in fact be measured directly.

Mathematically we might say that since s has the same value at every point in the reactor, it will not be affected by any averaging scheme we use, whether with a unit weighting (as with the ones vector, I) or with a weighting dependent on energy and position (ϕ^+). Hence, since

$$M\phi' = s V^{-1}\phi' , \qquad (1.37)$$

then

$$s \int \phi^{+} V^{-1} \phi' \, dv = \int \phi^{+} M \phi' \, dv. \tag{1.38}$$

for any weighting function, ϕ^{+}. We can write s as the ratio of these integral terms and it will then have the appearance of an average over the reactor

$$s = \frac{\int \phi^{+} M \phi' \, dv}{\int \phi^{+} V^{-1} \phi' \, dv} \tag{1.39}$$

We should remember that such an average is unique only if s is an eigenvalue. The average of any properties in a reactor, for example depends significantly on the weighting function, ϕ^{+}, or importance, employed.

The form of equation (1.39) is very convenient in the proof of the properties of the weighted form that improve approximation methods. By its means, we shall be able to express diffusion theory for several groups of neutrons and precursors in a single variational principle. The form which only displays one value of s, does have the disadvantage of concealing the additional values of s that we must expect when more than one equation is represented by the matrix notation. In the case of one group of neutrons and arbitrary groups of precursors, it becomes possible to express the multiple values explicitly by means of the algebraic coupling coefficients. For more than one group of neutrons, this is no longer true.

In addition to the possible oscillations within each mode due to the higher eigenfunctions which are solutions of equation (1.43), we anticipate additional solutions and eigenfunctions due to changes of sign in the coupling coefficient between the components within a flux. In a

bare reactor in two-group theory, even for the lowest algebraic buckling, B^2, there are two eigenfunction solutions. In one, the coupling coefficient is positive and the eigenfunction is the physically realisable asymptotic flux with the largest eigenvalue. The other coupling coefficient is negative; the eigenvalue is algebraically smaller, indeed it may well be smaller than one of the eigenvalues in the next harmonic where the flux within a component oscillates spatially.

We can group all those eigenfunctions in which each component has the same number of oscillations and call them a harmonic of the eigenfunctions. The number of oscillations in each component then characterises the harmonic. The asymptotic flux must lie in the zero or lowest harmonic. Within each harmonic there will be at least as many eigenfunctions or modes as there are simultaneous equations. In three-dimensional systems, as the number of the harmonic rises, there will be considerably more ways of getting the appropriate number of oscillations. In many cases, the eigenvalues corresponding to the eigenfunctions within an harmonic will all be different, but in cases of symmetry especially, it is often possible for a number of the eigenvalues to degenerate to the same value – an accidental degeneracy. Only in exceptional and trivial circumstances will there be a degeneracy in the lowest mode, the asymptotic eigenfunction (3). Because of the change of sign of the coupling coefficients, it is not always possible to use one approximate value to represent all the fluxes within a given harmonic. The limitations and some solutions overcoming this difficulty are taken up in Appendix C to Section 5.

1.9 Exact Solutions of Non-Critical Reactors

The kinetic behaviour of a one region reactor is well known to have the form of the inhour equation, usually written as

$$\rho = \frac{s\ell'}{k_{eff}} + \sum_i \frac{s\beta_i}{(\lambda_i + s)} \qquad (1.40)$$

We have primed the effective lifetime, ℓ', to emphasise that the actual absorption properties must be employed. Thus if the problem relates to the behaviour of the reactor when shut down by a control rod bank, then the correct lifetime would be calculated by using an effective absorption cross section to represent the rods as well as the rest of the reactor.

The characteristic property of equation (1.40) is the multiplicity of the roots which appear when the precursors and delayed neutrons are considered. This multiplicity of roots is a peculiarity of the spatially separable system. In general, the solution of a non-critical reactor involves finding not only the different roots but also the different flux shapes that correspond to them, one for one. The different flux solutions are all fundamental in the sense that they are all of the same sign throughout the reactor – they do not oscillate. Of course, one of the roots will be algebraically larger than the others and the corresponding flux solution is the one that dominates asymptotically in time, the remaining solutions decaying. In the one region case, the flux shapes for critical or non-critical reactors are determined by the external shape only and hence all the fundamental (i+1) flux solutions corresponding to equation (1.40) degenerate into the same shape. In so far as a multiregion reactor can be represented as a one region

reactor, the separate shapes of the general case cannot be very different if they are to degenerate to the same shape in the special case. They must also be very similar to the flux shapes in the reference reactor if this reactor is not too far different in size and properties from the actual reactor. Usually we choose to use a critical reference reactor and by ϕ we shall normally mean the flux shape corresponding to the asymptotic period (which is infinite if the reactor is critical).

It may be apparent that it is exceedingly tedious to obtain an exact analytic solution of even the simplest cases. Benedict (4) has given a solution for the reflected spherical reactor with one energy group, which leads basically to the solution of the transcendental equation

$$D_c(B_n R \cot B_n R - 1) = -D_r(K_n R \coth K_n T + 1), \tag{1.41}$$

where subscripts r, c refer to the reflector and core, R is the interface and T is the reflector thickness, while

$$D_c B_n^2 = \left[k - 1 - s_n k \sum_i \frac{\beta_i}{\lambda_i + s_n}\right] \Sigma_c - \frac{s_n}{V}, \tag{1.42}$$

$$D_r K_n^2 = \Sigma_r + \frac{s_n}{V}. \tag{1.43}$$

Two remarks are called for. First, we have neglected for the sake of clarity to represent the fact that there are higher modes in which the fluxes oscillate as in the simple one region case but with the corresponding splitting of eigenfunctions. Secondly, it should be noted that the root to be found, s_n, itself makes up part of the n^{th} buckling, B_n, which gives the shape of the flux solution in the core and part of K_n

giving the flux shape in the reflector. Also, s_n appears mostly in the form of an addition to the absorption cross section, $\frac{s_n}{V}$. That time dependent equations can be represented as equivalent steady state equations with a larger absorption is well known; one speaks of time absorption when the root, s_n, is positive since neutrons are used up not only in absorption but also in providing the net increase.

Even this set of equations represents only a very crude approximation to any real reactor. Few reactors are spherically symetric; no other fully reflected cases can be solved exactly and analytically. Furthermore, the one group model not only neglects the fast leakage of prompt neutrons but also the quite important effects due to the delayed neutrons having energies that differ appreciably from the mean fission neutron. Finally we would have to solve this system of equations for a number of different cases in order to be able to draw an inhour curve for the reactor. The task is indeed unrewarding.

For a just critical reactor, the precursor concentration is constant (asymptotically) and the precursors can be eliminated from the equations. Thus it is much easier to solve for the asymptotic flux in the reference reactor. An obvious advantage would be to find a form of approximation in which we could use the known flux of the critical reference reactor, ϕ, in place of the actual flux, ϕ'. This approximation of using ϕ for ϕ', is called the first order approximation. We must note however that it is required that the form in which we write our calculations must lead to a very accurate approximation. This requirement is due to the great sensitivity of reactors to such parameters as $k_{effective}$. For example, if k_{eff} is unity, we have a just

critical reactor. But a 1 per cent error in k_{eff} might lead to a prompt critical situation. Alternatively we can say that the time behaviour depends on the neutron balance or the difference between production and destruction. Thus the individual terms must be calculated with high accuracy to obtain a satisfactory value for their net effect. It is this stringent requirement for accuracy that rules out some simple approaches to our problem and forces us to use a more sophisticated technique.

1.10 One and Multiregion Reactors

Part of the difficulty in the task we have set ourselves lies in the difference between the nature of the solutions of the diffusion equation for the cases where the properties, D, etc., are uniform throughout the reactor and the cases where these properties vary.

We employ the following nomenclature: point systems, one-region systems and multiregion systems. Point systems describe cases where both the actual and the reference reactors have uniform properties. One-region systems imply a uniform reference reactor although the actual reactor is allowed to have variable properties. Finally, the multiregion system refers to those cases where both reference and actual reactors have variable composition. The distinction between these cases is not necessarily sharply drawn. A reflected reactor may be described adequately for some purposes by the concept of the equivalent bare reactor (e.g., for criticality calculations), while for other purposes (e.g., average neutron lifetimes) such an approximation may not be successful.

We must also distinguish between the terms homogeneous and

heterogeneous. Whereas many reactors have lattice or cell structures and are heterogeneous, they may often be represented, for the purpose of solving for the coarse flux distribution, as homogeneous systems. We shall not imply that such a homogeneous system necessarily has constant properties throughout the reactor, but we will admit macroscopic variations within the terminology of the multiregion case. Again, there is obviously no sharp division between heterogeneous and multiregion. If there is no variation from cell to cell, then the heterogeneous system is representable by a point system in our definition.

For the point system, the kinetic solutions are based on the following argument. The asymptotic flux in a critical or non-critical reactor behaves as an eigenfunction solution as previously discussed, the asymptotic inverse period being an eigenvalue. If we then equate the leakage to all other terms in the neutron balance, we find that the other terms by definition of the point system and by the eigenvalue property are each themselves independent of time and position. Hence the leakage term itself must be such an eigenvalue and can always be replaced by an expression containing the algebraic buckling

$$s V^{-1} \phi' + \Sigma' \phi' - F' \phi' = \nabla \cdot D' \nabla \phi' = - D' B^2 \phi' \qquad (1.44)$$

Since the left hand side of equation (1.44) is proportional to the flux when the flux has settled to the asymptotic shape, the right hand side must also be proportional to the flux. The point system is then representable by an equation whose terms have the same value point by point – i.e., all terms can be treated as eigenvalues and are independent of any averaging scheme employed. The fundamental difference between the special case and the general problem springs from

this property peculiar to the point system (or uniform property reactor).

1.11 Measures of Criticality in the Point System

Because of the reduction of the neutron and precursor balance to a simple algebraic equation involving only the properties and the reactor buckling, it is easy to develop apparently unambiguous measures of criticality. We have the conventional effective multiplication factor, the excess multiplication factor and the reactivity. In the one-group theory:

$$k_{eff} = \frac{k}{1 + L^2 B^2} \tag{1.45}$$

The effective multiplication factor is better displayed however as the ratio of production probability to destruction probability (destruction includes leakage as well as absorption), with the probability in terms of neutrons per neutron per second:

$$k_{eff} = \frac{V F'}{V(\Sigma' + D' B^2)} = \frac{\text{production}}{\text{destruction}} \tag{1.46}$$

Then the related concepts of excess multiplication and reactivity have the following significance

$$k_{ex} = k_{eff} - 1 = \frac{\text{increase}}{\text{destruction}} \tag{1.47}$$

$$\rho = \frac{k_{ex}}{k_{eff}} = \frac{\text{increase}}{\text{production}} \tag{1.48}$$

If we consider a point system for which the reference and the actual reactors have the same over-all size (and hence the same geometric buckling, B^2), and where the reference reactor has properties D etc., and the actual reactor D', etc., the measures of criticality have the

forms displayed in Table 1. 1. In this table, we have expressed the measures explicitly in terms of $\delta\tau$; the forms given simplify considerably when $\delta\tau$ is zero, i. e., when only thermal properties differ between reference and actual reactors. We might observe that the two-group and Fermi age expressions differ from the one-group expressions in that the production term is consistently reduced to a smaller effective value by the fast non-leakage probability. The expressions given for two-group theory are the commonly accepted ones. We will develop a slight modification in Section 5.

1. 12 Non-uniqueness of Criticality Measures and Lifetimes

We next discuss an important difference between the natures of measures of criticality and the inverse period, s. The asymptotic inverse period corresponds to a real, physically measurable property of the reactor – this statement is the physical counterpart of the mathematical property of the eigenvalue. Neither the effective multiplication nor the other measures of criticality derived from it have any such fundamental meaning when the reactor is not critical. Nor is there any way of measuring either the reactivity or the lifetime separately in a non-critical reactor. For the definition of both the measure of criticality and of the neutron lifetime are arbitrary in the sense that the definitions of the destruction process and the production process are arbitrary. For example, leakage is conventionally regarded as an additional destruction term. In a reflector, however, the net leakage is negative and the process might equally well be regarded as a production term of the opposite sign. Again, a common representation of scattering events, widely used in transport

	k_{eff}	k_{ex}	ρ
One Group	$\dfrac{F'}{\Sigma' + B^2 D'}$	$\dfrac{\delta F - (\delta\Sigma + B^2 \delta D)}{\Sigma' + B^2 D'}$	$\dfrac{\delta F - (\delta\Sigma + B^2 \delta D)}{F}$
Two Group	$\dfrac{\dfrac{F'}{(1+B^2\tau')}}{\Sigma_2' + B^2 D_2'}$	$\dfrac{\dfrac{F'}{1+B^2\tau'} - \dfrac{F}{1+B^2\tau} - (\delta\Sigma_2 + B^2\delta D_2)}{\Sigma_2' + B^2 D_2'}$ $\dfrac{\dfrac{\delta F}{1+B^2\tau'} - \dfrac{F B^2 \delta\tau}{(1+B^2\tau')(1+B^2\tau)} - (\delta\Sigma_2 + B^2\delta D_2)}{\Sigma_2' + B^2 D_2'}$	$\dfrac{\dfrac{F'}{1+B^2\tau'} - \dfrac{F}{1+B^2\tau} - (\delta\Sigma_2 + B^2\delta D_2)}{\dfrac{F'}{1+B^2\tau'}}$ $\dfrac{\dfrac{\delta F}{1+B^2\tau'} - \dfrac{F B^2 \delta\tau}{(1+B^2\tau')(1+B^2\tau)} - (\delta\Sigma_2 + B^2\delta D_2)}{\dfrac{F'}{1+B^2\tau'}}$
Fermi-Age	$\dfrac{F' e^{-B^2\tau'}}{\Sigma' + B^2 D'}$ approximately $(B^2\delta\tau)^2 \ll 1$	$\dfrac{F' e^{-B^2\tau'} - F e^{-B^2\tau} - (\delta\Sigma + B^2\delta D)}{\Sigma' + B^2 D'}$ $\dfrac{\delta F e^{-B^2\tau'} - F B^2 \delta\tau\, e^{-B^2\tau'} - (\delta\Sigma + B^2\delta D)}{\Sigma' + B^2 D'}$	$\dfrac{F' e^{-B^2\tau'} - F e^{-B^2\tau} - (\delta\Sigma + B^2\delta D)}{F' e^{-B^2\tau'}}$ $\dfrac{\delta F e^{-B^2\tau} F e^{-B^2\tau'} B^2\delta\tau - (\delta\Sigma + B^2\delta D)}{F' e^{-B^2\tau'}}$

TABLE 1.1 POINT SYSTEM MEASURES OF CRITICALITY

theory, is to consider the event as both an absorption of one neutron and a birth of another, i.e., equal amounts are added to the production and destruction terms. Obviously this does not invalidate the neutron balance. But who is to say whether the neutron emerging from the scattering potential is the 'same' neutron that entered? So long as we are consistent in interpreting the scattering event as either a death and a birth, or as no death and no birth, our choice of description will not affect the neutron balance or the physical period predicted for a non-critical reactor.

The choice of description for production and destruction will make a considerable difference to the values we would report for both lifetime and measures of criticality. Since scattering produces only one neutron when regarded as a production event, while fission averages 2 1/2, the average yield of production events will drop and our effective multiplication will be decreased (if originally greater than unity) or increased (if originally less than unity). However, it is easy to see that the value cannot change from one side of unity to the other merely from our changed description (nor can the reactivity or the excess multiplication change sign), since adding equal amounts to both terms cannot change the sign of the difference between the terms. Similarly if a reactor is critical, production equals destruction whatever the (consistent) description.

Since no change in the description can affect the period, the change in the reported value of the excess multiplication for example must be paralleled by a corresponding change in the reported lifetime. As an example, consider the previous case where scattering is included as

an event. Table 1.2 compares the forms for the two descriptions. The first description would be the natural one for a diffusion analysis; the second would be natural for a transport analysis.

Thus both ℓ and k_{ex} (and k_{eff} and ρ of course) contain an arbitrary but common normalization; the physically measurable period leads only to values of the ratio k_{ex}/ℓ. Note that the reported lifetime in the reactor may vary just because of the description selected but that at criticality, k_{ex} is zero independent of the lifetime.

Other authors have pointed out the lack of uniqueness in employing reactivity, etc., as a measure of the departure from critical. Weinberg and Wigner (1) for instance quote the example of the n, 2n reaction as one where it is not possible to determine whether the same neutron emerges as is absorbed. Thus the effect of the n, 2n reaction on the multiplication and the lifetime depends, again, on the description.

A study of the results in Table 1.2 is worthwhile. Often a reactor system is called 'safer' than some other system if the lifetime is much longer for the former than for the latter. In fact, if the reactor cross sections were changed by the same amount, the inverse periods would be the same (for the same mean velocity, V), independent of the neutron lifetime. Thus what is important to compare in such cases is not the lifetime but rather the amount of uniform addition of material that the accident represents in both cases. Comparing the addition of the same amount of uranium to a large and to a small reactor, it is evident that the perturbation is relatively smaller in the large reactor. When the two reactors to be compared are the same size, though perhaps with very different reported lifetimes (such as heavy water and

Table 1.2 Alternative description of Non-critical Systems

(neglecting delayed neutrons)

	Scattering not a death nor a birth		Scattering is a death and a birth
k_{ex}	$\frac{\delta F - \delta\Sigma - B^2\delta D}{\Sigma' + B^2 D'}$	$\neq$	$\frac{\delta F - \delta\Sigma - B^2\delta D}{\Sigma' + \Sigma'_s + B^2 D'}$
ℓ	$\frac{1}{V(\Sigma' + B^2 D')}$	$\neq$	$\frac{1}{V(\Sigma' + \Sigma'_s + B^2 D')}$
$s=k_{ex}/\ell$	$V(\delta F - \delta\Sigma - B^2\delta D)$	$=$	$V(\delta F - \delta\Sigma - B^2\delta D)$

(Σ is absorption cross section, Σ_s is scattering cross section)

Example

The decay of a pulsed source in a pure moderating material is conventionally analysed by diffusion theory as decaying with the period of the thermal lifetime. The corresponding values of the k_{ex} is then minus one, and the reactivity is minus infinity.

When interpreted as having production from scattering, the values change. For the original description based on the diffusion basis with:

$B^2 = .002\ cm^{-2}$ $\quad L^2 = 100\ cm^2$ $\quad V = 2200$ m/sec

$D = 1.00$ cm $\quad \Sigma_s = .333\ cm^{-1}$ $\quad \Sigma = .01\ cm^{-1}$

we obtain

ℓ	(diffusion)	$.38 \times 10^{-3}$ sec
ℓ	(transport)	1.32×10^{-5} sec
k_{ex}	(transport)	$-.035$
ρ	(transport)	$-.036$

water), from this crude analysis we see that the effects of the same addition will be of the same magnitude.

1.13 Choice of Measure of Criticality: the Reactivity

We can also express the effect of our choice of description through the relations for the measures of criticality, plotted in Figures 1.1 and 1.2. These figures are parametric in destruction, d_1 etc., and production, p_1 etc. That is, if we choose a description that will maintain a constant value for the destruction terms, then the relations between the inverse period and the measures of criticality are given by Figure 1.1. These figures also illustrate the relative advantages and disadvantages of the three measures in different regimes. Evidently it is a better procedure to calculate or estimate ρ or k_{ex} rather than to find k_{eff} and subtract unity, especially in the range close to critical. For if all the parameters could be calculated with the same fractional uncertainty, $(k_{eff} - 1)$ would be vastly more uncertain than k_{ex} or ρ. And since one of these last two is the required value for the inhour equation, it is more sensible to calculate the differences, k_{ex} or ρ directly. Close to critical there is no essential difference between k_{ex} and ρ.

For highly supercritical systems, ρ tends to unity while k_{ex} can be any value, so that in calculating the reactor period, it is probably more accurate to find k_{ex}. For highly subcritical systems, k_{ex} goes to minus unity (since k_{eff} is never negative); for subcritical systems especially in calculating control rods, it is probably better to calculate ρ. For further reasons advanced in Section 6 we shall base most of our discussion on the reactivity, ρ.

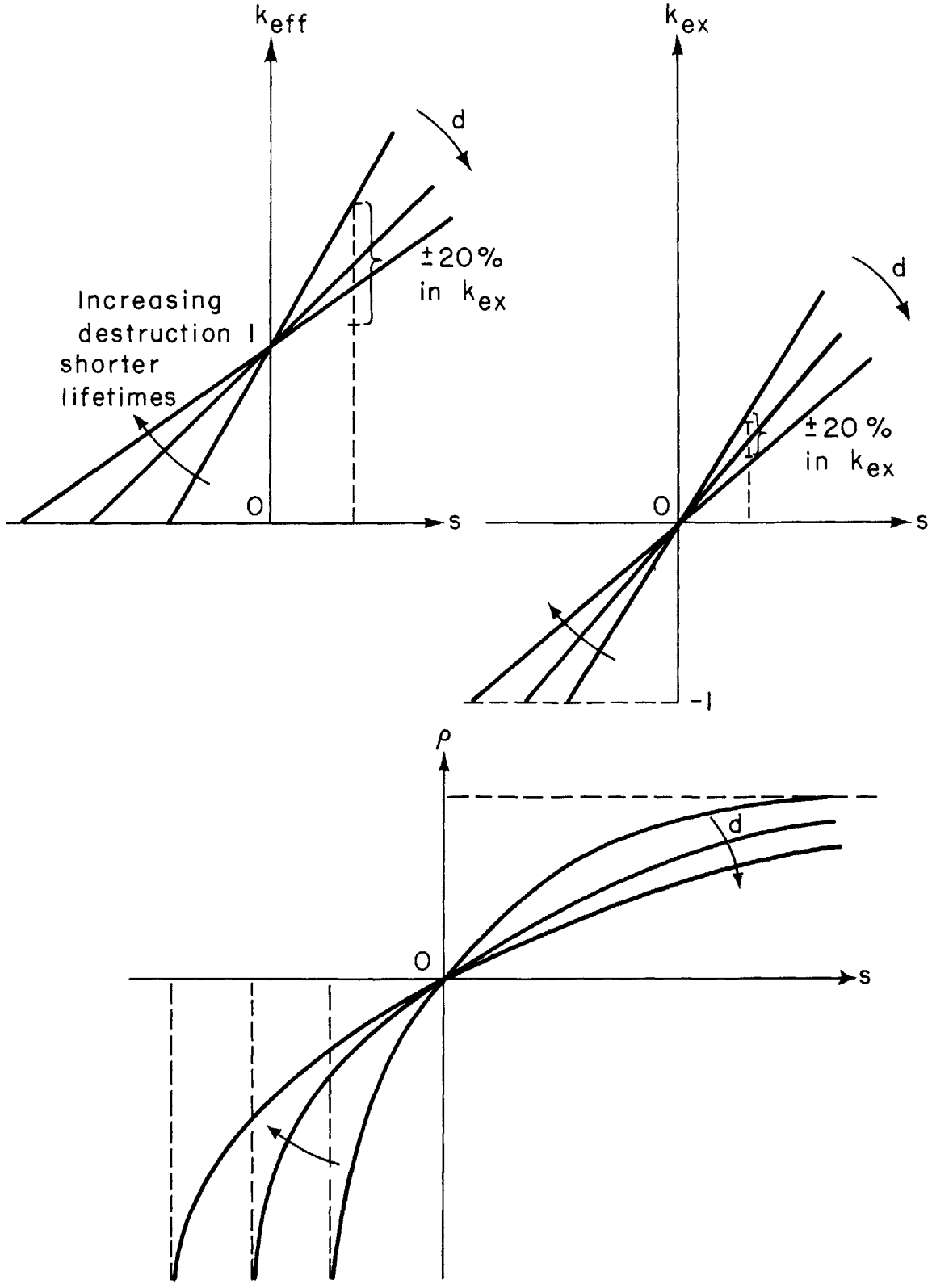
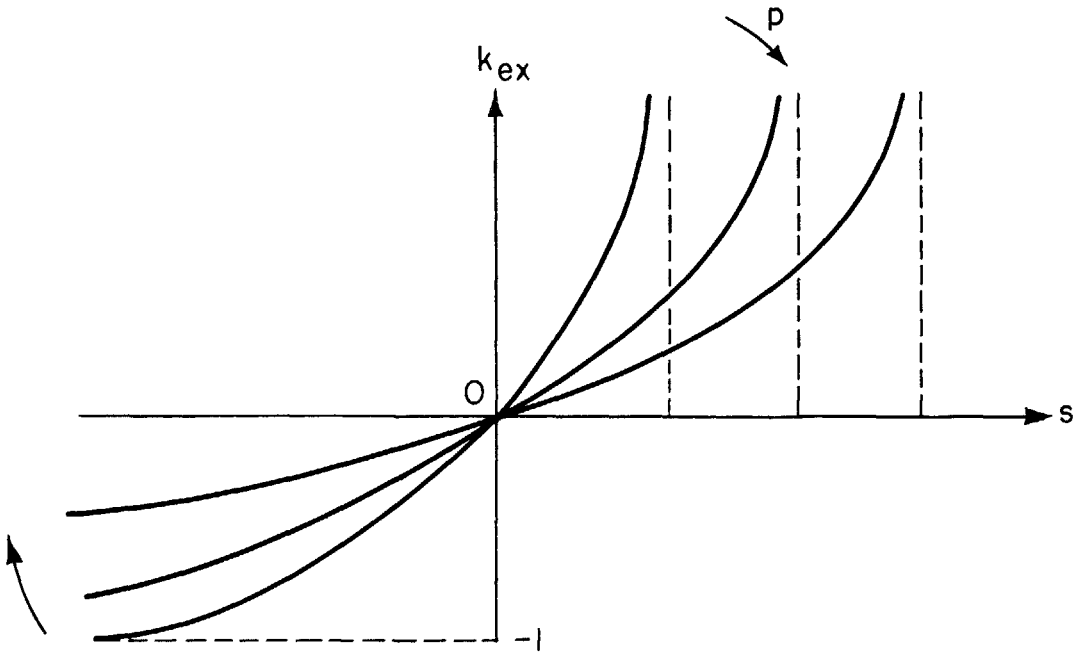


FIG. 1.1a MEASURES OF CRITICALITY AS A FUNCTION OF REACTOR INVERSE PERIOD, s, PARAMETRIC IN DESTRUCTION, d.



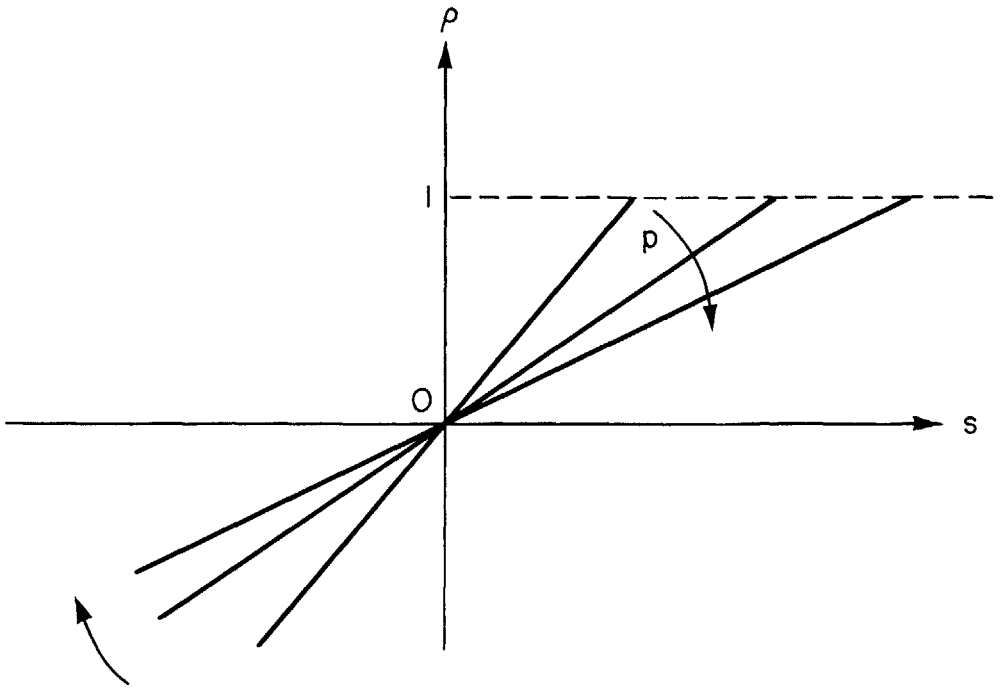


FIG.1.1bMEASURES OF CRITICALITY AS A FUNCTION OF REACTOR INVERSE PERIOD, s, PARAMETRIC IN PRODUCTION, p.

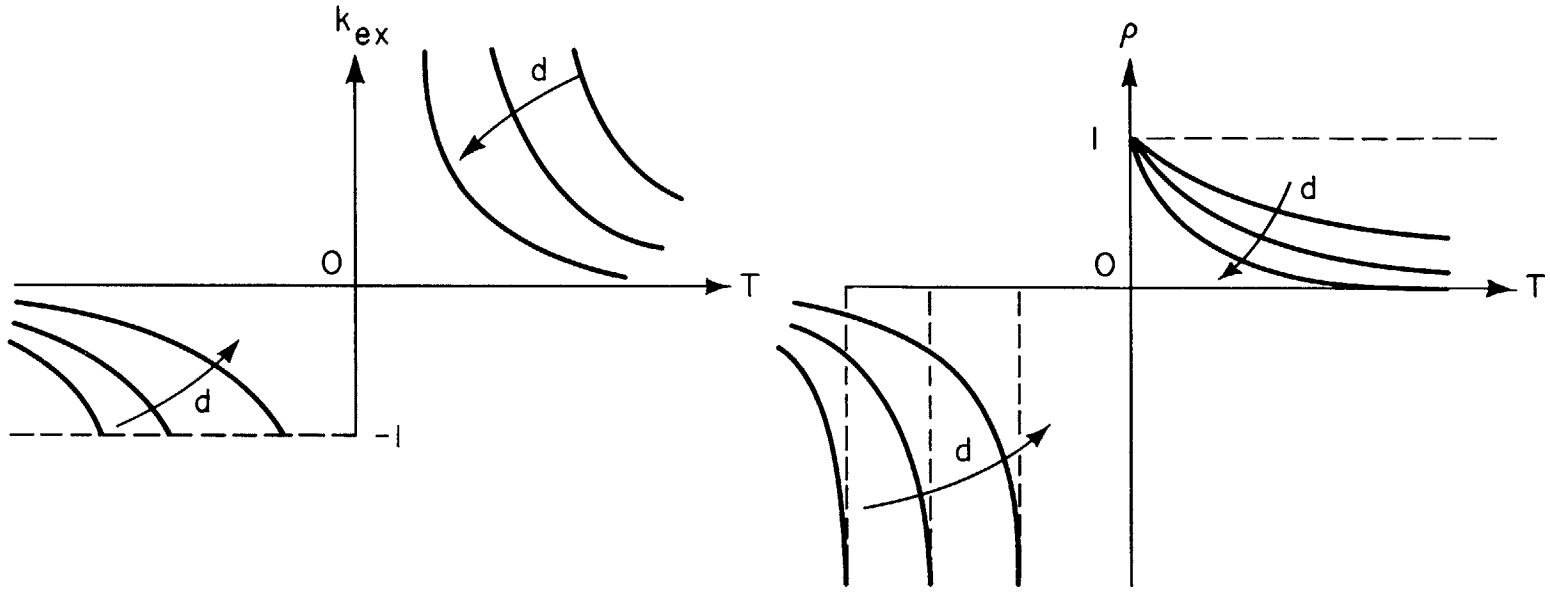


FIG.1.2a- MEASURES OF CRITICALITY AS A FUNCTION OF REACTOR PERIOD, T, PARAMETRIC IN THE DESTRUCTION, d.

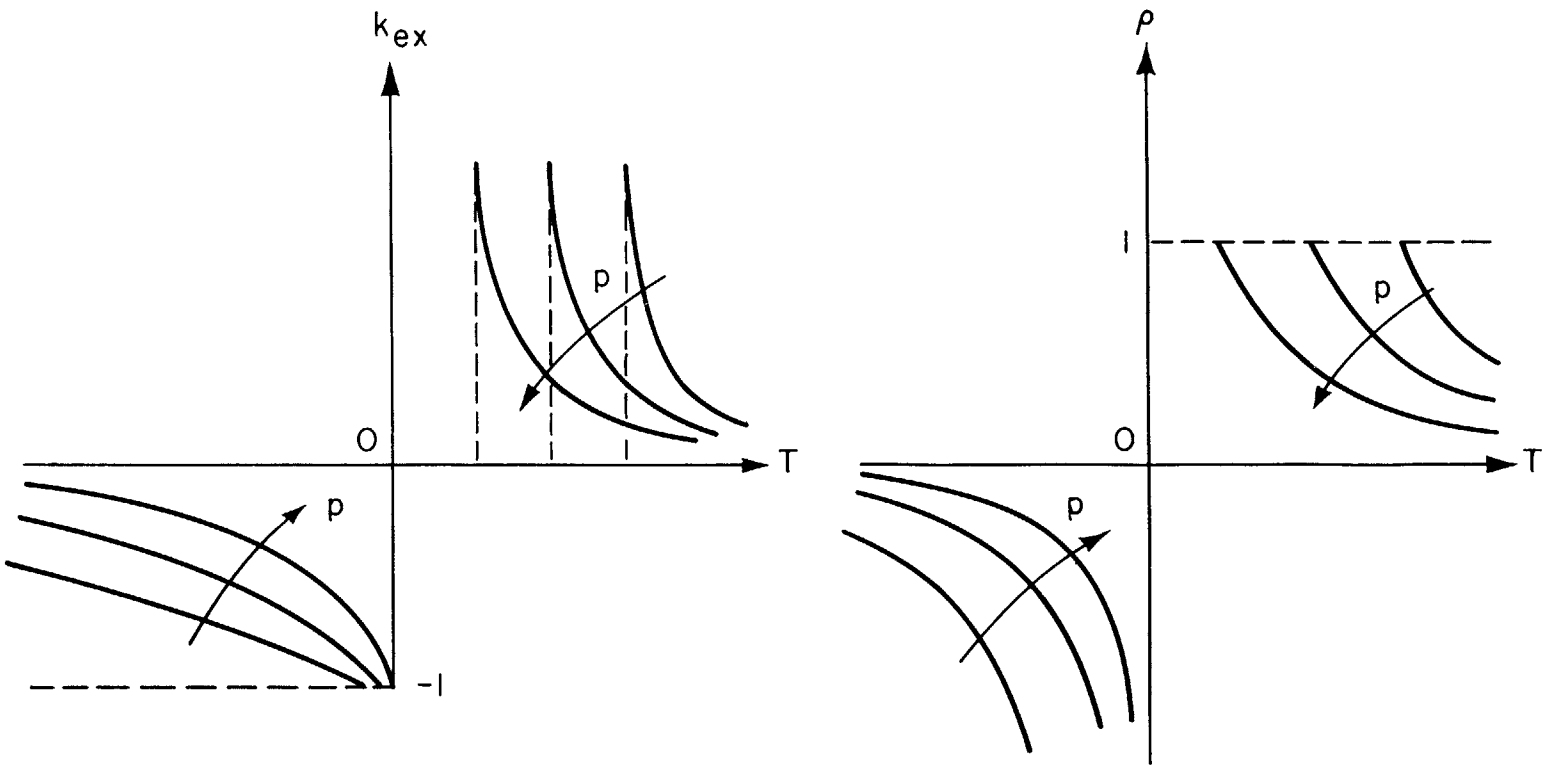


FIG.1.2b- MEASURES OF CRITICALITY AS A FUNCTION OF REACTOR PERIOD, T, PARAMETRIC IN THE PRODUCTION, p.

Example: Either the properties or the description of a reactor is changed in such a way as to decrease the destruction, e.g., the loading changed from natural to enriched uranium, as in the Brookhaven Reactor (5). Then for a given period, k_{ex} and ρ are increased. For a given value of k_{ex} or ρ, however, the inverse period decreases and the period increases with the decreased destruction and longer reported lifetime.

The Brookhaven Reactor provides a second example where the reported value of the lifetime is not in itself a sufficient criterion of the change of safety conditions. Although the reported lifetime was increased by a factor of about five, the k_{ex} of a given control rod or piece of experimental apparatus increased by approximately the same ratio (the discrepancy is attributable to the fact that the reloaded pile was actually slightly smaller so that a fixed experiment, further out from the core, was worth less than if it occupied the original relative position). As a result, despite the factor of five, the failure of a given control rod would lead to the same accident condition.

1.14 Role of Generating Time and Lifetime

Our discussion has served to show that the lifetime, ℓ, like any of the other parameters in the kinetics equation has no operational physical significance. It is purely a matter of convenience to solve the kinetics problems on the assumption that the lifetime is constant. In fact, the inverse period, the operational result, is independent of how the lifetime is defined, i.e., how destruction is defined, so long, of course, as production is correspondingly defined.

It will be observed from Figures 1. 1a and 1. 1b that when the inverse period is plotted against constant destruction, i. e., constant lifetime, then plots of s against k_{ex} are straight lines. Were we to plot s in units of the lifetime then these straight lines would reduce to one line in such a non-dimensional plot for the inverse period neglecting delayed neutrons. Mathematically, this single straight line solution at 45° to the axes, can be continued past the region where k_{ex} is not less than minus one. Thus to obtain a shut down period smaller than the lifetime, when the lifetime is held constant, would require negative production. In the first place, such negative production is admissable once we have accepted the fact that the definition of production is arbitrary and non-physical. In the second place, if a reactor is indeed shut down heavily in an attempt to get a shut down period less than the lifetime, the addition of control rods implies that the lifetime will not be a constant parameter in the solution. In a plot of s against ρ for constant lifetime, we have a corresponding solution for ρ larger than unity when the production becomes negative.

In a plot of s against ρ for constant production, we now have the straight lines. Again it is possible to prepare a universal plot with a straight line through the origin at 45° to the axes. Thus we are led to introduce the generating time, Λ, in analogy to the lifetime. Whereas the lifetime is the reciprocal of the destruction, the generating time is the reciprocal of the production probability. Table 1. 3 displays the six chief kinetics parameters, three based on destruction and three based on production.

The generating time is like the lifetime in that it has no operational

significance and depends on the arbitrary definition of the production processes. The advantages of using the generating time, Λ, arise when delayed neutrons are considered. Here the only natural parameter to describe the effect of delayed neutrons is the precursor yield as a fraction of the total production, β. The resulting equations are more simply expressed when use is made of ρ and Λ, the other two parameters also based on the production probability.

A detailed analysis of the effect of using the generating time, Λ, rather than the lifetime, ℓ, is given in Appendix A to Section 5. We should point out that a number of authors have employed the concept without we feel appreciating the full advantages and implications of the generating time (6, 7, 8) which is more than just an approximate form for the lifetime.

When delayed neutrons are neglected, we can write a solution directly in terms of the production and destruction probabilities

$$s\,n = \frac{n}{\Lambda} - \frac{n}{\ell} \tag{1.49}$$

The logarithmic derivative, s, can be interpreted as a probability of gain, neutrons per second per neutron. Then from Table 1.3 we have

$$\frac{1}{\ell}(k_{eff} - 1) = \frac{k_{ex}}{\ell}$$

$$s = \frac{1}{\Lambda} - \frac{1}{\ell} = \tag{1.50}$$

$$\frac{1}{\Lambda}\left(1 - \frac{1}{k_{eff}}\right) = \frac{\rho}{\Lambda}$$

Table 1.3 Kinetics Parameters Based on Production and Destruction

Based On Production		
Generating Time		
Λ	$\frac{1}{\text{Production Rate}}$	$\frac{1}{V F'}$
Effective Precursor Yield		
β_i'	$\frac{\text{Production Rate of i-th Precursor}}{\text{Production Rate}}$	$\frac{V \beta_i F'}{V F'}$
Reactivity		
ρ	$\frac{\text{Increase Rate}}{\text{Production Rate}}$	$\frac{V(\delta F - \delta \Sigma - \delta D B^2)}{V F'}$
Based On Destruction		
Lifetime		
ℓ	$\frac{1}{\text{Destruction Rate}}$	$\frac{1}{V(D' B^2 + \Sigma')}$
Effective Multiplication		
k_{eff}	$\frac{\text{Production Rate}}{\text{Destruction Rate}}$	$\frac{V F'}{V(D' B^2 + \Sigma')}$
Excess Multiplication		
k_{ex}	$\frac{\text{Increase Rate}}{\text{Production Rate}}$	$\frac{V(\delta F - \delta \Sigma - \delta D B^2)}{V(D' B^2 + \Sigma')}$

Production includes production of prompt neutrons and precursors

Destruction refers to destruction of neutrons

Increase is arbitrarily defined to be production minus destruction

1.15 The Linear Approximation

We have seen that even in the one region reactor in one-group theory there is no unique definition of any of the kinetics parameters. In the general problem of the multiregion reactor we have the further arbitrary manner of defining an average for these parameters.

There are a number of ways in which reactor averages can be set up for k_{eff} and Λ, say, which can be shown to fail. The failure is not that they do not predict the inverse period correctly when properly evaluated, but that they depend in a too sensitive fashion on the value to be employed for the flux. Thus, since it is essential to use the first order approximation, these averaging methods fail.

The most typical and the most plausible of these methods of averaging will be investigated in detail. The linear approximation attempts to calculate the rate of change of the neutron population, N, or of the population of neutrons and precursors in general. Certainly if ϕ' is the correct asymptotic flux, then the asymptotic period is indeed obtainable from the population behaviour.

$$s = \frac{1}{N}\frac{dN}{dt} = \frac{\int I M \phi' \, dv}{\int I V^{-1} \phi' \, dv} \tag{1.51}$$

Such an approach is often used in Monte Carlo calculations for example. For our purposes however, we must consider the accuracy of the first order approximation, putting $\phi' = \phi$ where ϕ is the solution of a critical reference reactor. We have

$$s \int I V^{-1} \phi' \, dv = \int (I R \phi + I R \delta\phi + I P \phi + I P \delta\phi) \, dv \tag{1.52}$$

Now by definition, $I R \phi$ is zero at every point. Similarly we might well

neglect the term $I P \delta\phi$ if P and hence $\delta\phi$ are both small terms, since the error is then of second order. There remains

$$s \int (I V^{-1} \phi + I V^{-1} \delta\phi)\, dv \approx \int (I R \delta\phi + I P \phi)\, dv \qquad (1.53)$$

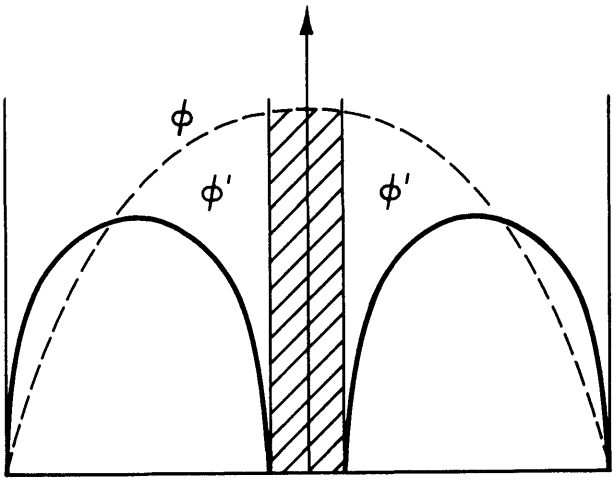
In equation (1.53) we might be justified in neglecting $I V^{-1} \delta\phi$ in the left hand side if $(\delta\phi/\phi)$ is small. But on the right hand side, to neglect $I R \delta\phi$ would be equivalent to throwing away half the effects of the perturbation since $I R \delta\phi$ is of the same magnitude as $I P \phi$.

Physically this approximation evaluates the population balance. Mathematically it would be called a linear averaging of the probabilities, linear in the flux which acts as a weighting function. In the linear form, the effect of the neutrons lost in the perturbed material is of the same order of magnitude as the change in the neutron balance in the remainder of the reactor due to the alteration of the flux; hence both terms must be included and the first order approximation is inadmissable for the linear form.

An example of a case where the flux change is of equal importance as the material change is afforded by control rods. In addition to the neutrons absorbed in the rod, the new flux distribution illustrated in Figure 1.3 leads to an increased external leakage and an added reactivity effect for the rod.

1.16 Summary

At the beginning of this introductory section, we discussed briefly the aims and outlines of this thesis: to give a complete physical interpretation of the variational approach to problems in reactor physics, to follow up certain mathematical consequences and then to give a number



Reactivity effect of rod due to:

(a) Neutrons absorbed in rod.

(b) Change of flux giving new neutron balance in reactor.

(c) Change of flux increasing external leakage.

FIG. 1.3- INCREASED LEAKAGE IN A SLAB REACTOR ON INSERTION OF A CONTROL ROD.

of new interpretations and applications in the realms of kinetics and statics. The remainder of the introduction served to prepare the matrix nomenclature to be employed and to prove the need to introduce the variational methods into the subject.

In the next Section, we shall be able to give an historical account of the Variational methods and their introduction into reactor physics, which will lead us to our main contribution, the physical exposition.

SECTION TWO: HISTORICAL

2. 1 Introduction

The development of perturbation methods for reactor physics and in particular the physical interpretation of the mathematical techniques employed was largely pragmatic under the stress of war time research and development. As a result the interpretations that are now accepted in the field bear little relation to the interpretations of the same variational methods used in a number of branches of mathematical physics.

That the reactor development lacked what Weinberg and Wigner (1) call a scholarly tradition is no outright condemnation; the interpretations given had a number of advantages over classical tradition. For example, whereas Morse and Feshbach (9) can admit the sacrifice of a certain reality in the variational treatment of mechanically dissipative systems, the reactor physicist's interpretation of the adjoint flux as a neutron importance or ultimate fissioning probability, is very much germane to the criticality of reactors.

Part of the present work is a reconciliation of the classical and reactor physics viewpoints in which we put forward a generalized physical principle that retains the advantages of both and serves to unify the field of reactor physics with the larger body of classical physics in the use of variational calculations. Our historical account consists first of a description of the development and outlook of the variational form in classical physics and second a historical and critical account of the introduction and interpretation of the adjoint flux in neutron diffusion and transport calculations.

CLASSICAL MATHEMATICS AND PHYSICS

2.2 Ordinary Linear Differential Equations

The adjoint function for ordinary linear differential equations was introduced by Lagrange (c 1762) as an extension of the concept of the integrating factor (10). The term adjoint itself means connected or related. Though the term was not introduced until Fuchs (1873), Lagrange's aim was to obtain a formal reduction of the order of an ordinary linear differential equation. By means of the adjoint function the original linear equation can be transformed to a new equation, the bilinear concomitant, linear in both functions. By suitable restrictions on the adjoint function the bilinear concomitant can be made immediately integrable or of one less order than the original equation. The gain is at the expense of the restriction on the adjoint function, that it satisfy an adjoint equation, similar and of the same order as the original equation. By a rearrangement, the bilinear concomitant is related (concomitant) to the two functions and their equations. The two functions are each others adjoints.

The most striking feature of the adjoint equation is perhaps that compared to the original equation, all odd powered derivatives are reversed in sign, all even powered derivatives being unchanged. There is a mathematical antisymmetry for the pairs of equations.

Since the bilinear concomitant is reducible to one less order, we can say the bilinear concomitant is stationary, though the expression stationary has a further meaning to be developed. Concomitant means related or existing together and is used in the sense that the original

equation and the adjoint equation form the bilinear concomitant (Lagrange identity). In Section 4 a detailed example is worked out to show how perturbation calculations can be related to the classical adjoint system. A further use of the adjoint system (which is not utilized in perturbation theory) arises when the adjoint equation is easier to solve than the original equation.

2.3 Rayleigh's Principle

The principles of Hamilton and Lagrange and the related principle of Least Action had proved to be powerful methods for the treatment of discontinuous mechanical conservative systems (1, 11). Typically, quadratic expressions for kinetic energy ($\frac{1}{2} m \dot{r}^2$) and potential energy ($\frac{1}{2} kr^2$) are involved. The laws of motion of the system are derived as extremum or variational principles. Thus Hamilton's principle might be stated as "for all possible paths of motion, the actual paths minimize the average difference between the kinetic and potential energies". Since this extremum principle considers all possible varieties of paths, the average energy difference is being minimized for all possible variations of paths – i.e., we have a variational principle. Furthermore, if this difference is a minimum, its rate of change with respect to a variation of any of the paths must be zero – i.e., the difference is stationary.

In a series of papers on acoustics, Lard Rayleigh (12) extended the work of Hamilton, Lagrange, etc., on discrete bodies to systems of continuous bodies. For example, the behavior of an elastic beam can be expressed in terms of the displacement, y. The energies of the beam as a system are integrals over suitable functions involving y,

such as $1/2 \int m(x)\, y^2(x)\, dx$ and $1/2 \int \left(\frac{\partial y}{\partial x}\right)^2 k(x)\, dx$. From the corresponding stationary property Rayleigh was able to show that variations of the paths corresponded to shapes of the beam other than the shape actually taken up by the beam in free vibration. Since Rayleigh could relate the frequency of vibration to the stationary integrals, quite crude guesses for the beam shapes led to good estimates of the vibration frequency. Although the beam shape, y, appears everywhere in quadratic form, it is usual to use the same guess, (y_1) for both values of y, e.g., $(y_1)^2$ rather than $(y_1)\,(y_2)$ and $\left(\frac{\partial y_1}{\partial x}\right)^2$ rather than $\left(\frac{\partial y_1}{\partial x}\right)\left(\frac{\partial y_2}{\partial x}\right)$.

A further development, due to Ritz (1), greatly enhanced the value of Rayleigh's energy method by utilizing the fact that the energy integrals are not only stationary to variations of the beam shape but are actually minima. For now guesses for the shape can be adjusted until they are forced to give a minimum approximation value to the frequency, the true frequency being known to have a value lower than any of the guessed frequencies. One should note that the Ritz development is an additional step which requires that the same guesses for the shape are inserted into each of the quadratic forms. The Ritz technique is not available to the reactor physicist except in the mono-energetic approximations, for reasons to be discussed.

The contribution made by Rayleigh is something more than just the specific methods for approximate solutions. The inclusion of continuous functions for the description of systems profoundly affected the thinking of physicists, leading to the general ideas of field theory (a field, mathematically, being one or more independent functions of space and time). Rather than the older ideas of expressing, say, electrostatic

effects through an action at a distance of two charged particles, the field theorist would consider the behavior of one particle in an electric field. The change of view has led to significant advances in many branches of physics, these advances being of two sorts. The field formulation is suitable to expression as a variational form. Then not only do we gain a useful approximation tool but we have a concise expression that leads to the equations of motion for all parts of the system. The satisfaction of these equations, the Euler equations, is the sufficient and necessary condition that the variational principle is satisfied. In the general variational principle, it should be noted that the terms appearing in the quadratic forms are considered to be separately variable. This aspect will be enlarged in the following discussion.

2.4 Non-Conservative Systems

Rayleigh's energy method and the variational principle justifying the approximate methods are applicable only to conservative systems, with no friction, dissipation, etc. To obtain the advantages of either the conciseness of expression or the approximation form, we postulate an adjoint system in which all dissipative effects are reversed. In such an adjoint system, for example heat 'undiffuses' and flows up the gradient of temperature – cause and effect are reversed. Then the combination of the real and adjoint systems exactly cancels out all dissipative effects and renders the pair of systems conservative.

In the example quoted for the heat conduction, the real system is described by a temperature field, T, and the Fourier equation

$$c \frac{\partial T}{\partial t} = \nabla \cdot k \nabla T \tag{2.1}$$

The adjoint system has a temperature field we can call T^+, the adjoint temperature. The equation describing the physical reverse of cause and effect invoked is just the mathematical adjoint of equation (2.1), i.e.,

$$c \frac{\partial T^+}{\partial t} = - \nabla \cdot k \nabla T^+ \tag{2.2}$$

We should note that we must take the same size system, the same properties and the same boundary conditions if indeed the two systems are to compensate each other and give a jointly conservative system.

In this example the adjoint equation is easily solved for the given boundary conditions if the solution of the original equation is known in terms of eigenfunctions and eigenvalues. It is evident from the equations and from physical reasoning, that the time behavior of the two systems is inverse. As the temperature falls in the real systems, the adjoint temperature rises. Mathematically we have $s_n = - s_n^+$ where s is the eigenvalue, while $T_n(\underline{r}, o) = T_n^+(\underline{r}, o)$ where T_n is the eigenfunction. An alternative expression of this relationship is that the systems have the same equations in the Laplace transform space or the transformed equations are self adjoint.

The preceding arguments lead to variational expressions involving now not quadratic forms but bilinear forms in T and T^+, or more generally, forms linear in the two fields. The two fields are both functions of all the coordinates. The full specification of the equation and boundary conditions (i.e., the system) for either the real or the adjoint system is sufficient to determine the solution of both systems. Although the classical interpretation of the adjoint system is perfectly valid in

terms of the reversal of cause and effect, the resulting properties and indeed the usefulness of the concept arises from the purely mathematical relations of two such systems. Thus, the mathematical properties are present whether any physical interpretation is given to the adjoint system or not. There is no good reason to suppose that there is only one physical interpretation of the adjoint system. Certainly the imaginary system with cause and effect reversed has only limited physical usefulness.

2.5 Perturbation Methods

The science of astronomy led to the development of perturbation methods for finite or discontinuous systems. The problem of the path of heavenly bodies under the influence of each others gravitational fields is in general only soluble for the two body case. The influence of additional bodies must be treated as a correction to the initial solution – the influence depending on the path of the bodies. When the path has been corrected then the new path can be used as the basis of more accurate corrections in an iterative fashion.

A corresponding development of perturbation methods appears in the solution of simultaneous algebraic equations where use can be made of solution estimates or approximations to speed the convergence and accuracy of machine computations.

The important developments in perturbation theory from our viewpoint arose from Rayleigh's extension of variational methods to continuous systems, discussed above. The emphasis in perturbation theory is now not to calculate the frequency of vibration of a loaded beam from scratch but rather to employ the known solution of the

unloaded beam. The difference in properties between the loaded and unloaded beam is a perturbation. The shape known to be taken up by the unloaded beam is the first estimate of the true shape. From this knowledge we can calculate with considerable accuracy the change of frequency produced by the perturbation. Analogous methods are used for problems in acoustics.

The techniques of perturbation theory have been advanced most by the problems of quantum theory. In the realm of quantum theory or wave mechanics there are only some five or six problems to which an exact solution is known (analogous to the slab, cylinder and sphere of reactor physics). Techniques have been developed to derive not only the eigenvalues but also the wave solutions or eigenfunctions by sophisticated iterative techniques based on a soluble problem.

NUCLEAR REACTOR PHYSICS

2.6 Mathematical

Perturbation methods were introduced into reactor physics by Wigner (13) in a treatment of the Fermi-age equations. Much of the formalism follows quantum mechanical usage – indeed the method is referred to by Wigner as the Rayleigh-Schroedinger perturbation method.

The usefulness of the adjoint equations and their solutions was quickly realized by other workers in the field. Nordheim, (14, 15)(31) et alia, applied the methods to the two-group approximation to develop the generalized inhour equation for a multiregion reactor (30) Soodak (16) and Feynman did more work in the field of reactor statics

and kinetics.

The extension of the mathematics to the Boltzmann equation was undertaken by Fuchs (17) Tait (18) Wilson (19) Brooks (20) and Pendlebury (21). More recently Ussachoff (22) gave a combined mathematical and physical account of neutron importance and its applications to reactor kinetics. The most useful mathematical account is probably Davison's textbook (3) on neutron transport problems. This text contains much material on the use of variational methods in general for reactor calculations, perturbation theory being only one application of the general variational techniques. There is however, practically no treatment of reactor kinetics in that text.

2.7 Physical Interpretation

Shortly after the introduction of the mathematical method by Wigner, a number of workers gave a physical account (including Wigner himself). Weinberg (23) has done much to disseminate the concept of the adjoint flux as the contribution made by one neutron to the asymptotic population, though the term importance is due to Soodak (24). Hurwitz (25) developed a parallel interpretation in terms of the iterated fission probability (see 2.8, Nomenclature).

This type of physical account of the adjoint equations has certain advantages over the interpretations of classical physics. In the treatment of heat conduction for example, a new, nonphysical system was invoked. In the treatment of neutron diffusion however, we are able to give physical meaning to the adjoint flux in terms of the behavior (or potential behavior) of neutrons in the real system. This advantage,

however, is offset by the increased difficulty in deriving the adjoint equations from the physical principle.

Unfortunately, the concept employed in the literature for the interpretation of the adjoint flux is of limited usefulness since it is confined to the just critical reactor. Thus the various authors have been able to justify only the fundamental mode or eigenfunction and have not been able to account for the higher modes. Furthermore no resolution has been made on physical grounds of the interesting mathematical property of the eigenvalues; that real and adjoint eigenvalues can be matched one to one in their magnitude.

The general viewpoint of field theory would require that the adjoint flux is as good a representation of the system as the actual flux – i. e., the adjoint flux should be a function of time as well as space with appropriate boundary and initial conditions. It is not possible to extend the accepted physical interpretation to encompass this generalization because of the two time variables involved – the time at which one neutron is selected for study and the time at which the neutrons progeny are counted. In Section 3 we put forward a new principle which overcomes this difficulty. We shall achieve a representation which enables us to use the large body of classical physics while retaining the advantages of the interpretations already given by reactor physicists.

Although a number of authors had given an a posteriori justification of the mathematical form, the first attempt at a derivation of the adjoint equations from a physical principle was given by Ussachoff in his paper on the adjoint Boltzmann flux (22). Ussachoff succeeds in

deriving the steady-state equation (though actually employing the corresponding eigenvalue equation). Nowak (28) has reviewed some of Ussachoff's work in deriving the transport adjoint equation though the corresponding diffusion adjoint equation is only derived by a mathematical reduction of the transport equation.

2.8 Some Common Usage

Owing, perhaps, to the secrecy and pressure of work surrounding the various war time projects, there is some confusion about terminology in the literature. The following notes give an account of the controversial terms.

(a) Adjoint Flux

The bilinear form of variational theory requires the introduction of a second mathematical field, ϕ^+, in addition to the flux ϕ. The function ϕ^+ is referred to as the adjoint flux and appears typically in expressions such as $\int \phi^+ V^{-1} \phi \, dv$. Since, in fact, $V^{-1} \phi$ is the neutron density, it might be more apt to call ϕ^+ the adjoint neutron density – indeed the physical interpretations given to ϕ^+ all imply a relation to the neutron density rather than the flux. The formal appearance of V^{-1} in the expression for the neutron density has been misinterpreted in the literature and has sometimes led to errors.

By rearranging the equations of the neutron balance (in $V^{-1} \frac{\partial \phi}{\partial t}$) into equations in the flux balance ($\frac{\partial \phi}{\partial t}$) it is possible to derive a corresponding adjoint that is in fact strictly adjoint to the neutron flux. Evidently this adjoint would be just the $\phi^+ V^{-1}$ appearing in the bilinear expression, $\int \phi^+ V^{-1} \phi \, dv$.

(b) Self-Adjoint

The general time dependent equations for diffusing systems are not self-adjoint in the strictest sense, i. e., $\phi^+(\underline{r}, t) \neq \phi(\underline{r}, t)$ even with an arbitrary normalization. It will be shown that the equation for the one-group approximation with no delayed neutrons has the property that the shapes of ϕ and ϕ^+ are identical to within a normalization (though of course they have an inverse time behavior). For many purposes it is only the shapes of the fluxes and not their behavior that is required and the custom has sprung up of calling this one group equation self-adjoint.

The multigroup equations lead to solutions that are not even self-adjoint in the exact or the approximate sense – e. g., the adjoint fast flux has a shape different from that of the fast flux. However, when the reactor is uniform, and has only one region, it can be shown that the shapes within each flux and adjoint flux group are the same (e. g., cosine in a slab). Thus Sangren (27) refers to the one region problem for any number of energy groups as self-adjoint. It should be noted that the coupling coefficients between the components of the fluxes and the adjoint fluxes are not the same even in this one region case. For this reason we shall not use the term self-adjoint for the multigroup cases.

(c) Importance

The physical interpretation of the adjoint flux, ϕ^+, is most commonly given in terms of neutron importance. The literature contains references to the importance of neutrons to a just critical reactor;

we will develop a more general time dependent concept. Thus, where the literature refers to importance, what is usually meant is the concept of our equivalence, to be discussed in more detail in the body of the work.

Even the importance (equivalence) has been given two slightly different meanings in the literature. Ussachoff (22) derives the steady state equation for the importance (equivalence) from a physical principle for the transport equation and then assumes the corresponding eigenvalue equation. His normalization of the importance (equivalence) is through the steady power level resulting from the introduction of one neutron in a just critical reactor. Ussachoff quotes dimensions of the importance (equivalence) as neutrons per second; they might be better expressed as neutrons per second per one neutron or $(\text{seconds})^{-1}$. The bilinear from $\int \phi^{+} V^{-1} \phi \, dv$ would then have the dimensions neutrons per second.

A common American usage (28) is to define the importance (equivalence) as the steady state neutron population produced by the introduction of one neutron. Hence the importance (equivalence) is non-dimensional. The bilinear form $\int \phi^{+} V^{-1} \phi \, dv$ however would then have the dimensions of neutrons.

We might remark here that the concept of the fractional importance we develop has the dimensions of $(\text{neutrons})^{-1}$ so that the integral $\int \phi^{+} V^{-1} \phi \, dv$ is non-dimensional.

Still other authors have used importance in the sense of the statistical weight (q. v.) or for the integral $\int \phi^{+} V^{-1} \phi \, dv$ itself (e. g., Soodak refers to a total importance (29)).

(d) Iterated Fission Probability

Hurwitz introduced the term "iterated fission probability" as a physical concept corresponding to the adjoint flux, referring to the probability of a neutron causing fission not in itself but through its progeny at some very much later generation. The dimensions can be made to correspond to either the American or the Russian usage.

(e) Statistical Weight

The bilinear form $\int \phi^+ V^{-1} \phi \, dv$ has corresponding expressions for the various reaction probabilities, Σ, F, etc., such as $\int \phi^+ \Sigma \phi \, dv$, $\int \phi^+ F \phi \, dv$. The statistical weight for a process Σ is then $\phi^+ \phi$. There is a different statistical weight for leakage processes represented by the integral form $\int \phi^+ \nabla \cdot D \nabla \phi \, dv$; we will show that the corresponding statistical weight is $\nabla \phi^+ \cdot \nabla \phi$.

In the multigroup case we must also distinguish between the energy of the flux from which the neutrons come, ϕ_i and the energy of the importance into which the neutrons go, ϕ_j^+, so that the statistical weight of a process occurring at a point $\underline{r}$ is more correctly $\phi_j^+(\underline{r}) \, \phi_i(\underline{r})$.

The use of "statistical weight" is usually confined to the first order approximation, where the flux and the importance are both taken from the solutions of the arbitrary reference reactor.

Certain British authors use 'statistical weight theorem' to include the whole of perturbation theory (3).

SECTION 3: PHYSICS OF NEUTRON IMPORTANCE

3. 1 Introduction

In the present section we shall define a physical concept, the neutron importance, which is the basis of our physical account of perturbation theory. The concept leads to a general interpretation of the variational form of calculation; the interpretation of the special application of perturbation theory will be given in Section 5 on kinetics. The first part of the present section derives, from physical reasoning, certain properties which must be associated with our definition of the neutron importance. The second part derives the equations and boundary conditions for the importance of neutrons and precursors.

DEFINITIONS AND PROPERTIES

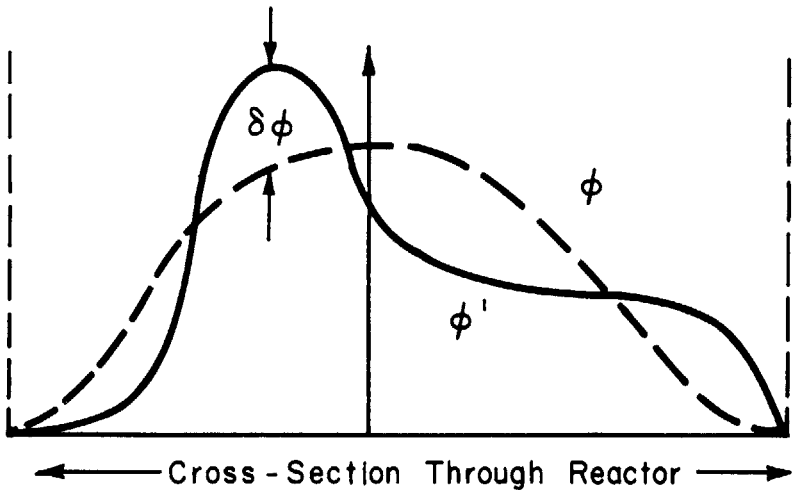
3. 2 Resident and Transient Fluxes

Experience with reactors at low power, e. g. , (39), and analogy with other branches of physics, gives support to the following statement: after a transient period, the neutron (and precursor) population of a reactor settles into a resident distribution or shape with a constant logarithmic derivative, s, at every point in the reactor. The semilog plot of flux, ϕ', at any point leads asymptotically to a straight line; the slope of the line is the same at every point in the reactor. Extrapolation of this line back to the origin of the initial condition clearly defines an asymptotic flux component for all times. We shall speak of the resident flux, ϕ, and the resident population π (the population being the total number of neutrons and precursors in the reactor at any time,

independent of position). The difference between the resident and actual values at any time can be called a transient flux, $\delta\phi = \phi' - \phi$, and a transient population, $\delta N = N' - \pi$, respectively. By definition $\delta\phi$ and δN die away. Physically the initial flux diffuses outwards until the shape is such that leakage of neutrons just balances all other processes. The decay of typical initial conditions is shown in Figures 3. 1 and 3. 2.

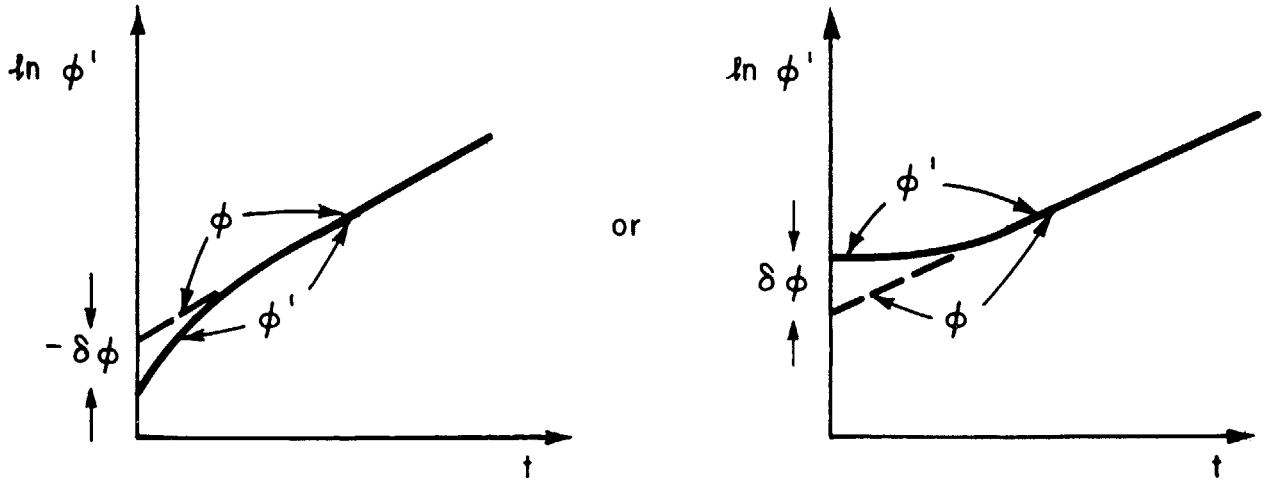
On physical grounds both the actual and resident fluxes and populations must be positive. This restriction does not apply to the transient flux and populations which are merely convenient mathematical representations. Indeed it will be shown mathematically that the transient flux must always change sign in the reactor. Whether the transient population is positive or negative depends on whether the initial flux is predominantly towards the outside of the reactor or towards the inside respectively. The definition of the resident flux by extrapolation is clear, yet leads to the difficulty of a non-physical transient flux. An alternative operational definition is to consider a second reactor, the duplicate reactor, whose properties are identical with the actual reactor. We further suppose some device which is capable of introducing into the duplicate reactor an initial condition of any required magnitude but one which is distributed in the shape of the asymptotic flux or precursor density. Then the resident flux, and population, are given by the values of the flux and population in the duplicate reactor that will lead to the same asymptotic values of flux and population in both reactors.

Since the resident population changes everywhere with the constant logarithmic derivative s (and it is just this asymptotic behavior which



FLUX IN A CRITICAL REACTOR, ϕ'
TRANSIENT FLUX, $\delta\phi$ RESIDENT FLUX, ϕ

FIGURE 3.1a



TIME BEHAVIOUR OF A FLUX AT A POINT IN A REACTOR

FIGURE 3.1b

characterizes the reactor) we can certainly write

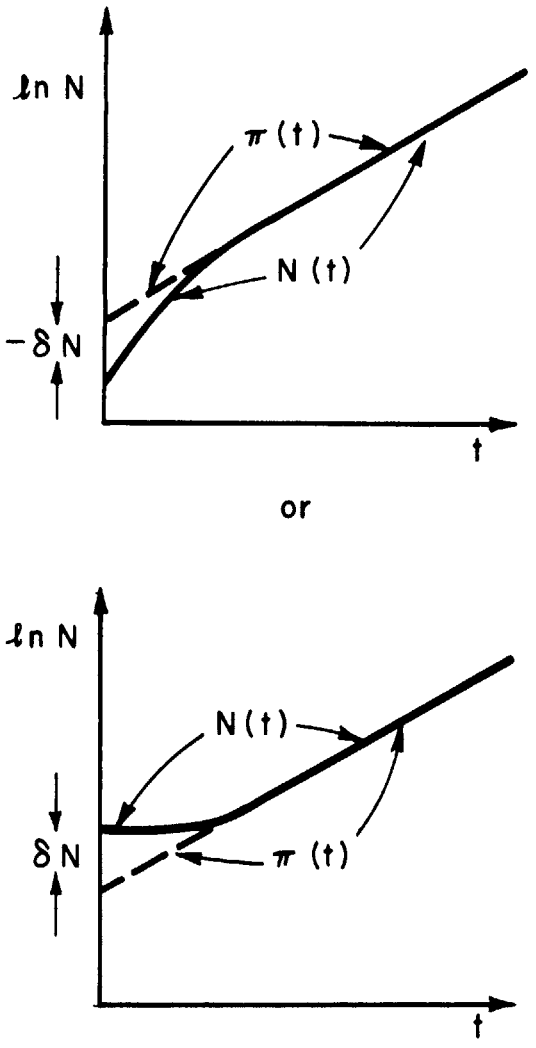
$$s = \frac{1}{\pi}\frac{d\pi}{dt} = \frac{\int I M \phi \, dv}{\int I V^{-1} \phi \, dv} \qquad (3.1)$$

So far, however, we have not found out how to obtain the magnitude and shape of the resident flux, ϕ, associated with an arbitrary initial flux, ϕ'. Even if the shape of ϕ is known from other considerations, we still have the problem of the normalization of ϕ.

3.3 Definition of Fractional Importance

One neutron, selected for study at time t and position $\underline{r}$, will have some probability of causing fission and hence having some progeny in the next neutron generation. These progeny themselves have further chances of causing fission. After a number of generations, the progeny of the one initial neutron will have diffused outwards from $\underline{r}$ and have taken up the resident flux shape. They will be some fraction of the total resident population at that time. As time proceeds still further, both the resident population and the fraction of it attributable to the one neutron will change on the steady asymptotic period. Thus the fractional contribution of the original neutron to the resident population is a constant number, independent of whether the reactor is above, below, or at critical. By either an extrapolation back to the time t or by considering a duplicate reactor, we can define this fractional contribution for all times after the initial selection at time t.

The fractional importance, or more briefly, the importance of a neutron, $\phi^+(\underline{r}, t)$, selected for study at position $\underline{r}$ and time t, is defined to be its fractional contribution to the resident population. The

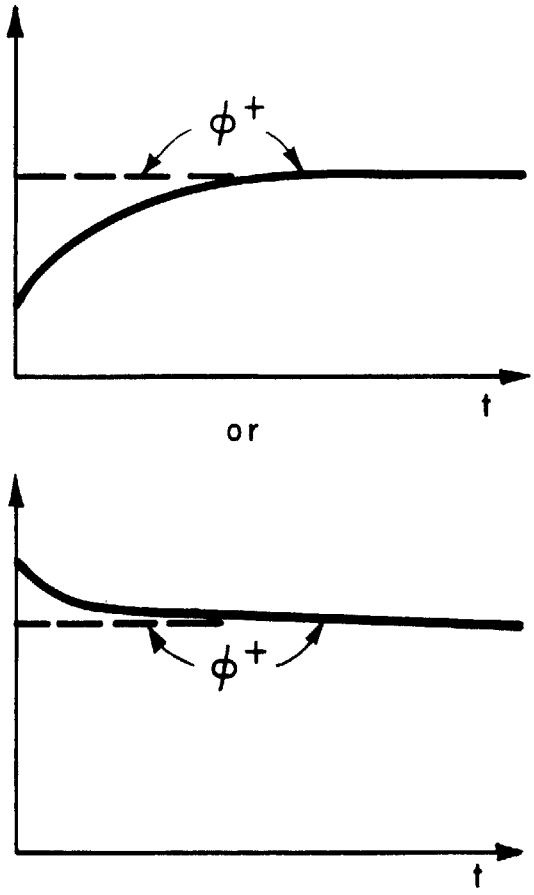


TIME BEHAVIOUR OF A REACTOR POPULATION, N
RESIDENT POPULATION, π. TRANSIENT
POPULATION, δN Figure 3.2

importance has been shown to be independent of the time at which the resident progeny are counted (see Figure 3. 3). The importance will depend on the initial position of the one neutron. A neutron near the edge of the reactor is likely to leak out and hence has a low probability of causing further fission and a corresponding small importance. A neutron at the centre of a reactor has a much better chance of causing fission and a corresponding high importance. In addition, a neutron of a higher energy may have a different importance due to a probability of leaking away rather than slowing down to the lower energy at the same point.

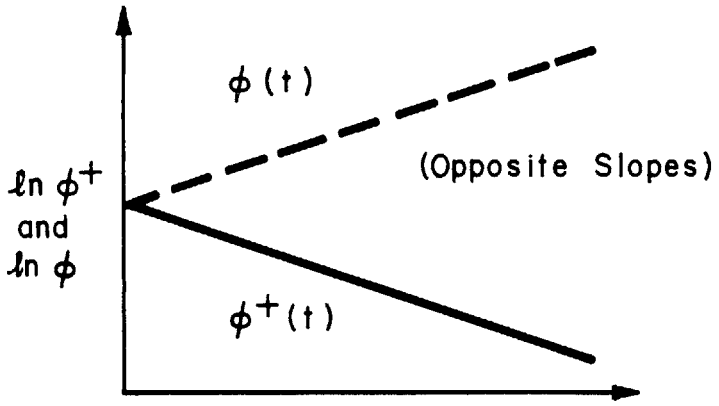
The importance does have a time dependence in a second sense, in the sense of the time at which a neutron is selected for study. If a second neutron is selected for study at the same point in a reactor but at a later time, $t + T$, during which the resident population has dropped by a half, then the fractional contribution made by the second neutron will be twice that of the first. The less neutrons around, the more important is one neutron. Thus from our definition the importance displays an inverse behavior to the population or flux (see Figure 3. 4). When a reactor is critical, the resident population and hence the importance must be time independent.

The importance is not a property of a neutron carried everywhere with it, such as mass. In this, importance is similar to the neutron density which is not a property of individual neutrons. The importance however describes the neutron behavior a posteriori. It is a signpost for the neutron which says by definition what is the likely fate of a neutron at this point in time, space and energy. If we commit one



FRACTIONAL PROGENY OF A NEUTRON SELECTED FOR OBSERVATION AT TIME ZERO, IN A REACTOR WHICH IS NOT NECESSARILY JUST CRITICAL ϕ^+ IS THE FRACTION OF THE RESIDENT POPULATION, π.

Figure 3.3



FRACTIONAL IMPORTANCE OF ONE NEUTRON SELECTED AT DIFFERENT TIMES BUT AT THE SAME POINT IN A REACTOR WITH AN INCREASING RESIDENT FLUX Figure 3.4

neutron to have say 5 progeny in the n^{th} generation, we have imposed certain limitations on the behavior of the intermediate progeny at the $(n-1)^{th}$ generation. Considerations of this sort lead to a balance of importance and the importance equations, as in Figure 3.5. Before deriving these we consider some integral properties of the importance.

3.4 Integral Properties of Importance

(a) Stationary

From the inverse time behaviors of the resident flux and importance

$$\int \phi^{+}(\underline{r}, t)\ V^{-1}\ \phi(\underline{r}, t)\ dv = \text{constant.} \tag{3.2}$$

(b) Normalization

The total contribution of all the neutrons actually in a reactor when measured as a fraction of the resident population must be unity and non-dimensional:

$$\int \phi^{+}(\underline{r}, t)\ V^{-1}\ \phi'(\underline{r}, t)\ dv = 1 \tag{3.3}$$

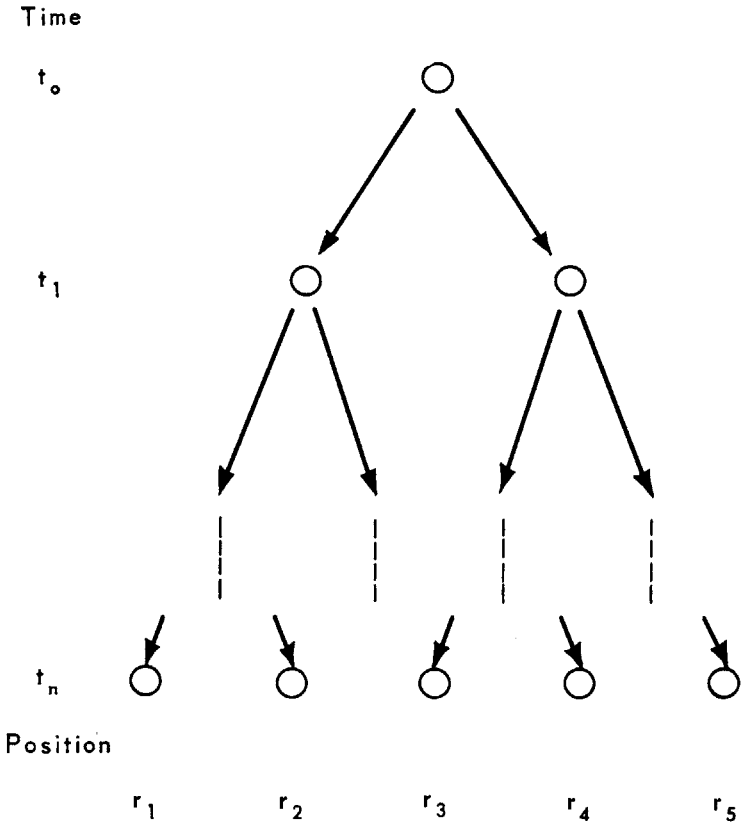
If we consider the initial values we have a normalization of ϕ^{+}:

$$\int \phi^{+}(\underline{r}, o)\ V^{-1}\ \phi'(\underline{r}, o)\ dv = 1 \tag{3.4}$$

(c) Orthogonality

Notice that ϕ^{+} always has the same period whereas ϕ' only takes up the corresponding inverse period after the transients have died away. Yet equation (3.3) is constant for all times. Thus $\phi^{+}(\underline{r}, t)$ acts as a type of filter on the actual flux, $\phi'(\underline{r}, t)$, and the resulting integral behaves as if only the resident component of ϕ' (i.e., ϕ) were present.

NEUTRON GENERATIONS AND CONSERVATION OF IMPORTANCE



Suppose the probable progeny of one neutron is to be 5 at time t_n, where there are then a total of say 50 neutrons in the reactor. Then $\phi^+(r_3, t_o)$ is 5/50. For consistency, the importance of any intermediate generation at time t_1 say, must also sum to the same result.

$$\phi^+(r_2, t_1) + \phi^+(r_4, t_1) = \phi^+(r_3, t_o)$$

The idea of this conservation of importance is the basis for the derivation of the importance equations.

Figure 3.5

Indeed, of all the possible initial conditions that lead to the same asymptotic population, one of them is certainly $\phi(\underline{r}, o)$ itself – i. e., that the initial condition was established in the resident flux shape. Then equation (3. 2) must also have the value unity. Alternatively we can apply our argument to the duplicate asymptotic reactor to show that equation (3. 3) is unchanged if the resident flux replaces the actual flux. Since

$$\delta \phi = \phi' - \phi \tag{3.5}$$

we have

$$\int \phi^+ (\underline{r}, t)\, V^{-1}\, \phi' (\underline{r}, t)\, dv = 1 = \int \phi^+ (\underline{r}, t)\, V^{-1}\, \phi(\underline{r}, t)\, dv\ , \tag{3.6}$$

$$\int \phi^+ (\underline{r}, t)\, V^{-1}\, \delta\, \phi(\underline{r}, t)\, dv = 0\ . \tag{3.7}$$

Equations (3. 6) and (3. 7) represent the orthogonality and normalization of ϕ^+ and ϕ. For a given initial condition, ϕ', equation (3. 4) normalizes the importance in a reactor. Equation (3. 6) then normalizes the resident population or tells us how large the resident population is for the given initial conditions. Equation (3. 7) expresses the definition that the transient flux dies away and can therefore make no contribution to the resident population.

It is to be noted that all these relations hold whether or not the reactor is critical. Owing to the stationary property introduced in the bilinear form, $\int \phi^+ V^{-1} \phi\, dv$, by the inverse time behavior, it is immaterial at what time such forms are evaluated.

(d) Approximation properties

The actual rate of increase of neutrons in a reactor is given by $M\phi'$ per unit volume so that the total rate of increase of the resident population normalized to the present resident population is given by the integral of the importance of each new neutron:

$$\frac{1}{\pi}\frac{d\pi}{dt} = s = \frac{\int \phi^{+} \; M \; \phi' \; dv}{\int \phi^{+} V^{-1} \phi' \, dv} \tag{3.8}$$

Consider first that the reactor has taken up its asymptotic behavior so that ϕ' in equation (3.8) becomes ϕ. Then the neutron increase $M\phi$ at every point is supplying exactly the $s V^{-1} \phi$ neutrons demanded by the asymptotic period $1/s$. There are no excess (or deficit) neutrons to be accounted for at any point in the reactor. Hence, in this special case it makes no difference what importance was assigned to neutrons. Thus, if the correct asymptotic flux is employed in equation (3.8) the asymptotic period is given correctly for any guessed value of the importance. Of course we have to be consistent and use the same guess top and bottom.

Now consider that the correct importance is used but that a guessed value of the asymptotic flux is employed. This guessed flux corresponds to some arbitrary initial flux in the reactor which must finally decay to the resident flux shape. Whatever the initial (guessed) flux, its contribution to the resident population evaluated through the use of the importance leads to the correct resident inverse period being given by equation (3.8).

Finally consider a calculation of s via equation (3.8), in which errors are made in both flux and importance. Instead of s we obtain s plus some error, δs, and we have:

$$s + \delta s = \frac{\int (\phi^+ + \delta\phi^+)\, M\, (\phi + \delta\phi)\, dv}{\int (\phi^+ + \delta\phi^+)\, V^{-1} (\phi + \delta\phi)\, dv} \tag{3.9}$$

We can still normalize our guesses to unity so that without loss of generality we have:

$$s + \delta s = \int \phi^+ M \phi dv + \int \phi^+ M \delta\phi dv + \int \delta\phi^+ M \phi dv + \int \delta\phi^+ M \delta\phi dv \tag{3.10}$$

From our discussion, the two middle integrals vanish exactly; they are the transient terms that do not contribute to the resident population when the correct flux or the correct importance is employed. The first integral is just the correct value of the inverse period, s. We have finally

$$\delta s = \frac{\int \delta\phi^+ M \delta\phi\, dv}{\int \phi^{+\prime} V^{-1} \phi'\, dv} \tag{3.11}$$

where we have reintroduced the unity normalization integral in the denominator.

Equation (3.11) is the fundamental basis of the variational form of calculation. If the two fields, ϕ^+ and ϕ, can be estimated for example within 10 per cent of their true value, then the error in the eigenvalue, s, calculated from the variational or bilinear form of equation (3.8), is only of the order of the product of the error in the two fields, or only 1 per cent in this example.

3.5 The Equivalence of Neutrons

Consider again the concept of the duplicate reactor whose population is distributed at all times in the fundamental mode. How many neutrons

need to be distributed in the duplicate reactor to be equivalent to one neutron at $\underline{r}$ in the actual reactor as far as the one neutron contributes ultimately to the population? The answer to this question is called the equivalence of one neutron, $w(\underline{r})$. The equivalence of one neutron is independent of time and of the actual population. It is not a fractional quantity like the importance. As a concrete example, the equivalence of one neutron in the centre of a bare one group spherical reactor is exactly two (2). Two neutrons in the resident flux shape are needed to be equivalent to one neutron at the centre. The equivalence of one neutron at the extrapolated boundary of the sphere is zero, however. These numbers are given now to supply a concrete example. They will be justified in Section 4, Mathematics.

The equivalent neutrons once introduced into the duplicate reactor will then grow or decrease with the period of the population. The population in the duplicate reactor equivalent to one neutron at $(\underline{r}, o)$ will afterwards change as $w e^{st}$. The concept of equivalence does not exhibit the inverse time behavior of the importance.

The total equivalence of all the neutrons in a reactor is the resident population, again whether the actual population has completed its transient or not. The equivalence and importance of one neutron at $\underline{r}$ and t can be related to the resident population since

$$w(\underline{r}) = \pi(t)\ \phi^{+}(\underline{r}, t). \qquad (3.12)$$

We emphasize that w is not time-dependent. In a reactor with constant properties, the fate of a neutron does not depend on the time it is introduced or on how many other neutrons are present. In contrast, the importance of one neutron is essentially governed by the number of

other neutrons present.

If the resident flux, ϕ, is known and has been correctly normalized, the resident population is given by

$$\pi = \int I V^{-1} \phi \, dv \quad , \tag{3.13}$$

and equation (3.12) becomes

$$w(\underline{r}) = \phi^{+}(\underline{r}, t) \int I V^{-1} \phi(\underline{r}, t) \, dv \tag{3.14}$$

3.6 Sources

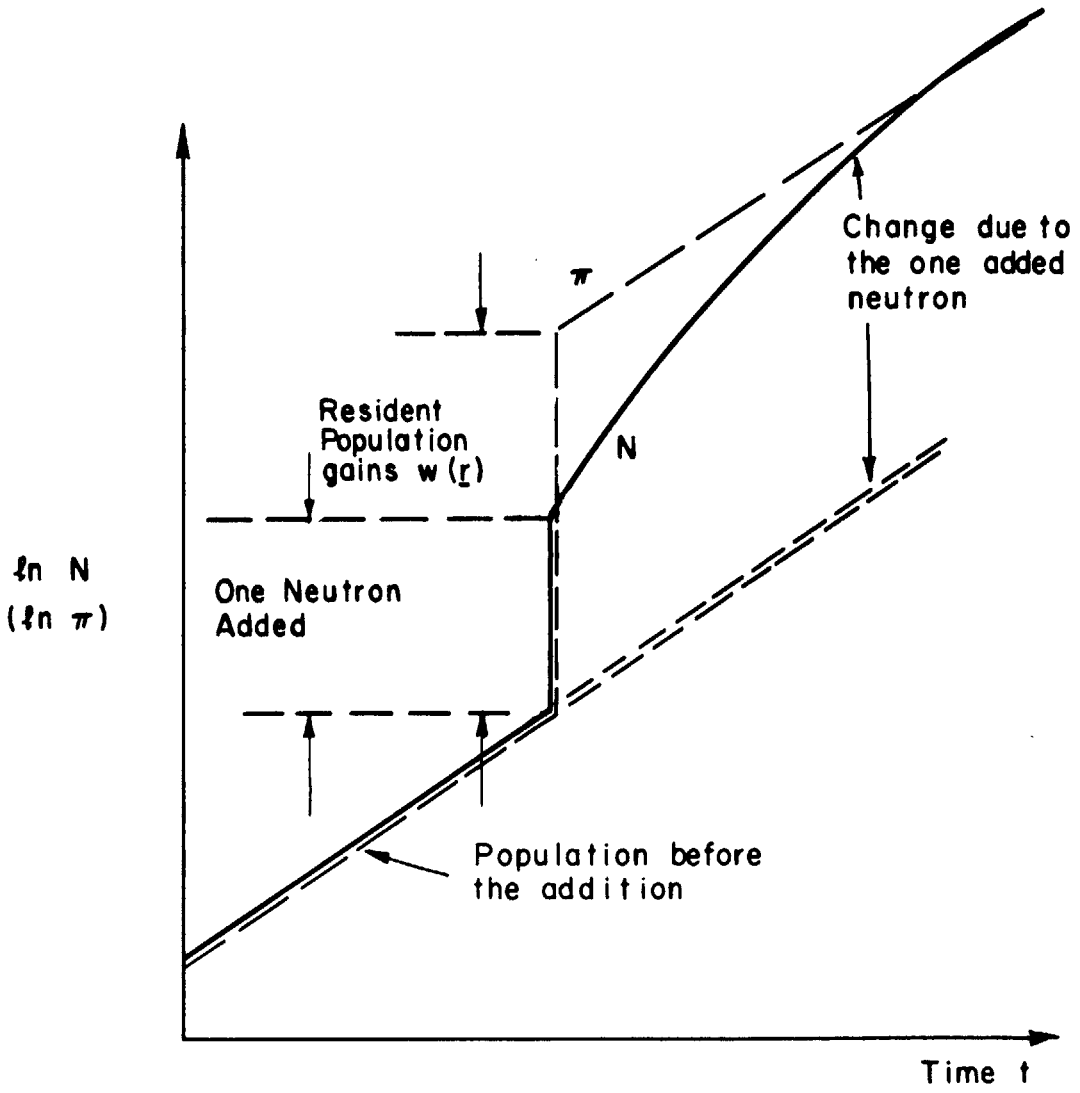
If we can neglect, as the diffusion approximation does neglect, any neutron-neutron interactions, we can estimate the effect of an external source neutron in a reactor in terms of what one neutron already in the reactor population is known to do, i.e., in terms of ϕ^{+} and w, the importance and equivalence of one neutron.

(a) One Added Neutron

On the average, a neutron added at $\underline{r}$ and t will not only increase the actual population instantly by unity but will increase the resident population by $w(\underline{r})$, its equivalence. At a later time, the increase of the resident population due to the source neutron is governed by $w e^{st}$. After a transient period, the actual flux and population will have recovered from the disturbance of shape induced by the source neutron and will return to the resident flux shape. The flux, however, will be the amount $w e^{st}$ higher because of the source neutron as shown in Figure 3.6.

(b) Constant Source Strength

If the source is not one neutron but is of constant strength, S neutrons per second at $\underline{r}$, then, although the population is being increased



CHANGE IN RESIDENT POPULATION DUE TO ONE ADDED NEUTRON

FIGURE 3.6

by S neutrons per second, the resident population is being increased by w S neutrons per second. If S is not a point source but rather a distributed volume source, measured in neutrons per second per unit of volume, we integrate over the source to obtain the rate of increase of the resident population.

The resident population of the reactor may be increasing or decreasing with the asymptotic period of the reactor. The additional change and corresponding rate of change due to the source terms will be distinguished by calling the rate of change:

$$\left.\frac{d\pi}{dt}\right)_S = \int w(\underline{r})\, S(\underline{r})\, dv \qquad (3.15)$$

In the integral of equation (3.15) we can say that we integrate over the whole reactor and that the source function, $S(\underline{r})$, is zero wherever there are no source neutrons. In this equation, which applies to the constant source strength, we can replace w from equation (3.12). Since w is independent of time, while π and ϕ^+ in equation (3.12) are time dependent, we must use the values of π and ϕ^+ appropriate to the same time.

$$\left.\frac{d\pi}{dt}\right)_S = \pi(t) \int \phi^+(\underline{r}, t)\, S(\underline{r})\, dv \qquad (3.16)$$

(c) Homogeneous Source

If the source strength is not of constant strength but rather is proportional to the actual flux, ϕ', the rate of increase of resident population is no longer a constant. We can write such a homogeneous source, $\overline{S}$, as a product function:

$$\overline{S} = S(\underline{r})\, \phi'(\underline{r}, t) \qquad (3.17)$$

Then

$$\left.\frac{d\pi}{dt}\right)_{\bar{S}} = \pi(t)\frac{\int \phi^{+}(\underline{r},t)\ S(\underline{r})\ \phi'(\underline{r},t)\ dv}{\int \phi^{+}(\underline{r},t)\ V^{-1}\ \phi'(\underline{r},t)\ dv} \tag{3.18}$$

In equation (3.18) we have reintroduced the unity normalization in the denominator, so that if we rearrange equation (3.18) into an equation for the inverse period due to the homogeneous source, we can see that in fact the normalization of ϕ^{+} is arbitrary, appearing top and bottom of the equation:

$$s\Big)_{\bar{S}} = \frac{1}{\pi}\frac{d\pi}{dt} = \frac{\int \phi^{+}(\underline{r},t)\ S(r)\ \phi'(\underline{r},t)\ dv}{\int \phi^{+}(\underline{r},t)\ V^{-1}\ \phi'(\underline{r},t)\ dv} \tag{3.19}$$

From the discussion of Section 3.3 and the inverse time dependence of the pairs ϕ^{+}, ϕ', equation (3.19) is independent of time. Thus a homogeneous source puts a reactor on a period. This change in the reactor's inverse period due to the source is given by

$$s\Big)_{\bar{S}} = \frac{\int \phi^{+}\ S\ \phi'\ dv}{\int \phi^{+}\ V^{-1}\ \phi'\ dv} \tag{3.20}$$

This result is the basis of the interpretation of the perturbation formula, where one reactor is interpreted as if it were another reactor, the differences between the reactors being expressible as source terms.

3.7 Modal Populations and Initial Conditions

The mathematical expression of the neutron equations as eigenvalue equations leads to the well known concept of the set of solutions ϕ_n, each with its eigenvalue or logarithmic time derivative, s_n. The higher modes above the fundamental mode (our resident flux) will decay more quickly

(the s_n are algebraically smaller than our s). Furthermore, each eigenfunction solution oscillates at least once within the reactor, i.e., it predicts negative and positive neutrons or precursors. If despite this non-physical behavior we are content to speak of modal populations,

$$N_n = \int I\, V^{-1}\, \phi_n \, dv \quad , \tag{3.21}$$

we can give an interpretation to the corresponding eigenvalues and eigenfunctions of the equations for neutron importance, ω_n and ϕ_n^+ say. First, the importance of one neutron to the modal population, N_n, will increase more rapidly than the importance of one neutron to the resident population, π. Furthermore, the importance of the neutrons to the modal population must exhibit an inverse time behavior just as for the resident population. Thus the eigenvalues, ω_n, of the importance can be matched in magnitude one for one with the eigenvalues of the flux and population, s_n, but with opposite sign.

The eigenfunctions of the importance equation will give the spatial variation of the importance of one neutron to the corresponding modal populations. The magnitude of these importance eigenfunctions for any particular case is determined from the initial conditions by the normalizing integral corresponding to equation (3.4):

$$\int \phi_n^+(\underline{r}, o)\, V^{-1}\, \phi'(\underline{r}, o)\, dv = 1. \tag{3.22}$$

Then the modal flux, ϕ_n, can be found by the further normalization corresponding to equation (3.6),

$$\int \phi_n^+(\underline{r}, o)\, V^{-1}\, \phi_n(\underline{r}, o)\, dv = 1 \tag{3.23}$$

The modal population, N_n, decays with its characteristic period, $1/s_n$, independently of the other modal populations. Thus the total importance of neutrons in the n^{th} modal population as members of any other mode, must be zero. That is, the total contribution of the n^{th} mode to the m^{th} mode is zero, or

$$\int \phi_m^+(\underline{r}, o)\, V^{-1}\, \phi_n(\underline{r}, o)\, dv = \delta_{nm}. \qquad (3.24)^*$$

The modal populations above the fundamental (the resident population, π) may be either positive or negative according to the initial conditions. The corresponding flux shape oscillations have a parallel in the oscillations of the ϕ_n^+, the eigenfunctions of the importance. Which modal importance goes with which modal flux can be determined in principle by matching the magnitudes of the eigenvalues, s_n to ω_n.

We can argue that oscillations are to be expected in the modal importances, ϕ_n^+, above the resident importance (the importance of neutrons to the resident mode, ϕ^+) on the following grounds. Each modal population has a value dependent on the initial conditions. In particular the N_n other than π may all be zero if in fact the flux is in the resident shape and only the resident population is present. If the contribution of one neutron to the n^{th} mode were everywhere of the same sign, then the resident population would make a finite contribution to the n^{th} modal population, i.e., $\int \phi_n^+ V^{-1} \phi\, dv$ could not vanish as it must for any physical value of ϕ and $n > 0$.

The concept of the equivalence can also be extended to a modal equivalence, $w_n(\underline{r})$, the number of neutrons distributed in the n^{th} flux mode that is equivalent to one neutron at $\underline{r}$. We have

*δ_{nm} is the usual Kronicker delta.

$$w_n(\underline{r}) = \phi_n^+(\underline{r}, t)\, N_n(\underline{r}, t) \qquad (3.25)$$

THE EQUATIONS FOR THE IMPORTANCE

The equations for neutron (and precursor) importance in the diffusion approximation are naturally derived using the same approximations and procedure as would be used in deriving the equations of the neutron balance. It is always possible to derive these latter as a mathematical consequence of transport theory and a reduction of the Boltzmann equation. However, such a method loses some of the physical insight of the diffusion theory and this would also be true for the corresponding equations for the importance. We shall, therefore, follow the simplest possible mathematical derivation (33) and merely note that transport corrections can be applied to the diffusion equations. The Boltzmann adjoint equation is derived from the same physical principles in Appendix A.

Following this simple derivation, we treat the problem in three parts. First, we derive a current of importance and a Fick's law behavior for a steady state condition in a pure scattering medium. Second, we draw a time-dependent balance of fractional resident population for a small element of volume of a reactor, allowing for energy transfer processes in multigroup theory. Third, we consider the appropriate boundary conditions.

3.8 Current of Importance

Consider an infinite, pure scattering medium with cross section Σ_s. At some point $\underline{r}$ let there be unit flux and consider the probability of neutrons from the elementary volume around $\underline{r}$ reaching any point $\underline{r}'$ and being scattered there in a further distance $d\underline{r}'$. Once scattered,

the neutron can be considered to be in the next generation, i.e., the progeny of the original neutrons are represented by the probabilities of neutrons being emitted from scattering events throughout the medium.

Although, by assumption, a constant density of neutrons is maintained in the elementary volume, there is a continual flow of the neutrons away from this region; these are replaced by some unspecified process. With isotropic scattering, equal numbers of neutrons leave in all directions. Yet those neutrons going towards the region of high importance (say the centre of the reactor) will have a better chance of producing more progeny than the neutrons going towards the region of low importance (say the edge of the reactor). The ultimate progeny associated with the neutrons originally in the element of volume flow towards the region of high importance. This flow leads to a steady current of ultimate progeny across any surface within the reactor. When normalised to the resident population itself, this flow of fractional resident population is called the current of importance. The current of importance will be derived for the case of unit flux in the element of volume. In the derivation of the equations for the importance we shall employ the divergence of the current of importance per unit density in the volume.

We shall derive the importance current under the assumption of time independent importance. A similar assumption is made in diffusion theory in deriving the neutron current. When the time dependent equations are derived later, we shall be making an error in both derivations. We might correct the neutron and importance equations by allowing for the finite time taken by neutrons to go from $\underline{r}$ to $\underline{r}'$. This correction would lead to the adjoint telegrapher's equation; we prefer to derive

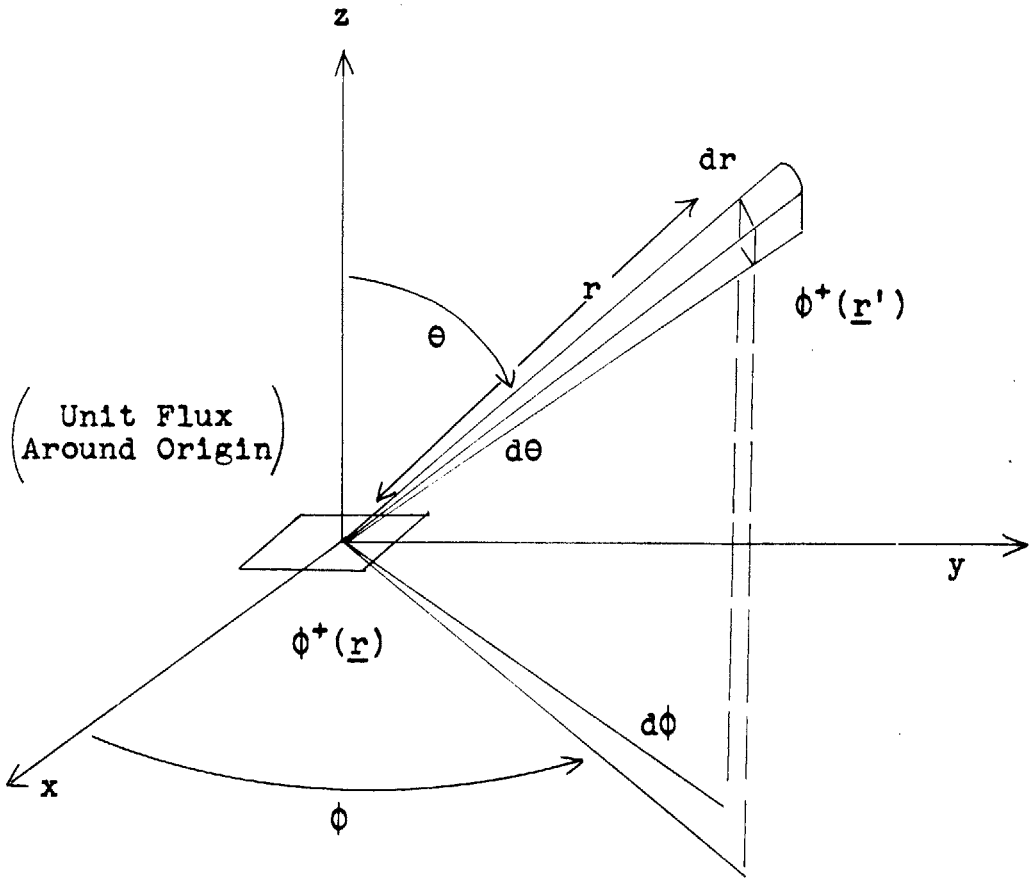
the adjoint Boltzmann equation in Appendix A, which takes this effect into account.

When the neutron is emitted from the scattering event at $\underline{r}'$, it must by definition have the importance $\phi^+(\underline{r}', t)$ or just $\phi^+(\underline{r}')$ in the steady state case we are considering. The potential ultimate progeny associated with one neutron at $\underline{r}$ diffuses outwards and is distributed partly in accordance with the importance of the regions to which the neutron is going. We count the total flow of potential progeny (the resident population, π) in the +z direction. The fraction of the resident population crossing a unit area in the x, y plane in unit time, is the current of fractional resident population or the current of importance in the +z direction through the point $\underline{r}$ due to unit flux around $\underline{r}$. If the importance varies in space the neutrons going to the region of high importance are more important than those going to the region of low importance. If there is a gradient of importance, we anticipate a net current of importance or fractional resident population even in the steady state considered.

With the usual r^2 and exponential form for the probability of a neutron going from $\underline{r}$ to $\underline{r}'$ (where $r = |\underline{r}' - \underline{r}|$), the current is

$$i_{+z} = \int_0^{\infty} \int_0^{2\pi} \int_0^{\pi/2} \frac{\phi^+(\underline{r}')\, e^{-\Sigma_s r}}{4\pi\, r^2} \Sigma_s\, r \sin\theta\, r\, d\theta \cos\theta\, d\phi\, dr. \tag{3.27}$$

If the importance, $\phi^+(\underline{r}')$, is a slowly varying function of $\underline{r}'$ (compared to the transmission probability kernel, $e^{-\Sigma_s r}/4\pi\, r^2$) then it may be expanded in a Taylor series about the origin at $\underline{r}$. The procedure follows the development of Fick's law for the neutron current except that we have



Neutrons around the origin (the point $\underline{r}$) with importance, $\phi^+(\underline{r})$, travel a distance r to scatter in the further distance dr. The scattered neutron has the importance, $\phi^+(\underline{r}')$, of the point of scattering, $\underline{r}'$.

Figure 3.7 Derivation of the Importance Current

a current of importance going from the origin in the +z direction rather than a current of neutrons being scattered towards the origin in the -z direction. Whereas the neutron current, j, was given as:

$$j_{-z} = \frac{\phi}{4} + \frac{1}{6\Sigma_s}\frac{\partial\phi}{\partial z}, \tag{3.28}$$

the current of importance is given by:

$$i_{+z} = \frac{\phi^+}{4} + \frac{1}{6\Sigma_s}\frac{\partial\phi^+}{\partial z}, \tag{3.29}$$

or

$$i_{-z} = \frac{\phi^+}{4} - \frac{1}{6\Sigma_s}\frac{\partial\phi^+}{\partial z}, \tag{3.29a}$$

the difference appearing in the sign of the gradient term.

It is shown in Appendix A that those corrections that are conventionally applied to diffusion theory are equally applicable to the adjoint system. The net current of neutrons and importance can be written as

$$\underline{j} = -D\nabla\phi,$$

but (3.30)

$$\underline{i} = +D\nabla\phi^+ .$$

As anticipated, the current of importance is running up the gradient of importance towards the region of high importance.

The current of importance, $\underline{i}$, has been derived under the assumption of (a) time independence (b) slowly spatially varying importance and (c) infinite uniform scattering medium. The last restriction can be dropped in applications to finite reactors with some absorption, if the boundary is sufficiently far away (i.e., the transmission kernel makes

the effect of the boundary or changed value of Σ_s negligible.) Just as for the diffusion neutron equations, the solution of the importance equations in the neutron approximation cannot be expected to give an accurate result close to boundaries, in regions of rapidly varying properties, or in highly absorbing regions. If we are willing to accept the defects of the diffusion approximation, however, then the above derivation of the current of importance introduces no errors beyond those already present in the expression for the neutron current.

3.9 The Importance Equations

Our second step is to derive the importance equations allowing for absorption, production, diffusion and transfer. Just as the conservation of neutrons leads to equations for the flux, conservation of fractional resident population leads to equations for the importance. Once an initial condition has been established, the resident population is determined. The local probabilities of neutron behavior then govern the way in which the resident population is shared per neutron actually present.

(a) One Group

Consider an element of volume with unit density of neutrons at position $\underline{r}$ and time t. The neutrons in this element have the importance $\phi^+(\underline{r}, t)$ each. After a time interval δt the same fraction of the resident population must be accounted for amongst all the progeny of the original neutrons. Within the element only $(1 - V\Sigma\delta t + V F \delta t)$ neutrons remain, owing to the absorption and production processes. In addition, there is a net loss of resident population due to the scattering out of neutrons (and scattering in) which is given by the divergence of the importance

current. The remaining neutrons in the element have the importance $\phi^+(\underline{r}, t + \delta t)$; the accounting gives the equation for the importance,

$$\phi^+(\underline{r}, t) = (1 - V\Sigma\,\delta t + VF\,\delta t)\,\phi^+(\underline{r}, t + \delta t) + V\nabla\cdot D\nabla\phi^+\,\delta t \qquad (3.31)$$

or

$$\phi^+(t + \delta t) - \phi^+(t) = V(\Sigma\phi^+ - F\phi^+ - \nabla\cdot D\nabla\phi^+)\,\delta t \qquad (3.32)$$

On dividing by $V\,\delta t$ and taking the limit as δt goes to zero, we have the partial differential equation for the importance:

$$\frac{1}{V}\frac{\partial\phi^+}{\partial t} = -(\nabla\cdot D\nabla - \Sigma + F)\,\phi^+ \qquad (3.33)$$

Note that absorption tends to concentrate the fractional importance of the remaining neutrons; production tends to dilute their importance.

(b) Multigroup Theory

In multigroup expressions, in addition to the leakage and true absorption terms appearing in each equation, we have removal or transfer processes to consider. The transfer process in multigroup diffusion theory is usually interpreted along the following lines. A neutron scatters within the group until it has scattered or travelled the average amount for neutrons of this group. At the end of this process it will suddenly be changed in energy and appear in the new group. When the balance of fractional resident population is drawn up, the neutrons about to be transferred leave fewer neutrons behind, just as absorption does. However the transferred neutrons must have, as they leave the group, the importance of neutrons in the group in which they are about to appear. Hence the share per neutron in the original group is not increased as much by transfer as by true absorption since some of the fractional

resident population is transferred out of the group with the transferred neutrons. Scattering transfer leads to terms $V_i \Sigma_{ij} (\phi_i^+ - \phi_j^+)$ for the rate of increase of ϕ_i^+.

When fission transfers neutrons to high energy groups without conserving neutrons, the net expression is modified. There is still the apparent absorption term, $V_2 \Sigma_2 \phi_2^+$ in a two group model, for the effect on the share of the remaining neutrons. The portion of the fractional resident population available to be shared amongst the remaining thermal neutrons is decreased by the $k \Sigma_2 V_2$ neutrons per neutron in the element of volume taking a share, ϕ_1^+ each, away, or decreasing the effect by $-k \Sigma_2 V_2 \phi_1^+$. Thus the two group example is

$$\begin{aligned} V_1^{-1} \frac{\partial \phi_1^+}{\partial t} &= -\nabla \cdot D_1 \nabla \phi_1^+ + \Sigma_1 \phi_1^+ - \Sigma_1 \phi_2^+ , \\ V_2^{-1} \frac{\partial \phi_2^+}{\partial t} &= -F \phi_1^+ - \nabla \cdot D_2 \nabla \phi_2^+ + \Sigma_2 \phi_2^+ \end{aligned} \qquad (3.34)$$

The general expression governing the formation of the adjoint equations for neutron importance is easily induced:

(a) ϕ is replaced by the corresponding ϕ^+,

(b) all probabilities are reversed in sign,

(c) all transfer probabilities (the off diagonal terms) are interchanged, row for column.

We can note that rule (b) is equally well satisfied by leaving the sign of the probabilities unchanged and changing the sign of the time derivatives. Evidently in the one group case rule (c) is not applicable.

In the matrix notation given in Section One, Introduction, the

equations for the neutron importance are very easily expressed. We have

$$V^{-1} \frac{\partial \phi^+}{\partial t} = -M^T \phi^+ = -\nabla \cdot D \nabla \phi^+ + \Sigma^T \phi^+ - F^T \phi^+$$

3.10 Boundary Conditions

The definition of ϕ^+, the fractional importance of neutrons to the resident population, requires that ϕ^+ be positive. The magnitude may range from zero to infinity however, if the resident population is infinite or zero respectively.

Consider the current of importance as given by equations (3.27) and (3.28). The presence of a source of neutrons tends to dilute the importance of neutrons. Thus a neutron source acts as a sink of importance. In the absence of sources and sinks, the current of importance in any direction (i_+ and i_- say) is continuous across any surface. Alternative addition and subtraction of the equations for the currents leads to the requirements

$$\phi^+_A = \phi^+_B \tag{3.36}$$

$$D \nabla \phi^+_A = D \nabla \phi^+_B \tag{3.37}$$

where A and B denote the two sides of the boundary.

Neutrons leaking from the outer surface of the reactor (or into a black body) do so for good and can have no further probability of causing fission. Although the surface condition on the neutron current is that no neutrons return from a vacuum:

$$j_- = 0 , \tag{3.38}$$

outer surface

the corresponding condition on the importance current is that no importance can be lost to a vacuum:

$$i_+ = 0, \qquad \text{outer surface.} \tag{3.39}$$

The change of sign in the expression for the importance current as a gradient of the importance makes the expression corresponding to equation (3.39) formally identical with the expression for the neutron current at a vacuum boundary;

$$\begin{aligned} 0 &= \frac{\phi}{4} + \frac{D}{2}\nabla\phi \\ 0 &= \frac{\phi^+}{4} + \frac{D}{2}\nabla\phi^+ \end{aligned} \qquad \text{outer surface.} \tag{3.40}$$

Then, whenever we wish to replace the strict neutron boundary condition with the concept of the extrapolated flux going to zero, we can employ the identical concept for the importance. This concept is a mathematical device with no physical meaning.

$$\begin{aligned} \phi &= 0 \\ \phi^+ &= 0 \end{aligned} \qquad \text{extrapolated surface.} \tag{3.41}$$

The appropriate initial conditions are determined by the initial conditions of the problem. If an eigenfunction solution is attempted, the initial conditions lead to a normalizing of the eigenfunctions along the lines of Section 3.4.

The one-group approximation is a special case since the eigenequations are identical for the flux and the importance;

$$\begin{aligned} s\,V^{-1}\,\phi &= M\,\phi \\ -\omega\,V^{-1}\,\phi^+ &= s\,V^{-1}\phi^+ = M\,\phi^+ \end{aligned} \qquad \text{(one group)} \tag{3.42}$$

In addition, the boundary conditions for equations (3.40) or (3.41) are identical. Hence the eigenfunction solutions have exactly the same shape. The one-group equation is sometimes called self-adjoint. The homogeneous form of solution leads to an arbitrary normalization. When the normalization appropriate to the initial conditions is inserted, the modal importance is no longer identical to the corresponding modal flux. Furthermore, the time behaviors are inverse as has been discussed. Of course, when the fluxes and importances are used only in ratio form, we may dispense with the normalization and for computational purposes take $\phi^+ = \phi$ in the one-group case.

3.11 Importance of Delayed Neutron Precursors

(a) The Equations of Precursor Importance

So far we have neglected the important role played by precursors and delayed neutrons in the kinetics of reactors. Our results can be extended to include these phenomena quite easily.

The importance of precursors, $C^+(\underline{r}, t)$, is the fractional contribution made by one precursor at $\underline{r}$ and t to the resident population. It is to be anticipated that the importance of a precursor will depend somewhat on the energy of the neutron which it will emit, if this energy differs from that of the prompt fission neutrons.

As before, the total importance of all neutrons and precursors in the reactor is unity;

$$\int \left[\phi^+(\underline{r}, t)\, V^{-1}\, \phi'(\underline{r}, t) + C^+(\underline{r}, t)\, C'(\underline{r}, t)\right] dv = 1 \qquad (3.43)$$

The equations for precursor importance are derived along the lines given for the neutron equations. In the case of one delayed neutron

group, the importance balance allowing for the leakage of importance, and the absorption and production of neutrons, leads to the partial balance:

$$\frac{\partial \phi^+}{\partial t} = -V\nabla \cdot D\nabla \phi^+ + V\Sigma\phi^+ - (1-\beta)\, V F \phi^+ \ldots \qquad (3.44)$$

In addition, however, $V\beta F$ precursors have been introduced into the precursor group where they must have the importance of precursors, $C^+(\underline{r})$. The potential progeny represented by these new precursors decrease the total available to be shared amongst the remaining neutrons. Hence, the correct importance balance for the neutrons has an additional term, to give

$$V^{-1}\frac{\partial \phi^+}{\partial t} = -\nabla \cdot D\nabla \phi^+ + \Sigma \phi^+ - (1-\beta) F \phi^+ - \beta F C^+ . \qquad (3.45)$$

The balance of precursor importance follows a similar argument. Consider one precursor per unit volume in a small element of volume, with importance $C^+(\underline{r}, t)$. After a time δt, the importance changes to $C^+(\underline{r}, t+\delta t)$. During that interval the one precursor has been reduced to $(1-\lambda\,\delta t)$ precursors by decay. This decay supplied neutrons to the neutron group with the importance $\phi^+(\underline{r}, t)$. The balance of importance for the remaining precursors is

$$C^+(\underline{r}, t) = (1-\lambda\,\delta t)\, C^+(\underline{r}, t+\delta t) + \lambda\,\delta t\, \phi^+(\underline{r}, t), \qquad (3.46)$$

whence

$$C^+(t+\delta t) - C^+(t) = (-\lambda\phi^+ + \lambda C^+)\,\delta t. \qquad (3.47)$$

On dividing by δt and taking the limit as δt goes to zero we have the definition of the partial derivative;

$$\frac{\partial C^+}{\partial t} = -\lambda\phi^+ + \lambda C^+ . \qquad (3.48)$$

The generalisation of equation (3.48) to several groups of precursors is obvious. We write equations such as (3.48) with subscript i for each of the i precursors. For the neutron importance we interpret β as the sum of β_i and equation (3.45) is unchanged.

The simple example of one-group of delayed neutrons used here can be written as:

$$\begin{aligned} V^{-1}\frac{\partial \phi^+}{\partial t} &= -\nabla\cdot D\nabla\phi^+ + \Sigma\phi^+ - (1-\beta)F\phi^+ - \beta F C^+ \\ \frac{\partial C^+}{\partial t} &= -\lambda\phi^+ \qquad\qquad\qquad + \lambda C^+ \end{aligned} \tag{3.49}$$

The general rules for writing the equations for neutron and precursor importance are induced to be identical with the rules given after equation (3.34). In particular, we change the sign of all the probabilities (or of all the time derivatives) and transpose the transfer probabilities row for column.

(b) Physical Interpretation

First let us consider the steady state case, when there is considerable simplification. Equation (3.49) gives

$$\begin{aligned} 0 &= -\nabla\cdot D\nabla\phi^+ + \Sigma\phi^+ - (1-\beta)F\phi^+ - \beta F C^+ \\ 0 &= -\lambda\phi^+ \qquad\qquad\qquad + \lambda C^+ . \end{aligned} \tag{3.50}$$

Hence,

$$C^+ = \phi^+, \tag{3.51}$$

and

$$0 = \nabla\cdot D\nabla\phi^+ - \Sigma\phi^+ + F\phi^+ \tag{3.52}$$

Equation (3.51) signifies that in the steady state, the importance of

precursors is just the importance of the delayed neutrons they will become. Equation (3.52) shows that in addition, in the steady state, the importance of neutrons including the effects of precursors is the same as the importance of neutrons that would be calculated if delayed neutrons were neglected. We might note that the precursors have an importance even in the reflector, i.e., C^+ gives the contribution that a precursor would make to the population. The physical limitation that there are no precursors in the reflector is expressed through the flux equations rather than the precursor equations.

For the non-critical case, we can take the corresponding eigenvalue problem, replacing $\frac{\partial \phi^+}{\partial t}$, $\frac{\partial C^+}{\partial t}$ with $\omega_n \phi_n^+$, $\omega_n C_n^+$. As before we find mathematically that $\omega_n = -s_n$ of the flux equations; the precursor importance has the inverse time behavior to the population as it must from the physical definition.

Corresponding to the critical case, we have coupling equations for the general case which are now

$$C^+ = \frac{\lambda}{\lambda - \omega} \phi^+ = \frac{\lambda}{\lambda + s} \phi^+ \tag{3.53}$$

while

$$-\omega V^{-1} \phi^+ = \nabla \cdot D \nabla \phi^+ - \Sigma \phi^+ + (1-\beta) F \phi^+ + \frac{\beta F \lambda}{\lambda - \omega} \phi^+ \tag{3.54}$$

or

$$s V^{-1} \phi^+ = \nabla \cdot D \nabla \phi^+ - \Sigma \phi^+ + F \phi^+ - \frac{s \beta F}{\lambda + s} \phi^+$$

Equation (3.59) shows that on a rising period, ω negative and s positive, the importance of a precursor is less than that of neutron it will become. This is due, of course, to the time hold-up of the neutron in its precursor state and reflects the tendency of a rising flux to outrun

the precursors. For a falling period, s negative, the precursor importance increases over the neutron importance until at $s = -\lambda$, the precursor importance becomes infinitely greater than neutron importance. For $s < -\lambda$, the ratio becomes negative and is evidently non-physical. The singularity and change of sign reflect the fact that a reactor cannot be shut down on a steady period faster than its precursors decay. Values of s less than $-\lambda$ cannot correspond to the dominant asymptotic solution but only to a transient solution.

3.12 Matrix Representation

The general equations for neutron and precursor importance are easily represented by the transposed matrix equations

$$-A \frac{\partial \psi^+}{\partial t} = +(M^T + T^T)\, \psi^+ \tag{3.56}$$

$$-\omega A \psi^+ = s A \psi^+ = (M^T + T^T)\, \psi^+ \tag{3.57}$$

where ψ^+ is the generalized importance vector.

One advantage of extending the matrix representation as we have done is that the previously given results for the case with no delayed neutrons are applicable to the general case with delayed neutrons. The distinction between resident and transient fluxes and populations, the normalization and orthogonality are all derivable by the same arguments as given in Section 3.2 to 3.7. In particular, the inverse period calculated by the bilinear form

$$s = \frac{\int \psi^+ (M+T)\, \psi \, dv}{\int \psi^+ A \psi \, dv} , \tag{3.58}$$

is correct to second order for first order guesses of importance, ψ^+, and

flux ψ. Equation (3.58) suffers a serious disadvantage in that s appears explicitly in only one place. Actually, since the coupling coefficients between the multigroup fluxes and between those and the precursor concentrations, are functions of the inverse period, equation (3.58) really represents a polynomial expression for s with as many solutions as there are equations even in the fundamental harmonic.

APPENDIX A TO SECTION THREE

THE ADJOINT BOLTZMANN EQUATION

The equation for importance is derived in the Boltzmann approximation on physical considerations. The share that a neutron has in the resident population (i.e., the fractional resident population) is conserved. Once one neutron has been specified to have a share $\phi^+(\underline{r}, \underline{\Omega}, V, t)$, this can be equated to the sum of the importances of the neutron's progeny after an interval δt. In this interval, the neutron travels $\delta r = V\delta t$, in the direction $\underline{\Omega}$.

By the time the neutron has travelled δr however, it will have had a probability of removal $\delta r/\ell$ where ℓ is the mean free path, i.e., the reciprocal of the removal probability, Σ, per unit path. Hence only $(1 - \frac{\delta r}{\ell})$ neutrons are present with the original direction and speed to contribute $\phi^+(\underline{r} + \delta\underline{r}, \underline{\Omega}, V, t + \delta t)$ each. In addition however, neutrons scattered or fissioned into other elements of the phase space have there the local value of the importance. We need only specify the distribution probability, $f(V\underline{\Omega} \rightarrow V' \underline{\Omega}')$ and the yield or multiplicity, $c(\underline{r}, t)$, to be able to integrate over the distribution probability and sum these additional contributions. Let us call the integral over the distribution probability weighted with the importance of each neutron produced X^+, where

$$X^+ = \iint \phi^+(\underline{r}, \underline{\Omega}', V', t)\, f(\underline{\Omega}, V, t \rightarrow \underline{\Omega}', V', t)\, dV'\, d\underline{\Omega}' \qquad (3A.1)$$

Then the balance of resident population share per neutron is

$$\phi^+(\underline{r}, \underline{\Omega}, V, t) = (1 - V\Sigma\,\delta t)\,\phi^+(\underline{r} + \delta\underline{r}, \underline{\Omega}, V, t + \delta t) + c(\underline{r}, \underline{\Omega}, V, t) V\Sigma X^+ \delta t \qquad (3A.2)$$

Then rearranging

$$\phi^+(\underline{r} + \delta\underline{r}, \underline{\Omega}, V, t + \delta t) - \phi^+(\underline{r}, \underline{\Omega}, V, t) = V \Sigma \phi^+(\underline{r} + \delta\underline{r}, \underline{\Omega}, V, t + \delta t)\, \delta t - c V \Sigma X^+ \delta t \tag{3A. 3}$$

Hence dividing by δt and taking the limit as δt goes to zero,

$$\frac{d\phi^+}{dt} = V \Sigma \phi^+ - c V \Sigma X^+ \tag{3A. 4}$$

But for this geometry, the total derivative, $\frac{d\phi^+}{dt}$, can be expressed in terms of the local partial derivatives, $\frac{\partial \phi^+}{\partial t}$ and $V\underline{\Omega} \cdot \nabla \phi^+$.

$$\frac{1}{V} \frac{\partial \phi^+}{\partial t} = - \underline{\Omega} \cdot \nabla \phi^+ + \Sigma \phi^+ - c \Sigma X^+ \tag{3A. 5}$$

We have derived the equation for importance or the adjoint time dependent Boltzmann equation. This equation does not seem to have been developed or used in neutron transfer though it has been given on mathematical grounds in the realm of radiation transfer, (38).

One important application of the Boltzmann adjoint is to show that all those corrections customarily made to the diffusion equations on the basis of Milne's problem are duplicated on considering the corresponding adjoint problem. First let us consider the transport boundary conditions.

(1) ϕ^+ positive

(2) No important neutrons are lost. Hence the outward flowing neutrons at a vacuum or black body surface have no importance

$$\phi^+(\underline{S}, \underline{\Omega}+, V, t) = 0 \tag{3A. 6}$$

(3) In the absence of sources or sinks of neutrons with their corresponding importance, the importance is continuous across an interface:

$$\phi^{+}(\underline{r} + R\underline{\Omega}, \underline{\Omega}, V, t + \frac{R}{V}) ,$$

is a continuous function of R where $\underline{r} + R\underline{\Omega}$ describes the interface.

(4) Condition at infinity. On physical grounds, no important neutrons can have travelled infinitely far. Hence we take the condition at infinity to be in agreement with this restriction in every particular case.

(5) The initial conditions are specified by the initial actual population which itself determines the resident population and the share of each neutron to the resident population.

A comparison with a standard text in transport theory (3) shows that the boundary conditions (as opposed to the initial conditions) are paralleled in the (forward) Boltzmann equation for the neutron flux, with one exception. The boundary condition at a vacuum surface is zero for $\Omega-$ in the forward equation but for $\Omega+$ in the adjoint equation.

In the treatment of the Milne problem, we consider the one energy steady state form of the Boltzmann equation. Let $\psi(\underline{r}, \underline{\Omega}) = \int VN(\underline{r}, V\underline{\Omega})\, dv$. The eigenvalue equation is then

$$\underline{\Omega} \cdot \nabla\psi + \psi\left(\Sigma + \frac{s}{V}\right) = c\,\Sigma \int \psi(\underline{r}, \underline{\Omega}')\, f(\underline{\Omega}' \rightarrow \underline{\Omega})\, d\underline{\Omega}' \qquad (3A.\,7)$$

where s is the eigenvalue.* The corresponding adjoint equation for the importance of neutrons in the Milne problem is

$$-\underline{\Omega} \cdot \nabla\psi^{+} + \psi^{+}\left(\Sigma - \frac{\omega}{V}\right) = c\,\Sigma \int \psi^{+}(\underline{r}, \underline{\Omega}')\, f(\underline{\Omega} \rightarrow \underline{\Omega}')\, d\underline{\Omega}' \qquad (3A.\,8)$$

Firstly the eigenvalue ω can be replaced by $-s$; the property of the

*The assumptions involved and the significance of these assumptions are discussed by Davison (3) whose nomenclature we follow.

eigenvalue means that the solution values are not changed by the formal change of representation. Secondly, if $f(\underline{\Omega} \rightarrow \underline{\Omega}')$ depends only on $\underline{\Omega} \cdot \underline{\Omega}'$, we have $f(\underline{\Omega} \rightarrow \underline{\Omega}') = f(\underline{\Omega}' \rightarrow \underline{\Omega})$. The equation and boundary condition for the importance is identical with the equation and boundary condition for the flux with $(\underline{\Omega})$ replacing $(-\underline{\Omega})$. Thus if for any pair of eigen solutions, ψ_n^+, ψ_n,

$$\psi_n^+ (\underline{r}, \underline{\Omega}) = \psi_n (\underline{r}, -\underline{\Omega}) \tag{3A. 9}$$

at some particular point $\underline{r}$, $\underline{\Omega}$, then this relation holds for every $\underline{r}$, $\underline{\Omega}$. Such a relation is assured by a suitable normalization. We have derived an optical reciprocity theorem in the Laplace transform space.

From the above argument it follows that solutions for the Milne problem can be used without change for the adjoint Milne problem of neutron importance. The important conclusion is that all those correction factors derived from the Milne problem now apply to the adjoint flux. It is for this reason that we can, in the diffusion applications, take;

(a) $D^+ = D$

(b) linear extrapolation distance $\lambda^+ = \lambda$

(c) extrapolated end point $z_o^+ = z_o$

SECTION 4: MATHEMATICS

In the present section, we shall rederive some well known relations in order to be able to prove and justify on mathematical grounds, some of the statements made on physical grounds in Section 3. To this end we briefly review the use of the adjoint function in ordinary linear differential systems and show how the requirement of the stationary property dictates the mathematical formulation of the equations for the importance.

In the second part of this section, we investigate some of the results of our physical definitions for the equivalence, the initial values of the importance and the bilinear concomitant. A contribution is made to the question of the completeness of the eigenfunction solutions to the multigroup equations. Finally, from our extension of the importance to a time-dependent concept, we are able to give a variational principle which succinctly expresses multigroup diffusion theory. The principle, which renders a Lagrangian stationary, is shown to lead to the equations of the system via the conventional Euler equations.

4.1 Review of Ordinary Linear Differential Systems

The first use of an adjoint function was made by Lagrange (c. 1762) who sought to extend the concept of the integrating factor to ordinary linear differential equations. The term adjoint itself means connected or related, but dates only from Fuchs (1873).

To indicate what is involved, we use a second order, ordinary linear differential equation with time as the independent variable; the equation corresponds roughly to the telegrapher's equation for the

diffusion of neutrons with finite velocity:

$$p \frac{d^2\phi}{dt^2} + V^{-1} \frac{d\phi}{dt} + \Sigma \phi = 0 \tag{4.1}$$

where p is some coefficient we need not specify.

In general, the coefficients in this equation are also functions of time. How could we reduce this second order equation to a first order equation, using an integrating factor or function, ϕ^*? Consider each term in the equation (4.1) multiplied by such an integrating function:

$$\phi^* p \frac{d^2\phi}{dt^2} + \phi^* V^{-1} \frac{d\phi}{dt} + \phi^* \Sigma \phi = 0 . \tag{4.2}$$

Each term can be transformed to give

$$\phi^* p \frac{d^2\phi}{dt^2} = \frac{d}{dt}(\phi^* p \frac{d\phi}{dt}) - \frac{d\phi^* p}{dt} \frac{d\phi}{dt} = \frac{d}{dt}(\phi^* p \frac{d\phi}{dt}) - \frac{d}{dt}(\frac{d\phi^* p}{dt} \phi) + \frac{d^2\phi^* p}{dt^2} \phi$$

$$\phi^* V^{-1} \frac{d\phi}{dt} = \frac{d}{dt}(\phi^* V^{-1} \phi) - \frac{d\phi^* V^{-1}}{dt} \tag{4.3}$$

$$\phi^* \Sigma \phi = \phi^* \Sigma \phi$$

If we sum the transformed terms, equation (4.2) has been rearranged to

$$\frac{d}{dt}\left[\phi^* p \frac{d\phi}{dt} - \frac{d(\phi^* p)}{dt} \phi + \phi^* V^{-1}\right]\phi + \left[\frac{d^2(\phi^* p)}{dt^2} - \frac{d(\phi^* V^{-1})}{dt} + \phi^* \Sigma\right] \phi = 0 \tag{4.4}$$

In equation (4.4), the left hand bracket is a perfect differential. We can then reduce that equation to one of first order if and only if, the second bracket is identically zero. This leads to the so-called adjoint equation for ϕ^* as the necessary and sufficient condition that ϕ^* is an integrating function for the equation (4.1):

$$\frac{d^2(\phi^* p)}{dt^2} = \frac{d(\phi^* V^{-1})}{dt} + \phi^* \Sigma = 0 \qquad (4.5)$$

From the symmetric form of the perfect differential, we see that we could have started with the adjoint equation and, working backwards, have derived the original equation as its adjoint. Thus the equations are each others adjoints. When both equations can be expressed in the same form, they are called self-adjoint. More strictly, the system of equations and boundary conditions is only self-adjoint when both equations and boundary conditions are identical.

The expression in brackets in the first term of equation (4.4) is called the bilinear concomitant, b, (bilinear since it is linear in both ϕ and ϕ^* , concomitant since it is related to the two (adjoined) equations). The coefficient, Σ, of the flux itself rather than its derivatives (i.e., the lowest coefficient) does not appear in the bilinear concomitant. By shifting the order of the differential equation from 2 to 1, we have eliminated explicit reference to the lowest coefficient. This coefficient does appear implicitly, since ϕ^* has to be a solution of the adjoint equation which does contain Σ. The perfect differential can be integrated to give a constant; we can say that b is stationary, or rendered stationary by the adjoint function, ϕ^*. This stationary property has further meaning to be developed.

If, for the moment, we denote the two equations, (4.1) and (4.5), by the short-hand notation

$$\begin{aligned} L\phi &= 0, \\ L^*\phi^* &= 0, \end{aligned} \qquad (4.6)$$

we have the so-called Lagrange identity:

$$\frac{db}{dt} = \phi^* L\phi - \phi L^* \phi^* \tag{4.7}$$

If both equations (4.6) are indeed satisfied by ϕ and ϕ^* respectively, then b is a constant.

Instead of deriving the bilinear concomitant for each problem, it is possible to give rules to form the adjoint equation for any given equation and then find the bilinear concomitant from the Lagrange identity. The rules for finding the adjoint equation for an ordinary linear differential equation are two:

(a) Take the coefficient of each term inside the differential. If the coefficients are independent, and are not variable parameters, this step is unnecessary.

(b) Change the sign of the odd powered derivatives. This is the same, of course, as changing the sign of all the even powered derivatives or terms.

In our present example, these two rules do indeed lead to

$$p\frac{d^2\phi}{dt^2} + V^{-1}\frac{d\phi}{dt} + \Sigma\phi = 0,$$

$$\frac{d^2(\phi^* p)}{dt^2} - \frac{d(\phi^* V^{-1})}{dt} + \phi^* \Sigma = 0. \tag{4.8}$$

So far we have reduced the order of our original equation, though at the expense of introducing a second equation, of the original order, which ϕ^* must satisfy. There are classes of problems where it is easier to solve the adjoint equation than the original equation, but

these will not be developed here.

The bilinear form offers two other advantages, however, which we shall employ. We note, first, that a judicious selection of the bilinear concomitant in the first place, and the requirement that it should be stationary, will lead to both equations as the necessary and sufficient condition. The derivation of both equations from a single principle (that a bilinear form should be stationary) is a very powerful and succinct technique. The extension of this method for the multigroup treatment of neutrons and precursors is undertaken in Section 4. 10. While still treating the ordinary differential equation, we illustrate, in the next section, a second use of the bilinear form in its application to perturbation calculations.

4. 2 A Perturbation Formula for Ordinary Linear Differential Equations

So far we have not had to specify the boundary conditions on either our original equation or function, or on the adjoint function. Satisfaction of the appropriate equation is all that is required. Mathematically, the equation together with its boundary conditions is called the system; we can also talk of an adjoint system.

A common situation arises when it is difficult to solve the given problem and its boundary conditions, but where a similar problem can be solved in which there is only a small difference between the lowest coefficients of the two problems. Thus, we may seek to solve

$$V^{-1} \frac{d\phi'}{dt} + (\Sigma + \delta\Sigma)\, \phi' = 0 \tag{4. 9}$$

knowing the solution of the equation and boundary conditions for ϕ:

$$V^{-1} \frac{d\phi}{dt} + \Sigma\, \phi = 0 \tag{4. 10}$$

This equation is, in a sense, a trivial example since a first order equation can always be integrated by quadratures from the given initial conditions. However, the formalism of this example is the same as for certain more useful cases with partial differential equations.

In addition, it will be supposed that the real problem is not to find ϕ' so much as to find $\frac{d\phi'}{dt}$. Thus we may be more interested in finding the rate of change of the reactor population than the actual level or magnitude.

As an approximate solution, to the real problem, (equation 4. 9), we might try to use the known solution to equation (4. 10). We would calculate the rate of change approximately by writing equation (4. 9) as:

$$V^{-1} \frac{d\phi'}{dt} = -(\Sigma + \delta\Sigma)\,\phi' = -(\Sigma + \delta\Sigma)(\phi + \delta\phi)$$

or

$$V^{-1} \frac{d\phi'}{dt} = \underset{\text{known}}{-(\Sigma\,\phi)} - \underset{\text{known}}{(\delta\Sigma\,\phi)} - \underset{\text{unknown}}{(\Sigma\,\delta\phi)} - \underset{\text{small}}{(\delta\Sigma\,\delta\phi)} \qquad (4.11)$$

In equation (4. 11), if the perturbation, $\delta\Sigma$, is small we might argue that the difference in the solution is also small, $\delta\phi = \phi' - \phi$. This can only be true however if the initial conditions on ϕ' and ϕ are in fact the same. It is at this point that we must start considering the system of equations and boundary conditions and not just the equations alone.

If indeed $\delta\phi$ and $\delta\Sigma$ are both small (compared to ϕ and Σ respectively) we might reasonably neglect their product in equation (4. 11). Of the other three terms in the equation, the first is known exactly from the solution of the known problem – it is just the rate of change, $V^{-1} \frac{d\phi}{dt}$. The remaining terms however are both of the same order of

magnitude and we cannot neglect one in comparison to the other. To do so would be to throw away half the answer, the answer depending essentially on the net effect of the two terms. But one term contains the unknown, $\Sigma\,\delta\phi$. Hence, in this formalism, we cannot afford to use the approximation that $\phi' \approx \phi$.

In developing the bilinear concomitant we saw that the lowest coefficient did not appear. It is suggestive, therefore, to use the bilinear concomitant as an alternative form for an approximation purpose. We may anticipate that there will then be no term in $\Sigma\,\delta\phi$. It is generally true that, when the known problem has been solved it is comparatively easy to find the solution to the corresponding adjoint problem. We assume that we know the solution, ϕ^*, to the problem

$$-\frac{d(\phi^* V^{-1})}{dt} + \phi^* \Sigma = 0 \qquad (4.12)$$

We call this known problem with the known solutions, ϕ and ϕ^*, the reference problem. To some extent, the choice of the reference problem is arbitrary, though (1) it should be soluble, and (2) it should be as near to the actual problem as practical in order to improve subsequent approximations.

Our present results do not require the specification of adjoint boundary conditions. We need only specify that the boundary conditions on ϕ' should be the same as on ϕ if we want to neglect $\delta\phi$ later on.

If, as before, we seek to find the derivative of our actual problem, we want to approximate

$$V^{-1}\frac{d\phi'}{dt} = -(\Sigma + \delta\Sigma)\,\phi' \qquad (4.13)$$

We form the mixed bilinear expression from equation (4. 13), our actual problem, but using the adjoint, ϕ^*, of our reference problem:

$$\frac{d(\phi^* V^{-1} \phi')}{dt} = \phi^* V^{-1} \frac{d\phi'}{dt} + \frac{d(\phi^* V^{-1})}{dt} \phi'$$

$$= - \phi^*(\Sigma + \delta\Sigma) \phi' + \phi^* \Sigma \phi'$$

$$= - \phi^* \delta\Sigma \phi' = - \phi^* \delta\Sigma(\phi + \delta\phi) \approx - \phi^* \delta\Sigma \phi . \qquad (4. 14)$$

In equation (4. 14) we know ϕ^* and $\frac{d\phi^*}{dt}$ by assumption. Hence a solution of this equation gives the behaviour of ϕ', the problem in hand. Yet on writing the known equations for the derivatives, we have reduced to the simpler form of equation (4. 14) in which it is possible to make the approximation of neglecting $\delta\phi$.

The general procedure to take advantage of the stationary form is as follows. Form the mixed bilinear expression, b', ($\phi^* V^{-1} \phi'$ of the example) by an expression analogous to the Lagrange identity. That is, multiply the actual problem by the reference adjoint and the reference equation by the unknown function, ϕ'.

$$\frac{db'}{dt} = \phi^* L' \phi' - \phi' L^* \phi^* \qquad (4. 15)$$

The bilinear concomitant, $\phi^* V^{-1} \phi$, is stationary. A change that may produce a severe change in one of the functions (ϕ to ϕ' say) will lead to a much smaller change in the bilinear concomitant. Thus the bilinear form is amenable to approximation techniques. The error is only of second order although a first order approximation is made. This advantage of the bilinear concomitant comes about directly from the disappearance of the lowest order coefficient. This enables us to form analogous expressions for more complicated problems in partial

differential systems.

4.3 Partial Differential Equations

So far we have derived the bilinear form only for ordinary linear differential equations. The requirement that the bilinear concomitant be stationary to changes of the variable (time) led to the two adjoint equations. The bilinear form was then shown to be less sensitive to changes in ϕ (or ϕ^*) than ϕ(or ϕ^*) forms alone. From this we derived a perturbation formula which enabled us to calculate $\frac{d\phi'}{dt}$ with accuracy when using the first order approximation, $\phi' \approx \phi$.

The problem of neutron diffusion is described by spatial as well as temporal effects, i.e., we must deal with a partial differential equation. On the other hand, the basic problem is to calculate either the time derivative of the flux (or population) or the logarithmic derivative. We would certainly like to retain the two advantages of the bilinear form just derived for the ordinary differential equation.

In the one-group case we might expect (quite correctly it happens) that the adjoint of the one group equation is given by our two rules for finding adjoint equations. If we apply these two rules to

$$V^{-1} \frac{\partial \phi}{\partial t} = \nabla \cdot D \nabla \phi - \Sigma \phi + F \phi = M \phi \tag{4.16}$$

we note that (a) there are no time dependent parameters to consider (by assumption) and (b) we need only change the sign of the time derivative. The one-group adjoint equation is

$$-V^{-1} \frac{\partial \phi}{\partial t} = \nabla \cdot D \nabla \phi^+ - \Sigma \phi^+ + F \phi^+ = M \phi^+ \tag{4.17}$$

In the discussion of Section 2, it was shown that the adjoint equation

is often interpreted as the equation describing a system where cause and effect are reversed, where probabilities have opposite signs. Equation (4. 17) does indeed bear this relation to equation (4. 16). For this reason, it is common to speak of the actual and adjoint fluxes as the forward and backward fluxes respectively. It is possible to interpret the multigroup adjoint equations as the description of a system with reversal of cause and effect; then with this assumption the derivation of the adjoint equations becomes trivial. Although the combination of the two systems is understandably stationary in the time sense, it is not apparent why the remaining properties of the stationary form should apply, or the meaning of such expressions as $\phi^+ F \phi$. For this reason, we employ the interpretation of ϕ^+ as a fractional importance. It is useful however to retain the nomenclature of the forward and backward fluxes and equations.

In the example on the ordinary differential equation, the bilinear concomitant did not contain the lowest coefficient explicitly. This property improved the accuracy of approximations. By analogy, we require the terms in M for the present system to disappear in the bilinear concomitant. From the Lagrange identity, we have

$$\frac{db}{dt} = \phi^+ L\phi - \phi L^+ \phi^+ = 0$$

$$= \phi^+ \left(V^{-1} \frac{\partial \phi}{\partial t} - M\phi\right) - \phi\left(-V^{-1} \frac{\partial \phi^+}{\partial t} - M\phi^+\right) \qquad (4.18)$$

With the equations postulated, equations (4. 16) and (4. 17), $\frac{db}{dt}$ is indeed zero. If $-\phi^+ M\phi + \phi M\phi^+$ is to vanish in the one group representation we would have to have

$$\phi^+ \nabla \cdot D\nabla\phi - \phi^+ \Sigma \phi + \phi^+ F \phi = \phi^+ \nabla \cdot D\nabla\phi^+ - \phi\Sigma\phi^+ + \phi F \phi^+ \qquad (4.19)$$

From their algebraic nature, the terms in Σ and F do cancel. If we are to satisfy the corresponding condition in the multigroup representation where Σ and F represent algebraic matrices, the adjoint equation must be modified to contain the transpose of Σ and F. Then if we take as a generalization of equation (4.17),

$$-V^{-1}\frac{\partial\phi^+}{\partial t} = \nabla\cdot D\nabla\phi^+ - \Sigma^T\phi^+ + F^T\phi^+ = M^T\phi^+ , \tag{4.20}$$

which is the correct multigroup adjoint equation, the terms $\phi^+\Sigma\phi$, $\phi^+F\phi$ commute and cancel with $\phi\Sigma^T\phi^+$, $\phi F^T\phi^+$ respectively.

There remain the terms in $\phi^+\nabla\cdot D\nabla\phi$ and $\phi\nabla\cdot D\nabla\phi^+$. Although these terms cannot be made to commute in general, we still have at our disposal the hitherto unused adjoint boundary condition. The following lemmas demonstrate that the divergence terms can be made to commute on the average over the reactor. Under the conditions of these lemmas, the fields ϕ and ϕ^+ and the equations given that define them can be used in a bilinear form in which the first order approximation can be used. These lemmas are well known in the form of Green's theorem in the first form and are given here for the sake of completeness.

4.4 Two Lemmas

Lemma One: The reactor averages of $\phi^+\nabla\cdot D\nabla\phi$ and $\phi\nabla\cdot D\nabla\phi^+$ commute for any ϕ, ϕ^+ with the appropriate boundary conditions.

$$\int\phi^+\nabla\cdot D\nabla\phi\,dv = \int\phi\nabla\cdot D\nabla\phi^+\,dv . \tag{4.21}$$

Proof: Since $\nabla\cdot\theta\underline{\mu} = \theta\nabla\cdot\underline{\mu} + \underline{\mu}\cdot\nabla\theta$ (4.22)

for any vector, $\underline{\mu}$ and scalar θ, we have

$$\phi^+ \nabla \cdot D \nabla \phi = \nabla \cdot \phi^+ D \nabla \phi - \nabla \phi^+ \cdot D \nabla \phi \,, \tag{4.23}$$

and

$$\phi \nabla \cdot D \nabla \phi^+ = \nabla \cdot \phi D \nabla \phi^+ - \nabla \phi \cdot D \nabla \phi^+ \,. \tag{4.24}$$

Now the dot product of two vectors commutes, being a scalar,

$$\nabla \phi^+ \cdot D \nabla \phi = \nabla \phi \cdot D \nabla \phi^+ \,. \tag{4.25}$$

Hence subtracting equation (4.24) from equation (4.23) and integrating over the reactor, we get:

$$\int \nabla \cdot (\phi^+ D \nabla \phi - \phi D \nabla \phi^+) \, dv = \int (\phi^+ \nabla \cdot D \nabla \phi - \phi \nabla \cdot D \nabla \phi^+) \, dv \,. \tag{4.26}$$

The divergence integral can be changed to a surface integral over the reactor boundary:

$$\int \nabla \cdot (\phi^+ D \nabla \phi - \phi D \nabla \phi^+) \, dv = \oiint (\phi^+ D \nabla \phi - \phi D \nabla \phi^+) \cdot d\underline{n} \tag{4.27}$$

If this surface integral vanishes, the lemma is proved. Thus the condition for the satisfaction is

$$-\oiint (\phi^+ D \nabla \phi - \phi D \nabla \phi^+) \cdot d\underline{n} = 0 = -\oiint \phi^+ \phi \left(\frac{D \nabla \phi}{\phi} - \frac{D \nabla \phi^+}{\phi^+} \right) \cdot d\underline{n} \tag{4.28}$$

The boundary conditions on the flux are assumed either of the form $\phi(S) = 0$ or $\left.\frac{D \nabla \phi}{\phi}\right|_S = \frac{2}{3} \lambda$. Thus the sufficient condition for the lemma is that the same boundary conditions are taken for the adjoint, ϕ^+, as are employed for the flux, ϕ. The necessary condition is that the surface integral vanish in equation (4.28).

Lemma Two: The reactor averages of the matrix expressions, $\phi \nabla \cdot D \nabla \phi^+$ and $\phi^+ \nabla \cdot D \nabla \phi$ commute for the same boundary conditions

as Lemma One and for arbitrarily many groups.

Proof: $\phi^+ \nabla \cdot D \nabla \phi$ and $\phi \nabla \cdot D \nabla \phi^+$ are scalar expressions. Since $\nabla \cdot D \nabla$ is a diagonal matrix we have to prove:

$$\int (\phi_1^+ \nabla \cdot D_1 \nabla \phi_1 + \phi_2^+ \nabla \cdot D_2 \nabla \phi_2 + \ldots \phi_n^+ \nabla \cdot D_n \nabla \phi_n \ldots) \, dv$$

$$= \int (\phi_1 \nabla \cdot D_1 \nabla \phi_1^+ + \phi_2 \nabla \cdot D_2 \nabla \phi_2^+ + \ldots \phi_n \nabla \cdot D_n \nabla \phi_n^+ \ldots) \, dv \tag{4.29}$$

From Lemma One, each group expression can be equated if the boundary conditions of each group are selected according to that lemma. It is then sufficient for the second lemma to hold if either of the boundary conditions are satisfied:

$$\begin{matrix} \phi_n(S) = 0 \\ \phi_n^+(S) = 0 \end{matrix} \quad \text{or} \quad \frac{D_n \nabla \phi_n}{\phi_n} = \frac{D_n \nabla \phi_n^+}{\phi_n^+} \quad \text{for all n groups} \tag{4.30}$$

The necessary condition is that the following relation hold (where the matrix notation is used):

$$(\phi^+ D \nabla \phi - \phi D \nabla \phi^+) \cdot d\underline{n} = 0 \tag{4.31}$$

4.5 Integral Properties

The previous discussion of the classical adjoint serves to relate the body of mathematics and classical physics to the present application. For completeness, we also gave a proof of the two Lemmas, though of course their content is well known. We can now use these lemmas to prove the properties assigned to the importance on physical grounds in section 3 and to develop further original theorems for the bilinear form.

(a) Commutation. From the Lemmas and from the usual algebraic properties of matrices, we have for the appropriate boundary conditions:

$$\psi^+ F \psi = \psi F^T \psi^+$$

$$\psi^+ \Sigma \psi = \psi \Sigma^T \psi^+ \tag{4.32}$$

$$\psi^+ T \psi = \psi T^T \psi^+$$

$$\psi^+ A \psi = \psi A \quad \psi^+ \tag{4.33}$$

$$\nabla \psi^+ \cdot D \nabla \psi = \nabla \psi \cdot D \nabla \psi^+$$

$$\int \psi^+ \nabla \cdot D \nabla \psi \, dv = \int \psi \nabla \cdot D \nabla \psi^+ \, dv \tag{4.34}$$

(b) To show that s is given to first order by the expression:

$$s = \frac{\int \psi^+ (M+T) \psi \, dv}{\int \psi^+ \quad A \quad \psi \, dv} \tag{4.35}$$

where

$$s A \psi = (M+T) \psi \, , \tag{4.36}$$

$$s A \psi^+ = (M+T)^T \psi^+ \tag{4.37}$$

We have for any ψ^+:

$$s \psi^+ A \psi = \psi^+ (M+T) \psi \tag{4.38}$$

Hence integration over the reactor leads to equation (4.35) for any ψ^+. Similarly, from equation (4.37), for any weighting function, ψ:

$$s \int \psi A \psi^+ \, dv = \int \psi (M+T)^T \psi^+ \, dv = \int \psi^+ (M+T) \psi \, dv \tag{4.39}$$

Then the error in s for errors in both ψ^+ and ψ is given by:

$$\delta s \int \psi^{+'} A \psi' \, dv = \int \delta \psi^+ (M+T) \, \delta \psi \, dv \tag{4.40}$$

4.6 Importance of One Neutron or Precursor

To show that ϕ_i^+ gives the fractional importance of one neutron of the i th group, we consider taking away a single neutron from the flux, at $\underline{r}'$ and at time t. Before the removal (at time t-), the flux was, say,

$$\phi'(\underline{r}, t) .$$

After the removal of one neutron, the flux immediately changes to (at time t+):

$$\phi'(\underline{r}) - V_i \, \delta(\underline{r} - \underline{r}') \, \delta(V - V_i) \, I \tag{4.41}$$

where we have employed the delta functions to localise the one neutron. The fraction of the resident population contributed by the one neutron at time t- can be evaluated by multiplying the neutron density by the importance of neutrons at t- and integrating over the reactor. The fractional loss of resident population is immediately evaluated to be:

$$\frac{\phi_i(\underline{r}', t)}{\int \psi^+ A \psi' \, dv} \tag{4.42}$$

Then, when the denominator is correctly normalised, ϕ^+ indeed gives the contribution of one neutron. The contribution of neutrons to higher modal populations, or the contribution of precursors is obtained in an exactly analogous manner.

4.7 Some Theorems Concerning the Bilinear Concomitant

Theorem 1: Conservation of Fractional Resident Population

Consider a generalisation of the Lagrange identity, equation (4.7), in which we introduce the partial differential equations. Let the identity be represented by G so that:

$$G = \psi^+ L\psi - \psi L^+ \psi^+ = 0 \tag{4.43}$$

and

$$G = \psi^+ A \frac{\partial \psi}{\partial t} - \psi^+ (M+T)\psi + \psi A \frac{\partial \psi^+}{\partial t} + \psi (M+T)^T \psi^+ \tag{4.44}$$

The quantity $\psi^+ A\psi$, which has the form of a density of fractional resident population, can be called d. From equation (4.34), the terms in Σ, F, and T cancel in equation (4.44) at every point. Furthermore, from the vector transformation of the Lemmas, regardless of boundary conditions on ψ and ψ^+, it follows that:

$$G = \frac{\partial d}{\partial t} + \nabla \cdot (\psi D \nabla \psi^+ - \psi^+ D \nabla \psi) = 0 \tag{4.45}$$

Whereas $-\psi^+ D\nabla\psi$ is the actual current of neutrons weighted with the importance of each neutron, $\psi D \nabla \psi^+$ is the current of importance per unit flux weighted by the flux. The sum of these two currents is the total current of fractional resident population, to be called $\underline{c}$. Then equation (4.45) is a conservation theorem which takes the form:

$$\frac{\partial d}{\partial t} + \nabla \cdot \underline{c} = 0 \tag{4.46}$$

Then theorem 1 states that the fractional resident population is conserved.

Theorem 2: Stationary Property of the Asymptotic Reactor.

In the expression for the first theorem, equation (4.46), we can introduce the asymptotic solutions for the flux and the importance in a given reactor. More generally, we can consider any pair of modal fluxes and importances, ψ_n and ψ_n^+. It has already been shown in Section 3 that such eigenfunctions have equal and opposite eigenvalues, s_n and $\omega_n = -s_n$, say. Then

$$\frac{\partial d}{\partial t} = \psi_n A \frac{\partial \psi_n^+}{\partial t} + \psi_n^+ A \frac{\partial \psi_n}{\partial t}$$

$$= \omega_n \psi_n A \psi_n^+ + s_n \psi_n^+ A \psi_n$$

$$= 0 \qquad (4.47)$$

Then Theorem 2 states that the asymptotic (or modal) fractional population is stationary or independent of time. Introducing equation (4.47) into equation (4.46) yields:

$$\nabla \cdot (\psi_n D \nabla \psi_n^+ - \psi_n^+ D \nabla \psi_n) = 0 \qquad (4.48)$$

or, in an obvious nomenclature,

$$\nabla \cdot \underline{c}_{nn} = 0 \qquad (4.49)$$

It cannot be concluded from Theorem 2 or equation (4.49), that $\underline{c}_{nn}$ vanishes everywhere in the reactor even though the surface integral of equation (4.49) over the reactor leads to the vanishing of $\underline{c}_{nn} \cdot \underline{n}$, the normal component of $\underline{c}_{nn}$ at the outer surface (from the boundary conditions on the flux and the importance, Lemma 2). Such a conclusion can only be justified under the condition that $\underline{c}_{nn}$ itself can be derived from a potential, i.e., if:

$$\underline{c}_{nn} = \nabla \Omega \qquad (4.50)$$

where Ω is some scalar function. In general however, there will be some circulating component of $\underline{c}_{nn}$ that will not vanish. There is an analogous difficulty pointed out by Casimir (37) in irreversible thermodynamic heat flow.

Theorem 3: Equivalence of the Forward and Adjoint Buckling.

The concept of the geometric buckling, B^2, which is a constant

algebraic number for a one region reactor, can be extended to the more general problem by defining a B^2 which is a function of position:

$$\nabla \cdot D_i \nabla \phi_i = - B_i^2 D_i \phi_i \tag{4.51}$$

Similarly, we can define an adjoint buckling:

$$\nabla \cdot D_i \nabla \phi_i^+ = - B_i^{+2} D_i \phi_i^+ \tag{4.52}$$

On introducing these definitions into Theorem 2, equation (4.48), we obtain, via Lemma One:

$$B_i^{+2} D_i \phi_i^+ = B_i^2 D_i \phi_i \, , \tag{4.53}$$

or the more general vector equation:

$$B^{+2} = B^2 \tag{4.54}$$

Thus Theorem 3 states that the adjoint generalised buckling is equal to the forward generalised buckling in each group and point by point in the reactor. It is to be noted that like Theorem 2, Theorem 3 holds for the asymptotic flux shape and the resident importance (or corresponding pairs of eigenfunctions) and not for transient terms with higher values of the buckling.

The results of Theorem 3 are used in an operational derivation of the adjoint equation in Appendix B to this section. In this Appendix, the starting point is the energy relation between the importance of neutrons in each group, which is taken to be of the form of the non-leakage probability or $1/(1 + \tau B^2)$. The use of the generalised buckling leads to the adjoint equations in the general case. Furthermore, it is a well known property of the two-group, two-region problem, whose solution is given in Appendix A, that the individual component solutions

are coupling coefficients which are also of the form $1/(1 + \tau B^2)$ where B^2 takes the values μ^2 and $-\nu^2$, the two compatible roots for B^2.

4.8 Initial Conditions

So far we have found it necessary, for the sake of accuracy, to impose boundary conditions on the adjoint as well as the neutron flux. The initial conditions were imposed on physical grounds, that the total fractional contribution of all neutrons to the resident population is unity.

To demonstrate how the initial flux determines the importance of neutrons and the modal populations, we consider a bare sphere in a one and two-group approximation. Case one treats an initial condition in which N neutrons are distributed throughout the reactor uniformly in either group. Case two treats one neutron concentrated at a point, i.e., it leads to the equivalence of one neutron as well as the importance of neutrons for this initial condition. It should be noted that in these examples we do not need to know the period the reactor is on although this is implicitly implied by the coupling coefficients, C_n and C_n^+, since these are functions of s. We require the following functions:

$$\phi_n^+ = a_n^+ \frac{\sin n\,Br}{n\,Br} \qquad \text{(one group)},$$

or (4.55)

$$\phi_n^+ = \begin{bmatrix} 1 \\ C_n^+ \end{bmatrix} a_n^+ \frac{\sin n\,Br}{n\,Br} \qquad \text{(two group)},$$

where $B = \frac{\pi}{R}$,

$$\phi_n = a_n \frac{\sin n\,Br}{n\,Br} \qquad \text{(one group)},$$

or (4.56)

$$\phi_n = \begin{bmatrix} 1 \\ C_n \end{bmatrix} a_n \frac{\sin n\,Br}{n\,Br} \qquad \text{(two group)},$$

where a_n, etc., are time dependent coefficients to be determined.

Case One: Uniform Initial Distribution

(a) N neutrons in group one initially:

$$\int \phi_n^+ V^{-1} \phi' \, dv = 1 \tag{4.57}$$

Hence

$$\int a_n^+ \frac{\sin n\,Br}{n\,Br} \frac{N 4\pi r^2}{\frac{4}{3}\pi R^3} \, dr = 1 ,$$

and

$$a_n^+ = \frac{1}{N}(-1)^{n+1} \frac{(n\pi)^2}{3} . \tag{4.58}$$

$$\int \phi_n^+ V^{-1} \phi_m \, dv = \delta_{nm} . \tag{4.59}$$

Hence

$$\int a_n^+ \frac{\sin n\,Br}{n\,Br} \left[\frac{1}{V_1} + \frac{C_n^+ C_n}{V_2} \right] a_n \frac{\sin n\,Br}{n\,Br} 4\pi r^2 \, dr = 1 ,$$

and

$$a_n = N(-1)^{n+1} \frac{3}{2\pi R^3} \frac{V_1 V_2}{V_1 C_n C_n^+ + V_2} \tag{4.60}$$

Therefore

$$N_n = \int I V^{-1} \phi_n \, dv = \frac{6N}{(n\pi)^2} \frac{V_1 C_n + V_2}{V_1 C_n^+ C_n + V_2} \tag{4.61}$$

(b) N neutrons in group two initially:

Similar expressions lead to:

$$a_n^+ = \frac{(-1)^{n+1}\,(n\pi)^2}{N\;{}_3C_n^+}\,, \tag{4.62}$$

$$a_n = N(-1)^{n+1}\,\frac{3}{2\pi R^3}\,\frac{V_1 V_2\, C_n^+}{V_1 C_n^+ C_n + V_2}\,, \tag{4.63}$$

and therefore,

$$N_n = \frac{6N}{(n\pi)^2}\,\frac{V_1 C_n^+ C_n + V_2 C_n^+}{V_1 C_n^+ C_n + V_2} \tag{4.64}$$

(c) One group of neutrons only, N neutrons uniformly distributed.

Reduction of the two group case yields:

$$a_n^+ = \frac{(-1)^{n+1}}{N}\,\frac{(n\pi)^2}{3}\,, \tag{4.65}$$

$$a_n = N(-1)^{n+1}\,\frac{3V}{2\pi R^3}\,, \tag{4.66}$$

and

$$N_n = \frac{6N}{(n\pi)^2}\,. \tag{4.67}$$

From equation (4.67) we have $\pi = 0.61$, the resident population initial value.

Case Two: One Neutron at Centre

(a) one fast neutron at centre initially:

$$a_n^+ = 1\,, \tag{4.68}$$

and

$$\int a_n^+ \frac{\sin n\,Br}{n\,Br}\left[\frac{1}{V_1} + \frac{1}{V_2\, C_n^+ C_n}\right] a_n \frac{\sin n\,Br}{n\,Br}\, 4\pi r^2\, dv = 1 . \qquad (4.69)$$

Hence

$$a_n = \frac{\pi n^2}{2R^3} \frac{V_1 V_2}{V_1 C_n^+ C_n + V_2} , \qquad (4.70)$$

and

$$N_n = \int I V^{-1} \phi_n\, dv = (-1)^{n+1}\, 2\, \frac{V_1 C_n + V_2}{V_1 C_n^+ C_n + V_2} \qquad (4.71)$$

(b) one slow neutron at centre initially:

$$a_n^+ = \frac{1}{C_n^+} , \qquad (4.72)$$

$$a_n = \frac{\pi n^2}{2R^3} \frac{V_1 V_2 C_n^+}{V_1 C_n^+ C_n + V_2} \qquad (4.73)$$

whence

$$N_n = (-1)^{n+1}\, 2\, \frac{V_1 C_n^+ C_n + V_2 C_n^+}{V_1 C_n^+ C_n + V_2} \qquad (4.74)$$

(c) one group

$$a_n^+ = 1 , \qquad (4.75)$$

$$a_n = \frac{\pi n^2 V}{2R^3} , \qquad (4.76)$$

and therefore

$$N_n = (-1)^{n+1}\, 2 \qquad (4.77)$$

Equation (4.77) is a typical result of the delta function initial value that

was assumed. On putting n = 1 we obtain the value of the resident population due to one neutron at the centre of the reactor:

$$\pi = 2 \tag{4.78}$$

This value is also, by definition, the equivalence of the neutron.

In these derivations we have assumed on the basis of symmetry that the antisymmetric eigenfunctions in the sphere, involving Legendre polynomials and Bessel functions, are not present in the modal population – i.e., the a_n of the corresponding antisymmetric functions is zero. It should be noted that the corresponding a_n^+ are infinite. If there are no neutrons at all in a modal population, one neutron is infinitely important. Secondly we have given only the initial value of time dependent coefficients; their time behaviour is as $\exp(s_n t)$.

Although we have derived the importance and equivalence of one neutron only at the centre of a sphere, we can induce the resident equivalence at any other point in the sphere without the necessity of introducing the antisymmetric eigenfunctions. For if in one group a number of neutrons were distributed in a thin spherical shell at radius r, the equivalence would be given by $2 \frac{\sin Br}{Br}$ per neutron, instead of 2 as at the centre. Yet it is apparent, since the neutrons in the shell do not interact with each other, that the neutrons distributed in the shell at distance r from the centre could be concentrated at a point at r from the centre without changing the physical effect. The equivalence in the higher modes in $\frac{\sin n Br}{n Br}$ can be similarly induced. This will not enable us however to determine the equivalence in the antisymmetric modes.

It is useful to note that the corresponding problems in rectangular and cylindrical geometry lead to analogous solutions. Thus, the equivalence of one neutron in a one group model at the centre of the following reactors has the values:

Sphere	2.00
Cylinder	2.04
Box	2.06

4.9 Completeness of the Eigenfunction Representation

The two common methods of solving partial differential equations are by integral formation and by eigenfunction expansion. It is probably true to say that the latter method is the more suitable in obtaining analytical numerical results. A basic problem in employing the method is to justify the completeness of the eigenfunction set. That is, if the initial conditions and boundary conditions are fitted by some suitable summation of eigenfunctions, can we be sure that these functions give the correct solution at all later times? Even for the much studied self adjoint sets of eigenfunctions, the completeness can only be proved in a (mathematically) limited sense. It is found that the set of such eigenfunctions can match any initial condition that has at the most a finite number of discontinuities, i.e., that is piecewise continuous. Then the set of eigenfunctions is said to be complete when the mean square error in representing such an initial condition over the range is zero. Fortunately the limitation to piecewise continuous functions still includes all the reasonable physical initial conditions. Furthermore from either the theory of partial differential equations, or on grounds

Table 4. 1a Summary of Properties of the Importance of Neutrons and Precursors

Fluxes and Importances

Flux $\quad \varphi' = \varphi + \delta\varphi = \varphi + \sum_1 \varphi_n$

Flux and Precursors $\quad \psi' = \psi + \delta\psi = \psi + \sum_1 \psi_n$

$$\varphi^{+\prime} = \varphi^+ + \delta\varphi^+ = \varphi^+ + \sum_1 \varphi_n^+$$

$$\psi^{+\prime} = \varphi^+ + \delta\psi^+ = \psi^+ + \sum_1 \psi_n^+$$

Population $\quad N = \pi + \sum_1 N_n$

$$N = \int IV^{-1}\varphi' dv = \int IA\psi' dv$$

$$N_n = \int IV^{-1}\varphi_n dv = \int IA\psi_n dv$$

$$\pi = \int IV^{-1}\varphi dv = \int IA\psi dv$$

Normalization and Orthogonality

$$\int \varphi^+ V^{-1}\varphi' dv = \int \varphi^+ V^{-1}\varphi dv = 1 = \int \psi^+ A\psi' dv = \int \psi^+ A\psi dv$$

$$\int \varphi_n^+ V^{-1}\varphi_n dv = 1 = \int \psi_n^+ A\psi_n dv$$

$$\int \varphi^+ V^{-1}\delta\varphi dv = \int \varphi_n^+ V^{-1}\varphi_m dv = 0 = \int \psi^+ A\delta\psi dv = \int \psi_n^+ A\psi_m dv$$

$$(n \neq m)$$

Equivalence

$$w(\underline{r}) = \pi(t)\varphi^+(\underline{r},t) = \pi(t)\psi^+(\underline{r},t) \; ; \; w_n(\underline{r}) = N_n(t)\varphi_n^+(\underline{r},t) = N_n(t)\psi_n^+(\underline{r},t)$$

Inverse Periods

$$\frac{\int \varphi^+ M\varphi\, dv}{\int \varphi^+ V^{-1}\varphi' dv} = s = \frac{\int \psi^+ (M+T)\psi\, dv}{\int \psi^+ A\psi\, dv} \; ; \; \frac{\int \delta\varphi^+ M\delta\varphi\, dv}{\int \varphi^{+\prime} V^{-1}\varphi' dv} = \delta s = \frac{\int \delta\psi^+ (M+T)\delta\psi}{\int \psi^+ A\psi' dv}$$

Homogeneous Source , $\bar{S}$

$$\frac{\int \varphi^+ S\varphi' dv}{\int \varphi^+ V^{-1}\varphi' dv} = s)_{\bar{S}} = \frac{\int \psi^+ S\psi' dv}{\int \psi^+ A\psi' dv}$$

Current of Importance

$$i_+ = \frac{\varphi^+}{4} + \frac{D}{2}\nabla\phi^+$$

$$i = D\nabla\varphi^+$$

Vacuum Boundary Conditions

$$i_+ = 0 \quad \text{or} \quad \phi+ = 0 \text{ extrapolated surface}$$

Importance Balance

(a) Two Group without Precursors

$$V_1^{-1}\frac{\partial\phi_1^+}{\partial t} = -\nabla D_1\nabla\phi_1^+ + \Sigma_1\phi_1^+ - \Sigma_1\phi_2^+$$

$$V_2^{-1}\frac{\partial\phi_2^+}{\partial t} = -F\phi_1^+ - \nabla D_2\nabla\phi_2^+ + \Sigma_2\phi_2^+$$

In General

$$V^{-1}\frac{\partial\phi^+}{\partial t^+} = -M^T\phi^+ = (-\nabla D\nabla + \Sigma^T - F^T)\phi^+$$

$$\omega V^{-1}\phi^+ = -s V^{-1}\phi^{+\prime} = -M^T\phi^+$$

(b) One Group with One Group Precursors

$$V^{-1}\frac{\partial\phi^+}{\partial t} = -\nabla D\nabla\phi^+ + \Sigma\phi^+ - (1-\beta)F\phi^+ - \beta F C^+$$

$$\frac{\partial C^+}{\partial t} = -\lambda\phi^+ + \lambda C^+$$

In General

$$A\frac{\partial\psi^+}{\partial t} = -(M^T + T^T)\psi^+ = (-\nabla D\nabla + \Sigma^T - F^T + T^T)\psi^+$$

$$\omega A\psi^+ = -sA\psi^+ = -(M^T + T^T)\psi^+$$

One Group Neutrons Coupling Coefficients

$$C_\iota = \frac{\beta_\iota F}{\lambda_\iota + s}\phi \quad ; \quad C_\iota^+ = \frac{\lambda_\iota}{\lambda_\iota + s}\phi^+$$

Table 4. 1b Summary of Equations for the Importance

of the uniqueness of the physical solution, we claim that if the eigenfunction set, each term of which is an admissible mathematical solution to the equation though not to all the boundary conditions, is a solution at any one time (i. e., initially) then the set provides the unique solution for all later times.

For non-self adjoint systems, the requirements on completeness still exist but from the non positive definite nature of the system, it is not possible to give a proof of completeness at this time. It is for this reason that we do not take up in this thesis the general problem of higher order theory and the convergence of iterative solutions. We are able to make some contribution to the problem of completeness on three points. To understand the contributions, we review the methods of proving completeness for the self adjoint system.

The first step in proving completeness of a self adjoint system of eigenfunctions is to investigate the nature of the eigenvalues. In general it is found that there is a finite largest eigenvalue, the eigenvalue of the lowest mode. The remaining eigenvalues form an infinite set, which may be discontinuous or part discontinuous and part continuous, but having no finite algebraically smallest eigenvalue. The proof of these properties is generally considered to belong to the realm of higher algebra. The necessity of proving them arises from the fact that the error in an expansion based on a given number of eigenfunctions can be related to the reciprocal of the last eigenvalue employed. Thus if the expansion set employed is infinite, the error can be shown to go to zero.

The second step in the proof is to relate the eigenfunctions to some

variational principle; in particular, the principle must be that some functional is a true minimum (or maximum) in order that the square error can be shown to be positive definite.

Finally, the above arguments can be pieced together to show that the mean square error in the representation of a function by the eigenfunction set does indeed vanish for the infinite set.

From this discussion, we can appreciate some of the difficulties introduced by the non-self adjoint nature of the multigroup functions. The one group system, with no precursors, lies within the conventional self adjoint systems and proof of its completeness follows conventional lines (9). For the general problem we make the following contributions:

(a) Relation between the eigenvalues of the forward and adjoint equations

(b) The joint error as a concept to replace the mean square error

(c) A variational principle for the eigenfunctions of both the forward and adjoint equations.

The first contribution, that the eigenvalues of the forward and adjoint equations can be matched one for one in magnitude has been given as the result of physical arguments in Section 3. Mathematical proofs of this theorem have been given by Clark (40) by comparing the functional determinants of both equations, and by Martino, (36), who casts the equations into integral form and employs the theory of Fredholm equations. The extension of Clark's proof to the case including precursors is trivial and will not be given here. We point out, however, that if the eigenvalues are to be given our physical interpretation, then the

eigenvalues for the adjoint equation are of opposite sign to those of the forward equation.

We now discuss the second contribution to the problem, the joint error. The general viewpoint of the field theory would demand that the system is equally well represented by either field function. Hence it is suggestive to investigate the joint error in representing both fields by expansion sets. This joint error when integrated over the range corresponds to the error in counting the resident population and its rate of change. Since this is the fundamental problem that we seek to solve, we can claim on physical grounds that the vanishing of such an integrated joint error is a sufficient criterion of completeness of the eigenfunctions. Our contribution is to show that the coefficients of the eigenfunction expansion that we have employed so far, i.e., the orthogonal of Fourier expansion coefficients, are then in a certain sense the best coefficients since they minimise the joint error.

Consider the assumed expansions of the actual flux and actual importance in a reactor in terms of some set of eigenfunctions:

$$\psi' \sim \sum_n a_n \theta_n \; ; \qquad \psi^{+\prime} \sim \sum_n a_n^+ \theta_n^+ \tag{4.79}$$

Here the θ_n and θ_n^+ are some solution to the equations of the reference reactor. They are normalised to the extent only that:

$$\int \theta_n^+ A \, \theta_m \, dv = \delta_{nm} \tag{4.80}$$

The orthogonal coefficients, a_n and a_n^+, are defined in the conventional way by multiplying equations (4.79) by the bilinear term and integrating over the range. If the set of eigenfunctions are not complete,

(4.79) is not a true equality. However, the definition of the orthogonal (or Fourier) coefficients holds in general:

$$\int \psi^{+\prime} A \theta_m \, dv = \sum_n \int a_n^+ \theta_n^+ A \theta_m \, dv = a_n^+ , \tag{4.81}$$

$$\int \theta_m^+ A \psi' \, dv = \sum_n \int a_n \theta_m^+ A \theta_n \, dv = a_n . \tag{4.82}$$

Now suppose that we employed some other expansion coefficients, c_n and c_n^+ say, rather than the orthogonal coefficients just described. The joint error, $\bar{E}$, in representing the two functions ψ' and $\psi^{+\prime}$ over the range is defined by:

$$\bar{E} = \int (\psi^{+\prime} - \sum_n c_n^+ \theta_n) A (\psi' - \sum_m c_m \theta_m) \, dv \tag{4.83}$$

Then

$$\bar{E} = 1 - \sum_n c_n^+ a_n - \sum_n c_n a_n^+ + \sum_n \sum_m c_n^+ c_m \delta_{nm} , \tag{4.84}$$

where we have employed the unity normalisation of $\psi^{+\prime} A \psi'$ and of the $\theta_n^+ A \theta_n$. Furthermore, from the orthogonal properties of the θ^+, θ, the double summation in the last term of equation (4.84) reduces to a single summation over the terms in $n = m$. We can now adjust the separate c_n^+, c_n, to give the minimum joint error by setting the partial derivatives of the error to zero:

$$\frac{\partial E}{\partial c_n} = -a_n^+ + c_n^+ = 0 , \tag{4.85}$$

$$\frac{\partial E}{\partial c_n^+} = -a_n + c_n = 0 .$$

We have shown that the orthogonal coefficients are those that give the least mean error in the representation.

In the above development, we have only allowed the normalising coefficient of the vector flux, ψ, ψ^+, to be adjusted; we did not allow any freedom to change the coupling coefficients between the components of any vector flux. If this restriction is relaxed, we would have the general form of expansion to be:

$$\psi_i' = \sum_n c_{ni}\, \theta_{ni} \tag{4.86}$$

with the corresponding orthogonal coefficients, a_{ni}. On setting the partial derivative of the error to zero we now obtain the condition that $c_{ni} = a_{ni}$, etc. It would appear at first sight that we have a better representation of the original function by allowing these extra degrees of freedom and hence that the original expansion set cannot be considered complete. We must consider the nature of the θ_{ni} more carefully however. In the first place, the component solutions in each group are not in themselves eigenfunction solutions to the original equation and do not have an eigenvalue time behaviour. Secondly, the expansion set of the components within a group is no longer a linearly independent set. Consider for example, a slab reactor in the two group approximation. In the lowest harmonic, there are two eigenfunction solutions; both solutions have components which are cosines but differ in the sign of the coupling coefficient between the group components. If we expand the component initial shape in the set of the component solutions, each group has only the single linearly independent term, the cosine. The expansion in the original eigenfunction set however provides the two

linearly independent terms within the first harmonic.

It follows that in general, the number of linearly independent terms in the set of the group solutions is only the number of the harmonic. The total number of terms to match the initial condition for all groups is the number of harmonics times the number of groups. Yet we have seen that this number is exactly the number of eigenfunctions, so that no advantage is gained by separate adjustment of the coefficients. We conclude that the set of eigenfunctions itself supplies the best expansion set and the orthogonal coefficients supply the best coefficients.

4.10 A Variational Principle

The adjoint equations are derivable in a number of ways. Physically we can define an importance; the consequences of our definition led to the adjoint equations in Section 3. We can more intuitively interpret the adjoint equations as the equations describing a system with cause and effect reversed. (Appendix A to this Section). In the present section we have shown how the adjoint equation and boundary conditions can be formed to ensure the property that the bilinear form is suitable for approximation purposes. In Appendix B to this Section, we give a further mathematical method of deriving the adjoint equations by a purely artificial device.

We now give a method whereby the adjoint system can be derived from a single variational principle. This principle itself expresses the stationary properties of the bilinear form. In addition, it leads as sufficient and necessary consequences, to the equations for the neutron balance as well as the importance. Such a principle is evidently an

elegant summary of the whole of the diffusion system.

We define a Lagrange density including the effects of precursor transfer to neutrons, (T):

$$L = \nabla\psi^+ \cdot D\nabla\psi + \psi^+ (\Sigma - T - F)\psi + \psi^+ V^{-1} \frac{\partial\psi}{\partial t}$$

$$+ \nabla\psi \cdot D\nabla\psi^+ + \psi(\Sigma^T - T^T - F^T)\psi^+ - \psi V^{-1} \frac{\partial\psi^+}{\partial t} \qquad (4.87)$$

L is to be regarded as a function not only of each component, ψ_i and ψ_i^+, but of each spatial and temporal derivative, $\frac{\partial\psi_i}{\partial x}$, $\frac{\partial\psi_i^+}{\partial t}$, etc. As a short hand notation we can write the 3 spatial derivatives in the form $\nabla\psi_i$, $\nabla\psi_i^+$ etc. Otherwise we follow the nomenclature of Morse and Feshbach (9).

A Lagrangian, $\mathcal{L}$, is found by integrating the Lagrange density, L, over space and time. We need not specify the limits of the integration or the values taken up by ψ, ψ^+, etc., at these limits. The limits are understood to be fixed however and not a variable in the subsequent treatment. We have:

$$\mathcal{L} = \iint dv\, dt\, L \qquad (4.88)$$

The principle is that for all possible variations of the functions, ψ, ψ^+, $\nabla\psi$, $\frac{\partial\psi^+}{\partial t}$, etc., the Lagrangian is stationary. We write this as:

$$\delta(\mathcal{L}) = 0 \qquad (4.89)$$

Consider ψ_i changing to $\psi + \epsilon_i \eta_i$ where the ϵ_i are some numbers and each η_i is an arbitrary function of the parameters that does not alter the boundary and initial conditions for the ith component of ψ. L can be expanded in a Taylor series about the value ψ. If L is stationary

as required, then at least the first terms in such an expansion, in ϵ_i and ϵ_i^+, must all vanish for arbitrary values of ϵ_i, ϵ_i^+.

$$\delta(\mathcal{L}) = \iint dv\,dt \sum_i \epsilon_i \left[\frac{\partial L}{\partial \psi_i} \eta_i + \frac{\partial L}{\partial \psi_i} \frac{\partial \eta_i}{\partial t} + \frac{\partial L}{\partial \nabla \psi_i} \nabla \eta_i \right]$$

$$+ \sum_i \epsilon_i^+ \left[\frac{\partial L}{\partial \psi_i^+} \eta_i^+ + \frac{\partial L}{\partial \psi_i^+} \frac{\partial \eta_i^+}{\partial t} + \frac{\partial L}{\partial \nabla \psi_i^+} \nabla \eta_i^+ \right] + \ldots = 0 \qquad (4.90)$$

Hence a necessary and sufficient condition for $\mathcal{L}$ to be stationary to arbitrary small values of ϵ_i, ϵ_i^+, is for each of the 2 i brackets to vanish simultaneously. These are the Euler equations for the system:

$$\frac{\partial L}{\partial \psi_i} \eta_i + \frac{\partial L}{\partial \psi_i} \frac{\partial \eta_i}{\partial t} + \frac{\partial L}{\partial \nabla \psi_i} \nabla \eta_i = 0$$

$$\frac{\partial L}{\partial \psi_i^+} \eta_i^+ + \frac{\partial L}{\partial \psi_i^+} \frac{\partial \eta_i^+}{\partial t} + \frac{\partial L}{\partial \nabla \psi_i^+} \nabla \eta_i^+ = 0 \qquad (4.91)$$

We can make use of the boundary conditions to express these equations in more recognizable form. From an integration by parts we have:

$$\left[\frac{\partial L}{\partial \psi_i} \frac{\partial \eta_i}{\partial t} \right] - \int \frac{\partial}{\partial t} \frac{\partial L}{\partial \psi_i} \eta_i \, dt = \int \frac{\partial L}{\partial \psi_i} \frac{\partial \eta_i}{\partial t} \, dt \qquad (4.92)$$

But from the stated conditions at the boundary, the bracketed term vanishes at the boundary. Similarly for the spatial derivatives:

$$\left[\frac{\partial L}{\nabla \psi_i} \nabla \eta_i \right] - \int \nabla \cdot \frac{\partial L}{\partial \nabla \psi_i} \eta_i \, dv = \int \frac{\partial L}{\partial \nabla \psi_i} \nabla \eta_i \, dv \qquad (4.93)$$

Hence the Euler equations become:

$$\left[\frac{\partial L}{\partial \psi_i} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \psi_i} - \nabla \cdot \frac{\partial L}{\partial \nabla \psi_i} \right] \eta_i = 0 \,,$$

$$\left[\frac{\partial L}{\partial \psi_i^+} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\psi}_i^+} - \nabla \cdot \frac{\partial L}{\partial \nabla \psi_i^+} \right] \eta_i^+ = 0 \tag{4.94}$$

But η_i and η_i^+ are defined as arbitrary functions of the parameters. Hence the condition for the principle is expressible as two sets of equations:

$$\frac{\partial L}{\partial \psi_i} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\psi}_i} - \nabla \cdot \frac{\partial L}{\partial \nabla \psi_i} = 0 ,$$

$$\frac{\partial L}{\partial \psi_i^+} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\psi}_i^+} - \nabla \cdot \frac{\partial L}{\partial \nabla \psi_i^+} = 0 \tag{4.95}$$

It is now necessary to carry out the partial differentiation of the assumed Lagrangian. We have:

$$\frac{\partial L}{\partial \dot{\psi}_i} = \psi_i^+ V_i^{-1} ; \quad \frac{\partial L}{\partial \dot{\psi}_i^+} = V_i^{-1} \psi_i \tag{4.96}$$

These are the canonical momenta of the system. Then

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\psi}_i} = \frac{\partial \psi_i^+}{\partial t} V_i^{-1} ; \quad \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\psi}_i^+} = - V_i^{-1} \frac{\partial \psi_i}{\partial t} . \tag{4.97}$$

Also

$$\frac{\partial L}{\partial \nabla \psi_i} = 2 D_i \nabla \psi_i^+ ; \quad \frac{\partial L}{\partial \nabla \psi_i^+} = 2 D_i \nabla \psi_i , \tag{4.98}$$

So that:

$$\nabla \cdot \frac{\partial L}{\partial \nabla \psi_i} = 2 \nabla \cdot D_i \nabla \psi_i^+ ; \quad \nabla \cdot \frac{\partial L}{\partial \nabla \psi_i^+} = 2 \nabla \cdot D_i \nabla \psi_i , \tag{4.99}$$

and finally

$$\frac{\partial L}{\partial \psi_i} = 2 \sum_j (\Sigma_{ji} - T_{ji} - F_{ji}) \psi_j^+ - V_i^{-1} \frac{\partial \psi_i^+}{\partial t} , \tag{4.100}$$

$$\frac{\partial L}{\partial \psi_i^+} = 2 \sum_j (\Sigma_{ij} - T_{ij} - F_{ij}) \psi_j + V_i^{-1} \frac{\partial \psi_i}{\partial t} \qquad (4.101)$$

Hence the Euler equations become the two sets of equations:

$$\begin{aligned} 2 \left[\nabla \cdot D_i \nabla \psi_i - \sum_j (\Sigma_{ji} - T_{ji} - F_{ji}) \psi_j^+ + V^{-1} \frac{\partial \psi_i^+}{\partial t} \right] = 0 \\ 2 \left[\nabla \cdot D_i \nabla \psi_i - \sum_j (\Sigma_{ij} - T_{ij} - F_{ij}) \psi_j - V^{-1} \frac{\partial \psi_i}{\partial t} \right] = 0 \end{aligned} \qquad (4.102)$$

These equations are just the two matrix equations:

$$\begin{aligned} \nabla \cdot D \nabla \psi^+ + (F - \Sigma + T)^T \psi^+ = - V^{-1} \frac{\partial \psi^+}{\partial t} , \\ \nabla \cdot D \nabla \psi + (F - \Sigma + T) \psi = V^{-1} \frac{\partial \psi}{\partial t} , \end{aligned} \qquad (4.103)$$

and are the consequence of the principle that the bilinear form is stationary to small variations of the functions.

Corresponding to the Lagrange density we have a Hamiltonian defined by:

$$H = \sum_i p_i^+ \psi_i^+ + \sum_i p_i \psi_i - L \qquad (4.104)$$

where the p_i are the canonical momenta, already derived. We find that:

$$H = -2 \nabla \psi^+ \cdot D \nabla \psi - \psi^+ (\Sigma - T - F) \psi - \psi (\Sigma - T - F)^T \psi^+ \qquad (4.105)$$

This Hamiltonian can be put into the more recognisable form by the transformation of the terms $\nabla \psi^+ \cdot D \nabla \psi$ to $\psi^+ \nabla \cdot D \nabla \psi$, by means of Lemmas One and Two. We have:

$$\begin{aligned} H &= 2 \psi^+ \nabla \cdot D \nabla \psi + \psi^+ (F - \Sigma + T) \psi + \psi (F - \Sigma + T)^T \psi^+ \\ &= \psi^+ (M+T) \psi + \psi (M+T)^T \psi^+ \end{aligned} \qquad (4.106)$$

The Poynting vector, $\underline{S}$, is given by:

$$\underline{S} = 2\left[\frac{\partial \psi^+}{\partial t} D \nabla \psi + \nabla \psi^+ D \frac{\partial \psi}{\partial t}\right], \tag{4.107}$$

and the usual relationship holds;

$$\nabla \cdot \underline{S} + \frac{\partial H}{\partial t} = 0 \tag{4.108}$$

We have already derived the current of the system, $\underline{c}$, as the spatial term in the bilinear concomitant

$$\underline{c} = \psi D \nabla \psi^+ - \psi^+ D \nabla \psi \tag{4.109}$$

We have now proved our assertion, that the diffusion approximation could be expressed as a variational principle. This principle not only gives the stationary properties that are used in applications of the bilinear form, but also leads to the equations for the balance of neutrons and the importance of neutrons as a sufficient and necessary condition for the principle to be satisfied.

APPENDIX A TO SECTION 4

TWO GROUP TWO REGION PROBLEM

We will show that by making a simple and physically meaningful rearrangement of the adjoint equations for the two group, two region problem, that the rearranged adjoint system has exactly the form of a two group, two region flux problem. Hence the adjoint solutions are found with ease if the corresponding flux problem is solved.*

Before doing this, we wish to present the two group two region flux solution itself in a form that differs somewhat from that given by Glasstone and Edlund say (33). We shall consider the general case of two multiplying regions. The inner region will be called the core and the outer region the blanket except in the special (but more common case) when the multiplication factor of the blanket is zero. Then this case can be obtained by a suitable limiting process. When in fact k of the blanket is zero, we shall call this outer region the reflector.

In addition, we show how a reidentification of the symbols in the flux solution for the one dimensional, critical system enables the form of solution for the general time dependent system and for two and three dimensional separable systems to be found. These results are presented as an addendum since they need not modify the basic solution to the simple problem.

Two Region Flux Solutions

For either the core or the blanket, which are assumed separately

*Wolfe (3) has pointed out a similar rearrangement so that machine computations can be made to yield adjoint solutions.

homogeneous, we have

$$(D_1 \nabla^2 - \Sigma_1)\,\phi_1 + \frac{k}{p}\,\Sigma_2\,\phi_2 = 0 \qquad \text{core}$$

$$p\,\Sigma_1\,\phi_1 + (D_2 \nabla^2 - \Sigma_2)\,\phi_2 = 0 \qquad \text{blanket} \tag{4A. 1}$$

For a reflector, the same equations hold except that k has the value zero.

Assuming separable solutions, B^2, as usual leads to the conventional pair of roots for B^2 in the equation expressing the compatibility of eq. (4A. 1)

Compatibility equation:

$$k = (1 + \tau B^2)(1 + L^2 B^2) \tag{4A. 2}$$

with the corresponding roots

$$B^2 = \mu^2 = \frac{1}{2}\left[-\left(\frac{1}{\tau} + \frac{1}{L^2}\right) + \sqrt{\left(\frac{1}{\tau} - \frac{1}{L^2}\right)^2 + \frac{4k}{\tau L^2}}\right] \qquad \text{core}$$

or

$$= -\nu^2 = -\frac{1}{2}\left[+\left(\frac{1}{\tau} + \frac{1}{L^2}\right) + \sqrt{\left(\frac{1}{\tau} - \frac{1}{L^2}\right)^2 + \frac{4k}{\tau L^2}}\right] \qquad \text{blanket} \tag{4A. 3}$$

Naturally to find the μ^2 or ν^2 solution for the blanket say, we introduce the properties of the blanket. The theory of equations indicates that μ^2 will not be positive unless k (for the region) is greater than unity. Negative μ^2 indicates that the corresponding solution will be of the hyperbolic rather than the trigonometric form.

For the special case of k being zero, in the reflector, the roots reduce to

$$\mu^2 = -\frac{1}{L^2}; \qquad \nu^2 = -\frac{1}{\tau} \qquad \text{reflector} \tag{4A. 4}$$

Region	Slab	Sphere	Cylinder
Core X	$\frac{\cos \mu x}{\cos \mu X_o}$	$\frac{R_o \sin \mu r}{r \sin \mu R_o}$	$\frac{J_o(\mu r)}{J_o(\mu R_o)}$
Y	$\frac{\cosh \nu x}{\cosh \nu X_o}$	$\frac{R_o \sinh \nu r}{r \sinh \nu R_o}$	$\frac{I_o(\nu r)}{I_o(\nu R_o)}$
Finite Blanket X	$\frac{\sin \mu (X-x)}{\sin \mu (X-X_o)}$	$\frac{R_o \sin \mu (R-r)}{r \sin \mu (R-R_o)}$	$\frac{J_o(\mu r) - y\, Y_o(\mu r)}{J_o(\mu R_o) - y\, Y_o(\mu R_o)}$ **
(μ^2 pos)* Y	$\frac{\sinh \nu (X-x)}{\sinh \nu (X-X_o)}$	$\frac{R_o \sinh \nu (R-r)}{r \sinh \nu (R_o-r)}$	$\frac{I_o(\nu r) -]< K_o(\nu r)}{I_o(\nu R_o) -]< K_o(\nu R_o)}$ **
Infinite Blanket* Y	$e^{-\nu(x-X_o)}$	$\frac{R_o}{r} \cdot$	$\frac{K_o(\nu r)}{K_o(\nu R_o)}$

* For μ^2_{neg}, X is given by Y with μ replacing ν (μ being $|\sqrt{\mu^2}|$)

** $y = J_o(\mu R) / Y_o(\mu R)$, $]< = I_o(\nu R) / K_o(\nu R)$

For a relector (blanket with k=o), $\mu = 1/L$ (μ^2 neg), $\nu = 1/\sqrt{\tau}$

Inteface at X_o, R_o, Total extrapolated half-thickness, of both regions, X, R

Table 4A.1 FLUX FUNCTIONS FOR 2 GROUP 2 REGION, NORMALIZED TO UNITY AT INTERFACE.

When μ^2 is in fact negative, we shall use μ in the same sense as ν, i.e., as the magnitude of the imaginary root of ν^2.

Thus one of the solutions in the outer region will vary with different values of k, depending on the sign of the μ^2 root. As usual the fast and slow fluxes are not independent but are linked by coupling coefficients. We write

$$\begin{aligned} \phi_1 &= S_\mu A X + S_\nu C Y, && \text{(core)} \quad = T_\mu F X + T_\nu G Y \quad \text{(blanket)} \\ \phi_2 &= \quad A X + \quad C Y, && = F X \quad + G Y \end{aligned} \tag{4A.5}$$

These equations hold for both regions; Table 1 gives values X and Y appropriate to each region. Note that the coupling coefficients, which are called S_μ etc., to emphasize their dependence on the roots of the compatibility equation, are not in the conventional position but give the fast flux related to the thermal flux.

Coupling Coefficients

In the following expressions, the appropriate region values for the roots and properties are to be used,

$$S_\mu,\ T_\mu = \frac{\Sigma_2}{p\Sigma_1}(1 + \mu^2 L^2) = \frac{D_2}{D_1}\frac{\tau}{pL^2}(1 + \mu^2 L^2) \tag{4A.6}$$

but for reflector, = 0

Similarly

$$S_\nu,\ T_\nu = \frac{\Sigma_2}{p\Sigma_1}(1 - \nu^2 L^2) = \frac{D_2}{D_1}\frac{\tau}{pL^2}(1 - \nu^2 L^2) \tag{4A.7}$$

for reflector, $= \dfrac{D_2}{D_1 p}\left(\dfrac{\tau - L^2}{L^2}\right)$

	Actual	Adjoint
Core Blanket	Flux Solutions $\phi_1 = \begin{pmatrix} S_\mu & A \\ T_\mu & F \end{pmatrix} X + \begin{pmatrix} S_\nu C \\ T_\nu G \end{pmatrix} Y$ $\phi_2 = \begin{pmatrix} A \\ F \end{pmatrix} X + \begin{pmatrix} C \\ G \end{pmatrix} Y$	$\phi_2^+ = \begin{pmatrix} S_\mu^+ & A^+ \\ T\mu^+ & F^+ \end{pmatrix} + \begin{pmatrix} S_\nu^+ & C^+ \\ T_\nu^+ & G^+ \end{pmatrix} Y$ $\phi_1^+ = \begin{pmatrix} A^+ \\ F^+ \end{pmatrix} X + \begin{pmatrix} C^+ \\ G^+ \end{pmatrix} Y$
Core Blanket Reflecter	Compatability $\quad k = (1 + r B^2)(1 + L^2 B^2)$ $B^2 = \mu^2 = \frac{1}{2}\left[-\left(\frac{1}{r} + \frac{1}{L^2}\right) + \sqrt{\left(\frac{1}{r} - \frac{1}{L^2}\right)^2 + \frac{4k}{rL^2}}\right]$ $B^2 = -\nu^2 = -\frac{1}{2}\left[+\left(\frac{1}{r} + \frac{1}{L^2}\right) + \sqrt{\left(\frac{1}{r} - \frac{1}{L^2}\right)^2 + \frac{4k}{rL^2}}\right]$ $\mu^2 = -1/L^2 \; ; \; -\nu^2 = -1/r$	
Core Blanket	Coupling Coefficients $\begin{pmatrix} S_B \\ T_B \end{pmatrix} = \frac{D_2}{D_1} \quad \frac{r}{pL^2} \quad (1 + L^2 B^2)$	$\begin{pmatrix} S_B^+ \\ T_B^+ \end{pmatrix} = \frac{(1 + r B^2)}{p}$
Reflecter	$\begin{pmatrix} S_\mu \\ T_\mu \end{pmatrix} = 0 \; ; \begin{pmatrix} S_\nu \\ T_\nu \end{pmatrix} = \frac{D_2}{D_1} \quad \frac{r - L^2}{pL^2}$	$\begin{pmatrix} S_\mu^+ \\ T_\mu^+ \end{pmatrix} = \frac{L^2 - r}{pL^2} \; ; \begin{pmatrix} S_\nu^+ \\ T_\nu^+ \end{pmatrix} = 0$
	Relations $\begin{pmatrix} S_B & S_B^+ \\ T_B & T_B^+ \end{pmatrix} = \frac{\Sigma_2}{\Sigma_1} \quad \frac{k}{p^2} = \frac{D_2}{D_1} \quad \frac{r\,k}{p^2 L^2}$ $\begin{pmatrix} S_\mu/S_\nu^+ \\ T_\mu/T_\nu^+ \end{pmatrix} = \begin{pmatrix} S_\nu/S_\mu^+ \\ T_\nu/T_\mu^+ \end{pmatrix} = -\frac{D_2}{D_1}$	

Table 4A.2a RELATIONS FOR THE 2 GROUP 2 REGION PROBLEM
(FLUXES NORMALIZED TO UNITY AT THE INTERFACE)

Interface Conditions

Write the currents at the interface as

$$\begin{aligned} a_1 &= D_1 \nabla X, \quad \beta_1 = D_1 \nabla Y \qquad \text{(core)} \\ \gamma_1 &= D_1 \nabla X, \quad \delta_1 = D_1 \nabla Y \qquad \text{(blanket)} \\ a_2 &= D_2 \nabla X \qquad \text{etc.:} \end{aligned} \tag{4A. 8}$$

then the second compatibility or critical equation is given by the vanishing of the determinant of the flux and current equations of continuity at the interface. This determinant can be written

$$\Delta = \begin{vmatrix} (A) & (C) & (-F) & (-G) \\ S_\mu & S_\nu & T_\mu & T_\nu \\ 1 & 1 & 1 & 1 \\ S_\mu a_1 & S_\nu \beta_1 & T_\mu \gamma_1 & T_\nu \delta_1 \\ a_2 & \beta_2 & \gamma_2 & \delta_2 \end{vmatrix} = 0 \tag{4A. 9}$$

From Chio's theorem, Δ may be reduced to give

$$\Delta = \begin{vmatrix} (S_\mu - T_\mu) & (S_\nu - T_\mu) & (T_\nu - T_\mu) \\ (S_\mu a_1 - T_\mu \gamma_1) & (S_\nu \beta_1 - T_\mu \gamma_1) & (T_\nu \delta_1 - T_\mu \gamma_1) \\ (a_2 - \gamma_2) & (\beta_2 - \gamma_2) & (\delta_2 - \gamma_2) \end{vmatrix} = 0 \tag{4A. 10}$$

It is convenient to define net interface currents as follows:

$$\begin{aligned} i_1 &= (S_\nu - T_\mu)(a_2 - \gamma_2); \quad j_1 = (S_\nu - T_\mu)(\delta_2 - \gamma_2); \quad k_1 = (S_\nu \beta_1 - T_\mu \gamma_1) \\ i_2 &= (T_\nu - T_\mu)(\beta_2 - \gamma_2); \quad j_2 = (T_\nu - T_\mu)(a_2 - \gamma_2); \quad k_2 = (T_\nu \delta_1 - T_\mu \gamma_1) \\ i_3 &= (S_\mu - T_\mu)(\delta_2 - \gamma_2); \quad j_3 = (S_\mu - T_\mu)(\beta_2 - \gamma_2); \quad k_3 = (S_\mu a_1 - T_\mu \gamma_1) \end{aligned} \tag{4A. 11}$$

	Actual	Adjoint
Core Blanket Outer Reflecter	Criticality (1) interface currents $\alpha_1 = D_1 \nabla X \quad \beta_1 = D_1 \nabla Y \quad \gamma_1 = D_1 \nabla X \quad \delta_1 = D_1 \nabla Y$ (core) (blanket) $\alpha_2 = D_2 \nabla X \quad \beta_2 = D_2 \nabla Y \quad \gamma_2 = D_2 \nabla X \quad \delta_2 = D_2 \nabla Y$	
	$i_1 = (S_\nu - T_\mu)(\alpha_2 - \gamma_2) \quad j_1 = (S_\nu - T_\mu)(\delta_2 - \gamma_2) \quad k_1 = (S_\nu \beta_1 - T_\mu \gamma_1) \quad l_1 = S_\mu (\alpha_1 - \delta_1)$ $i_2 = (T_\nu - T_\mu)(\beta_2 - \gamma_2) \quad j_2 = (T_\nu - T_\mu)(\alpha_2 - \gamma_2) \quad k_2 = (T_\nu \delta_1 - T_\mu \gamma_1) \quad l_2 = S_\nu (\beta_1 - \delta_1)$ $i_3 = (S_\mu - T_\mu)(\delta_2 - \gamma_2) \quad j_3 = (S_\mu - T_\mu)(\beta_2 - \gamma_2) \quad k_3 = (S_\mu \alpha_1 - T_\mu \gamma_1) \quad l_3 = T_\nu (\delta_1 - \beta_1)$	
	(2) Criticality $k_1 (i_3 - j_2) + k_2 (i_1 - j_3) + k_3 (i_2 - j_1) = 0$ $i_1 l_3 + i_2 l_1 + i_3 l_2 = 0$	
	Flux Coefficients for Blanket as Reflecter	
	$S_\mu A = \delta_1 - \beta_1$ $S_\nu C = \alpha_1 - \delta_1$ $T_\nu G = \alpha_1 - \beta_1$ $F = \frac{\delta_1 - \beta_1}{S_\mu} + \frac{\alpha_1 - \delta_1}{S_\nu} - \frac{\alpha_1 - \beta_1}{T_\nu}$	$S_\mu^+ A^+ = \gamma_2 - \beta_2$ $S_\nu^+ C^+ = \alpha_2 - \gamma_2$ $T_\mu^+ F^+ = \alpha_2 - \beta_2$ $G^+ = \frac{\gamma_2 - \beta_2}{S_\mu^+} + \frac{\alpha_2 - \gamma_2}{S_\nu^+} - \frac{\alpha_2 - \beta_2}{T_\mu^+}$

Table 4A.2b RELATIONS FOR THE 2 GROUP 2 REGION PROBLEM
(FLUXES NORMALIZED TO UNITY AT THE INTERFACE.)

Then we have the critical equation

$$\Delta = k_1(i_3 - j_2) + k_2(i_1 - j_3) + k_3(i_2 - j_1) = 0 \qquad (4A.\ 12)$$

The variables corresponding to each column in equation (4A. 9) are in parenthesis at the head. When the system is critical and equations (4A. 9) satisfied, then these variables are proportional to the cofactors in any one row. It is convenient to choose the fourth row for this purpose, e. g. ,

$$A S_\mu \propto \begin{vmatrix} 1 & 1 & 1 \\ S_\nu^{-1} & T_\mu^{-1} & T_\nu^{-1} \\ \beta_1 & \gamma_1 & \delta_1 \end{vmatrix} \qquad (4A.\ 13)$$

Reflector

For the special case when the blanket is a pure reflector, the previous results reduce to the conventional simpler form. We have $T_\mu = 0$, whence:

$$\begin{aligned} &i_1 \to S_\nu(\alpha_2 - \gamma_2) \ ; \ j_1 \to S_\nu(\delta_2 - \gamma_2) \ ; \ k_1 \to S_\nu \beta_1 \\ &i_2 \to T_\nu(\beta_2 - \gamma_2) \ ; \ j_2 \to T_\nu(\alpha_2 - \gamma_2) \ ; \ k_2 \to T_\nu \delta_1 \\ &i_3 \to S_\mu(\delta_2 - \gamma_2) \ ; \ j_3 \to S_\mu(\beta_2 - \gamma_2) \ ; \ k_3 \to S_\mu \alpha_1, \end{aligned} \qquad (4A.\ 14)$$

so that putting

$$\begin{aligned} \ell_1 &= S_\mu(\alpha_1 - \delta_1) \\ \ell_2 &= S_\nu(\beta_1 - \alpha_1) \\ \ell_3 &= T_\nu(\delta_1 - \beta_1), \end{aligned} \qquad (4A.\ 15)$$

the criticality for the special case of the reflector becomes:

COEFFICIENTS		$S_\mu^{(+)}$	$S_\nu^{(+)}$	$T_\mu^{(+)}$	$T_\nu^{(+)}$
$A^{(+)}$	$S_\mu^{(+)}$	0	$\gamma-\delta$	$\delta-\beta$	$\beta-\gamma$
$C^{(+)}$	$S_\nu^{(+)}$	$\delta-\gamma$	0	$\alpha-\delta$	$\gamma-\alpha$
$F^{(+)}$	$T_\mu^{(+)}$	$\delta-\beta$	$\alpha-\delta$	0	$\beta-\alpha$
$G^{(+)}$	$T_\nu^{(+)}$	$\beta-\gamma$	$\gamma-\alpha$	$\alpha-\beta$	0

1. Coefficients A, A^+ etc. given by sum of terms in the corresponding row. Each term however is to be multiplied by the two coupling coefficients that do not appear in its row or column.
2. For actual flux, use α_1, β_1 etc. For adjoint, use α_2, β_2 etc.
3. For the pure relefctor, $T_\mu = T_\nu^+ = 0$

Example (1) $G^+ = S_\nu^+ T_\mu^+ (\beta_2 - \gamma_2) + S_\mu^+ T_\mu^+ (\gamma_2 - \alpha_2) + S_\mu^+ S_\nu^+ (\alpha_2 - \beta_2)$

(2) G (reflector) = $S_\mu S_\nu (\alpha_1 - \beta_1)$

Table 4A.3 ACTUAL AND ADJOINT COEFFICIENTS FOR TWO GROUP, TWO REGION PROBLEM.

$$i_1 \ell_3 + i_2 \ell_1 + i_3 \ell_2 = 0 \qquad (4A.16)$$

The coefficients can be expressed directly in the reflector case, as

$$S_\mu A = (\delta_1 - \beta_1) \; ; \quad S_\nu C = (a_1 - \delta_1) \; ; \quad T_\nu G = (a_1 - \beta_1)$$

$$F = \frac{(\delta_1 - \beta_1)}{S_\mu} + \frac{(a_1 - \delta_1)}{S_\nu} - \frac{(a_1 - \beta_1)}{T_\nu} \qquad (4A.17)$$

Adjoint Problem

We write the adjoint equations in the inverse order, that is, we start with the balance of thermal neutrons. Then for each of the two regions we have:

$$\begin{aligned} (D_2 \nabla^2 - \Sigma_2)\, \phi_2^+ + \frac{k}{p}\, \Sigma_2\, \phi_1^+ &= 0 \qquad \text{core or} \\ p\, \Sigma_1\, \phi_2^+ + (D_1 \nabla^2 - \nabla_1)\, \phi_1^+ &= 0 \qquad \text{blanket} \end{aligned} \qquad (4A.18)$$

These equations can be given the same form as the flux equations by the further device of defining an adjoint resonance escape probability, p^+, by, $p^+ \Sigma_2 = p \Sigma_1$. For then the equations become:

$$\begin{aligned} (D_2 \nabla^2 - \Sigma_2)\, \phi_2^+ + \frac{k}{p^+}\, \Sigma_1\, \phi_1^+ &= 0 \qquad \text{core or} \\ p^+ \Sigma_2\, \phi_2^+ + (D_1 \nabla^2 - \Sigma_1)\, \phi_1^+ &= 0 \qquad \text{blanket} \end{aligned} \qquad (4A.19)$$

We need only remember that the adjoint resonance escape is to be used and that fast and slow properties are interchanged. The boundary and interface conditions are of course the same for the actual and adjoint fluxes. It is very useful to realize both in this problem and in general, that the adjoint fluxes tend to behave as their opposite real fluxes. Thus when the thermal flux tends to peak at the interface, the fast importance

has a similar behavior. Indeed the adjoint equations themselves can be interpreted as the description of an adjoint system where cause and effect are reversed, neutrons undiffuse and unfissioning and unfastening up transfer processes occur. For this reason, the actual flux is often called the forward flux and the adjoint flux called the backward flux.

The flux solutions have the corresponding form

$$\phi_2^+ = S_\mu^+ A^+ X + S_\nu^+ C^+ Y \underset{\text{(core)}}{\quad}, \quad = T_\mu^+ F^+ X + T_\nu^+ G^+ Y \underset{\text{(blanket)}}{\quad}$$
$$\phi_1^+ = A^+ X \quad + C^+ Y \quad , \quad = F^+ X \quad + G^+ Y \qquad (4A.20)$$

The coupling coefficients are, by inspecting the actual flux values, interchanging fast and slow properties and replacing p with p^+,

$$S_\mu^+, T_\mu^+ = \frac{\Sigma_1}{p^+ \Sigma_2} (1 + \mu^2 \tau) = \frac{(1 + \mu^2 \tau)}{p} \quad ; \qquad (4A.21)$$

for reflector, $= \dfrac{L^2 - \tau}{L^2 p}$

While

$$S_\nu^+, T_\nu^+ = \frac{\Sigma_1}{p^+ \Sigma_2} (1 - \nu^2 \tau) = \frac{(1 - \nu^2 \tau)}{p} \quad ; \qquad (4A.22)$$

for reflector, = 0

It is to be noted that for each solution, the ratio of the importance of a fast neutron to a slow neutron at the same position is just the non-capture, non-leakage probability. The negative roots correspond to leakage of neutrons into the region diluting the importance of a fast neutron which is slowed down at that point.

Interface Conditions

The criticality condition is of course identical for both actual and adjoint systems, since the time behavior of the importance is determined by the flux fundamental mode. Thus k, μ^2 and ν^2 are common to both systems. The actual coefficients themselves are obtainable from the solutions given for the flux coefficients, making the necessary interchange of fast and slow properties and fluxes. Thus in equation (4A. 13) we interchange fast and slow gradients to find

$$A^+ S_\mu^+ \propto \begin{vmatrix} 1 & 1 & 1 \\ S_\nu^{+-1} & T_\mu^{+-1} & T_\nu^{+-1} \\ \beta_2 & \gamma_2 & \delta_2 \end{vmatrix} \qquad (4A.23)$$

Reflector: The special case of the reflector is found by observing that whereas $T_\mu = 0$, now we have $T_\nu^+ = 0$. Hence interchanging the columns of equation (9) corresponding to T_μ and T_ν, we have

$$S_\mu^+ A^+ = (\gamma_2 - \beta_2);\ S_\nu^+ C^+ = (\alpha_2 - \gamma_2);\ T_\mu^+ F^+ = (\alpha_2 - \beta_2)$$

$$G^+ = \frac{\gamma_2 - \beta_2}{S_\mu^+} + \frac{\alpha_2 - \gamma_2}{S_\nu^+} - \frac{\alpha_2 - \beta_2}{T_\mu^+} \qquad (4A.24)$$

Relations between Coupling Coefficients

Certain convenient relations exist between the forward and backward coupling coefficients. Thus

$$\begin{pmatrix} S_\mu & S_\mu^+ \\ T_\mu & T_\mu^+ \end{pmatrix} = \begin{pmatrix} S_\nu & S_\nu \\ T_\nu & T_\nu^+ \end{pmatrix} = \frac{\Sigma_2}{\Sigma_1} \frac{k}{p^2} = \frac{D_2}{D_1} \frac{\tau}{L^2} \frac{k}{p^2} \qquad (4A.25)$$

where the parenthesis denote that the coupling coefficients correspond to the region whose properties are employed. Similarly

$$\begin{pmatrix} S_\mu & / S_\nu^+ \\ T_\mu & / T_\nu^+ \end{pmatrix} = \begin{pmatrix} S_\nu & / S_\mu^+ \\ T_\nu & / T_\mu^+ \end{pmatrix} = -\frac{D_2}{D_1} \qquad (4A.26)$$

Table 4A.4 Matrix Representation of Flux and Adjoint Coefficients

1. Let the 4×4 table of coefficients in Table 3 be represented by the matrix η. (Let η imply fast properties, η^+ slow properties.)

2. Let $Q = \begin{bmatrix} S_\mu & 0 & 0 & 0 \\ 0 & S_\nu & 0 & 0 \\ 0 & 0 & T_\mu & 0 \\ 0 & 0 & 0 & T_\nu \end{bmatrix}$, $Q^+ = \begin{bmatrix} S_\mu^+ & 0 & 0 & 0 \\ 0 & S_\nu^+ & 0 & 0 \\ 0 & 0 & T_\mu^+ & 0 \\ 0 & 0 & 0 & T_\nu^+ \end{bmatrix}$

3. Let $R = \begin{bmatrix} \bar{A} \\ \bar{C} \\ \bar{F} \\ \bar{G} \end{bmatrix}$ $R^+ = \begin{bmatrix} \bar{A}^+ \\ \bar{C}^+ \\ \bar{F}^+ \\ \bar{G}^+ \end{bmatrix}$

where $\bar{A}$, the normalized coefficient is given by $\bar{A} = A/(S_\mu S_\nu T_\mu T_\nu)$, etc.

4. The coefficients of flux and adjoint are given by

$$R = Q^{-1} \eta Q^{-1} I$$

$$R^+ = Q^{+-1} \eta^+ Q^{+-1} I$$

5. For the reflector, $\lim_{k_r \to 0} T_\mu = T_\nu^+ = 0$. Limiting process is to be carried out after evaluation of coefficients.

ADDENDUM 1 TO APPENDIX A TO SECTION 4

TWO AND THREE DIMENSIONAL GEOMETRY

The solutions given so far have been for a one dimensional geometry. Except for the spherical case, fully reflected systems are not amenable to separation of variables and recourse would be made to approximate methods of solution. However, certain system reflected in one dimension only can be solved. These are

(1) Cylinder with top and bottom or radial reflector.

(2) Rectangular with reflector on two opposite faces only. The asymmetric cases will not be dealt with here; they can be solved by including the extra functions (such as sine in the slab core) that are normally excluded on grounds of symmetry.

In the directions of the bare core, other than in the blanket or reflector direction, the equation resulting from the separation of variables results in a uniform (positive) B_t^2, the one region transverse geometric buckling. The solution of the core material compatibility equation

$$k = (1 + L^2 B^2)(1 + \tau B^2) \tag{4A.27}$$

must provide a positive root, μ^2 which is larger than B_t^2. The effective value of μ^2 to be used in the solution of the equation for the reflected direction as given, is then

$$\mu_{eff}^2 = \mu^2 - B_t^2 \tag{4A.28}$$

Similarly, the second root is made more negative – i.e., the side leakage corresponds to an additional absorption term

$$-\nu_{eff}^2 = -(\nu^2 + B_t^2) \tag{4A.29}$$

Even in the reflector, the roots are no longer given by the material properties alone, but become

$$\mu^2 \text{ reflector effective} = -(1/L^2 + B_t^2)$$
$$\nu^2 \text{ reflector effective} = -(1/\tau + B_t^2) \qquad (4A.30)$$

Note however that the coupling coefficients, which are based on the complete flux solution, are still correctly given by the μ^2 and ν^2 solutions of the material equations. The leakage at a point is not changed in any one solution, but only redistributed in other directions.

ADDENDUM 2 TO APPENDIX A TO SECTION 4

TIME DEPENDENT SOLUTION

It may happen that it is necessary to solve the non-critical system corresponding to the problem of the critical two group, two region system. Since the time behaviour of a reactor is usually governed by the delayed neutrons, it is unlikely that the time dependent problem is usefully answered by such a solution. On the other hand, it may be of interest to find the higher eigenfunctions and eigenvalues of the critical problem for use in a higher order perturbation theory.

We show how the corresponding flux and adjoint problems can be reduced to the same form as the critical problem by expressing time behaviour as a pseudo-absorption term. In general, the properties of the system are fixed by solution of the criticality equation. Then the procedure would be to guess values of the negative eigenvalues, s, of the higher orders that will still satisfy the determinantal equations. It will evidently be advantageous to start with judicious guesses, based on

solutions for the one region two group or for the one group two region case. Iteration is made about these guesses until consistent values of s are found. The flux functions must be checked for oscillation however to ensure that no required solutions have been omitted. Naturally it will only be feasible to find a finite number of solutions, especially if a problem of more than one dimension is attempted with its corresponding degeneracies.

The general flux equations, for the eigen problem, are

$$\begin{aligned} (\nabla \cdot D_1 \nabla - \Sigma_1)\,\phi_1 + \frac{k}{p}\,\Sigma_2\,\phi_2 &= V_1^{-1}\,\frac{\partial \phi_1}{\partial t} = \frac{s}{V_1}\,\phi_1 \\ p\,\Sigma_1\,\phi_1 + (\nabla \cdot D_2 \nabla - \Sigma_2)\,\phi_2 &= V_2^{-1}\,\frac{\partial \phi_2}{\partial t} = \frac{s}{V_2}\,\phi_2 \end{aligned} \qquad (4A.31)$$

Define a group lifetime by

$$\ell_1 = \frac{1}{V_1\,\Sigma_1} \quad \text{and} \quad \ell_2 = \frac{1}{V_2\,\Sigma_2} \qquad (4A.32)$$

and an effective cross section by

$$\begin{aligned} \overline{\Sigma}_1 &= \Sigma_1 + \frac{s}{V_1} = \Sigma_1(1 + s\ell_1) \\ \overline{\Sigma}_2 &= \Sigma_2 + \frac{s}{V_2} = \Sigma_2(1 + s\ell_2) \end{aligned} \qquad (4A.33)$$

Then the equations become

$$(\nabla \cdot D_1 \nabla - \overline{\Sigma}_1) + \frac{k}{p(1 + s\ell_2)}\,\overline{\Sigma}_2\,\phi_2 = 0 \qquad (4A.34)$$

$$\frac{p}{(1 + s\ell_1)}\,\overline{\Sigma}_1\,\phi_1 + (\nabla \cdot D_2 \nabla - \overline{\Sigma}_2)\,\phi_2 = 0$$

Let us now define an effective resonance escape probability

$$\bar{p} = \frac{p}{(1 + s\ell_1)} \tag{4A. 35}$$

This is smaller than the actual value, p, for positive periods, since some neutrons are left behind in the fast group to supply the assumed increase in fast flux. We also define an effective multiplication constant

$$\bar{k} = \frac{k}{(1 + s\ell_1)(1 + s\ell_2)} \tag{4A. 36}$$

$\bar{k}$ is also less than k for positive s, since fission has now to supply the extra neutrons for the growing fast and slow fluxes in addition to making up the leakage and absorption.

Our time dependent equations are now in the same form as a critical system, only with effective values.

$$(\nabla \cdot D_1 \nabla - \bar{\Sigma}_1)\, \phi_1 + \frac{\bar{k}}{\bar{p}} \bar{\Sigma}_2\, \phi_2 = 0$$

$$\bar{p}\, \bar{\Sigma}_1\, \phi_1 + (\nabla \cdot D_2 \nabla - \bar{\Sigma}_2)\, \phi_2 = 0$$

We may define the related parameters by

$$\bar{\tau} = D_1/\bar{\Sigma}_1 = \tau/(1 + s\ell_1)$$

$$\bar{L^2} = D_2/\bar{\Sigma}_2 = L^2/(1 + s\ell_2)$$

and then the solution follows the critical form. We recollect that there will be different effective values and hence different roots, μ^2, $-\nu^{-2}$, for each value of s assumed.

The adjoint solution may be derived from the flux solution exactly as before.

ADDENDUM 3 TO APPENDIX A TO SECTION 4

TWO GROUP, ONE REGION PROBLEM AND SOLUTION

From the general two region solution we may obtain the solution for the case when there is only one region. The interface condition is now replaced by the requirement that the flux solutions in the core go to zero at the boundary. We now take the flux functions, X, Y, etc., to refer to the appropriate function without the normalization at the interface. Thus

X (one region)	$\cos \mu x$	$\frac{\sin \mu x}{\mu x}$	$J_o(\mu x)$
Y	$\cosh \nu x$	$\frac{\sinh \nu x}{x}$	$I_o(\nu x)$

The boundary conditions are then of the form, to be evaluated at the boundary,

$$\begin{aligned} S_\mu A X + S_\nu C Y &= 0 \\ A X + C Y &= 0 \end{aligned} \qquad (4A.39)$$

whence the determinantal equation is

$$A X C Y (S_\mu - S_\nu) = 0 \qquad (4A.40)$$

Now $S_\mu \neq S_\nu$, and $Y \neq 0$ on surface. Hence either A, X, or C must be zero. If A is zero then C must be zero to satisfy the thermal flux equation, a trivial solution. If X is zero C must also be zero, but the solution is non-trivial. Similarly C zero leads to a non-trivial solution with X being zero. Thus the non-trivial solution is that only the X (μ positive type) solution can exist and the critical equation reduces to the requirement that μ^2 is equal to the geometric buckling of the region.

APPENDIX B TO SECTION FOUR

ALTERNATIVE DERIVATION OF IMPORTANCE EQUATIONS

We start from the premise that in a critical reactor the importance of a fast neutron at $\underline{r}$ is less than that of a slow neutron at the same position, due only to the probability of the fast neutron becoming the slow neutron being less than unity. This probability, the fast non-leakage probability, is known to have the following form in the two group two region flux solutions. In the core there are two admissible solutions with corresponding probabilities $1/(1 + \tau\mu^2)$ and $1/(1 - \tau\nu^2)$. Actually these terms are merely a special case of the general form, which can be written operationally as $1/(1 - \tau\nabla^2)$. This is to be evaluated at each point, the ∇^2 operator acting on the fast flux. The result can also be expressed as $1/(1 + \tau B^2)$ where B^2 is now understood to be a function of position, giving the local curvature. Only for a one region reactor will B^2 be independent of position.

Symbolically, we have:

$$\phi_1^+/\phi_2^+ = 1/(1 + \tau B^2) = 1/(1 - \tau\nabla^2)$$

$$= 1\Big/\left(1 - \frac{\nabla \cdot D_1 \nabla}{\Sigma_1}\right), \qquad (4B.\ 1)$$

leading to

$$\Sigma_1 \phi_1^+ - \nabla \cdot D_1 \nabla \phi_1^+ = \Sigma_1 \phi_2^+ , \qquad (4B.\ 2)$$

or

$$(\nabla \cdot D_1 \nabla - \Sigma_1)\ \phi_1^+ + \Sigma_1 \phi_2^+ = 0 \qquad (4B.\ 3)$$

We also know from the flux solutions that in such a one region critical reactor $k = (1 + \tau B^2)(1 + L^2 B^2)$. With a similar operational

symbolism we have:

$$\phi_1^+/\phi_2^+ = \frac{1 + L^2B^2}{k} = \frac{1}{k} - \frac{\nabla \cdot D_2 \nabla}{\Sigma_2 k} \quad , \tag{4B. 4}$$

leading to:

$$k\Sigma_2 \phi_1^+ = \Sigma_2 \phi_2^+ - \nabla \cdot D_2 \nabla \phi_2^+ \quad , \tag{4B. 5}$$

or

$$k \Sigma_2 \phi_1^+ + (\nabla \cdot D_2 \nabla - \Sigma_2) \phi_2^+ = 0 \tag{4B. 6}$$

These two equations are the required adjoint equations for the critical system. For the non-critical system we can again use a symbolic approach. Each neutron supplied to the fast group now has two side processes before it can return to the slow group by slowing down. Firstly there is the fast leakage and secondly there is the requirement of excess neutrons to keep the fast group increasing on the assumed rising period. The additional term gives rise to a new non-leakage probability:

$$1 \Big/ \left|1 + \tau B^2 + \frac{s}{V_1 \Sigma_1}\right| = 1/(1 + \tau B^2)(1 + s\ell_1) \quad , \tag{4B. 7}$$

where $\ell_1 = \frac{1}{V_1 \Sigma_1} \quad \frac{1}{(1 + \tau B^2)}$. (4B. 8)

From our discussion of the inverted time behaviour of the importance, the time operator $s \left(= \frac{\partial}{\partial t}\right)$ for the flux is replaced by $-\omega$. Then, symbolically:

$$\phi_1^+/\phi_2^+ = \frac{1}{(1 + \tau B^2)(\ -\omega\ell_1)}$$

$$= \frac{1}{\Sigma_1 - \nabla \cdot D_1 \nabla - \frac{1}{V_1} \frac{\partial}{\partial t}} \tag{4B. 9}$$

Then we have the time dependent equation:

$$(\nabla \cdot D_1 \nabla - \Sigma_1)\ \phi_1^+ + \Sigma_1\ \phi_2^+ = \frac{1}{V_1}\ \frac{\partial \phi_1^+}{\partial t} \qquad (4B.\ 10)$$

For the second equation, we have from the flux solution:

$$k = (1 + \tau B^2)(1 + s\ell_1)(1 + L^2 B^2)(1 + s\ell_2)\ , \qquad (4B.\ 11)$$

and replacing τB^2 by the operator $-\frac{\nabla \cdot D_1 \nabla}{\Sigma_1}$ and s by $\omega = -\frac{\partial}{\partial t}$, we are led to the corresponding equation:

$$k \Sigma_2\ \phi_1^+ + (\nabla \cdot D_2 \nabla - \Sigma_2)\ \phi_2^+ = -\frac{1}{V_2}\ \frac{\partial \phi_2^+}{\partial t} \qquad (4B.\ 12)$$

SECTION 5: APPLICATIONS TO KINETICS

5. 1 Introduction

In the present section, we shall develop and discuss the perturbation form of the reactor kinetics equations as a generalisation of the point system discussed in Section 1, Introduction. After giving the generalised inhour relation and the interpretation of the perturbation result in terms of our physical development, we discuss in detail the interpretation and calculation of the generalised reactor parameters: generating time, $\bar{\Lambda}'$, and effective precursor yields, $\bar{\beta}_i'$.

In the development of Section 3 and 4, we introduced the concept of the time dependent importance of neutrons and precursors. The development of the inhour relation utilises a reference reactor and its importance. We could choose to give the relation based on the non-critical reference reactor and then specialise to the particular case of the critical reference reactor. However, by far the most significant and useful result is the special case itself so that we choose to give a preliminary treatment in terms of the critical reference reactor and delay the introduction of the non-critical reactor to a later section. We follow the conventional treatment of Nordheim (15) and Henry (7) in this first part, giving our own commentary. After a discussion of the calculation of the parameters, we give a general account of the properties of the reactivity and the time dependent reference reactor in Section 5. 14 et seq. In the main we shall only consider the first order approximation and its validity; a treatment of some solutions where the first order approximation is inadmissible is reserved for an appendix to this section.

5.2 One Group of Neutrons and Precursors Example

Consider the equations representing the kinetic behaviour of one group of neutrons and one group of precursors:

$$V^{-1}\frac{\partial \phi'}{\partial t} = \left[\nabla \cdot D'\nabla - \Sigma' + (1-\beta)F'\right]\phi' + \lambda C' \ ,$$
$$\frac{\partial C'}{\partial t} = \beta F'\phi' - \lambda C' \qquad (5.1)$$

These equations are multiplied by the importance of neutrons and precursors respectively, the importance being taken from some arbitrary reference reactor. We shall assume at this point however, that the reference reactor properties are chosen to be critical and the importance employed corresponds to the fundamental mode. It follows that there is no time dependency and, from Section 3, that the importance of the precursors, $C^+(\underline{r})$, is equal to the importance of neutrons, $\phi^+(\underline{r})$, in the reference reactor. We might equally well interpret the operation as one of multiplying by the equivalence of neutrons, since the normalisation will prove immaterial. The equations are then integrated over the reactor volume and the order of integration and differentiation changed. We obtain

$$\frac{d}{dt}\int \phi^+ V^{-1}\phi' \, dv = \int \phi^+\left[\nabla \cdot D'\nabla - \Sigma' + (1-\beta)F'\right]\phi' \, dv + \int \phi^+ \lambda C' \, dv \ ,$$
$$\frac{d}{dt}\int \phi^+ C' \, dv = \int \phi^+ \beta F'\phi' \, dv - \int \phi^+ \lambda C' \, dv \qquad (5.2)$$

We may define the effective volume averages for the neutron density, $\bar{n}'$, and the precursor density, $\bar{C}'$:

$$\bar{n}' = \int \phi^+ V^{-1}\phi' \, dv \ ,$$
$$\bar{C}' = \int \phi^+ C' \, dv \qquad (5.3)$$

Then equation (5. 3) becomes

$$\frac{d\bar{n}'}{dt} = \int \phi^+ [\nabla \cdot D' \nabla - \Sigma' + (1-\beta) F'] \phi' \, dv + \lambda \bar{C}' ,$$

$$\frac{d\bar{C}'}{dt} = \int \phi^+ \beta F' \phi' \, dv - \lambda \bar{C}' \qquad (5.4)$$

In passing we note that the addition of the equations in (5. 4) leads to the rate of change of the weighted population:

$$\frac{d\bar{n}'}{dt} + \frac{d\bar{C}'}{dt} = \int \phi^+ [\nabla \cdot D' \nabla - \Sigma' + F'] \phi' \, dv \qquad (5.5a)$$

Such an equation is equivalent to the matrix form

$$\frac{d}{dt} \int \psi^+ A \psi' \, dv = \int \psi^+ M \psi' \, dv \qquad (5.5b)$$

We note that the population in equations (5. 5) or (5. 6) is not the π of Section 3 and 4, since we are weighting the actual flux with the importance in a different reactor. Furthermore, the right hand side of these equations contains no reference to the precursor densities but has the same value as would be obtained in an approximation where all neutrons were assumed to be prompt.

So far, one arbitrary parameter (ϕ^+) has been introduced. A second arbitrary step is now employed and both sides of equation (5. 5) are divided by the total weighted production. This second step corresponds to the development of the reactor parameters in Section 1. To ensure that the generalised parameters will reduce to their one energy point system prototype, we actually divide by the production probability per weighted population of neutrons:

$$\frac{\int \phi^+ F' \phi' \, dv}{\int \phi^+ V^{-1} \phi' \, dv}$$

Then we obtain

$$\frac{\int \phi^+ V^{-1} \phi' \, dv}{\int \phi^+ F' \phi' \, dv} \left[\frac{d\bar{n}'}{dt} + \frac{d\bar{C}'}{dt} \right] = \frac{\int \phi^+ [\nabla \cdot D' \nabla - \Sigma' + F] \phi' \, dv}{\int \phi^+ F' \phi' \, dv} \int \phi^+ V^{-1} \phi' \, dv \tag{5.6}$$

We define the generalized generating time, $\bar{\Lambda}'$, where the bar indicates the bilinear volume average and the prime indicates the use of the actual flux, by

$$\bar{\Lambda}' = \frac{\int \phi^+ V^{-1} \phi' \, dv}{\int \phi^+ F' \phi' \, dv} \tag{5.7}$$

Similarly the generalized reactivity, $\bar{\rho}'$, is defined by

$$\bar{\rho}' = \frac{\int \phi^+ (\nabla \cdot D' \nabla - \Sigma' + F') \, dv}{\int \phi^+ F' \phi' \, dv} \tag{5.8}$$

It is seen that $\bar{\Lambda}'$ has the form of the reciprocal production probability and $\bar{\rho}'$ the form of the increase of prompt neutrons and precursors as a fraction of the total weighted production. For a given flux, ϕ', and properties, the values of $\bar{\Lambda}'$ and $\bar{\rho}'$ will in general vary with the reference reactor or weighting function, ϕ^+, employed. Equation (5.6) can now be written as

$$\bar{\Lambda}' \left[\frac{d\bar{n}'}{dt} + \frac{d\bar{C}'}{dt} \right] = \bar{\rho}' \bar{n}' \tag{5.9}$$

We can make a further modification of equation (5.7) by assuming an eigenfunction solution for ϕ' and C'. In this case we can write an algebraic coupling relation in the form:

$$C' = \frac{\beta F' \phi'}{\lambda + s} \tag{5.10}$$

and equation (5.9) becomes

$$\bar{\Lambda}' \left[s\bar{n}' + \frac{s}{\lambda + s} \int \phi^+ \beta F' \phi' \, dv \right] = \bar{\rho}' \, \bar{n}' \tag{5.11}$$

By a further rearrangement we can write equation (5.11) as

$$\bar{\Lambda}' \, s \, \bar{n}' + \frac{s}{\lambda + s} \frac{\int \phi^+ \beta F' \phi' \, dv}{\int \phi^+ F' \phi' \, dv} \, \bar{n}' = \bar{\rho}' \, \bar{n}' \, , \tag{5.12}$$

so that we are led to define an effective precursor yield, $\bar{\beta}'$ by

$$\bar{\beta}' = \frac{\int \phi^+ \beta F' \phi' dv}{\int \phi^+ F' \phi' \, dv} \tag{5.13}$$

to give the desired inhour relation,

$$\bar{\Lambda}' s + \frac{s}{\lambda + s} \bar{\beta}' = \bar{\rho}' \tag{5.14}$$

The effective yield in this example reduces to the measured yield, β, since no energy differences have been represented and there is only one value of ϕ^+ implied by equation (5.13). In general, when energy effects are included, even in one region $\bar{\beta}$ will differ from β.

5.3 Kinetics Parameters

The generalization of the kinetics parameters, which are the bilinear averages of the parameters of the Introduction, are displayed in Table 5.1 (a) and (b). These parameters are defined as a convenient grouping of terms appearing in the kinetics equations. The bilinear form results mainly from the desire to be able to employ the first order approximation of using ϕ for ϕ'. When this approximation is made, we

shall drop the prime e.g., $\bar{\rho}'$ becomes $\bar{\rho}$ in the first order approximation. A second reason for the use of the bilinear form is that $\bar{\rho}'$ may be related to the perturbation in properties, i.e., in terms of δD, δF, $\delta\Sigma$ rather than the definition in terms of $D_j'F'$, and Σ'.

As in the case of the point system prototypes, the value of the parameter depends on the value assigned to production and destruction. In addition, the values depend on the importance ϕ^+ employed and hence on the choice of reference reactor. For the point system, the properties become independent of position and hence the averages become independent of the weighting function as far as its spatial dependence is concerned. The parameters will still depend upon any energy differences and corresponding variations of ϕ^+. In the point system, the shape of each flux component will similarly be unchanged. The coupling coefficients behaviour flux groups are functions of the period however. Hence only if the reactor period is left unchanged (or in the one group model) can we say that ϕ' is identically equal to ϕ in the point system. Some further investigations are carried out in Appendix C.

An additional advantage to be gained by the given kinetics parameters is the concentration of the terms involving the effects of delayed neutron energy differences in the single parameter $\bar{\beta}_i'$. The remaining kinetics parameters are very well calculated by prompt models neglecting the energy differences involved. The error is of the order of $(\bar{\beta}-\beta)$, or perhaps 0.03 per cent in the M.I.T.R.

We have discussed at length the fact that neither the generating time, $\bar{\Lambda}'$, nor the lifetime, $\bar{\ell}'$, are operational quantities associated with a reactor. Kinetics solutions obtained parametrically for fixed

Generating Time $\bar{\Lambda}'$	$\dfrac{1}{\text{production rate}}$	$\dfrac{\int \phi^+ V^{-\prime} \phi' dv}{\int \phi^+ F' \phi' dv}$
Effective Precursor Yield $\bar{\beta i}'$	$\dfrac{\text{production rate of } i^{th} \text{ precursor}}{\text{production rate}}$	$\dfrac{\int \phi^+ \beta i F' \phi' dv}{\int \phi^+ F' \phi' dv}$
Reactivity $\bar{\rho}'$	$\dfrac{\text{increase rate}}{\text{production rate}}$	$\dfrac{\int \phi^+ (\nabla \cdot D' \nabla - \Sigma' + F') \phi' dv}{\int \phi^+ F' \phi' dv}$

TABLE 5.1a Kinetics Parameters based on Production
Production refers to production of prompt neutrons and precursors.
Primes indicate actual properties.

Lifetime $\bar{\ell}'$	$\dfrac{1}{\text{destruction rate}}$	$\dfrac{\int \phi^+ V^{-\prime} \phi' dv}{\int \phi^+ (-\nabla \cdot D' \nabla + \Sigma') \phi' dv}$
Effective Multiplication $\bar{k}_{eff}'$	$\dfrac{\text{production rate}}{\text{destruction rate}}$	$\dfrac{\int \phi^+ F' \phi' dv}{\int \phi^+ (-\nabla \cdot D' \nabla + \Sigma') \phi' dv}$
Excess Multiplication $\bar{k}_{ex}'$	$\dfrac{\text{increase rate}}{\text{destruction rate}}$	$\dfrac{\int \phi^+ (\nabla \cdot D' \nabla - \Sigma' + F') \phi' dv}{\int \phi^+ (-\nabla \cdot D' \nabla + \Sigma') \phi' dv}$

TABLE 5.1b Kinetics Parameters based on Destruction
Increase refers to production minus destruction.

values of $\bar{\Lambda}'$ or $\bar{\ell}'$ do not necessarily mean that any given reactor has a constant value of either. In selecting which parameter to use we can be guided by two criteria.

(a) Which parameter gives the more convenient solution? The answer to this question is that $\bar{\Lambda}'$ is more convenient than $\bar{\ell}'$. It is easier to demonstrate this for a point system with no spatial dependence rather than for the present generality of perturbation theory. Thus in Appendix A, we solve some of the elementary kinetics problems in terms of the generating time.

(b) Which parameter is more nearly constant in an actual reactor? Again the answer is generally in favor of the generating time. Where a reactor is controlled by alterations of the destruction probability (control rods, variable reflector) the lifetime is bound to change correspondingly. The generating time however will be essentially unchanged (only changed through changes of ϕ'). Even if control is exercised by changes of fuel concentration, it is probable that both destruction and production probabilities will change. In the MTR and ETR (47), however, control is achieved with grey absorbing rods replacing fuel rods. To a good approximation, there is no change in the destruction processes. Hence in this last example it is $\bar{\ell}'$ that will vary less than $\bar{\Lambda}'$. In any case, since the variation in either parameter is small, we can be governed by the simplicity obtained from the employment of $\bar{\Lambda}'$.

5.4 General Kinetics Equations

The general bilinear form for multigroups of neutrons and precursors is obtained by multiplying each balance equation by the appropriate

importance and integrating over the reactor. In the matrix notation of Section 1, we obtain

$$\frac{d}{dt}\int \psi^+ A\psi' \, dv = \int \psi^+ (M+T)\psi' \, dv \tag{5.15}$$

The term $\psi^+ T\psi'$ vanishes for the critical reference reactor (where C_i^+ is equal to the importance of the neutron it will become, ϕ_i^+), since $\psi^+ T\psi'$ consists of the sum of all terms $(\phi_i^+ - C_i^+)\lambda_i C_i'$. Then we can write equation (5.15) as

$$\frac{d}{dt}\int \phi^+ V^{-1}\phi' \, dv + \sum_i \frac{d}{dt}\int \phi_i^+ C_i' \, dv = \int \psi^+ M\psi' \, dv \tag{5.16}$$

Here we have decomposed the generalized vectors ψ' and ψ^+ into the vector flux components and the individual precursor components. Without loss of generality, the term $\psi^+ M\psi'$ can be written as $\phi^+ M\phi'$ since all terms in C_i' are related only to the T matrix which vanished on assuming $C_i^+ = \phi_i^+$. Equation (5.16) is now divided by the weighted average production probability to obtain

$$\frac{\int \phi^+ V^{-1}\phi' \, dv}{\int \phi^+ F'\phi' \, dv}\left[\frac{d}{dt}\int \phi^+ V^{-1}\phi' \, dv + \sum_i \frac{d}{dt}\int \phi_i^+ C_i' \, dv\right] =$$

$$= \frac{\int \phi^+ M\phi' \, dv}{\int \phi^+ F'\phi' \, dv}\int \phi^+ V^{-1}\phi' \, dv \tag{5.17}$$

We can now employ the kinetics parameters of Table 5.1a. We obtain

$$\bar{\Lambda}'\left[\frac{d}{dt}\int \phi^+ V^{-1}\phi' \, dv + \sum_i \frac{d}{dt}\int \phi_i^+ C_i' \, dv\right] = \bar{\rho}'\int \phi^+ V^{-1}\phi' \, dv \tag{5.18}$$

We can also write the effective neutron and precursor densities as

$$\bar{n}' = \int \phi^+ V^{-1} \phi' \, dv$$

$$\bar{C}_i' = \int \phi_i^+ C_i' \, dv \tag{5.19}$$

We obtain

$$\bar{\Lambda}' \left[\frac{d\bar{n}'}{dt} + \sum_i \frac{d\bar{C}_i'}{dt} \right] = \bar{\rho}' \, \bar{n}' \tag{5.20}$$

It is to be remembered that $\bar{n}'$ is actually a sum over all the neutron groups represented. In general these groups are not changing at the same rate. Asymptotically, however, (or if we are seeking the eigenfunction solutions) all groups of neutrons and precursors are changing on the same inverse period, s. With this assumption, equation (5.20) becomes

$$\bar{\Lambda}' s \left[\bar{n}' + \sum_i \bar{C}_i' \right] = \bar{\rho}' \, \bar{n}' \tag{5.21}$$

We can now substitute for C_{ij}' in terms of ϕ_j' where by ϕ_j' we whall mean the j^{th} flux giving rise to the production of the i^{th} precursor. If the production of precursors is from fast and thermal fission for example, there would be a term in C_{i1}' and a term in C_{i2}'. We obtain for equation (5.21) the result that

$$\bar{\Lambda}' s \left[\bar{n}' + \sum_i \sum_j \frac{\int \phi_i^+ \beta_i \phi_j' \, dv}{\lambda_i + s} \right] = \bar{\rho}' \bar{n}' \tag{5.22}$$

and by a further manipulation equation (5.22) becomes

$$\bar{\Lambda}' s \bar{n}' + \sum_i \sum_j \frac{\int \phi_i^+ \beta_i F' \phi_j \, dv}{(\lambda_i + s) \int \phi^+ F' \phi \, dv} \int \phi^+ V^{-1} \phi' dv = \bar{\rho}' \, \bar{n}' \tag{5.23}$$

The double summation over i and j refers to the summation over the i characteristic species and the j fluxes producing each characteristic species. The double summation is conveniently contracted to a single summation by the renumbering of all the possible ij combinations against the single index i. Then by employing the definition of the effective precursor yield, $\bar{\beta}_i'$, we have

$$\bar{\Lambda}' s \bar{n}' + \sum_i \frac{s}{\lambda_i + s} \bar{\beta}_i' \bar{n}' = \bar{\rho}' \bar{n}' \tag{5.24}$$

or the conventional inhour relation,

$$s \bar{\Lambda}' + \sum_i \frac{s}{\lambda_i + s} \bar{\beta}_i' = \bar{\rho}' \quad . \tag{5.25}$$

Equation (5.24) can be rearranged in a meaningful way to give

$$s \bar{n}' = \frac{(\bar{\rho}' - \bar{\beta}')}{\bar{\Lambda}'} \bar{n}' + \sum_i \frac{\bar{\beta}_i'}{\bar{\Lambda}'} \frac{\lambda_i}{(\lambda_i + s)} \bar{n}' \tag{5.26}$$

where we have defined

$$\bar{\beta}' = \sum_i \bar{\beta}_i' \tag{5.27}$$

In equation (5.26), $\bar{\rho}'$ is the increase of precursors and prompt neutrons over the destruction of neutrons as a fraction of the total production. $(\bar{\rho}' - \bar{\beta}')$ is the increase of prompt neutrons over the destruction of neutrons. Divided by the production, this term represents the direct contribution to the probability of population increase, s, due to prompt neutrons. The rate of formation of precursors is $\bar{\beta}'/\bar{\Lambda}'$. In steady state, one delayed neutron is returned for every precursor formed. On the inverse period s, however, only a fraction λ_i neutrons are returned

for every $\lambda_i + s$ precursors formed. Thus the second term on the right of equation (5.26) is an indirect contribution to the probability of population increase, s.

5.5 The Perturbation Formula

The kinetics equations have been grouped into parameters one of which, the reactivity, has useful properties. Firstly, since our expression for the reactivity does not contain precursor decay terms and corresponding importance, the reactivity is well approximated by using the flux of a critical calculation that neglects delayed neutron energies. Secondly, the bilinear form makes the use of the reference reactor flux for the actual flux sufficiently accurate for most cases. Thirdly, the reactivity can be expressed explicitly in terms of the difference of properties between the actual and reference reactors. This third property is useful in interpreting the effects of an inserted sample or in calculating the effect of a small change in reactor properties.

The reactivity is given (for a critical reference reactor) by

$$\bar{\rho}' = \frac{\int \phi^+ M \phi' \, dv}{\int \phi^+ F' \phi' \, dv} = \frac{\int \phi^+ (\nabla \cdot D' \nabla - \Sigma' + F') \phi' \, dv}{\int \phi^+ F' \phi' \, dv} \tag{5.28}$$

From the discussion in the Introduction, Section One, and in Section Four, this can be expressed as

$$\bar{\rho}' = \frac{\int \phi^+ R \phi' \, dv}{\int \phi^+ F' \phi' \, dv} + \frac{\int \phi^+ P \phi' \, dv}{\int \phi^+ F' \phi' \, dv}, \tag{5.29}$$

where R, the reference increase matrix, is $\nabla \cdot D \nabla - \Sigma + F$ and P the perturbation, M-R, is $\nabla \cdot \delta D \nabla - \delta \Sigma + \delta F$. Then with the use of the reference importance, the reactivity reduces to

$$\bar{\rho}' = \frac{\int \phi^+ P \phi' \, dv}{\int \phi^+ F' \phi' \, dv} = \frac{\int \phi^+ (\nabla \cdot \delta D \nabla - \delta \Sigma + \delta F) \phi' \, dv}{\int \phi^+ F' \phi' \, dv} \tag{5.30}$$

To interpret equation (5.30) term by term, consider for simplicity a system with no precursor holdup (i.e., $\lambda \to \infty$) in which we could write:

$$s = \frac{\int \phi^+ M \phi' \, dv}{\int \phi^+ V^{-1} \phi' \, dv} = \frac{\int \phi^+ (\nabla \cdot D' \nabla - \Sigma' + F') \phi' \, dv}{\int \phi^+ V^{-1} \phi' \, dv} \tag{5.31}$$

Such an equation is to be interpreted as the inverse period resulting from the introduction of a homogeneous source ($M \phi'$) in a just critical reactor with resident population, π. In Section 3, we derived the effect of such a homogeneous source:

$$s\Big|_{\bar{S}} = \frac{\int \phi^+ S \phi' \, dv}{\int \phi^+ V^{-1} \phi' \, dv} \tag{5.32}$$

Thus if equation (5.31) is written out as

$$\frac{\int \phi^+ M \phi' \, dv}{\int \phi^+ V^{-1} \, dv} = \frac{\int \phi^+ R \phi \, dv}{\int \phi^+ V^{-1} \phi' \, dv} + \frac{\int \phi^+ R \delta \phi \, dv}{\int \phi^+ V^{-1} \phi' \, dv} + \frac{\int \phi^+ P \phi' \, dv}{\int \phi^+ V^{-1} \phi' \, dv}, \tag{5.33}$$

we can make the following comment on the three terms on the right hand side:

(a) $\dfrac{\int \phi^+ R \phi \, dv}{\int \phi^+ V^{-1} \phi' \, dv}$, whatever the normalization of ϕ, is a just critical flux shape in a reactor with the corresponding critical properties. Indeed ϕ is specifically the solution of the equation $R\phi = 0$. Hence the term vanishes.

(b) $\dfrac{\int \phi^+ R \delta \phi \, dv}{\int \phi^+ V^{-1} \phi' \, dv}$ has the form of an initial transient flux in the

critical reactor. The neutrons resulting from the increase $R\,\delta\phi$ can be regarded as just such a homogeneous source as described in equation (5. 32). Yet since they result from a transient, by definition of ϕ^+ they can make no contribution to the resident population. Hence the term vanishes.

(c) $\dfrac{\int \phi^+ P\ \phi'\ dv}{\int \phi^+ V^{-1}\ \phi'\, dv}$ has the form of a homogeneous source term due to neutrons that are generated by the actual flux in the perturbed or added material itself. This is the only term not to vanish: we have as proved before, that

$$s = \frac{\int \phi^+ P\ \phi'\ dv}{\int \phi^+ V^{-1}\ \phi'\, dv} \tag{5.34}$$

Thus the real reactor is representable as the reference reactor plus two sources. The first source corresponds to a transient flux in the critical reference reactor and must vanish. The second source corresponds to the actual flux in the added materials and leads to the change of reactor period as a homogeneous source.

It will be noted that the previous discussion involved only the numerator and not the denominator of the three terms. Thus in the general problem with precursors decaying we can make a corresponding interpretation of the expression for the reactivity

$$\bar{\rho}' = \frac{\int \phi^+ R\ \phi\, dv}{\int \phi^+ F'\, \phi'\, dv} + \frac{\int \phi^+ R\delta\phi\ dv}{\int \phi^+ F'\, \phi'\ dv} + \frac{\int \phi^+ P\ \phi'\, dv}{\int \phi^+ F'\, \phi'\, dv} \tag{5.35}$$

We have for the three terms:

(a) $R\,\phi$, increase in reference reactor, zero by definition.

(b) $R\,\delta\,\phi$, transient in reference, can lead to no production of resident

population, hence no reactivity.

(c) Source production term in added material as a fraction of total production is actual reactivity.

Expanding the term $P\phi'$, we have

$$\bar{\rho}' \int \phi^+ F' \phi' \, dv = \int \phi^+ \nabla \cdot \delta D \nabla \phi' \, dv - \int \phi^+ \delta \Sigma \phi' \, dv + \int \phi^+ \delta F \phi' \, dv \tag{5.36}$$

Again the three right hand terms have physical interpretation:

(a) $\int \phi^+ \delta F \phi' \, dv$ is the sum of all the neutrons produced in the added material weighted by their contribution to the reference resident population increase rate.

(b) $\int \phi^+ \delta \Sigma \phi' \, dv$ is the sum of all the neutrons absorbed or transferred in the added material similarly weighted.

(c) $\int \phi^+ \nabla \cdot \delta D \nabla \phi' \, dv$ is the weighted sum of all the neutrons lost in leakage in the new material. This leakage term can be usefully transformed on the basis of Lemmas One and Two of Section Four.

It was there shown that with the boundary conditions employed,

$$\int \phi^+ \nabla \cdot \delta D \nabla \phi' \, dv = - \int \nabla \phi^+ \cdot \delta D \nabla \phi' \, dv \tag{5.37}$$

In this transformed expression, $(-\delta D \nabla \phi')$ is the current of neutrons in the added material. Such a current of neutrons is not in itself a gain or loss to the population balance. Rather the balance gains if a neutron going from A to B goes from a region of low importance to a region of high importance. Then the weighting by $\nabla \phi^+$ of $(-\delta D \nabla \phi')$ gives just this change of importance seen by each neutron in the current. Equation (5.37) can be transposed and a further application of the

Lemmas yields

$$-\int \nabla \phi^+ \cdot \delta D \nabla \phi' \, dv = -\int \nabla \phi' \cdot \delta D \nabla \phi^+ \, dv = \int \phi' \nabla \cdot \delta D \nabla \phi^+ \tag{5.38}$$

The expression in the form $\nabla \phi' \cdot \delta D \nabla \phi^+$ can be understood as follows: $\delta D \nabla \phi^+$ is the additional current of importance, per unit flux in the reference reactor. Again if the actual flux were uniform, such an exchange of importance leads to no net leakage. Only when there is a gradient, $\nabla \phi'$, in the opposite direction to the importance current will the resident population lose by leakage of importance. The last form is easily interpreted as the increased leakage of importance due to the added material, per unit flux.

The above transformations have some practical advantages. Firstly, by integrating over the unknown flux as much as possible before making the first order approximation for the flux, we can improve the accuracy of our approximation. For example, consider a one region system in which the reference reactor is one region, with an algebraic buckling B^2. Then if the change in the diffusion coefficient, δD, is uniform throughout, whatever the other property changes and hence whatever the actual flux, we have

$$\nabla \cdot \delta D \nabla \phi^+ = \delta D \nabla \cdot \nabla \phi^+ = -\delta D B^2 \phi^+ \tag{5.39}$$

Thus the contribution to the reactor period neglecting delayed neutrons, of just the change δD, is found exactly:

$$\frac{\int \phi^+ \nabla \cdot \delta D \nabla \phi dv}{\int \phi^+ \, V^{-1} \, \phi' \, dv} = \frac{-\int \phi' \, \delta D B^2 \, \phi^+ dv}{\int \phi' \, V^{-1} \, \phi^+ \, dv} = -V(\delta D B^2) \quad \text{(one group)} \tag{5.40}$$

This result is exact for the model whatever the other perturbations of

properties, $\delta\Sigma$ and δF.

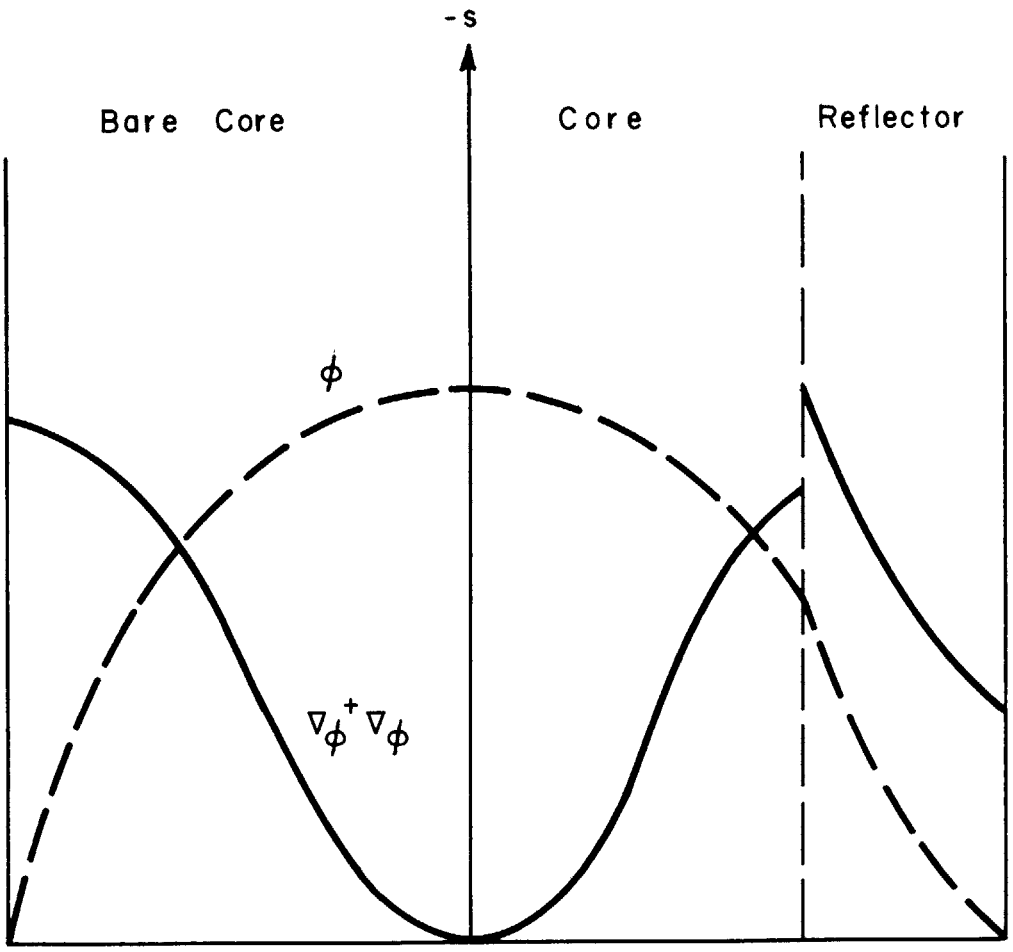
Secondly the form $\nabla\phi^+ \cdot \delta D \nabla\phi'$ has advantages since when the first order approximation $\phi' = \phi$ is made, we can evaluate the scalar dot product $\nabla\phi^+ \cdot \nabla\phi$ once and for all. Then for different perturbations δD, we need only integrate the perturbation with the weighting function $\nabla\phi^+ \cdot \nabla\phi$.

5.6 Statistical Weight

Such a weighting function, $\nabla\phi^+ \cdot \nabla\phi$, is well known as the statistical weight for perturbations of the diffusion coefficient. The ratio involved in the expression for the reactivity make it unnecessary to carry out the normalization of ϕ^+. Then in one group we can take $\phi^+ = \phi$ and the statistical weight becomes $(\nabla\phi)^2$. In multigroup we must consider the separate $\nabla\phi_i^+ \cdot \nabla\phi_i^+$ and talk of the statistical weight for scattering in the i^{th} group. Figure 5.1 is a plot of the one group statistical weight for a slab reactor, bare and reflected. Naturally at the centre of the reactor, where there is a flat flux and no leakage, the change of the diffusion coefficient resulting from a change of scattering properties has no effect. The effect is larger travelling towards the outer surface except that once in the reflector, scattering becomes less and less important as the reflector becomes nearer being infinitely thick.

There is of course a statistical weight for other processes than change of diffusion coefficient. In one group we can again take $\phi^+ = \phi$, so that absorption and production in equation (5.36) are weighted by ϕ^2. The shape of this weighting function is markedly different from that for the change of diffusion coefficient, $(\nabla\phi)^2$. A typical slab leads to a $\cos^2$

STATISTICAL WEIGHT FOR A SCATTERING SAMPLE



Relative effect on inverse period, s, for a small sample at different places in a one group slab reactor, where the sample changes only the diffusion coefficient, by δD. Note discontinuity in $\nabla\phi$ at interface due to change of D in reference reactor.

FIG. 5.1

weighting function of the form of Figure 5.2

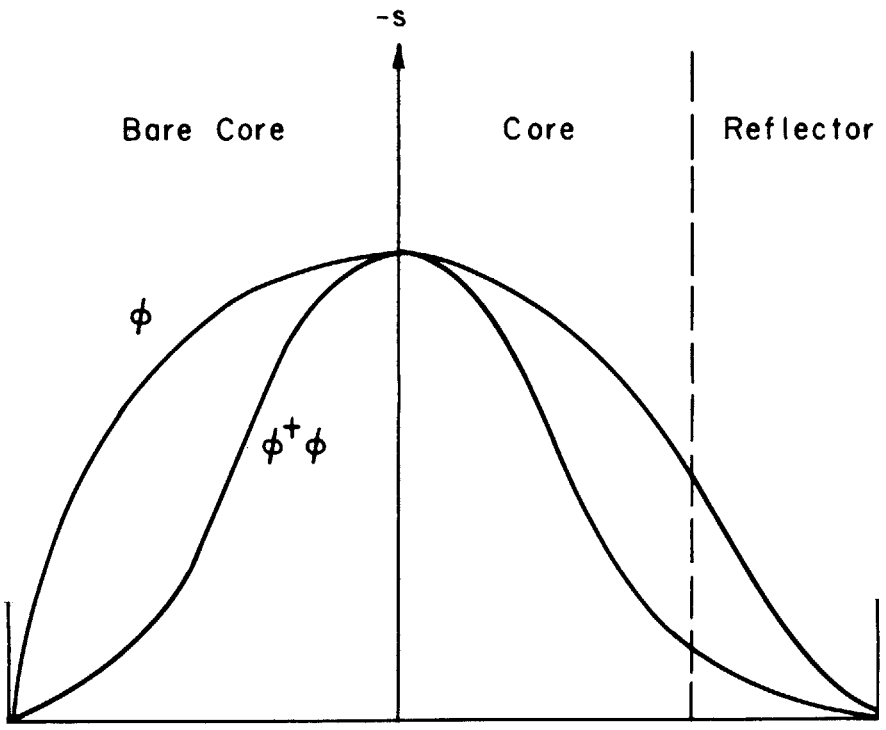
The result for the absorption and production process is well known. It is not so well known that the statistical weight for scattering changes discontinuously between regions in which the reference diffusion coefficient changes – compare Figures 5.1 and 5.2.

For multigroup we must specify the process in terms of where the neutron is from (which flux) and where it is going to (which importance). Thus we have to speak of the statistical weight of the $i \rightarrow j$ process, $\phi_j^+ \phi_i$.

Since the statistical weight for various processes varies differently in a reactor, it is possible to obtain information on the contents of a sample by positioning it at various places in the reactor and measuring the period. When the effect is greatest at the centre and negligible towards the edge of the reactor, we obviously do not have a good scattering sample. By previous calibration of known samples it is possible to estimate the ratio of scattering to absorption cross-section. Some practical applications to the analysis of hydrocarbons has been reported (46).

Similar experiments enabled us to make an experimental determination of the importance of fast and slow neutrons in the MIT Reactor. A sample of boron was moved along a vertical sample tube – the reactivity effect as a function of distance from the centre is plotted in Figure 5.3. By dividing through by the measured thermal flux, we can obtain a relative plot of the thermal importance, Figure 5.4, since $\phi_2^+ \propto -\bar{\rho}/\phi_2$.

A fissioning sample calculated to have the same thermal absorption cross-section was also measured. Correction from the previous



RELATIVE EFFECT OF A SMALL ABSORBING SAMPLE IN A SLAB REACTOR - STATISTICAL WEIGHT $\phi^+\phi$

FIG. 5.2

experiment led to a measure of the relative effect of the production of fast neutrons from thermal and hence to the fast importance.

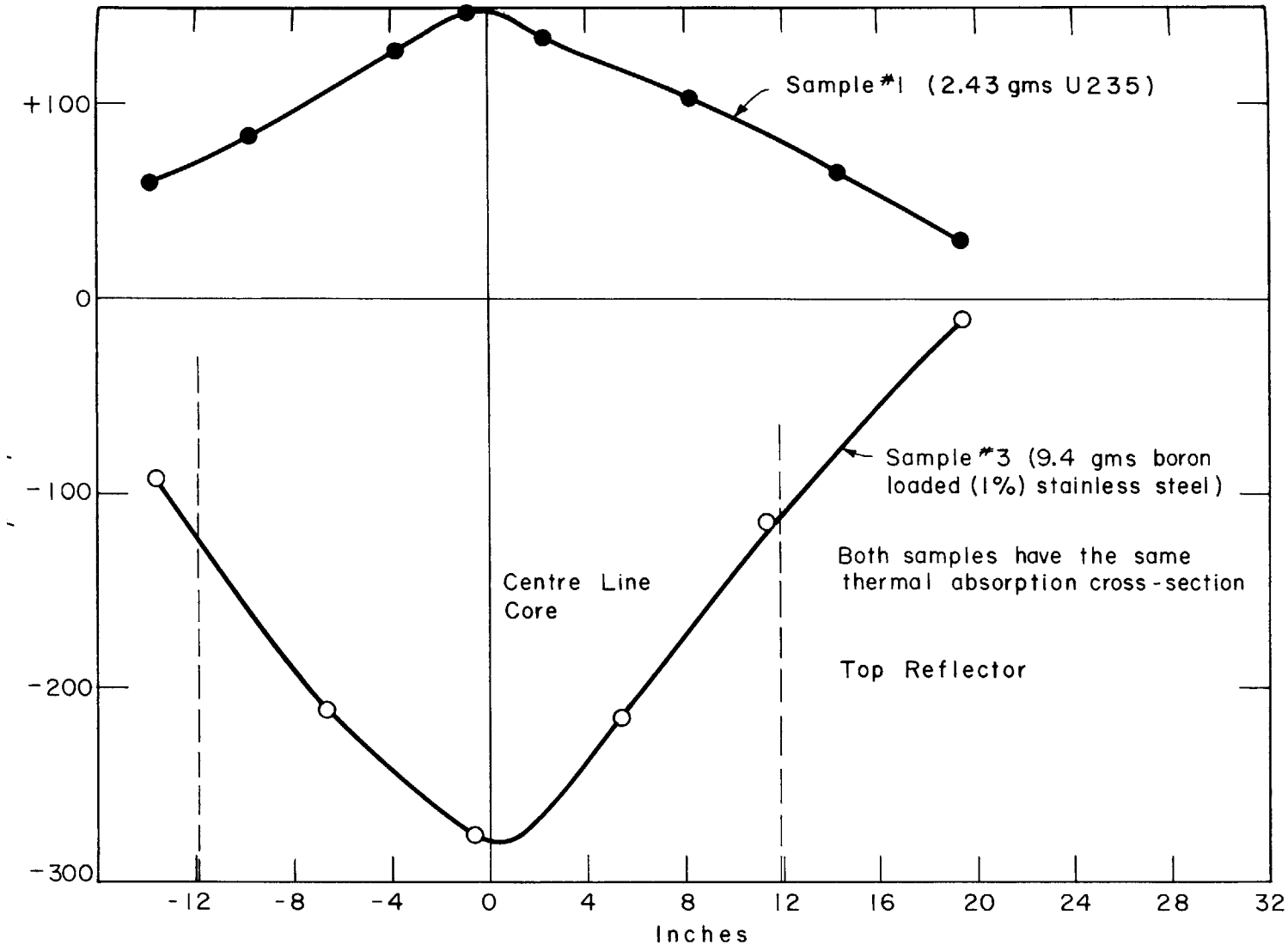
A plot of the ratio of fast to thermal importance bears out theoretical prediction. (Appendix A to Section Four.) Just as the thermal flux tends to rise at a core-reflector interface, the fast importance tends to dip relative to the thermal importance. This variation makes it possible to distinguish experimentally between fast and slow processes and hence add to the information obtainable in analysis.

A number of experiments designed to measure reactor parameters such as η by the effect of a small sample, can in fact only be interpreted as measuring $(\eta^* - 1)$ where η^* is an effective η. Although $\nu\Sigma_f$ neutrons are emitted in fission, their effect is weighted by the local fast importance. Similarly the neutrons absorbed, Σ, are weighted by the thermal importance. Hence the effect of the sample can only be interpreted as

$$\bar{\rho}' \sim \phi_1^+ \nu\Sigma_f - \phi_2^+ \Sigma_a \sim \frac{\phi_1^+}{\phi_2^+} \eta - 1 \qquad (5.41)$$

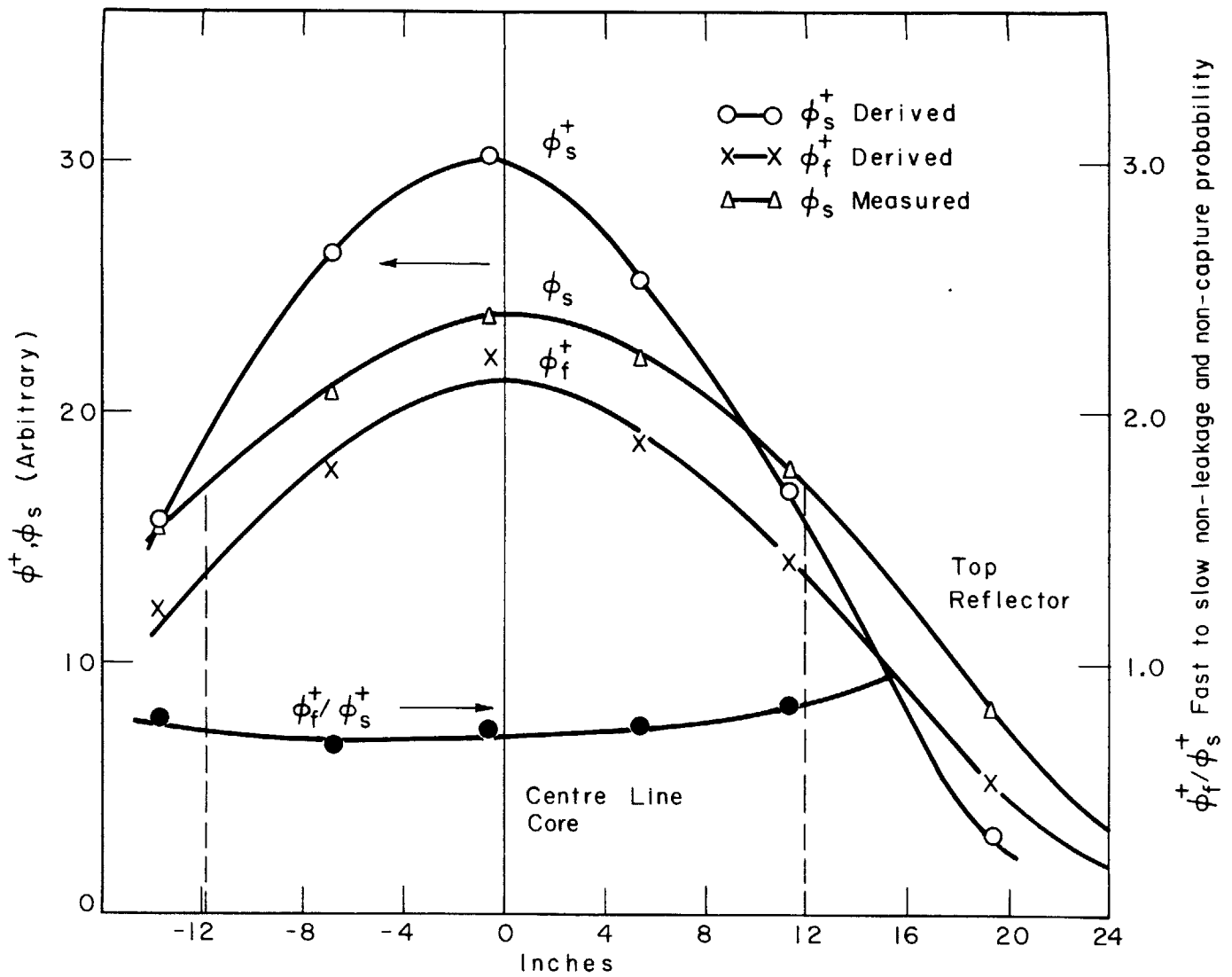
Therefore there are advantages in performing such an experiment away from the centre of the reactor towards the interface, where the ratio ϕ_1^+/ϕ_2^+ is closest to unity, as shown experimentally for the MITR in Figure 5.4. Under these conditions any error in the estimate of the ratio will lead to the smallest resulting error in η.

The idea of interpreting the reactivity effects in different regions of a test reactor has been applied to new and burnt-up fuel elements, in order to determine relative burn-up and fission product effects in thermal, epi-thermal, and fast groups (47).



REACTIVITY WORTHS OF SAMPLES IN VERTICAL SAMPLE TUBE - CENTRAL POSITION OF MITR

FIG. 5.3



DISTRIBUTION OF THERMAL FLUX AND IMPORTANCES IN VERTICAL SAMPLE TUBE - CENTRAL POSITION OF MITR.

FIG. 5.4

In contrast to the source free experiments represented by Figures 5.3 and 5.4, it is also possible to determine the equivalence of neutrons experimentally in a sub-critical reactor with a source. In the experiments briefly reported* in Figures 5.5 and 5.6, a source is moved in an exponential pile with a detector placed in the conventional source position in the pedestal. Also shown are the results of a conventional experiment in the pile with the same source and detector now interchanged, so that we have an experimental determination of the reciprocity relation for a loaded and unloaded pile. The preliminary results indicate that not only is the reciprocal relation valid in both cases but that the novel procedure can offer worthwhile advantages over the conventional use of an exponential pile.

The discussion of reactivity given in this section has been on the basis of its definition in the kinetics equation and its experimental interpretation. In summary, the reactivity is an arbitrary kinetics parameter which together with the effective precursor yield and the generating time, are related to the operational inverse period. The bilinear form enables the first order approximation to be used in calculations. It also enables the reactivity to be related to the perturbation explicitly, e.g., to the added sample alone.

The discussion of detailed calculations of the reactivity together with a new concept, the static reactivity, and the interpretation of

*J. F. Pearson, Jr., and R. B. Sims, An Investigation of Reciprocity in the Exponential Assembly, to be submitted in partial fulfillment for the degree of Master of Science in Naval Architecture, Massachusetts Institute of Technology, June 1959.

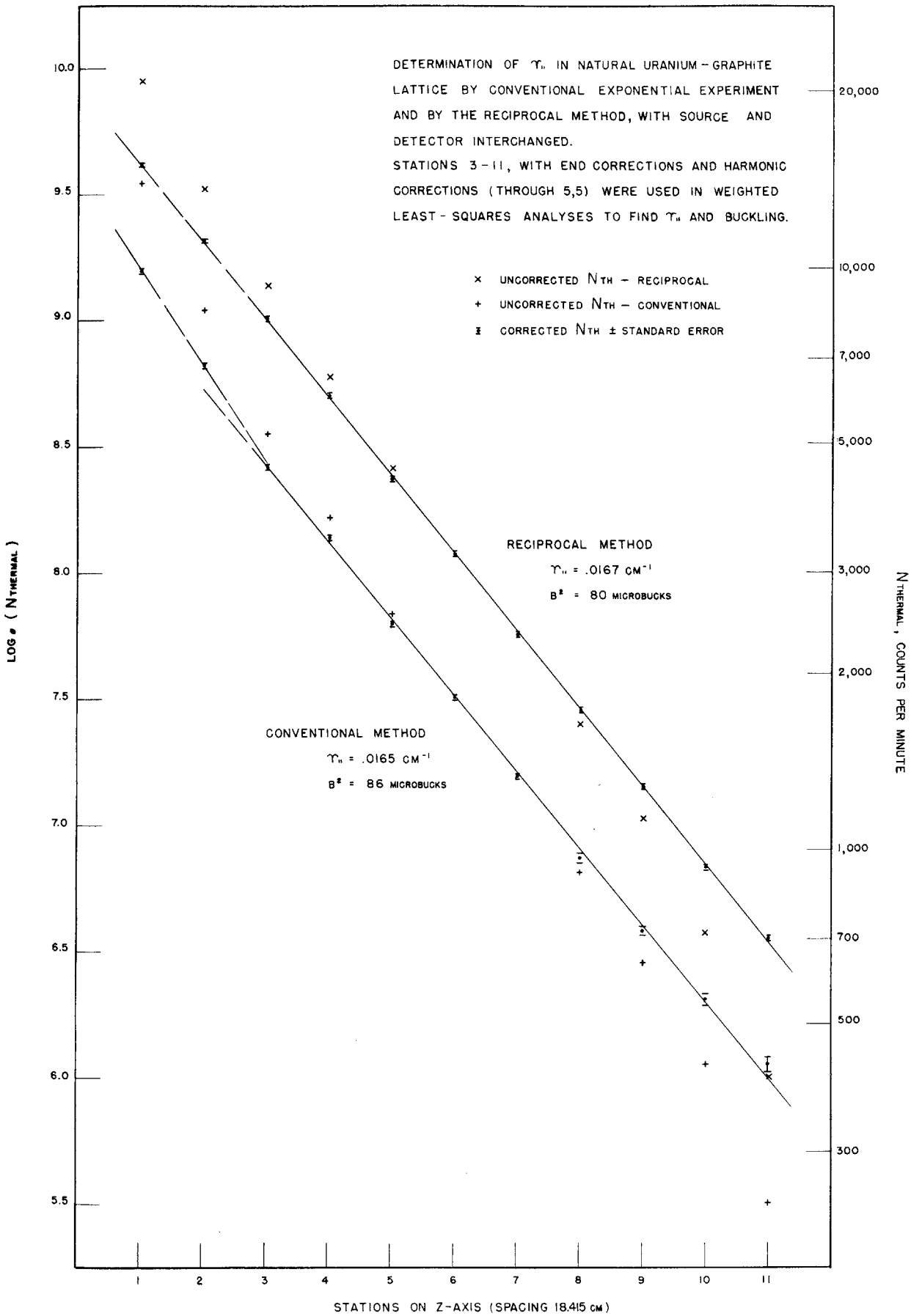


FIGURE 5.6 COMPARISON OF SOURCE MOVED IN LOADED EXPONENTIAL PILE (DETECTOR UNMOVED) WITH CONVENTIONAL RESULT. NOTE THE HIGHER COUNT RATE IN THE NOVEL METHOD DUE TO THE BETTER UTILIZATION OF THE FAST SOURCE, AND ALSO NOTE THE BETTER STRAIGHT LINE FIT OVER THE WHOLE RANGE OF STATIONS.

equivalent perturbations, is treated in Section 6. We shall also develop the enhancement, a concept that expresses through the statistical weight, the geometric effects of various samples or reactor processes.

THE GENERATING TIME

5.7 Interpretation

In the general kinetics expression we have the formal term:

$$\text{Generating time,} \quad \overline{\Lambda}' = \frac{\int \phi^+ V^{-1} \phi' \, dv}{\int \phi^+ F' \phi' \, dv} \tag{5.42}$$

The primes are retained to emphasize that the exact solution of the kinetics problems requires the use of the exact values of the fluxes, ϕ', and the production properties, F'. It is one of the purposes of the bilinear form that we can utilize the first order approximation of $\phi' \approx \phi$ reference.

Corresponding to the generating time, we have an alternative form, the lifetime, $\overline{\ell}'$, based on the destruction probability.

$$\text{Lifetime,} \quad \overline{\ell}' = \frac{\int \phi^+ V^{-1} \phi' \, dv}{\int \phi^+ (-\nabla \cdot D' \nabla + \Sigma') \phi' \, dv} \tag{5.43}$$

Consider the completely artificial case where all the destruction processes in the reactor (including decay from precursors) is stopped. Initially the flux is ϕ' though evidently it will rapidly change in shape as well as increase in magnitude. Instantaneously the logarithmic derivative of the population is given by equation (5.5) with M replaced by F' if there are no destruction processes.

Hence, with a critical reference reactor,

$$s \int \phi^+ V^{-1} \phi' \, dv + s \int \phi^+ C' \, dv = \int \phi^+ F' \phi' \, dv \tag{5.44}$$

This time however, under the assumed conditions that λ is zero,

$$C_i' = \frac{\beta_i F' \phi'}{s} \tag{5.45}$$

Hence

$$s \frac{\int \phi^+ V^{-1} \phi' dv}{\int \phi^+ F' \phi' dv} + \sum_i \frac{s}{s} \frac{\int \phi^+ \beta_i F' \phi' dv}{\int \phi^+ F' \phi' dv} = 1 , \tag{5.46}$$

or

$$s \bar{\Lambda}' + \sum_i \bar{\beta}' = 1 \tag{5.47}$$

Hence

$$s = \frac{1 - \bar{\beta}'}{\bar{\Lambda}'} \tag{5.48}$$

The instantaneous period, $1/s$, is therefore $\bar{\Lambda}'/(1-\bar{\beta}')$ or quite closely, the generating time gives the instantaneous period if all destruction processes were stopped (including precursor decay). If delayed neutrons are neglected, this interpretation yields T (instantaneous) is $\bar{\Lambda}'$ exactly.

A corresponding interpretation can be made for the lifetime, $\bar{\ell}'$. Consider the instantaneous period when all production processes (prompt neutrons and precursors) is mysteriously stopped. Then the precursors decay with their separate periods, $s = -\lambda_i$. The neutron population is governed by

$$s \int \phi^+ V^{-1} \phi' dv = \int \phi^+ (+\nabla \cdot D' \nabla - \Sigma') \phi' dv + \lambda \int \phi^+ C' dv \tag{5.49}$$

Equation (5.49) cannot be solved without knowledge of the initial conditions, due to the uncoupling that we have assumed between the precursor equations and the neutron equations. If we further assume that

the decay probability, λ, is zero we have, however,

$$s\bar{\ell}' = 1 \tag{5.50}$$

Thus the lifetime can be interpreted as the instantaneous period after all production and precursor decay is stopped.

It will be evident from the above discussion that the two interpretive concepts given are not operationally measurable. The flux and population may finally settle on a resident and measurable period. The new asymptotic flux shape however will in general be different from ϕ'.

For a critical system the production and destruction probabilities are equal and $\bar{\Lambda}' = \bar{\ell}'$. For most purposes it is sufficient to calculate either $\bar{\Lambda}$ or $\bar{\ell}$ for the critical reference reactor. Allowance must be made for the actual properties. For instance in calculating the lifetime, proper allowance must be made for the effective absorption cross-section of control rods. Alternatively, if a clean cold reactor has been calculated using a reduced fictitious value of ν, the proper production probability should employ the true physical value of ν. This is equivalent to allowing for hot operating conditions, control rods, etc.

A common hand calculation uses the two group formulation in which

$$\Lambda' = \frac{\int \left(\frac{\phi_1^+ \phi_1'}{V_1} + \frac{\phi_2^+ \phi_2'}{V_2} \right) dv}{\int \phi^+ F' \phi' \, dv} \tag{5.51}$$

The two group representation of the fast fission effect is discussed in Appendix B to this section. The form of equation (5.51) has been given or utilized by a number of authors (e.g., 52). It is to be noted that the literal interpretation lends to our generating time: the lifetime is given

by the expression containing the weighted destruction terms rather than the production.

It might be remarked that the two group formulation of reactor kinetics has not always been borne out experimentally. Rumsey (53) has given a treatment of slowing down theory which has been verified in heavy water reactors. Other multigroup approaches have been employed (e.g., 55). Experience in the MIT Reactor so far is that the two group calculated values of generating time and the effective yield (58) lead to a set of consistent relations for the reactivity/period relations.

Commonly the generating time will be calculated for a critical reactor, i.e., the reference reactor itself. Equation (5.51) can be expressed in two parts:

$$\bar{\Lambda} = \bar{\Lambda}_1 + \bar{\Lambda}_2 , \tag{5.52}$$

where we define the group generating time, Λ_i, by

$$\bar{\Lambda}_i = \frac{\int \frac{\phi_i^+ \phi_i \, dv}{V_i}}{\int \phi^+ F \phi \, dv} \tag{5.53}$$

Evidently these group generating times are also equal to the group lifetimes based on destruction, when the reactor is critical.

5.8 Point System Generating Time

The point system values for Λ_1 and Λ_2 are easily evaluated:

$$\Lambda_1 = \ell_1 = \frac{1}{V_1 \Sigma_1 (1 + \tau B^2)} \tag{5.54}$$

$$\Lambda_2 = \ell_2 = \frac{1}{V_2 \Sigma_2 (1 + L^2 B^2)} \qquad (5.55)$$

These values are appropriate for a 'true' point system and have often been applied to a reflected reactor using the concept of reflector savings and the equivalent bare core.

The use of an equivalent bare core is not always valid in calculating lifetimes, even though a good estimate of the critical mass may be obtained from the reactor savings concept. For a small reactor, especially in a D_2O reactor like the MITR with its low reflector absorption (and hence smaller required production probability), the discrepancy can be appreciable. In the multiregion calculation, the reflector tends to increase the effective lifetime, even though it has the bilinear weighting (i.e., roughly ϕ^2, low in the reflector). Calculations for the MITR showed an appreciable effect on the two group generating time when the bilinear rather than the linear form was employed – see Table 5.2.

Table 5.2

Calculated Generating Time for an MITR Type Reactor
(Enriched uranium/D_2O moderated and reflected)

1.	Λ_o, ℓ_o	Core infinite medium generating time	0.46 milli-sec
2.	Λ, ℓ	Core finite medium generating time	0.37
3.	Λ_r, ℓ_r	Reflector finite medium generating time	7.91
4.	$\overline{\Lambda}, \overline{\ell},$	Bilinear weighted generating time for reactor	1.24
5.	$\mathcal{L}$	Linear weighted generating time for reactor	1.49

The values of Table 5.2 were calculated for cold clean criticality

and not the operating condition. In items 2 and 3, the equivalent bare core non-leakage probability, $1/(1 + L^2B^2)$, was employed. The curtailment of the production integral at the true core brings up the generating time from 0.37 to 1.24 milli-secs. Alternatively we can say the low absorption in the reflector brings up the average lifetime the same amount.

The bilinear value reported here for $\bar{\Lambda}$ is some 20 per cent lower than the linear averaged lifetime based on $\int I F \phi \, dv$. Although for small reactivities, the error in the lifetime may not affect the inhour equation appreciably, it should be noted that calculations of reactivities using a linear averaging scheme will differ from the bilinear form by exactly the same amount in this reactor.

5.9 Group Speeds V

In the expressions given for either the general or the one region case, the inverse group speeds, V_1^{-1}, and V_2^{-1} appear and values are required for numerical applications. There has been a certain amount of confusion in the literature (41) on how appropriate values are obtained. First it should be recollected that the kinetics expression are for the rate of change of population or of neutron density. The reciprocal speeds appear in the form $V^{-1} \phi$ to represent such a neutron density. The flux ϕ can here be regarded as a non-operational, arbitrarily defined concept which together with the cross-sections, Σ, correctly represent the probability of neutron interaction. Thus the question of the appropriate speeds centres around the definitions of flux and cross section.

For the thermal group speed, Weinberg and Wigner (1) have shown that for a fair scattering medium or better, the use of the Maxwell Boltzmann averaged cross-sections and 'true' flux requires the use of the Maxwell-Boltzmann averaged speed, $\overline{V}$, for consistency. If however thermal cross-sections based on Westcott's effective cross-section treatment (59) are employed, then by definition the thermal flux is V_o times the thermal neutron density. Hence we should use V_o, 2200 m/sec to find the neutron density from the flux.

For a two group calculation, the value of the fast group speed, V_1, can be found as follows. The conventional derivation of multigroup removal cross-sections is based on a concept of 'true' flux [see Glasstone and Edlund, for example, (33)]. That is, the fast flux, ϕ_1, that we calculate is related to the neutron density by

$$\phi_1 = \int_{V_t}^{V_o} n(V)\, V\, dV \tag{5.56}$$

From experiment and calculation, the flux spectrum is 1/E or

$$\phi(E)\, dE = \frac{\phi_o}{E}\, dE \tag{5.57}$$

As a function of velocity however, we have

$$\phi(V) = \phi(E)\, \frac{dE}{dV} = \frac{\phi_o}{V}\,, \tag{5.58}$$

and

$$n(V) = \frac{\phi_o}{V^2} \tag{5.59}$$

Consistency requires that

$$\frac{\phi_1}{V_1} = n = \int_{V_t}^{V_o} n(V)\, dV \,, \tag{5.60}$$

or

$$V_1 = \frac{\int_{V_t}^{V_o} \frac{dV}{V}}{\int_{V_t}^{V_o} \frac{dV}{V^2}} \tag{5.61}$$

We have

$$V_1 = \frac{\ell n \frac{V_o}{V_t}}{\left[\frac{1}{V}\right]_{V_t}^{V_o}} \tag{5.62}$$

For large V_o corresponding to 2 Mev, we can approximate equation (5.62) by

$$V_1 \approx V_t \ln \frac{V_o}{V_t} = \frac{V_t}{2} \ln \frac{E_o}{E_t} \tag{5.63}$$

We can take V_t to correspond to a 5kT cut-off energy (or to whatever cut-off energy was employed in calculating Σ_1 and D_1) and V_o as the 2 Mev mean fission energy. Then for kT = 0.025 ev,

$$V_1 = \frac{2200}{2} \quad 5 \ln \frac{2 \times 10^6}{0.026 \times 5} \tag{5.64}$$

$$= 40,600 \text{ metres/sec} \,,$$

compared with

$$V_2 = 2,200 \text{ metres/sec} \tag{5.65}$$

Fortunately, for most thermal reactors, the slowing down contribution is significantly smaller than the thermal contribution so that an

approximate calculation of this nature suffices. This might not be the case for hydrogeneous reactors or for intermediate and fast reactors. However, the basic principle remains, that we must maintain a consistent definition of the group speeds such that the neutron density is correctly represented.

5.10 Time Absorption

A further aspect of the generating time arises from the interpretation of the time variation of density as a type of absorption term, the time absorption. If a reactor population is increasing, the production must supply more neutrons than are actually destroyed. This extra requirement can be thought of as an additional absorption. Since on an inverse period, s, the extra neutrons are sn for a neutron density n, or $sV^{-1}\phi$ in terms of the flux, the equivalent absorption cross-section is (sV^{-1}). If the period is unity, $s = 1$, the effective absorption is just V^{-1}. This effective absorption behaves as a perturbation consisting of the addition of a 1/V absorption cross-section to each group. Thus the generating time can be interpreted as the reactivity which would give a unit period:

$$\frac{\int \phi^{+} V^{-1} \phi' \, dv}{\int \phi^{+} F' \phi' \, dv} = \bar{\rho}' \qquad (5.66)$$

in a system in which the perturbation, P, is the added equivalent 1/V absorber, $P = V^{-1}$. Thus generating time can be measured experimentally by the uniform insertion of a 1/V absorber. To prevent severe distortion of the flux, it is useful to introduce a fractional increment only, XV^{-1}, where X can be made smaller and smaller until there is no flux perturbation. Then if ρX is the reactivity effect of the added XV^{-1} absorber,

we have

$$\bar{\rho} = \mathop{\mathcal{L}}_{X\to 0} \frac{1}{\bar{X}} \frac{\int \phi^+ X V^{-1} \phi \, dv}{\int \phi^+ F \phi \, dv} \tag{5.67}$$

The technique of utilising the perturbation equivalent to the time absorption is well known (e. g., 51). It is to be noted however that measurements of reactivity lead strictly to our generating time; the lifetime is obtained by measurements of the excess multiplication.

EFFECTIVE PRECURSOR YIELDS

5.11 Interpretation

The kinetics equation yielded convenient parameters for the precursor yield that we called the effective yield, $\bar{\beta}_i'$:

$$\bar{\beta}_i' = \frac{\int \phi^+ \beta_i F' \phi' \, dv}{\int \phi^+ F' \phi' \, dv} \tag{5.68}$$

Although Nordheim et al in (15) first gave the 2 group inhour equation, the important role of the effective precursor yield was apparently first recognised by de Hoffman (54). Again we emphasise that the 'effectiveness' is the result of the bilinear weighting used to improve the accuracy when the first order approximation is used, i. e., when $\bar{\beta}_i$ of the reference reactor is employed. Only in calculating the $\bar{\beta}_i$ do we need to take into account the energy differences between delayed neutrons and prompt neutrons. In calculating $\bar{\rho}'$ and $\bar{\Lambda}'$ we could use flux distributions which neglected these differences. As some compensation for the equations now needed, it is often possible to calculate the $\bar{\beta}_i'$ (or more strictly, the $\bar{\beta}_i$) using an effective bare core. This approximation will be appropriate for a uniformly loaded core when the flux

shape in the core is well given by the equivalent bare core, since the integrals of equation (5.68) are only evaluated over the true core. We consider the bare core approximation first and later discuss techniques that can be employed when the approximation is invalid.

Consider a one region reactor with three groups of neutrons, the minimum number of equations with which it is possible to represent the fast energy differences. The logical basis for the calculation of the Fermi-age for the two fast groups is discussed in Appendix B to this Section. For a critical system, where the precursor densities can be eliminated, we have the remaining system of equations

$$\begin{aligned} -(D_1 B^2 + \Sigma_1)\,\phi_1 + (1-\beta)\,F\,\phi_3 &= 0 \\ -(D_2 B^2 + \Sigma_2)\,\phi_2 + \beta\,F\,\phi_3 &= 0 \\ p_1\,\Sigma_1\,\phi_1 + p_2\,\Sigma_2\,\phi_2 - (D_3 B^2 + \Sigma_3)\,\phi_3 &= 0 \end{aligned} \qquad (5.69)$$

We have a thermal feed into the prompt and delayed groups. Both fast groups however feed directly into the thermal group. This partial uncoupling loses very little accuracy. Although fully coupled, we would have a three group system, the energy difference between the first and second groups is so small that little accuracy would be gained. By the partial uncoupling we introduce many zeros into the matrices representing the equations and simplify the results appreciably.

It is convenient to use the actual critical reactor as the reference reactor. The adjoint equations of the reference reactor are then

$$-(D_1 B^2 + \Sigma_1)\,\phi_1^+ + 0 + p_1\,\Sigma_1\,\phi_3^+ = 0,$$

$$0 - (D_2 B^2 + \Sigma_2)\,\phi_2^+ + p_2\,\Sigma_2\,\phi_3^+ = 0,$$

and

$$(1-\beta)\,F\,\phi_1^+ + \beta\,F\,\phi_2^+ - (D_3 B^2 + \Sigma_3)\,\phi_3^+ = 0$$

Hence we have the coupling coefficients:

$$\phi_1^+ = \frac{p_1}{(1 + \tau_1 B^2)}\,\phi_3^+,$$

$$\phi_2^+ = \frac{p_2}{(1 + \tau_2 B^2)}\,\phi_3^+. \qquad (5.71)$$

The interpretation of these coupling coefficients is obvious. A neutron in group one will have some probability of leakage or capture before reaching the thermal group. Hence one neutron in group one is only as important as the $p_1/(1 + \tau_1 B^2)$ neutrons it will become in group 3. Similarly $p_2/(1 + \tau_2 B^2)$ is just the probability a delayed neutron has of reaching thermal.

The production matrices required for this example are, explicitly,

$$\begin{bmatrix} 0 & 0 & (1-\beta)\,F \\ 0 & 0 & \beta\,F \\ 0 & 0 & 0 \end{bmatrix} = F \qquad (5.72)$$

and

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta\,F \\ 0 & 0 & 0 \end{bmatrix} = \beta\,F \qquad (5.73)$$

Thus the effective yield is

$$\bar{\beta} = \frac{\int \phi^+ \beta F \phi \, dv}{\int \phi^+ F \phi \, dv} = \frac{\int \phi_2^+ \beta F \phi_3 \, dv}{\int \phi_1^+ (1-\beta) F \phi_3 + \phi_2^+ \beta F \phi_3 \, dv} \tag{5.74}$$

$$= \frac{\dfrac{p_2 \beta}{1 + \tau_2 B^2}}{\dfrac{p_1 (1-\beta)}{1 + \tau_1 B^2} + \dfrac{p_2 \beta}{1 + \tau_2 B^2}} \tag{5.75}$$

This expression for $\bar{\beta}$ can be interpreted as follows: The effective yield is the fraction of neutrons returning to thermal that have come from precursors.

Under the conditions that β and $(\tau_1 - \tau_2)$ are both small compared to unity, we may write the approximation that

$$\gamma = \frac{\bar{\beta}}{\beta} = \frac{p_2}{1 + \tau_2 B^2} \Big/ \frac{p_1}{1 + \tau_1 B^2} \tag{5.76}$$

In general, the ratio $\bar{\beta}/\beta$ is called the effectiveness, γ. Strictly there will be a different effectiveness, γ_i, for each group. Often it is practical to calculate just one value for an average delayed neutron value.

5.12 Mixed Species and Fast Fission

Quite significant effects on the value of the effective yields arise when mixed fissionable fuels are present. These effects may be present at the start of the reactor life, when the loading varies across the reactor, or may build-up due to the presence of fertile material. In addition, the occurrence of fast fission effects may complicate the dependence, especially if the fast flux varies in shape from the thermal flux. The most convenient approach is to write separate equations for

each type of production and for each resulting decayed neutron flux. Thus although we may have an i^{th} precursor group characterized by the decay constant, λ_i, if there are j fissionable materials, consider each yield β_{ij} as a separate yield for the purpose of finding the $\bar{\beta}_{ij}$. Then the total effective yield of the i^{th} species is just the sum over the j materials. Of course in evaluating the $\bar{\beta}_{ij}$, these will be the fractional yield in the pure material multiplied by the fraction of that fissionable material present. When fission takes place in more than one group we can again treat the yields by writing separate equations for each and finally summing the i^{th} decay type to give the i^{th} effective yield. A discussion of these effects has been given by Henry (67) and a more detailed set of calculations by Jarvis (56).

5.13 Multiregion Reactors

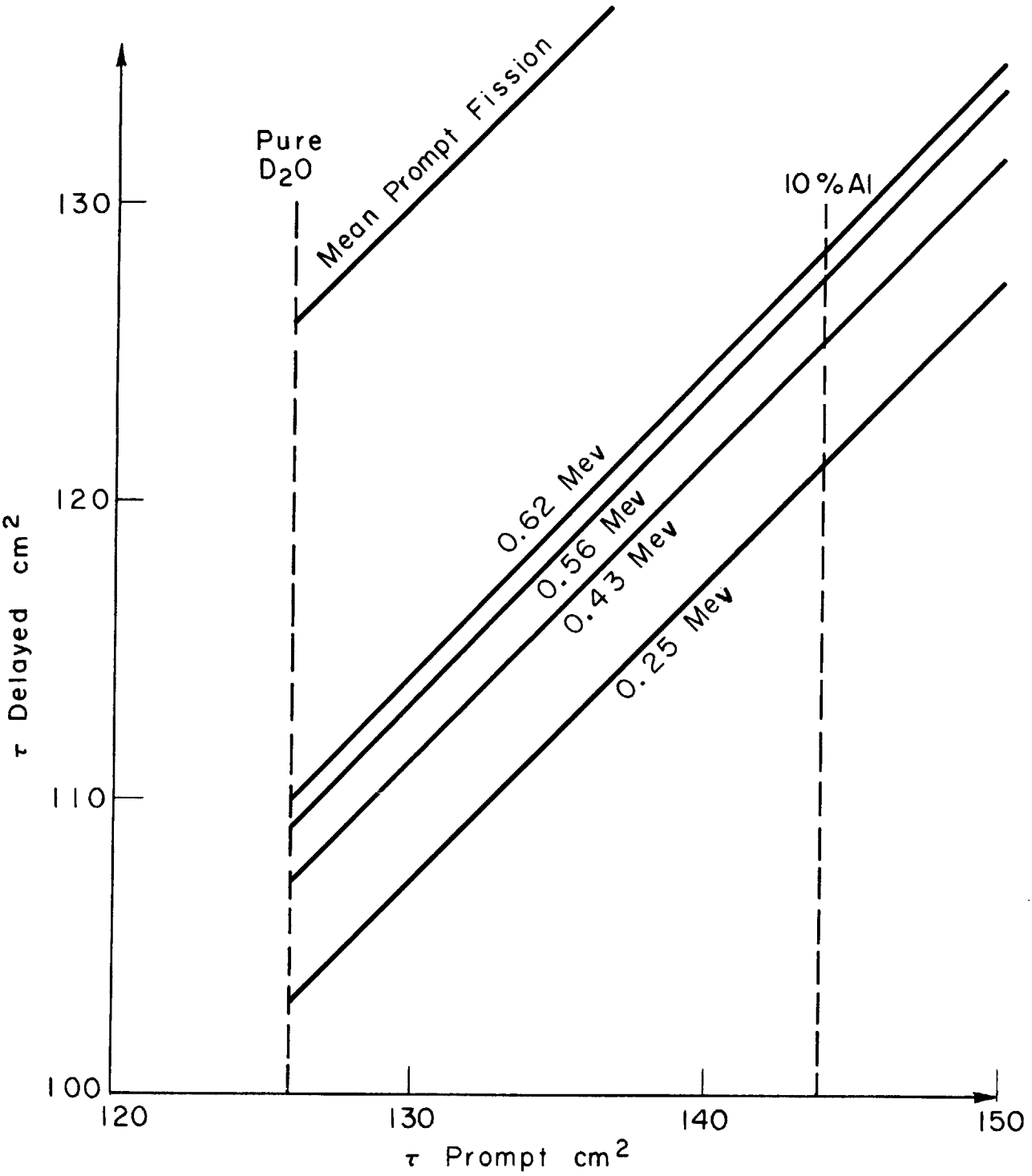
For most cases it is sufficient to evaluate the effective yield on the basis of a one region reactor. We then have expressions of the form $p_i P_i$ to evaluate, where p_i is the resonance escape probability and P_i the fast non-leakage probability. Since resonance absorption takes place at energies below the energies of both delayed neutrons and prompt neutrons, the escape probability is essentially the same for both groups. On the other hand, the two and a half group representation includes all fast and resonance absorption in the term $p\Sigma_1$. Thus in some systems, the difference in true fast absorption may lead to effects equivalent to differences in resonance escape.

The fast leakage effects are reducible to finding the appropriate differences in the Fermi-age for the delayed neutrons in the slowing down

medium and finding the appropriate B^2 to represent the actual core behaviours. For the MIT Reactor, it was necessary to calculate (τ delayed/ τ prompt) for various mixtures of heavy water to aluminum. The calculations were made along the lines of the original calculation for the age of fission neutrons in these mixtures, (48). The results are summarized in Figure 5.7.

The selection of the appropriate B^2 for the bare reactor approximation can be based on considerations of the fast leakage. In a heavy water reactor with no appreciable resonance or fast absorption, it is of course the fast leakage that governs the difference of importance between the energy groups. Thus as an approximation to a bare reactor, it is possible to use the B^2 corresponding to the fast flux shape in the core, i.e., mid-way between the material and geometric bucklings as in Figure 5.8. For the MIT Reactor this approximation had an error of the same order as the basic data when compared with a three group calculation of a reflected reactor.

The general equations can be solved analytically if there are no more than about two regions and three energy groups. Thus calculations for two or three energies can be done to obtain plots of a one group $\bar{\beta}$ parametric in energy. From such a plot the appropriate values of the $\bar{\beta}_i$ can be interpolated. Even this limited number of regions leads to a six by six determinant as a minimum. This can be very tedious. An alternative analytic approach is based on the fact that the fast importance differs for prompt and delayed neutrons essentially in the coupling coefficient to the thermal importance. Thus if a prompt calculation (neglecting delayed neutrons) has been undertaken, to find the generating



FERMI-AGE FOR DELAYED NEUTRONS IN D_2O -AL MIXTURES
(Contours in Delayed Neutron Energies, Data of Hughes)

FIGURE 5.7

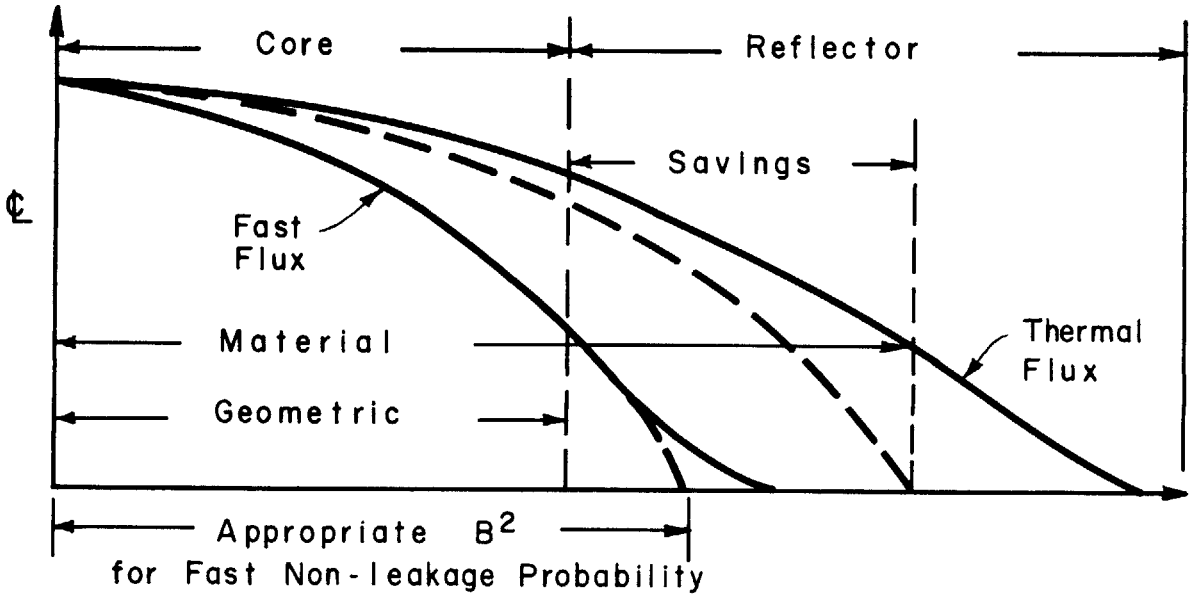


FIG5.8 TYPICAL U235/D_2O FLUX DISTRIBUTION. FAST NON-LEAKAGE PROBABILITY GIVEN BY A BUCKLING SQUARED MID-WAY BETWEEN CORE GEOMETRIC AND MATERIAL BUCKLINGS

time, we have available the solution to the prompt importance. In a two group, two region slab in particular, we have in the core:

$$\phi_1^+ = A^+ \frac{p_1}{(1 + \tau_1 \mu^2)} \cos \mu x + C^+ \frac{p_1}{(1 - \tau_1 \nu^2)} \cosh \nu x \qquad (5.77)$$

Then an excellent approximation, which is exact in the point system, is to take

$$\phi_2^+ = A^+ \frac{p_2}{(1 + \tau_2 \mu^2)} \cos \mu x + C^+ \frac{p_2}{(1 - \tau_2 \nu^2)} \cosh \nu x \qquad (5.78)$$

evaluating the effectiveness, γ, on the basis of the original two group solutions.

For more complicated systems, where hand calculations become impractical, recourse can be had to either machine calculations or to certain experiments. Certain machine codes are available that will calculate the reactivity worth of samples and the effective yield can indeed be interpreted as the contribution made by the delayed neutrons to the reactivity. Alternatively, consider the $\bar{k}_{ex}$ resulting from the removal of the precursor contribution to the population balance. We have

$$\bar{k}''_{ex} = \frac{-\int \phi^+ \beta_i F \phi'' \, dv}{\int \phi^+ (-\nabla \cdot D \nabla + \Sigma) \phi'' \, dv} = \frac{-\int \phi^+ \beta_i F \phi'' \, dv}{\int \phi^+ F \phi'' \, dv} = \bar{\beta}''_i \qquad (5.79)$$

where we have employed the relation that

$$(-\nabla \cdot D \nabla + \Sigma^T) \phi^+ = F^T \phi^+ \qquad (5.80)$$

in the reference reactor. We see that the effective yield, to first order (ϕ for ϕ''), is the $\bar{k}_{ex}$ that would result on removing the particular precursor before it could decay to a neutron. This result differs slightly

from that of Abrams (49) who interprets $\bar{\beta}$ as a change of reactivity rather than a change of $\bar{k}_{ex}$. In interpreting machine codes, it is necessary to make sure that the calculation in fact gives the bilinear form of the reactivity and not say a linear form, since the difference between these results (20 % for the MITR) could lead to an entirely wrong result for $\bar{\beta}$. Alternatively, the code should compute the static reactivity, to be discussed in Section 6. Attempts have been made in the literature to compare the one region approximation with the true value for $\bar{\beta}'$ be computing the 'true' value as the linearly weighted non-leakage probability (49). It will be appreciated from our discussion that although the energy aspects of the effectiveness are the most important, when spatial variations have to be taken into account, it is essential, for a consistent result, to employ the bilinear weighting.

Corresponding to machine calculations of the reactivity, it is possible to interpret 'substitution' experiments in which the reactivity effect of 1/V absorbers is measured as a determination of the total effective yield, $\bar{\beta}$. Such experiments depend for their interpretation on a calculation of $\bar{\rho}$ and $\bar{\Lambda}$ in order that inverse periods can be directly related to $\bar{\beta}$. (50, 54, etc.) The inhour relation, equation (5.25), can be manipulated to give

$$\frac{s\bar{\Lambda}'}{\bar{\beta}'} + \sum_i \frac{s\bar{\Lambda}'}{\bar{\beta}'} \quad \frac{\bar{\beta}_i'/\bar{\beta}'}{\frac{\lambda_i\bar{\Lambda}'}{\bar{\beta}'} + \frac{s\bar{\Lambda}'}{\bar{\beta}'}} = \frac{\bar{\rho}'}{\bar{\beta}'} = \rho^* \qquad (5.81)$$

where $\bar{\rho}'/\bar{\beta}'$ is a fiducially measured reactivity corresponding to the well known dollars and cents unit. In this form, it is apparent that since the kinetic parameters appear only in ratio form, it is impossible to

determine any one of them directly from experiment unless some value is assumed for the remainder. If the common, and usually apt, assumption is made, that the effectiveness, γ_i, of all the delayed neutron groups has the same value, then $\bar{\beta}_i/\bar{\beta}$ reduces to the experimentally measurable relative yield, a_i. It is then possible to plot the left hand side of equation (5.81) as a universal function of $(s\bar{\Lambda}'/\bar{\beta}')$ for a particular fuel, where the decay constants and relative yields are known, independent of the effectiveness in the particular reactor under consideration. Such a plot is an extension of the Smets Z function (45, 60). Some further considerations are given in Appendix A to this Section.

It is seen from equation (5.81), that if the left hand side is indeed a function determined only by the value of $s\bar{\Lambda}'/\bar{\beta}'$, then measurements or calculations of $\bar{\rho}'$ and $\bar{\Lambda}'$ lead to the value of $\bar{\beta}'$.

THE GENERAL NON-CRITICAL REFERENCE REACTOR

5.14 Generalised Inhour Relation

We now investigate the nature of the results when a reference reactor is employed that is not necessarily critical. First we consider the use of arbitrary importances, i.e., that the importance of a precursor is not necessarily equal to the importance of the neutron it will become. We have that

$$\frac{d}{dt}\int \psi^+ A\psi' \, dv = \int \psi^+ (M+T)\,\psi' \, dv \qquad (5.82)$$

or

$$\frac{d}{dt}\int \phi^+ V^{-1} \phi' \, dv + \frac{d}{dt}\int C^+ C' \, dv =$$

$$\int \phi^+ [\nabla \cdot D'\nabla - \Sigma' + (1-\beta) F'] \phi' \, dv + \sum_i \int C_i^+ \beta_i F' \phi' \, dv + \sum_i \int (\phi_i^+ - C_i^+) \lambda_i C_i' \, dv$$

$$= \int \phi^+ M \phi' \, dv + \sum_i \int (C_i^+ - \phi_i^+)(\beta_i F' \phi' - \lambda_i C_i') \, dv \qquad (5.83)$$

Then introducing the precursor equations themselves,

$$\frac{d}{dt} \int (C_i^+ - \phi_i^+) C_i' \, dv = \int (C_i^+ - \phi_i^+) (\beta_i F' \phi' - \lambda_i C_i') \, dv \, , \tag{5.84}$$

equation (5.83) becomes

$$\frac{d}{dt} \int \phi^+ V^{-1} \phi' \, dv + \frac{d}{dt} \int \phi^+ C' \, dv = \int \phi^+ M \phi' \, dv \tag{5.85}$$

Equation (5.85) has the same form as the previously derived kinetics relation. The result is not surprising since the kinetic behaviour of the actual reactor is not fundamentally dependent on the arbitrarily selected reference reactor. When we attempt to express the inverse period in terms of a perturbation however, there will be a fundamental dependence on the reference properties, the reference inverse period, and the perturbation as far as the value of the kinetics parameters is concerned.

Equation (5.84) can be reduced to an inhour equation of the conventional form by the introduction of the eigenvalues and the division by the weighted production rate per neutron. We distinguish between the inverse period of the actual reactor, s', (with multiple solutions to be anticipated in the lowest harmonic) and the eigenvalue of the importance and reference reactor, s. The corresponding equations are

$$s' A \psi' = (M+T) \psi' \, , \tag{5.86}$$

and

$$s A \psi^+ = (R+T)^T \psi^+ \tag{5.87}$$

We neglect any question of varying the group speeds (A unchanged) or the precursor decay probability (T unchanged). Nominally equation (5.87) implies that precursors are included in the solution for the reference reactor. If for mathematical purposes, we prefer to base the

reference reactor on a model neglecting precursors, we can take the fractional yield of precursors to be zero in the matrix R. In this case, however, the subsequent perturbation must include the introduction of the fractional yield to precursors.

From equations (5.86) and (5.87), we may form the bilinear integral and express the results in terms of perturbations only:

$$s' \int \psi^+ A \psi' \, dv = \int \psi^+ (R+T) \psi' \, dv + \int \psi^+ P \psi' \, dv$$

$$= s \int \psi^+ A \psi' \, dv + \int \psi^+ P \psi' \, dv \qquad (5.88)$$

or

$$(s-s') \int \psi^+ A \psi' \, dv = \int \psi^+ P \psi' \, dv \qquad (5.89)$$

Then the kinetic equation (5.85) can be expressed in terms of the period of the reference reactor and the perturbation in properties. To compare the two forms we write both:

$$(s'-s) \left[\int \phi^+ V^{-1} \phi' \, dv + \int C^+ C' \, dv \right] = \int \psi^+ P \psi' \, dv , \qquad (5.90)$$

$$s' \left[\int \phi^+ V^{-1} \phi' \, dv + \int \phi^+ C' \, dv \right] = \int \phi^+ M \phi' \, dv \qquad (5.91)$$

The choice of the dividing factor to turn either of these equations into conventional form is arbitrary. The best choice is probably to use that value of weighted production such that in the one energy case, $\bar{\beta}'$ reduces to β. We obtain

$$(s'-s)\ \frac{\int \phi^+ V^{-1} \phi'\, dv}{\int \psi^+ F' \psi'\, dv} \int \phi^+ V^{-1} \phi'\, dv + \frac{\int C^+ C'\, dv}{\int \psi^+ F'\, \psi' dv} \int \phi^+ V^{-1} \phi'\, dv$$

$$= \frac{\int \psi^+ P \psi'\, dv}{\int \psi^+ F' \psi'\, dv} \int \phi^+ V^{-1} \phi'\, dv \qquad (5.92)$$

Then if the kinetic parameters are generalised to the form

$$\tilde{\Lambda}' = \frac{\int \phi^+ V^{-1}\, \phi'\, dv}{\int \psi^+ F'\, \psi'\, dv} = \frac{\int \phi^+ V^{-1}\, \phi'\, dv}{\int \psi^+ F'\, \phi'\, dv}\ ,$$

$$\tilde{\beta}_i' = (\lambda_i + s') \frac{\int C_i^+ C_i'\, dv}{\int \psi^+ F' \psi' dv} = \frac{\int C_i^+ \beta_i F' \phi' dv}{\int \psi^+ F'\, \psi'\, dv}\ ,$$

and

$$\tilde{\rho}' = \frac{\int \psi^+ P \psi'\, dv}{\int \psi^+ F' \psi' dv} \qquad (5.93)$$

we have

$$(s'-s)\ \left[\tilde{\Lambda}' + \sum_i \frac{\tilde{\beta}_i'}{\lambda_i + s'}\right] \bar{n}' = \tilde{\rho}'\, \bar{n}' \qquad (5.94)$$

or the modified inhour equation:

$$s'\breve{\Lambda}' = s\left[\breve{\Lambda}' + \sum_i \frac{\breve{\beta}_i'}{\lambda_i + s'}\right] + \breve{\rho}' - \breve{\beta}' + \sum_i \frac{\lambda_i \breve{\beta}_i'}{\lambda_i + s'} \tag{5.95}$$

This form of the inhour equation differs from the conventional form through the appearance of the term $\left(\breve{\Lambda}' + \sum_i \frac{\breve{\beta}_i'}{\lambda_i + s'}\right) s$, which would evidently vanish for the critical reference reactor. The relation between the new form of the reactivity and the original form is given by

$$\bar{\rho}' = \breve{\rho}' + s\left(\breve{\Lambda}' + \sum_i \frac{\breve{\beta}_i'}{\lambda_i + s'}\right) \tag{5.96}$$

An inhour form is also obtained by the operation of dividing equation (5.91) by the production term $\int \phi^+ F' \phi' \, dv$. Although a perfectly admissible operation, the divisor is not the weighted total production of precursors and neutrons, for the general non-critical reactor.

Furthermore, the form of equation (5.91) does not include explicit reference to either a perturbation, or to the inverse period of the reference reactor. With the conventional definition of the kinetics parameters therefore we obtain the usual inhour form:

$$s'\bar{\Lambda}' = \bar{\rho}' - \bar{\beta}' + \sum_i \frac{\lambda_i \bar{\beta}_i'}{\lambda_i + s'} \tag{5.97}$$

5.15 Application of Non-critical Reference Reactors

The previous section served to derive the kinetic equations for the general case of the non-critical reference reactor. Two forms of results were forthcoming. In the first, the properties

of the reference reactor were not employed and only the importance of neutrons, not precursors, entered the kinetics equation as a weighting function. Indeed in this form the weighting function can be arbitrary and the properties of the reference reactor unknown. For such a degree of arbitrariness however, we would be unable to claim the important approximation properties. In the second form, the reactivity was related to the inverse period of the reference reactor through the perturbation of properties between the actual and reference reactors. If the perturbation is small, the approximation properties follow as in Section 4.

Monte Carlo calculations provide an example of the use of the first form of results, albeit sometimes unknowingly. Quite commonly such calculations are based on a linear weighting scheme - i.e., reactivity is determined via the number of neutrons produced or destroyed, each with the same importance. Occasionally the so-called strong focusing technique is utilized, playing the role of a sharply peaked weighting function. Rarely is this weighting function, analogous to ϕ^+ given an energy dependence that would approximate it to the true adjoint function of the problem. However we should mention the excellent discussion given by Goertzel, (57), of the role of the importance function in stochastie processes, together with some related remarks on reactor kinetics by Weinberg and Wigner (1). The general result of Monte Carlo calculations is a measure of the deviation from critical as a reactivity, effective multiplication, etc.,

that may differ significantly in value from the bilinear forms put forward here and from the static reactivity, etc.* The number that results is not necessarily wrong; it has been stressed that in themselves the reactor parameters have no operational significance and cannot be directly verified. To obtain correct predictions of the periods however, it is necessary to employ generating times and precursor yields that are consistently determined, using the same weighting function. In particular, if no energy weighting has been employed, the proper precursor yield to be employed is the "true" measured yield. This general limitation on the Monte Carlo method is very inconvenient where attempts are made to correlate the results of the Monte Carlo calculations with other information.

This second general form, involving perturbations about a non-critical react, may find future applications in the theoretical or experimental studies of very subcritical (highly shut-down) or highly super-critical systems. Such applications will not be developed here.

5.16 Some Theorems for the Reactivity

Our discussion of the reactor parameters, in particular the reactivity, has been related to integrals or averages over the reactor. We should remark that we do not consider the idea of a spatially varying "reactivity" or "effective multiplication" to have any operational significance. In the steady state, the ratio of production to destruction is the same at all places in

*See Section 6.

the reactor; at critical, destruction then equals production point by point independent of any weighting scheme. It follows that when the actual flux is employed in calculating the reactor parameters (or the actual as opposed to the reference importance), the reactivity is positive for a supercritical system, zero for a critical system, and negative for a sub-critical system. This systematic property is an advantage of the defined reactivity; we investigate in the following theorems the nature of the parameter when the correct flux is not employed.

Theorem 5.1. Sign of Reactivity at Critical: In general, the reactivity, $\bar{\rho}'$, is not zero unless the asymptotic flux shape is used. Consider as an example the use of the ones vector as the weighting function or

$$\bar{\rho}' = \frac{\int I M \psi' \, dv}{\int I F' \psi' \, dv} \tag{5.98}$$

For any all positive and hence physically acceptable ψ', the denominator is everywhere positive so long as F' is defined to contain only positive elements. The sign of the reactivity is determined by the numerator. This numerator will only vanish point by point if ψ' is the actual critical asymptotic flux.

Furthermore, the reactivity depends in sign as well as magnitude on the flux employed. If a flux corresponding to one neutron at the center of a reactor is employed, $\int IM\psi' \, dv$ and hence $\bar{\rho}'$ is positive, the actual population increasing from unity in this sample. If the flux employed corresponds to one neutron at the edge of a reactor, $\int IM \psi' \, dv$ and hence the reactivity is initially negative since the actual population is diminishing.

If a critical reactor is used as its own reference reactor to give the importance or weighting function, then the bilinear form for the reactivity will vanish for any value of the flux employed:

$$\bar{\rho}' = \frac{\int \psi^+ (M+T)\,\psi'\,dv}{\int \psi^+ F'\,\psi'\,dv} = \frac{\int \psi' (M+T)^T \psi^+\,dv}{\int \psi^+ F'\,\psi'\,dv} = 0 \qquad (5.99)$$

If, however, some other critical system is used as the reference reactor, such that

$$\begin{aligned} (M+T)\,\psi &= 0 \\ (R+T)\,\psi^{+'} &= 0 \end{aligned} \qquad (5.100)$$

then for any flux, ψ' , we have

$$\begin{aligned} \bar{\rho}' &= \frac{\int \psi^{+'} (M+T)\,\psi'\,dv}{\int \psi^{+'} F'\psi'\,dv} \\ &= \frac{\int (\psi^{+'}(M+T)\,\psi + \psi^{+'}(R+T)\,\delta\psi + \psi^{+'} P\,\delta\psi)\,dv}{\int \psi^{+'} F'\psi'\,dv} \\ &= \frac{\int \psi^{+'} P\,\delta\psi\,dv}{\int \psi^+ F'\,\psi'\,dv} \end{aligned} \qquad (5.101)$$

Hence the reactivity of such a critical reactor, evaluated with some other critical reference state and the non-asymptotic flux, is zero to first order.

The same conclusions can be drawn for the non-critical reactor using its actual time dependent importance as a weighting function: the period will be predicted correctly whatever flux is employed. However, the result is of no practical value since if the actual importance were known, the problem would be already solved.

Theorem 5.2. Reactivity of a Non-Critical Reactor: The discussion of Theorem 5.1 indicates that the sign of the reactivity of a non-critical reactor is indeterminate if the flux used is not the actual asymptotic flux or if the weighting used is not the actual asymptotic importance. For any physically realisable reference reactor and hence all positive ψ^+, the reactivity when correctly evaluated has the sign of the resident period:

$$\bar{\rho}' = \frac{\int \psi^+ M \psi' \, dv}{\int \psi^+ F' \psi' \, dv}$$

$$= \frac{s \int \psi^+ A \psi' \, dv - \int \psi^+ T \psi' \, dv}{\int \psi^+ F' \psi' \, dv}$$

$$= \frac{s \int \psi^+ A \psi' \, dv}{\int \psi^+ F' \psi' \, dv} \qquad (5.102)$$

where we have assumed $\phi_i^+ = C_i^+$. Since the final ratio of integrals is positive under the assumptions of the theorem, the sign of the reactivity is that of the asymptotic period.

When non-physical importances, corresponding to higher modes, are employed, the sign of the reactivity is indeterminate. In any case, if neither the actual flux nor the actual importance is employed, the sign of the reactivity is again indeterminate.

5.17 Reactivity in the Point System

We reduce the general perturbation expressions for the kinetics parameters, in particular for the reactivity, to the one region case for various group models. For one group of neutrons only we obtain:

$$\bar{\rho}' = \frac{\int \phi^+ P \phi' \, dv}{\int \phi^+ F' \phi' \, dv}$$

$$= \frac{(\delta F - \delta \Sigma - B^2 \delta D) \int \phi^+ \phi' \, dv}{F' \int \phi^+ \phi' \, dv}$$

$$= \frac{\delta F - \delta \Sigma - B^2 \delta D}{F'} \qquad (5.103)$$

The result is seen to be identical with the one group expression given in the introduction, Table 1.1 of Section 1.

The two-group expression, however, does not reduce identically to the previously quoted result. We have, for the thermal fissioning case, that

$$\bar{\rho}' \int \phi_2^+ \frac{p}{1 + \tau B^2} F' \phi_2' \, dv =$$

$$\int \phi_2^+ \left[\frac{p}{1 + \tau B^2} (-\delta \Sigma_1 - B^2 \delta D_1) \phi_1' + \delta p \Sigma_1 \phi_1' + \left(\frac{p}{1 + \tau B^2} \delta F - \delta \Sigma_2 - B^2 \delta D_2 \right) \phi_2' \right] dv \qquad (5.104)$$

The coupling coefficient between the fast and slow flux can be expressed as

$$\frac{\phi_1'}{\phi_2'} = \frac{F'}{(1 + \tau' B^2)(1 + s \ell_1')} , \qquad (5.105)$$

where

$$\ell_1' = \frac{1}{V_1 \Sigma_1' (1 + \tau B^2)} . \qquad (5.106)$$

Then

$$\bar{\rho}' = \frac{-\frac{p}{1 + \tau B^2} \frac{(\delta \Sigma_1 + B^2 \delta D_1)}{(1 + \tau' B^2)(1 + s \ell_1')} + \frac{\delta p \Sigma_1}{(1 + \tau' B^2)(1 + s \ell_1')} + \frac{p \delta F}{(1 + \tau B^2)} - \delta \Sigma_2 - B^2 \delta D_2}{\frac{p F'}{1 + \tau B^2}} \qquad (5.107)$$

To reduce this form to that given in Section One, we make the first order approximation, that $1 + s \ell_1' = 1$. Then taking p

as unity in accordance with Section One, we have

$$\bar{\rho}' = \frac{\dfrac{F'}{\Sigma_1(1+B^2\tau')}\,\delta\Sigma_1 - \dfrac{\delta\Sigma_1 + B^2\,\delta D_1}{1+\tau B^2} + \dfrac{\delta F}{1+\tau B^2} - B^2\delta D_2 - \delta\Sigma_2}{\dfrac{pF'}{1+\tau B^2}} \tag{5.108}$$

But since

$$\delta\tau = \frac{\delta D_1}{\Sigma_1'} - \frac{\delta\Sigma_1}{\Sigma_1'}\frac{D_1}{\Sigma_1}, \tag{5.109}$$

we have

$$\bar{\rho}' = \frac{\dfrac{-F'B^2\,\delta\tau}{(1+B^2\tau')(1+B^2\tau)} + \dfrac{\delta F}{(1+B^2\tau)} - B^2\,\delta D_2 - \delta\Sigma_2}{\dfrac{F'}{(1+B^2\tau)}} \tag{5.110}$$

This expression for the reactivity differs from that of Section One, only in the denominator, where τ' replaces τ. However, it is to be noted that when the reactivity is combined with the generating time and the effective yield of neutrons, the denominator is cancelled out. Hence, both expressions give the operationally observable behavior to first order, if a consistent use is made of the kinetic parameters.

To extend the comparison to include delayed neutron effects, we consider the one region reactor with fast non-leakage characterized by $\overline{P}$ for the prompt group and $\overline{P}_i$ for each delayed neutron group. Then the effective yield reduces to an expression representing the fraction of neutrons returning to thermal:

$$\bar{\beta}_i = \frac{\nu\Sigma_f\overline{P}_i\beta_i}{\nu\Sigma_f[(1-\beta)\overline{P} + \sum_i \overline{P}_i\beta_i]} = \frac{\overline{P}_i\beta_i}{[(1-\beta)\overline{P} + \sum_i \overline{P}_i\beta_i]} \tag{5.111}$$

The perturbation expression for the effective multiplication then becomes:

$$\bar{k}_{eff} = \frac{\nu\Sigma_f[(1-\beta)\,\bar{P} + \sum_i \bar{P}_i\beta_i]}{\Sigma + B^2 D} \tag{5.112}$$

while

$$\bar{\Lambda} = \frac{1}{\nu\Sigma_f[(1-\beta)\,\bar{P} + \sum_i \bar{P}_i\beta_i]} \tag{5.113}$$

It will be realized that the various $\bar{P}$ are strictly functions of the period. The first order approximation consists of using the critical value.

The bracketed terms in equations (5.111), (5.112), and (5.113) can be approximated as:

$$\begin{aligned}[(1-\beta)\,\bar{P} + \sum_i \bar{P}_i\beta_i] &= \bar{P}[1-\beta + \sum_i \frac{\bar{P}_i}{\bar{P}}\beta_i] \\ &\approx \bar{P}[1-\beta + \sum_i \gamma_i\beta_i] \\ &= \bar{P}[1-\beta+\bar{\beta}] = \bar{P}[1+\beta(\gamma-1)]\end{aligned} \tag{5.114}$$

where

$$\gamma = \bar{\beta}_i/\beta \;;\quad \gamma = \bar{\beta}/\beta \;;\quad \beta = \sum_i \gamma_i\beta_i \tag{5.115}$$

Then the error on neglecting the changed spectrum due to the delayed fission energies in calculations of reactivity and generating time is of the order of $\beta(\gamma-1)$ or say 0.04% for the MITR - evidently negligible.

APPENDIX A TO SECTION FIVE

THE USE OF THE GENERATING TIME IN REACTOR KINETICS

5A. 1 Introduction

In the main body of the work we have derived the general inhour equation for multiregion systems in terms of effective parameters, $\bar{\rho}'$, $\bar{\beta}_i'$ and $\bar{\Lambda}'$. The use of the generating time, $\bar{\Lambda}'$, rather than the lifetime, $\bar{\ell}'$, is not very common in the literature; we believe that $\bar{\Lambda}'$ is a more useful parameter than $\bar{\ell}'$. To demonstrate the advantages of $\bar{\Lambda}'$, we give an account of the simple one region analysis of the step change of reactivity to show how the exact solution is easily developed together with certain approximations. Although we shall not here deal with spatial variations, the results are indeed valid for the general case when the perturbation expressions are substituted for the reactivity, ρ, precursor yield, β_i, and generating time, Λ, that we employ here.

The introduction, Section One, served to demonstrate that the lifetime, ℓ, given in the one group, one region model by

$$\ell = \frac{1}{V(\Sigma + D B^2)} \tag{5A. 1}$$

is not an operational concept, but depends on the arbitrary definition of the destruction processes represented by Σ and $D B^2$ in equation (5A. 1).

Analogous to ℓ, we have a generating time, Λ, based on the production probability rather than the destruction probability:

$$\Lambda = \frac{1}{V F} \tag{5A. 2}$$

where F in the one group model is just $\nu \Sigma_f$. The generating time, Λ, has been introduced in perturbation theory by Henry (6, 7) who regards Λ

as an approximate form of the lifetime, while realizing that its use simplifies the resulting equations. A generation time, τ_o, has been introduced into kinetics by Feiner, Frost and Hurwitz (43), though no definition is given and the equations utilized are not derived. It would seem that τ_o is in fact identical with our Λ. Other authors have attempted to simplify the kinetics equations by defining approximate lifetimes— the ℓ^* of Schultz (8)— and prompt multiplication factors, k_p, of Frost et alia, op cit., and Dietrich (44). Again Schultz employs the form of kinetics equations used in the present work but regards them as an approximation only. In fact the results of Schultz have greater validity. The prompt multiplicity is defined by Dietrich as $k_p = (1+\beta)\, k_{eff}$; the k_p of Frost, et alia, is not defined but for consistency should be given by $k_p = 1 + \rho - \beta$.

Hurwitz (65) has derived the kinetic equations for the diffusion approximation in a form in which the "generation" time is introduced. The final result is an equation formally identical to our own, so that the generation time must be taken to be identical with our generating time. However, the definition of Hurwitz (via the bilinear average of the reactor period) differs considerably from our own (the reciprocal average production probability). Furthermore, the approximations made by Hurwitz imply that the originally defined generation time could equally well be reduced to the lifetime.

We believe that although not operational quantities, the reciprocal of the average production and destruction probabilities (i.e., the generating time and the lifetime) are conceptually clear. Since neutrons are not divided into discrete generations however, it is not conceptually

clear what is meant by the time between generations (except in the limit of the critical reactor where the generating time equals the lifetime). It is for this reason that we eschew the Hurwitz approach.

The main body of the work emphasises that neither the lifetime nor the generating time are true physical, operational quantities. However, if the lifetime is regarded as the mean time of existence as a neutron, then the generating time may be regarded as the mean time before a neutron produces a further neutron, or the mean birth time.

The generating time and lifetimes are both reactor parameters. Neither are operationally determinable; their ratios with the measures of criticality determine the reactor period. Only the reactor period can be measured or thought of as an operational concept characterizing the reactor.

5A. 2 Criticality Measures

Starting with the effective multiplication, k_{eff}, as the ratio of production to destruction, we have:

$$k_{eff} = \frac{\text{production probability}}{\text{destruction probability}} = \frac{F\,V}{(\Sigma + B^2 D)\,V} \quad ,$$

$$k_{ex} = k_{eff} - 1 = \frac{\text{increase}}{\text{destruction}} = \frac{F - (\Sigma + B^2 D)}{\Sigma + B^2 D} \quad , \qquad (5A.\ 3)$$

$$\rho = \frac{k_{ex}}{k_{eff}} = \frac{\text{increase}}{\text{production}} = \frac{F - (\Sigma + B^2 D)}{F}$$

When delayed neutrons are to be considered, it is convenient to regard production as the production of prompt neutrons and precursors whilst the destruction refers to the removal of thermal neutrons. In the multigroup representation, each production term is weighted with

the probability that the precursor or the prompt neutron will eventually yield a thermal neutron. None of these measures of criticality are true balances and by themselves do not determine the reactor behaviour. They are merely useful parameters.

The yield of delayed neutron emitters of type i is given by the fraction of precursors of type i formed for every neutron ultimately produced whether prompt or via precursors. Thus β_i is a fraction of the production rate of prompt neutrons and precursors F as employed in the measures of criticality. Thus we have a natural grouping of the reactor parameters according to whether they are based on production or destruction.

	Production		Destruction
β_i	$\frac{\beta_i F}{F}$	k_{eff}	$\frac{F}{\Sigma + BD^2}$
ρ	$\frac{F - (\Sigma + DB^2)}{F}$	k_{ex}	$\frac{F - (\Sigma + DB^2)}{\Sigma + DB^2}$
Λ	$\frac{1}{VF}$	ℓ	$\frac{1}{V(\Sigma + DB^2)}$

Table 5A. 1 Normalization of Reactor Parameters.

From Table 5A. 1 we have the relation:

$$k_{eff} = \frac{k_{ex}}{\rho} = \frac{\ell}{\Lambda} \qquad (5A.\ 4)$$

It will be evident that the terms based on production are the natural parameters to express the behaviour with delayed neutrons. Their use

leads to significant simplification.

In addition to the simplified results, the use of Λ, the reciprocal production probability, rather than ℓ, the reciprocal destruction probability, has a further advantage. The majority of reactor control is achieved with absorbing control rods. Therefore the destruction terms are more likely to vary in a reactor than the production terms; ℓ will vary more than Λ. An estimate of the magnitude of this effect can be gained from an estimate of the change of k_{eff} from shutdown to operation of a reactor. A reasonable estimate might be $0.7 < k_{eff} \leq 1$. Thus ℓ might change by a factor of 1.4. This variation is not very important when delayed neutron effects govern the reactor behaviour but might be significant for the analysis of an accident.

5A.3 Kinetics Equations

Neutron balance:

$$\frac{dn}{dt} = [(1-\beta)\ F - \Sigma - D\,B^2]\ V n + \sum_i \lambda_i C_i\ , \qquad (5A.5)$$

where $\beta = \sum_i \beta_i$. Rearranging we have:

$$\frac{dn}{dt} = \frac{F - \Sigma - D\,B^2}{F}\ FV\,n - \beta\,F\,V\,n + \sum_i \lambda_i C_i\ , \qquad (5A.6)$$

or from the definitions of ρ and β,

$$\frac{dn}{dt} = \frac{\rho}{\Lambda}\,n - \frac{\beta}{\Lambda}\,n + \sum_i \lambda_i C_i = \frac{\rho-\beta}{\Lambda}\,n + \sum_i \lambda_i C_i \qquad (5A.7)$$

Consider the case of no delayed neutrons and precursors when equation (5A.7) reduces to:

$$\frac{dn}{dt} = \frac{\rho}{\Lambda}\, n\,, \tag{5A. 8}$$

whence

$$n/n_o = \exp \frac{\rho t}{\Lambda} = \exp \frac{k_{ex} t}{\ell} \tag{5A. 9}$$

Corresponding to equation (5A. 8) we can write:

$$\frac{dn}{dt} = FV\, n - (\Sigma + B^2 D)\, V\, n$$

$$= \frac{1}{\Lambda}\, n - \frac{1}{\ell}\, n$$

$$= \left[\frac{1}{\Lambda} - \frac{1}{\ell}\right] n = \frac{1}{\Lambda}\left[1 - \frac{\Lambda}{\ell}\right] n \tag{5A. 10}$$

But $(1 - \frac{\Lambda}{\ell})$ is indeed the reactivity ρ. If we interpret $s = \frac{1}{n}\frac{dn}{dt}$ as the neutron gain probability we can write:

$$s = \frac{1}{\Lambda} - \frac{1}{\ell} \tag{5A. 11}$$

$$\text{probability of gain} = \text{probability of production} - \text{probability of destruction}$$

The maximum reactivity is +1, when there is negligible destruction. Then Λ is the period on which the reactor would rise if the destruction was not taking place. This result is similar to the better known interpretation of ℓ as the period on which the reactor would fall if there were no production processes ($k_{ex} = -1$).

Returning to the delayed neutron case, we have the precursor balance equations:

$$\frac{dC_i}{dt} = \beta_i\, F\, V_n - \lambda_i\, C_i\,, \tag{5A. 12}$$

whence:

$$\frac{dC_i}{dt} = \frac{\beta_i}{\Lambda}\, n - \lambda_i\, C_i \tag{5A. 13}$$

5A. 4 Step Problem

The classic problem of the step change in reactivity is soluble on taking the eigen equations corresponding to (5A. 12) and (5A. 13):

$$\frac{dn}{dt} = sn = \frac{\rho - \beta}{\Lambda} n + \sum_i \lambda_i C_i \,, \tag{5A. 14}$$

$$\frac{dC_i}{dt} = sC_i = \frac{\beta_i}{\Lambda} n - \lambda_i C_i \tag{5A. 15}$$

Substitution of the i equations similar to (5A. 15) in (5A. 14) leads to:

$$sn = \frac{\rho - \beta}{\Lambda} n + \sum_i \frac{\lambda_i}{\lambda_i + s} \frac{\beta_i}{\Lambda} n \,, \tag{5A. 16}$$

or

$$\Lambda s = \rho - \beta + \sum_i \frac{\lambda_i \beta_i}{\lambda_i + s} \tag{5A. 17}$$

The roles played by the direct increase of prompt neutrons and the indirect supply of delayed neutrons, a fraction $\lambda/(\lambda + s)$ of the precursor production, are clearly distinguishable.

5A. 5 Solution by Inspection

(a) Around zero

Certain properties of equation (5A. 15) in the (ρ, s) plane can be found by inspection. Evidently there are poles at each $s = -\lambda_i$. Near s=0 however, equation (5A. 17) becomes:

$$\Lambda s = \rho - \sum_i \frac{s\beta_i}{\lambda_i + s}$$

$$\mathop{\mathcal{L}}_{s \to 0} \rho = s \left[\Lambda + \sum_i \frac{\beta_i}{\lambda_i} \right] \tag{5A. 18}$$

Hence the path is through the origin with slope $\Lambda + \sum_i \frac{\beta_i}{\lambda_i}$

(b) Asymptotic

For very large magnitudes of s however we have:

$$\mathop{\mathcal{L}}_{s\to\infty} \Lambda s = \rho - \beta \qquad (5A.19)$$

Thus the asymptotic solution away from the poles is also a line of slope Λ but passing through the point $\rho = \beta$ on the ρ axis.

The asymptotic solution is well known in an approximate form though shown here to be the exact solution for very large magnitudes of s. We have asymptotically

$$s = \frac{\rho-\beta}{\Lambda} = \frac{k_{ex} - k_{eff}\beta}{\ell} \qquad (5A.20)$$

(c) Around the poles:

Consider the behaviour of:

$$\Lambda s = \rho - \beta + \sum_i \frac{\beta_i \lambda_i}{\lambda_i + s} , \qquad (5A.21)$$

in the region around $s = -\lambda_1$. Then $|\rho| \gg 1$ and certain terms can be neglected. We have approximately:

$$\mathop{\mathcal{L}}_{s\to-\lambda_1} \rho - \beta = \frac{\beta_1\lambda_1}{s + \lambda_1} \qquad (5A.22)$$

Equation (5A. 22) gives a rectangular hyperbola in the (ρ, s) plane with asymptotes through $s = -\lambda_1$ and $\rho=\beta$. The curve passes through $\rho - \beta_1 + \beta$ on the ρ axis.

5A.6 Interpretation of Prompt Root

The interpretation of the asymptotic solution above prompt critical

is also well known. In fact on either side of prompt critical, the reactor tends to behave as a prompt multiplying or prompt critical system with a prompt reactivity β below ρ.

Let us consider for a moment a just critical system in steady state as a sub prompt critical multiplying system together with a number of sources of delayed neutrons from the precursors. The total source strength, proportional to the flux and precursor yield, is $\sum_i \beta_i \phi$ in steady state. Since the flux level is given by the source strength times the multiplication, M, we have

$$\phi = M \sum_i \beta_i \phi = M \beta \phi \tag{5A. 23}$$

Hence the multiplication is $1/\beta$. We can see that for every neutron introduced, a further $(1-\beta)$ prompt neutrons are created in the next generation. The multiplication of one neutron is given by

$$M = 1 + (1-\beta) + (1-\beta)^2 + \ldots = \frac{1}{1-(1-\beta)} = \frac{1}{\beta} \tag{5A. 24}$$

Equation (5A. 24) is the multiplication as anticipated. If now the reactivity is increased by ρ above delayed critical, instead of being β below prompt critical, the system is now only $(\beta-\rho)$ below. Thus the total yield for one neutron (M) is increased from $1/\beta$ to $1/(\beta-\rho)$. Evidently the yield becomes infinite at prompt criticality, $\rho=\beta$.

From the above argument we can derive the coefficient of the so-called prompt jump. Although the total yield of equation (5A. 24) nominally requires an infinite time, it is in fact virtually completed after a few prompt generating times. Thus the full yield appears before the precursor density or the source strength has increased appreciably.

Then the flux level 'jumps' by

$$\phi = \frac{1}{\beta-\rho}\sum_i \beta_i \phi_o = \frac{\beta}{\beta-\rho}\phi_o \qquad (5A.25)$$

This expression for the jump is valid for any number of groups.

5A.7 Non-dimensional Representation

A certain simplification of equation (5A.17) can be obtained by expressing the variables in a non-dimensional form. The use of $\rho^* = \rho/\beta$, as a fiducial unit is well known in terms of dollars and cents, or betas and milli-betas. One advantage of this reduced unit is that the dangerous condition of prompt criticality ($\rho=\beta$, $\rho^*=1$) is easily recognized whatever the value of β.

We write equation (5A.17) as:

$$\frac{\Lambda s}{\beta} = \frac{\rho}{\beta} - 1 + \sum_i \frac{\Lambda\lambda_i \beta_i/\beta}{\Lambda(\lambda_i + s)} \qquad (5A.26)$$

Then if s and λ_i are all measured in units of the generating time, Λ, we can define these non-dimensional units by

$$\rho^* = \frac{\rho}{\beta}\ ;\ s^* = \frac{\Lambda s}{\beta}\ ;\ \lambda_i^* = \frac{\Lambda\lambda_i}{\beta} \qquad (5A.27)$$

Remembering that the relative precursor yield is given by:

$$a_i = \frac{\beta_i}{\beta} \quad , \qquad (5A.28)$$

equation (5A.26) becomes:

$$s^* = \rho^* - 1 + \sum_i \frac{\lambda_i^* a_i}{\lambda_i^* + s^*} \qquad (5A.29)$$

The roles of the direct contribution from prompt fission and the indirect contribution from precursor decay are still clearly distinguishable. We have achieved a useful simplification and our equations are now independent of the values of Λ and β.

The asymptotic results achieved by inspection are expressible in the new units of course. For large magnitudes of s^*, equation (5A. 29) becomes

$$s^*_{(\text{asymptotic})} = \rho^* - 1 \tag{5A. 30}$$

Near $s^* = 0$, we have the slope of ρ^* against s^*:

$$\frac{\rho^*}{s^*} = 1 + \sum_i \frac{a_i}{\lambda_i^*} \tag{5A. 31}$$

In addition there are the poles at $s^* = -\lambda_i^*$. Figure (5A. 1) gives a sketch of the form of solution for a three group of precursors approximation.

5A. 8 Physical Interpretation of Non-Dimensional Parameters

(a) Neutron-precursor ratio:

The non-dimensional units we have introduced have interesting physical significance. Consider the ratio of neutron to precursor density given by equation (5A. 13)

$$\frac{dC_i}{dt} = sC_i = \frac{\beta_i}{\Lambda} n - \lambda_i C_i \tag{5A. 32}$$

Hence

$$\frac{n}{C_i} = \frac{\Lambda(s+\lambda_i)}{\beta_i} = \frac{\beta}{\beta_i} \frac{\Lambda(s+\lambda_i)}{\beta} = \frac{(s^* + \lambda_i^*)}{a_i} \tag{5A. 33}$$

Thus $(s^* + \lambda_i^*)$ is related to the ratio of neutrons to precursors. When

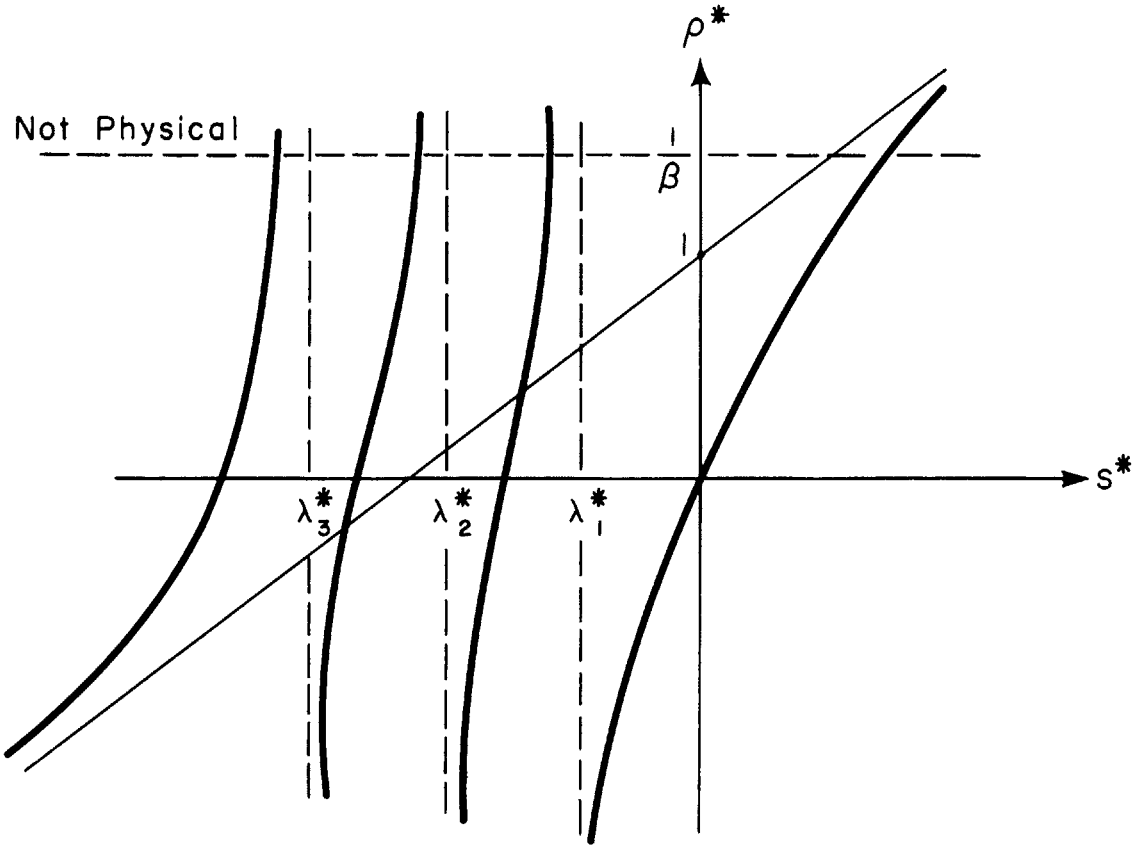


FIG. 5A.1 KINETICS SOLUTION FOR A THREE GROUP OF PRECURSORS APPROXIMATION

the reactor is on an inverse period, s, the ratio of total precursor, C, to neutrons is:

$$\frac{C}{n} = \frac{1}{n}\sum_i C_i = \sum_i \frac{a_i}{s^* + \lambda_i^*} \qquad (5A.34)$$

A typical value of the ratio of precursors to neutrons in a critical heavy water reactor is 99 to 1, i.e., only one percent of the population is neutrons, the remainder being precursors.

This expression for the precursor neutron ratio brings out the physical importance of the delayed neutrons to the reactor behaviour. In addition we can see that the reactor can never be on a stable or dominant period of less than $-\lambda_j^*$ where λ_j refers to the longest lived precursors, or else the ratio of densities would be negative – a physically impossible situation. In the long run a reactor cannot be shut down faster than its delayed neutrons. This point can be significant when long lived photo neutrons are to be considered since there may be an appreciable shut down flux from this source.

(b) Cycle Time:

The kinetics solution without delayed neutrons, equation (5A. 9), took the form

$$s = \frac{\rho}{\Lambda} \; ; \; s^* = \rho^* \qquad (5A.35)$$

We might ask what is the average cycle time, L, to replace Λ in equation (5A. 35) such that L would give the behaviour with delayed neutrons included. There will of course be as many values of L for each value of the reactivity as there are roots of the equations.

We can write the solution for L formally in terms of s or s^* as:

$$L = \frac{\rho}{s} \; ; \quad \frac{L}{\Lambda} = \frac{\rho^*}{s^*} \tag{5A. 36}$$

Thus ρ^*/s^* gives the number of times the average cycle time is greater than the prompt generation time, a measure of the delaying effect of delayed neutrons on the reactor.

If we write equation (5A. 29), the general multigroup solution, in terms of ρ^*/s^*, we have:

$$s^* = \rho^* - \sum_i \frac{a_i s^*}{\lambda_i^* + s^*} \tag{5A. 37}$$

or

$$\frac{\rho^*}{s^*} = 1 + \sum_i \frac{a_i}{\lambda^* + s^*} = \frac{L}{\Lambda} \tag{5A. 37}$$

5A. 9 One Group Approximation

A one group of delayed neutrons approximation is commonly employed either to simplify kinetics problems or to present the essential physics involved. Although Smets (45) has put forward several methods of selecting the group constants $\bar{\beta}$ and $\bar{\lambda}$ of the one group representation, we shall only consider the conventional representation.

We wish to match the two representations:

$$s^* = \rho^* - 1 + \sum_i \frac{a_i \lambda_i^*}{\lambda_i^* + s^*} \tag{5A. 38}$$

and

$$s^* = \rho^* - 1 + \frac{\lambda^*}{\lambda^* + s^*} \tag{5A. 39}$$

with appropriate selection of $\bar{\beta}$ and $\bar{\lambda}$ in the one group representation. To match the asymptotic root, $s^* = \rho^* - 1$, we need only take:

$$\bar{\beta} = \sum_i \beta_i = \beta \tag{5A.40}$$

This procedure ensures that at very large magnitudes of ρ^*, the reactivity, ρ^*, is correctly given. Actually, the asymptotic root is a fair representation of the solution at intermediate values of s^* so long as s^* is well away from the poles at $-\lambda_i^*$.

The second region of main interest is probably around delayed critical. Both exact and one group representations pass through the point (0, 0). Hence we are at liberty to match the slope of the exact solution, given by equation (5A. 18) as:

$$\frac{\rho^*}{s^*} = 1 + \sum_i \frac{a_i}{\lambda_i^*} \tag{5A.41}$$

(as $s^* \to 0$)

Hence we can ensure a correct solution for small magnitudes of s^*, by taking:

$$\frac{1}{\bar{\lambda}^*} = \sum_i \frac{a_i}{\lambda_i^*}\,; \quad \frac{\bar{\beta}}{\bar{\lambda}} = \sum_i \frac{\beta_i}{\lambda_i} \tag{5A.42}$$

We can now drop the bars on β and λ in the one group representation.

Although the solution has been matched asymptotically, the intermediate solution is not necessarily correct. Indeed in the left hand plane in the region of the poles at the $-\lambda_i^*$, it is evident that this region is not well represented. Physically the flux can never decrease on a stable or resident period at a rate faster than the decay of the

slowest precursor group. Thus part of the dominant root predicted by the one group approximation is in fact unrealisable when it crosses the right hand pole of the correct representation.

The one group equation forms a hyperbola in the s^*, ρ^* plane, with one vertical asymptote at $s_1^* = -\lambda^*$ and 45° asymptote through (0, 1), $s^* = \rho^* - 1$. Further special solutions are easily obtainable by inspection.

The quadratic form in s^* is obtainable by a rearrangement of equation (5A. 39):

$$s^{*2} + s^*(\lambda^* + 1 - \rho^*) - \lambda^* \rho^* = 0 \qquad (5A.\ 43)$$

The second root corresponding to the asymptotic roots are obtainable from the final coefficient which is the product of roots:

$$s_1^* s_2^* = -\lambda^* \rho^*$$

Hence we have for the asymptotic value of s_1^*:

$$s_1^* = \rho^* - 1$$

$$s_2^* = \frac{\lambda^* \rho^*}{1 - \rho^*} \qquad (5A.\ 44)$$

This corresponding solution $s_2^* = \frac{\lambda^* \rho^*}{1 - \rho^*}$ takes the form of a rectangular hyperbola which intersects the true solution at (0, 0) and has asymptotes horizontally through $\rho^* = 1$ and vertically at $s^* = -\lambda^*$. The slope at (0, 0) is $1/\lambda^*$ (cf. $1 + 1/\lambda^*$ of the correct multigroup solution.) Since λ^* is so small, the slope is well represented by this second hyperbola, which in addition is easier to sketch. It is thus a fair approximation to the

dominant root in the right half plane. This root, incidentally, is identical with a commonly used approximation derived by expansion of the equation solution:

$$\frac{\lambda^* \rho^*}{1 - \rho^*} = \frac{\lambda\rho}{\beta - \rho} \tag{5A.45}$$

The physical interpretation of this root will be given.

Alternatively from the second coefficient we can derive the sum of the roots:

$$(s_1^* + s_2^*) = -(\lambda^* + 1 - \rho^*) \tag{5A.46}$$

Hence for the asymptotic values:

$$\begin{aligned} s_1^* &= \rho^* - 1 \\ s_2^* &= -\lambda^* \end{aligned} \tag{5A.47}$$

Since the particular solution, (0, 0), is known, there is a corresponding solution at:

$$\begin{aligned} s_1^* &= \rho^* = 0 \\ s_2^* &= -(\lambda^* + 1) \end{aligned} \tag{5A.48}$$

By considering the case where the sum of the roots is zero we find:

$$s^* = \pm\sqrt{\lambda^* \rho^*} = \pm\sqrt{\lambda^*(1 + \lambda^*)} \tag{5A.49}$$

For $\lambda^* \ll 1$ this pair become

$$s^* = \pm\sqrt{\lambda^*} \tag{5A.50}$$

The asymptotes intersect at the point $(-\lambda^*, 1 - \lambda^*)$ in the s^*, ρ^* plane. All particular solutions give rise to second solutions as mirror images

about this intersection. In particular the origin of coordinates as a solution leads to the mirror solution $(-2\lambda^*, 2 - 2\lambda^*)$. Some of these results are sketched in Figure 5A. 2.

By an expansion for $\rho^* > 1$ and assuming $\lambda^* \ll 1$, we can find roots to compare with the asymptotic root, $s^* = \rho^* - 1$:

reduced reactivity:	ρ^*	2	3	4	5
expansion solution:	s^*	$1+\lambda^*$	$2 + \frac{\lambda^*}{2}$	$3 + \frac{\lambda^*}{3}$	$4 + \frac{\lambda^*}{4}$
asymptotic solution:	s^*	1	2	3	4

Hence the fractional error in the asymptotic root for $|\rho^*| > 1$ goes rapidly to zero as

$$\frac{\delta s^*}{s^*} = \frac{\lambda^*}{(\rho^* - 1)^2} \tag{5A.51}$$

The equation can be rearranged to yield a rectangular hyperbola in a form suitable for plotting as a universal equation

$$\frac{\rho^*}{s^*} = 1 + \frac{1}{\lambda^* + s^*} \tag{5A.52}$$

Figure 5A. 3 is such a plot of ρ^*/s^* against $\lambda^* + s^*$ valid for all values of yield, β, generating time Λ, and precursor decay probability, λ. The physical interpretation of $(\lambda^* + s^*)$ as the ratio of neutron to precursor probability has already been discussed. We proceed with a physical description of $\rho^*/s^* = L/\Lambda$, the cycle time, for the one group approximation.

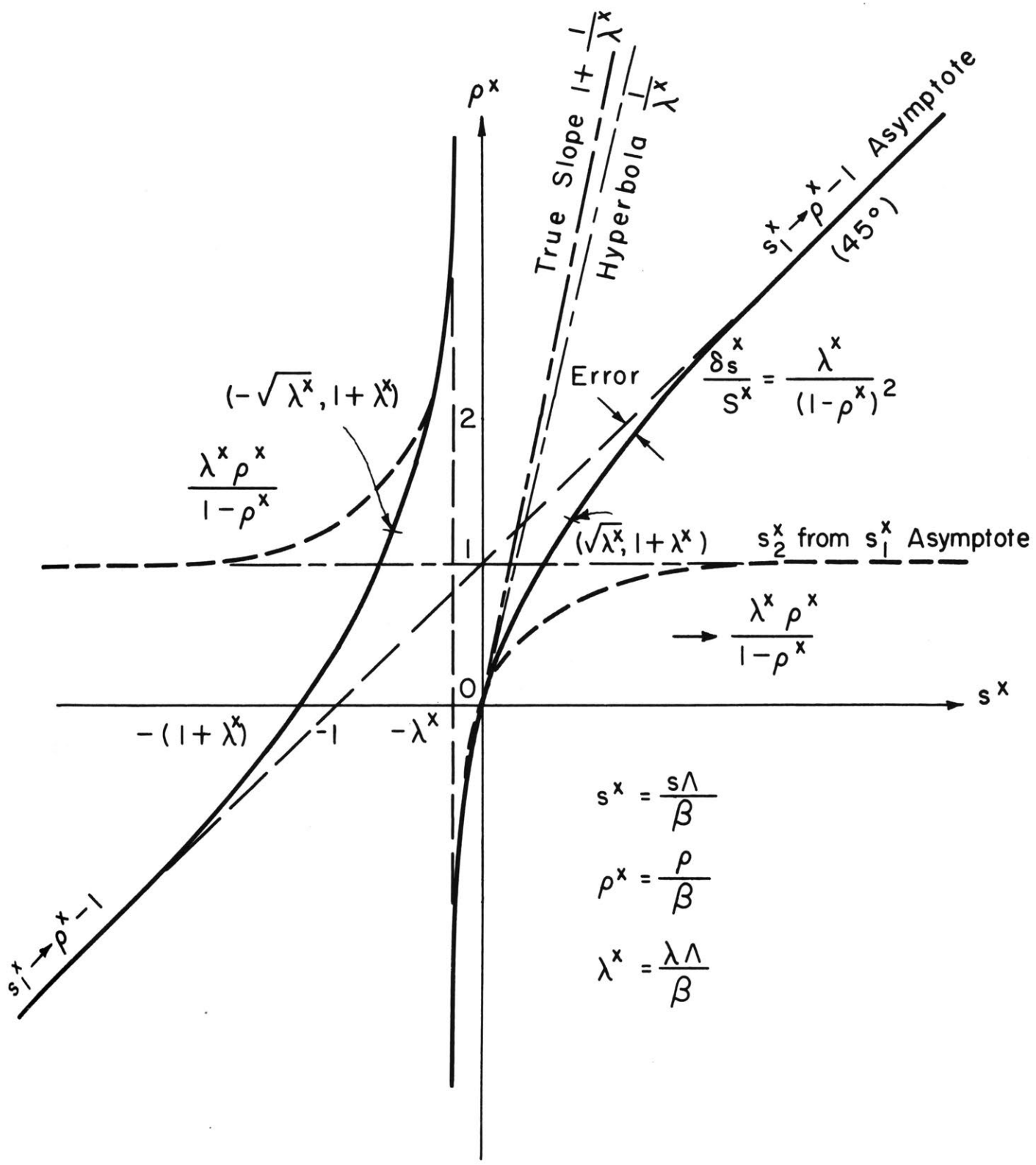


FIG. 5A.2 STEP REACTIVITY CHANGE. ONE GROUP OF DELAYED NEUTRONS

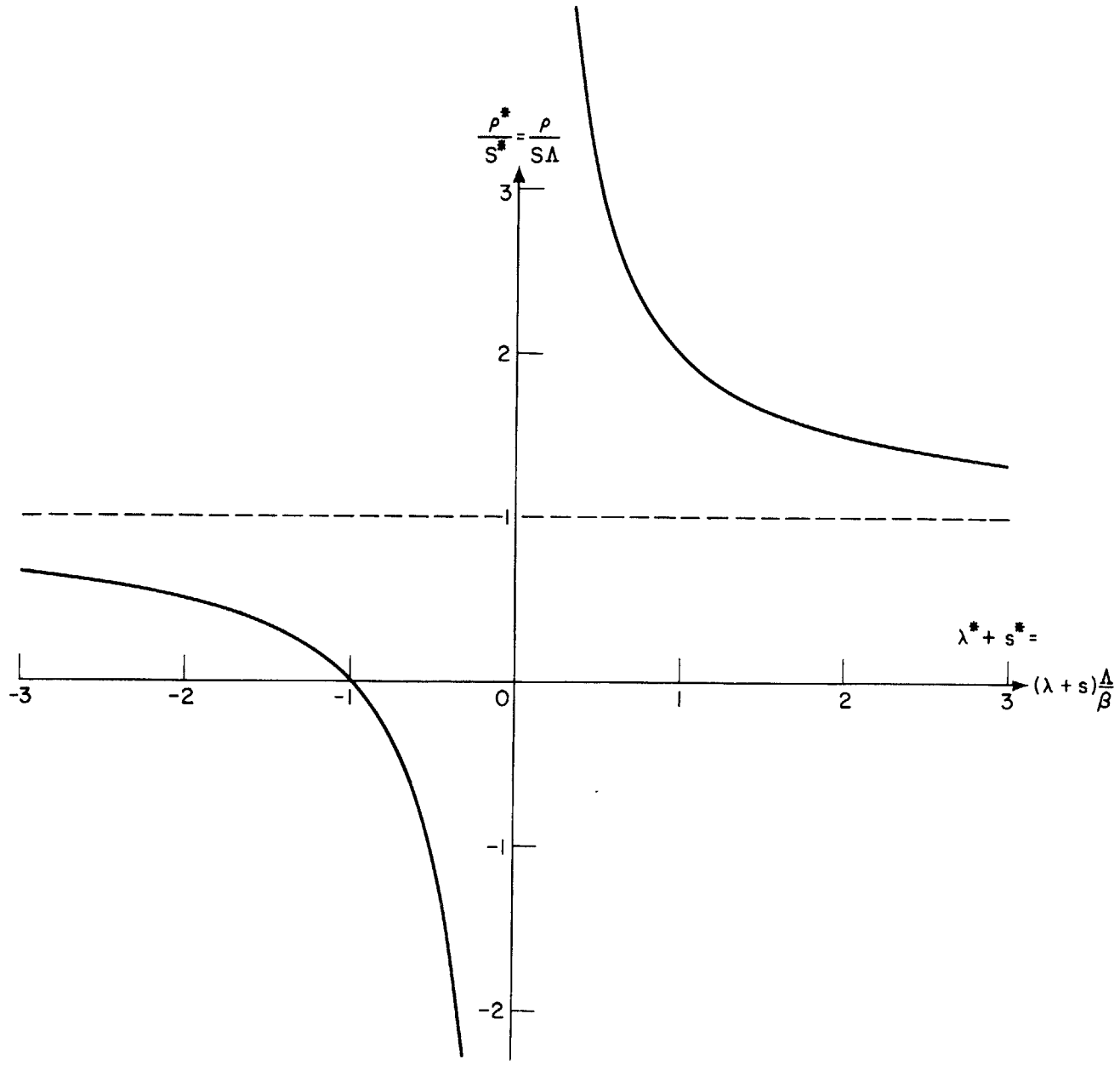


FIG. 5A.3 – ONE GROUP DELAYED NEUTRONS. Universal Inhour Relation
Reduced reactivity versus neutron/precursor ratio. (Valid for all values of delayed neutron yield prompt generation time and precursor decay time.)

5A. 10 One Group Cycle Time

(a) Exact Value:

The cycle time L in the one group approximation is given by our universal equation, equation (5A. 52):

$$\frac{L}{\Lambda} = \frac{\rho^*}{s^*} = 1 + \frac{1}{\lambda^* + s^*} \tag{5A. 53}$$

or:

$$L = \Lambda + \frac{\beta}{\lambda + s} \tag{5A. 54}$$

$$= \Lambda(1-\beta) + \beta(\Lambda + \frac{1}{\lambda + s})$$

The term $\Lambda(1-\beta)$ is the contribution of the $(1-\beta)$ prompt fraction with generating time Λ. The term $\beta(\Lambda + \frac{1}{\lambda+s})$ is the contribution of the β delayed neutrons which have a generating time Λ as prompt neutrons and $1/(\lambda+s)$ as precursor. Just as neutron destruction probability is: $1/\ell$ and precursor destruction probability is: $1/\tau = \lambda$, so then neutron production probability is $1/\Lambda$ and precursor production probability is $\lambda + s$.

(b) Approximate Value:

We have justified the asymptotic root, $s^* = \rho^* - 1$, as the behaviour of a prompt system sufficiently far away from prompt critical that the multiplication effects are completed before the precursors have changed their concentration appreciably. The corresponding roots of the equation, in particular for the one group approximation, can also be justified by consideration of the prompt multiplying effect and the average cycle time, L.

The behaviour of the additional roots of the equation is governed

basically by the precursor decay time, $\tau = 1/\lambda$. The neutron released initially on decay is multiplied within a few generation times to the value $\frac{1}{\beta-\rho}$ as derived in equation (5A. 24) et seq. Thus more neutrons are released per precursor decaying. The probability of a delayed neutron appearing is greater than the decay probability by $\frac{\lambda}{\beta-\rho}$ to λ. Thus the effective delayed neutron lifetime is not τ but only $(\beta-\rho)\tau$. Thus if we take the cycle time, L, to be this effective time, we have

$$L = \frac{(\beta-\rho)}{\lambda} \; ; \; \frac{L}{\Lambda} = \frac{\rho^*}{s^*} = \frac{1-\rho^*}{\lambda^*} \tag{5A.55}$$

This interpretation does indeed give the second root corresponding to the solution given by equation (5A. 44):

$$s^* = -\frac{\lambda^* \rho^*}{(\rho^* - 1)} \tag{5A.56}$$

5A. 11 Comparison of Solutions Parametric in Generating Time Λ and Lifetime, ℓ

From the previous discussion we are able to sketch the solution to the one group approximation with considerable accuracy. It is interesting to compare this solution with the better known form parametric in ℓ rather than Λ. There is however an additional pole at $-1/\ell$ where k_{eff} goes to zero and ρ to minus infinity when ℓ is considered constant. One can say that in practice this root cannot exist since it would not be possible to lower the reactivity without indeed changing the destruction probability and hence the lifetime.

The slope of the asymptotic root in a ρ, s plot is Λ. Both representations follow this solution in the region of k_{eff} unity. The divergence as k_{eff} leaves unity is not noticeable around the poles where the precursor

behaviour is the dominating influence. These results are sketched in Figure 5A. 4.

5A. 12 Exact Non-linear Solution

In this Appendix, we have given a detailed account of the application of the generating time to the simplest problem of reactor kinetics, the step change of reactivity. We have shown the nature of the solution on using Λ rather than the lifetime, ℓ. The simplifications that were there present hold for more difficult problems and can be applied with advantage where exact solutions are sought (42, 45). As a single example, it will be sufficient to discuss Akcasu's solution to the non-linear problem (42), and show that three approximations made in the treatment can be removed on utilising the generating treatment.

In Akcasu's original treatment, three approximations were made:

(a) the lifetime, ℓ, was taken to be a constant parameter,

(b) the precursor equation was simplified by the assumption that:

$$k_{ex} \ll 1,$$

(c) the neutron equation was simplified by the assumption that:

$$\beta \ll 1$$

In our treatment, assumptions (b) and (c) are removed entirely while assumption (a) is replaced by the assumption that the generating time, Λ, is a constant. As we have seen, for many practical cases, this latter assumption is better than assuming ℓ to be constant. Assumption (a) is generally in error by the same magnitude as assumptions (b) and (c).

We start with the previously derived kinetics equations:

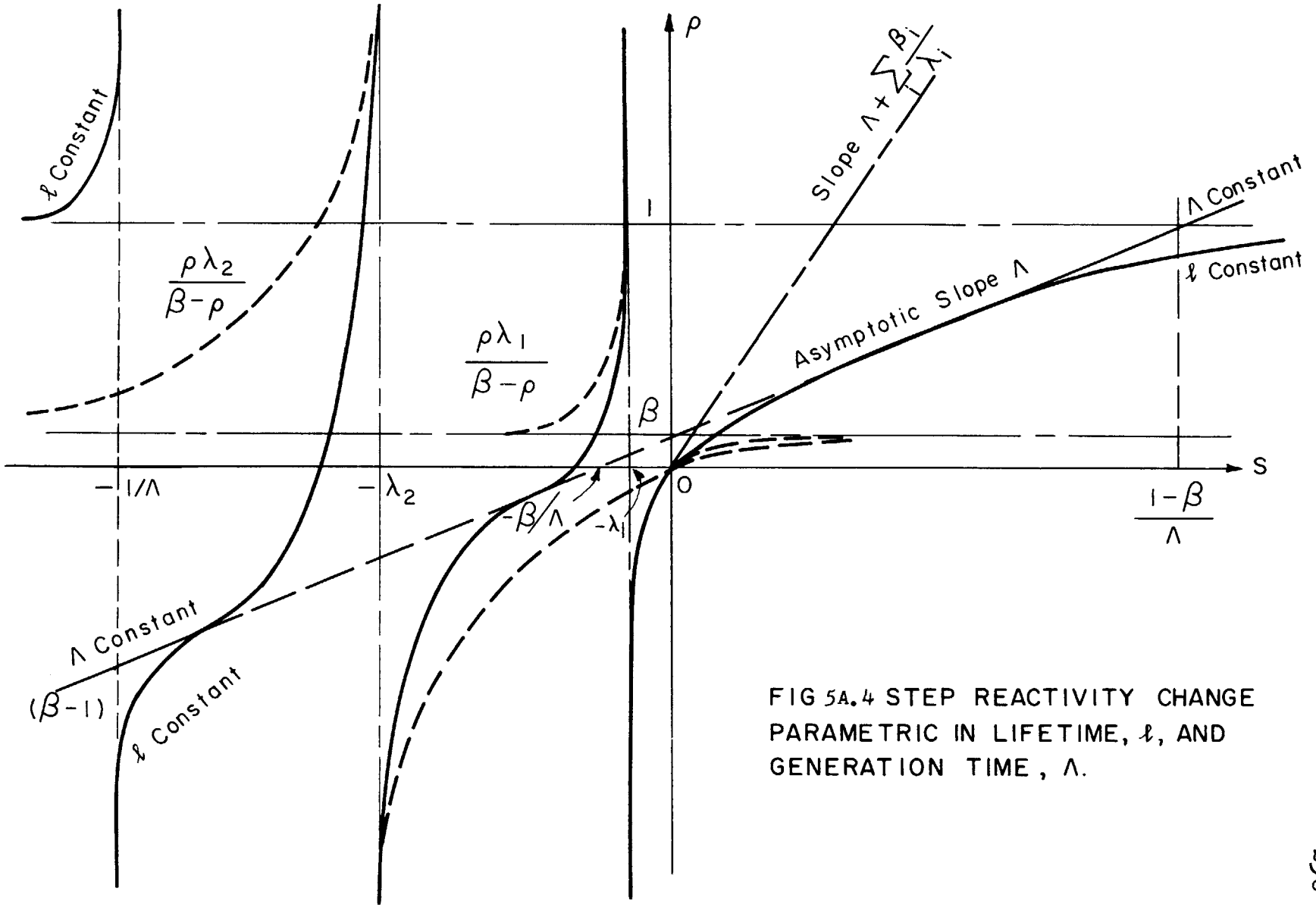


FIG 5A.4 STEP REACTIVITY CHANGE PARAMETRIC IN LIFETIME, ℓ, AND GENERATION TIME, Λ.

$$\frac{dn}{dt} = \frac{\rho-\beta}{\Lambda} n + \sum_i' p_i \lambda_i C_i \,, \tag{5A.57}$$

$$\frac{dC_i}{dt} = \frac{\beta_i}{p\Lambda} n - \lambda_i C_i \tag{5A.58}$$

We put $p_i C_i = \overline{C}_i$ and we again measure time in units of the generating time, Λ:

$$\lambda_i^* = \frac{\lambda_i \Lambda}{\beta} \; ; \; t^* = \frac{t}{\Lambda} \; ; \; \rho^* = \frac{\rho}{\beta} \; ; \; \frac{\beta_i}{\beta} = a_i \tag{5A.59}$$

Then

$$\frac{dn}{dt^*} = (\rho^* - 1)\, n + \sum_i \lambda_i^* \overline{C}_i \,, \tag{5A.60}$$

and

$$\frac{d\overline{C}_i}{dt^*} = a_i n - \lambda_i^* \overline{C}_i \tag{5A.61}$$

Let $\rho^* = \rho_o^* + \mu z(t^*)$ where μ and ρ_o^* are constant. We seek solutions of the form

$$n(t^*) = \exp \int_0^{t^*} m(t^*)\, dt^* \tag{5A.62}$$

$$\overline{C}_i(t^*) = g_i(t^*) \exp \int_0^{t^*} m(t^*)\, dt^* \tag{5A.63}$$

Substitution gives:

$$m(t^*) = (\rho_o^* - 1) + \mu z(t^*) + \sum_i \lambda_i^* g_i(t^*) \tag{5A.64}$$

$$\frac{dg_i(t^*)}{dt^*} + g_i(t^*)\, m(t^*) = a_i - \lambda_i^* g_i(t^*) \tag{5A.65}$$

These equations are identical in form with these given by Akcasu, except

that ρ replaces his k_{ex} and Λ replaces ℓ. Thus the solution to the present form follows exactly as given by Akcasu on making these substitutions, but without the two approximations indicated above. Similar solutions given by Smets (45) can be made rigorous in exactly the same fashion.

APPENDIX B TO SECTION FIVE

FAST FISSION AND FAST LEAKAGE PARAMETERS

5B. 1 Two Group Representation of Fast Fission

It is reasonable to demand that a 2 1/2 group representation including fast fission will reduce, in a one region reactor, to the modified one group four factor formula when criticality is achieved. Such a definition however makes the relationship between ϵ and Σ_f a function of geometry.

In practice the correction factor is of the order of ϵ, so that some uncertainty is admissable. The uncertainty is essentially that involved in what Weinberg and Wigner (1) call intensive and extensive reactors properties, ϵ and ϵ^*. The 'four factor formula' requires that

$$\frac{\nu\epsilon p\Sigma_{f2}}{1+\tau B^2} = \frac{\epsilon p F_2}{1+\tau B^2} = \Sigma_2(1 + L^2 B^2) , \tag{5B. 1}$$

since $F_1 = \nu\Sigma_{f1}$; $F_2 = \nu\Sigma_{f2}$. The group criticality expression is given by

$$\begin{aligned} &-\Sigma_1(1+\tau B^2)\,\phi_1 + F_1\phi_1 + F_2\phi_2 = 0 , \\ &p\Sigma_1\phi_1 - \Sigma_2(1+L^2B^2)\,\phi_2 = 0 \end{aligned} \tag{5B. 2}$$

whence

$$\frac{\phi_1}{\phi_2} = \frac{F_2}{\Sigma_1(1+\tau B^2)}\left[1 - \frac{F_1}{\Sigma_1(1+\tau B^2)}\right]^{-1} = \frac{\Sigma_2(1+L^2B^2)}{p\Sigma_1} \tag{5B. 3}$$

and also

$$\frac{\phi_1^+}{\phi_2^+} = \frac{p}{(1+\tau B^2)}\left[1 - \frac{F_1}{\Sigma_1(1+\tau B^2)}\right]^{-1}$$

$$= \frac{\Sigma_2(1+L^2B^2)}{F_2} \qquad (5B.4)$$

From equation (5B. 1) we have

$$\frac{pF_2}{\Sigma_2(1+L^2B^2)} = (1+\tau B^2)\left[1 - \frac{F_1}{\Sigma_1}\frac{1}{(1+\tau B^2)}\right] \qquad (5B.5)$$

Hence comparing equation (5B. 1) with equation (5B. 5), we take

$$\frac{1}{\epsilon} = 1 - \frac{F_1}{\Sigma_1}\frac{1}{1+\tau B^2}\,, \qquad (5B.6)$$

and the coupling coefficients become:

$$\frac{\phi_1}{\phi_2} = \frac{\epsilon F_2}{\Sigma_1(1+\tau B^2)} \qquad (5B.7)$$

$$\frac{\phi_1^+}{\phi_2} = \frac{p}{(1+\tau B^2)} \qquad (5B.8)$$

On inserting these results into the one region generating time for a critical system, we obtain:

$$\Lambda_1 = \frac{1}{V_1\Sigma_1(1+\tau B^2)}\,,$$

$$\Lambda_2 = \frac{1/\epsilon}{V_2\Sigma_2(1+L^2B^2)} \qquad (5B.9)$$

It is seen that with this representation, the effect of fast fission is to reduce the thermal generating time by the fast fission factor. Other results have been given in the literature (41). Since the effect is small

there is no practical difference between them; equation (5B. 9) is perhaps the easiest and simplest representation.

5B. 2 Consistent Definition of Resonance Escape and Fast Leakage

In evaluating the effective precursor yield, we assumed values for the resonance escape probabilities for fast prompt and fast delayed neutrons and assumed values for the Fermi-age for these neutron groups. This appendix investigates the consistent relation of such assumed values to the results obtained by experiment or calculation for a single lumped fast group.

Since resonance absorption takes place in the kV region, well below the energies of both groups, it is unlikely that there is in fact any variation of the resonance escape probability. On the other hand, the symbol p is used to represent not only the true resonance escape probability between the groups, but also any true absorption within the fast group. Thus situations may arise where p differs from the prompt and delayed group.

The consistent definition of the parameters is again based on the requirement that the one group representation predicts criticality at the same loading as the multigroup representation. To determine the appropriate values of the p, we consider an infinite system; to obtain the appropriate τ we consider a finite system. As in the case of the fast fission, the Fermi-age will be an extensive parameter dependent (in theory) on the geometry of the system.

For the resonance escape we require consistency between the sets of equations:

$$-\left(D_f B^2 + \Sigma_f\right)\phi_f + \frac{k}{p}\Sigma_s \phi_s = 0$$

$$\left(p\Sigma_f \phi_f - D_s B^2 + \Sigma_s\right)\phi_s = 0 \qquad (5B.\ 10)$$

and

$$-(D_1 B^2 + \Sigma_1)\phi_1 + 0 + (1-\beta)\frac{k}{p_3}\Sigma_3 \phi_3 = 0$$

$$0 - (D_2 B^2 + \Sigma_2) + \beta\frac{k}{p_3}\Sigma_3 \phi_3 = 0 \qquad (5B.\ 11)$$

$$p_1 \Sigma_1 \phi_1 + p_2 \Sigma_2 \phi_2 - (D_3 B^2 + \Sigma_3)\phi_3 = 0$$

where for an infinite system we take B^2 to be zero. The consistency of equation (5B. 10) requires that k = 1. This result can be substituted in equation (5B. 11) to obtain a relation between the resonance escape probabilities:

$$p_3 = (1-\beta)p_1 + \beta p_2 \qquad (5B.\ 12)$$

At the same time, equation (5B. 12) provides a logical definition of the average resonance escape probability, p, used in equation (5B. 10). Thus if we take $p = (1-\beta)p_1 + \beta p_2$, we have a consistent definition of the resonance escape parameters to be used in the calculation of effective yields. It is likely that our knowledge of p_2 and p_3 comes in the form of the difference, $p_2 - p_3$. Then this knowledge and the value of p serve to fix consistent values of p_2 and p_3.

To determine the consistent values of the fast properties, we write equation (5B. 11) in the form:

$$-(\tau_1 B^2 + 1)\,\phi_1 + 0 + (1-\beta)\,\frac{k}{p}\,\frac{\Sigma_3}{\Sigma_1}\,\phi_3 = 0$$

$$0 - (\tau_2 B^2 + 1)\,\phi_2 + \beta\,\frac{k}{p}\,\frac{\Sigma_3}{\Sigma_2}\,\phi_2 = 0 \qquad (5B.\ 13)$$

$$p_1\,\frac{\Sigma_1}{\Sigma_3}\,\phi_1 + p_2\,\frac{\Sigma_2}{\Sigma_3}\,\phi_2 - (L^2 B^2 + 1)\,\phi_3 = 0$$

From the determinant of the coefficients of equation (5B. 13), we obtain the condition

$$(1+\tau_1 B^2)(1+\tau_2 B^2)(1+L^2 B^2) = \frac{k}{p}[(1-\beta)\,p_1(1+\tau_1 B^2) + \beta\,p_2(1+\tau_2 B^2)] \qquad (5B.\ 14)$$

On introducing the criticality condition, that

$$k = (1 + \tau B^2)\ , \qquad (5B.\ 15)$$

it follows that

$$\frac{1}{1+\tau B^2} = \frac{(1-\beta)\,p_1}{(1+\tau_2 B^2)\,p} + \frac{\beta p_2}{(1+\tau_1 B^2)\,p} \qquad (5B.\ 16)$$

For the practical case where, $p = p_1 = p_2$, equation (5B. 16) reduces to

$$\frac{1}{1+\tau B^2} = \frac{1-\beta}{1+\tau_2 B^2} + \frac{\beta}{1+\tau_1 B^2} \qquad (5B.\ 17)$$

Again we presume that τ is known from a combination of theory and experiment, while our knowledge of τ_2 and τ_1 is in the form of a known difference. Then equation (5B. 16) or equation (5B. 17), serve to determine consistent values of τ_1 and τ_2.

5B. 3 The Effect of Fast Fission on Precursor Effective Yields

A number of authors have pointed out that the difference in yields between the fissionable species, the difference in delayed neutron

energies and the difference in fission thresholds all combine to a complicated inter-relation (40, 6, 7, 49, 56). Krasik (61) concludes that the effectiveness of delayed neutrons is increased by the fast fission factor, ϵ, in rather the same way as we have shown the thermal generating time to be decreased in the presence of fast fission. Our previous result for the generating time was eminently reasonable; when the neutron group is partly 'short circuited' within the fast group, the generating time is decreased. The result quoted for the change in effective yields is by no means apparent. Since ϵ can be as high as 1.05 say in light water lattices, the effect is worth investigating in more detail.

We give an account of a much simplified system. As before, the relation of the multigroup constants to the fast fission factor (or the four factor formula) is somewhat arbitrary. In particular, for a homogeneous system, fast fission is naturally related to a fast group cross-section. For a heterogeneous system however, the fast fission effect takes place in the fuel elements themselves and is spatially proportional to the first collision density. In a two group scheme, the thermal flux provides a better spatial estimate of the density than the fast flux. Consider for example the edge of a highly reflected core, where the thermal flux will be peaked and the fast fission effect still high. The fast flux will be depressed around the interface however as far as it is given by the two group approximation.

The assumed equations for our system at critical are

$$\begin{aligned} -(\Sigma_1 + D_1 B^2)\,\phi_1 + (1-\beta)\,(F_1 + F_3)\phi_3 &= 0 \\ -(\Sigma_2 + D_2 B^2)\,\phi_2 + \beta\,(F_1 + F_3)\,\phi_3 &= 0 \\ p\,\Sigma_1\,\phi_1 + p\,\Sigma_2\,\phi_2 - (\Sigma_3 + D_3 B^2)\,\phi_3 &= 0 \end{aligned} \qquad (5B.18)$$

We have made a number of assumptions:

(a) One group of precursors with $\beta(U238) = \beta(U235)$ or no distinction to be made between species.

(b) Common resonance escape probability, $p_1 = p_2 = p$.

(c) One region system, with a unique B^2.

(d) Delayed neutron energies, approximately 0.5 Mev, do not lead to fast fission in U238 (a good approximation) or in U235.

(e) All fast fission effects are included in a thermal probability even though U235 undergoes fission over the full slowing down spectrum.

It is apparent that with these assumptions, the introduction of fast fission makes no formal difference to the results; our previous results hold with F replaced by $F_1 + F_3$, the fast fission probability plus the thermal fission probability, or ϵF_3 of the four factor formula. We conclude that the fast fission factor does not appear explicitly in the precursor effectiveness.

APPENDIX C TO SECTION 5

SOME CONDITIONS FOR SOLUTIONS WHERE THE FIRST ORDER APPROXIMATION DOES NOT HOLD

We investigate a system described by two groups of neutrons and one group of precursors, to demonstrate the limitation of the first order approximation in the flux. The system considered has application to the description of a reactor excited by a pulsed neutron source.

The general vector equation for the three equation system results in the form

$$s = \frac{\int \psi^{+}(M+T)\,\psi'\,dv}{\int \psi^{+} A\,\psi'\,dv} \tag{5C.1}$$

where the vector fluxes are, in this example,

$$\psi^{+} = \begin{bmatrix} \phi_1^{+} \\ \phi_2^{+} \\ C^{+} \end{bmatrix} \qquad \text{and} \qquad \psi' = \begin{bmatrix} \phi_1' \\ \phi_2' \\ C' \end{bmatrix} \tag{5C.2}$$

As usual we employ a just critical reference reactor so that

$$(R+T)^{T}\,\psi^{+} = 0; \quad C_i^{+} = \phi_i^{+} \tag{5C.3}$$

For a one region reactor, the shapes of the flux components ϕ_1' , ϕ_2' and C', are determined only by the geometry and can therefore be characterized as the shape of the reference fluxes. The coupling coefficients, however, are not yet determined; these coefficients will be functions of the inverse period, s, itself.

If the one region reactor is just critical, though with transient fluxes whose nature is to be determined, it is most convenient to use the reactor itself as its own reference state; The reactivity will then be identically zero for any transient flux. Equation (5C.1) becomes:

$$s = \frac{\int \psi^+ (M+T)\psi' \, dv}{\int \psi^+ A \psi' \, dv} = \frac{\int \psi \, (M+T)^T \psi^+ \, dv}{\int \psi^+ A \psi' \, dv} = 0 \qquad (5C.4)$$

Then for any flux employed in evaluating s, including the first order approximation, we obtain the value zero from Equation (5C.4). However, in view of the three simultaneous equations, we anticipate two other roots for s in the first harmonic. It is clear that the single matrix equation is not suitable for finding the other roots, s_2 and s_3 say, but only for finding small changes in the asymptotic root, s_1. The unperturbed flux, ψ , is only close to that ψ' corresponding to the resident population. Higher eigenfunctions, even within the harmonic, may be appreciably different to ψ .

A second root may be obtained by casting Equation (5C.4) into the form of the inhour equation, i.e., by using the algebraic coupling coefficient for C/ϕ_2. If the equations for the lowest harmonic are

$$\begin{aligned} (B^2 D_1 + \Sigma_1)\,\phi_1 + (1-\beta)\,F\,\phi_2 + \lambda C &= \frac{s}{V_1}\,\phi_1 \\ p\,\Sigma_1\,\phi_1 - (B^2 D_2 + \Sigma_2)\,\phi_2 &= \frac{s}{V_2}\,\phi_2 \\ \beta F\,\phi_2 - \lambda C &= sC \end{aligned} \qquad (5C.5)$$

then we have the coupling coefficients

$$C = \frac{\beta F}{\lambda + s}\ \phi_2$$

$$\phi_2 = \frac{p\Sigma_1}{\frac{s}{V_2} + B^2 D_2 + \Sigma_2}\ \phi_1 = \frac{p\Sigma_1}{\Sigma_2}\ \frac{1}{(1 + s\ell_2)(1 + L^2B^2)}\ \phi_1 \qquad (5C.6)$$

where as before,

$$\ell_2 = \frac{1}{V_2 \Sigma_2 (1 + L^2 B^2)} \qquad (5C.7)$$

It is seen that the coupling coefficient between the fast and slow flux is modified by the factor $(1 + s\ell_2)$. The coupling coefficient for the importance is independent of s, or rather, is the coefficient appropriate to the value s = 0:

$$C^+ = \phi_1^+$$

$$\phi_1^+ = \frac{p\Sigma_1}{B^2 D_1 + \Sigma_1}\ \phi_2^+ = \frac{p}{(1 + \tau B^2)}\phi_2^+ \qquad (5C.8)$$

The inhour equation obtained from Equation (5C.1) by substituting for the precursor density is

$$s\bar{\Lambda}' = \bar{\rho}' - \bar{\beta}' + \frac{\lambda}{\lambda + s}\ \bar{\beta}' \qquad (5C.9)$$

In the present example, the reactivity is zero for any flux shape. Since no account of energy differences for prompt and delayed neutrons is made in this representation, $\bar{\beta}'$ reduces to β, the measured or linear yield. Hence

$$s\bar{\Lambda}' = \frac{s}{\lambda + s} \qquad (5C.10)$$

Whereas the first order approximation when employed in Equation (5C.1) yielded only the root zero, the first order approximation ($\bar{\Lambda}$ for $\bar{\Lambda}'$) in Equation (5C.10), yields a quadratic form for s.

One root is immediately zero and the second found to be

$$s_2 = -(\lambda + \frac{\beta}{\Lambda}) \tag{5C.11}$$

Since $1/\Lambda$ is large, the second root is approximately $-\beta/\Lambda$, the behavior of the reactor as a prompt critical system,$-\beta$ below prompt critical.

If there had been only one group of neutrons considered, the second solution of Equation (5C.11) would be exact. The actual error in the second root, and the undetermined third root, can be evaluated in this particular case of the one region reactor, since the coupling coefficients between the neutron fluxes are algebraic and can be substituted into the expression for the generating time. It is seen that the error in the first order approximation corresponds to taking $1 + s\ell_2$ to be unity in the coupling coefficient of Equation (5C.6). Since this is of the order of β to 1, it is well neglected.

The third root, s_3, is obtained by substituting the coupling coefficient in the exact value of the generating time. This will only be possible in one region cases; in general, there will be no unique value of B^2 and hence no unique coupling coefficient to substitute. At best we could investigate the equivalent one region reactor to obtain a rough estimate of the missing root, or to estimate the error in the first order approximation. The exact generating time in the present problem becomes

$$\bar{\Lambda}' = \frac{\frac{\phi_1^+ \phi_1'}{V_1} + \frac{\phi_2^+ \phi_2'}{V_2}}{\phi_1^+ F \phi_2'} = \frac{(1 + \tau B^2)\frac{1}{V_1 p \Sigma_1}\frac{\Sigma_2}{}(1 + L^2 B^2)(1 + s\ell_2) + \frac{1}{V_2}}{\frac{F}{1 + \tau B^2}} \tag{5C.12}$$

$$= (1 + s\ell_2)\ell_1 + \ell_2$$

Then the exact inhour equation becomes

$$s[\ell_1(1 + s\ell_2) + \ell_2] + \frac{s\beta}{\lambda + s} = 0 \qquad (5C.13)$$

Again we have the immediate root zero and the quadratic:

$$\ell_1\ell_2 s^2 + (\ell_1 + \ell_2 + \ell_1\ell_2\lambda)s + \beta + (\ell_1 + \ell_2)\lambda = 0 \qquad (5C.14)$$

Further simplification occurs on using typical values for the parameters of a thermal reactor, listed in Table 5C.1. We obtain

$$s^2\ell_1\ell_2 + s(\ell_1 + \ell_2) + \beta = 0 \qquad (5C.15)$$

TABLE 5C.1 TYPICAL VALUES OF KINETICS PARAMETERS

$\lambda = 0.08$	$\beta = 0.007$
$\ell_1 = 5 \times 10^{-5}$ sec	$\ell_2 = 1 \times 10^{-3}$ sec

The roots of Equation (5C.15) can be obtained by expansion and the neglection of terms in β compared to unity:

$$s_2 = \frac{-\beta}{\ell_1 + \ell_2} = -\beta/\Lambda$$

$$s_3 = \frac{-\ell_1 + \ell_2}{\ell_1\ell_2} = \frac{-\Lambda}{\ell_1\ell_2} \qquad (5C.17)$$

It will be seen that the second root has the value obtained from the inhour equation with the first order approximation. To a good approximation, the third root is given by the reciprocal slowing down time. On introducing this value of s into the coupling coefficient, Equation (5C.6), we find that $1 + s\ell_2$ is approximately - 19, certainly not close to unity.

Having found good approximate values for the eigenvalues, it is now possible to express the first three eigenfunction solutions explicitly, in terms of the coupling coefficient between the vector flux components and in terms of the common spatial distribution. It is easily verified that only the lowest mode has coupling coefficients that are both positive. The higher, non-physical or transient modes, have components of opposite signs. The second and third solutions and their eigenvalues are easily understood to provide the mechanism whereby an initial distribution that is not in the steady state distribution, can decay away. The neutrons out of balance with the precursors decay away with the second or prompt root, $-\beta/\Lambda$. The fast neutrons not in balance with the thermal neutrons decay away with the third root, the reciprocal of the fast or slowing down time.

Consider the initial condition that all the neutrons are introduced into the system as fast neutrons, with a central flux a. Then the thermal flux decays in time as the sum of the three terms: $a_1\exp(s_1 t) + a_2\exp(s_2 t) + a_3\exp(s_3 t)$. These coefficients, after some manipulation, are found to be given by the vector equation

$$a\begin{bmatrix}1\\0\\0\end{bmatrix} = a_1\begin{bmatrix}\frac{pV_2\Sigma_1}{\ell_2}\\1\\\frac{\beta\Sigma_2}{\lambda}\end{bmatrix} + a_2\begin{bmatrix}\frac{pV_2\Sigma_1}{\ell_2}\\1\\-\Lambda\Sigma_2\end{bmatrix} + a_3\begin{bmatrix}\frac{-pV_2\Sigma_1\ell_1}{\ell_2^2}\\1\\-\beta\Sigma_2\ell_1\end{bmatrix} \qquad (5C.18)$$

Any other initial conditions can be substituted for the left-hand vector to enable the expansion coefficients to be determined.

SECTION SIX: APPLICATIONS TO STATICS

6.1 Introduction

The focus in Section Five, Applications to Kinetics, has been on the definition and interrelation of the parameter, $\bar{\rho}'$, $\bar{\Lambda}'$ and $\bar{\beta}'$ which relate to the measurable inverse period, s. Through the reactivity, $\bar{\rho}'$, was defined and discussed from the viewpoint of interpreting experiments, no detailed account was given of how perturbation methods can be used to calculate reactivity. It will be remembered that there is no unique value of the reactivity in a reactor but that its value depends on (a) definition of production and destruction and (b) selection of the reference reactor. The use of the bilinear averaging in our definition of $\bar{\rho}'$ not only improved the accuracy of subsequent first order approximations, but also enabled the effect of the reactivity to be related directly to a perturbation or the difference between the actual and reference reactors.

The peculiar advantages of the grouping of the reactor parameters adopted for reactivity, is that with considerable accuracy we can base the calculation of $\bar{\rho}'$ on a flux spectrum neglecting the division of production between precursors and prompt neturons.* In this section we discuss several ways in which perturbation methods can be used to calculate reactivity

*Hurwitz, (65), calls such an approximate calculation of the reactivity the "prompt reactivity." We find the term misleading since it is more commonly used in the sense that precursors are assumed to make no contribution at all.

effects in reactors. Before undertaking this survey, we must investigate the basis on which reactivity effects can be compared.

6.2 Equivalence of Perturbations

(a) Kinetic

Often the situation arises when a more complicated effect is to be represented by a simple approximation and the question is then asked, how can the approximation be made to represent the actual situation (or represent perhaps a more accurate and sophisticated representation of the actual situation) ? For example, what uniform change of absorption cross-section is equivalent to a spatially dependent absorbing sample. The basic question to resolve however is in what way two such samples are to be equivalent.

First, we might ask that the actual reactor and an equivalent reactor should have the same asymptotic period. We can say that the two reactors are kinetically equivalent. In terms of our general matrix notation we demand that

$$\frac{\int \psi_a^+ (M_a + T_a)\,\psi_a'\,dv}{\int \psi_a^+ A_a\,\psi_a'\,dv} = \frac{\int \psi_b^+ (M_b + T_b)\,\psi_b'\,dv}{\int \psi_b^+ A_b\,\psi_b'\,dv} \qquad (6.1)$$

Here we have arbitrary importances, ψ_a^+ and ψ_b^+, for the reactors with properties M_a, A_a and M_b, A_b respectively. We must in addition demand that ψ_a' and ψ_b' are the resident flux distributions

corresponding to the two reactors.

If we wish to equate two perturbations, i.e., parts of M_a and M_b, we will obviously select a common reference reactor, $\psi_a^+ = \psi_b^+$. Furthermore since the importance is defined by

$$(R+T)^T \psi^+ = sA\psi^+ \tag{6.2}$$

we have the cancellation in equation (6.1) of the terms

$$\frac{\int \psi^+ (R+T)\psi_a' \, dv}{\int \psi^+ A_a \psi_a' \, dv} = \frac{\int \psi^+ (R+T)\psi_b' \, dv}{\int \psi^+ A_b \psi_b' \, dv}, \tag{6.3}$$

to leave only the requirement for kinetic equivalence, that

$$\frac{\int \psi^+ (P_a + \delta T_a)\psi_a' \, dv}{\int \psi^+ A_a \psi_a' \, dv} = \frac{\int \psi^+ (P_b + \delta T_b)\psi_b' \, dv}{\int \psi^+ A_b \psi_b' \, dv} \tag{6.4}$$

Unless we are looking specifically at changes in the precursor decay probabilities, we will have $\delta T_a = \delta T_b = 0$ in equation (6.4), leaving differences between P_1 and P_2, in the production and the destruction terms. If in addition the group speeds are unchanged, we have $A_a = A_b$. Then in these circumstances there will be negligible differences in either the $\bar{\beta}_i'$ or the $\bar{\Lambda}'$.

To a good approximation, then for the kinetic comparison of most perturbations, it is sufficient then to equate the corresponding reactivities:

$$\bar{\rho}'_a = \frac{\int \phi^+ P_a \phi'_a \, dv}{\int \phi^+ F'_a \phi'_a \, dv} = \bar{\rho}'_b = \frac{\int \phi^+ P_b \phi'_b \, dv}{\int \phi^+ F'_b \phi'_b \, dv} \tag{6.5}$$

When equation (6.5) is satisfied we can say that the actual reactor and the reference reactor plus the equivalent perturbation will have the same resident period and be kinetically equivalent. There is no reason to suppose in equation (6.5) that the resident fluxes ϕ'_a and ϕ'_b are the same nor the production probabilities, F'_a and F'_b. Furthermore there is no reason to suppose that there is any unique value of the equivalent perturbation, since we have not specified whether it is to be a perturbation in the production terms or any of the destruction terms. It may be convenient to consider standard types of equivalent perturbation, e.g., an addition of a uniform 1/V absorber. Even when the equivalent perturbation is defined to be of a unique type, it may have a value dependent on the reference reactor employed. That is to say, the equivalence of the reactivities as given in equation (6.5) is not sufficient to give kinetic equivalence unless the lifetimes and effective precursor yields are exactly equal.

We could demand that the reactivity effects of perturbations in each group should be separately equated to standard equivalent perturbations. This concept is taken up in more detail in Section 6.6. In that section we consider equivalent cross-sections and their relation to equivalent perturbations.

(b) Static

A conceptually different definition of the equivalence of perturbations is based on the static equivalence. The static equivalence of two perturbations does not require that two reactors are put on the same period. Rather it demands that a critical reference reactor which has been made non-critical by a perturbation P should be returned to critical by a second, subsequent equivalent static perturbation, P_s. Again we have no unique answer for P_s if we do not define a standard type of perturbation to return an actual reactor to critical. Once a standard static perturbation has been defined in a suitable form however, there is only one value of that perturbation which will make an actual reactor critical. The proof of this theorem is given by Birkhoff and Varga (62).

Two common examples of such standard static perturbations are (a) uniform addition (or removal) of absorption cross-section and (b) uniform fractional change of ν until the reactor is critical. Thus, if we assume some properties of a reactor, M, we calculate the spatially dependent equation $M\phi_s = 0$ for varying values of ν until the equation is in fact satisfied at every point in the reactor. For a single type of fuel loading, we have changed the true physical value of ν at every point in the reactor. Even in the reflector, ν can be thought to have the common uniform value assigned; it is the fission probability, Σ_f, that is zero. Hence $\delta\nu/\nu$, the

uniform fractional change of ν that we have imposed on our assumed reactor, is an eigenvalue independent of any averaging system. Physically it is certainly reasonable to suppose that there is only one value of the fractional change of the production in the reactor that will exactly balance the destruction. We must note that the flux distribution ϕ_s, calculated for this artificially critical case, is not necessarily the flux distribution, ϕ', that the reactor would have if it were built with properties M and the true physical value ν_o.

The static equivalence of perturbations, especially using changes in ν as a standard, is convenient for a number of reasons. First, the resulting $\delta\nu/\nu_o$ is unique (once the standard of equivalent perturbation has been defined). Secondly, since ν appears in very few terms in the reactor balance, it makes calculations considerably easier. Thirdly, both for hand and machine calculations, the treatment of only the three spatial dimensions rather than the spatial dimensions plus the fourth time dimensions of the kinetic analysis, is a great saving. It is usually much easier to do a few calculations in 3 dimensions and by iteration or interpolation find the value of ν to satisfy $M\phi_s = 0$ than it is to find the solution of $M\phi' = V^{-1} \frac{\partial \phi'}{\partial t}$. This is partly because ν appears in so few elements of the matrix comprising M.

It is also true to say that the question in a reactor physicist's mind is often what will make this reactor critical

rather than what will this reactor do if left as it is. However, it will be realized that calculations of $\delta\nu/\nu_o$ are only an approximate (though useful) answer for the question that asks how many control rods or how much extra fuel is to be put in a reactor.

Due to its comparative simplicity, the calculated $\delta\nu/\nu_o$ is often used to characterise control rod calculations. It is also dmployed quite widely to express the calibration of control rods from kinetic measurements, e.g., Benedict (4). We now discuss the relation between the measure of what the reactor is doing (the kinetic reactivity) and the measure of what needs to be changed to stop the reactor doing it (the static reactivity).

Calculations of the fractional change of ν required for criticality are not necessarily done using either perturbation theory or even any of the variational methods. The fractional change of ν is an eigenvalue for any mixed composition of fuels and is independent of any weighting scheme. However, we can use the perturbation scheme to relate the static perturbation, P_s, to the requirement that the actual reactor, with properties, M, is made critical. The condition is that some physically admissible ψ_s is found such that at every point in the reactor

$$(M + T - P_s)\psi_s = 0 \qquad (6.6)$$

i.e., that there is a second perturbation, P_s, that renders the reference reactor plus actual perturbation critical.

For any reference reactor and hence for any importance, ψ^+, or weighting function, it follows that

$$\int \psi^+ (M + T + P_s)\, \psi_s \, dv = 0 \tag{6.7}$$

or

$$\int \psi^+ P_s \, \psi_s \, dv = - \int \psi^+ (M+T)\, \psi_s \, dv \tag{6.8}$$

Consider the standard equivalent static perturbation, where a fractional change in ν is made. Each term in F', the actual production probability, is changed by $\delta\nu/\nu_o$ or the standard perturbation P_s is $\frac{\delta\nu}{\nu_o} F'$. Then equation (6.8) becomes

$$\int \psi^+ \frac{\delta\nu}{\nu_o} F' \, \psi_s \, dv = (\frac{\delta\nu}{\nu_o}) \int \psi^+ F' \, \psi_s \, dv = - \int \psi^+ (M+T) \, \psi_s \, dv \tag{6.9}$$

Hence

$$- (\frac{\delta\nu}{\nu_o}) = \frac{\int \psi^+ (M+T)\, \psi_s \, dv}{\int \psi^+ F' \psi_s \, dv} \tag{6.10}$$

6.3 The Static Reactivity

We should compare equation (6.10) with our definition for the kinetic reactivity, $\bar{\rho}'$:

$$\bar{\rho}' = \frac{\int \psi^+ (M+T) \, \psi' \, dv}{\int \psi^+ F' \, \psi' \, dv} \tag{6.11}$$

We see that the integrals in equation (6.10) and (6.11) differ only in having either ψ_s or ψ'. The former is the flux

on returning the reactor to critical by changing ν ; the latter is the flux in the actual reactor. Evidently from the ratios involved, the normalization or power level of the fluxes is immaterial at this point. From the close resemblance of the two equations, $(-\frac{\delta\nu}{\nu_o})$ will be called the static reactivity, ρ_s. The negative sign appears since it is the change of ν to return the reactor to critical rather than the departure from critical that ρ_s measures. The kinetic reactivity, $\bar{\rho}'$, is either the value arrived at from the interpretation of a reactor period or else the value to be inserted, together with $\bar{\Lambda}'$ and the $\bar{\beta}_i'$, in an inhour equation to predict the period. The static reactivity, ρ_s, is a mathematical computation device. In certain circumstances, the magnitude of the two is closely the same.

Since ρ_s is an eigenvalue, the magnitude given by equation (6.10) is independent of the reference reactor or importance employed. The kinetic reactivity does depend to some degree on the selection of the reference reactor, however. Corresponding to the static flux, ψ_s, there is a static importance, ψ_s^+, in the reactor that has been rendered critical by the change in ν. This importance is defined via

$$(M + T + P_s)^T \psi_s^+ = 0 = (M + T - \rho_s F')^T \psi_s^+ \qquad (6.12)$$

We may employ this importance and reactor as the reference in calculating the kinetic reactivity. To denote

the special, unique value of the reference reactor, we write this kinetic reactivity as ${}^{s}\bar{\rho}'$; the superscripts indicating the appropriate flux averaging:

$$ {}^{s}\bar{\rho}' = \frac{\int \psi_s^+ (M+T)\, \psi'\, dv}{\int \psi_s^+ F'\, \psi'\, dv} = \frac{\int \psi' (M+T)^T \psi_s^+\, dv}{\int \psi'\, F'^T \psi_s^+\, dv} $$

$$ = \frac{\int \psi'\, F'^T \psi_s^+\, dv}{\int \psi'\, F'^T \psi_s^+\, dv}\ \rho_s \qquad (6.13) $$

Hence we have ${}^{s}\bar{\rho}' = \rho_s$, or the two reactivities are identical when ψ_s^+ is employed as the importance in the reference reactor.

If our calculation of the kinetic reactivity is based on a reference reactor that is reasonably close to the static reactor, then $(\psi^+ - \psi_s^+)$ will be small and there will be a negligible difference between $\bar{\rho}'$ and ρ_s. This might not apply to a calculation of $\bar{\rho}'$ based on a linear weighting. There is therefore an extra reason to favour the use of the bilinear weighting in calculating the kinetic reactivity, above the considerations of accuracy and explicit appearance of the perturbation.

Corresponding to the kinetic reactivity employing the static importance, ψ_s^+, we must use values of $\bar{\Lambda}'$ and $\bar{\beta}_i'$ which consistently employ the same static importance, i.e.,

${}^{s}\bar{\Lambda}'$ and ${}^{s}\bar{\beta}_i'$. With these parameters, the kinetic behaviour will be correctly predicted via the static reactivity.

Just as we can have a static reactivity, there are analogous expressions for the static excess multiplication and the static effective multiplication. These are related to the fractional change of ν by $\delta\nu$ from the physical value ν_o by the expressions

$$\rho_s = -\frac{\delta\nu}{\nu_o} \tag{6.14}$$

$$(k_{ex})_s = -\frac{\delta\nu}{\nu} \tag{6.15}$$

$$(k_{eff})_s = \frac{\nu_o}{\nu} \tag{6.16}$$

In these expressions, ν_o gives the actual production, ν the production at static critical which is then equal to the actual destruction. Often ν_o/ν is written as an eigenvalue, $1/\lambda$, so that the static equation appears as,

$$-(\nabla \cdot D'\nabla - \Sigma')\phi_s = \lambda F'\phi_s \tag{6.17}$$

In the present work, we have delayed the introduction of the static reactivity in order to emphasise that it plays no direct role in reactor kinetics. It is the kinetic reactivity as we have defined it, that determines the kinetic behaviour or alternatively, is determined from period

measurements in the reactor. Under certain conditions, the numerical values of the two reactivities may be the same to first order. In special cases (the one energy, one region reactor) the two reactivities are identical since both importances have the same shape. This identity for the special case has resulted in some misunderstanding of the different roles and the essential conceptual difference. Thus Hurwitz (66) and Henry (6, 7) introduce the static reactivity directly into reactor kinetics, while Webster (63) concludes that the two reactivities are identical. The latter's proof however supposes that there is a unique fast non-leakage probability in the general reactor, which we have shown is not true (Section 5). Weinberg and Wigner (1) have pointed out that in general, the static and kinetic reactivities are not identical.

Control Rod Calibration: Our general ideas have immediate application to the question of the calibration of control rods. We have discussed the fact that the effect of control rods is most easily calculated in terms of the static reactivity. When the reactor is built and the effect of the control rods measured experimentally, the calibration is commonly of a kinetic nature, i.e., in terms of the kinetic reactivity. For obvious reasons the change of period associated with the pulling of a rod is to be limited and the rod can only be calibrated in a number of steps. Each

step yields so small a value for the reactivity worth that it may equally well be regarded as the excess multiplication value for the step. The question arises as to how the individual measurements are to be combined to represent the over-all worth of the rod, if it were withdrawn in one step.

Two methods of combining the individual effects are commonly in use. In one, the effective multiplication of each step is multiplied together to give a product effective multiplication. In the second (used in the MITR) the reactivity worths are summed. These two methods differ only when the total worth is large; the error is of the order $\frac{nx^2}{2}$ where n is the number of steps and x the fractional worth of each step. An alternative to the second method would be to sum the worths of all steps but to interpret the result as an excess multiplication. For a bank of rods worth say 20 percent, this alternative method differs by some one part in four or five. This serious difference is to be resolved.

The resolution comes on considering the reactivity as the increase as a fraction of the production. When the production in a reactor is essentially unchanged during the over-all change or rod configurations, then the individual reactivities are additive. When the control rod change results in the destruction being constant, then the excess multiplication, based on the destruction, is additive. The condition of constant production, identical with the condition that the

generating time be constant, is in fact well met by control systems using black absorbing rods. Such rods operate predominantly by altering the leakage probability and hence the destruction rate. Where grey rods are used to displace fuel, it is the production probability alone that is basically altered and hence the lifetime is constant and the excess multiplication is additive.

Intermediate cases can be handled if some estimate of the change of production to the change of destruction is available. Then an intermediate measure of criticality can be constructed which has the additive property; the final summed worth can then be expressed back as a reactivity or as an excess multiplication as desired. We have not been able to adduce a case for the product of effective multiplications.

Using the foregoing prescription, it is then possible to obtain a fair estimate of the total kinetic worth of a control rod bank in a reactor from experimental determinations. This value of reactivity is then to be compared with the static reactivity worth of the unrodded reactor using the mathematical device of changing ν. If such a computation is undertaken, it becomes comparatively simple to determine the static importance and to use this weighting function in calculations of the generating time and the

effective yields. The unrodded reactor is chosen to enable the calculations to be carried out with clear boundary conditions and a feasible reactor geometry. Under the conditions of the experiment, the effective yields and the generating time will only vary through variations of the flux shape. Although this variation of the flux may itself be large, the effect on $\bar{\beta}_i'$ and $\bar{\Lambda}'$ is very small due to the cancellation of similar integrals in the numerator and the denominator. Thus the one calculation of generating time and effective yield will serve to determine an accurate relation between the observed period and the kinetic reactivity. However, due to the use of the static importance, the kinetic reactivity is now identical with the static reactivity. We can thus put considerable confidence in comparing experimentally determined kinetic reactivities with calculated static reactivities for the testing of theories, etc.

6.4 The Enhancement

In Section 5.6, the statistical weight was discussed with a view to interpreting the effect of samples or perturbations. It was there pointed out that if relative effects only were to be considered, then the normalization of either the importance or the flux was immaterial. In the present section we put forward a normalised form of the statistical weight that brings out more clearly the effect of spatial

variations. It is useful for many purposes to be able to estimate the increased effect on reactivity when various samples are not distributed uniformly through the reactor but are concentrated or distributed in different characteristic fashions. For example, irradiation samples are small and can often be regarded as being concentrated at a point; fuel burn-up takes place over the reactor according to the flux distribution; fission product poisons are formed in a fashion dependent on flux but with a final concentration that is affected by decay.

The enhancement, a , is defined as the ratio of the actual reactivity effect of a sample to the effect of the sample spread uniformly over the reactor. Where the first order approximation is valid, the enhancement is evidently the ratio of the statistical weight averaged over the actual sample to the average of the statistical weight over the whole reactor. We shall calculate values of the enhancement for several illustrative cases based on the first order approximation for simple one region systems. The results bring out the effects of the geometric distribution which can change the uniform reactivity worth by factors up to 8. The results for the enhancement are useful for rough calculations of many types even when the first order approximation does not hold, as for a small black sample say. For in such cases, it is often possible to combine the results for the enhancement

with correction factors that give the effective cross-section or effective perturbation in the unperturbed flux. Such factors are discussed in more detail in 6.6. We shall consider only absorbing and fissioning processes as the most typical reactor calculations. In the remainder of Section 6, the use of $\bar{\rho}$ rather than $\bar{\rho}'$ will indicate that the first order approximation is being made.

Consider a given amount of absorbing material distributed uniformly over various sizes of volumes, v_o, in a reactor. The product $v_o \delta\Sigma$ is constant, $\delta\Sigma$ being the absorption cross-section of the sample when occupying the volume v_o. The reactivity effect of the sample is then proportional to the weighted integral over the volume v_o:

$$\bar{\rho} \sim \int_{v_o} \phi^+ \delta\Sigma \, \phi \, dv \sim \delta\Sigma \int_{v_o} \phi^+ \phi \, dv \tag{6.18}$$

If the average value of the statistical weight over v_o is given by

$$v_o \langle \phi^+ \phi \rangle = \int_{v_o} \phi^+ \phi \, dv \quad , \tag{6.19}$$

then it follows that

$$\bar{\rho} \sim \delta\Sigma \, v_o \langle \phi^+ \phi \rangle \sim \langle \phi^+ \phi \rangle \tag{6.20}$$

Since $v_o \delta\Sigma$ is constant, the ratio of reactivity effects

is given by the ratio of average statistical weights in the two volumes:

$$\frac{\bar{\rho}_a}{\bar{\rho}_b} = \frac{\langle \phi^+ \phi \rangle_a}{\langle \phi^+ \phi \rangle_b} \tag{6.21}$$

For the particular case when the first volume corresponds to the concentration of the sample at a point, $\underline{r}$, compared to the sample spread over the whole reactor, we have the enhancement, a , to be

$$a = \frac{\bar{\rho}_a}{\bar{\rho}_b} = \frac{\phi^+(\underline{r})\ \phi(\underline{r})}{\overline{(\phi^+ \phi)}} \tag{6.22}$$

the barred $\overline{\phi^+ \phi}$ indicating the reactor average. In Table 6.1, the enhancement is evaluated for a centrally located sample in three simple geometries using the one group approximation. The results apply to either fissioning or absorbing samples, but contain the inherent error of the first order approximation. In many cases, the concentration of the sample at a point is only a mathematical fiction that leads to little error in evaluating the average statistical weight of the small region. The very large enhancement on concentrating a sample at the centre is brought out by the results of Table 6.1. The value 8 for the rectangular reactor is related to the average of $\cos^2$ over the range (1/2) in the three axial directions. Similarly the cylindrical results

Table 6.1 The Enhancement – One Group, Bare Reactors For Absorbing and Fissioning Samples at the Reactor Centre

Box	8
Cylinder	7.42
Sphere	6.57

contain a factor of 2 from the concentration along the axis and the remaining 3.71 from the concentration radially to the centre (1/3.71 is the average of J_o^2 over the range).

A further application of the enhancement, α, is to the field of fuel burn-up. The importance of allowing for the spatial distribution of the burnt fuel in evaluating the reactivity loss is fairly well known both for high flux research reactors (66) and for power reactors, e.g., (67). Again we give simple results for the enhancement in three simple geometries that indicate the magnitude of the spatial effects.

Consider a one region system where the flux shape is approximately unchanged through the lifetime of the reactor. Hence the flux can be written as a constant flux, $\bar{\phi}$, (in shape and magnitude) and the exposure expressed as some average time,* c(T), for which the reactor has been exposed

*This time differs from the average time at a given power output due to the burn-up of fuel; the flux must increase to maintain steady power.

to the assumed constant flux up to time T. This average time is related to the flux-time, θ, at any point r, by

$$\theta(\underline{r}, t) = \int_0^T \phi'(\underline{r}, t)\, dt \approx \bar{\phi}(\underline{r})\, c(T) \tag{6.23}$$

Wolfe, in a more detailed account of the present method (67), has shown how the average time can be related to the total delivered thermal energy in Mw-days.

The burn-up of fuel, $\delta\Sigma_u$, in time δt is given by

$$\delta\Sigma_u = \sigma_u\, \delta n_u = -\sigma_u\, \phi'\, \Sigma_u\, \delta t = -\sigma_u \Sigma_u\, \delta\theta \tag{6.24}$$

where σ_u is the microscopic cross-section and n_u is the atomic concentration of the fuel, atoms / cms^3. Equation (6.24) is in fact easily solved in terms of exponentials and the initial value $\Sigma_u(o)$ for finite exposure.* We shall only consider the short exposure effects and indicate the long term departures qualitatively. The perturbation caused by the burn-up of fuel along – neglecting fission product poisons – is well represented by the change in thermal absorption and fission production probabilities:

*If $\Sigma_u(o)$ is the initial value of Σ_u, then

$$\Sigma_u = \Sigma_u(o)\, e^{-\sigma_u \theta} = \Sigma_u(o)\, e^{-\sigma_u \bar{\phi} c}$$

$$\delta\Sigma = -\Sigma_u \sigma_u \delta\theta \tag{6.26}$$

and

$$\delta F = -\epsilon p \Sigma_f \sigma_u \delta\theta \tag{6.27}$$

Then since we have $\nu\Sigma_f = \eta\Sigma_u$, the perturbation (neglecting minor changes in fast fission factor, etc.) is given for a one group model by

$$P = -(\epsilon\eta - 1) \Sigma_u \delta\theta \sigma_u \tag{6.28}$$

For the one region system, the difference between the importance of thermal neutrons absorbed in the fuel and fast neutrons produced by fission is just the fast non-leakage probability, $\overline{P}$. Taking the spatial distribution of the importance to be the same as for the thermal flux, the reactivity effect of the burn-up is given by

$$\overline{\rho} = \frac{\int \phi^+ P\phi \, dv}{\int \phi^+ F' \phi \, dv} = -\frac{\int \phi(\epsilon\eta \overline{P} - 1) \Sigma_u \sigma_u \delta\theta \, \phi \, dv}{\int \phi(1 - \sigma_u \delta\theta)\epsilon\eta \overline{P} \Sigma_u \phi \, dv} \tag{6.29}$$

Substituting $\delta\theta = \phi \, \delta c$, we obtain

$$\overline{\rho} = \frac{\int (\epsilon\eta \overline{P} - 1) \sigma_u \delta c \, \phi^3 \, dv}{\int \epsilon\eta \overline{P}(\phi^2 - \sigma_u \delta c \, \phi^3) \, dv}$$

$$= -\frac{(\epsilon\eta \overline{P} - 1) \sigma_u \delta c \int \phi^3 \, dv}{\epsilon\eta \overline{P} \int \phi^2 \, dv - \epsilon\eta \overline{P} \sigma_u \delta c \int \phi^3 \, dv} \tag{6.30}$$

To obtain the enhancement we should compare the reactivity loss given by equation (6.30) with the loss in a duplicate reference reactor with uniform removal of the same total amount of fuel. Denoting such a reactivity by $\langle \rho \rangle$ and the spatial average flux by $\langle \phi \rangle$, we have

$$\langle \rho \rangle = \frac{(\epsilon\eta P - 1)\, \sigma_u\, \delta c\, \langle \phi \rangle}{\epsilon\eta P(1 - \sigma_u\, \delta c\, \langle \phi \rangle)} \tag{6.31}$$

Then the enhancement is given by

$$a = \bar{\rho}/\langle \rho \rangle = \frac{\langle \phi^3 \rangle}{\langle \phi^2 \rangle \langle \phi \rangle} \tag{6.32}$$

Table 6.2 lists results for the three bare geometries.

Table 6.2 The Enhancement for Fuel Burn-up. Short Burn-up Times and Bare One Group Reactors.

Box	2.37
Cylinder	2.23
Sphere	2.03

For a number of reasons, the results listed in Table 6.2 are pessimistic. In the first place, larger burn-up times will decrease the enhancement as the burn-up flattens the spatial distribution of the product of

flux and fuel. In addition, the flux will be further flattened by the build up of fission products. Furthermore, the intelligent programming of control rods can flatten the distribution throughout the life of the charge. Finally, the enhancement is actually a strong function of reflector savings. Quantitative results for reflected reactors will be given in the ensuing section in a discussion of Xenon poisoning.

6.5 Fission Products-Xenon

The build-up of fission products can be treated in a fashion similar to the treatment of fuel burn-up. We have an additional complication in the case of those fission products, such as Xenon, that undergo radio-active decay. This time we shall derive an expression for the reactivity loss for a system with uniform production and burn-out of Xenon to show how the results differ from some of the conventional expressions. We shall then give the geometric correction factor to allow for the spatial dependence as an extension of the work of Chernick (68), (69).

(a) Point System Xenon Reactivity Loss (No Spatial Dependence): The introduction of the Xenon cross-section into the Fermi-age expression for reactivity (Section I, Table 1.1) gives

$$\rho = \frac{\delta\Sigma}{F\, e^{-\tau B^2}} = -\frac{\Sigma_{xe}}{\Sigma(1 + L^2 B^2)} \tag{6.33}$$

where we have assumed the reactor originally critical and the only perturbation to be the build-up of the Xenon. The expression for the reactivity can be expressed in a number of ways. First, with Glasstone and Edlund, we define a poisoning, P_o, by

$$P_o = \frac{\Sigma_{xe}}{\Sigma_a}, \tag{6.34}$$

and writing

$$\gamma = \frac{\Sigma_m}{\Sigma_u} \quad \begin{matrix}\text{(moderator)} \\ \text{(uranium)}\end{matrix}, \tag{6.35}$$

we have

$$\rho = \frac{-P_o}{(1 + \gamma)(1 + L^2 B^2)} \tag{6.36}$$

where L^2 has its original value. The expression differs from that of Glasstone and Edlund by the term in $(1 + L^2 B^2)$. Because the Xenon cross-section does not change the leakage, DB^2, the relative thermal leakage is decreased when the thermal absorption cross-section increases. Hence the reactivity loss is slightly less when this effect is considered.

A second way to write the expression for the reactivity loss is to introduce the steady state expression for the Xenon:

$$\Sigma_{xe} = \frac{y\,\Sigma_f\,\phi\sigma_{xe}}{\phi\sigma_{xe} + \lambda_{xe}} \tag{6.37}$$

where y is the total Xenon yield. With Chernick we define $\phi_{xe} = \lambda_{xe}/\sigma_{xe}$, the flux at which burn-out equals decay. Then

$$\rho = -\frac{\Sigma_{xe}}{F e^{-B^2\tau}} = \frac{-y}{\nu}\,\frac{1}{e^{-B^2\tau}}\,\frac{\phi}{\phi_{xe}+\phi}$$

$$= -\frac{y}{\nu}\,\frac{I'}{e^{-B^2\tau}} \tag{6.38}$$

where $I' = \frac{\phi}{\phi_{xe}+\phi}$. For a model other than the Fermi-age we could replace $\exp(-B^2\tau)$ by the appropriate fast non-leakage probability, $\bar{P}$, to obtain $\rho = -\frac{y\ I'}{\nu\ \bar{P}}$.

A third firm for the reactivity loss relates to that given by Kaplan (5). This form is given in reference (69) but is here modified to allow for the change of thermal non-leakage probability. Starting with Kaplan's expression we have

$$\rho \approx \frac{\delta k}{k} \approx -f \frac{\sigma_f}{\sigma_{ao}} \frac{y}{(1 + L^2 B^2)} I' = -\frac{\Sigma_f}{\Sigma_a} \frac{y I'}{(1 + L^2 B^2)} \qquad (6.39)$$

But in the original critical reactor, $\Sigma_a(1 + L^2 B^2) = \nu\Sigma_f \bar{P}$. Hence ρ (Kaplan) $= -\frac{y I'}{\nu \bar{P}}$, our second form of the reactivity loss.

We should also mention a difference in the conventional treatment of Xenon poisoning (Glasstone and Edlund say), the treatment in a one group model in the useful tabulation of Xenon effects by Clark and English (70) for transient Xenon and Iodine concentration as a function of local thermal power density, and the treatment given by us, which allows for fast leakage and the effect of Xenon on the thermal non-leakage probability. For one region systems we have:

Conventional	Clark and English	Kaplan/Lewins
$\rho = -\frac{y}{\nu} I' k_\infty$	$-\frac{y}{\nu} I'$	$-\frac{y I'}{\nu \bar{P}}$

In these expressions, k_∞ and $\bar{P}$ refer to the original values in the critical reactor. It is seen that the allowance made by changing the thermal diffusion length takes the form of reducing the reactivity by the thermal non-leakage probability. In large reactors there is evidently little change made by the correction; in particular, the one group expression of Clark and English gives closely the same value as our multigroup models. In small reactors, however, the values of k_∞ and $\bar{P}$ are such as to

make a significant difference. Typical numbers that correct the conventional estimate of reactivity loss are given in Table 6.3

(k_∞)	One Group	Multigroup
(a) Natural Uranium	0.93	0.96
BNL 1.07		
(b) Enriched Uranium	0.61	0.79
MITR 1.65		

Table 6.3 Typical Correction Factors to Conventional Xenon Reactivity Loss

(b) Spatial Dependence of Xenon Poisoning: For the general case of reflected reactors, it is necessary to evaluate the perturbation expression for the reactivity loss:

$$\bar{\rho}' = \frac{\int \phi^+ P \phi' \, dv}{\int \phi^+ F' \phi' \, dv} \tag{6.40}$$

However, for certain large power reactors, with an original uniform core leading, the flux and adjoint flux have the form of chopped cosines, etc. For such a reactor we can develop an extension of Chernick's method.

We assume

$$\phi_1^+ = \bar{P} \phi_2^+ \tag{6.41}$$

and that the steady state Xenon cross-section is given by

$$\Sigma_{xe} = \frac{y\,\Sigma_f\,\phi_2\,\sigma_{xe}}{\phi_2\,\sigma_{xe} + \lambda_{xe}} = \frac{y\,\Sigma_f\,\phi_2}{\phi_2 + \phi_{xe}} \tag{6.42}$$

Then

$$\bar{\rho} = -\frac{\int \phi_2^+\,\Sigma_{xe}\,\phi_2\,dv}{\int \phi_1^+\,\nu\Sigma_f\,\phi_2\,dv} = -\frac{y\int \frac{\phi_2^+\,\phi_2^2}{\phi_2+\phi_{xe}}\,dv}{\nu P \int \phi_2^+\,\phi_2\,dv} \tag{6.43}$$

With Chernick we define the ratio of integrals by

$$I = \frac{\int \frac{\phi_2^+\,\phi_2^2}{\phi_2+\phi_{xe}}\,dv}{\int \phi_2^+\,\phi_2\,dv} \tag{6.44}$$

Then

$$\bar{\rho} = -\frac{y\,I}{\nu P} \tag{6.45}$$

To evaluate I over the core rather than over the equivalent bare reflector, we express the core size as a fraction, δ, of the bare extrapolated dimension. Then the integrals in equation (6.44) are to be taken over

0	to	$\frac{\pi}{2}\,\delta$	slab
0	to	$2.405\,\delta$	cylinder
0	to	$\pi\delta$	sphere

Consider first the sphere, where

$$\phi = \phi_o \frac{\sin \frac{\pi r}{R}}{\frac{\pi r}{R}} \tag{6.46}$$

Substituting $\theta = \frac{\pi r}{R}$, and $\psi = \frac{\phi_o}{\phi_o + \phi_{xe}}$ we then have Chernick's expansion in the form

$$I \int_0^{\pi\delta} \left(\frac{\sin\theta}{\theta}\right)^2 \theta^2 \, \delta\theta = \sum_{n=0}^{\infty} \psi^n \int_0^{\pi\delta} \left(\frac{\sin\theta}{\theta}\right)^3 \left(1 - \frac{\sin\theta}{\theta}\right)^n \theta^2 \, \delta\theta \tag{6.47}$$

We define $\overline{\theta^n}$ in spherical geometry by

$$\overline{\theta^n} = \int_0^{\pi\delta} \left(\frac{\sin\theta}{\theta}\right)^n \theta^2 \, \delta\theta \tag{6.48}$$

The right hand integral of equation (6.47) can be expanded in terms of the binomial coefficients, ${}_nC_r$, to give

$$I\,\overline{\theta^2} = \psi\overline{\theta^3} + \sum_{n=1}^{\infty} \psi^{n+1} \sum_{r=1}^{n} (-1)^r \, {}_nC_r \, \overline{\theta^{r+3}} \tag{6.49}$$

The explicit expansion for the first five terms is

$$n = 0 \qquad \psi \ \overline{\theta^3}$$

$$n = 1 \qquad \psi^2\left[\overline{\theta^3} - \overline{\theta^4}\right]$$

$$n = 2 \qquad \psi^3\left[\overline{\theta^3} - 2\overline{\theta^4} + \overline{\theta^5}\right]$$

$$n = 3 \qquad \psi^4\left[\overline{\theta^3} - 3\overline{\theta^4} + 3\overline{\theta^5} - \overline{\theta^6}\right]$$

$$n = 4 \qquad \psi^5\left[\overline{\theta^3} - 4\overline{\theta^4} + 6\overline{\theta^5} - 4\overline{\theta^6} + \overline{\theta^7}\right]$$

For a finite cylinder, or a box reactor, it is necessary to form a product expansion. We define $\overline{\theta_i^n}$ as

$$\overline{\theta_x^n} = \int_0^{\frac{\pi}{2}\delta} \cos^n \theta \, d\theta \qquad \text{slab},$$

$$\overline{\theta_c^n} = \int_0^{2.405\,\delta} J_o^n(\theta)\, \theta \, d\theta \qquad \text{cylinder}, \tag{6.50}$$

while as before,

$$\overline{\theta_s^n} = \int_0^{\pi\delta} \left(\frac{\sin\theta}{\theta}\right)^n \theta^2 \, d\theta \qquad \text{sphere}$$

On substituting product fluxes, $\cos\theta_x \, J_o(\theta_r)$, etc., into the expression for I, the integrals can be expressed as

products in the separate dimensions. Thus for a finite cylinder we obtain

$$I\,\overline{\theta_x^2}\,\overline{\theta_c^2} = \psi\,\overline{\theta_x^3}\,\overline{\theta_c^3} + \sum_{n=1}^{\infty} \psi^{n+1} \sum_{r=1}^{n} (-r)^r\, {}_nC_r\, \overline{\theta_x^{r+3}}\,\overline{\theta_c^{r+3}} \tag{6.51}$$

For a box reactor with averages defined in the x, y, and z directions, we have

$$I\,\overline{\theta_x^2}\,\overline{\theta_y^2}\,\overline{\theta_z^2} = \psi\,\overline{\theta_x^3}\,\overline{\theta_y^3}\,\overline{\theta_z^3} + \sum_{n=1}^{\infty} \psi^{n+1} \sum_{r=1}^{n} (-1)^r\, {}_nC_r\, \overline{\theta_x^{r+3}}\,\overline{\theta_y^{r+3}}\,\overline{\theta_z^{r+3}} \tag{6.52}$$

The product coefficients can be written in the form $\overline{{}^i\theta^n}$ where i takes the values 1, 2, 3, for spherical cylindrical and box geometry respectively. The corresponding number of components of the separated variables are then to be multiplied together. With this nomenclature all these geometries can be expressed as

$$I\,\overline{{}^i\theta^2} = \psi\,\overline{{}^i\theta^3} + \sum_{n=1}^{\infty} \psi^{n+1} \sum_{r=1}^{n} (-1)^r\, \overline{{}^i\theta^{r+3}}\, {}_nC_r \tag{6.53}$$

For asymmetric reactors, we can no longer take all the coefficients over the half range or just the integral from zero to one edge of the core. The coefficients must be taken as the sum of the two half-range coefficients and then the appropriate coefficients in the separable geometries multiplied together.

Fortunately for the evaluation of the coefficients, a related table has recently been formed by Murray and Mink (64) who tabulate $\overline{\phi^n}$, where

$$\overline{\phi^n} = \frac{3}{(\pi\delta)^3}\,\overline{\theta_s^n} \qquad \text{sphere } *$$

$$\overline{\phi^n} = \frac{2}{(2.405\,\delta)^2}\,\overline{\theta_c^n} \qquad \text{cylinder}$$

$$\overline{\phi^n} = \frac{1}{\frac{\pi}{2}\,\delta}\,\overline{\theta_x^n} \qquad \text{slab}$$

We have given a formal expression for I in terms of the central or maximum flux, ϕ_o. Although this is the more convenient for evaluation, the specifically geometric effects are better understood in relation to the average flux or in terms of our enhancement. From the results and tables therefore, we have calculated the enhancement for a spherical reactor with varying reflector thicknesses, i.e., with a range of δ. The enhancement is calculated from its definition as the ratio of the true value of I to the

* The renormalisation is immaterial when, as in the calculation of I, only ratios of terms are required.

average based on the average flux:

$$a = \frac{I(\langle\phi\rangle + \phi_{xe})}{\langle\phi\rangle} \tag{6.54}$$

The results plotted in Figure 6.1 show that in the limit of very high flux, i.e., when burn-out makes the distribution of Xenon more uniform, the enhancement goes to unity. The enhancement is seen also to a rather strong function of the amount of reflector savings. The MITR is characterised by a δ of about 0.6, so that a is unity for the MITR from about a 3×10^{12} n/cm^2 sec central flux upwards (i.e., in the range of about 0.1 to 1 Mw). Hence it would be a very good approximation to base the calculation of I on the average flux. This result is useful since the non-uniform loading of the reactor core makes an analytical determination of I difficult. The approximations involved in finding a separable flux are probably of less validity than those inherent in using the average core flux.

Even a large natural uranium pile such as the BNL reactor, may have a reflector savings corresponding to $\delta = 0.8$. Hence there is a slight error in such medium flux reactors in using a value of I calculated for the equivalent bare pile (of the order of 10 percent in the Brookhaven pile).

The limiting values of a at low fluxes for different bare geometries are the same as for the short-time burn-up

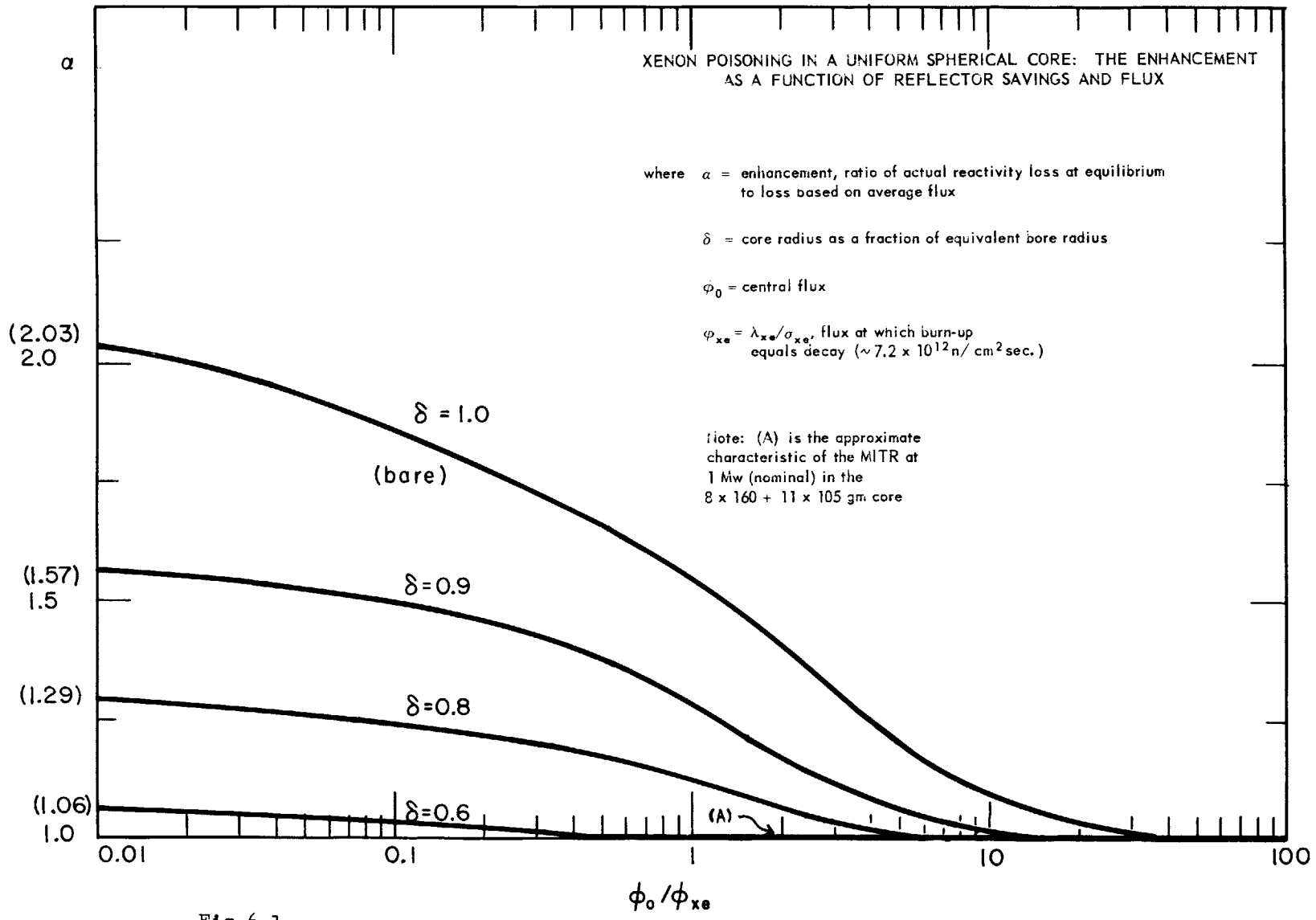

α
XENON POISONING IN A UNIFORM SPHERICAL CORE: THE ENHANCEMENT AS A FUNCTION OF REFLECTOR SAVINGS AND FLUX
where α = enhancement, ratio of actual reactivity loss at equilibrium to loss based on average flux
δ = core radius as a fraction of equivalent bore radius
φ0 = central flux
φxe = λxe/σxe, flux at which burn-up equals decay (~7.2 x 10^12 n/cm^2 sec.)
Note: (A) is the approximate characteristic of the MITR at 1 Mw (nominal) in the 8 x 160 + 11 x 105 gm core
(2.03)
2.0
δ = 1.0
(bare)
(1.57)
1.5
δ=0.9
(1.29)
δ=0.8
(1.06)
1.0
δ=0.6
(A)
0.01
0.1
1
10
100
φ0/φxe

Fig 6.1

results, Table 6.1 It is seen that α itself is fairly sensitive to the geometry. However, Chernick has shown that the value of I is insensitive to geometry in bare core reactors and hence is probably insensitive in reflected cases.

6.6 Control Rod Movement

Perturbation theory has had many applications to the general realm of control rod calculations. The first order approximation, even in one group, is often very successful in predicting the relative worths of rods in different reactor positions or with only partial insertion. An experimental verification in the MITR is offered in this section. The calculation of the absolute reactivity worth is considerably more difficult due to the large perturbations associated with black rods. It is necessary to introduce additional methods to allow for the perturbation. One such method is introduced in the next section.

Control rods are often calculated by methods other than perturbation theory, for ideal positions in a reactor, e.g., at the centre, in order to supply symmetry and ease of computation. The relative change in reactivity on placing the same control rod in a different position can be estimated quite well in practice by comparing the statistical weights in the first order, one group approximation:

$$\frac{\bar{\rho}_a}{\bar{\rho}_b} = \frac{\int \phi^+(\underline{r}_a)\,\phi(\underline{r}_a)\,dv}{\int \phi^+(\underline{r}_b)\,\phi(\underline{r}_b)\,dv} = \frac{\phi^2(\underline{r}_a)}{\phi^2(\underline{r}_b)} \qquad (6.55)$$

It would seem that the errors in the first order approximation cancel top and bottom or that the relative flux depression caused by the black rod is about the same for any position.

A similar problem concerns the partial insertion of a given rod. Here the flux shapes vary considerably as the rod is withdrawn. It is not immediately clear perhaps that the first order approximation in one group should give a good result even for the relative effect of different rod insertions. However, a detailed variational calculation by Schwindler, (73), has justified the use of the unperturbed flux as the first term in a variational calculation, the second term in Schwindler's expression providing a very small correction. From our own development of the nature of perturbation theory, the identification with the variational calculation is not surprising.

The one group bare reactor perturbation treatment, leading to an S-shaped curve, is well known (26); we apply the method to the regulating rod of the MITR and compare with the measured data of Larson (79). This is a hard test case since a one group equivalent bare reactor is not a good

representation of the MITR. The reactor configuration is sketched in Figure 6.2.

There is the additional complication that the active length of the absorber does not fill the complete length of the guide when fully inserted. The flux distribution is assumed to have a vertical component unaffected by the rod position and given by

$$\phi = \phi_o \cos \frac{\pi \Sigma}{40} \tag{6.56}$$

where the height of the equivalent bare core is 40 inches. The radial contribution will cancel in the relative expression.

We wish to derive an expression for the relative worth of the rod at different depths of insertion. That is to say, we want to integrate the statistical weight of the region along the axis of the rod from the top of the equivalent reactor down to the tip of the rod. Such an integral is then a function of Z, f(Z) say, where Z is the position of the tip. Then if $f(Z_o)$ is the worth of the fully inserted rod, with tip at Z_o (13 1/2 inches), the fractional worth of the rod over its range of movement is given by $f(Z)/f(Z_o)$. Thus we calculate

$$f(Z) = \int_{Z}^{Z+27} \cos \frac{\pi Z}{40} \, \Sigma \cos \frac{\pi Z}{40} \, dz \tag{6.57}$$

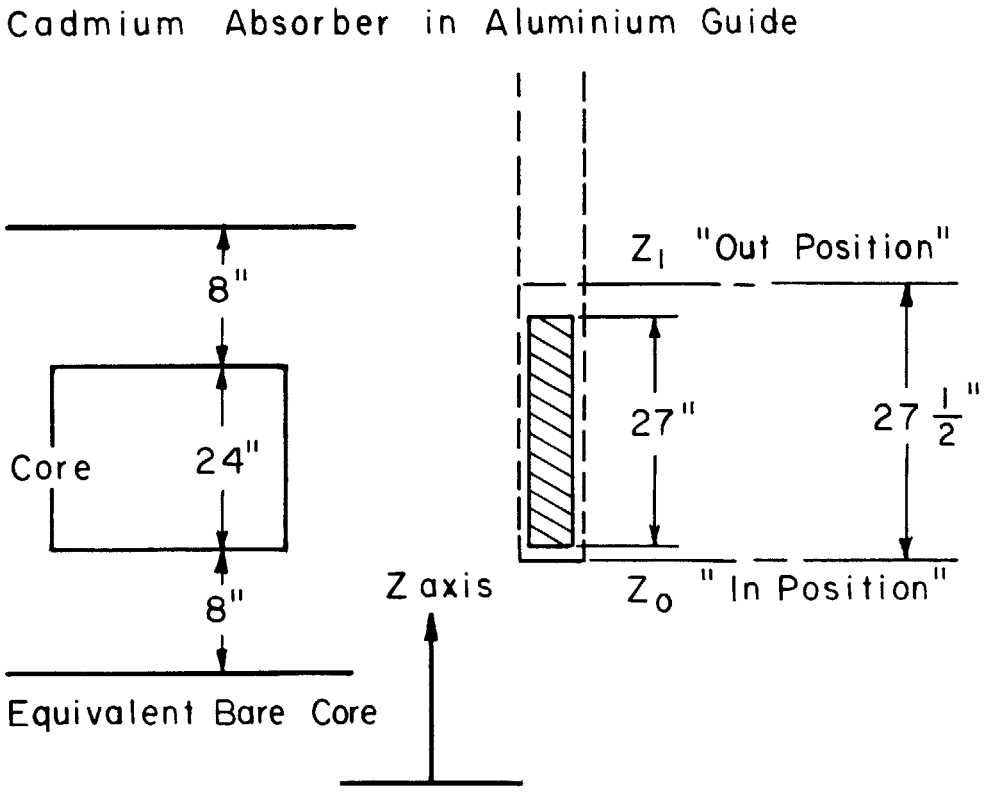


FIG. 6.2 – MITR REGULATING ROD CONFIGURATION

Note that the upper limit is given as Z + 27, 27 inches being the length of the rod. However, the effective length of the rod is only up to the top of the reactor of course and the (Z + 27) is to be not more than 20 inches.

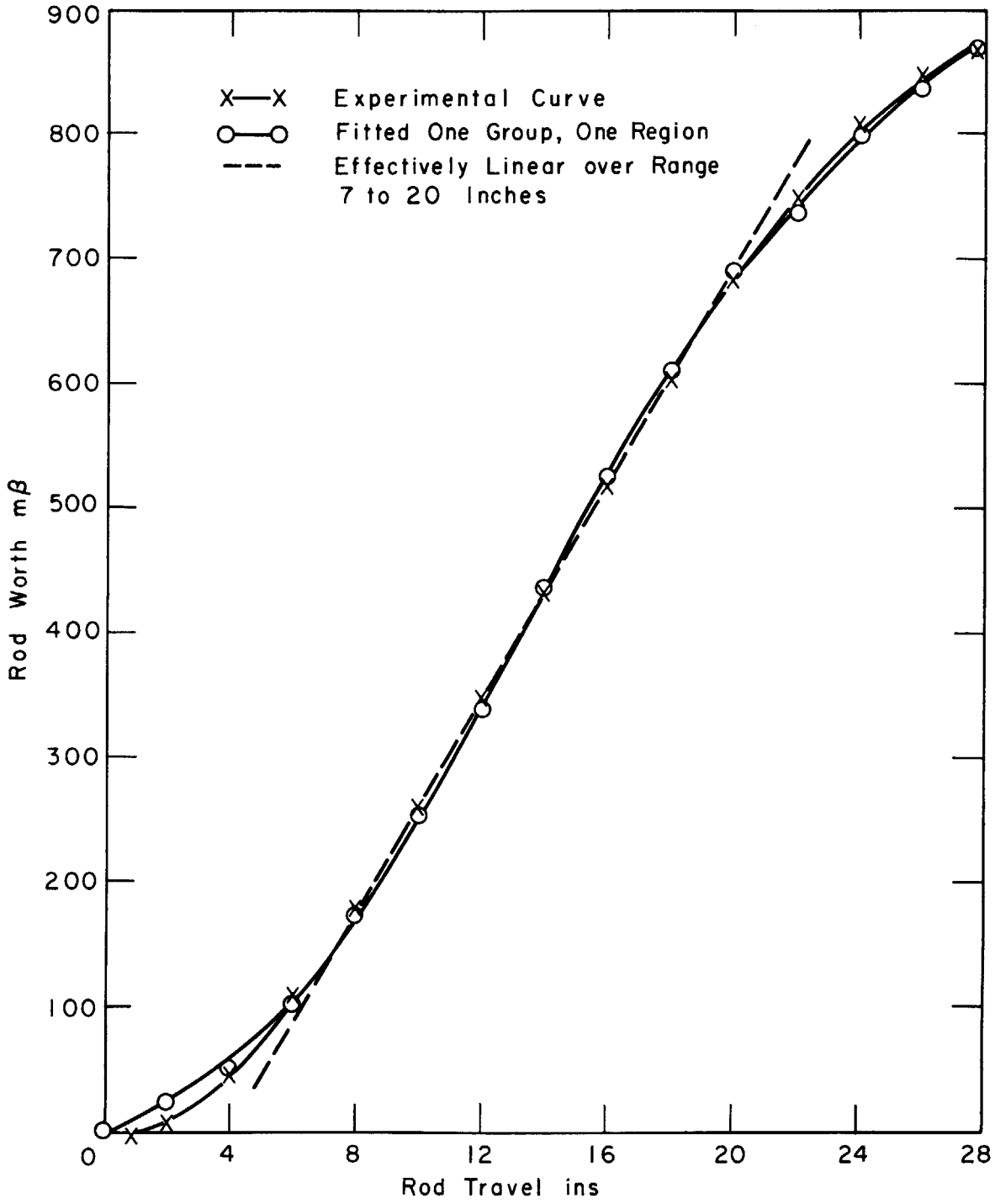
The expression for f(Z) is easily integrated. In practice, the rod is not fully withdrawn from the reactor (i.e., f does not go to zero) but the tip can be raised to a maximum height of 27 1/2 inches, to Z_1 say. Then Z_1 = 13 3/4 inches. The experimental value for the total rod worth is actually the change of worth between these positions. The fractional worth, $\overline{F}(Z)$, for this limited range of travel is given for some intermediate tip position Z, by

$$\overline{F}(Z) = \frac{f(Z) - f(Z_1)}{f(Z_o) - f(Z_1)} \tag{6.58}$$

$$\text{Since } f(Z) = \Big[\, 2\theta + \sin 2\theta \,\Big]_{\theta = \frac{Z\pi}{40}}^{\theta = (\frac{Z+27}{40})\pi \leq \frac{\pi}{20}} \tag{6.59}$$

we calculate the value of the fractional worth, equation (6.58), to be

$$\overline{F}(Z) = \frac{f(Z) - 1.229}{2.121 - 1.229} \tag{6.60}$$



MITR REGULATING ROD CALIBRATION AND FITTED ONE GROUP, ONE REGION EXPRESSION. (UNIFORM 19 ELEMENT CORE AUG 1958)

Fig 6.3

The expression for f(Z) with the component $(2\theta + \sin 2\theta)$ is a typical result and gives a fair approximation for the S-shaped curve of the experimental results, (see Figure 6.3). The agreement between prediction and theory is good but could be improved by an a posteriori selection of the equivalent bare reactor height.

6.7 Control Rod Worth

To calculate the absolute reactivity worth of a rod from perturbation theory, it is necessary to find some method to allow for the severe depression in the thermal flux caused by black rods (naturally the problem does not arise if the control system is designed 'grey', with no highly perturbing effects). Arnold (71) has suggested that an equivalent cross-section for the rod can be employed that in the unperturbed flux will absorb the same number of neutrons as are actually absorbed in the black rod in the actual flux. Such equivalent cross-sections are fairly easily calculated for different configurations and geometries in the diffusion approximation (76). It is not clear however that such an equivalent cross-section as defined is indeed an equivalent perturbation. Thus we investigate the functional relation between the two equivalent forms (72). We follow with a comparison of the diffusion and exact transport solution for one particular equivalent cross-section.

The equivalent cross-section is not in itself an equivalent perturbation, i.e., its use does not necessarily lead to the same reactivity effect. The two can be related in the following fashion. Consider a two group model and an equivalent cross-section, $\overline{\Sigma}$, in each group that will lead to the same reactor average cross-section in the group in the unperturbed flux as the actual property perturbation in the group gives in the actual flux. In the thermal group it is sufficient to take

$$\frac{P_{22}\,\phi_2' + P_{21}\,\phi_1'}{\int \phi_2'\,dv} = \frac{\overline{\Sigma}_2\,\phi_2}{\int \phi_2\,dv} \tag{6.61}$$

Equation (6.61) defines such an equivalent group cross-section point by point in terms of the removal of thermal neutrons ($+P_{22}$) and the production of thermal neutrons from fast neutrons ($-P_{21}$). On taking the volume average of the cross-sections over the whole reactor, i.e., by integrating both sides of equation (6.61) we have the same reactor average cross-section for the perturbation and its equivalent cross-section.

For a perturbation to be equivalent, i.e., to give the same kinetic reactivity, we follow the method of Section 6.2. It is sufficient in defining an equivalent perturbation, $\overline{P}$, to require the equivalence, in the thermal

group, of

$$\frac{\phi_2^+ P_{22} \phi_2' + \phi_2^+ P_{21} \phi_1'}{\int \phi^+ V^{-1} \phi' \, dv} = \frac{\phi_2^+ \bar{P}_2 \phi_2}{\int \phi^+ V^{-1} \phi \, dv} \tag{6.62}$$

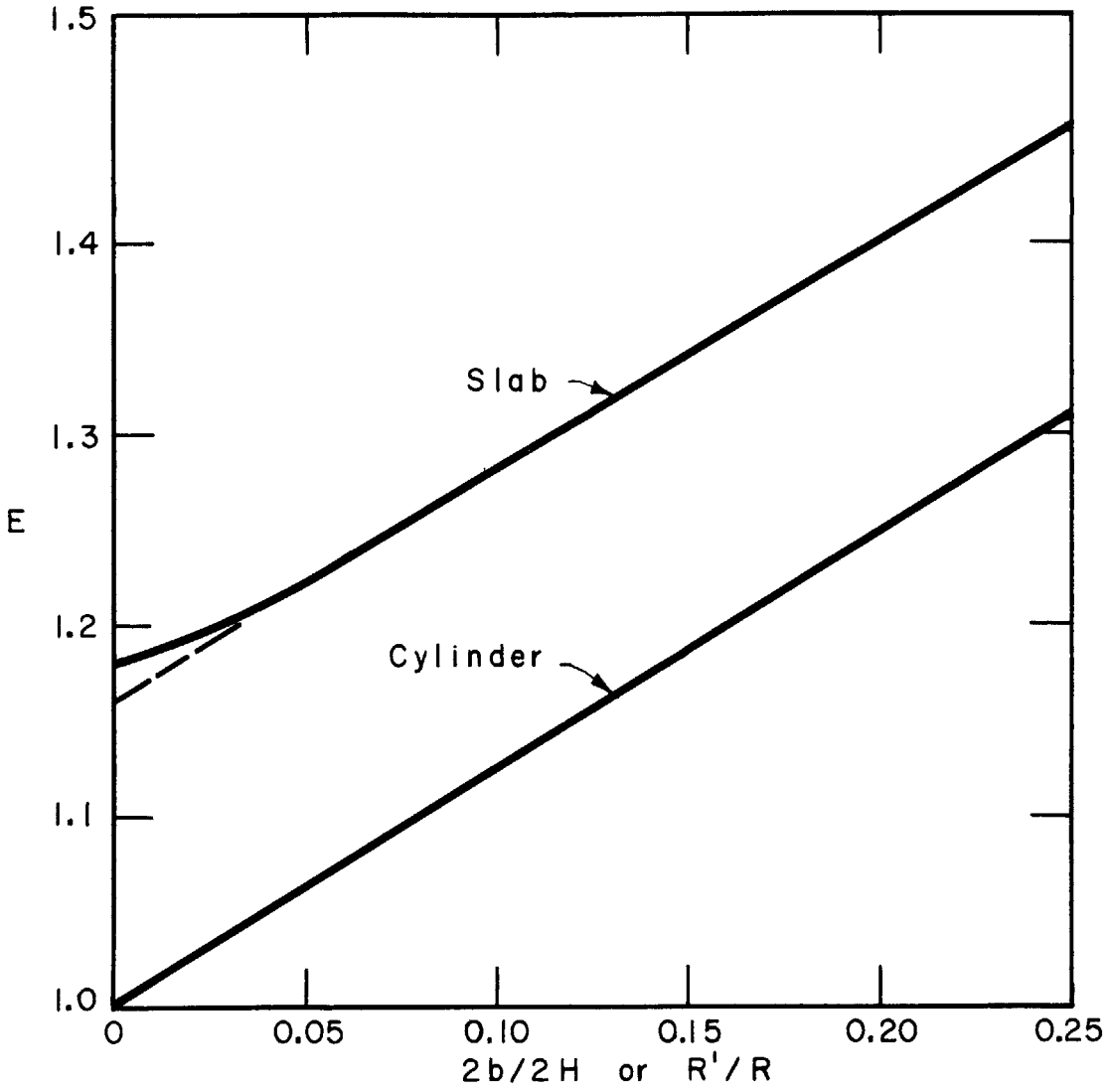
On integration over the reactor, the inverse period without delayed neutrons (and hence very closely the reactivity) is given by the expression employing the equivalent perturbations, P_1 and P_2, together with the unperturbed flux. Comparing equations (6.61) and (6.62), we find

$$\bar{P}_2 = E_2 \bar{\Sigma}_2 , \tag{6.63}$$

where we have defined

$$E_2 = \frac{\int \phi_2' \, dv}{\int \phi_2 \, dv} \cdot \frac{\int \phi^+ V^{-1} \phi \, dv}{\int \phi^+ V^{-1} \phi' \, dv} \tag{6.64}$$

If the physical normalization to unity is employed for the importance integrals in equation (6.64), then each E reduces to the ratio of the perturbed to unperturbed neutron populations. In general the E ratios will be larger than unity and have the following significance: the reactivity is affected by the flux change as well as by the property perturbation. For example, the leakage out of a control rod perturbed reactor (Figure 1.3) is likely to be increased. Thus the perturbation is something more than just the



Central Black Slab: Control Slab to Reactor Extrapolated Thickness, 2b/2H

Coaxially Rodded Cylinder: Inner to Outer Extrapolated Radii R'/R

E FACTORS FOR 1 GROUP 1 REGION DIFFUSION THEORY

Fig 6.4

neutrons absorbed in the control rod. The ratio depends on the velocity group since a control rod is likely to perturb the thermal flux much more than the fast flux.

We can calculate values of E for some simple geometries to find the correction to be employed to an equivalent cross-section. Figure 6.4 shows the results of such calculations for slab and cylindrical geometries for various values of the ratio of rod thickness to reactor thickness. These values are all calculated on the basis of simple one group diffusion theory with the rod thickness measured in units of the extrapolated thickness at which the flux goes to zero. These E factors can then be used to correct the equivalent cross-section. (See Appendix 6A)

The next problem is to calculate the effective cross-sections themselves such that combined with the unperturbed flux they will correctly indicate the absorption rate of neutrons. We shall consider only the case of a black slab in a medium with a uniform source of thermal neutrons (Figure 6.5). Then the governing equation is

$$D \frac{d^2\phi}{dx^2} - \Sigma_a \phi = -S \,, \tag{6.65}$$

where S is the source strength. The unperturbed solution, infinitely far from the black slab is

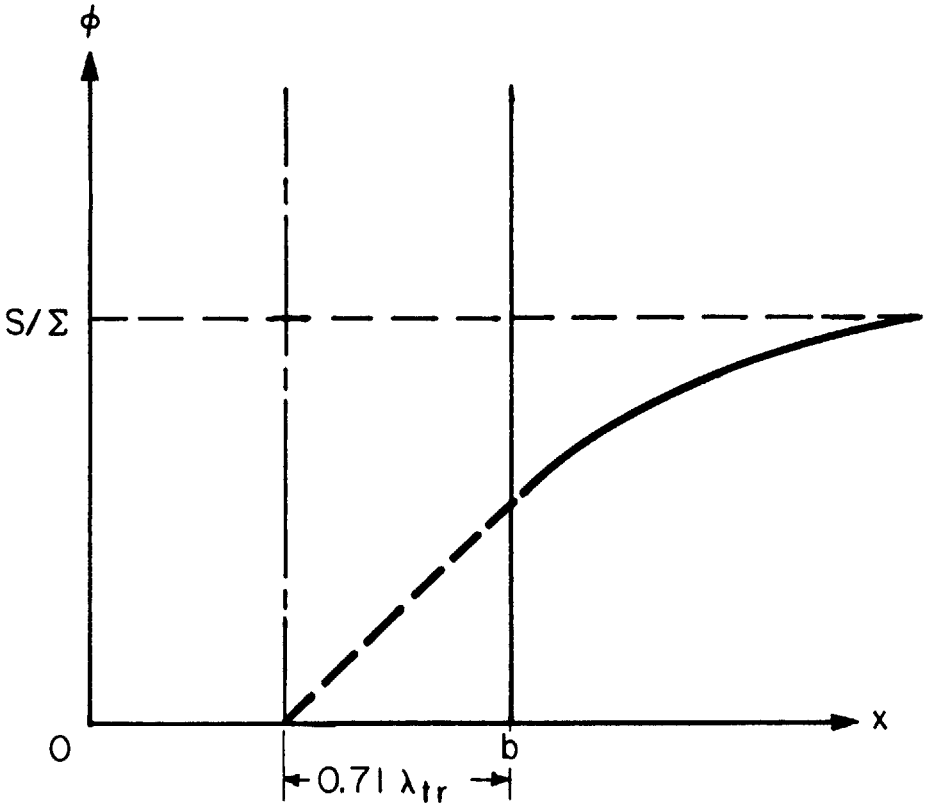


FIG. 6.5 - BLACK SLAB IN AN INFINITE MEDIUM WITH UNIFORM SOURCE, S.

$$\phi = \frac{S}{\Sigma_a} \tag{6.66}$$

The actual solution has the form

$$\phi = \frac{S}{\Sigma_a} (1 - A\, e^{-\kappa x}) \tag{6.67}$$

where

$$\kappa^2 = \frac{D}{\Sigma_a} \tag{6.68}$$

If we take ϕ to vanish at the extrapolated boundary, we would have

$$A = e^{\kappa b}\, e^{-0.71 \kappa \lambda_{tr}} \tag{6.69}$$

The current of neutrons into both sides of the slab would be

$$j = 2D \left.\frac{d\phi}{dx}\right)_b = \frac{2D}{\Sigma_a} SA\, \kappa\, e^{-\kappa b} \tag{6.70}$$

which on expanding the exponential gives approximately

$$j = \frac{2S}{\kappa} (1 - 0.71 \lambda_{tr} \kappa) \tag{6.71}$$

Actually the more exact transport solution can be shown to be, (Appendix 6A),

$$j_{tr} = \frac{2S}{c\kappa} \left(1 - \frac{0.7104 \kappa}{\Sigma}\right) \tag{6.72}$$

where Σ is the total cross-section in the surrounding medium, $(\Sigma_s + \Sigma_a)$, and c the multiplicity, $(\nu\Sigma_f + \Sigma_s)/\Sigma$. Since the thermal group does not produce thermal neutrons, we take $c = \Sigma_s/\Sigma$ in evaluating this example. For the MIT Reactor we have approximately, $c = 0.95$.

The equivalent absorption cross-section is now found by requiring that the equivalent cross-section in the unperturbed flux (S/Σ_a) absorbs the actual number of neutrons. Hence

$$\bar{\Sigma}_2 = \frac{\Sigma_a}{bc\kappa}\left(1 - 0.7104\,\frac{\kappa}{\Sigma}\right) \tag{6.73}$$

where the properties refer to the thermal group. We are now in a position to calculate a control rod worth using the unperturbed flux, ϕ, and the perturbation in the thermal group, $E\bar{\Sigma}_2$. Calculations of control rods for the Yankee Power Reactor based on this simple method (71) gave results that were about 1 part in 10 too low, which is quite satisfactory for a preliminary study.

6.8 Surface Perturbations

Considerable work has been done in applying perturbation techniques to the calculation of the effect of reflectors, (74). Holway (75) has expressed the effect of a reflector as a perturbation of a bare uniform core. This method is elegant since the simple Fermi-age analysis

of a bare uniform reactor can be employed, which is potentially more accurate than the two group approach.

Another class of problem is to consider variations in the thickness of a reflector. For example, the MITR has an upper reflector capable of being dumped or even maintained at intermediate positions. This type of problem is more one of the variation of the surface or boundary conditions than the perturbation of properties within the region. We can develop a corresponding surface perturbation treatment, analogous to certain problems in acoustics (9).

The range of validity of such perturbation methods is quite severely limited when the first order approximation is used. Experimental comparisons will be given. The insertion of a black control rod can also be expressed as a surface perturbation though again of such a severe nature that the first order approximation fails. Fairly recently however, Wolfe and Fisher (77) have developed a higher order treatment of control rods which can be related to the surface perturbation treatment to be developed here.

The reference reactor is chosen such that its boundary surface, S, lies outside the actual reactor surface, S'. For convenience the two surfaces should coincide where possible to reduce the subsequent evaluation. The reference reactor is chosen to have the same properties as the actual reactor point by point - i.e., there is no

perturbation of properties within S'. The properties outside of S' determine the criticality of the reference reactor but do not otherwise affect the treatment. We shall, of course, use a critical reference reactor, with flux ϕ and importance ϕ^+. The actual flux, ϕ' is not the same as ϕ however since the external leakage and hence the neutron balance has changed. The appropriate equations are therefore

$$M^T \phi^+ = 0 \,, \tag{6.75}$$

and

$$M\phi' = s\, V^{-1}\, \phi' \tag{6.76}$$

together with the appropriate boundary conditions. These conditions are that $j_-(\underline{S}') = 0$ and $i_+(\underline{S}) = 0$. The result is that we have

$$\left(\frac{\phi'}{4} + \frac{D}{2}\,\nabla\phi'\right)_{S'} = 0 \quad \text{and} \quad \left(\frac{\phi^+}{4} + \frac{D}{2}\,\nabla\phi^+\right)_S = 0 \tag{6.77}$$

As usual equation (6.75) is multiplied by the importance and equation (6.76) by the perturbed flux. On subtraction, the terms in Σ and F will cancel - there is no perturbation of these properties in the region within S'. Hence within the actual reactor

$$\phi^+ \nabla \cdot D \nabla \phi' - \phi' \nabla \cdot D \nabla \phi^+ = s \phi^+ V^{-1} \phi' \tag{6.78}$$

By the usual vector transformation we cancel terms in $\nabla \phi^+ \cdot D \nabla \phi'$:

$$\nabla \cdot (\phi^+ D \nabla \phi' - \phi' D \nabla \phi^+) = s \phi^+ V^{-1} \phi' \tag{6.79}$$

On integrating equation (6.78) over the region of validity, i.e., the actual reactor, we can transform the divergence to a surface integral:

$$s \int \phi^+ V^{-1} \phi' \, dv = \oiint (\phi^+ D \nabla \phi' - \phi' D \nabla \phi^+) \cdot d\underline{S}$$

$$= \oiint \phi^+ \phi' \left(\frac{D \nabla \phi'}{\phi'} - \frac{D \nabla \phi^+}{\phi^+} \right) \cdot d\underline{S}' \tag{6.80}$$

From the boundary condition, equation (6.77), equation (6.80) will vanish wherever the two surfaces, S' and S, are coincident. Hence the surface integral in equation (6.80) need only be evaluated over the perturbed surface. The loss to the resident population due to the perturbation is seen to be given by the current of neutrons escaping weighted by their importance plus the current of importance per unit flux weighted by the flux:

$$\oiint(\phi^+ D\nabla\phi' - \phi' D\nabla\phi^+) \cdot d\underline{S}' = - \oiint(\phi^+ \underline{j}' + \phi' \underline{i}) \cdot d\underline{S}' \qquad (6.81)$$

As shown in Section 5, the calculation of an inverse period neglecting delayed neutrons can be related to the reactivity of the actual reactor by

$$\bar{\rho}' \approx s \frac{\int \phi^+ V^{-1} \phi' dv}{\int \phi^+ F' \phi' dv} = - \frac{\oiint(\phi^+ D\nabla\phi' - \phi' D\nabla\phi^+) \cdot d\underline{S}'}{\int \phi^+ F' \phi' dv} \qquad (6.82)$$

The above theory was applied to an analysis of the reactivity worth of the upper reflector in the MITR, using first of all, a two group approximation. For simplicity and with little loss of accuracy, the boundary conditions employed were those of the extrapolated surface, so that ϕ' in equation (6.79) can be taken to be zero everywhere.

$$\bar{\rho}' = - \frac{\oiint \phi^+ D\nabla\phi' \cdot d\underline{S}'}{\int \phi^+ F' \phi' dv}$$

$$= - \frac{\oiint(\phi_1^+ D_1 \nabla\phi_1' + \phi_2^+ D_2 \nabla\phi_2') \cdot d\underline{S}'}{\int \phi_1^+ F \phi_2' dv} \qquad (6.83)$$

In this equation, the reactivity is the loss of neutrons at the extrapolated surface weighted by the value of the

importance of neutrons to the reference reactor at that surface, expressed, as usual, as a fraction of the total production.

The severe limits on the first order approximation can be seen by considering a perturbation that removes all of the reflector, leaving the bare core. There will certainly be an actual current of neutrons out of the core. In the two group reference reactor however, the peaking of the thermal flux at the core interface may well result in the first order value of the thermal current being into the core. Evidently the first order expression will be severely in error for such a severe perturbation in the two group approximation.

The results of such an analysis are compared (Figure 6.6) with experimental determinations of the reflector worth in the MITR and in general show the failure of the first order approximation near the core interface. For small perturbations of the reflector, the fit is fairly good however.

Apart from the gross result near the core interface, it is not too easy to apportion the rest of the departure of the prediction from observation. The experimental results themselves depend on a perturbation analysis of the measured periods to obtain a value for the reactivity. The presence of the delayed photo-neutron contribution (with the correspondingly poor data) makes the uncertainty of the calculated

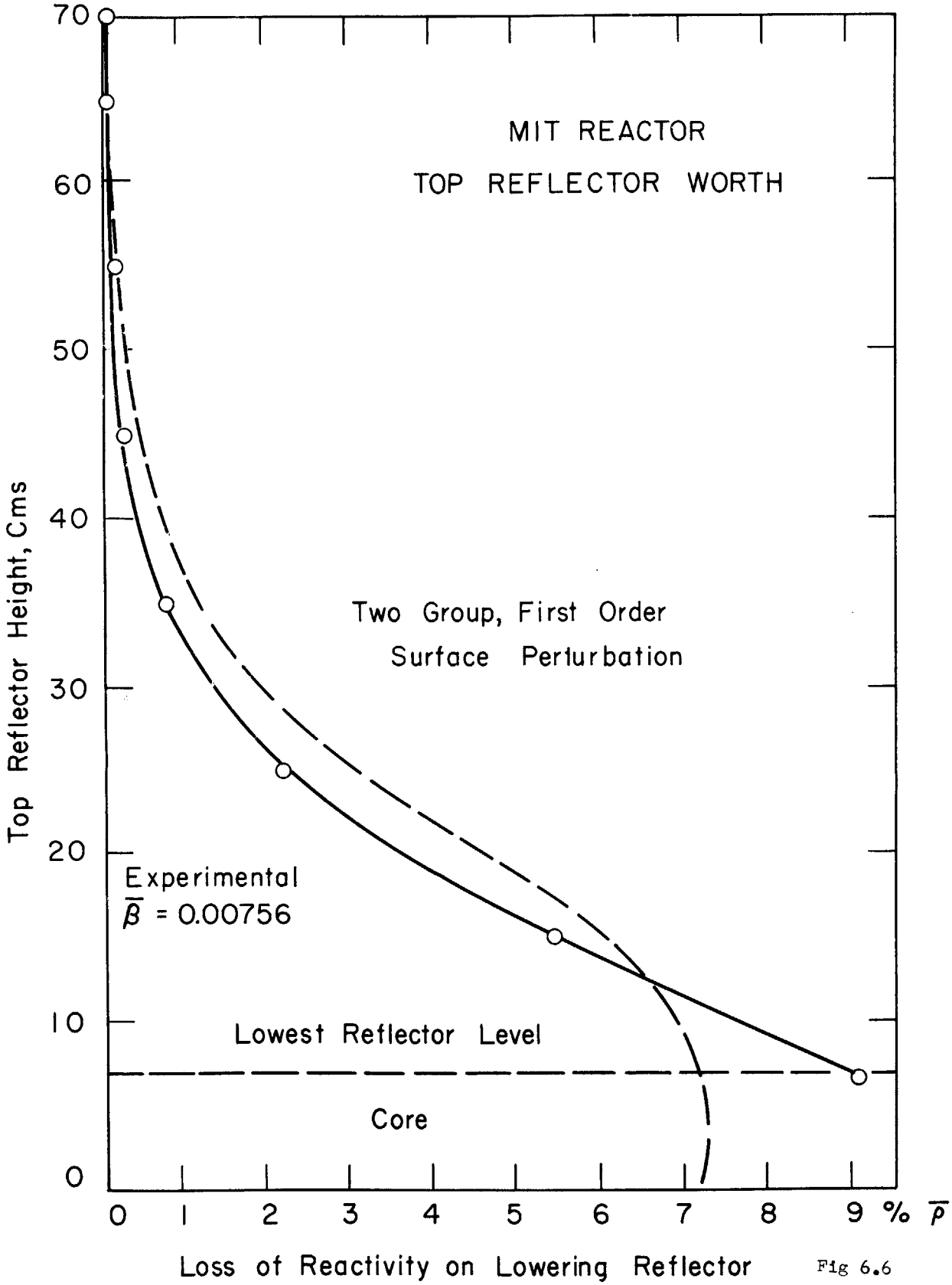

MIT REACTOR
TOP REFLECTOR WORTH
Two Group, First Order
Surface Perturbation
Experimental
$\bar{\beta}$ = 0.00756
Lowest Reflector Level
Core
Top Reflector Height, Cms
Loss of Reactivity on Lowering Reflector
70
60
50
40
30
20
10
0
0 1 2 3 4 5 6 7 8 9 % $\bar{\rho}$

Fig 6.6

value of $\bar{\beta}$ to be up to 1 part in 12. The use of two group theory to calculate neutron currents in the reflector is open to doubt, even when a two group analysis correctly predicts the critical mass of the unperturbed reactor (78).

Consideration must be given to the geometry of the MITR, sketched in Figure 6.7. The presence of a permanent cylindrical annulus of graphite means that the central heavy water reflector forms part of a reentrant cavity, so that strictly, the vacuum boundary condition does not apply. In view of the low flux in the graphite areas it seems reasonable to neglect this complication. In addition, as the heavy water is lowered, the aluminum structural components, supporting fuel elements and control rods, are unchanged, further complicating the boundary condition.

These comments on the approximations and assumptions show that it is not too surprising that the perturbation calculation failed to give a perfect fit. It should be noted that as far as the first order approximation alone is concerned, that the effect of the perturbation should be underestimated at all levels, since the leakage currents must always be increased over the reference values on removing the reflector.

The two group expression does not predict the magnitude of the reflector worth over the whole range of thickness. In many cases it is comparatively easy to obtain,

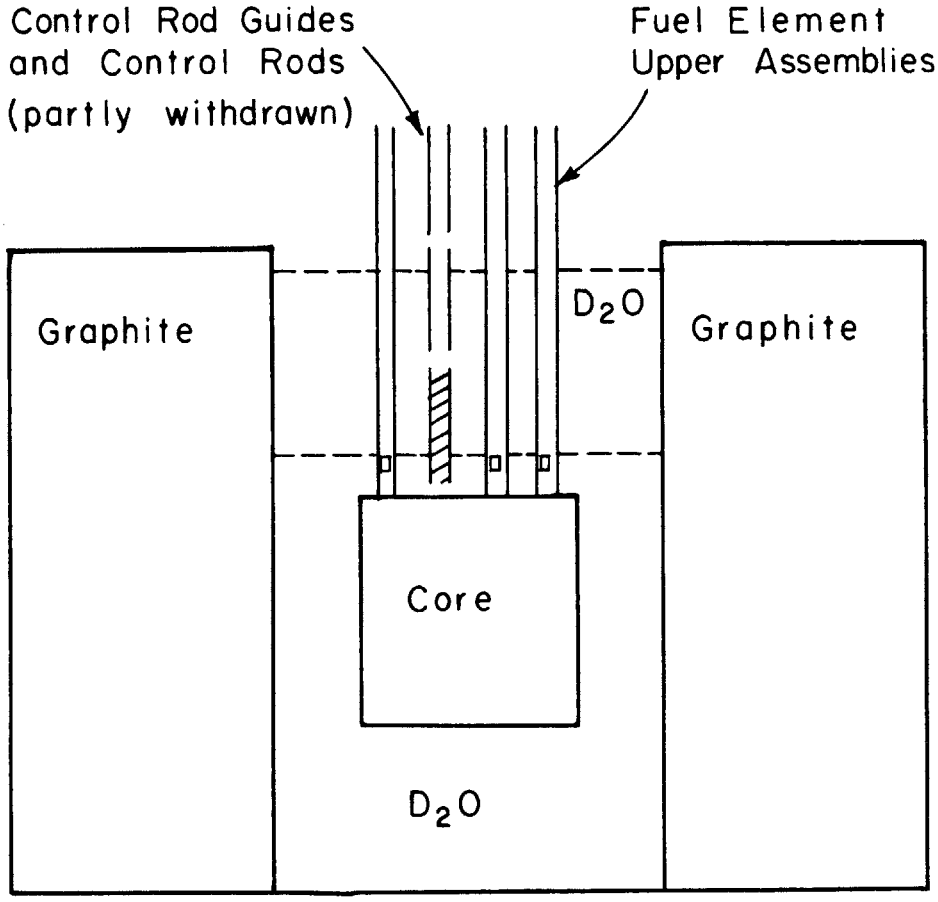


FIG. 6.7 - SKETCH OF THE VARIABLE UPPER REFLECTOR OF THE MITR.

by a separate calculation, the reactivity worth of the bare core. In those cases, the use of a simple one group prediction of the shape of the reflector worth curve (rather than the magnitude) enables a prediction to be made of intermediate thickness values with ease. The one group expression does not suffer from the defect of predicting the wrong sign for the actual bare core neutron current and can fit the whole range very adequately. The technique is analogous to the S-curve fitting of control rod theory.

For a one group model, the flux in the reflector at a point x measured from the outside (extrapolated) surface, is given by

$$\phi = \phi_o \sinh \kappa x \tag{6.84}$$

The value of ϕ_o, is determined by matching the flux at the core-reflector interface, x = T:

$$\phi_T = \phi_o \sinh \kappa T \tag{6.85}$$

Hence

$$\phi = \phi_T \frac{\sinh \kappa x}{\sinh \kappa T} \tag{6.86}$$

In the one group approximation, the importance has the same shape as the flux. Consider the removal of the reflector

to take place along the axis of a right cylinder (as is the case for the MITR) so that the evaluation of the surface integral is trivial. If $\bar{\rho}(T)$ is the worth of the fully removed reflector, the worth at some other depth, $\bar{\rho}(x)$, is given by equation (6.83) to be

$$\bar{\rho}(x) = \bar{\rho}(T) \frac{\sinh \kappa x}{\sinh \kappa T} \frac{D\frac{\partial}{\partial x}(\sinh \kappa x)_x}{D\frac{\partial}{\partial x}(\sinh \kappa x)_T}$$

$$= \bar{\rho}(T) \frac{\sinh 2\kappa x}{\sinh 2\kappa T} \qquad (6.87)$$

For thick reflector, $\kappa T > 3$ say, the sinh term can be replaced by the exponential to give

$$\frac{\bar{\rho}(x)}{\bar{\rho}(T)} = e^{2\kappa(x-T)} \qquad (6.88)$$

Such a one group expression for the relative worth still retains the reciprocal diffusion length, κ, as an adjustable parameter. The heavy curve of Figure 6.6, fitted to the experimental results, then yields the value of κ for the best fit of the data. It is interesting to compare this experimental value with predictions that do or do not take into account the presence of the control rods in the upper reflector.

In view of the one group assumption, we should expect κ to be given by the modified one group expression, $\sqrt{L^2 + \tau}^{-1}$. The value calculated for the upper reflector ignoring control rods is then $1/\sqrt{1835 + 132} = 0.0257$ cms^{-1}. The value obtained by Larson (79) by a calculation including the effective cross-section of the rods, as described in Section 6.7, was 0.0543 cms^{-1}, a considerable change. The result derived from the experiment agrees with Larson's result better however and is 0.0454 cms^{-1}. It would appear that the method has some usefulness.

APPENDIX A TO SECTION 6

CALCULATION OF THE E FACTORS FOR CONTROL ROD WORTH

(a) Slab

Reactor extrapolated width, 2H. Symmetric slab of width 2b. One group of neutrons.

$$\phi^+ = \phi = \cos\frac{\pi x}{2H}\,; \quad \phi' = \sin\frac{\pi(x-b)}{H-b} \tag{6A.1}$$

$$\frac{\int \phi' \, dv}{\int \phi \, dv} = \frac{\int_b^H \sin\frac{\pi(x-b)}{H-b}\,dx}{\int_0^H \cos\frac{\pi x}{2H}\,dx} = \frac{H-b}{2H} \tag{6A.2}$$

$$\int \phi^+ \phi \, dv = \int_0^H \cos^2\frac{\pi x}{2H}\,dx = \frac{H}{2} \tag{6A.3}$$

$$\int \phi^+ \phi' \, dv = \int_b^H \cos\frac{\pi x}{2H}\sin\frac{\pi(x-b)}{H-b}\,dx$$

$$= \frac{4H^2(H-b)}{\pi(3H-b)(H+b)}\cos\frac{\pi b}{2H} \tag{6A.4}$$

$$\text{Therefore } E\text{ (slab)} = \frac{\pi}{8}\frac{(3H-b)(H+b)}{H^2\cos\frac{\pi b}{2H}} \tag{6A.5}$$

which in the limit of b going to zero gives $\frac{3}{8}\pi = 1.18$.

(b) Infinite Cylinder

Extrapolated radius R, control rod surface at R'.

$$\phi = \phi^+ = J_o(\alpha r)\,; \quad \phi' = J_o(\beta r) - \frac{J_o(R')}{Y_o(R')}Y_o(\beta r) \tag{6A.6}$$

$$\int \phi \, dv = \int_0^R J_o(\alpha r)\, r\, dr = 0.5191\frac{R}{\alpha} \tag{6A.7}$$

$$\int \phi' \, dv = \int_{R'}^{R} \left[J_o(\beta r) - \frac{J_o(R')}{Y_o(R')} Y_o(\beta r) \right] r \, dr$$

$$= \frac{R}{\beta} J_1(\beta R) - \frac{J_o(\beta R)}{Y_o(\beta R)} \frac{R}{\beta} Y_1(\beta R) - \frac{R'}{\beta} J_1(\beta R') +$$

$$+ \frac{J_o(\beta R)}{Y_o(\beta R)} \frac{R'}{\beta} Y_1(\beta R') \qquad (6A.8)$$

$$\int \phi^+ \phi \, dv = \int_0^R J_o^2(ar) \, r \, dr = \frac{R^2}{2} (0.5191)^2 \qquad (6A.9)$$

$$\int \phi^+ \phi' \, dv = \int_{R'}^{R} J_o(a r) \left[J_o(\beta r) - \frac{J_o(\beta R')}{Y_o(\beta R')} Y_o(\beta r) \right] r \, dr$$

$$= \frac{\beta R' J_o(a R')}{a^2 - \beta^2} \left[J_1(\beta R') - \frac{J_o(\beta R')}{Y_o(\beta R')} Y_1(\beta R') \right] \qquad (6A.10)$$

Solutions of the critical equation to determine β were obtained from Janke und Emde (35). Evaluation of the E function at 6 points over the range $R'/R = 0.05$ to 0.25 and the observation that the E function has the limit of unity in cylindrical geometry, leads to the straight line result.

APPENDIX B TO SECTION 6

TRANSPORT SOLUTION FOR CURRENT INTO A BLACK SLAB

Consider an infinite scattering and producing medium in the one energy transport approximation. A uniformly distributed volume source of neutrons supplies a current of neutrons to a central black slab. We have a modified Milne problem for which Davison (3) has given a solution in terms of the solution to the Milne problem itself.

If

S is the source of neutrons per second per unit of volume

L is the diffusion length

ℓ is the mean free path

c is the multiplicity $\dfrac{\nu\Sigma_f + \Sigma_s}{\Sigma_a + \Sigma_s}$

$\psi_m(o,\mu)$ is the angular flux at the interface for the conventional Milne problem

$j_m(o)$ is the Milne problem current at the interface

$j_t(o)$ is the current of the present problem

$$j_t(o) = S \int_{-1}^{0} \frac{L + \ell\mu}{c j_m(o)} \mu \psi_m(o,\mu)\, d\mu \tag{6B.1}$$

whence

$$j_t = -\frac{L}{c}\left[1 - \frac{L}{\ell}\,\frac{\int_{-1}^{0} \mu^2 \psi_m(o,\mu)\, d\mu}{\int_{-1}^{0} \mu\, \psi_m(o,\mu)\, d\mu}\right] \tag{6B.2}$$

Placzek (34) has given a detailed numerical solution of the Milne problem from which we can derive the required result. It is only necessary to integrate the numerator of equation (6B. 2) since it is known that the integral in the denominator should be unity. However, we used Simson's rule for a numerical integration of both functions, in ten steps over the range, to obtain an estimate of the accuracy of our result.

We obtain the results

$$\int_{-1}^{0} \mu\, \psi_m(o,\mu)\, d\mu = 1.00001$$

$$\int_{-1}^{0} \mu^2 \psi_m(o,\mu)\, d\mu = 0.71044$$

We conclude that our result will be accurate to within one part in, say 10^4. The result for the second moment is of course the classic value of the Milne problem result. We found no a priori demonstration of this assertion however.

Introducing these numerical results, we have

$$j_t(o) = \frac{L}{c}\left[1 - 0.7104\,\frac{\ell}{L}\right] \qquad (6B.3)$$

SECTION 7 : REFERENCES AND NOMENCLATURE

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7.2 NOMENCLATURE

The symbols are defined according to their Section and Page of introduction.

Section 1		Page
A	Generalised inverse speed matrix	10
B	One region algebraic buckling	18
C	Precursor density	6
C^+	Importance of one precursor	12
D	Diffusion Coefficient	3
d	Destruction rate parameter	30
dv	Element of volume	14
F	Production rate (replaces $k\Sigma$), of prompt neutrons and precursors	6
I	Ones vector, (1,.....1)	14
k_{eff}	Effective multiplication	24
k_{ex}	Excess multiplication	24
ℓ	Lifetime, inverse destruction rate	19
L	Diffusion length	29
M	Increase matrix, production less destruction	8
N	Neutron (and precursor) population	14
p	Resonance escape probability (as $p\Sigma$)	5
p	Production rate parameter	30
P	Perturbation, M-R	14
R	Value of M in reference reactor	13
$\underline{r}$	Position vector	4

Section 1, Cont.

		Page
s	Inverse period	4
t	Time	3
T	Precursor Transfer matrix	10
V	Group speed, V^{-1}, inverse group speed	3
β	Precursor fractional production rate	6
δ	Increment or perturbation	13
λ	Precursor decay rate	6
ρ	Reactivity	19
Λ	Generating time	36
ϕ	Flux or flux vector, $(\phi_1 \ldots \phi_n)$	3,8
ϕ^+	Importance of one neutron	11
ψ	Generalised flux vector, $(\phi_1 \ldots C_1 \ldots)$	10
ψ^+	Generalised importance, $(\phi_1^+ \ldots C_1^+ \ldots)$	12
τ	Fermi-age	25
Σ	Absorption rate, or absorption and transfer between neutron groups	3,8
κ	Reciprocal diffusion length	20

Superscripts

'	Primes indicate perturbed or actual properties or actual fluxes	13
+	Importance of neutrons and precursors	4
T	Transposed matrix	11

Subscripts

1,2..n	Neutron energy groups in decreasing energies	5
i	Precursor groups	6

Subscripts, Cont.		Page
c	Core	20
r	Reflector	20
n	Eigenfunction or eigenvalue numbering	16
Section 2		
c	Reciprocal specific heat per unit volume	48
k	Spring constant	45
k	Thermal conductivity	48
m	Mass	45
T	Temperature field	47
T^+	Adjoint temperature field	48
Section 3		
A	Side of interface	85
B	Side of interface	85
c	Multiplicity	93
f	Scattering and fissioning distribution function	93
i_+, i_-	Directional current of importance	79
$\underline{i}$	Vector current of importance	81
j_+, j_-	Directional current of neutrons	81
$\underline{j}$	Vector current of neutrons	81
ℓ	Mean free path	93
r	Distance from position vector $\underline{r}$ to $\underline{r}'$	79
R	Distance to boundary surface	95

Section 3, Cont.

		Page
S	Constant source distribution	71
$\overline{S}$	Homogeneous source	73
$w(\underline{r})$	Equivalence of one neutron or precursor	71
z_o	Extrapolation distance in transport theory	96
δ_{nm}	Kronicker delta	76
λ	Free surface flux slope	96
λ	Precursor decay constant	88
θ	Azimuthal angle	79
π	Resident population	57
ψ	Isotropic transport flux	57
Σ_s	Scattering cross-section	77
ω	Eigenvalue of importance mode	86
$\underline{\Omega}$	Solid angle vector	79
ϕ	Polar angle	79

Section 4

		Page
A, A^+	Flux, adjoint flux coefficients core	137
a_n, a_n^+	Fourier expansion coefficients	115
b	Bilinear concomitant	100
B^2, B^{+2}	Generalised buckling	113
$\underline{c}$	Current of fractional resident population	112
c_n, c_n+	Arbitrary expansion coefficients	126

Section 4, Cont.

		Page
C_n, C_n^+	Coupling Coefficients	115
C, C^+	Flux, adjoint flux coefficients core	137
d	Density of fractional resident population	112
$\overline{E}$	Joint error	126
F, F^+	Flux, adjoint flux coefficients blanket	137
G	Lagrange identity for partial differentials	111
G, G^+	Flux, adjoint flux coefficients blanket	137
H	Hamiltonian	132
I_o, J_o	Bessel functions of first kind	136
K_o	Bessel function of second kind	136
i,j,k,l	Interface currents	139
ℓ_1, ℓ_2	Group lifetimes	149
$k\Sigma_2$	Thermal fission production probability	135
L, L^*	Forward and adjoint operators	99
L	Lagrange density	129
$\mathcal{L}$	Lagrangian	129
$\underline{n}$	Normal vector	108
p	Ordinary linear differential equation coefficient	98
p, p^+	Canonical momenta	132
Q, Q^+	Matrix of coefficients	146
R, R^+	Matrix of coefficients	146
R	Extrapolated radius of sphere	115
r	Radial distance	115

Section 4, Cont.

Section 4, Cont.

		Page
Subscripts		
t	Transverse buckling	147
μ, ν	Positive and negative type bucklings	137

Section 5

$\overline{C}', \overline{C}$	Spatial integral of precursor density	156
$\overline{k}_{eff}$	Bilinear average effective multiplication	161
$\overline{k}_{ex}$	Bilinear average excess multiplication	161
$\overline{\ell}'$	Bilinear average lifetime	161
$\overline{\ell}_I'$	Bilinear average group lifetime	184
$\overline{n}$	Spatial integral of neutron density	156
$\overline{P}$	Fast non-leakage probability	211
s', s	Eigenvalue of actual and reference systems	201
$\overline{V}$	Maxwell-Boltzmann averaged group speed	187
V_o	Standard group speed (2200 m/sec)	187
$\overline{\beta}'$	Total bilinear averaged yield of precursors	159
$\overline{\beta}_i'$	Bilinear averaged yield of precursors of i th type	161
ϵ	Fast fission factor	239
γ	Ratio of effective to actual precursor yield	193
$\overline{\Lambda}'$	Bilinear average generating time	158
$\hat{\Lambda}'$	Generating time for non-critical reference reactor	203
$\overline{\rho}'$	Bilinear average reactivity	158

Section 5 Cont.

		Page
$\hat{\rho}'$	Reactivity for non-critical reference reactor	203
η	Neutron yield per absorption	175
η^*	Effective η	175

Subscripts

i	i th decay characteristic of precursor	194
j	j th formation characteristic of ith precursor species	194

Section 6

A	An equation constant	298
b	Half-thickness of slab	298
c(T)	An average exposure time	270
c	Multiplicity	299
${}_nC_r$	Binomial coefficient	280
E	Function relating equivalent cross-section and perturbations	294
$\overline{F}(Z)$	Fractional worth of partially inserted control rod	290
I	Chernick's Xenon function	279
I'	Xenon function based on average flux	276
n	Integer number of rod calibration steps	264
$\overline{P}$	Fast to thermal non-leakage and non absorption probability, Equivalent perturbation	276 293
P_o	Poisoning	275
r	Radial coordinate, also an integer	280

Section 6, Cont.

		Page
S	Source strength, n/ cms^3 sec	296
S,S'	Reference and actual surfaces	300
T	Reflector thickness	308
x,y,z	Cartesian coordinates	287
x	Worth of a step rod movement	264
y	Total fission product yield	276
Z	Position of rod tip.	288
α	Enhancement	267
γ	Ratio of Moderator to Fuel absorption cross-section	275
δ	Ratio of core dimension to equivalent bare dimension	279
ν	Yield of neutrons per fission	256
ν_o	True physical value of ν	257
$\overline{\phi}$	Time invariant flux	270
ϕ_{xe}	Flux at which burn-out of Xenon equals decay	276
ϕ_o	Central flux value	280
ψ	Ratio $\phi_o/(\phi_o+\phi_{xe})$	280
θ	Spatial flux variable	280
$\overline{\theta^n}$	N th moment of flux	280

Superscripts

i	Index of dimension	282
n	Integer index of spatial moment	280

Section 6, Cont.

7.3 BIOGRAPHICAL SKETCH

Captain Jeffery Lewins, Royal Engineers (British Army).
Born 30 Nov 1930

Jeffery Lewins is the only son of Mrs. Jessie Lewins and the late Capt. F. H. Lewins, R. A. Born in London, he went to school at the Hove Grammar School, Sussex. He graduated from the Royal Military Academy, Sandhurst in 1952, First Order of Merit, with the Advanced Science Prize, the King George VI Gold Medal and the Queen's Commission into the Royal Engineers. At the Institute of Royal Engineers, Chatham, Lewins was awarded the Renée Anderson Prize for Science and the Silver Medal of the Institute.

Captain Lewins served in the Commonwealth Division in Korea in 1952-53. He returned in September 1953 to take up a place at Cambridge, England, from which he graduated in 1955 with the degree of B. A. in Mechanical Sciences. Having taken honours in this Tripos, he was able to stay a third year to read a Part II of the Tripos in Thermodynamics. In the summer of 1956, he held a scholarship at the Atomic Energy Research Establishment, Harwell.

Jeffery Lewins was awarded a King George VI Memorial Fellowship by the trustees of the English Speaking Union, for one year's advanced study in the United States, which he elected to take up at the Massachusetts Institute of Technology. He received his Master's Degree in Nuclear Engineering in September 1957. He was able to proceed with a Doctoral programme in the Nuclear Engineering Department by serving as a half-time Research Assistant and an operator of the MIT Research Reactor. During the current year, Captain Lewins holds a Department Scholarship and is lecturing in the Department on perturbation theory.

Captain Lewins married Sabrina, daughter of Mr. and Mrs. Lloyd Moss of Salem, Mass. Publications include:

An Approximate Plant Characteristic, Soc Inst Tech (1956)
A Contribution to Black Control Rod Worth, Nuclear Sci and Engr 4, 500 (1958)
Effective Heat Resistances in Reactors, with W. Rohsenow, Nucleonics, 5, (1959)
Lecture Notes in the Nuclear Engineering Department:
(a) Reactor Temperature Distribution, with W. Rohsenow, and J. P. Barger
(b) Lecture Notes on Perturbation Theory
MIT Reactor Memo's: Five, including MIT Inhour Relation.
Master's thesis, MIT: A Comparison of End-fed and Centre-fed Cooling in Nuclear Reactors.

Captain Lewins is a Member of the Institute of Royal Engineers, Overseas Member of the American Nuclear Society, a Member of Tau Beta Pi and Member of Sigma Xi.