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# Classification of gapped quantum liquid phases of matter

Xiao-Gang Wen

ABSTRACT. Quantum phases of matter correspond to patterns many-body entanglement in quantum many-body systems. This article summaries some recent results on classification of gapped quantum liquid phases of matter in low dimensions in terms of unitary braid fusion categories.

## 1. Introduction

Quantum condensed matter physics deals with local quantum systems (see Fig. 1). Similar quantum systems form a phase of matter. Quantum phases of matter are phases of matter at zero temperature. They can be divided into two classes: gapped quantum phases (see Fig. 2) and gapless quantum phases. For a long time, we believe that the gapped quantum phases of matter are described by Landau symmetry breaking theory. In 1989, it was realized that Landau symmetry breaking theory cannot describe all gapped quantum phases of matter. The new kind of gapped quantum phases were called topological orders [36, 37, 41].

At the beginning, topological order was defined by (a) the topology-dependent ground-state degeneracy [36, 41] and (b) the non-Abelian geometric phases of the degenerate ground states [14, 37], where both of them are *robust against any local perturbations* that can break any symmetry [41]. This is just like that superfluid order is defined by zero-viscosity and quantized vorticity, which are robust against any local perturbations that preserve the  $U(1)$  symmetry. Chiral spin liquids [13, 42], integral/fractional quantum Hall states [21, 33, 34],  $\mathbb{Z}_2$  spin liquids [25, 29, 38], non-Abelian fractional quantum Hall states [26, 28, 39, 43] etc, are examples of topologically ordered phases.

Microscopically, superfluid order is originated from boson or fermion-pair condensation. So it is natural for us to ask: what is the microscopic origin of topological order? What is the microscopic origin of robustness against *any* local perturbations? Recently, it was found that, microscopically, topological order is related to long-range entanglement [16, 22]. In fact, we can regard topological order as patterns of long-range entanglement [5, 32, 45] defined through generalized local unitary (LU) transformations.

In fact, the topologically ordered states are not arbitrary gapped states, but belong to a special kind of gapped quantum states, called *gapped quantum liquids* [32, 45]. 3D gapped states formed by stacking 2D fractional quantum Hall states

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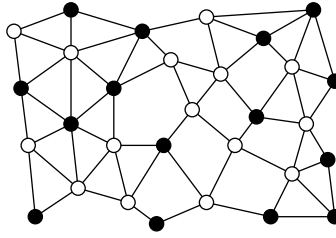


FIGURE 1. A local quantum system is given by a pair  $\mathcal{V}_N, H_N$  plus a graph  $\mathcal{G}$ . There is a finite dimensional Hilbert space  $\mathcal{V}_{\text{site}}^i$  associated with each of the  $N$  sites of the graph. The total Hilbert space  $\mathcal{V}_N$  is given by the tensor product  $\mathcal{V}_N = \otimes_{i=1}^N \mathcal{V}_{\text{site}}^i$ . The hermitian operator  $H_N$  in the total Hilbert space  $\mathcal{V}_N$  (called a Hamiltonian) have the following local form  $H_N = \sum_i H_i + \sum_{ij} H_{ij} + \sum_{ijk} H_{ijk} + \dots$ , where, for example,  $H_{ijk}$  acts on the Hilbert space  $\mathcal{V}_{\text{site}}^i \otimes \mathcal{V}_{\text{site}}^j \otimes \mathcal{V}_{\text{site}}^k$ , and the distance between the sites  $i, j, k$  are bounded by the constant.

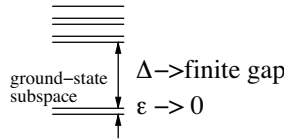


FIGURE 2. The eigenvalue spectrum of  $H_N$  in  $N \rightarrow \infty$  limit for a gapped quantum system. It is also called energy spectrum in physics.

and Haah's cubic model are examples of gapped quantum states which is not a gapped quantum liquid [10]. Those gapped non-liquid states do not have a quantum field description (with a finite number of fields).

The mathematical foundation for many-body quantum physics is infinite dimensional linear algebra with an infinite tensor product decomposition. The mathematical foundation for topological order is the braided fusion category theory. This gives us a classification of gapped quantum phases in 1+1D and 2+1D, and a partial classification of gapped quantum liquid phases in 3+1D.

## 2. Classification of gapped quantum liquids

In history, we have completely classified some large classes of matter states only for a few times. The first time is the classification of all spontaneous symmetry breaking orders, which can be classified by a pair of groups:

$$(2.1) \quad (G_\Psi \subset G_H),$$

where  $G_H$  is the symmetry group of the system and  $G_\Psi$ , a subgroup of  $G_H$ , is the symmetry group of the ground state. This includes the classification of all 230 crystal orders in 3-dimensions.

TABLE 1. A braided fusion category BFC can be viewed as a set of point-like excitations, with fusion and braiding properties. The centralizer of a BFC is the set of point-like excitations in BFC that have trivial double-braiding with all the point-like excitations in BFC. A BFC becomes a MTC if its centralizer contains only trivial point-like excitations. The following table lists the correspondences between point-like excitations and braided fusion category. Here, point-like excitations are created as the ground state subspace of Hamiltonian  $H_N$  with traps  $\delta H(\mathbf{x}_i)$  (see Fig. 3).

|  |  |
|--|--|
| Point-like excitations                                       | Braided fusion category (BFC)  |
| Ground state subspace of $H_N + \sum \delta H(\mathbf{x}_i)$ | Fusion space of BFC: $\text{Hom}(i \otimes j \otimes \dots, \mathbf{1})$ |
| Stable excitations   | Simple objects   |
| Excitations with accidental degeneracy                       | Composite objects  |
| Deformation of excitations (i.e. $\delta H(\mathbf{x}_i)$ )  | Morphism of objects  |
| Bound state of excitations                                   | Fusion of objects (Grothendieck ring)                                    |
| Statistics of excitations                                    | Braiding of objects: $i \otimes j \rightarrow j \otimes i$               |
| Mutual statistics of excitations                             | Double braiding of objects   |
| Local excitations  | Transparent objects (trivial double braiding with all objects)           |
| Topological excitations                                      | Non-transparent objects (non-trivial double braiding with some objects)  |
| Internal degrees of freedom                                  | Quantum dimension $d_i$  |
| Angular momentum of an excitation                            | Topological spin $s_i$   |

The second time is the classification of all gapped 1-dimensional quantum states:

RESULT 2.1. *Gapped 1-dimensional quantum states with on-site symmetry  $G_H$  can be classified by a triple: [6, 31]*

$$(2.2) \quad [G_\Psi \subset G_H; \text{pRep}(G_\Psi)],$$

where  $\text{pRep}(G_\Psi)$  is a projective representation of  $G_\Psi$  [27].

The third time is the classification of all gapped quantum phases in 2+1D. Since early on, it was conjectured that

RESULT 2.2. *All 2+1D bosonic topological orders (without symmetry) are classified by  $S, T$  modular matrices (plus other gauge connections) [37], or more precisely by a pair [15, 30, 35]:*

$$(2.3) \quad (\text{MTC}, c),$$

where MTC is a unitary modular tensor category (see Table 1) and  $c = c_{\text{MTC}} \bmod 8$  is the chiral central charge  $c$  of the edge states.

Here,  $c_{MTC}$  is the central charge of the MTC given by  $e^{i2\pi c_{MTC}/8} = \frac{\sum_i e^{2\pi i s_i d_i^2}}{\sqrt{\sum_i d_i^2}}$ , and  $d_i, s_i$  are quantum dimensions and topological spins in MTC (see Table 1). We like to pointed out that the above classification include the invertible topological orders [8, 17], which described by the central charge  $c$ .

For bosonic quantum systems with symmetry, we have the following result:

RESULT 2.3. *All 2+1D gapped bosonic phases with a finite unitary on-site symmetry  $G_H$ , are classified by [19]*

$$(2.4) \quad [G_\Psi \subset G_H; \text{Rep}(G_\Psi) \subset \text{BFC} \subset \text{MTC}; c].$$

where  $c = c_{MTC} \bmod 8$ .

Here  $\text{Rep}(G_\Psi)$  is the SFC formed by the representations of  $G_\Psi$  where all representations are assigned Bose statistics, In addition,  $\text{Rep}(G_\Psi)$  is the centralizer of BFC. Also MTC is a modular tensor category that contains BFC such that  $\text{Rep}(G_\Psi)$  is the centralizer of BFC in MTC, i.e.  $\text{Rep}(G_\Psi)$  is formed by the point-like excitations in MTC that that have trivial double braiding with all the point-like excitations in BFC. Such a MTC is called a minimal modular extension of the pair  $\text{Rep}(G_\Psi) \subset \text{BFC}$ .

Such a result is equivalent to an earlier result that all 2+1D gapped bosonic phases with a finite unitary on-site symmetry  $G_H$ , are classified by [1]

$$(2.5) \quad [G_\Psi \subset G_H; G_\Psi\text{-crossed MTC}; c],$$

The above classification includes symmetry breaking orders, symmetry protected topological (SPT) orders [4, 7, 9], topological orders, and *symmetry-enriched topological orders* (SET) described by *projective symmetry group* [40]. SET orders of time-reversal/reflection symmetry are classified by Ref. [2]. Some more discussions on SET orders can be found in [3, 11, 12, 23, 24, 44].

The above results for bosonic systems can be generalized to fermion systems:

RESULT 2.4. *2+1D fermionic topological orders are classified by a triple:[18]*

$$(2.6) \quad [s\text{Rep}(Z_2^f) \subset \text{BFC}; c],$$

where  $s\text{Rep}(Z_2^f)$  is the symmetric fusion category (SFC) formed by the representations of the fermion-number-parity symmetry  $Z_2^f$  where the non-trivial representation is assigned Fermi statistics, and BFC is a unitary braided fusion category whose centralizer is  $s\text{Rep}(Z_2^f)$ .

Here,  $c = c_{MTC} \bmod \frac{1}{2}$ , and  $c_{MTC}$  is the central charge of a modular extension of the BFC.

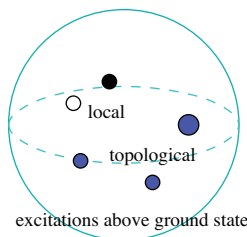


FIGURE 3. Point-like excitations at  $\mathbf{x}_i$  correspond to the punctures in the space created the trap Hamiltonians  $\delta H(\mathbf{x}_i)$  which is non-zero only near  $\mathbf{x}_i$ .

We also have a similar result for fermion systems:

RESULT 2.5. All 2+1D gapped fermionic phases with unitary finite on-site symmetry  $G_H^f$  are classified by [19]

$$(2.7) \quad [G_\Psi^f \subset G_H^f; \text{sRep}(G_\Psi^f) \subset \text{BFC} \subset \text{MTC}; c],$$

where  $\text{sRep}(G_\Psi^f)$  is the SFC formed by the representations of  $G_\Psi^f$  where some representations are assigned Fermi statistics, and  $c = c_{\text{MTC}} \bmod 8$ . Also  $\text{sRep}(G_\Psi^f)$  is the centralizer of BFC in MTC.

In 3+1D, we only have a partial result [20]:

RESULT 2.6. Consider 3+1D bosonic gapped quantum liquid phases without symmetry and with all the point-like excitations to be bosons. All such phases are classified by unitary pointed fusion 2-categories, which are one-to-one labeled by a finite group  $G$  and its group 4-cocycle  $\omega_4 \in \mathcal{H}^4[G; U(1)]$  up to group automorphisms. Furthermore, all such 3+1D topological orders can be realized by Dijkgraaf-Witten gauge theories.

The above classification results come from the realization that the point-like excitations in 2+1D topological orders are described by unitary braided fusion categories (see Table 1). Therefore, a classification of unitary braided fusion categories gives rise to a classification of possible point-like excitations, which in turn leads to a classification of 2+1D topological orders.

### 3. Summary

We have seen that to understand the symmetry breaking states, physicists need to learn group theory. From the above results, it appears that to understand patterns of many-body entanglement that correspond to topological order and SPT order, physicists will have to learn tensor category theory and group cohomology theory. In modern quantum many-body physics and in modern condensed matter physics, tensor category theory and group cohomology theory will be as useful as group theory. The days when physicists need to learn tensor category theory and group cohomology theory are coming, may be soon.

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