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Citation: Edelman, Alan et al. "Beyond Universality in Random Matrix Theory." The Annals of Applied Probability 26, 3 (June 2016): 1659–1697 © 2016 Institute of Mathematical Statistics

Published Version: <http://dx.doi.org/10.1214/15-AAP1129>

Publisher: Institute of Mathematical Statistics

Permanent Link: <http://hdl.handle.net/1721.1/115297>

Version: Author's final manuscript: final author's manuscript post peer review, without publisher's formatting or copy editing

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BEYOND UNIVERSALITY IN RANDOM MATRIX THEORY¹

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In order to have a better understanding of finite random matrices with non-Gaussian entries, we study the $1/N$ expansion of local eigenvalue statistics in both the bulk and at the hard edge of the spectrum of random matrices. This gives valuable information about the smallest singular value not seen in universality laws. In particular, we show the dependence on the fourth moment (or the kurtosis) of the entries. This work makes use of the so-called complex Gaussian divisible ensembles for both Wigner and sample covariance matrices.

1. Beyond universality. The desire to assess the applicability of universality results in random matrix theory has pressed the need to go beyond universality, in particular the need to understand the influence of finite n and what happens if the matrix deviates from Gaussian normality. In this article, we provide exact asymptotic correction formulas for the smallest singular value of complex matrices and bulk statistics for complex Wigner matrices.

“Universality,” a term encountered in statistical mechanics, is widely found in the field of random matrix theory. The universality principle loosely states that eigenvalue statistics of interest will behave asymptotically as if the matrix elements were Gaussian. The spirit of the term is that the eigenvalue statistics will not care about the details of the matrix elements.

It is important to extend our knowledge of random matrices beyond universality. In particular, we should understand the role played by:

- finite n and
- non-Gaussian random variables.

Received July 2014; revised July 2015.

¹Supported by NSF Grants DMS-13-12831, DMS-10-16125, DMS-10-16086, DMS-13-07704 and of the Simons foundation.

AMS 2000 subject classifications. 15A52.

Key words and phrases. Random matrix, singular value, universality, Wigner matrix, bulk, hard edge.

<p>This is an electronic reprint of the original article published by the Institute of Mathematical Statistics in <i>The Annals of Applied Probability</i>, 2016, Vol. 26, No. 3, 1659–1697. This reprint differs from the original in pagination and typographic detail.</p>

From an application viewpoint, it is very valuable to have an estimate for the departure from universality. Real problems require that n be finite, not infinite, and it has long been observed computationally that ∞ comes very fast in random matrix theory. The applications beg to know how fast. From a theoretical viewpoint, there is much to be gained in searching for proofs that closely follow the underlying mechanisms of the mathematics. We might distinguish “mechanism oblivious” proofs whose bounds require n to be well outside imaginably useful ranges, with “mechanism aware” proofs that hold close to the underlying workings of random matrices. We encourage such “mechanism aware” proofs.

In this article, we study the influence of the fourth cumulant on the local statistics of the eigenvalues of random matrices of Wigner and Wishart type.

On one hand, we study the asymptotic expansion of the smallest eigenvalue density of large random sample covariance matrices. The behavior of smallest eigenvalues of sample covariance matrices when p/n is close to one (and more generally) is somewhat well understood now. We refer the reader to [6, 7, 14, 17, 34]. The impact of the fourth cumulant of the entries is of interest here; we show its contribution to the distribution function of the smallest eigenvalue density of large random sample covariance matrices as an additional error term of order of the inverse of the dimension (see Theorem 3.1).

On the other hand, we consider the influence of the fourth moment in the local fluctuations in the bulk. Here, we consider Wigner matrices and discuss a conjecture of Tao and Vu [32] that the fourth moment brings a correction to the fluctuation of the expectation of the eigenvalues in the bulk of order of the inverse of the dimension. We prove (cf. Theorem 3.3) that the quantiles of the one point correlation function fluctuate according to the formula predicted by Tao and Vu for the fluctuations of the expectation of the eigenvalues.

In both cases, we consider the simplest random matrix ensembles that are called Gaussian divisible, that is whose entries can be described as the convolution of a distribution by the Gaussian law. To be more precise, we consider the so-called Gaussian-divisible ensembles, also known as Johansson–Laguerre and Johansson–Wigner ensembles. These ensembles, defined hereafter, have been first considered in [24] and have the remarkable property that the induced joint eigenvalue density can be computed. It is given in terms of the Itzykson–Zuber–Harich–Chandra integral. From such a formula, saddle point analysis allows to study the local statistics of the eigenvalues. In [20], this idea was used to bound the rate of convergence of the partition function of Gaussian divisible ensemble toward their limit by the inverse of the dimension to the power $2/3$. We precise this study by showing that at the hard edge or in the bulk, this error is in fact of the order of the inverse of the dimension and give the explicit form of this error, and in particular

its dependency on the fourth moment. It turns out that in both cases under study, the contribution of the fourth moment to the local statistics can be inferred from the fluctuations of the one-point correlation function, that is of the mean linear statistics of Wigner and Wishart random matrices. The covariance of the latter is well known, since [25], to depend on the fourth moments, from which our results follow.

2. Discussion and simulations.

2.1. *Preliminaries: Real kurtosis.* We will only consider distributions whose real and imaginary parts are independent and are identically distributed.

DEFINITION 1. The *kurtosis* of a distribution is

$$\gamma = \frac{\kappa_4^{\Re}}{\sigma_{\Re}^4} = \frac{\mu_4}{\sigma_{\Re}^4} - 3,$$

where κ_4^{\Re} is the fourth cumulant of the real part, σ_{\Re}^2 is the variance of the real part, and μ_4 is the fourth moment about the mean. The fourth cumulant of a centered complex distribution P with i.i.d. real and complex part with variance σ_{\Re}^2 , is given by

$$\kappa_4 = \int |zz^*|^2 dP(z) - 8\sigma_{\Re}^4 = 2\kappa_4^{\Re} = 2\gamma\sigma_{\Re}^4.$$

Note. From a software viewpoint, commands such as `randn` make it natural to take the real and the imaginary parts to separately have mean 0, variance 1, and also to consider the real kurtosis.

Example of Kurtoses γ for distributions with mean 0, and $\sigma^2 = 1$ is provided in Table 1.

For the matrices themselves, we compute the smallest eigenvalues of the Gram matrix constructed from $(n + \nu) \times n$ complex random matrices with Julia [8] code provided for the reader's convenience in Table 2.

TABLE 1
Standard Kurtoses and codes

Distribution	γ	Univariate code
Normal	0	<code>randn</code>
Uniform $[-\sqrt{3}, \sqrt{3}]$	-1.2	<code>(rand-0.5)*sqrt(12)</code>
Bernoulli	-2	<code>sign(randn)</code>
Gamma	6	<code>rand(Gamma()) - 1</code>

TABLE 2
Matrix codes

RM	Complex matrix code
Normal	<code>randn(n+ν,n)+im*randn(n+ν,n)</code>
Uniform	<code>((rand(n+ν,n)-0.5)+im*(rand(n+ν,n)-0.5))*sqrt(12)</code>
Bernoulli	<code>sign(randn(n+ν,n))+im*sign(randn(n+ν,n))</code>
Gamma	<code>(rand(Gamma(),n+ν,n)-1)+im*(rand(Gamma(),n+ν,n)-1)</code>

2.2. *Smallest singular value experiments.* Let A be a random $n + \nu$ by n complex matrix with i.i.d. real and complex entries all with mean 0, variance 1 and kurtosis γ . In the next several subsections, we display special cases of our results, with experiment vs. theory curves for $\nu = 0, 1$ and 2.

We consider the cumulative distribution function

$$F(x) = \mathbb{P}\left(\lambda_{\min}(AA^*) \leq \frac{x}{n}\right) = \mathbb{P}\left((\sigma_{\min}(A))^2 \leq \frac{x}{n}\right),$$

where $\sigma_{\min}(A)$ is the smallest singular value of A . We also consider the density

$$f(x) = \frac{d}{dx}F(x).$$

In the plots to follow, we took a number of cases when $n = 20, 40$ and sometimes $n = 80$. We computed 2,000,000 random samples on each of 60 processors using Julia [8], for a total of 120,000,000 samples of each experiment. The runs used 75% of the processors on a machine equipped with 8 Intel E7-8850-2.0 GHz-24M-10 Core Xeon MP Processors. This scale experiment, which is made easy by the Julia system, allows us to obtain visibility on the higher order terms that would be hard to see otherwise. Typical runs took about an hour for $n = 20$, three hours for $n = 40$ and twelve hours for $n = 80$.

We remark that we are only aware of two or three instances where parallel computing has been used in random matrix experiments. Working with Julia is pioneering in showing just how easy this can be, giving the random matrix experimenter a new tool for honing in on phenomena that would have been nearly impossible to detect using conventional methods.

2.3. *Example: Square complex matrices ($\nu = 0$).* Consider taking, a 20 by 20 random matrix with independent real and imaginary entries that are uniformly distributed on $[-\sqrt{3}, \sqrt{3}]$.

$$((\text{rand}(20,20) - 0.5) + \text{im} * (\text{randn}(20,20) - 0.5)) * \text{sqrt}(12).$$

This matrix has real and complex entries that have mean 0, variance 1 and kurtosis $\gamma = -1.2$.

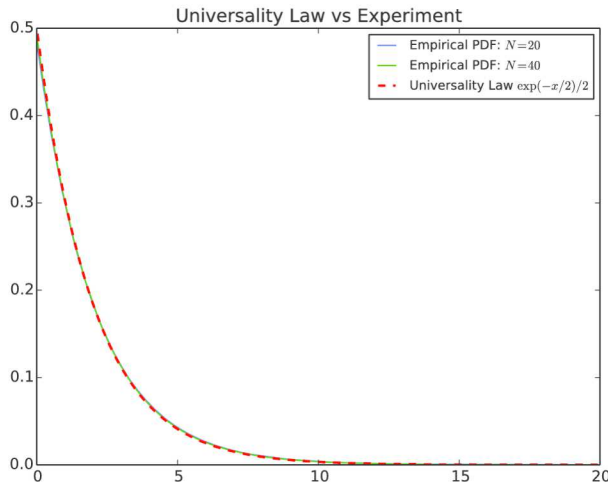


FIG. 1. *Universality law vs. experiment: $n = 20$ and $n = 40$ already resemble $n = \infty$.*

An experimenter wants to understand how the smallest singular value compares with that of the complex Gaussian matrix

$$\text{randn}(20, 20) + \text{im} * \text{randn}(20, 20).$$

The law for complex Gaussian matrices [13, 14] in this case valid for all finite sized matrices, is that $n\lambda_{\min}(AA^*) = n\sigma_{\min}^2(A)$ is exactly exponentially distributed: $f(x) = \frac{1}{2}e^{-x/2}$. Universality theorems say that the uniform curve will match the Gaussian in the limit as matrix sizes go to ∞ . The experimenter obtains the curves in Figure 1 (taking both $n = 20$ and $n = 40$).

Impressed that $n = 20$ and $n = 40$ are so close, he or she might look at the proof of the universality theorem only to find that no useful bounds are available at $n = 20, 40$.

The results in this paper give the following correction in terms of the kurtosis (when $\nu = 0$):

$$f(x) = e^{-x/2} \left(\frac{1}{2} + \frac{\gamma}{n} \left(\frac{1}{4} - \frac{x}{8} \right) \right) + O\left(\frac{1}{n^2}\right).$$

On the bottom of Figure 1, with the benefit of 60 computational processors, we can magnify the departure from universality with Monte Carlo experiments, showing that the departure truly fits $\frac{\gamma}{n} \left(\frac{1}{4} - \frac{x}{8} \right) e^{-x/2}$. This experiment can be run and rerun many times, with many distributions, kurtoses that are positive and negative, small values of n , and the correction term works very well. Figure 2 shows that the corrections converge as predicted for uniform, Bernoulli and Gamma distributed entries.

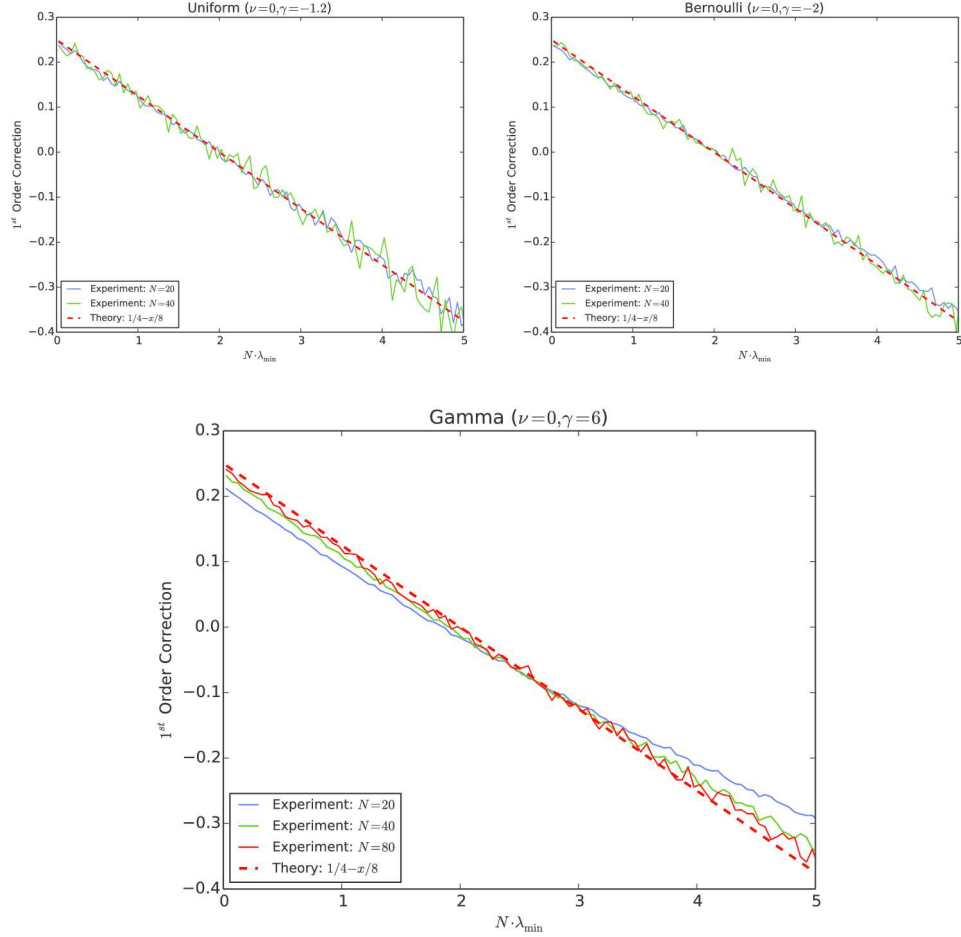


FIG. 2. Correction for square matrices *Uniform*, *Bernoulli* ($\nu=0$). Monte Carlo simulations are histogrammed, 0th order term subtracted and result multiplied by $ne^{x/2}/\gamma$. Bottom curve shows convergence for $n=20, 40, 80$ for a distribution with positive kurtosis.

2.4. *Example: $n+1$ by n complex matrices ($\nu=1$).* The correction to the density can be written as

$$f(x) = e^{-x/2} \left(\frac{1}{2} I_2(s) + \frac{1+\gamma}{8n} (sI_1(s) - xI_2(s)) \right) + O\left(\frac{1}{n^2}\right),$$

where $I_1(x)$ and $I_2(x)$ are Bessel functions and $s = \sqrt{2x}$.

Simulations are shown in Figure 3.

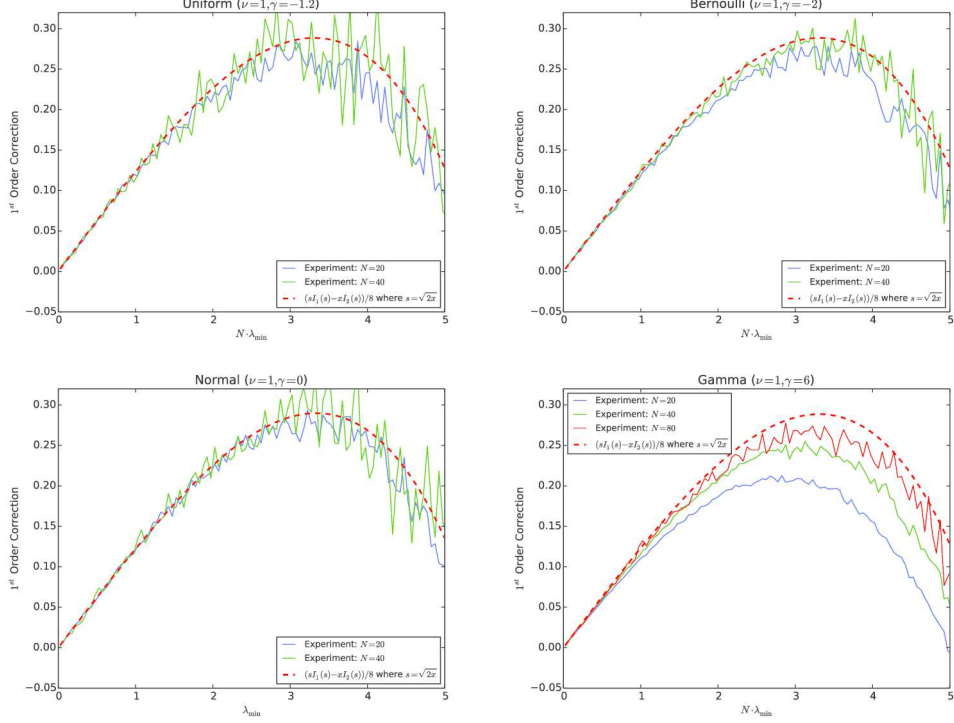


FIG. 3. Correction for $\nu = 1$. Uniform, Bernoulli, Normal and Gamma; Monte Carlo simulations are histogrammed, 0th order term subtracted and result multiplied by $ne^{x/2}/(1 + \gamma)$. Bottom right curve shows convergence for $n = 20, 40, 80$ for a distribution with positive kurtosis.

2.5. *Example: $n + 2$ by n complex matrices ($\nu = 2$).* The correction to the density for $\nu = 2$ can be written

$$f(x) = \frac{1}{2}e^{-x/2} \left([I_2^2(s) - I_1(s)I_3(s)] + \frac{2 + \gamma}{2n} [(x + 4)I_1^2(s) - 2sI_0(s)I_1(s) - (x - 2)I_2^2(s)] \right),$$

where I_0, I_1, I_2 , and I_3 are Bessel functions, and $s = \sqrt{2x}$.

Simulations are given in Figure 4.

3. Models and results. In this section, we define the models we will study and state the results. Let some real parameter $a > 0$ be given. Consider a matrix M of size $p \times n$:

$$M = W + aV,$$

where:

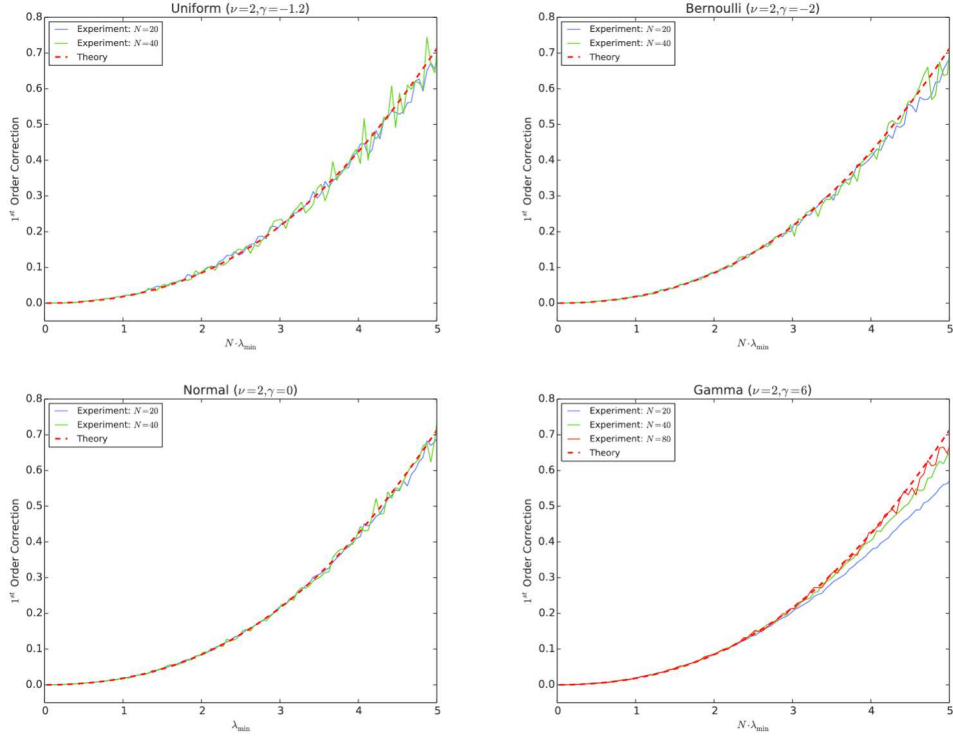


FIG. 4. Correction for $\nu = 2$. Uniform, Bernoulli, Normal and Gamma; Monte Carlo simulations are histogrammed, 0th order term subtracted and result multiplied by $ne^{x/2}/(2 + \gamma)$. Bottom right curve shows convergence for $n = 20, 40, 80$ for a distribution with positive kurtosis.

- $V = (V_{ij})_{1 \leq i \leq p; 1 \leq j \leq n}$ has i.i.d. entries with complex $\mathcal{N}_{\mathbb{C}}(0, 1)$ distribution, which means that both $\Re V_{ij}$ and $\Im V_{ij}$ are real i.i.d. $\mathcal{N}(0, 1/2)$ random variables,
- $W = (W_{ij})_{1 \leq i \leq p; 1 \leq j \leq n}$ is a random matrix with entries being mutually independent random variables with distribution $P_{ij}, 1 \leq j \leq n$ independent of n and p , with uniformly bounded fourth moment,
- W is independent of V ,
- $\nu := p - n \geq 0$ is a fixed integer independent of n .

We then form the Gaussian divisible ensemble (also known as the Johansson–Laguerre matrix):

$$(1) \quad \frac{1}{n} M^* M = \left(\frac{1}{\sqrt{n}} (W + aV) \right)^* \left(\frac{1}{\sqrt{n}} (W + aV) \right).$$

When W is fixed, the above ensemble is known as the *deformed Laguerre ensemble*.

We assume that the probability distributions $P_{j,k}$ satisfy

$$(2) \quad \int z dP_{j,k}(z) = 0, \quad \int |zz^*| dP_{j,k}(z) = \sigma_C^2 = \frac{1}{4}.$$

Here, the complex $\sigma_C^2 = 2\sigma_{\Re}^2$ represents the complex variance. Hypothesis (2) ensures the convergence of the spectral measure of $\frac{1}{n}W^*W$ to the Marchenko–Pastur distribution with density

$$(3) \quad \rho_{\text{PM}}(x) = \frac{2}{\pi} \frac{\sqrt{1-x}}{\sqrt{x}}, \quad 0 \leq x \leq 1.$$

Condition (2) implies also that the limiting spectral measure of $\frac{1}{n}M^*M$ is then given by Marchenko–Pastur’s law with parameter $1/4 + a^2$; we denote ρ_a the density of this probability measure, that is, $\rho_a(x) = (1 + 4a^2)^{-1/2} \rho_{\text{PM}}(x/\sqrt{1 + 4a^2})$.

For technical reasons, we assume that the entries of W have subexponential tails: There exist $C, c, \theta > 0$ so that for all $j, \in \mathbb{N}$, all $t \geq 0$

$$(4) \quad P_{j,k}(|z| \geq t) \leq Ce^{-ct^\theta}.$$

This hypothesis could be weakened to requiring enough finite moments.

Finally, we assume that the fourth moments do not depend on j, k and let κ_4 be the difference between the fourth moment of $P_{j,k}$ and the complex Gaussian case,

$$\kappa_4 = \int |zz^*|^2 dP_{j,k} - 8^{-1}.$$

(Thus, with the notation of Definition 1, $\kappa_4 = 2\gamma\sigma_{\Re}^4 = 2\kappa_4^{\Re}$.)

Then our main result is the following. Let $\sigma := \sqrt{4^{-1} + a^2}$ and for an Hermitian matrix A denote $\lambda_{\min}(A) = \lambda_1(A) \leq \lambda_2(A) \leq \dots \leq \lambda_n(A)$ the eigenvalues of A .

THEOREM 3.1. *Let F_n^ν be the cumulative density function of the hard edge of a Gaussian $p \times n$ matrix V , $\nu = p - n$, with entries with complex variance σ^2 :*

$$F_n^\nu(s) = \mathbb{P}\left(\sigma^2 \lambda_{\min}(VV^*) \leq \frac{s}{n}\right).$$

Then, for all $s > 0$, if our distribution has complex fourth cumulant $\kappa_4 = 2\kappa_4^{\Re}$,

$$\mathbb{P}\left(\lambda_{\min}(MM^*) \leq \frac{s}{n}\right) = F_n^\nu(s) + \frac{s(F_n^\nu)'(s)}{\sigma^4 n} \kappa_4 + o\left(\frac{1}{n}\right).$$

We note that this formula is scale invariant. It is equivalent to

$$\mathbb{P}\left(\lambda_{\min}(MM^*) \geq \frac{s}{n}\right) = 1 - F_n^\nu(s) + \frac{s(1 - F_n^\nu)'(s)}{\sigma^4 n} \kappa_4 + o\left(\frac{1}{n}\right).$$

Let $F_\infty^\nu(s) = \lim_{n \rightarrow \infty} F_n^\nu(s)$ be the limiting cumulative distributive function in the Gaussian case. F_∞^ν is well known: we provide a few of its definitions in Section 4; see also (12). The difference $F_n^\nu - F_\infty^\nu$ was derived by Schehr [31] and Bornemann [10], after it was conjectured in the first version of this paper. We can then deduce the following.

COROLLARY 3.2. *For all integer number ν ,*

$$\mathbb{P}\left(\lambda_{\min}(MM^*) \leq \frac{s}{n}\right) = F_\infty^\nu(s) + \left(\nu + \frac{\kappa_4}{\sigma^4}\right) \frac{s(F_\infty^\nu)'(s)}{n} + o\left(\frac{1}{n}\right).$$

Note. Corollary 3.2 is the convenient formulation we used for $\nu = 0, 1, 2$ in our showcase examples in Sections 2.3, 2.4 and 2.5, respectively.

Note. Corollary 3.2 is remarkable because it states that the correction term for n being a finite Gaussian as opposed to being infinite, and the correction term for n being non-Gaussian as opposed to Gaussian “line up,” in that either way the corrections are multiples of $s(F_\infty^\nu)'(s)$. This could not be predicted by Theorem 3.1 alone.

Note. It is worth taking more of a close look between the formulation in Theorem 3.1 and Corollary 3.2. The first term in Theorem 3.1 is n dependent, while in Corollary 3.2 the first term has reached its $n \rightarrow \infty$ limit. Also of note is that the F_n formulation in Theorem 3.1 involves Laguerre polynomials (and exponentials.) The F_∞ formulation in Corollary 3.2 involves Bessel functions (and exponentials).

For the Wigner ensemble, we consider the matrix

$$M_n = \frac{1}{\sqrt{n}}(W + aV),$$

where W is a Wigner matrix with complex (resp., real) independent entries above (resp., along) the diagonal $W_{ij}, 1 \leq i \leq j \leq N$ with law P_{ij} . We assume that the distributions P_{ij} have subexponential moments: there exists $C, c > 0$, and $\alpha > 0$ such that for all $t \geq 0$ and all $1 \leq i \leq j \leq N$

$$(5) \quad P_{ij}(|x| \geq t) \leq C \exp\{-ct^\alpha\},$$

and satisfy

$$(6) \quad \int x dP_{ij}(x) = 0, \quad \int |x|^2 dP_{ij}(x) = 1/4, \quad \int x^3 dP_{ij}(x) = 0.$$

Again we assume that the fourth moments do not depend on i, j and let κ_4 be the difference between the fourth moment of P_{ij} ($j \neq i$) and the Gaussian case,

$$\kappa_4 = \int |zz^*|^2 dP_{ij} - 1/8.$$

The other matrix V is a GUE random matrix with i.i.d. $\mathcal{N}_{\mathbb{C}}(0, 1)$ entries. We denote by $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ the ordered eigenvalues of M_n . By Wigner's theorem, it is known that the spectral measure of M_n

$$\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}$$

converges weakly to the semicircle distribution with density

$$(7) \quad \sigma_{\text{sc}}^{2\sigma}(x) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} \mathbb{1}_{|x| \leq 2\sigma}; \quad \sigma^2 = 1/4 + a^2.$$

This is the Gaussian-divisible ensemble studied by Johansson [24]. We study the dependency of the one point correlation function ρ_n of this ensemble, given as the probability measure on \mathbb{R} so that for any bounded measurable function f

$$\mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n f(\lambda_i) \right] = \int f(x) \rho_n(x) dx$$

as well as the localization of the quantiles of ρ_n with respect to the quantiles of the limiting semicircle distribution. In particular, we study the $1/n$ expansion of this localization, showing that it depends on the fourth moment of μ . Define $N_n(x) := \frac{1}{n} \#\{i, \lambda_i \leq x\}$, and $N_{\text{sc}}(x) = \int_{-\infty}^x d\sigma_{\text{sc}}(u)$, with σ_{sc} defined in (35). Let us define the quantiles $\hat{\gamma}_i$ (resp., γ_i) by

$$\hat{\gamma}_i := \inf \left\{ y, \mathbb{E} N_n(y) = \frac{i}{n} \right\} \quad \text{respectively } N_{\text{sc}}((-\infty, \gamma_i]) = \frac{i}{n}.$$

We shall prove the following.

THEOREM 3.3. *Let $\varepsilon > 0$. There exists a function D on $[-2 + \varepsilon, 2 - \varepsilon]$, independent of the distributions $P_{i,j}$, such that for all $x \in [-2 + \varepsilon, 2 - \varepsilon]$*

$$\rho_n(x) = \sigma_{\text{sc}}(x) + \frac{1}{n} \kappa_4 D(x) + o\left(\frac{1}{n}\right).$$

For all $i \in [n\varepsilon, n(1 - \varepsilon)]$ for some $\varepsilon > 0$, one has that

$$(8) \quad \hat{\gamma}_i - \gamma_i = \frac{\kappa_4}{2n} (2\gamma_i^3 - \gamma_i) + o\left(\frac{1}{n}\right).$$

This is a version of the rescaled Tao–Vu conjecture 1.7 in [32] where $\mathbb{E}[\lambda_i]$ is replaced by $\hat{\gamma}_i$. A similar result could be derived for Johansson–Laguerre ensembles. We do not present the details of the computation here, which would resemble the Wigner case. The function D is computed explicitly in Proposition 6.1.

4. Smallest singular values of $n + \nu$ by n complex Gaussian matrices.

Theorem 3.1 depends on the partition function for Gaussian matrices, which itself depends on ν and n . In this section, we investigate these dependencies.

4.1. *Known exact results.* It is worthwhile to review what exact representations are known for the smallest singular values of complex Gaussians.

We consider the finite n density $f_n^\nu(x)$, the finite n cumulative distribution $F_n^\nu(x)$ (we stress the ν -dependency in this section), and their asymptotic values $f_\infty^\nu(x)$ and $F_\infty^\nu(x)$. We have found the first form in the list below useful for symbolic and numerical computation. In the formulas to follow, we assume $\sigma_{\Re}^2 = 1$ so that a command such as `randn()` can be used without modification for the real and imaginary parts. All formulas concern $n\lambda_{\min}(AA^*) = n\sigma_{\min}^2(A)$ and its asymptotics. We present in the array below eight different formulations of the exact distribution F_n^ν .

Some of these formulations allow one or both of ν or n to extend beyond integers to real positive values. Assuming ν and n are integers ([14], Theorem 5.4), the probability density $f_n^\nu(x)$ takes the form $x^\nu e^{-x/2}$ times a polynomial of degree $(n-1)\nu$ and $1 - F_n^\nu(x)$ is $e^{-x/2}$ times a polynomial of degree $n\nu$.

REMARK. A helpful trick to compare normalizations used by different authors is to inspect the exponential term. The 2 in $e^{-x/2}$ denotes total complex variance 2 (twice the real variance of 1). In general the total complex variance $\sigma^2 = 2\sigma_{\Re}^2$ will appear in the denominator.

In the next paragraphs, we discuss the eight formulations introduced above.

4.1.1. *Determinant: ν by ν determinant.* The quantities of primary use are the beautiful ν by ν determinant formulas for the distributions by Forrester and Hughes [19] in terms of Bessel functions and Laguerre polynomials. The infinite formulas also appear in [18], equation (8.98). Hereafter, I_j denotes the modified Bessel functions and L_j the Laguerre polynomials:

$$\begin{aligned} F_\infty^\nu(x) &= e^{-x/2} \det[I_{i-j}(\sqrt{2x})]_{i,j=1,\dots,\nu}, \\ f_\infty^\nu(x) &= \frac{1}{2} e^{-x/2} \det[I_{2+i-j}(\sqrt{2x})]_{i,j=1,\dots,\nu}, \\ F_n^\nu(x) &= e^{-x/2} \det[L_{n+i-j}^{(j-i)}(-x/2n)]_{i,j=1,\dots,\nu}, \\ f_n^\nu(x) &= \left(\frac{x}{2n}\right)^\nu \frac{(n-1)!}{2(n+\nu-1)!} e^{-x/2} \\ &\quad \times \det[L_{n-1+i-j}^{(j-i+2)}(-x/2n)]_{i,j=1,\dots,\nu}. \end{aligned}$$

Recall that $I_j(x) = I_{-j}(x)$. To facilitate reading of the relevant ν by ν determinants, we provide expanded views:

$$\det[I_{i-j}(\sqrt{2x})]_{i,j=1,\dots,\nu} = \begin{vmatrix} I_0 & I_1 & I_2 & \cdots & I_{\nu-1} \\ I_1 & I_0 & I_1 & \cdots & I_{\nu-2} \\ I_2 & I_1 & I_0 & \cdots & I_{\nu-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{\nu-1} & I_{\nu-2} & I_{\nu-3} & \cdots & I_0 \end{vmatrix},$$

Bessel functions evaluated at $\sqrt{2x}$

$$\det[I_{2+i-j}(\sqrt{2x})]_{i,j=1,\dots,\nu} = \begin{vmatrix} I_2 & I_1 & I_0 & \cdots & I_{\nu-3} \\ I_3 & I_2 & I_1 & \cdots & I_{\nu-4} \\ I_4 & I_1 & I_2 & \cdots & I_{\nu-5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{\nu+1} & I_\nu & I_{\nu-1} & \cdots & I_2 \end{vmatrix},$$

Bessel functions evaluated at $\sqrt{2x}$

$$\det\left[L_{n+i-j}^{(j-i)}\left(-\frac{x}{2n}\right)\right]_{i,j=1,\dots,\nu} = \begin{vmatrix} L_n & L_{n-1}^{(1)} & L_{n-2}^{(2)} & \cdots & L_{n-\nu+1}^{(\nu-1)} \\ L_{n+1}^{(-1)} & L_n & L_{n-1}^{(1)} & \cdots & L_{n-\nu+2}^{(\nu-2)} \\ L_{n+2}^{(-2)} & L_{n+1}^{(-1)} & L_n & \cdots & L_{n-\nu+3}^{(\nu-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{n+\nu-1}^{(1-\nu)} & L_{n+\nu-2}^{(2-\nu)} & L_{n+\nu-3}^{(3-\nu)} & \cdots & L_n \end{vmatrix},$$

evaluated at $-x/2n$

$$\det\left[L_{n-1+i-j}^{(j-i+2)}\left(-\frac{x}{2n}\right)\right]_{i,j=1,\dots,\nu} = \begin{vmatrix} L_{n-1}^{(2)} & L_{n-2}^{(3)} & L_{n-3}^{(4)} & \cdots & L_{n-\nu}^{(\nu+1)} \\ L_n^{(1)} & L_{n-1}^{(2)} & L_{n-2}^{(3)} & \cdots & L_{n-\nu+1}^{(\nu)} \\ L_{n+1} & L_n^{(1)} & L_{n-1}^{(2)} & \cdots & L_{n-\nu+2}^{(\nu-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{n+\nu-2}^{(3-\nu)} & L_{n+\nu-3}^{(4-\nu)} & L_{n+\nu-4}^{(5-\nu)} & \cdots & L_{n-1}^{(2)} \end{vmatrix}.$$

evaluated at $-x/2n$

The following Mathematica code symbolically computes these distributions:

```
M[x_, v_] := Table[BesselI[Abs[i-j], x], {i, v}, {j, v}];
```

```

m[x_,v_] := Table[BesselI[Abs[2+i-j],x],{i,v},
                 {j,v}];
M[x_,n,v_] := Table[LaguerreL[n+i-j,j-i,-x/(2*n)],
                    {i,v},{j,v}];
m[x_,n_,v_] := Table[LaguerreL[n-1+i-j,j-i+2,-x/(2*n)],
                    {i,v},{j,v}];
F[x_,v_] := 1-Exp[-x/2]*Det [M[Sqrt[2 x],v]];
f[x_,v_] := (1/2)*Exp[-x/2]*Det [m[Sqrt[2 x],v]];
F[x_,n_,v_] := 1-Exp[-x/2]*Det [M[x,n,v]];
f[x_,n_,v_] := (x/(2n))^v*((n-1)!/(2(n+v-1)!))
               *Exp[-x/2]*Det [m[x,n,v]].

```

4.1.2. *Painlevé III.* According to [18], equation (8.93), [9], pages 814–815, [33, 34], we have the formula valid for all $\nu > 0$:

$$F_\infty^\nu(x) = 1 - \exp\left(-\int_0^{2x} \sigma(s) \frac{ds}{s}\right),$$

where $\sigma(s)$ is the solution to a Painlevé III differential equation. Please consult the references taking care to match the normalization.

4.1.3. *n by n determinant.* Following standard techniques to set up the multivariate integral and applying a continuous version of the Cauchy–Binet theorem (Gram’s formula) [28], for example, Appendix A.12 or [35], for example, equations (1.3) and (5.2) one can work out an $n \times n$ determinant valid for any ν , so long as n is an integer [26]:

$$F_n^\nu(x) = 1 - \frac{\det(M(m, \nu, x/2))}{\det(M(m, \nu, 0))},$$

where, if Γ denotes the incomplete gamma function,

$$M(m, \nu, x) = \begin{pmatrix} \Gamma(\nu+1, x) & \Gamma(\nu+2, x) & \Gamma(\nu+3, x) & \cdots & \Gamma(\nu+m, x) \\ \Gamma(\nu+2, x) & \Gamma(\nu+3, x) & \Gamma(\nu+4, x) & \cdots & \Gamma(\nu+m+1, x) \\ \Gamma(\nu+3, x) & \Gamma(\nu+4, x) & \Gamma(\nu+5, x) & \cdots & \Gamma(\nu+m+2, x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Gamma(\nu+m, x) & \Gamma(\nu+m+1, x) & \Gamma(\nu+m+2, x) & \cdots & \Gamma(\nu+2m-1, x) \end{pmatrix}.$$

4.1.4. *Remaining formulas in Table 3.* The Fredholm determinant is a standard procedure. The multivariate integral recurrence was computed in the real case in [14] and in the complex case in [19]. Various hypergeometric representations may be found in [12], but to date we are not aware of the complex representation of the confluent representation in [30] which probably is worth pursuing.

TABLE 3

Exact results for smallest singular values of complex Gaussians (smallest eigenvalues of complex Wishart or Laguerre ensembles)

1. Determinant: ν by ν	[18, 19]
2. Painlevé III	[18], equation (8.93)
3. Determinant: n by n	[26]
4. Fredholm determinant	[9, 33]
5. Multivariate integral recurrence	[14, 19]
6. Finite sum of schur polynomials (evaluated at \mathbb{I})	[12]
7. Hypergeometric function of matrix argument	[12]
8. Confluent hypergeometric function of matrix argument	[30]

4.2. *Asymptotics of smallest singular value densities of complex Gaussians.* A very useful expansion extends a result from [19], (3.29).

LEMMA 4.1. *As $n \rightarrow \infty$, we have the first two terms in the asymptotic expansion of scaled Laguerre polynomials whose degree and constant parameter sum to n :*

$$L_{n-k}^{(k)}(-x/n) \sim n^k \left\{ \frac{I_k(2\sqrt{x})}{x^{k/2}} - \frac{1}{2n} \left(\frac{I_{k-2}(2\sqrt{x})}{x^{(k-2)/2}} \right) + O\left(\frac{1}{n^2}\right) \right\}.$$

PROOF. We omit the tedious details but this (and indeed generalizations of this result) may be computed either through direct expansion of the Laguerre polynomial or through the differential equation it satisfies. \square

We can use the lemma above to conjecture asymptotics of the distribution $F_n^\nu(x)$. This conjecture was recently proved independently by Anthony Perret and Grégory Schehr [31] and by Folkmar Bornemann [10]. The first uses properties of the Jacobi matrix associated with modified Laguerre polynomials and its implication to Painlevé while the second takes a close look at the Fredholm determinant and the asymptotics of Laguerre polynomials. This result states as follows.

THEOREM 4.2. *Let ν be an integer number. Let $F_n^\nu(x)$ be the distribution of $n\sigma_{\min}^2(A)$ of an $n + \nu$ by n complex Gaussian A . We have the expansion*

$$F_n^\nu(x) = F_\infty^\nu(x) + \frac{\nu}{2n} x f_\infty^\nu(x) + O\left(\frac{1}{n^2}\right).$$

Note. The above is readily checked to be scale invariant, so it is not necessary to state the particular variances in the matrix as long as they are equal.

5. The hard edge of complex Gaussian divisible ensembles. The hard edge denotes the location of the smallest eigenvalues of sample covariance matrices when $\nu = p - n$ is a fixed integer.

5.1. *Reminder on Johansson–Laguerre ensemble.* We here recall some important facts about the Johansson–Laguerre ensemble, that we use in the following.

Notation. We call $\mu_{n,p}$ the law of the sample covariance matrix $\frac{1}{n}M^*M$ defined in (1). We denote by $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ the ordered eigenvalues of the random sample covariance matrix $\frac{1}{n}M^*M$. We also set

$$H = \frac{W}{\sqrt{n}},$$

and denote the distribution of the random matrix H by P_n . The ordered eigenvalues of HH^* are denoted by $y_1(H) \leq y_2(H) \leq \dots \leq y_n(H)$.

We can now state the known results about the joint eigenvalue density (j.e.d.) induced by the Johansson–Laguerre ensemble. Propositions 5.1, 5.2 and 5.3 are derived in [7], Sections 3 and 7. By construction, this is obtained as the integral w.r.t. P_n of the j.e.d. of the deformed Laguerre ensemble. We recall that the deformed Laguerre ensemble denotes the distribution of the covariance matrix $n^{-1}MM^*$ when H is given. The latter has been first computed by [21] and [23].

We now set

$$s = \frac{a^2}{n}.$$

PROPOSITION 5.1. *The symmetrized eigenvalue measure on \mathbb{R}_+^n induced by $\mu_{n,p}$ has a density w.r.t. Lebesgue measure given by*

$$(9) \quad g(x_1, \dots, x_n) = \int dP_n(H) g(x_1, \dots, x_n; y(H))$$

with

$$\begin{aligned} & g(x_1, \dots, x_n; y(H)) \\ &= \frac{\Delta(x)}{\Delta(y(H))} \det \left(\frac{e^{-(y_i(H)+x_j)/(2t)}}{2t} I_\nu \left(\frac{\sqrt{y_i(H)x_j}}{t} \right) \left(\frac{x_j}{y_i(H)} \right)^{\nu/2} \right)_{i,j=1}^n, \end{aligned}$$

where $t = \frac{a^2}{2n} = \frac{s}{2}$, and $\Delta(x) = \prod_{i < j} (x_i - x_j)$.

From the above computation, all eigenvalue statistics can in principle be computed. In particular, the m -point correlation functions of $\mu_{n,p}$ defined

by

$$R_m(u_1, \dots, u_m) = \frac{n!}{(n-m)!} \int_{\mathbb{R}_+^{n-m}} g(u_1, \dots, u_n) \prod_{i=m+1}^n du_i$$

are given by the integral w.r.t. to $dP_n(H)$ of those of the deformed Laguerre ensemble. Let

$$R_m(u_1, \dots, u_m; y(H)) = \frac{n!}{(n-m)!} \int_{\mathbb{R}_+^{n-m}} g(u_1, \dots, u_n; y(H)) \prod_{i=m+1}^n du_i$$

be the m -point correlation function of the deformed Laguerre ensemble (defined by the fixed matrix H). Then we have the following.

PROPOSITION 5.2.

$$R_m(u_1, \dots, u_m) = \int_{M_{p,n}(\mathbb{C})} dP_n(H) R_m(u_1, \dots, u_m; y(H)).$$

In particular,

$$P\left(\lambda_{\min}\left(\frac{MM^*}{n}\right) \geq a\right) = \int_a^\infty R_1(u) du = \int_a^\infty \int_{M_{p,n}(\mathbb{C})} dP_n(H) R_1(u; y(H)) du.$$

The second remarkable fact is that the deformed Laguerre ensemble induces a determinantal random point field, that is all the m -point correlation functions are given by the determinant of a $m \times m$ matrix involving the same *correlation kernel*.

THEOREM 5.3. *Let m be a given integer. Then one has that*

$$R_m(u_1, \dots, u_m; y(H)) = \det(K_n(u_i, u_j; y(H)))_{i,j=1}^m,$$

where the correlation kernel K_n is defined by

$$\begin{aligned} K_n(u, v; y(H)) &= \frac{e^{\nu i\pi}}{i\pi s^3} \int_{\Gamma} \int_{\gamma} dw dz w z K_B\left(\frac{2z\sqrt{u}}{s}, \frac{2w\sqrt{v}}{s}\right) \left(\frac{w}{z}\right)^\nu \exp\left\{\frac{w^2 - z^2}{s}\right\} \\ &\quad \times \prod_{i=1}^n \frac{w^2 - y_i(H)}{z^2 - y_i(H)} \left(1 - s \sum_{i=1}^n \frac{y_i(H)}{(w^2 - y_i(H))(z^2 - y_i(H))}\right), \end{aligned}$$

where the contour Γ is symmetric around 0 and encircles the $\pm\sqrt{y_i(H)}$, γ is the imaginary axis oriented positively $0 \rightarrow +\infty$, $0 \rightarrow -\infty$, and K_B is the kernel defined by

$$(10) \quad K_B(x, y) = \frac{xI'_\nu(x)I_\nu(y) - yI'_\nu(y)I_\nu(x)}{x^2 - y^2}.$$

There are two important facts about this determinantal structure. The fundamental characteristic of the correlation kernel is that it depends only on the spectrum of HH^* and more precisely on its spectral measure. Since we are interested in the determinant of matrices with entries $K_n(x_i, x_j; y(H))$, we can consider the correlation kernel up to a conjugation: $K_n(x_i, x_j; y(H)) \frac{f(x_i)}{f(x_j)}$. This has no impact on correlation functions and we may use this fact later.

For ease of exposition, we drop from now on the dependency of the correlation kernel K_n on the spectrum of H and write $K_n(u, v)$ for $K_n(u, v; y(H))$. The goal of this section is to deduce Theorem 3.1 by a careful asymptotic analysis of the above formulas. Set

$$(11) \quad \alpha = \sigma^2/4,$$

with $\sigma = \sqrt{1/4 + a^2}$.

We recall that it was proved in [7] that

$$(12) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left(\lambda_{\min} \left(\frac{MM^*}{n} \right) \geq \frac{\alpha s}{n^2} \right) = \det(I - \tilde{K}_B)_{L^2(0, s)},$$

where \tilde{K}_B is the usual Bessel kernel

$$(13) \quad \tilde{K}_B(u, v) := e^{\nu i \pi} K_B(i\sqrt{u}, i\sqrt{v})$$

with K_B defined in (10). This is a universality result as the limiting distribution function is the same as that of the smallest eigenvalue of the Laguerre ensemble (see, e.g., [17]).

5.2. *Asymptotic expansion of the partition function at the hard edge.* The main result of this section is to prove the following expansion for the partition function at the hard edge: recall that α is given by (11).

THEOREM 5.4. *Assume that the distributions P_{j_k} satisfy the assumptions (4) and (2). Then there exists a nonnegative function g_n^0 , depending on n , so that*

$$\mathbb{P} \left(\lambda_{\min} \left(\frac{MM^*}{n} \right) \geq \frac{\alpha s}{n^2} \right) = g_n^0(s) + \frac{1}{n} \partial_\beta g_n^0(\beta s) \Big|_{\beta=1} \int dP_n(H) [\Delta_n(H)] + o\left(\frac{1}{n}\right),$$

where

$$\Delta_n(H) = \frac{-1}{v_c^\pm m'_{\text{MP}}(v_c^+)} X_n(v_c^+)$$

with $X_n(z) = \sum_{i=1}^n \frac{1}{y_i(H) - z} - nm_{\text{MP}}(z)$, $m_{\text{MP}}(z)$ is the Stieltjes transform of the Marchenko–Pastur distribution ρ_{MP} , $(y_i(H))_{1 \leq i \leq n}$ are the eigenvalues of H , and $v_c^+ = (w_c^\pm)^2$ where

$$(14) \quad w_c^\pm = \pm i(R - 1/R)/2, \quad R := \sqrt{1 + 4a^2}.$$

We will estimate the term $\int dP_n(H)[\Delta_n(H)]$ in terms of the kurtosis in the next section.

REMARK 5.5. The function g_n^0 is universal, in the sense that it does not depend on the detail of the distributions P_{jk} .

5.2.1. *Expansion of the correlation kernel.* Let z_c^\pm be the critical points of

$$(15) \quad E_n(w) := w^2/a^2 + \frac{1}{n} \sum_{i=1}^n \ln(w^2 - y_i(H)),$$

where the $y_i(H)$ are the eigenvalues of H^*H . Then we have the following lemma. Let K_n be the kernel defined in Theorem 5.3.

LEMMA 5.6. *There exists a smooth function A which is independent of κ_4 such that for all u, v*

$$\begin{aligned} & \frac{\alpha}{n^2} K_n(u\alpha n^{-2}, v\alpha n^{-2}; y(H)) \\ &= \tilde{K}_B(u, v) + \frac{A(u, v)}{n} + \left(\left(\frac{z_c^+}{w_c^+} \right)^2 - 1 \right) \frac{\partial}{\partial \beta} \Big|_{\beta=1} \beta \tilde{K}_B(\beta u, \beta v) + o\left(\frac{1}{n}\right), \end{aligned}$$

for H in a set with probability greater than $1 - e^{-n^{1/2}}$, and where \tilde{K}_B has been defined in (13). Note that $\frac{z_c^+}{w_c^+} = \frac{z_c^-}{w_c^-}$.

PROOF. To focus on local eigenvalue statistics at the hard edge, we consider

$$u = \left(\frac{a^2}{2nr_0} \right)^2 x; \quad v = \left(\frac{a^2}{2nr_0} \right)^2 y \quad \text{where } r_0 \text{ will be fixed later.}$$

As $\nu = p - n$ is a fixed integer independent of n , this readily implies that the Bessel kernel shall not play a role in the large exponential term of the correlation kernel. In other words, the large exponential term to be considered is E_n defined in (15). The correlation kernel can then be rewritten as

$$(16) \quad \begin{aligned} K_n(u, v) &= \frac{1}{i\pi s^3} e^{\nu i\pi} \int_{\Gamma} \int_{\gamma} dw dz w z K_B \left(\frac{zx^{1/2}}{r_0}, \frac{wy^{1/2}}{r_0} \right) \left(\frac{w}{z} \right)^\nu \\ &\quad \times \exp \{ nE_n(w) - nE_n(z) \} \tilde{G}(w, z), \end{aligned}$$

where

$$\begin{aligned} \tilde{G}(w, z) &:= a^2 G(w, z) = 1 - s \sum_{i=1}^n \frac{y_i(H)}{(w^2 - y_i(H))(z^2 - y_i(H))} \\ &= \frac{a^2 w E_n'(w) - z E_n'(z)}{2(w^2 - z^2)}. \end{aligned}$$

We note that $E_n(w) = H_n(w^2)$ where $H_n(w) = w/a^2 + \frac{1}{n} \sum_{i=1}^n \ln(w - y_i(H))$.

We may compare the exponential term E_n to its “limit,” using the convergence of the spectral measure of $H^*H = \frac{1}{n}W^*W$ to the Marchenko–Pastur distribution ρ_{MP} . Set

$$E(w) := w^2/a^2 + \int \ln(w^2 - y) d\rho_{\text{MP}}(y).$$

It was proved in [7] that this term has two conjugated critical points satisfying $E'(w) = 0$ and are given by w_c^\pm defined in (14). Let us also denote by z_c^\pm the true nonreal critical points (which can be seen to exist and be conjugate [7]) associated to E_n . These critical points do depend on n but for ease of notation we do not stress this dependence. These critical points satisfy

$$E'_n(z_c^\pm) = 0, \quad z_c^+ = -z_c^-$$

and it is not difficult to see that they are also on the imaginary axis.

We now refer to the results established in [7] to claim the following facts:

- there exists a constant C so that for any $\xi \in (0, 1)$

$$(17) \quad |z_c^\pm - w_c^\pm| \leq Cn^{-\xi}$$

with probability greater than $1 - e^{-n^{2-2\xi}}$ for n large enough. In the sequel we will take $\xi = 3/4$. This comes from concentration results for the spectral measure of H established in [22] and [2] and formula (30).

- Fix $\theta > 0$. By the saddle point analysis performed in [7], the contribution of the parts of the contours γ and Γ within $\{|w - z_c^\pm| \geq n^\theta n^{-1/2}\}$ is $O(e^{-cn^\theta})$ for some $c > 0$. This contribution “far from the critical points” is thus exponentially negligible. In the sequel, we will choose $\theta = 1/11$. The choice of $1/11$ is arbitrary.
- We can thus restrict both the w and z integrals to neighborhoods of width $n^{1/11}n^{-1/2}$ of the critical points z_c^\pm .

Also, we can assume that the parts of the contours Γ and γ that will contribute to the asymptotics are symmetric w.r.t. z_c^\pm . This comes from the fact that the initial contours exhibit this symmetry and from the location of the critical points. A plot of the oriented contours close to critical points is given in Figure 5.

Let us now make the change of variables

$$w = z_c^1 + sn^{-1/2}, \quad z = z_c^2 + tn^{-1/2},$$

where z_c^1, z_c^2 are equal to z_c^+ or z_c^- depending on the part of the contours γ and Γ under consideration and s, t satisfy $|s|, |t| \leq n^{1/11}$. Then we perform

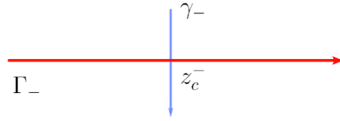
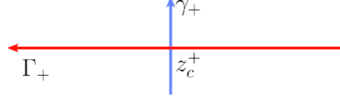


FIG. 5. Contours close to the critical points.

the Taylor expansion of each of the terms arising in both z and w integrands. Then one has that

$$\begin{aligned}
& e^{nE_n(z_c^\pm + sn^{-1/2}) - nE_n(z_c^\pm)} \\
&= e^{E_n''(z_c^\pm)s^2/2 + \sum_{i=3}^5 E_n^{(i)}(z_c^\pm)s^i/(i!n^{i/2-1})} (1 + O(n^{-23/22})) \\
(18) \quad &= e^{E_n''(z_c^\pm)s^2/2} + \underbrace{\frac{1}{n^{1/2}} e^{E_n''(z_c^\pm)s^2/2} \frac{E_n^{(3)}(z_c^\pm)}{6} s}_{e_1(s)} \\
&+ \underbrace{\frac{1}{n} e^{E_n''(z_c^\pm)s^2/2} \left(\frac{E_n^{(4)}(z_c^\pm)s^4}{4!} + \left(\frac{E_n^{(3)}(z_c^\pm)}{6} \right)^2 \frac{s^6}{2} \right)}_{e_2(s)} + o\left(\frac{1}{n}\right) e^{E_n''(z_c^\pm)s^2/2},
\end{aligned}$$

as $|s| \leq n^{1/11}$. For each term in the integrand, one has to consider the contribution of equal or opposite critical points. In the following, we denote by z_c, z_c^1, z_c^2 any of the two critical points (allowing z_c to take different values with a slight abuse of notation). We then perform the Taylor expansion of each of the functions arising in the integrands. This yields the following four expansions:

$$(19) \quad wz = z_c^1 z_c^2 + n^{-1/2} \underbrace{(sz_c^2 + tz_c^1)}_{v_1(s,t)} + \frac{1}{n} \underbrace{st}_{v_2(s,t)},$$

and

$$\begin{aligned}
& g\left(z_c^1 + \frac{s}{n^{1/2}}, z_c^2 + \frac{t}{n^{1/2}}\right) \\
(20) \quad &= \frac{E_n''(z_c)}{2} \mathbb{1}_{z_c^1=z_c^2} + \frac{1}{\sqrt{n}} \underbrace{\left(s \frac{\partial}{\partial x_1} + t \frac{\partial}{\partial x_2}\right) G(x_1, x_2)}_{g_1(s,t)} \Big|_{z_c^1, z_c^2}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{n} \underbrace{\left(\left(\frac{s^2}{2} \frac{\partial^2}{\partial x_1^2} G(x_1, x_2) + \frac{t^2}{2} \frac{\partial^2}{\partial x_2^2} + st \frac{\partial^2}{\partial x_2 \partial x_1} \right) G(x_1, x_2) \right)}_{g_2(s,t)} \Big|_{z_c^1, z_c^2} \\
& + o\left(\frac{1}{n}\right).
\end{aligned}$$

One also has

$$\begin{aligned}
\left(\frac{w}{z}\right)^\nu & = (z_c^1/z_c^2)^\nu + n^{-1/2} \underbrace{(z_c^1/z_c^2)^\nu \left(\frac{\nu s}{z_c^1} - \frac{\nu t}{z_c^2}\right)}_{r_1(s,t)} \\
(21) \quad & + \frac{1}{n} \underbrace{(z_c^1/z_c^2)^\nu \left(\frac{\nu(\nu-1)s^2}{(z_c^1)^2} + \frac{\nu(\nu+1)t^2}{(z_c^2)^2} - \frac{\nu^2 st}{z_c^1 z_c^2}\right)}_{r_2(s,t)} + o\left(\frac{1}{n}\right).
\end{aligned}$$

Last, one has that

$$\begin{aligned}
& K_B\left(\frac{zx^{1/2}}{r_0}, \frac{wy^{1/2}}{r_0}\right) \\
& = K_B\left(\frac{z_c x^{1/2}}{r_0}, \frac{z_c y^{1/2}}{r_0}\right) \\
(22) \quad & + \frac{1}{\sqrt{n}} \underbrace{\left(s \frac{\partial}{\partial x_1} + t \frac{\partial}{\partial x_2}\right) \Big|_{z_c, z_c} K_B\left(\frac{x_1 x^{1/2}}{r_0}, \frac{x_2 y^{1/2}}{r_0}\right)}_{h_1(s,t)} \\
& + \frac{1}{n} \underbrace{\left(\frac{s^2}{2} \frac{\partial^2}{\partial x_1^2} + \frac{t^2}{2} \frac{\partial^2}{\partial x_2^2} + st \frac{\partial^2}{\partial x_1 \partial x_2}\right) \Big|_{z_c, z_c} K_B\left(\frac{x_1 x^{1/2}}{r_0}, \frac{x_2 y^{1/2}}{r_0}\right)}_{h_2(s,t)} + o\left(\frac{1}{n}\right).
\end{aligned}$$

In all the lines above, $z_c^1/z_c^2 = \pm 1$ as critical points are either equal or opposite. Also one can note that the o are uniform as long as $|s|, |t| < n^{1/11}$.

We now choose

$$r_0 = |w_c^+|.$$

Combining the whole contribution of neighborhoods of a pair of equal critical points, for example, denoted by $K_n(u, v)_{\text{equal}}$, we find that it has an expansion of the form

$$\frac{a^4}{4n^2 r_0^2} K_n(u, v)_{\text{equal}}$$

$$\begin{aligned}
(23) &= \sum_{z_c = z_c^\pm} \frac{\pm}{4i\pi} e^{\nu i\pi} \int_{\mathbb{R}} \int_{i\mathbb{R}} ds dt \frac{|z_c|^2}{r_0^2} \left(K_B \left(\frac{z_c x^{1/2}}{|w_c^+|}, \frac{z_c y^{1/2}}{|w_c^+|} \right) \right. \\
&\quad \left. + \sum_{i=1}^2 \frac{h_i(s, t)}{n^{i/2}} + o\left(\frac{1}{n}\right) \right) \\
&\quad \times \left(\frac{E_n''(z_c)}{2} + \sum_{i=1}^2 \frac{g_i(s, t)}{n^{i/2}} + o\left(\frac{1}{n}\right) \right) \left(1 + \sum_{i=1}^2 \frac{r_i(s, t)}{n^{i/2}} + o\left(\frac{1}{n}\right) \right) \\
&\quad \times \left(1 + \sum_{i=1}^2 v_i(s, t) n^{-i/2} z_c^{-2} \right) \left(\exp \{E_n''(z_c)(s^2 - t^2)/2\} \left(1 + o\left(\frac{1}{n}\right) \right) \right. \\
&\quad \left. + n^{-1/2}(e_1(s) - e_1(t)) + \frac{1}{n}(-e_1(s)e_1(t) + e_2(s) - e_2(t)) \right) \Big),
\end{aligned}$$

where h_i, e_i, r_i, v_i and g_i defined above have no singularity.

It is not difficult also to see that h_1, g_1, r_1, e_1 are odd functions in s as well as in t : because of the symmetry of the contour, their contribution will thus vanish. The first nonzero lower order term in the asymptotic expansion will thus come from the combined contributions $h_1 g_1, g_1 r_1, r_1 h_1, h_1 e_1, g_1 e_1, r_1 e_1, r_1 v_1, \dots$ and those from h_2, g_2, r_2, e_2, v_2 . Therefore, one can check that one gets the expansion

$$\begin{aligned}
&\frac{\alpha}{n^2} K_n \left(\frac{\alpha x}{n^2}, \frac{\alpha y}{n^2} \right)_{\text{equal}} \\
&= \frac{e^{i\nu\pi}}{2} \left(\frac{|z_c^\pm|}{|w_c^\pm|} \right)^2 K_B \left(\frac{z_c^\pm x^{1/2}}{|w_c^+|}, \frac{z_c^\pm y^{1/2}}{|w_c^+|} \right) + \frac{a_1(z_c^\pm; x, y)}{n} + o\left(\frac{1}{n}\right),
\end{aligned}$$

where a_1 is a function of z_c^\pm, x, y only. a_1 is a smooth and nonvanishing function a priori.

We can write the first term above as $(\frac{z_c^\pm}{w_c^\pm})^2 \tilde{K}_B((\frac{z_c^\pm}{w_c^\pm})^2 x, (\frac{z_c^\pm}{w_c^\pm})^2 y)$ so that we deduce that

$$\begin{aligned}
&e^{i\nu\pi} \left(\frac{z_c^\pm}{w_c^\pm} \right)^2 K_B \left(\frac{z_c^\pm x^{1/2}}{|w_c^+|}, \frac{z_c^\pm y^{1/2}}{|w_c^+|} \right) \\
&= \tilde{K}_B(x, y) + \left(\left(\frac{z_c^\pm}{w_c^\pm} \right)^2 - 1 \right) \partial_\beta (\beta \tilde{K}_B(\beta x, \beta y)) \Big|_{\beta=1} + o(z_c^\pm - w_c^\pm).
\end{aligned}$$

One can do the same thing for the combined contribution of opposite critical points and get a similar result. We refer to [7] for more detail about this fact. Summing these terms yield a contribution of order $\frac{1}{n}$. However, it is clear that, as a_1 is smooth and using (17),

$$(24) \quad a_1(z_c^\pm; x, y) = a_1(w_c^\pm; x, y) + o(1).$$

Note that w_c^\pm does not depend on the exact distributions P_{jk} , but only on the limiting Marchenko–Pastur distribution ρ_{MP} . As a consequence, there is no fourth moment contribution in this $\frac{1}{n}$ terms. We denote the contribution of the deterministic error from all the combined (equal or not) critical points by $A(x, y)/n$. This completes the proof of the lemma. \square

5.2.2. *Asymptotic expansion of the density.* The distribution of the smallest eigenvalue of M_n is defined by

$$\mathbb{P}\left(\lambda_{\min}\left(\frac{MM^*}{n}\right) \geq \frac{\alpha s}{n^2}\right) = \int dP_n(H) \det(I - \tilde{K}_n(y(H)))_{L^2(0,s)},$$

for H in a set with overwhelming probability and where \tilde{K}_n is the rescaled correlation kernel $\frac{\alpha}{n^2}K_n(x\alpha n^{-2}, y\alpha n^{-2}; y(H))$. In the above we choose $\alpha = (a^2/2r_0)^2$. The limiting correlation kernel is then, at the first order, the Bessel kernel:

$$\tilde{K}_B(x, y) := e^{\nu i\pi} K_B(i\sqrt{x}, i\sqrt{y}).$$

Hereafter we drop the dependency in $y(H)$ to simplify the notations. The error terms are ordered according to their order of magnitude: the first-order error term, in the order of $O(n^{-1})$, can thus come from two terms, namely:

- The deterministic part that is $A(x, y)/n$, which we have seen is independent of κ_4 .
- The kernel (arising 4 times due to the combination of critical points)

$$(25) \quad \begin{aligned} & e^{\nu i\pi} \left(\frac{z_c^+}{|w_c^+|}\right)^2 K_B\left(\frac{z_c^+}{|w_c^+|}(\sqrt{x}, \sqrt{y})\right) \\ &= \tilde{K}_B(x, y) + \int_1^{(z_c^+/w_c^+)^2} \frac{\partial}{\partial \beta} \beta \tilde{K}_B(\beta x, \beta y) d\beta. \end{aligned}$$

Lemma 5.6 and the arguments above (24), (25) give the following:

$$\begin{aligned} & \frac{\alpha}{n^2} K_n(x\alpha n^{-2}, y\alpha n^{-2}) \\ &= \tilde{K}_B(x, y) + \frac{A(x, y)}{n} + ((z_c^+/w_c^+)^2 - 1) \frac{\partial}{\partial \beta} \Big|_{\beta=1} \beta \tilde{K}_B(\beta x, \beta y) + o\left(\frac{1}{n}\right). \end{aligned}$$

We insist that the kernel A is universal in the sense that it does not depend on the detail of the distributions P_{jk} .

The Fredholm determinant can be developed to obtain that

$$\begin{aligned} & \det(I - \tilde{K}_n)_{L^2(0,s)} \\ &= \sum_k \frac{(-1)^k}{k!} \int_{[0,s]^k} \det(\tilde{K}_n(x_i, x_j))_{i,j=1}^k \prod dx_i \end{aligned}$$

$$\begin{aligned}
(26) \quad &= \sum_k \frac{(-1)^k}{k!} \int_{[0,s]^k} \det(\tilde{K}_B(x_i, x_j))_{i,j=1}^k \det(I + G(x_i, x_j))_{i,j=1}^k \prod dx_i \\
&\quad + o\left(\frac{1}{n}\right),
\end{aligned}$$

where we have set

$$G(x_i, x_j) = (\tilde{K}_B(x_i, x_j))_{1 \leq i, j \leq k}^{-1} (B(x_i, x_j))_{i,j=1}^k$$

with

$$(27) \quad B(x_i, x_j) = \frac{A(x_i, x_j)}{n} + \left(\left(\frac{z_c^+}{w_c^+} \right)^2 - 1 \right) \frac{\partial}{\partial \beta} \Big|_{\beta=1} \beta \tilde{K}_B(\beta x_i, \beta x_j) + o\left(\frac{1}{n}\right).$$

The matrix $(\tilde{K}_B(x_i, x_j))_{i,j=1}^k$ is indeed invertible for any k .

Therefore, up to an error term in the order $o(\frac{1}{n})$ at most,

$$\begin{aligned}
&\det(I - \tilde{K}_n)_{L^2(0,s)} \\
&= \det(I - \tilde{K}_B) \\
&\quad + \sum_k \frac{(-1)^k}{k!} \int_{[0,s]^k} \det(\tilde{K}_B(x_i, x_j))_{i,j=1}^k \operatorname{Tr}(G(x_i, x_j))_{i,j=1}^k dx + o\left(\frac{1}{n}\right).
\end{aligned}$$

Now if we just consider the term which is linear in $((z_c^+/w_c^+)^2 - 1)$ which will bring the contribution depending on the fourth cumulant, we have that the correction is

$$\begin{aligned}
L &:= \sum_k \frac{(-1)^k}{k!} \int_{[0,s]^k} \det(\tilde{K}_B(x_i, x_j))_{i,j=1}^k \\
&\quad \times \operatorname{Tr}(\tilde{K}_B^{-1} \partial_\beta \beta \tilde{K}_B(\beta x_i, \beta x_j))_{i,j=1}^k dx \Big|_{\beta=1} \\
&= \partial_\beta \sum_k \frac{(-1)^k}{k!} \int_{[0,s]^k} \det(\tilde{K}_B(x_i, x_j))_{i,j=1}^k \\
&\quad \times \operatorname{Tr}(\log \beta \tilde{K}_B(\beta x_i, \beta x_j))_{i,j=1}^k dx \Big|_{\beta=1}.
\end{aligned}$$

As \tilde{K}_B is trace class, we can write

$$\operatorname{Tr}(\log \beta \tilde{K}_B(\beta x_i, \beta x_j))_{i,j=1}^k = \log \det(\beta \tilde{K}_B(\beta x_i, \beta x_j))_{i,j=1}^k.$$

Therefore, we have

$$L = \partial_\beta \sum_k \frac{(-1)^k}{k!} \int_{[0,s]^k} \det(\tilde{K}_B(x_i, x_j))_{i,j=1}^k$$

$$\begin{aligned}
& \times \log \det(\beta \tilde{K}_B(\beta x_i, \beta x_j))_{i,j=1}^k dx|_{\beta=1} \\
(28) \quad & = \partial_\beta \sum_k \frac{(-1)^k}{k!} \int_{[0,s]^k} \det(\beta \tilde{K}_B(\beta x_i, \beta x_j))_{i,j=1}^k dx|_{\beta=1} \\
& = \partial_\beta \sum_k \frac{(-1)^k}{k!} \int_{[0,s\beta]^k} \det(\tilde{K}_B(y_i, y_j))_{i,j=1}^k dy_i|_{\beta=1} \\
& = \partial_\beta \det(I - \tilde{K}_B)_{L^2(0,s\beta)}|_{\beta=1}.
\end{aligned}$$

Hence, since $\det(I - \tilde{K}_B)_{L^2(0,s\beta)}$ is the leading order in the expansion of $\mathbb{P}(\lambda_{\min}(\frac{MM^*}{n}) \geq \frac{\alpha s}{n^2})$ plugging (27) into (28) shows that there exists a function g_n^0 [whose leading order is $\det(I - \tilde{K}_B)_{L^2(0,s\beta)}$] so that

$$\begin{aligned}
(29) \quad & \mathbb{P}\left(\lambda_{\min}\left(\frac{MM^*}{n}\right) \geq \frac{\alpha s}{n^2}\right) \\
& = g_n^0(s) + \partial_\beta g_n^0(\beta s)|_{\beta=1} \int dP_n(H) \left[\left(\frac{z_c^+}{w_c^+}\right)^2 - 1 \right] + o\left(\frac{1}{n}\right).
\end{aligned}$$

5.2.3. *An estimate for $(\frac{z_c^+}{w_c^+})^2 - 1$.* Let

$$X_n(z) = \sum_{i=1}^n \frac{1}{y_i(H) - z} - nm_{\text{MP}}(z),$$

where $z \in \mathbb{C} \setminus \mathbb{R}_+$. Let us express $(z_c^+)^2 - (w_c^+)^2$ in terms of X_n . The critical point z_c^+ of E_n lies in a neighborhood of the critical point w_c^+ of E . So $u_c^+ = (z_c^+)^2$ is in a neighborhood of $v_c^+ = (w_c^+)^2$. These points are the solutions with nonnegative imaginary part of

$$\frac{1}{a^2} + \frac{1}{n} \sum_{i=1}^n \frac{1}{u_c^+ - y_i(H)} = 0, \quad \frac{1}{a^2} + \int \frac{1}{v_c^+ - y} d\rho_{\text{MP}}(y) = 0.$$

Therefore, it is easy to check that

$$- \int \frac{u_c^+ - v_c^+}{(v_c^+ - y)^2} d\rho_{\text{MP}}(y) + \frac{1}{n} X_n(v_c^+) = o\left(\frac{1}{n}, (z_c^+ - w_c^+)\right)$$

which gives

$$(30) \quad \left(\frac{z_c^+}{w_c^+}\right)^2 - 1 = \frac{1}{v_c^+ m'_{\text{MP}}(v_c^+)} \frac{1}{n} X_n(v_c^+) + o\left(\frac{1}{n}\right).$$

The proof of Theorem 5.4 is therefore complete. In the next section, we estimate the expectation of $X_n(v_c^+)$ to get the correction in (29).

5.3. *The role of the fourth moment.* In this section, we compute $\mathbb{E}[X_n(v_c^+)]$, which with Theorem 5.4, will allow to complete the proof of Theorem 3.1.

5.3.1. *The expected value $\mathbb{E}[X_n(v_c^+)]$.* In this section, we give the asymptotics of the mean of $X_n(z)$. Such type of estimates is now well known, and can for instance be found in Bai and Silverstein book [3] for either Wigner matrices or Wishart matrices with $\kappa_4 = 0$. We refer to [3], Theorem 9.10, for a precise statement. In the more complicated setting of F -matrices, we refer the reader to [36]. In the case where $\kappa_4 \neq 0$, the asymptotics of the mean have been computed in [29] and [5].

PROPOSITION 5.7. *Let $z \in \mathbb{C} \setminus \mathbb{R}_+ \cap \{Z \in \mathbb{C}, \Im Z \geq 0\}$. Then*

$$\lim_{n \rightarrow \infty} \mathbb{E}[X_n(z)] = A(z) - \kappa_4 B(z)$$

with A independent of κ_4 , and if $m_{\text{MP}}(z) = \int (x - z)^{-1} d\rho_{\text{MP}}(x)$,

$$(31) \quad B(z) = \frac{m_{\text{MP}}(z)^2}{(1 + (m_{\text{MP}}(z))/4)^2 (z + (zm_{\text{MP}}(z))/2)}.$$

Note that the above result follows from a simple expansion (up to the $1/N$ order) of the normalized trace of the resolvent $\frac{1}{n} \text{Tr}(\frac{WW^*}{n} - zI)^{-1}$ for a complex number z with nonzero imaginary part. We recall [27] that

$$m_{\text{MP}}(z) := \lim_{n \rightarrow \infty} \frac{1}{n} \text{Tr} \left(\frac{WW^*}{n} - zI \right)^{-1}$$

is uniquely defined as the solution with nonnegative imaginary part of the equation

$$(32) \quad \frac{1}{1 + 1/4m_{\text{MP}}(z)} = -zm_{\text{MP}}(z).$$

5.3.2. *Estimate at the critical point.* Since $m_{\text{MP}}(v_c^+) = a^{-2}$ and $v_c^+ < 0$ so that Proposition 5.7 applies, we deduce from (31) that there exists a constant $c(v_c^+)$ independent of κ_4 such that

$$\mathbb{E}[X_n(v_c^+)] = c(v_c^+) - \kappa_4 \frac{a^{-4}}{(1 + (1/4)a^{-2})^2} \frac{1}{v_c^+ (1 + (1/2)a^{-2})} + o(1).$$

Moreover, we have

$$(33) \quad v_c^+ = -\frac{a^4}{1/4 + a^2} = -\frac{4a^4}{1 + 4a^2}$$

and by (32), after taking the derivative, we find

$$m'_{\text{MP}}(z) = -\frac{m_{\text{MP}}(z)(1 + m_{\text{MP}}(z)/4)}{z(1 + m_{\text{MP}}(z)/2)},$$

so that at the critical point we get

$$(34) \quad \begin{aligned} m'_{\text{MP}}(v_c^+) &= \frac{(4a^2 + 1)^2}{16a^6(a^2 + 1/2)}, \\ v_c^+ m'_{\text{MP}}(v_c^+) &= -\frac{(1 + 4a^2)}{4(1/2 + a^2)} = -\frac{a^{-2}(1 + 1/(4a^2))}{1 + 1/(2a^2)}. \end{aligned}$$

Therefore, with the notation of Theorem 5.4, and using (33), (33) and (34), we find constants C independent of κ_4 (and which may change from line to line) so that

$$\begin{aligned} & \int dP_n(H)[\Delta_n(H)] \\ &= \frac{1}{v_c^+ m'_{\text{MP}}(v_c^+)} \mathbb{E}[n(m_n(v_c^+) - m_{\text{MP}}(v_c^+))] + o(1) \\ &= \frac{1 + 1/(2a^2)}{a^{-2}(1 + 1/(4a^2))} \kappa_4 \frac{a^{-4}}{(1 + 1/(4a^2))^2} \frac{1}{1 + 1/(2a^2)} \frac{1 + 4a^2}{4a^4} + C + o(1) \\ &= \frac{16\kappa_4}{(1 + 4a^2)^2} + C + o(1) = \frac{\kappa_4}{(1/4 + a^2)^2} + C + o(1). \end{aligned}$$

Rescale the matrix M by dividing it by σ so as to standardize the entries. Combining Theorem 5.4 and the above, we have therefore found that the deviation of the smallest eigenvalue are such that

$$\mathbb{P}\left(\lambda_{\min}\left(\frac{MM^*}{n\sigma^2}\right) \geq \frac{s}{n^2}\right) = \mathbf{g}_n(s) + \frac{\gamma}{2n} s \mathbf{g}'_n(s) + o\left(\frac{1}{n}\right),$$

where γ is the kurtosis defined in Definition (1). At this point \mathbf{g}_n is identified to be the distribution function at the hard edge of the Laguerre ensemble with variance 1, as it corresponds to the case where $\gamma = 0$. Theorem 3.1 follows.

6. The bulk of Gaussian divisible ensembles. We here choose to consider the deformed GUE instead of the deformed Laguerre ensemble. Indeed, while the arguments are completely similar, the technicalities in the deformed Laguerre ensemble are more involved. To ease the reading, we here present the simplest ensemble.

6.1. *Deformed GUE in the bulk.* Let $W = (W_{ij})_{i,j=1}^n$ be a Hermitian Wigner matrix of size n . The entries W_{ij} $1 \leq i < j \leq n$ are i.i.d. with distribution P_{ij} . The entries along the diagonal are i.i.d. real random variables with law P_{ii} independent of the off diagonal entries. We assume that $P_{ij}, P_{ii}, 1 \leq i \leq j \leq N$ have subexponential tails and satisfy (5) and (6). The fourth moment of the P_{ij} 's is also assumed not to depend on i, j . Let also V be a GUE random matrix with i.i.d. $\mathcal{N}_{\mathbb{C}}(0, 1)$ entries and consider the rescaled matrix

$$M_n = \frac{1}{\sqrt{n}}(W + aV).$$

We denote by $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ the ordered eigenvalues of M_n . By Wigner's theorem, it is well known that the spectral measure of M_n

$$\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}$$

converges weakly to the semicircle distribution with density

$$(35) \quad \sigma_{\text{sc}}^{2\sigma}(x) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} 1_{|x| \leq 2\sigma}; \quad \sigma^2 = 1/4 + a^2.$$

This is the deformed GUE ensemble studied by Johansson [24]. In this section, we study the localization of the eigenvalues λ_i with respect to the quantiles of the limiting semicircle distribution. We study the $\frac{1}{n}$ expansion of this localization, showing that it depends on the fourth moment of P_{ij} , and prove Theorem 3.3.

The route we follow is similar to that we took in the previous section for Wishart matrices: we first obtain a $\frac{1}{n}$ expansion of the correlation functions of the Deformed GUE. The dependency of this expansion in the fourth moment of P_{ij} is then derived.

6.2. *Asymptotic analysis of the correlation functions.* Let ρ_n be the one point correlation function of the Deformed GUE. We prove in this subsection the following result, with z_c^\pm, w_c^\pm critical points similar to those of the last section, which we will define precisely in the proof.

PROPOSITION 6.1. *For all $\varepsilon > 0$, uniformly on $u \in [-2\sigma + \varepsilon, 2\sigma - \varepsilon]$, we have*

$$\rho_n(u) = \sigma_{\text{sc}}^{2\sigma}(u) + \mathbb{E} \left[\left(\frac{\Im z_c^+(u)}{\Im w_c^+(u)} - 1 \right) \right] \sigma_{\text{sc}}^{2\sigma}(u) + o\left(\frac{1}{n}\right),$$

where z_c^+ depends on the eigenvalues of W .

PROOF. Denote by $y_1(\frac{W}{\sqrt{n}}) \leq y_2(\frac{W}{\sqrt{n}}) \leq \dots \leq y_n(\frac{W}{\sqrt{n}})$ the ordered eigenvalues of W/\sqrt{n} . Johansson [24], (2.20) and (2.21) (see also [11]) proves that, for a fixed W/\sqrt{n} , the eigenvalue density of M_n induces a determinantal process with correlation kernel given by

$$\begin{aligned} K_n\left(u, v; y\left(\frac{W}{\sqrt{n}}\right)\right) \\ = \frac{n}{(2i\pi)^2} \int_{\Gamma} dz \int_{\gamma} dw e^{n(E_v(w) - E_v(z))} \frac{1 - e^{((u-v)zn)/a^2}}{z(u-v)} g_n(z, w), \end{aligned}$$

where

$$E_v(z) = \frac{(z-v)^2}{2a^2} + \frac{1}{n} \sum_{i=1}^n \ln\left(z - y_i\left(\frac{W}{\sqrt{n}}\right)\right),$$

and

$$g_n(z, w) = E'_u(z) + z \frac{E'_u(z) - E'_u(w)}{z-w}.$$

The contour Γ has to encircle all the y_i 's and γ is parallel to the imaginary axis.

We now consider the asymptotics of the correlation kernel in the bulk, that is close to some point $u_0 \in (-2\sigma + \delta, 2\sigma - \delta)$ for some $\delta > 0$ (small). We recall that we can consider the correlation kernel up to conjugation: this follows from the fact that $\det(K_n(x_i, x_j; y(\frac{W}{\sqrt{n}}))) = \det(K_n(x_i, x_j; y(\frac{W}{\sqrt{n}})) \frac{h(x_i)}{h(x_j)})$, for any nonvanishing function h . We omit some details in the next asymptotic analysis as it closely follows the arguments of [24] and those of Section 5.2.

Let then u, v be points in the bulk with

$$u = u_0 + \frac{\alpha x}{n}, \quad v = u_0 + \frac{\alpha \tilde{x}}{n}; \quad u_0 = \sqrt{1 + 4a^2} \cos(\theta_0), \quad \theta_0 \in (2\varepsilon, \pi - 2\varepsilon).$$

The constant α will be fixed afterward. Then the approximate large exponential term to lead the asymptotic analysis is given by

$$\tilde{E}_v(z) = \frac{(z-v)^2}{2a^2} + \int \ln(z-y) d\sigma_{sc}^1(y).$$

In the following, we note $R_0 = \sqrt{1 + 4a^2} = 2\sigma$.

We recall the following facts from [24], Section 3. Let $u_0 = \sqrt{1 + 4a^2} \cos(\theta_0)$ be a given point in the bulk.

- The approximate critical points, that is the solutions of $\tilde{E}'_{u_0}(z) = 0$ are given by

$$w_c^{\pm}(u_0) = \left(R_0 e^{i\theta_c} \pm \frac{1}{R_0 e^{i\theta_c}} \right) / 2.$$

The true critical points satisfy $E'_{u_0}(z) = 0$. Among the solutions, we disregard the $n - 1$ real solutions which are interlaced with the eigenvalues $y_1(\frac{W}{\sqrt{n}}), \dots, y_n(\frac{W}{\sqrt{n}})$. The two remaining solutions are complex conjugate with nonzero imaginary part and we denote them by $z_c^\pm(u_0)$. Furthermore [22] and [1] prove that

$$|z_c^\pm - w_c^\pm| \leq Cn^{-\xi}$$

with probability greater than $1 - e^{-n^{2-2\xi}}$ for n large enough and any point u_0 in the bulk of the spectrum. In the sequel we will take $\xi = 3/4$.

- We now fix the contours for the saddle point analysis. The steep descent/ascent contours can be chosen as

$$\begin{aligned} \gamma &= z_c^+(v) + it, \quad t \in \mathbb{R}, \\ \Gamma &= \{z_c^\pm(r), r = R_0 \cos(\theta), \theta \in (\varepsilon, \pi - \varepsilon)\} \cup \{z_c^\pm(R_0 \cos(\varepsilon)) + x, x > 0\} \\ &\quad \cup \{z_c^\pm(-R_0 \cos(\varepsilon)) - x, x > 0\}. \end{aligned}$$

It is an easy computation [using that $\Re E''_{u_0}(w) > 0$ along γ] to check that the contribution of the contour $\gamma \cap \{|w - z_c^\pm(v)| \geq n^{1/12-1/2}\}$ is exponentially negligible. Indeed there exists a constant $c > 0$ such that

$$\left| \int_{\gamma \cap \{|w - z_c^\pm(v)| \geq n^{1/12-1/2}\}} e^{n\Re(E_{u_0}(w) - E_{u_0}(z_c^\pm(v)))} dw \right| \leq e^{-cn^{1/6}}.$$

Similarly, the contribution of the contour $\Gamma \cap \{|w - z_c^\pm(v)| \geq n^{1/12-1/2}\}$ is of order $e^{-cn^{1/6}}$ that of a neighborhood of $z_c^\pm(v)$.

For ease of notation, we now denote $z_c(v) := z_c^+(v)$. We now modify slightly the contours so as to make the contours symmetric around $z_c^\pm(v)$. To this aim, we modify the Γ contour as follows: in a neighborhood of width $n^{1/12-1/2}$ we replace Γ by a straight line through $z_c^\pm(v)$ with slope $z'_c(v)$. This slope is well defined as

$$z'_c(v) = \frac{1}{E''_v(z_c(v))} \neq 0,$$

using that $|z_c^\pm(v) - w_c^\pm(u_0)| \leq n^{-\xi}$. We refer to Figure 6, to define the new contour Γ' which is more explanatory. Denote by E the leftmost point of $\Gamma \cap \{w, |w - z_c(v)| = n^{1/12-1/2}\}$. Then there exists v_1 such that $E = z_c(v_1)$. We then define e by $e = z_c(v) + z'_c(v)(v_1 - v)$. We then draw the segment $[e, z_c(v)]$ and draw also its symmetric to the right of $z_c(v)$. Then it is an easy fact that

$$|E - e| \leq Cn^{2(1/12-1/2)} \quad \text{for some constant } C.$$

We then perform the same Taylor expansion as in Section 5.2 of all the terms in the integrands. As the contours are symmetric around $z_c(u_0)$, the first nonzero term in the expansion is in the scale of $\frac{1}{n}$. Furthermore, apart from constants, one has that

$$\begin{aligned} \frac{\alpha}{n} K_n \left(u, v; y \left(\frac{W}{\sqrt{n}} \right) \right) &= \frac{e^{(x-\tilde{x})(\alpha/a^2)\Re z_c^+}}{2i\pi(x-\tilde{x})} (e^{i(x-\tilde{x})(\alpha/a^2)\Im z_c^+} - e^{-i(x-\tilde{x})(\alpha/a^2)\Im z_c^+}) \\ &\quad + \frac{C(x, \tilde{x})}{n} + o\left(\frac{1}{n}\right). \end{aligned}$$

The function $C(x, \tilde{x})$ does not depend on the detail of the distributions of the entries of W . We now choose $\alpha = \sigma_{\text{sc}}^{2\sigma}(u_0)^{-1}$ where $\sigma_{\text{sc}}^{2\sigma}$ is the density of the semicircle distribution defined in (35). It has been proved in [24] that $\Im w_c^+(u_0) = \pi a^2 \sigma_{\text{sc}}^{2\sigma}(u_0)$. Setting then

$$\beta := \Im z_c^+(u_0) / \Im(w_c^+(u_0))$$

we then obtain that

$$\frac{\alpha}{n} K_n \left(u, v; y \left(\frac{W}{\sqrt{n}} \right) \right) e^{-(x-\tilde{x})(\alpha/a^2)\Re z_c^+} = \frac{\sin \pi \beta (x - \tilde{x})}{\pi (x - \tilde{x})} + \frac{C'(x, \tilde{x})}{n} + o\left(\frac{1}{n}\right).$$

The constant $C'(x, \tilde{x})$ does not depend on the distribution of the entries of W . This proves Proposition 6.1 since

$$\begin{aligned} \rho_n(x) &= \mathbb{E} \left[\frac{1}{n} K_n \left(u, u; y \left(\frac{W}{\sqrt{n}} \right) \right) \right] \\ &= \frac{1}{\alpha} \mathbb{E}[\beta] + \frac{C'(x, x)}{n} + o\left(\frac{1}{n}\right) \\ &= \sigma_{\text{sc}}^{2\sigma}(u_0) + \sigma_{\text{sc}}^{2\sigma}(u_0) \mathbb{E}[(\beta - 1)] + \frac{C'(x, x)}{n} + o\left(\frac{1}{n}\right). \end{aligned}$$

It can be checked, for example, in the case where W is Gaussian that $C'(x, x) = 0$ since moments expand as a series in $1/n^2$. This completes the proof of the proposition. \square

6.3. An estimate for $z_c - w_c$ and the role of the fourth moment. We follow the route developed for Wishart matrices, showing first that the fluctuations of $z_c^\pm(u_0)$ around $w_c^\pm(u_0)$ depend on the fourth moment of the entries of W .

PROPOSITION 6.2. *One has that*

$$\mathbb{E}[z_c^+(u_0) - w_c^+(u_0)] = \frac{\beta_4 m_{\text{sc}}(w_c^+(u_0))^4 / (16n)}{(a^{-2} - m'_{\text{sc}}(w_c^+(u_0)))(w_c + m_{\text{sc}}(w_c^+(u_0))/2)} + o\left(\frac{1}{n}\right).$$

As a consequence, for any $\varepsilon > 0$ uniformly on $u \in [-2\sigma + \varepsilon, 2\sigma - \varepsilon]$,

$$(36) \quad \rho_n(u_0) = \sigma_{\text{sc}}^{2\sigma}(u_0) + \kappa_4 \frac{D(u_0)}{n} + o\left(\frac{1}{n}\right),$$

where $D(u_0) = d(w_c^+(u_0))$ is given for $z \in \mathbb{C} \setminus \mathbb{R}$ by

$$(37) \quad d(z) = \frac{1}{16\pi a^2} \Im \left(\frac{m_{\text{sc}}(z)^4}{(a^{-2} - m'_{\text{sc}}(z))(z + m_{\text{sc}}(z)/2)} \right).$$

PROOF. We first relate the critical points z_c^+ and w_c^+ to the difference of the Stieltjes transforms $m_n - m_{\text{sc}}$. The true and approximate critical points satisfy the following equations:

$$\frac{z_c^+ - u_0}{a^2} - m_n(z_c^+) = 0; \quad \frac{w_c^+ - u_0}{a^2} - m_{\text{sc}}(w_c^+) = 0.$$

Hence,

$$(38) \quad z_c^+ - w_c^+ = \frac{(m_n(w_c^+) - m_{\text{sc}}(w_c^+))}{(1/a^2 - m'_{\text{sc}}(w_c^+))} + o\left(\frac{1}{n}\right),$$

where we have used that $m_n - m_{\text{sc}}$ is of order $\frac{1}{n}$. Indeed, the estimate will again rely on the estimate of the mean of the central limit theorem for Wigner matrices; see [3], Lemma 9.5. They find that for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\lim_{n \rightarrow \infty} n \mathbb{E}(m_n(z) - m_{\text{sc}}(z)) = (1 + \frac{1}{4} m'_{\text{sc}}(z)) \kappa_4 m_{\text{sc}}(z)^5 / 16.$$

Hence, we deduce that

$$\lim_{n \rightarrow \infty} n(z_c^+ - w_c^+) = \left(\frac{1}{a^2} - m_{\text{sc}}'(w_c^+) \right)^{-1} \left(1 + \frac{1}{4} m'_{\text{sc}}(w_c^+) \right) \kappa_4 m_{\text{sc}}(w_c^+)^5 / 16.$$

Using Proposition 6.1, the expansion for the one point correlation function follows as

$$\begin{aligned} \rho_n(x) &= \sigma_{\text{sc}}^{2\sigma}(u_0) + \sigma_{\text{sc}}^{2\sigma}(u_0) \frac{1}{\Im w_c^+(u_0)} \Im(\mathbb{E}[z_c^+(u_0) - w_c^+(u_0)]) + o(1/n) \\ &= \sigma_{\text{sc}}^{2\sigma}(u_0) + \frac{\sigma_{\text{sc}}^{2\sigma}(u_0)}{16 \Im w_c^+(u_0)} \Im \left(\frac{(1 + (1/4) m_{\text{sc}}'(w_c^+))}{((1/a^2) - m_{\text{sc}}'(w_c^+))} \kappa_4 m_{\text{sc}}(w_c^+)^5 \right) \\ &\quad + o(1/n) \end{aligned}$$

It is then an easy computation, using that $m_{\text{sc}}(z) + \frac{1}{z + m_{\text{sc}}(z)/4} = 0$, that this yields Proposition 6.2. \square

6.4. *The localization of eigenvalues.* We now use (36) to obtain a precise localization of eigenvalues in the bulk of the spectrum. A conjecture of Tao and Vu (more precisely Conjecture 1.7 in [32]) states that (when the variance of the entries of W is $\frac{1}{4}$), there exists a constant $c > 0$ and a function $x \mapsto C'(x)$ independent of κ_4 such that

$$(39) \quad \mathbb{E}(\lambda_i - \gamma_i) = \frac{1}{n\sigma_{\text{sc}}^{2\sigma}(\gamma_i)} \int_0^{\gamma_i} C'(x) dx + \frac{\kappa_4}{2n}(2\gamma_i^3 - \gamma_i) + O\left(\frac{1}{n^{1+c}}\right),$$

where γ_i is given by $N_{\text{sc}}(\gamma_i) = i/n$ if $N_{\text{sc}}(x) = \int_{-\infty}^x d\sigma_{\text{sc}}^{2\sigma}(u)$. We do not prove the conjecture but another version instead. More precisely, we obtain the following estimate. Fix $\delta > 0$ and an integer i such that $\delta < i/n < 1 - \delta$. Define also

$$(40) \quad N_n(x) := \frac{1}{n} \#\{i, \lambda_i \leq x\} \quad \text{with } \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n.$$

Let us define the quantile $\hat{\gamma}_i$ by

$$\hat{\gamma}_i := \inf \left\{ y, \int_{-\infty}^y \rho_n(x) dx = \frac{i}{n} \right\}.$$

By definition $\mathbb{E}N_n(\hat{\gamma}_i) = i/n$. We prove the following result.

PROPOSITION 6.3. *Let $\varepsilon > 0$ and take $i \in [\varepsilon n, (1 - \varepsilon)n]$. There exists a constant $c > 0$ such that*

$$(41) \quad \hat{\gamma}_i - \gamma_i = \frac{\kappa_4}{2n}(2\gamma_i^3 - \gamma_i) + O\left(\frac{1}{n^{1+c}}\right).$$

The main step to prove this proposition is the following.

PROPOSITION 6.4. *Let $\varepsilon > 0$. Assume that $i \in [\frac{n}{2}, (1 - \varepsilon)n]$ without loss of generality. There exists a constant $c > 0$ such that*

$$(42) \quad \begin{aligned} & \hat{\gamma}_i - \gamma_i - \hat{\gamma}_{[n/2]} + \gamma_{[n/2]} \\ &= \frac{1}{\sigma_{\text{sc}}^{2\sigma}(\gamma_i)} \int_{\gamma_{[n/2]}}^{\gamma_i} [\rho_n(x) - \sigma_{\text{sc}}^{2\sigma}(x)] dx + O\left(\frac{1}{n^{1+c}}\right). \end{aligned}$$

Note here that $\gamma_{[n/2]} = 0$ when n is even.

PROOF. The proof is divided into Lemmas 1 and 2 below. \square

LEMMA 1. *For any $\varepsilon > 0$, there exists $c > 0$ such that uniformly on $i \in [\varepsilon N, (1 - \varepsilon)N]$*

$$(43) \quad \gamma_i - \hat{\gamma}_i = \frac{1}{\sigma_{\text{sc}}^{2\sigma}(\gamma_i)} \mathbb{E}(N_n(\hat{\gamma}_i) - N_{\text{sc}}(\hat{\gamma}_i)) + O\left(\frac{1}{n^{1+c}}\right).$$

PROOF. Under assumptions of subexponential tails, it is proved in [15] and [16] (see also Remark 2.4 of [32]) that given $\eta > 0$ for n large enough

$$(44) \quad \mathbb{P}\left(\max_{\varepsilon N \leq i \leq (1-\varepsilon)n} |\gamma_i - \lambda_i| \geq n^{\eta-1}\right) \leq n^{-\log n}.$$

This implies that

$$\mathbb{E}[N_n(\gamma_i + n^{\eta-1})] \geq \frac{i}{n} + n^{-\log n}, \quad \mathbb{E}[N_n(\gamma_i - n^{\eta-1})] \leq \frac{i}{n} - n^{-\log n},$$

from which it follows that for n large enough

$$(45) \quad \max_{\varepsilon N \leq i \leq (1-\varepsilon)n} |\gamma_i - \hat{\gamma}_i| \leq 2n^{\eta-1}.$$

From the fact that $\mathbb{E}N_n(\hat{\gamma}_i) = N_{\text{sc}}(\gamma_i)$, we deduce that

$$(46) \quad \begin{aligned} \mathbb{E}N_n(\hat{\gamma}_i) - N_{\text{sc}}(\hat{\gamma}_i) &= N_{\text{sc}}(\gamma_i) - N_{\text{sc}}(\hat{\gamma}_i) \\ &= N'_{\text{sc}}(\gamma_i)(\gamma_i - \hat{\gamma}_i) - \int_{\gamma_i}^{\hat{\gamma}_i} \int_{\gamma_i}^u N''_{\text{sc}}(s) ds du. \end{aligned}$$

Using that $N'_{\text{sc}}(x) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} \mathbf{1}_{|x| \leq 2\sigma}$ and that both γ_i and $\hat{\gamma}_i$ lie within $(-2\sigma + \varepsilon, 2\sigma - \varepsilon)$ for some $0 < \varepsilon < 2\sigma$, we deduce that

$$\mathbb{E}N_n(\hat{\gamma}_i) - N_{\text{sc}}(\hat{\gamma}_i) = \sigma_{\text{sc}}^{2\sigma}(\gamma_i)(\gamma_i - \hat{\gamma}_i) + O(\gamma_i - \hat{\gamma}_i)^2.$$

We now make the following replacement. \square

LEMMA 2. *Let $\varepsilon > 0$. There exist a constant $c > 0$ such that uniformly on $i \in [\varepsilon n, (1-\varepsilon)n]$,*

$$(47) \quad \mathbb{E}(N_n(\hat{\gamma}_i) - N_{\text{sc}}(\hat{\gamma}_i)) = \mathbb{E}(N_n(\gamma_i) - N_{\text{sc}}(\gamma_i)) + O\left(\frac{1}{n^{1+c}}\right).$$

PROOF. We write that

$$(48) \quad \begin{aligned} &\mathbb{E}(N_n(\hat{\gamma}_i) - N_{\text{sc}}(\hat{\gamma}_i)) \\ &= \mathbb{E}(N_n(\gamma_i) - N_{\text{sc}}(\gamma_i)) + \mathbb{E}(N_n(\hat{\gamma}_i) - N_n(\gamma_i) - N_{\text{sc}}(\hat{\gamma}_i) + N_{\text{sc}}(\gamma_i)). \end{aligned}$$

We show that the second term in (48) is negligible with respect to n^{-1} . In fact, by (36) and (45), for $\varepsilon > 0$, there exists $\delta > 0$ such that for any $i \in [\varepsilon n, (1-\varepsilon)n]$,

$$(49) \quad \begin{aligned} |\mathbb{E}(N_n(\hat{\gamma}_i) - N_n(\gamma_i) - N_{\text{sc}}(\hat{\gamma}_i) + N_{\text{sc}}(\gamma_i))| &\leq \left| \int_{\gamma_i}^{\hat{\gamma}_i} (\rho_n(x) - \sigma_{\text{sc}}^{2\sigma}(x)) dx \right| \\ &\leq n^{\eta-1} \frac{1}{n} \leq \frac{1}{n^{2-\eta}}. \end{aligned}$$

In the last line, we have used (36). This completes the proof of Lemma 2.

Combining Lemmas 1 and 2 yields Proposition 6.4:

$$\begin{aligned}
(50) \quad \delta_n(i) &:= \sigma_{\text{sc}}^{2\sigma}(\gamma_i)(\gamma_i - \hat{\gamma}_i) - \sigma_{\text{sc}}^{2\sigma}(\gamma_{[n/2]})(\gamma_{[n/2]} - \hat{\gamma}_{[n/2]}) \\
&= \int_{\gamma_{[n/2]}}^{\gamma_i} [\rho_n(x) - \sigma_{\text{sc}}^{2\sigma}(x)] dx + O\left(\frac{1}{n^{1+c}}\right) \\
&= \frac{1}{n} \int_{\gamma_{[n/2]}}^{\gamma_i} \kappa_4 D(x) dx + O\left(\frac{1}{n^{1+c}}\right) \\
&= \frac{1}{n} \int_0^{\gamma_i} \kappa_4 D(x) dx + O\left(\frac{1}{n^{1+c}}\right),
\end{aligned}$$

where we used that $\gamma_{[n/2]}$ vanishes or is at most of order $1/n$. This formula will be the basis for identifying the role of κ_4 in the $\frac{1}{n}$ expansion of $\hat{\gamma}_i$. We now write for a point x in the bulk $(-2\sigma(1-\delta), 2\sigma(1-\delta))$ that

$$x = 2\sigma \cos \theta.$$

We also write that $\gamma_i = 2\sigma \cos \theta_0$. We then have that

$$w_c(x) = \frac{\cos \theta}{2\sigma} + \frac{a^2}{\sigma} e^{\pm i\theta}; \quad m_{\text{sc}}(w_c(x)) = \pm i\pi \sigma_{\text{sc}}^{2\sigma}(x) - \frac{1}{2\sigma^2} x.$$

By combining Proposition 6.2 and (37), we have that

$$(51) \quad D(x) = \Im \left(\frac{m_{\text{sc}}(w_c(x))^4}{16(w_c(x) + m_{\text{sc}}(w_c(x)))\pi} (1 + o(1)) \right).$$

When $a \rightarrow 0$, we then have the following estimates:

$$\begin{aligned}
x &\sim \cos \theta; & m_{\text{sc}}(w_c(x)) &\sim -2e^{-i\theta}; \\
\sigma(x) &\sim \frac{2}{\pi} \sin \theta; & w_c + m_{\text{sc}}(w_c)/2 &\sim i \sin \theta.
\end{aligned}$$

Using (50) and identifying the term depending on κ_4 in the limit $a \rightarrow 0$, we then find that

$$\begin{aligned}
(52) \quad \delta_n(i) &= \frac{1}{n} \int_0^{\gamma_i} \kappa_4 D(x) dx + O\left(\frac{1}{n^{1+c}}\right) \\
&= \frac{\kappa_4}{n} \int_{\theta_0}^{\pi/2} \frac{\cos(4\theta)}{\pi} d\theta + O\left(\frac{1}{n^{1+c}}\right) \\
&= -\frac{\kappa_4}{2n} \sigma_{\text{sc}}^{2\sigma}(\gamma_i) \cos \theta_0 (2 \cos^2 \theta_0 - 1) + O\left(\frac{1}{n^{1+c}}\right),
\end{aligned}$$

where in the last line we used that $\frac{1}{4}\sin(4\theta) = \sin\theta \cos\theta \cos(2\theta)$. Thus, we have that

$$(53) \quad \delta_n(i) = -\frac{\kappa_4}{2n}\sigma_{\text{sc}}^{2\sigma}(\gamma_i)(2\gamma_i^3 - \gamma_i) + O\left(\frac{1}{n^{1+c}}\right).$$

We finally show that

$$\lim_{n \rightarrow \infty} n(-\gamma_{[n/2]} + \hat{\gamma}_{[n/2]}) = 0$$

which completes the proof of Proposition 6.3.

To that end, let us first notice that for any C^4 function f whose support is strictly included in that of $\sigma_{\text{sc}}^{2\sigma}$, we have by [4], Theorem 1.1, that

$$(54) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left[\sum_{i=1}^n f(\lambda_i) \right] = m(f) + \kappa_4 \int_{-1}^1 f(t) T_4(t) \frac{dt}{\sqrt{1-t^2}} := m_{\kappa_4}(f),$$

with T_4 the fourth Tchebychev polynomials and $m(f)$ a linear form independent of κ_4 .

Next, we can rewrite (54) in terms of the quantiles $\hat{\gamma}_i$ as

$$\begin{aligned} m_{\kappa_4}(f) &= n \int f(x) \rho_n(x) dx + o(1) \\ &= \sum_i f(\hat{\gamma}_i) + n \sum_i f'(\hat{\gamma}_i) \int_{\hat{\gamma}_i}^{\hat{\gamma}_{i+1}} (x - \hat{\gamma}_i) \rho_n(x) dx + o(1) \\ &= \sum_i f(\hat{\gamma}_i) + \frac{1}{2} \sum_i f'(\hat{\gamma}_i) (\hat{\gamma}_{i+1} - \hat{\gamma}_i) + o(1), \end{aligned}$$

where we used that $\hat{\gamma}_{i+1} - \hat{\gamma}_i$ is of order $n^{-1+\eta}$ by (45). Now, again by (45), we have

$$\sum_i f(\hat{\gamma}_i) = \sum_i f(\gamma_i) + \sum_i f'(\gamma_i) (\hat{\gamma}_i - \gamma_i) + O\left(\frac{1}{n^{-1+2\eta}}\right).$$

Moreover, since $\gamma_{i+1} - \gamma_i$ is at most of order $1/n$, we have

$$\sum_i f(\gamma_i) = n \int f(x) \sigma_{\text{sc}}^{2\sigma}(x) dx - \frac{1}{2} \sum_i f'(\gamma_i) (\gamma_{i+1} - \gamma_i) + o(1).$$

Noting that the first term in the right-hand side vanishes we deduce that

$$m_{\kappa_4}(f) = \sum_i f'(\gamma_i) \left[\frac{1}{2} (\hat{\gamma}_{i+1} - \hat{\gamma}_i - \gamma_{i+1} + \gamma_i) + \hat{\gamma}_i - \gamma_i \right] + o(1),$$

where $\hat{\gamma}_{i+1} - \hat{\gamma}_i - \gamma_{i+1} + \gamma_i$ is at most of order n^{-1-c} by (53) since it is approximately equal to $(\delta_n(i) - \delta_n(i+1))/\sigma_{\text{sc}}^{2\sigma}(\gamma_i)$. Hence, we find by (53)

that

$$\begin{aligned}
-m_{\kappa_4}(f) &= \sum f'(\gamma_i)(\gamma_i - \hat{\gamma}_i) + o(1) \\
&= \frac{1}{n} \sum_i f'(\gamma_i) \frac{\sigma_{\text{sc}}^{2\sigma}(\gamma_{[n/2]})}{\sigma_{\text{sc}}^{2\sigma}(\gamma_i)} [n(\gamma_{[n/2]} - \hat{\gamma}_{[n/2]})] \\
&\quad + \frac{1}{n} \sum_i \frac{f'(\gamma_i)}{\sigma_{\text{sc}}^{2\sigma}(\gamma_i)} \int_0^{\gamma_i} C'(x) dx + \frac{\kappa_4}{2n} \sum_i f'(\gamma_i)(2\gamma_i^3 - \gamma_i) + o(1) \\
&= \sigma_{\text{sc}}^{2\sigma}(\gamma_{[n/2]}) [n(\gamma_{[n/2]} - \hat{\gamma}_{[n/2]})] \int_{-2\sigma}^{2\sigma} f'(x) dx \\
&\quad + \int f'(x) \int_0^x C'(y) dy dx \\
&\quad + \frac{\kappa_4}{2} \int f'(x)(2x^3 - x) \sigma_{\text{sc}}^{2\sigma}(x) dx + o(1).
\end{aligned}$$

We finally take f' even, that is f odd in which case the last term in κ_4 vanishes, as well as the term depending on κ_4 in $m_{\kappa_4}(f)$ as T_4 is even and f odd. Moreover, $\sigma_{\text{sc}}^{2\sigma}(\gamma_{[n/2]})$ goes to $1/2$. Hence, we deduce that there exists a constant independent of κ_4 such that

$$\lim_{n \rightarrow \infty} n(\gamma_{[n/2]} - \hat{\gamma}_{[n/2]}) = C.$$

In fact, this constant must vanish as in the case where the distribution is symmetric, and n even, both $\gamma_{[n/2]}$ and $\hat{\gamma}_{[n/2]}$ vanish by symmetry. \square

Acknowledgments. We gratefully acknowledge the early work with Po-Ru Lu that investigated real versions of these results. We are indebted to Bernie Wang for so very much: the careful computations in Julia, incredibly helpful conversations and his checking the consistency of various formulations. We thank Peter Forrester for helping us track down results for the smallest singular value. We also thank Folkmar Bornemann for teaching us how to use the beautiful codes in [9] for drawing some of these distributions. We thank an anonymous referee for his careful reading and helpful remarks.

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