Stability and distributivity over orbital ∞-categories

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ABSTRACT
Let G be a finite group. The homotopy theory of topological spaces with an action of G has provided important applications in many parts of homotopy theory and geometry. An especially important role has been played by the so-called "norm maps". In this thesis we develop a characterization of the ∞-category of G-spectra and of its multiplicative structure in term of the behaviour with respect to equivariant colimits. This will allow us to give an alternative construction of the norm map.

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Introduction

Let G be a finite group. The homotopy theory of topological spaces with an action of G (where both maps and homotopies are equivariant) has provided important applications in many parts of homotopy theory and geometry. One important example is the seminal paper [22] by Hill, Hopkins and Ravenel, where they solve the Kervaire invariant one problem by making essential use of the properties of G-spectra (namely the norm functor and the slice spectral sequence) to show that some elements in the classical Adams spectral sequence are not permanent cycles. This shows that even if one is only interested in nonequivariant results, the proof might require to briefly step into the equivariant world. Doubtless, as time progresses many more examples will appear.

Another application of equivariant homotopy theory to "classical" phenomena is in the study of algebraic K-theory. In fact, if R is a commutative ring spectrum (for example the Eilenberg-MacLane spectrum of a commutative ring) we can construct another commutative ring spectrum $THH(R)$ receiving a trace map from $K(R)$. But $THH(R)$ has a much richer structure of $K(R)$: it is indeed a *cyclotomic spectrum*, in particular a S^1-spectrum[1], and this additional structure allow us to get a more refined trace map $K(R) \to TC(R)$ which is close to be an equivalence.

In the paper [1] it is shown that this important cyclotomic stucture can be easily recovered from a different description of $THH(R)$: it is simply $N^{S^1}(R)$ where N^{S^1} is the norm functor, which was also one the main ingredients in [22]. It is therefore obvious that these "parametrized multiplications" are important objects, with applications in different parts of homotopy theory.

It is thus disappointing that there is not any framework to construct and develop the theory of norm functors and G-commutative rings similarly to how symmetric monoidal model categories or symmetric monoidal ∞-categories can be used to study commutative ring spectra. This is the reason why C. Barwick, S. Glasman, E. Dotto, J. Shah and the author of this thesis decided to start a project developing just such a framework ([7]). In doing so it became increasingly clear that the same theorems, with the same proofs, show up not only in the case of equivariant homotopy theory, but in many other examples, among which the parametrized homotopy theory of May and Sigurdsson ([28])

[1] Here by S^1-spectrum we mean relative to the orbit category of S^1-orbits with finite stabilizers.

and global homotopy theory ([30]).

The common generalization of all these example is an **atomic orbital ∞-category** (definitions 1.2 and 2.7). In the case of a finite group G the corresponding category is $\mathbf{O}_G$, the category of transitive G-sets (hence the name), but there are many other ∞-categories that share just enough formal properties with it to allow us to develop a fully fledged "parametrized homotopy theory".

An important non-example of a homotopy theory not covered by orbital ∞-categories is the G-equivariant homotopy theory with respect to a complete G-spectrum, when G is a compact Lie group. While this homotopy theory has many features in common with the case of a finite group, there are other differences that make our approach not working there. For example, the fact that orbits are not self-dual anymore creates unexpected problems when one wants to define G-commutative monoids.

The basic theory of orbital ∞-categories can be found in [5], [32] and [9]. For the convenience of the reader we have summarized the most important results in chapter 1, without proof.

When working with atomic orbital ∞-categories, modelling parametrized spectra via Mackey functors becomes extremely convenient. This model has been first suggested in [20] and then fully developed in [14] and [12], but there is at the best of the author's knowledge no reference when the elementary results for G-spectra have been completely worked out in that context. Since we will need some of them, we have decided to include in appendix A the development of stable equivariant homotopy theory using Mackey functors. This has the double advantage of giving us some place to refer to and of reassuring the perhaps uncertain reader that spectral Mackey functor are indeed as good as orthogonal G-spectra in modelling G-spectra. We hope that, while no results therein are original, appendix A can be a useful addition to the literature.

Stability

It is often said that spectra are the same as homology theories. This is strictly speaking wrong when homology theories are interpreted as valued in graded abelian groups, due to the presence of phantom maps. Luckily, Goodwillie calculus provides us with an equivalence between spectra and linear functors from finite pointed spaces to spaces (that is space-valued homology theories). This allows us to state a universal property for the category of spectra: it is the universal source of a linear functor to spaces ([25, Pr. 1.4.2.22]).

One would imagine that a similar statement should be true for G-spectra, where G is a finite group. The category of G-spectra is not, however, the universal source of linear functors to G-spaces (that would be spectral presheaves over the orbit category of G). It has been an important insight in the solution to the Kervaire invariant one problem by Hill, Hopkins, and Ravenel ([22]) that in G-spectra one should ask for a stronger form of additivity: they should not only turn coproducts into products, but also coproducts indexed by a finite G-set into the corresponding product. This is merely a form of Atiyah duality for finite G-sets, but a highly suggestive one.

In order to speak of indexed products and coproducts it is necessary to be able to remember the notion of objects with an H-action for every subgroup H of G. So we need to move from the notion of ∞-category to the notion of G-∞-category, which is a presheaf of categories over $\mathbf{O}_G$, the orbit category of G. This sends G/H to the ∞-category of objects corresponding to the subgroup H (e.g. H-spaces, H-spectra etc.) and encodes all the functoriality of restriction to subgroups (corresponding to the map $G/H \to G/K$ for $H \subseteq K$) and of twisting the action by conjugation (corresponding to the isomorphism of G/H with G/gHg^{-1} in $\mathbf{O}_G$). More generally, for any orbital ∞-category T one can define a T-∞-category as a family of ∞-categories parametrized by the elements of T. The general theory of (co)limits indexed by a T-∞-category has been developed in [32]. The necessary results will be found in section 1.3.

Once the notion of T-(co)limit has been set up, one can try to mimic the whole theory of additive and stable ∞-categories in this equivariant setting. This works nicely and provides us with a universal property for the T-∞-category of G-spectra: it is the universal recipient of a T-linear functor from the T-∞-category of finite T-spaces (cf. theorem 2.36).

Theorem. *For any T-∞-category with finite T-colimits C the T-functor Ω^∞ : $\underline{\mathrm{Sp}}^T \to \underline{\mathrm{Top}}_T$ induces an equivalence*

$$\mathrm{Fun}_T^{T-\mathrm{rex}}(C, \underline{\mathrm{Sp}}^T) \to \mathrm{Lin}_T(C, \underline{\mathrm{Top}}_T)$$

between the category of T-functors $C \to \underline{\mathrm{Sp}}^T$ preserving finite T-colimits and the category of T-linear T-functors $C \to \underline{\mathrm{Top}}_T$.

This universal property is going to be one of the two crucial ingredients for our construction of the norm, and the main motivation for this chapter.

Another important result in the same spirit is the identification of connective spectra with group-like commutative monoids in spaces, as done in [31]. This too has an equivariant analogue (cf. corollary 2.41). In fact it turns out that G-commutative monoids are the same thing as product-preserving functors from the effective Burnside category of [14]. This explains the ubiquity of Mackey functors in equivariant homotopy theory and allows us to give an alternative proof of [20, Th. 0.1], identifying orthogonal G-spectra with spectral Mackey functors (see section 2.6).

Two important predecessors of this chapter are [15] and [17]. In the first a description of G-spectra as enriched functors from G-spaces to G-spaces is provided for a general compact Lie group, while the second contains a characterization of G-spectra as functors in term of an excisivity condition for a finite group G. While the approach taken here is different, the intuition behind it is very similar.

The results of this chapter can originally be found in [29].

Monoidality

We have seen that an important tool to describe T-stability is the definition of coproducts and products indexed by a finite T-set. It is natural, therefore, to try to define tensor products indexed by a finite T-set, that a "norm functor" $N^e : C_V \to C_{V'}$ for each edge $e : V \to V'$ in T. This has been done, in the case of $T = \mathbf{O}_G$ in [22], but the proof that the norm functors constructed there have homotopical significance is very complicated and, in a certain sense, unnatural.[2]

Norm functors are also inextricably linked to the notion of T-commutative algebra, which is the spectral analogue of the notion of Tambara functor (see [33]). Many commutative rings in G-spectra are naturally endowed with norm maps $N^G R \to R$ that represent the ability to "multiply a finite G-set of elements". The most famous model for G-commutative rings are the strictly commuative rings in orthogonal G-spectra.

As for the norm, constructing a model for the category of G-commutative ring spectra is rife with technical subtleties. For this reason we decided to develop a notion of T-symmetric monoidal T-∞-categories, which encodes the intuition of a family of symmetric monoidal ∞-categories parametrized by the objects of T, equipped with norm functors for each edge of T. We believe that our theory in the case $T = \mathbf{O}_G$ is fundamentally equivalent to the theory of N_∞-operads of Blumberg and Hill ([16]) and the theory of G-symmetric monoidal categories of Hill and Hopkins ([21]), but we have no comparison result yet.

This chapter is dedicated to the presentation of this theory, allowing to recover all important properties of norm functors and T-commutative rings from the combinatorics of finite T-sets. In particular, we prove the following result (corollary 3.28):

Theorem. *The moduli space of presentable T-symmetric monoidal structure on the T-∞-category of T-spectra where the sphere spectrum is the unit is contractible.*

This induces a norm functor that will automatically satisfy the properties required to make the proof of [22] work (in particular it is a symmetric monoidal and it is the left adjoint of the forgetful functor from G-commutative rings to H-commutative rings). To prove the above theorem we will use an approach similar to the one used in [25] to construct the symmetric monoidal category on the ∞-category of spectra: we categorify the problem by putting a T-symmetric monoidal structure on the T-∞-category of presentable T-∞-categories and we study the T-commutative algebra structures on $\underline{\mathrm{Sp}}^T$.

One would be let to conjecture, then, that the T-∞-category of T-spectra could be constructed simply by norming up the ∞-category of spectra. This is known to be true when $T = \mathbf{O}_G$ (where it is equivalent to the statement that G-spectra are obtained from G-spaces by inverting the regular representation). It is the belief of the author that the statement is true in full generality, thus

[2]An useful insight told us by H. Miller is that proposition B.105 in [22] can be seen as a strictification theorem, telling us that the "indexed point-set smash product" coincides, under favorable conditions, with the "homotopically-correct indexed smash product".

describing yet another sense in which the T-∞-category of T-spectra are canonically obtained from the ∞-category of spectra, but we do not know a proof at the present moment.

The generality in which these arguments have been developed here might seem offputting, but have the advantage of allowing a wide range of applications. For example in [3] these results allow to apply the methods of [4] to construct a T-commutative ring structure on the K-theory of a Waldhausen T-∞-category (which is, roughly speaking, a family of Waldhausen ∞-categories parametrized by the objects of T). For example, when T is the category of finite separable field extensions of a field F, this allows equip $K(F)$ with a $Gal(F)$-commutative structure.

The results of this chapter can be originally found in [6] and [13].

Table of Notation

$\mathrm{Map}_C(x,y)$	Mapping space between two objects x, y of an ∞-category C
$\mathrm{MapSp}_C(x,y)$	Mapping spectrum between two objects x, y of a stable ∞-category C
$\mathrm{Fun}_T(C,D)$	∞-category of T-functors between C and D
$\underline{\mathrm{Fun}}_T(C,D)$	T-∞-category of T-functors between C and D
$\underline{C}_T$	T-∞-category of T-objects in an ∞-category C
res_e, res^W_V	pushforward functor in a T-∞-category along a map $V \to W$ in T
$\coprod_e$, $\coprod_{W/V}$	left adjoint of res_e
$\prod_e$, $\prod_{W/V}$	right adjoint of res_e
$\underline{\mathrm{Sp}}_T(C)$	Fiberwise stabilization of a T-∞-category C
$\underline{\mathrm{Sp}}^T(C)$	T-stabilization of a T-∞-category C
$\underline{\mathrm{Fun}}_T^{T-\mathrm{lex}}(C,D)$:	subcategory of T-functors preserving finite T-colimits;
$\underline{\mathrm{Fun}}_T^{fb-\mathrm{lex}}(C,D)$:	subcategory of T-functors preserving finite fiberwise colimits;
$\underline{\mathrm{Fun}}_T^{\amalg}(C,D)$:	subcategory of T-functors preserving finite T-coproducts;
$\underline{\mathrm{Fun}}_T^{T-\mathrm{rex}}(C,D)$:	subcategory of T-functors preserving finite T-limits;
$\underline{\mathrm{Fun}}_T^{fb-\mathrm{rex}}(C,D)$:	subcategory of T-functors preserving finite fiberwise limits;
$\underline{\mathrm{Fun}}_T^{\times}(C,D)$:	subcategory of T-functors preserving finite T-products;
$\underline{\mathrm{Fun}}_T^{L}(C,D)$:	subcategory of T-functors preserving all T-colimits;
$\underline{\mathrm{Fun}}_T^{R}(C,D)$:	subcategory of T-functors preserving all T-limits.
$P_T(C)$	T-∞-category of presheaves over a T-∞-category C
$L_S C$	Bousfield localization of a presentable T-∞-category C at a small subcategory S
$\underline{\mathrm{Sp}}_T(C)$	fiberwise stabilization of a T-∞-category C
$\underline{\mathrm{CMon}}_T(C)$	T-∞-category of T-commutative monoids in a T-∞-category C
$\underline{\mathbf{A}}^{\mathrm{eff}}(T)$	Burnside T-∞-category
$\mathrm{Fun}_T^{\delta}\left(\prod_f C, D\right)$	∞-category of distributive functors from $\prod_f C$ to D

Chapter 1

The yoga of orbital ∞-categories

In this chapter we will rapidly survey the basic theory of orbital ∞-categories, as developed in [5], [32] and [9]. Many of these results are directly inspired by analogues in equivariant homotopy theory, but the ability of treating very different cases with a unified formalism will be useful. For example the definition of distributive functor (Df. 3.15 is greatly simplified by using the same language for talking about T-spaces and local systems of T-spaces.

1.1 Elmendorf theorem and orbital ∞-categories

The story begins with the following theorem by Elmendorf

Theorem 1.1. *Let G be a finite group*[1] *and Top_G be the ∞-category associated to the simplicial model category of genuine G-spaces. Then the inclusion of the orbits $\mathbf{O}_G \to \mathrm{Top}_G$ induces an equivalence*

$$\mathrm{Top}_G \xrightarrow{\sim} \mathrm{Fun}(\mathbf{O}_G^{\mathrm{op}}, \mathrm{Top})\,.$$

So we can identify G-spaces with presheaves over the orbit category of G. Starting from this, it is possible to develop stable equivariant homotopy theory using only the combinatorial properties of $\mathbf{O}_G$. An important result in this direction is the Guillou-May theorem ([20, Th. 0.1]), stating that the ∞-category of G-spectra (modelled e.g. by orthogonal G-spectra) is equivalent to the category of spectral Mackey functors for G.

From this one could expect that the theory of G-spectra can be applied to any category with properties similar enough to those of $\mathbf{O}_G$. This is indeed the case.

[1] It is worth noting that the theorem is also true in the more general case of a compact Lie group, although we are never going to use this fact.

Definition 1.2. *Let T be a small ∞-category and let $\mathbf{F}_T$ denote its finite coproduct completion, that is the subcategory of* $\mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Top})$ *spanned by finite coproducts of representables. We will sometimes call $\mathbf{F}_T$ the ∞-category of finite T-sets, since when $T = \mathbf{O}_G$ $\mathbf{F}_T$ is equivalent to the category $\mathbf{F}_G$ of finite G-sets. We say that T is* ***orbital*** *if $\mathbf{F}_T$ has all pullbacks and* ***cartesian orbital*** *if $\mathbf{F}_T$ has all finite limits.*

There are many examples of orbital ∞-categories, including some categories playing important roles in homotopy theory.

Example 1.3.

- *If G is a profinite group, the category of transitive finite G-sets is cartesian orbital;*
- *If G is a Lie group, the ∞-category of the transitive G-spaces with finite stabilizers is orbital (but it is cartesian if and only if G is a finite group).*
- *As a special case of the previous example when $G = S^1$, the cyclonic ∞-category of [10] is orbital, but not cartesian.*
- *Let X be a space, seen as an ∞-groupoid. Then X is an orbital ∞-category (but it is not cartesian orbital if X is not contractible). We will see that parametrized homotopy theory with respect to X is exactly parametrized homotopy theory in the sense of [28].*
- *Combining two previous examples, let $X : \mathbf{O}_G^{\mathrm{op}} \to \mathrm{Top}$ be a G-space, where G is a finite group. Then the ∞-category of points $\underline{X}$ (that is the total category of the right fibration classified by X) is orbital. More generally, if $U \to T$ is a right fibration where T is an orbital ∞-category, then U is also an orbital ∞-category.*
- *For a more exotic example, the category $\mathbf{F}_s$ of finite sets and surjections is cartesian orbital. The main result of [19] shows that parametrized homotopy theory with respect to $\mathbf{F}_s$ is strictly interrelated to the homotopy theory of Taylor towers of functors from spectra.*

Inspired by Elmendorf's theorem we define a T-space as a presheaf of spaces over T. We will denote the ∞-category of T-spaces with Top_T. But we will go a step further. We say that a **T-∞-category** is a cocartesian fibration over T^{op} $p : C \to T^{\mathrm{op}}$. That is, for every $t \in T$ we have a category C_t and for every edge $e : t \to t'$ in T we have a restriction functor $\mathrm{res}_e : C_{t'} \to C_t$. When there is no possible ambiguity, I will denote the restriction functor by $\mathrm{res}^t_{t'}$.

Example 1.4. *Let G be a finite group. Then a $\mathbf{O}_G$-∞-category (or G-∞-category for short) is the datum of*

- *For each subgroup $H < G$ a category C_H;*
- *For each inclusion of subgroups $H' < H$, a restriction functor $\mathrm{res}^H_{H'} : C_H \to C_{H'}$;*

- *For each $g \in G$ and each subgroup $H < G$ a conjugation map $c_g : C_H \to C_{gHg^{-1}}$.*

And $\mathrm{res}_H^{H'}$ *and c_g are required to satisfy the relations that come from identifying them as arrows in the orbit category. For example the presheaf*

$$G/H \mapsto \mathrm{Top}_H$$

*form a G-∞-category that we will call the G-∞-category of **G-spaces** and that we will denote $\underline{\mathrm{Top}}_G$. Similarly for pointed G-spaces, G-sets and G-spectra, that we will denote by $\underline{\mathrm{Top}}_{G,*}$, $\underline{\mathbf{F}}_G$ and $\underline{\mathrm{Sp}}^G$ respectively.*

In fact, as we will see, there are analogues of all the categories in the example for any orbital ∞-category T.

Example 1.5. *Let $X : T^{\mathrm{op}} \to \mathrm{Top}$ be a T-space. Then the associated left fibration is a T-∞-category, that we will call the **category of points** of X. We will often denote it with a bold version of the name of the space (in this case $\mathbf{X}$). It will used mainly when X is a finite T-set (that is a T-space which is a finite coproduct of orbits).*

If C is a T-∞-category and $e : V \to V'$ an arrow in T, we will denote the pushforward functor $e_! : C_{V'} \to C_V$ by res_e or $\mathrm{res}_V^{V'}$ when there is no ambiguity.

1.2 Constructions on T-∞-categories

If C, D are two T-∞-categories, we will say that a T-functor is a map $C \to D$ of cocartesian fibrations, that is a map of simplicial sets over T^{op} sending cocartesian arrows to cocartesian arrows. The ∞-category of T-functors will be denoted by $\mathrm{Fun}_T(C, D)$. Moreover the ∞-category $\mathrm{Cat}_{\infty,T}$ has finite products, given by the fiber product over T^{op}.

Theorem 1.6. *The ∞-category of T-∞-categories is cartesian closed, that is for any pair of T-∞-categories C, D there is a T-∞-category $\underline{\mathrm{Fun}}_T(C, D)$ such that*

$$\mathrm{Fun}_T(E, \underline{\mathrm{Fun}}_T(C, D)) \cong \mathrm{Fun}_T(E \times_{T^{\mathrm{op}}} C, D)$$

Many other construction from ∞-category theory can be carried on in this more general setting. We will need only three more of them: the fiberwise opposite category, the relative join and the relative slice category.

Definition 1.7. *Let C be a T-∞-category. Then the fiberwise opposite is the cocartesian fibration $C^{\vee\mathrm{op}} \to T^{\mathrm{op}}$ obtained by applying the construction of [11] to the cartesian fibration $C^{\mathrm{op}} \to T$.*

Definition 1.8. *Let C, D two simplicial sets over T^{op}. Then the **relative join** $C \star_T D$ is the simplicial set over $\Delta^1 \times T^{\mathrm{op}}$ such that*

$$\mathrm{Hom}_{\mathrm{sSet}/\Delta^1 \times T^{\mathrm{op}}}(K, C \star_T D) = \mathrm{Hom}_{\mathrm{sSet}/\partial\Delta^1 \times T^{\mathrm{op}}}(K \times_{\Delta^1 \times T^{\mathrm{op}}} \partial\Delta^1 \times T^{\mathrm{op}}, C \amalg D).$$

If C, D are T-∞-categories, so is $C \star_T D$.

Definition 1.9. *Let C be a T-∞-category and $c : T^{\mathrm{op}} \to C$ be a cocartesian section (that is a T-functor from T^{op} to C). Then the slice T-∞-category $C_{/_T c}$ is the simplicial set over T^{op} such that*

$$\mathrm{Hom}_{\mathrm{sSet}/T^{\mathrm{op}}}(K, C_{/_T c}) = \mathrm{Hom}_{\mathrm{sSet}/T^{\mathrm{op}}}(K \star_T T^{\mathrm{op}}, C) .$$

and similarly for $C_{c/_T}$. Both $C_{c/_T}$ and $C_{/_T c}$ are T-∞-categories.

Definition 1.10. *Let C be an ∞-category. Then the T-∞-category of T-objects is the cocartesian fibration $p : \underline{C}_T \to T^{\mathrm{op}}$ defined by*

$$\mathrm{Hom}_{\mathrm{sSet}/T^{\mathrm{op}}}(K, \underline{C}_T) = \mathrm{Hom}(K \times_{T^{\mathrm{op}}} \mathrm{Fun}(\Delta^1, T^{\mathrm{op}}), C) .$$

It is the T-∞-category that sends $t \in T$ to the ∞-category $\mathrm{Fun}(T_{/t}, C)$. In particular the T-∞-category of small T-∞-categories is just $\underline{\mathrm{Cat}}_{\infty,T}$.

Example 1.11. *Since taking the fiber over eH induces an equivalence $(\mathbf{O}_G)_{/G/H} \cong \mathbf{O}_H$, we see that both definition we gave of $\underline{\mathrm{Top}}_G$ coincide.*

The T-∞-category of T-objects has a very convenient universal property:

Theorem 1.12. *Let C be a T-∞-category and D be an ∞-category. Then there is a canonical equivalence*

$$\mathrm{Fun}_T(C, \underline{D}_T) \cong \mathrm{Fun}(C, D) .$$

In particular by the straightening-unstraightening theorem we have that the ∞-category of T-functors from C to $\underline{\mathrm{Cat}}_{\infty,T}$ is equivalent to the ∞-category of cocartesian fibrations over C.

Proof. See theorem 7.8 in [5]. $\square$

1.3 Relative limits and colimits

Definition 1.13. *A T-adjunction between two T-∞-categories C and D is an adjunction $FG : C \leftrightarrows D$ between the two total categories such that F and G are T-functors (that is they send cocartesian arrows to cocartesian arrows) and unit and counit lie above the identity natural transformation of the identity functor on T. This is the same thing as a relative adjunction in the sense of [25, Sec. 7.3.2] such that both functors are T-functors. Note that the left adjoint in a relative adjunction is is automatically a T-functor, but this is not true for the right adjoint.*

Definition 1.14. *Let C be a full T-subcategory of D. If the inclusion $C \subseteq D$ has a left T-adjoint, we say that C is a* ***reflective subcategory*** *of D. The left T-adjoint $L_C : D \to C$ is called the localization onto C and arrows that are sent to equivalences by L_C are called C-equivalences or L_C-equivalences.*

Definition 1.15. *Precomposition with the structure map $C \to T^{\mathrm{op}}$ induces a diagonal T-functor*

$$\Delta : D \cong \underline{\mathrm{Fun}}_T(T, D) \to \underline{\mathrm{Fun}}_T(C, D)\,.$$

When this T-functor has a left T-adjoint we say that D has all C-indexed T-colimits. Similarly, if it has a right T-adjoint we say that D has all C-indexed T-limits. If a T-∞-category D has all C-indexed T-colimits (respectively T-limits) for every small T-category C we say that D is T-cocomplete (respectively T-complete).

A T-colimit indexed by a T-category of the form $\mathrm{pr}_2 : K \times T^{\mathrm{op}} \to T^{\mathrm{op}}$ for K an ∞-category is called a fiberwise T-colimit. A T-colimit indexed by the category of points of a finite T-set (example 1.5) is called a finite T-coproduct. A T-∞-category is said to be pointed *if it has both a T-initial and a T-terminal object (that are cocartesian sections of the structure map that fiberwise select the initial and the terminal object respectively) and the canonical comparison map is an equivalence.*

The following proposition summarizes the results on T-(co)limits from [32] that will be needed in this paper.

Proposition 1.16. *Let C be a T-∞-category.*

- *C has all T-colimits indexed by $K \times T^{\mathrm{op}}$ if and only if for every $b \in T$ the fiber C_b has all colimits indexed by K and for every edge $e : b \to b'$ in T the pushforward functor $\mathrm{res}_e : C_{b'} \to C_b$ preserves colimits indexed by K.*
- *C has all (finite) T-coproducts if and only if the following two conditions are satisfied*
  1. *for every $b \in T$ the fiber C_b has all (finite) coproducts and for every edge $e : b \to b'$ the pushforward res_e preserves (finite) coproducts;*
  2. *For every edge $e : b \to b'$ the pushforward res_e has a left adjoint $\coprod_e$ satisfying the Beck-Chevalley condition: for every pair of edges $e : b \to b'$ and $e' : b'' \to b'$ the canonical base change natural transformation of functors from $C_{b''}$ to C_b*

     $$\mathrm{res}_e \coprod_{e'} \to \coprod_{o \in \mathrm{Orbit}(b \times_{b'} b'')} \coprod_{pr_1} \mathrm{res}_{pr_2}$$

     is an equivalence, where $pr_1 : o \to b$ and $pr_2 : o \to b'$ are the restrictions to o of the two projections from $b \times_{b'} b''$.
- *C has all T-colimits if and only if it has all fiberwise colimits and all finite T-coproducts.*

Similar statements hold for T-limits. When T-products exist the right adjoint of res_e will be denoted by $\prod_e$.

Proof. See [32]. □

Example 1.17. *Let $\underline{C}_T$ be a T-∞-category of T-objects (1.10). Then the diagonal functor*

$$\mathrm{Fun}(T^{\mathrm{op}}, C) \cong \mathrm{Fun}_T(T^{\mathrm{op}}, \underline{C}_T) \to \mathrm{Fun}_T(I, \underline{C}_T) \cong \mathrm{Fun}(I, C)$$

is just precomposition by $I \to T^{\mathrm{op}}$. Hence, we can compute T-colimits (resp. T-limits) just via the ordinary left (resp. right) Kan extension. It is instructive to check that in categories of T-objects T-coproducts and T-products behave as expected.

Definition 1.18. *There is also a notion of T-Kan extension, defined exactly as for the T-(co)limit: if we have a T-functor $j : I \to J$ there is a T-functor induced by precomposition with j:*

$$j^* : \underline{\mathrm{Fun}}_T(J, D) \to \underline{\mathrm{Fun}}_T(I, D)\,.$$

If j^* has a left T-adjoint we denote it by $j_!$ and call it the left T-Kan extension *along j. Similarly, when j^* has a right T-adjoint we call it the* right T-Kan extension *j_*.*

The T-Kan extensions behave exactly as one would expect from the theory of ordinary Kan extensions.

Proposition 1.19. *Let C be a T-∞-category with all T-colimits. Then for every map of small T-∞-categories $j : I \to I'$ the left Kan extension along j*

$$j_! : \underline{\mathrm{Fun}}_T(I, C) \to \underline{\mathrm{Fun}}_T(I', C)$$

exists. Moreover for each $F \in \mathrm{Fun}_T(I, C)$ the value of $j_!F$ at $i \in I'_t$ is given by the $T_{/t}$-colimit

$$j_!F(i) = T_{/t}\text{--}\mathrm{colim}\left(I_{/t} \times_{I'_{/t}} (I'_{/t})_{/T_{/t}i} \to I'_{/t} \to C_{/t}\right)$$

where $I_{/t}$, $I'_{/t}$ and $C_{/t}$ are the base change of I, I' and C along $(T_{/t})^{\mathrm{op}} \to T^{\mathrm{op}}$.
Similarly for T-limits and the right Kan extension j_*.

Proof. See [32]. □

Notation 1.20. *A T-functor is said to be* fiberwise left exact, T-left exact, fiberwise right exact, T-right exact *if it preserves finite fiberwise limits, finite T-limits, finite fiberwise colimits and finite T-colimits respectively. We will denote the full T-∞-subcategories of $\underline{\mathrm{Fun}}_T(C, D)$ preserving certain (co)limits will be denoted as in the following list:*

- $\underline{\mathrm{Fun}}_T^{T-\mathrm{lex}}(C,D)$: *finite T-colimits;*
- $\underline{\mathrm{Fun}}_T^{fb-\mathrm{lex}}(C,D)$: *finite fiberwise colimits;*
- $\underline{\mathrm{Fun}}_T^{\amalg}(C,D)$: *finite T-coproducts;*
- $\underline{\mathrm{Fun}}_T^{T-\mathrm{rex}}(C,D)$: *finite T-limits;*
- $\underline{\mathrm{Fun}}_T^{fb-\mathrm{rex}}(C,D)$: *finite fiberwise limits;*
- $\underline{\mathrm{Fun}}_T^{\times}(C,D)$: *finite T-products;*
- $\underline{\mathrm{Fun}}_T^{L}(C,D)$: *all T-colimits;*
- $\underline{\mathrm{Fun}}_T^{R}(C,D)$: *all T-limits.*

1.4 The Yoneda embedding and presentable T-∞-categories

Let C be a small T-∞-category. Then the T-∞-category of presheaves is

$$P_T(C) = \underline{\mathrm{Fun}}_T(C^{\vee\mathrm{op}}, \underline{\mathrm{Top}}_T)$$

There is a T-functor $j : C \to P_T(C)$ adjoint to the standard pairing of [11]

$$C \times_{T^{\mathrm{op}}} C^{\vee\mathrm{op}} \to \mathrm{Top}$$

that we call the Yoneda embedding.

Proposition 1.21. *The Yoneda embedding is fully faithful. Moreover the left T-Kan extension $j_!j$ of the Yoneda embedding along itself is the identity of $P_T(C)$. Finally, precomposition by the Yoneda embedding induces an equivalence for all cocomplete T-∞-categories D*

$$j^* : \mathrm{Fun}_T^L(P_T(C), D) \xrightarrow{\sim} \mathrm{Fun}_T(C, D)\,.$$

Proof. See [32]. □

Let $F : C \to D$ be a T-functor between T-cocomplete T-∞-categories. Then we say that T is **accessible** if it preserves all κ-filtered fiberwise colimits for some regular cardinal κ. A T-∞-category C is **presentable** if it is equivalent to an accessible localization of a presheaf category. That is, there is a fully faithful T-functor $C \to P_T(C_0)$, for some small C_0, that has an accessible left T-adjoint. If C is presentable it is T-complete and T-cocomplete, and for all $V \in T$ the fiber C_V is presentable.

Let $S \subseteq C$ be a small T-subcategory of C. Then an S-local object is an object $x \in C_V$ such that for every $W \in T_{/V}$ the functor

$$\mathrm{Map}_{C_W}(-, \mathrm{res}^V_W x) : C_W^{\mathrm{op}} \to \mathrm{Top}$$

sends all morphisms of S_V to equivalences. We will denote the T-subcategory of S-local objects by $L_S C$. If C is presentable, it is an accessible localization of C and so in particular it is presentable. Then we can express presentable T-∞-categories as those that admit a small presentation by generator and relations:

Proposition 1.22. *Let C be a presentable T-∞-category. Then there is a small T-∞-category C_0 and a small T-subcategory $S \subseteq P_T(C_0)$ such that C is equivalent to the T-subcategory of S-local objects in $P_T(C_0)$.*

Proof. See [32]. □

Finally we will need a universal property of $L_S C$.

Proposition 1.23. *Let C, D be presentable T-∞-categories and $S \subseteq C$ a small T-subcategory. Then the functor*

$$\mathrm{Fun}^L_T(L_S C, D) \to \mathrm{Fun}^L_T(C, D)$$

given by precomposition with L_S is fully faithful with essential image those functors sending every arrow in S to a cocartesian arrow.

Proof. See [32]. □

Chapter 2

Parametrized additivity and parametrized stability

In this chapter we will explain the important notion of T-stability (definition 2.33). This will allow us to describe spectral Mackey functors as the universal T-stable category. Important applications of this property will be a construction of the Guillou-May comparison map between spectral Mackey functors and orthogonal G-spectra (theorem 2.40) and the construction of norm functors (proposition 3.27).

The results of this chapter originally appeared in [29].

2.1 Fiberwise stability

Recollection 2.1. *Recall that if C is an ∞-category with finite colimits and D is an ∞-category with finite limits a functor $F : C \to D$ is called* linear *if it sends the initial object of C to the terminal object of D and pushout squares in C to pullback squares in D (this functors are called* pointed excisive *in [25]). In [25, Pr. 1.4.2.13] it is proven that a pointed functor is linear if and only if the natural transformation $F \to \Omega F \Sigma$ is an equivalence. The full subcategory of* $\mathrm{Fun}(C, D)$ *spanned by linear functors is denoted* $\mathrm{Lin}(C, D)$.

Definition 2.2. *Let C, D T-∞-categories and assume that C has all finite fiberwise colimits and D has all finite fiberwise limits. We say that a T-functor $F : C \to D$ is* fiberwise linear *if the restriction on the fiber $F_b : C_b \to D_b$ is linear for every $b \in T$. We denote the full T-subcategory of $\underline{\mathrm{Fun}}_T(C, D)$ spanned by fiberwise linear functors with $\underline{\mathrm{Lin}}_T(C, D)$.*

First we want to show that, if C is T-pointed, $\mathrm{Lin}_T(C, D)$ is a localization of the subcategory $\mathrm{Fun}_{T,*}(C, D)$ of functors sending the zero object to the terminal object in each fiber. To do so we introduce two additional functors $\Sigma_T : C \to C$

and $\Omega_T : D \to D$ which are the pushout (respectively pullback) of the diagrams

$$\begin{array}{ccc} id_C & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array} \quad \text{and} \quad \begin{array}{ccc} & & * \\ & & \downarrow \\ * & \longrightarrow & id_D \end{array} .$$

Since fiberwise linearity can be checked fiberwise it is clear that a functor $F \in \mathrm{Fun}_{T,*}(C, D)$ is in $\mathrm{Lin}(C, D)$ if and only if the canonical map $F \to \Omega_T F \Sigma_T$ is an equivalence.

Lemma 2.3. *Suppose that C is a pointed T-category. Then the ∞-category $\mathrm{Lin}_T(C, D)$ is stable.*

Proof. It is clear that $\mathrm{Lin}_T(C, D)$ has finite limits and that it is pointed. If we show that Ω is an equivalence we are done by proposition 1.4.2.24 of [25]. But Ω is just postcomposition with $\Omega_T : D \to D$ and then it is obvious that precomposition with $\Sigma_T : C \to C$ is an inverse. □

Definition 2.4. *We say that a T-∞-category D with all finite fiberwise limits and colimits is fiberwise stable if all fibers D_b are stable.*

Construction 2.5. *If C is a T-∞-category we want to construct a fiberwise stabilization, that is the universal source of a fiberwise linear T-functor to C. Let $\mathcal{E}(D)$ be the simplicial set over T^{op} such that*

$$\mathrm{Hom}_{/T^{\mathrm{op}}}(K, \mathcal{E}(D)) \cong \mathrm{Hom}_{/T^{\mathrm{op}}}(K \times \mathrm{Top}_*^{fin}, D)$$

(this is an instance of the pairing construction of [24, Cor. 3.2.2.13]). The fiber over $b \in T^{\mathrm{op}}$ is the category $\mathrm{Fun}(\mathrm{Top}_*^{fin}, D_b)$.

We let $\underline{\mathrm{Sp}}_T(D)$ be the simplicial subset of $\mathcal{E}(D)$ consisting of all simplices whose vertices are linear functors $\mathrm{Top}_*^{fin} \to D_b$. This is the same simplicial set denoted by Stab(D) in [25, Cn. 6.2.2.2]. It comes equipped with a natural map of simplicial sets $\Omega^\infty : \underline{\mathrm{Sp}}_T(D) \to D$ over T^{op} that on vertices is evaluation at S^0.

Proposition 2.6. *The map $\underline{\mathrm{Sp}}_T(D) \to T^{\mathrm{op}}$ is a fiberwise stable T-∞-category.*

Moreover the natural functor $\Omega^\infty \colon \underline{\mathrm{Sp}}_T(D) \to D$ is a fiberwise left exact T-functor and for every pointed T-∞-category C with finite T-colimits the induced map

$$\underline{\mathrm{Fun}}_T^{fb-\mathrm{rex}}(C, \underline{\mathrm{Sp}}_T(D)) \to \underline{\mathrm{Lin}}_T(C, D)$$

is an equivalence of categories.

Proof. From [24, Cor. 3.2.2.13] it follows immediately that $\mathcal{E}(D)$ is a cocartesian fibration whose cocartesian edges are those maps $(\Delta^1)^\sharp \times (\mathrm{Top}_*^{fin})^\flat \to D^\natural$ that are marked. So to prove that $\underline{\mathrm{Sp}}_T(D) \to T^{\mathrm{op}}$ is a cocartesian fibration we need only to prove that it contains all cocartesian edges whose source is in it (that is, that $\underline{\mathrm{Sp}}_T(D)$ is closed under pushforward). But, by our description of cocartesian edges, the pushforward functor along an edge $e : b \to b'$ of $\mathcal{E}$ is given by

$$(\mathrm{res}_e)_* : \mathcal{E}_{b'} = \mathrm{Fun}(\mathrm{Top}_*^{fin}, D_{b'}) \to \mathcal{E}_b = \mathrm{Fun}(\mathrm{Top}_*^{fin}, D_b)\,,$$

that is postcomposition with the pushforward in D. Since the pushforward in D preserves finite limits by definition, $(\mathrm{res}_e)_*$ preserves linear functors and so $\underline{\mathrm{Sp}}_T(D)$ is a T-∞-category.

Note that the fiber of $\underline{\mathrm{Sp}}_T(D)$ over $b \in T$ is exactly the stabilization of the fiber D_b and that the pushforward functors between fibers of $\underline{\mathrm{Sp}}_T(D)$ are the functors induced by the pushforward between the fibers of D. So the cocartesian fibration $\underline{\mathrm{Sp}}_T(D)$ has all finite fiberwise limits and colimits and is fiberwise stable. Moreover the functor $\underline{\mathrm{Sp}}_T(D) \to D$ is a T-functor preserving T-limits.

Finally let us prove the universal property. Since the fibers of

$$\underline{\mathrm{Fun}}_T^{fb-\mathrm{lex}}(C, \underline{\mathrm{Sp}}_T(D)) \text{ and } \underline{\mathrm{Lin}}_T(C, D)$$

over $b \in T$ are

$$\mathrm{Fun}_{T_{/b}}^{fb-\mathrm{lex}}(C \times_{T^{\mathrm{op}}} T_{b/}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{T_{/b}}(D \times_{T^{\mathrm{op}}} T_{b/}^{\mathrm{op}})) \text{ and } \mathrm{Lin}_{T_{/b}}(C \times_{T^{\mathrm{op}}} T_{b/}^{\mathrm{op}}, D \times_{T^{\mathrm{op}}} T_{b/}^{\mathrm{op}})$$

respectively, up to replacing T by its slice $T_{/b}$ it is enough to prove that the functor

$$(\Omega^\infty)_* : \mathrm{Fun}_T^{fb-\mathrm{lex}}(C, \underline{\mathrm{Sp}}_T(D)) \to \mathrm{Lin}_T(C, D)$$

is an equivalence of categories (since being an equivalence can be checked on every fiber). Observe that $\mathrm{Fun}_T^{fb-\mathrm{lex}}(C, \underline{\mathrm{Sp}}_T(D))$ and $\mathrm{Sp}(\mathrm{Lin}_T(C, D))$ are the same subcategory of $\mathrm{Fun}_T(C \times \mathrm{Top}^{fin}, D)$, because both are spanned by the functors $F : C \times \mathrm{Top}_T^{fin} \to D$ whose restriction to $C_b \times \mathrm{Top}^{fin}$ lie in $\mathrm{Fun}^{lex}(C_b, \mathrm{Sp}(D_b)) = \mathrm{Sp}(\mathrm{Lin}(C_b, D_b))$ for any $b \in T$. Then the thesis is obvious because $\mathrm{Lin}(C_b, D_b)$ is stable. □

2.2 Categories of finite T-sets

Definition 2.7. *An orbital ∞-category T is* atomic *if there are no nontrivial retracts, that is if every map with a left inverse is an equivalence. All the examples of orbital ∞-categories in example 1.3 are atomic. For the remainder of this chapter, T will be a fixed atomic orbital category. We will now construct T-∞-categories of finite T-sets that will be used to parametrize the various multiplications composing the structure of a T-commutative monoid.*

Definition 2.8. *We want to construct the T-category classified by the functor $T \to \mathrm{Cat}_\infty$ sending V to $\mathbf{F}_{T_{/V}}$. We contemplate the arrow ∞-category $\mathrm{Fun}(\Delta^1, \mathbf{F}_T)$ of the ∞-category $\mathbf{F}_T$ of finite T-sets. Since $\mathbf{F}_T$ admits all pullbacks, the target functor*

$$\mathrm{Fun}(\Delta^1, \mathbf{F}_T) \to \mathrm{Fun}(\{1\}, \mathbf{F}_T) \cong \mathbf{F}_T$$

is a cartesian fibration. We may pull it back along the fully faithful inclusion $T \hookrightarrow \mathbf{F}_T$ *to obtain a cartesian fibration*

$$\tau\colon \operatorname{Fun}(\Delta^1, \mathbf{F}_T) \times_{\operatorname{Fun}(\{1\},\mathbf{F}_T)} T \to T.$$

It is classified by the functor $T^{\mathrm{op}} \to \mathrm{Cat}_\infty$ *that carries an orbit* V *to the* ∞*-category* $\mathbf{F}_{T_{/V}}$*.*

We now write

$$p\colon \underline{\mathbf{F}}_T \to T^{\mathrm{op}}$$

for the dual cocartesian fibration $\tau^\vee$ *(constructed in [11]) to the cartesian fibration* τ*. This is now a* T*-*∞*-category, called the* T*-*∞*-category of finite* T*-sets, and once again it is classified by the functor* $T^{\mathrm{op}} \to \mathrm{Cat}_\infty$ *that carries an orbit* V *to the nerve of the* ∞*-category* $\mathbf{F}_{T_{/V}}$*. Its objects are arrows* $I = [U \to V]$ *with* $U \in \mathbf{F}_T$ *and* $V \in T$ *and an arrow* $[U \to V] \to [U' \to V']$ *is a diagram*

$$\begin{tikzcd} U \arrow[d] & W \arrow[l] \arrow[r] \arrow[d] & U' \arrow[d] \\ V & V' \arrow[l] \arrow[r, equal] & V' \end{tikzcd}$$

where the left square is cartesian. Composition is then defined by forming suitable pullbacks. The target functor

$$[U \to V] \mapsto V$$

is the structure map $p\colon \underline{\mathbf{F}}_T \to T^{\mathrm{op}}$*.*

Example 2.9. *If* $T = \mathbf{O}_G$ *is the orbit category of a profinite group* G*, then* $\underline{\mathbf{F}}_G \to \mathbf{O}_G^{\mathrm{op}}$ *is the cocartesian fibration classified by the functor sending an orbit* G/H *to the category of finite* H*-sets (under the canonical identification that sends a finite* G*-set over* G/H *to the fiber over* eH*).*

Example 2.10. *Suppose* V *is an object of* T*. Then there is an object*

$$I(V) = [\mathrm{id}\colon V \to V]$$

of $\underline{\mathbf{F}}_T$*, which enjoys the following property. For any object* $J = [g\colon X \to Y]$ *of* $\underline{\mathbf{F}}_T$*, one has an equivalence*

$$\operatorname{Map}_{\underline{\mathbf{F}}_T}(J, I(V)) \simeq \operatorname{Map}_T(V, Y).$$

In particular, the assignment $V \mapsto I(V)$ *defines a fully faithful right adjoint to the structure map* $p\colon \underline{\mathbf{F}}_T \to T^{\mathrm{op}}$*.*

In what follows it will be convenient to have at our disposal more general categories whose objects are finite T-sets. They will be all more easily manipulated as subcategories of the Burnside category of finite T-sets. The latter is the dual version of the main construction in [8], but we will repeat it here both because of its simplicity and because we will need to use some details in our main results.

Construction 2.11. *Let again us consider the cartesian fibration*

$$\tau\colon S_T := \mathrm{Fun}(\Delta^1, \mathbf{F}_T) \times_{\mathrm{Fun}(\{1\},\mathbf{F}_T)} T \to T.$$

It is also, for much easier reasons, a cocartesian fibration.

With this in mind, we now proceed to define triple structures on these ∞-categories. Denote by $\iota T \subset T$ the subcategory consisting of the equivalences of T. Then we can contemplate the triple structures

$$(T, \iota T, T) \quad \textit{and} \quad (S_T, S_T \times_T \iota T, S_T).$$

It is a simple matter to see that these triple structures are adequate in the sense of [14, Df. 5.2]. We may therefore construct their effective Burnside ∞-categories, and the projection induces a functor

$$t'\colon A^{\mathrm{eff}}(S_T, S_T \times_T \iota T, S_T) \to A^{\mathrm{eff}}(T, \iota T, T).$$

An object of $A^{\mathrm{eff}}(S_T, S_T \times_T \iota T, S_T)$ is a morphism $[U \to V]$ of finite T-sets in which $V \in T$. If

$$I = [U \to V] \quad \textit{and} \quad J = [X \to Y]$$

are two objects, then a morphism $I \to J$ of $A^{\mathrm{eff}}(S_T, S_T \times_T \iota T, S_T)$ is a commutative diagram

$$\begin{tikzcd} U \arrow[d] & W \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\ V & Z \arrow[l] \arrow[r, "\sim"'] & Y \end{tikzcd}$$

in which the morphism $Z \xrightarrow{\sim} Y$ is an equivalence in T.

Lemma 2.12. *The functor t' above is both a cartesian and a cocartesian fibration. Furthermore, any morphism of $A^{\mathrm{eff}}(S_T, S_T \times_T \iota T, S_T)$ represented as a commutative diagram*

$$\begin{tikzcd} U \arrow[d] & W \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\ V & Z \arrow[l] \arrow[r, "\sim"'] & Y \end{tikzcd}$$

is

- *t'-cartesian if the morphisms $W \xrightarrow{\sim} U$ and $W \xrightarrow{\sim} X$ are equivalences;*
- *t'-cocartesian if the left square is cartesian and $\equiv WX$ is an equivalence.*

Proof. This follows immediately from (the opposite of) the "omnibus theorem" for effective Burnside ∞-categories [14, Th. 12.2]. □

Definition 2.13. *We have an inclusion $T^{\mathrm{op}} \hookrightarrow A^{\mathrm{eff}}(T, \iota T, T)$, which is a weak equivalence. Write*

$$\underline{\mathbf{A}}^{\mathrm{eff}}(T) := A^{\mathrm{eff}}(S_T, S_T \times_T \iota T, S_T) \times_{A^{\mathrm{eff}}(T,\iota T,T)} T^{\mathrm{op}};$$

the projection $\underline{\mathbf{A}}^{\mathrm{eff}}(T) \to A^{\mathrm{eff}}(S_T, S_T \times_T \iota T, S_T)$ is thus an equivalence, and the projection

$$t\colon \underline{\mathbf{A}}^{\mathrm{eff}}(T) \to T^{\mathrm{op}}$$

is a cartesian and cocartesian fibration, so it is a T-∞-category.

It is classified by the functor sending $V \in T$ to the Burnside category $A^{\mathrm{eff}}(\mathbf{F}_{T/V})$.

Note that $\underline{\mathbf{F}}_T$ is naturally a T-subcategory of $\underline{\mathbf{A}}^{\mathrm{eff}}(T)$, consisting of all objects and all morphisms such that the left square is cartesian. This is the analogue of the classical inclusion of the category $\mathbf{F}$ of finite sets inside the Burnside category of finite sets $A^{\mathrm{eff}}(\mathbf{F})$ by considering only the egressive maps.

This can be extended to an inclusion of *pointed* finite sets $\mathbf{F}_*$ inside $A^{\mathrm{eff}}(\mathbf{F})$ as the subcategory containing all objects and as maps the spans $[I \leftarrow \tilde{I} \to I']$ such that the "left leg" is an inclusion ($\tilde{I}$ under this identification corresponds to the preimage of I' under the map $I_+ \to I'_+$, so that we can identify $\mathbf{F}_*$ with the category of finite sets and partially defined maps. It will be convenient for us to turn this into the definition of finite pointed T-sets.

Definition 2.14. *We'll say that a map $U \to U'$ of finite T-sets is a* summand inclusion *if there is $U'' \to U'$ such that the map $U \amalg U'' \to U'$ is an equivalence.*

Consider the subcategory of $\underline{\mathbf{A}}^{\mathrm{eff}}(T)$ containing all objects and whose morphisms are those diagrams

$$\begin{tikzcd} U \arrow[d] & \tilde{U} \arrow[l] \arrow[r] \arrow[d] & U' \arrow[d] \\ V & V' \arrow[l] \arrow[r, equal] & V' \end{tikzcd}$$

such that the arrow $\tilde{U} \to U \times_V V'$ is a summand inclusion (this is a condition of the left square of the diagram and does not depend on the particular choice of pullback $U \times_V V'$). This subcategory contains all cocartesian morphisms of $\underline{\mathbf{A}}^{\mathrm{eff}}(T) \to T^{\mathrm{op}}$ and so it is a T-subcategory. We will call it the T-∞-category of finite pointed T-sets *and denote it by $\underline{\mathbf{F}}_{T,*}$.*

An edge of $\underline{\mathbf{F}}_{T,*}$ is ***inert*** *if the map $\tilde{U} \to U'$ in the above diagram is an equivalence.*

Notation 2.15. *We will often decorate an object $I = [U \to V]$ of $\underline{\mathbf{F}}_{T,*}$ with a subscript $+$, to remind ourselves that we see it as living in $\underline{\mathbf{F}}_{T,*}$ rather than of $\underline{\mathbf{F}}_T$ or $\underline{\mathbf{A}}(T)$. The $+$ does not have any real meaning (in our construction there are no "basepoints") and it is only a mnemonic aid. The canonical inclusion $\underline{\mathbf{F}}_T \to \underline{\mathbf{F}}_{T,*}$ will be indicated by $(-)_+\colon I \mapsto I_+$.*

Lemma 2.16. *The cocartesian fibration $\underline{\mathbf{F}}_{T,*} \to T^{\mathrm{op}}$ is classified by the functor sending V to the category of pointed objects in $(\mathbf{F}_T)_{V/}$.*

Moreover the canonical inclusion $(-)_+ \colon \underline{\mathbf{F}}_T \to \underline{\mathbf{F}}_{T,*}$ has a right T-adjoint sending $[U \to V]$ to $[U \amalg V \to V]$.

Proof. The fiber of $\underline{\mathbf{F}}_{T,*}$ over V consists in the Burnside category of $\mathbf{F}_{T_{/V}}$ where the egressive morphisms are the summand-inclusions. We can identify this with the category of pointed objects by sending a span

$$U \leftarrow \tilde{U} \to U'$$

where $U = \tilde{U} \amalg W$ to the map

$$U \amalg V \to U' \amalg V$$

where the central map is $U \amalg V = \tilde{U} \amalg (W \amalg V) \to U' \amalg V$, given by $\tilde{U} \to U'$ on the first component and the structure map to V on the second component.

Now it is clear that the functor

$$[U \amalg V \to V] \mapsto [U \amalg V \to V]$$

is the right adjoint to the inclusion of $\mathbf{F}_{T_{/V}}$ into pointed objects. So in order to have a T-adjunction we need only to verify that the right adjoint provided by [25, Pr. 7.3.2.6] is a T-functor, but this follows from the universality of finite coproducts in $\mathbf{F}_T$ and the fact that coproducts therein are disjoint. □

2.3 T-semiadditive functors and T-semiadditive categories

Notation 2.17. *Let $I = [U \to V] \in \underline{\mathbf{F}}_T$ and $W \in \mathrm{Orbit}(U)$, then the canonical map $W \to U \times_V W$ must be a summand-inclusion, since it factors through an unique orbit, of which W is a retract. So we can define the* characteristic map

$$\chi_{[W \subseteq U]} \colon I_+ \to I(W)_+$$

(where $I(W)$ is the construction of example 2.10) as the map of pointed finite T-sets described by the following diagram

$$\begin{tikzcd} U \arrow[d] & W \arrow[l] \arrow[r, equal] \arrow[d, equal] & W \arrow[d, equal] \\ V & W \arrow[l] \arrow[r, equal] & W \end{tikzcd}$$

Note that this map is in $\underline{\mathbf{F}}_{T,*}$ due to the fact that, thanks to the atomicity of T, the map $W \to U \times_V W$ is a summand inclusion, since the orbit it factors through retracts onto W.

Construction 2.18. *Let C be a pointed T-∞-category with all finite T-coproducts, $I = [U \to V] \in \underline{\mathbf{F}}_T$ and $X \in \mathrm{Fun}_T(\mathbf{U}, C)$ be a diagram. Then there is a map induced on the colimits*

$$(\chi_{[W \subseteq U]})_* : \mathrm{res}_{W/V} \coprod_I X \to X_{[W \subseteq U]} .$$

above $\chi_{[W \subseteq U]}$, where $\coprod_I$ is the left adjoint to $\Delta_I : C_V \cong \mathrm{Fun}_T(\mathbf{V}, C) \to \mathrm{Fun}_T(\mathbf{U}, C)$ and $X_{[W \subseteq U]}$ is the value of X at $[W \subseteq U] \in \mathbf{U}$. We can describe it as follows: by the base change condition in proposition 1.16 we have

$$\mathrm{res}^W_V \coprod_I X \cong \coprod_{[U \times_V W/W]} \mathrm{res}^{U \times_V W}_U X$$

As before, the atomicity of T implies that we can write

$$U \times_V W \cong W \amalg \tilde{U}$$

where W on the right hand side is the diagonal copy. Hence

$$\mathrm{res}_{W/V} \coprod_I X \cong X_{[W \to U]} \amalg \coprod_{[\tilde{U} \to U]} X .$$

So we can define a map

$$(\chi_{[W \subseteq U]})_* : \mathrm{res}_{W/V} \coprod_I X \to X_{[W \subseteq U]}$$

which is the identity on the first summand and the zero map on the other.

If D is a T-∞-category with finite T-products and $F : C \to D$ a T-functor we will denote by $(\chi_{[W \subseteq U]})_*$ also the natural transformation

$$F\left(\coprod_I X\right) \to \prod_{W/V} F(X)$$

obtained by adjunction on $F(\chi_{[W \subseteq U]})_*$, where $\prod_{W/V}$ is the right adjoint of res^W_V.

Definition 2.19. *Let C be a pointed T-∞-category with all finite T-coproducts and D a T-∞-category with all finite T-products. Then a T-functor $F : C \to D$ is said to be* T-semiadditive *if for every $I = [U \to V] \in \underline{\mathbf{F}}_T$ and $X \in \mathrm{Fun}_T(\mathbf{U}, C)$ the map*

$$\prod_{W \in \mathrm{Orbit}(U)} (\chi_{[W \subseteq U]})_* : F\left(\coprod_I X\right) \to \prod_{W \in \mathrm{Orbit}(U)} \prod_{W/V} F(X_{[W \subseteq U]}) \cong \prod_I F(X) \tag{2.1}$$

is an equivalence. We will denote the T-∞-category of all T-semiadditive T-functors with $\underline{\mathrm{Fun}}^{\oplus}_T(C, D)$.

We say that a pointed T-∞-category with all finite T-products and T-coproducts is T-semiadditive *if the identity functor is T-semiadditive. That is, if the map*

$$\coprod_I X \to \prod_I X \tag{2.2}$$

is an equivalence for every I-uple X.

It is clear that if $F : C \to D$ preserves finite T-coproducts and $G : D \to E$ is T-semiadditive then the composition GF is T-semiadditive. Similarly if F is T-semiadditive and G preserves finite T-products.

Example 2.20. *Let C be a pointed T-∞-category with all finite T-coproducts and D an ∞-category with all finite products. Then the category of T-objects $\underline{D}_T$ has all finite T-products and a T-functor $C \to \underline{D}_T$ is T-semiadditive if and only if the associated functor $F : C \to D$ is such that for every $[U \to V] \in \underline{\mathbf{F}}_T$ the map*

$$F\left(\coprod_I X\right) \to \prod_{W \in Orbit(U)} F(X_{[W \subseteq U]})$$

is an equivalence. In particular if D is semiadditive (i.e. it has biproducts), then $\underline{D}_T$ is T-semiadditive.

Example 2.21. *The T-∞-category $\mathbf{A}^{\text{eff}}(T)$ is T-semiadditive. In fact every fiber is semiadditive by proposition 4.3 of [14], so it is sufficient to observe that for any arrow $W \to V$ in T the functor*

$$\mathrm{res}^W_V : A^{\text{eff}}(T_{/V}) \to A^{\text{eff}}(T_{/W})$$

has well behaved left and right adjoints and the canonical comparison map is an equivalence, with the same proof as A.15.

Construction 2.22. *Let C be a pointed T-∞-category with finite T-products. Then if $I = [U \to V] \in \mathbf{F}^T$ and $X : \mathbf{U} \to C$, let us consider for every $W \in \mathrm{Orbit}(U)$ the map of C_V*

$$\eta_{[W \subseteq U]} : \coprod_{W/V} X_{[W \to U]} \hookrightarrow \coprod_I X \to \prod_I X .$$

This can be described as the adjoint to the map in the fiber over W

$$X_{[W \subseteq U]} \to \mathrm{res}^W_V \prod_{I_W} X \cong \prod_{W' \in \mathrm{Orbit}(U \times_V W)} \prod_{W'/W} X_{[W' \to U]} ,$$

given by the identity map $X_W \to \prod_{W'/W} X_{W'}$ when W' is the diagonal copy W in $U \times_V W$ and the zero map on the other components.

Then C being T-semiadditive is equivalent to the fact that $\{\eta_W\}_{W \in \mathrm{Orbit}(U)}$ assemble to an equivalence

$$\coprod_I X \cong \coprod_{W \in \mathrm{Orbit}(U)} \coprod_{W/V} X_{[W \subseteq U]} \xrightarrow{\coprod \eta_{[W \subseteq U]}} \prod_I X .$$

The previous remark immediately yields the following criterion for determining when a category is T-semiadditive

Lemma 2.23. *Let C be a pointed T-∞-category with finite products and suppose that for every $I = [U \to V] \in \mathbf{F}^T$ there is a natural transformation*

$$\mu^I : \prod_I \Delta X \to X$$

of functors $C_V \to C_V$, where $\Delta : C_V \to \mathrm{Fun}_T(\mathbf{U}, C)$ is the functor of definition 1.15, such that for every $W \in \mathrm{Orbit}(U)$ the composition

$$\mu^I \circ \eta_{[W \subseteq U]} : \coprod_{W/V} \mathrm{res}^W_V X \to X$$

is homotopic to the counit of the adjunction $\coprod_{W/V} \dashv \mathrm{res}^W_V$. Then C is T-semiadditive.

Proof. We need to prove that for every $[U \to V] \in \underline{\mathbf{F}}_T$, $X \in \mathrm{Fun}_T(\mathbf{U}, C)$ and $Y \in C_V$, the map

$$\prod_{W \in \mathrm{Orbit}(U)} (\eta_{[W \subseteq U]})^* : \mathrm{Map}_{C_V}\left(\prod_I X, Y\right) \to \prod_{W \in \mathrm{Orbit}(U)} \mathrm{Map}_{C_V}\left(\coprod_{W/V} X_{[W \subseteq U]}, Y\right)$$

is an equivalence. But using the μ^I we can construct an inverse

$$(\mu^I_Y)_* \circ \prod_I : \prod_{W \in \mathrm{Orbit}(U)} \mathrm{Map}_{C_V}\left(\coprod_{W/V} X_{[W \subseteq U]}, Y\right) \cong \mathrm{Map}_{\mathrm{Fun}(\mathbf{U},C)}(X, \Delta Y) \to$$
$$\to \mathrm{Map}_{C_V}\left(\prod_I X, \prod_I \Delta I\right) \to \mathrm{Map}_{C_V}\left(\prod_I X, Y\right).$$

□

Proposition 2.24. *If C is a pointed T-∞-category with finite T-coproducts and D is a T-∞-category with finite T-products then $\underline{\mathrm{Fun}}^{\oplus}(C, D)$ is T-semiadditive*

Proof. First let us note that by I to be empty in 2.1 every T-semiadditive functor must send the zero object of C to the terminal object of D. Then for any additive functor F the left T-Kan extension of the restriction to the zero object of C is the constant functor at the terminal object, since the T-colimit of a constant functor at the zero object is the zero object. So, if $i : \{0\} \subseteq C$ is the inclusion of the zero object

$$\mathrm{Map}_{\mathrm{Fun}^{\oplus}_T}(*, F) = \mathrm{Map}(i_!(F|_0), F) = \mathrm{Map}(F|_0, F|_0) = *$$

Hence the constant functor at the terminal object is the zero object of $\underline{\mathrm{Fun}}^{\oplus}_T(C, D)$.

Then we need to prove that $\underline{\mathrm{Fun}}_T^{\oplus}(C, D)$ satisfies the hypothesis of the previous lemma. But this is easy: for F a T-semiadditive T-functor remember that $(\prod_I F)(-) \cong F(\coprod_I -)$ so we can choose

$$\mu^I : \left(\prod_I F\right)(-) \cong F\left(\coprod_I -\right) \to F(-)$$

given by precomposition with the canonical map $id_C \to \coprod_I$ provided by the universal property of the coproduct of C. Since the required identites are easily verified we are done. $\square$

Definition 2.25. *Let C be a T-∞-category with finite products. Then a T-commutative monoid is a T-semiadditive functor $\underline{\mathbf{F}}_{T,*} \to C$. We will indicate the T-∞-category of T-commutative monoids in C with $\underline{\mathrm{CMon}}_T(C)$. Precomposition with the cocartesian section $I(-)_+ : T^{\mathrm{op}} \to \underline{\mathbf{F}}_{T,*}$ induces a T-functor*

$$\underline{\mathrm{CMon}}_T(C) \to C\,.$$

In order to prove the universal property of $\underline{\mathrm{CMon}}_T(C)$ we will need the following lemma

Lemma 2.26. *Let C be a pointed T-∞-category with finite T-coproducts. Then the map*

$$\mathrm{Fun}_T^{\amalg}(\underline{\mathbf{F}}_{T,*}, C) \to C$$

given by precomposition with $I(-)_+$ is an equivalence

Proof. We can construct an inverse by sending every $c \in C_V$ to the left Kan extension of its cocartesian section $\mathbf{V} \to C$ along $I(-)_+ : T^{\mathrm{op}} \to \underline{\mathbf{F}}_{T,*}$. $\square$

Proposition 2.27. *Let C be a T-∞-category with finite T-products. The functor*

$$\underline{\mathrm{CMon}}_T(C) \to C\,.$$

induced by precomposition with the cocartesian section $I(-)_+ : T \to \mathbf{F}_*^T$ is an equivalence if and only if C is T-semiadditive.

Proof. If the map is an equivalence then C is T-semiadditive, since $\underline{\mathrm{CMon}}_T(C)$ is. Vice versa if C is T-semiadditive then

$$\underline{\mathrm{CMon}}_T(C) = \underline{\mathrm{Fun}}_T^{\oplus}(\underline{\mathbf{F}}_{T,*}, C) = \underline{\mathrm{Fun}}_T^{\amalg}(\underline{\mathbf{F}}_{T,*}, C) \to C$$

is an equivalence by lemma 2.26. $\square$

Corollary 2.28. *Let C be a T-∞-category with finite T-products and D a pointed T-∞-category with finite T-coproducts. Then the map*

$$\mathrm{Fun}_T^{\amalg}(D, \underline{\mathrm{CMon}}_T(C)) \to \mathrm{Fun}_T^{\oplus}(D, C)$$

is an equivalence of categories.

Proof. Observe that

$$\mathrm{Fun}^{\amalg}_{T}(D, \underline{\mathrm{CMon}}_T(C)) \cong \mathrm{Fun}^{\oplus}_{T}(D, \underline{\mathrm{CMon}}_T(C)) \cong \underline{\mathrm{CMon}}_T(\mathrm{Fun}^{\oplus}_{T}(D, C))$$

where the first equivalence comes from the T-semiadditivity of $\underline{\mathrm{CMon}}_T(C)$. Since $\mathrm{Fun}^{\oplus}_{T}(D, C)$ is T-semiadditive the thesis follows by the previous proposition. □

2.4 T-commutative monoids and Mackey functors

The notion of T-commutative monoid, while being the natural generalization of Γ-space to the parametrized setting, might seem abstract and difficult to work with. The aim of this section is that in fact T-commutative monoids are just objects very familiar in equivariant homotopy theory: Mackey functors.

Lemma 2.29. *Let $\underline{\mathbf{F}^{in}_{*}}_T$ be the T-subcategory of $\underline{\mathbf{F}}_{T,*}$ containing all objects and all the maps represented by spans in S_T*

$$I \leftarrow \tilde{I} \rightarrow I'$$

where the right arrow is an equivalence. Then a functor $M : \underline{\mathbf{F}}_{T,*} \to \mathrm{Top}$ is a T-commutative monoid if and only if its restriction to $\underline{\mathbf{F}^{in}_{*}}_T$ is a right Kan extension along $I(-)_+ : T^{\mathrm{op}} \to \mathbf{F}^{T}_{*}$.

Proof. Obvious from the limit description of right Kan extensions (see [24, Pr. 4.3.2.15]). □

Lemma 2.30. *The inclusion $j : \underline{\mathbf{F}}_{T,*} \to \underline{\mathbf{A}}^{\mathrm{eff}}(T)$ is a T-commutative monoid.*

Proof. Clear from the fact that $\underline{\mathbf{F}}_{T,*}$ contains all T-coproduct diagrams of $\underline{\mathbf{A}}^{\mathrm{eff}}(T)$ and example 2.21. □

Lemma 2.31. *Let C be an ∞-category and let E, F, D subcategories of C such that (C, E, D) and (C, E, F) are adequate triples in the sense of [14]. Consider the diagram of categories*

$$\begin{tikzcd} A^{\mathrm{eff}}(C, \iota E, D) \arrow[r, "f"] \arrow[d, "g'"] & A^{\mathrm{eff}}(C, E, D) \arrow[d, "g"] \\ A^{\mathrm{eff}}(C, \iota E, F) \arrow[r, "f'"] & A^{\mathrm{eff}}(C, E, F) \end{tikzcd}$$

Then if a right Kan extension along g' exists so does the right Kan extension along g' and the natural transformation

$$f'^{*} g_{*} \to g'_{*} f^{*}$$

is an equivalence.

Proof. Let us fix $I \in C$. We need to prove that the functor

$$A^{\text{eff}}(C, \iota E, D) \times_{A^{\text{eff}}(C,\iota E,F)} A^{\text{eff}}(C, \iota E, F)_{I/} \to A^{\text{eff}}(C, E, D) \times_{A^{\text{eff}}(C,E,F)} A^{\text{eff}}(C, E, F)_{I/}$$

is coinitial. This is equivalent to the fact that for every $J \in C$ with a map from I the category

$$\left(A^{\text{eff}}(C, \iota E, D) \times_{A^{\text{eff}}(C,\iota E,F)} A^{\text{eff}}(C, \iota E, F)_{I/}\right) \times_{\left(A^{\text{eff}}(C,E,D) \times_{A^{\text{eff}}(C,E,F)} A^{\text{eff}}(C,E,F)_{I/}\right)} \left(A^{\text{eff}}(C, E, D) \times_{A^{\text{eff}}(C,E,F)} A^{\text{eff}}(C, E, F)_{I/}\right)_{/J}$$

is weakly contractible. Let us start naming names. We have a fixed map $I \to J$ in $A^{\text{eff}}(C, E, F)$. This correspond to a span

$$I \xleftarrow{F} \tilde{J} \xrightarrow{E} J\,,$$

where we decorate every arrow with the subcategory it lives in. Now an object of our category is $T \in T$ together with an arrow in $A^{\text{eff}}(C, \iota E, F)$ from I and an arrow in $A^{\text{eff}}(C, E, D)$ to J. These correspond to spans

$$I \xleftarrow{F} T' \xrightarrow{\iota E} T \qquad \text{and} \qquad T \xleftarrow{D} \tilde{T} \xrightarrow{E} J\,.$$

The last piece of data needed is an homotopy of their composition with the given map $I \to J$, that is a diagram

$$\begin{tikzcd}
 & & J \\
 & \tilde{J} \arrow[ur, "E"] \arrow[r] \arrow[d] \arrow[dl, "F"'] & \tilde{T} \arrow[u, "E"'] \arrow[d, "D"] \\
I & T' \arrow[l, "F"] \arrow[r, "\iota E"] & T
\end{tikzcd}$$

where the central square is cartesian. But this is equivalent to the map $\tilde{J} \to \tilde{T}$ being an equivalence. Summing up, an object of our category is a factorization of $\tilde{J} \to I$

$$\tilde{J} \xrightarrow{D} T \xrightarrow{F} I\,.$$

Moreover a similar analysis on higher simplices shows that this is the opposite of the category of factorizations. It is easy to see that $\tilde{J} \xrightarrow{=} \tilde{J} \to I$ is a terminal object for this category, which is then weakly contractible. ☐

Theorem 2.32. *Let C be a T-∞-category with finite T-limits. Precomposition with the inclusion $j : \underline{\mathbf{F}}_{T,*} \to \underline{\mathbf{A}}^{\text{eff}}(T)$ induces an equivalence*

$$\underline{\text{Fun}}_T^{\times}(\underline{\mathbf{A}}^{\text{eff}}(T), C) \to \underline{\text{CMon}}_T(C)\,.$$

We denote the category on the left hand side by $\underline{\text{Mack}}^T(C)$ and call it the T-∞-category of T-Mackey functors valued in C.

Proof. Let $\underline{\mathrm{Psh}}_T(C)$ be the T-presheaf T-∞-category of C. Then, by the full faithfulness of the T-Yoneda embedding (see [5, Th. 10.4]) we have a pullback square

$$\begin{tikzcd}
\underline{\mathrm{Fun}}_T^{\times}(\underline{\mathbf{A}}^{\mathrm{eff}}(T), C) \arrow[r] \arrow[d] & \underline{\mathrm{CMon}}_T(C) \arrow[d] \\
\underline{\mathrm{Fun}}_T^{\times}(\underline{\mathbf{A}}^{\mathrm{eff}}(T), \mathrm{Psh}_T(C)) \arrow[r] \arrow[d, equal] & \underline{\mathrm{CMon}}_T(\underline{\mathrm{Psh}}_T C) \arrow[d, equal] \\
\underline{\mathrm{Fun}}_T\left(C^{\vee\mathrm{op}}, \underline{\mathrm{Fun}}_T^{\times}(\underline{\mathbf{A}}^{\mathrm{eff}}(T), \underline{\mathrm{Top}}_T)\right) \arrow[r] & \underline{\mathrm{Fun}}_T\left(C^{\vee\mathrm{op}}, \underline{\mathrm{CMon}}_T(\underline{\mathrm{Top}}_T)\right)
\end{tikzcd},$$

where $C^{\vee\mathrm{op}}$ is the fiberwise opposite of [11], so it is enough to show the thesis for $C = \underline{\mathrm{Top}}_T$.

We claim that sending every T-commutative monoid to its right Kan extension is the inverse of the restriction map. The first step is showing that the natural map

$$j_* M \circ j \to M$$

is an equivalence. By applying 2.31 with $C = F = S_T$, E the subcategory of fiberwise arrows and D the category of summand inclusions (that is the egressive maps in the definition of $\underline{\mathbf{F}}_{T,*}$) we see that it is enough to prove that the map

$$k_*(M|_{\underline{\mathbf{F}}_*^{in}{}_T}) \circ k \to M|_{\underline{\mathbf{F}}_*^{in}{}_T}$$

is an equivalence, where k is the inclusion of E^{op} in F^{op}. But this follows immediately from the fact that $M|_{\underline{\mathbf{F}}_*^{in}{}_T}$ is the right Kan extension of its restriction to T^{op}.

Hence $j_* M$ must be a product preserving functor (since the image of j contains all product diagrams in $\underline{\mathbf{A}}^{\mathrm{eff}}(T)$. Viceversa let suppose that N is a product preserving functor from $\underline{\mathbf{A}}^{\mathrm{eff}}(T)$ to C. Then there is a natural map

$$N \to j_*(N \circ j)\,.$$

But since j is essentially surjective we can check that this is an equivalence after precomposing with j, which follows immediately from the previous case. $\square$

With a similar proof it is possible to prove that if D is an ∞-category with finite products there is an equivalence

$$\mathrm{Fun}^{\times}(A^{\mathrm{eff}}(T), D) \cong \mathrm{CMon}_T(D^T)\,.$$

2.5 T-linear functors and T-stability

Recall the definition of fiberwise linear functor and fiberwise stable cocartesian fibration from section 2.1.

The following definition is inspired to hypothesis (A) of [15].

Definition 2.33. *Let C be a pointed T-∞-category with finite T-colimits and let D be a T-∞-category with finite T-limits. Then a T-functor $F : C \to D$ is T-linear if it is fiberwise linear and T-semiadditive. A T-∞-category with all finite T-limits and T-colimits is T-stable if it is fiberwise stable and T-semiadditive.*

We will denote the T-subcategory of $\underline{\mathrm{Fun}}_T(C, D)$ which on the fiber above V is spanned by $T_{/V}$-linear functors from $C \times_{T^{\mathrm{op}}} (T_{/V})^{\mathrm{op}}$ to $D \times_{T^{\mathrm{op}}} (T_{/V})^{\mathrm{op}}$ with $\underline{\mathrm{Lin}}^T(C, D)$.

Lemma 2.34. *Let D be a T-semiadditive T-∞-category. Then $\underline{\mathrm{Sp}}_T(D)$ is T-semiadditive (and hence T-stable) and the functor $\Omega^\infty : \underline{\mathrm{Sp}}_T(D) \to D$ preserves T-products (and so all T-limits).*

Proof. Recall that $\underline{\mathrm{Sp}}_T(D)$ is the cocartesian fibration classified by the functor $V \mapsto \mathrm{Sp}(D_V)$ so it is clearly fiberwise semiadditive and we just need to show that for every arrow $W \to V$ in T the pushforward functor

$$\mathrm{Sp}(D_V) \to \mathrm{Sp}(D_W)$$

has a coinciding left and right adjoints that satisfies the Beck-Chevalley condition. But since the left and right adjoint are clearly given by postcomposition of those of $D_V \to D_W$ the thesis follows. □

Definition 2.35. *Let D be a T-∞-category with all finite T-limits. Then the T-∞-category of T-spectra is*

$$\underline{\mathrm{Sp}}^T(D) = \underline{\mathrm{Sp}}_T(\underline{\mathrm{CMon}}_T(D)) .$$

By the previous lemma the latter category is T-stable

Note that there is a natural T-functor $\Omega^\infty : \underline{\mathrm{Sp}}^T(D) \to D$ given by the composition

$$\underline{\mathrm{Sp}}^T(D) = \underline{\mathrm{Sp}}_T(\underline{\mathrm{CMon}}_T(D)) \xrightarrow{\Omega^\infty} \underline{\mathrm{CMon}}_T(D) \xrightarrow{I(-)_+^*} D .$$

It is immediate by the previous lemma that it preserves all T-limits.

Theorem 2.36 (Universal property of T-spectra). *Let C be a pointed T-∞-category with finite T-colimits and D be a T-∞-category with finite T-limits. Then the functor*

$$(\Omega^\infty)_* \colon \underline{\mathrm{Fun}}_T^{T-\mathrm{rex}}(C, \underline{\mathrm{Sp}}^T(D)) \to \underline{\mathrm{Lin}}^T(C, D)$$

is an equivalence of T-∞-categories, where the source categories is the full subcategory of those functors preserving finite T-limits. In particular

$$\underline{\mathrm{Sp}}^T(D) \cong \underline{\mathrm{Lin}}_T(\underline{\mathrm{Top}}_{*T}^{fin}, D) ,$$

and the functor Ω^∞ is given by evaluation at the cocartesian section $I(-)_+ : T^{\mathrm{op}} \to \underline{\mathbf{F}}_{T,*}$.

Proof. Since the map is clearly a T-functor we just need to check that it is an equivalence fiberwise. But, remembering that finite T-colimits are generated by T-coproducts and finite fiberwise colimits, we can apply 2.28 and 2.6 and conclude

$$\mathrm{Fun}_T^{T-\mathrm{rex}}(C, \underline{\mathrm{Sp}}_T(\underline{\mathrm{CMon}}_T(D))) = \mathrm{Lin}_T^{\amalg}(C, \mathrm{CMon}_T(D)) = \mathrm{Lin}^T(C, D)\,.$$

□

Corollary 2.37. *Let D be an ∞-category with all finite limits. Then there is an equivalence*

$$\mathrm{Lin}^T(\underline{\mathrm{Top}_*}_T^{fin}, \underline{D}_T) \cong \mathrm{Fun}^{\oplus}(A^{eff}(T), \mathrm{Sp}(D))\,.$$

Proof. Both of those are equivalent to the global sections of $\underline{\mathrm{Sp}}^T(\underline{D}_T)$. □

2.6 A proof of the Guillou-May theorem

Let G be a finite group. In this appendix we will prove that our notion of G-spectra coincides with the orthogonal G-spectra developed in [27], thus reproving a theorem by Guillou and May ([20]). Fix once and for all a complete G-universe $\mathcal{U}$ (that is an isometric G-action on $\mathbb{R}^\infty$ such that every finite-dimensional representation can be isometrically embedded in $\mathbb{R}^\infty$ countably many times) and note that its restriction to a subgroup H of G is an H-universe, which we will take with indexing set given by the G-invariant subspaces. The category of orthogonal H-spectra with respect to U ([27, Df. II.2.6]) will be denoted by $\mathrm{Sp}^H_{(1)}$.

Definition 2.38. *Let $(\mathbf{O}_G)_*$ be the category of G-orbits with a distinguished basepoint but with any possible map.*[1]*, We will write an element of $(\mathbf{O}_G)_*$ as G/H where the distinguished basepoint is eH. It is clear that the functor $(\mathbf{O}_G)_* \to \mathbf{O}_G$ that forgets the basepoint is an equivalence of categories. A map $G/H \to G/K$ is the datum of $gK \in G/K$ such that $g^{-1}Hg \subseteq K$. We have a functor from $(\mathbf{O}_G)_*^{\mathrm{op}}$ to categories sending*

- *A pointed orbit G/H to the category $\mathrm{Sp}^H_{(1)}$ of orthogonal H-spectra with respect to U;*
- *A map $G/H \to G/K$ the composition of the functors*

  $$\mathrm{Sp}^K_{(1)} \to \mathrm{Sp}^{g^{-1}Hg}_{(1)} \to \mathrm{Sp}^H_{(1)}$$

  where the first functor is the restriction along the inclusion $g^{-1}Hg \subseteq K$ and the second functor is induced by the isomorphism $g^{-1}Hg \cong H$ given by conjugating by g^{-1}.

[1] Another way of thinking of this category is as the category of G-orbits together with an explicit isomorphism with an orbit of the form G/H. This is of course purely bookkeeping and has nothing to do with the use of basepoints when defining G-spectra.

If we equip every category $\mathrm{Sp}^H_{(1)}$ with the family of π_*-isomorphisms ([27, Df. III.3.2]) this becomes a functor from $(\mathbf{O}_G)^{\mathrm{op}}_*$ to the category of relative categories (since [27, Lm. V.2.2] implies that change of groups preserve π_*-isomorphisms). By precomposing with the equivalence $\mathbf{O}_G^{\mathrm{op}} \cong (\mathbf{O}_G)^{\mathrm{op}}_*$ and postcomposing with the localization functor from relative categories to ∞-categories we finally obtain a functor

$$\mathbf{O}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

that classifies a cocartesian fibration $\underline{\mathrm{Sp}}^G_{orth} \to \mathbf{O}_G^{\mathrm{op}}$. We call this cocartesian fibration the G-∞-category of orthogonal G-spectra. It comes equipped with a natural G-functor

$$\Omega^\infty : \underline{\mathrm{Sp}}^G_{orth} \to \underline{\mathrm{Top}}_G$$

induced by the natural transformation obtained by sending every orthogonal H-spectrum to its 0-th space.

Lemma 2.39. *The G-∞-category $\underline{\mathrm{Sp}}^G_{orth}$ is G-stable*

Proof. Since the fibers are obtained by localizing a stable model category at the weak equivalences Sp^G_{orth} is fiberwise stable. So we just need to check G-semiadditivity. But after unwrapping the definitions this is equivalent to the Wirthmüller isomorphism ([23, Th. II.6.2], which holds for orthogonal G-spectra by [27, Th. III.4.16]). □

We can now give a very simple proof of [20, Th. 0.1] along the outline in section 11 of [12].

Theorem 2.40 (Guillou-May). *The functor $\Omega^\infty : \mathrm{Sp}^G_{orth} \to \underline{\mathrm{Top}}_G$ lifts to an equivalence of G-∞-categories $\underline{\mathrm{Sp}}^G_{orth} \cong \underline{\mathrm{Sp}}^G$.*

Proof. Since the functor $\underline{\mathrm{Sp}}^G_{orth} \to \underline{\mathrm{Top}}_G$ preserves all finite G-limits (it has a left G-adjoint by proposition [25, Pr. 7.3.2.1]) it lifts uniquely to a functor $\Xi : \mathrm{Sp}^G_{orth} \to \mathrm{Sp}^G$.

For every orbit V the fibers $(\mathrm{Sp}^G_{orth})_V$ and $(\mathrm{Sp}^G)_V$ are both generated by suspension spectra of orbits. Moreover Ξ sends suspension spectra of orbits to suspension spectra of orbits and is fully faithful when restricted to those subcategories, since $\mathrm{Map}(\Sigma^\infty_+ G/H, \Sigma^\infty_+ G/K)$ is just $\Omega^\infty\left(\Sigma^\infty_+(G/H \times G/K)\right)^G$ and the tom Dieck splitting holds in both categories by corollary A.14 and [23, Th. V.11.1] respectively. Hence it is an equivalence by the Schwede-Shipley theorem [25, Th. 7.1.2.1]. □

From this description of G-spectra we immediately obtain a recognition principle for G-connective G-spectra

Corollary 2.41. *There is an adjuction*

$$B \dashv \Omega^\infty \colon \mathrm{CMon}_G(\underline{\mathrm{Top}}_G) \leftrightarrows \mathrm{Sp}^G$$

such that

- *the unit $X \to \Omega^\infty BX$ is an equivalence if and only X^H is a group-like monoid for every subgroup $H < G$;*
- *the counit $B\Omega^\infty E \to E$ is an equivalence if and only if E^H is connective for every subgroup $H < G$.*

Proof. After our identifications this is just the adjunction

$$\mathrm{Fun}^\times(A^{\mathrm{eff}}(G), \mathrm{Top}) \cong \mathrm{Fun}^\oplus(A^{\mathrm{eff}}(G), \mathrm{CMon}(\mathrm{Top})) \leftrightarrows \mathrm{Fun}^\oplus(A^{\mathrm{eff}}(G), \mathrm{Sp})$$

given by postcomposition with the adjunction for ordinary spectra, and the thesis follows from the classical recognition theorem. □

Chapter 3

Parametrized monoidal structures

The following chapter deals with the notion of parametrized multiplication. In the same way a symmetric monoidal ∞-category has an operation $\otimes : C^S \to C$ for each finite set S, a T-symmetric monoidal T-∞-category has one such operation for every finite T-set. This attempts to captures the intuition of [22], but in a way that is manifestly homotopy invariant, thus sidestepping the hard proofs required to show that the Hill-Hopkins-Ravenel norm can be left derived. This material (except for section 3.1 which is a brief review of [6]) will form part of [13].

The main result of this section is proposition 3.27, showing that the space of norm functors in $\underline{\mathrm{Sp}}^G$ is contractible and so that there is essentially only one way to define a homotopy invariant multiplicative norm in spectra.

3.1 Parametrized ∞-operads

We define a T-symmetric monoidal T-∞-category as a T-commutative monoid in the T-∞-category of T-∞-categories. Using ... we can identify it with a cocartesian fibration $C^\otimes \to \underline{\mathbf{F}_{T,*}}$ such that for each $I = [U \to V] \in \underline{\mathbf{F}_T}$ the map

$$\prod_{O \in \mathrm{Orbit}(U)} (\chi_{[O \subseteq U]})_! : C^\otimes_{I_+} \to \prod_{O \in \mathrm{Orbit}(U)} C^\otimes_{I(O)_+}$$

is an equivalence.

In order to have more flexibility in our construction, it will be useful to have a good notion of parametrized multicategory. The following definition is similar in spirit to definition 2.1.1.10 of [25]. Following the terminology that is now quickly becoming standard we will name our notion T-∞-operad.

Definition 3.1. *A T-∞-operad is an inner fibration $p : O^\otimes \to \underline{\mathbf{F}}_{T,*}$ such that*

- *For every inert morphism (definition 2.14) $e : I_+ \to J_+$ in $\underline{\mathbf{F}}_{T,*}$ and every $x \in O^{\otimes}_{I_+}$ there is a cocartesian morphism above e starting at x;*
- *For one (and so any) choice of pushforward functors along inert morphisms, and every $I = [U \to V] \in \underline{\mathbf{F}}_T$ the functor*

$$\prod_{W \in Orbit(U)} (\chi_{[W \subseteq U]})_! : C^{\otimes}_{I_+} \to \prod_{W \in Orbit(U)} C^{\otimes}_{I(W)_+}$$

  is an equivalence of categories.
- *For one (and so any) choice of pushfoward functors along inert morphisms and for every map $f : I_+ \to J_+$ in $\underline{\mathbf{F}}_{T,*}$ and every $x \in O^{\otimes}_{I_+}$ and $y \in O^{\otimes}_{J_+}$ the map*

$$\mathrm{Map}^{f}_{O^{\otimes}}(x,y) \to \prod_{W \in \mathrm{Orbit}(U)} \mathrm{Map}^{\chi_{[W \subseteq U]} f}_{O^{\otimes}} \left(x, (\chi_{[W \subseteq U]})_! y\right)$$

  is an equivalence, where $J = [U \to V]$ and $\mathrm{Map}^f(x,y)$ is the fiber over f of the map

$$\mathrm{Map}_{O^{\otimes}}(x,y) \to \mathrm{Map}_{\underline{\mathbf{F}}_{T,*}}(I_+, J_+)\,.$$

Let $p : O^{\otimes} \to \underline{\mathbf{F}}_{T,*}$ be a T-∞-operad. We say that an edge e is ***inert*** *if it is cocartesian and pe is inert. If $O^{\otimes}$ and $E^{\otimes}$ are two T-∞-operads, a map of T-∞-operads is a map of inner fibrations $g : O^{\otimes} \to E^{\otimes}$ sending inert edges to inert edges. A map of T-∞-operads $f : O^{\otimes} \to E^{\otimes}$ is also sometimes called an $O^{\otimes}$-algebra in $E^{\otimes}$ (especially when $E^{\otimes}$ is a T-symmetric monoidal T-∞-category).*

Example 3.2. *Every T-symmetric monoidal T-∞-category is a T-∞-operad.*

Example 3.3. *The identity $\underline{\mathbf{F}}_{T,*} \to \underline{\mathbf{F}}_{T,*}$ is a T-∞-operad. In fact it is the terminal T-∞-operad and it is called the* ***T-commutative*** *T-∞-operad. Algebras over $\underline{\mathbf{F}}_{T,*}$ are called T-commutative algebras.*

Warning 3.4. *One might imagine that if $C^{\otimes} \to \mathbf{F}_*$ is a symmetric monoidal ∞-category, the T-∞-category of T-objects $\underline{C}_T$ inherits a natural T-∞-symmetric monoidal structure. This is not in general true. For example, let $T = \mathbf{O}_G$ be the orbit category of a finite group and let us consider the symmetric monoidal ∞-category of spectra Sp. Then the associated G-∞-category $\underline{\mathrm{Sp}}_G$ is the G-∞-category of naive G-spectra. As we will see the presence of a G-symmetric monoidal structure would equip naive G-spectra with norm functors and so $S^{\rho} = N^G(S^1)$ would be invertible. But this is well-known to be false for naive G-spectra.*

If $p : O^{\otimes} \to \underline{\mathbf{F}}_{T,*}$ is a T-∞-operad, we will call the pullback $O^{\otimes} \times_{\underline{\mathbf{F}}_{T,*}} T^{\mathrm{op}}$ along $I(-)_+ : T^{\mathrm{op}} \to \underline{\mathbf{F}}_{T,*}$ the **underlying T-∞-category** of $O^{\otimes}$ and denote it by O. As the name suggests, it is a T-∞-category since the image of $I(-)_+$ lands in the subcategory of inert arrows.

Note that if $C^{\otimes}$ is a T-symmetric monoidal T-∞-category, we can form the fiberwise opposite $(C^{\otimes})^{\vee\mathrm{op}}$, which will still be a T-symmetric monoidal T-∞-category with underlying T-∞-category the fiberwise opposite of C, $C^{\vee\mathrm{op}}$.

Let us now see some of the structure that a T-symmetric monoidal T-∞-category possess. For every $V \in T$ let $I = [V \amalg V \to V] \in \mathbf{F}_T$ and let us consider the fold map $\nabla : I_+ \to I(V)_+$ given by the diagram:

$$\begin{tikzcd}
V \amalg V \arrow[d] & V \amalg V \arrow[l] \arrow[r] \arrow[d] & V \arrow[d] \\
V \arrow[r, equal] & V, \arrow[r, equal] & V
\end{tikzcd}$$

Then the pushforward along ∇ induces a functor

$$\otimes : C_V \times C_V \cong C^{\otimes}_{I_+} \to C^{\otimes}_{I(V)_+} \cong C_V \qquad (x, y) \mapsto x \otimes y\,,$$

which will be called the **tensor product**.

If we instead start with an arrow $e : V \to W$ in T, we can consider the arrow in $\underline{\mathbf{F}}_{T,*}$ given by the diagram

$$\begin{tikzcd}
V \arrow[r, equal] \arrow[d, "e"] & V \arrow[r, "e"] \arrow[d, "e"] & W \arrow[d, equal] \\
W \arrow[r, equal] & W \arrow[r, equal] & W
\end{tikzcd}$$

Pushforward along this arrow induces a functor

$$N^e : C_V \cong C^{\otimes}_{[V \to W]_+} \to C^{\otimes}_{I(W)_+} \cong C_W$$

which we will call the **norm along** e. When there is no ambiguity we will also denote this functor N_V^W and call it the norm from V to W.

The following proposition explains the behaviour of multiplicative structures under localization.

Proposition 3.5. *Let $O^{\otimes} \to \underline{\mathbf{F}}_{T,*}$ be a T-∞-operad and let $E \subseteq O$ be a reflective T-subcategory (definition 1.14). Define $E^{\otimes}$ to be the full subcategory of $O^{\otimes}$ spanned by those objects $x \in O^{\otimes}_{I_+}$ that lie in the subcategory $\prod_W E_{I(W)_+}$ under the identification $O^{\otimes}_{I_+} = \prod_W O_{I(W)_+}$. Then $E^{\otimes} \to \underline{\mathbf{F}}_{T,*}$ is a T-∞-operad.*

If $O^{\otimes}$ is a T-symmetric monoidal T-∞-category and the localization $E \subseteq O$ satisfies the following condition:

- *For every $V \in T$ and every object $x \in O_V$ the functor $x \otimes - : O_V \to O_V$ preserves E_V-equivalences;*
- *For every arrow $e : V \to V'$ in T the functor $N^e : O'_V \to O_V$ sends $E_{V'}$-equivalences to E_V-equivalences.*

Then $E^{\otimes}$ is a T-symmetric monoidal T-∞-category and the localization functor $C \to E$ is T-symmetric monoidal. In this case we say that the localization is compatible with the T-symmetric monoidal structure.

Proof. The fibration $E^{\otimes} \to \underline{\mathbf{F}}_{T,*}$ satisfies the conditions in the definition of T-∞-operad, since it is a full subcategory of $O^{\otimes}$, contains all inert arrows and the following square is a pullback square

$$\begin{tikzcd}
\mathrm{Map}^{f}_{O^{\otimes}}(x,y) \arrow[r] \arrow[d] & \prod_{W \in \mathrm{Orbit}(U)} \mathrm{Map}^{\chi_{[W\subseteq U]}f}_{O^{\otimes}}\left(x, (\chi_{[W\subseteq U]})_! y\right) \arrow[d] \\
\mathrm{Map}^{f}_{E^{\otimes}}(x,y) \arrow[r] & \prod_{W \in \mathrm{Orbit}(U)} \mathrm{Map}^{\chi_{[W\subseteq U]}f}_{E^{\otimes}}\left(x, (\chi_{[W\subseteq U]})_! y\right)
\end{tikzcd}.$$

So now let us assume that $O^{\otimes}$ is a T-symmetric monoidal T-∞-category and that the localization is compatible with the T-symmetric monoidal structure. We need to prove that $p : E^{\otimes} \to \underline{\mathbf{F}}_{T,*}$ is a cocartesian fibration. By lemma [25, 2.2.4.11] it is enough to see that for every $f : I_+ \to J_+$ in $\underline{\mathbf{F}}_{T,*}$ the pushforward functor

$$f_! : O^{\otimes}_{I_+} \to O^{\otimes}_{J_+}$$

sends $E^{\otimes}_{I_+}$-equivalences to $E^{\otimes}_{J_+}$-equivalences. First of all, we can assume without loss of generality that $J_+ = I(V)_+$. This is because $O^{\otimes}_{J_+} = \prod_V O^{\otimes}_{I(V)_+}$ and we can check the E-equivalences on each component separately. So now let f be represented by the diagram

$$\begin{tikzcd}
U \arrow[d] & T \arrow[l] \arrow[r] \arrow[d] & V \arrow[d, equal] \\
W & V \arrow[l] \arrow[r, equal] & V
\end{tikzcd}.$$

And let i be the arrow

$$\begin{tikzcd}
U \arrow[d] & T \arrow[l] \arrow[r] \arrow[d] & T \arrow[d] \\
W & V \arrow[l] \arrow[r, equal] & V
\end{tikzcd},$$

n be the arrow

$$\begin{tikzcd}
T \arrow[r, equal] \arrow[d] & T \arrow[r] \arrow[d] & \mathrm{Orbit}(T) \times V \arrow[d] \\
V \arrow[r, equal] & V \arrow[r, equal] & V
\end{tikzcd},$$

where the top right map is the coproduct of $O \to V$ where O runs through the orbits of T, and m be the arrow

$$\begin{tikzcd} \mathrm{Orbit}(T) \times V \arrow[r, equal] \arrow[d] & \mathrm{Orbit}(T) \times V \arrow[r] \arrow[d] & V \arrow[d] \\ V \arrow[r, equal] & V \arrow[r, equal] & V \end{tikzcd},$$

where the top right map is the fold map. Then $f = mni$, so $f_! = m_! n_! i_!$. Now $i_!$ preserves E-equivalences since i is inert and so $i_!$ is just a projection onto some factors, $m_!$ preserves the equivalences since it is a composition of functors of the form $x \otimes -$ and $n_!$ preserves the E-equivalences since it is a product of functors of the form N^e. $\square$

We will need the following technical result, the proof of which can be found in [29].

Theorem 3.6. *Let C be a T-∞-category. Then there is a T-∞-operad $C^{\amalg}$ such that for all $f : I_+ \to I(V)_+$ in $\underline{\mathbf{F}}_{T,*}$ and $x \in C^{\amalg}_{I_+}$, $y \in C_V$*

$$\mathrm{Map}^f_{C^{\amalg}}(x, y) = \prod_{W \in \mathrm{Orbit}(I)} \mathrm{Map}(x_W, \delta^V_W y)\,.$$

Moreover if C has all finite T-coproducts, $C^{\amalg}$ is a T-symmetric monoidal T-∞-category.

We will call $C^{\amalg}$ the cocartesian T-∞-operad of C. If C has finite T-products we can form $C^{\times} = \left((C^{\mathrm{Vop}})^{\amalg}\right)^{\mathrm{Vop}}$ and call it the cartesian T-symmetric monoidal structure.

3.2 Idempotents

Recollection 3.7. *Recall ([25, Df. 4.8.2.1]) that if $C^{\otimes}$ is a symmetric monoidal ∞-category, an idempotent is an object $(\eta : 1 \to A) \in C_{1/}$ such that the map*

$$A \xrightarrow{1_A \otimes \eta} A \otimes A$$

is an equivalence. A commutative algebra A is idempotent if and only if the multiplication map $A \otimes A \to A$ is an equivalence. Then the forgetful functor from idempotent algebras to idempotent objects is an equivalence and both ∞-categories are equivalent to posets (that is the mapping spaces are either contractible or empty) ([25, Pr. 4.8.2.9]). This poset is obviously equivalent to the poset of smashing localizations in C (that is those localizations $L : C \to C$ such that $L1 \otimes X \to LX$ is an equivalence)

We want to find a suitable notion of smashing localization for T-symmetric monoidal T-∞-categories.

Definition 3.8. *Let $C^{\otimes}$ be a T-symmetric monoidal ∞-category. Then a T-commutative algebra $A \in \mathrm{CAlg}_T(C^{\otimes})$ is* ***idempotent*** *if for every $V \in T$ the commutative algebra $A_V \in \mathrm{CAlg}(C^{\otimes}_V)$ is an idempotent algebra. Moreove a weakly idempotent algebra is* ***strongly idempotent*** *if for every $f : V \to V'$ in T the norm map $N^f A_V \to A_{V'}$ is an equivalence.*

The previous definition captures the intuition of what a smashing localization should look like in the parametrized setting, but it is hard to specify. Luckily the correspondence between idempotent algebras and idempotent objects carries through in our setting.

Definition 3.9. *Let $C^{\otimes}$ be a T-symmetric monoidal ∞-category and let $A : T^{\mathrm{op}} \to C$ be a cocartesian section with a map from the cocartesian section $1 : T^{\mathrm{op}} \to C$. Then we say that A is a* ***idempotent*** *if for every $f : V \to V'$ in T^{op} the map*

$$1_{A_{V'}} \otimes N^f \eta : A_{V'} \to A_{V'} \otimes N^f A_V$$

is an equivalence. Moreover a T-idempotent $(\eta : 1 \to A)$ is a ***strong idempotent*** *if for every $f : V \to V'$ the map*

$$\eta \otimes 1_{N^f A_V} : N^f A_V \to A_{V'} \otimes N^f A_V \cong A_{V'}$$

is an equivalence.

Example 3.10. *If A is an idempotent T-commutative algebra, the unit map $1 \to A$ is an idempotent. Moreover this is a strong idempotent if and only if A is a strongly idempotent T-commutative algebra. As we will see all idempotents arise this way.*

Lemma 3.11. *Let $(\eta : 1 \to A)$ be an idempotent. Then the T-functor $A \otimes - : C \to C$ is a T-symmetric monoidal localization, whose local objects are those $B \in C$ such that $B \to A \otimes B$ is an equivalence. In particular A has a natural T-commutative algebra structure and the inclusion of A-local objects into C is T-symmetric monoidal.*

Proof. Since $\eta_V : 1_V \to A_V$ is an idempotent object for every $V \in T^{\mathrm{op}}$, then $A \otimes -$ is a localization by [25, Pr. 4.8.2.4]. We only need to check that it is T-symmetric monoidal. The only nontrivial part is to check that if $A_V \otimes g$ is an equivalence, then $A_W \otimes N^f g$ is an equivalence. But

$$A_W \otimes N^f g \cong A_W \otimes N^f A_V \otimes N^f g \cong A_W \otimes N^f (A_V \otimes g)$$

and so it is an equivalence.

□

Theorem 3.12. *The forgetful functor from idempotent algebras to idempotent objects is an equivalence and both categories are equivalent to posets. Under this equivalence an algebra is strongly idempotent if and only if the underlying idempodent object is a strong idempotent. Moreover any T-idempotent object $\eta : 1 \to A$ has a unique T-commutative algebra structure where η is the unit, which is given by the corresponding T-idempotent algebra.*

Proof. By the previous lemma we know that the forgetful functor is essentially surjective, so it is enough to prove that it is fully faithful. We will prove that both ∞-categories are posets (that is the mapping spaces are either empty or contractible) and that in both categories there exists a map from A to A' if and only if $A' \to A \otimes A'$ is an equivalence.

Let A, A' be two idempotent objects and assume $\mathrm{Map}_{C_{1/}}(A, A')$ is nonempty. That is we can find a commutative diagram

$$\begin{tikzcd} & A \arrow[d, "f"] \\ 1 \arrow[ur, "\eta"] \arrow[r, "\eta'"'] & A' \end{tikzcd} .$$

If we tensor the diagram by A' we get that A' is a retract of $A \otimes A'$, hence it is A-local. But the map $\eta : 1 \to A$ is an A-equivalence and so

$$\mathrm{Map}_{C_{1/}}(A, A') \cong \mathrm{Map}_{C_{1/}}(1, A') = *$$

as required.

Now let A, A' be two idempotent algebras and suppose that there is a map $f : A \to A'$. Then, as before, A' is an A-local object (for the T-symmetric monoidal localization induced by A). So A' is a T-commutative algebra in the T-symmetric monoidal T-∞-category of A-local objects and so the space of maps from the unit is contractible.

Finally let A be an idempotent T-commutative algebra, and let us suppose that A' is another T-commutative algebra structure with the same unit. But A' is clearly an A-local T-commutative algebra, and so it is a T-commutative algebra in the T-symmetric monoidal T-∞-category of A-local objects where the unit map is an equivalence. So it has to be equivalent to A as a T-commutative algebra. □

3.3 Distributive functors

Recollection 3.13. *Let $U \in \mathbf{F}_T$ be a finite T-set. Then the category of points (example 1.5) of U is the left fibration $\mathbf{U} \to T^{\mathrm{op}}$ classified by U seen as a functor $T^{\mathrm{op}} \to \mathrm{Top}$. Now taking opposite we obtain a right fibration $\mathbf{U}^{\mathrm{op}} \to T$ over an orbital ∞-category. But then $\mathbf{U}^{\mathrm{op}}$ is again an orbital ∞-category (example 1.3). Moreover, by theorem 1.12 there is an equivalence of ∞-categories*

$$\mathrm{Cat}_{\infty, \mathbf{U}^{\mathrm{op}}} = \mathrm{Fun}(\mathbf{U}, \mathrm{Cat}_\infty) \cong \mathrm{Fun}_T(\mathbf{U}, \underline{\mathrm{Cat}}_{\infty, T})$$

We will often write $\mathrm{Cat}_{\infty,\mathbf{U}}$ *instead of* $\mathrm{Cat}_{\infty,\mathbf{U}^{\mathrm{op}}}$ *for simplicity.*

Construction 3.14. *Let $f : U \to U'$ be a map of finite T-sets. Then there is a functor*

$$f^* : \mathrm{Cat}_{\infty,\mathbf{U}'} = \mathrm{Fun}_T(\mathbf{U}', \underline{\mathrm{Cat}}_{\infty,T}) \to \mathrm{Fun}_T(\mathbf{U}, \underline{\mathrm{Cat}}_{\infty,T}) = \mathrm{Cat}_{\infty,\mathbf{U}}\,.$$

Since $\underline{\mathrm{Cat}_{\infty T}}$ is a T-∞-category with finite T-products, proposition 1.19 implies that this functor has a right adjoint $\prod_f$. That is for every $\mathbf{U}$-∞-category C there is an $\mathbf{U}'$-∞-category $\prod_f C$ such that for every $\mathbf{U}'$-∞-category D

$$\underline{\mathrm{Fun}}_{\mathbf{U}'}\left(D, \prod_f C\right) \cong \underline{\mathrm{Fun}}_{\mathbf{U}}\left(D|_{\mathbf{U}}, C\right)$$

This is an $\mathbf{U}'$-∞-category with fiber over V given by

$$\mathrm{Fun}_{\mathbf{U}}(\mathbf{U}_{\mathbf{V}}, C) \cong \prod_{O \in \mathrm{Orbit}(U \times_{U'} V)} C_O$$

where $\mathbf{U}_{\mathbf{V}}$ is the model for the category of points of $U \times_{U'} V$ whose fiber over $[W \to U]$ is given by the space of commutative squares in $\mathbf{F}_T$

$$\begin{tikzcd} W \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & U' \end{tikzcd}.$$

For example if $T = \mathbf{O}_G$ and $f : G/e \to G/G$ for any ∞-category C, the fiber of the G-∞-category $\prod_f C$ over an orbit V is C^V, that is the product of $\#V$ copies of C.

Recall that a functor $F : \prod_{i \in I} C_i \to D$ between ∞-categories is said to preserve colimits separately in each variable, if for every collection of colimit diagrams $\{p_i : K_i^{\triangleright} \to C_i\}_{i \in I}$ the diagram

$$\left(\prod_{i \in I} K_i\right)^{\triangleright} \to \prod_{i \in I} (K_i^{\triangleright}) \xrightarrow{\prod p_i} \prod_{i \in I} C_i \xrightarrow{F} D$$

is a colimit diagram. That is, if

$$\operatorname*{colim}_{(k_i) \in \prod_i K_i} F(p_i(k_i))_{i \in I} \cong F\left(\operatorname*{colim}_{k_i \in K_i} p_i(k_i)\right)_{i \in I}.$$

The following definition attempts to capture the same behaviour for indexed products.

Definition 3.15. *Let $f : U \to U'$ be a map of finite T-sets and let $C \to \mathbf{U}$ and $D \to \mathbf{U}'$ be respectively an $\mathbf{U}$-∞-category and an $\mathbf{U}'$-∞-category. Then an $\mathbf{U}'$-functor $F : \prod_f C \to D$ is said to be distributive if for every $[V \to U'] \in \mathbf{U}'$, every $\mathbf{U_V}$-∞-category K and every $\mathbf{U_V}$-colimit diagram*

$$\bar{p} : K \star_{\mathbf{U_V}} \mathbf{U_V} \to C|_{\mathbf{U_V}}$$

the diagram

$$\left(\prod_f K\right) \star_{\mathbf{V}} \mathbf{V} \to \prod_f (K \star_{\mathbf{U_V}} \mathbf{U_V}) \xrightarrow{\prod \bar{p}} \prod_f C|_{\mathbf{U_V}} \xrightarrow{F} D|_{\mathbf{V}}$$

is a $\mathbf{V}$-colimit diagram. We will denote the full subcategory spanned by distributive functors with $\mathrm{Fun}^{\delta}_{\mathbf{U}'}(\prod_f C, D)$

Example 3.16. *Let $T = \Delta^0$. Then a functor $F : \prod_{u\in U} C_u \to D$ is distributive if and only if it preserves colimits separately in each variable.*

Example 3.17. *Let $T = \mathbf{O}_{C_2}$ and let C be an ∞-category, D be a C_2-∞-category and $F : \prod_{C_2/e \to C_2/C_2} C \to D$ be a distributive C_2-functor. Then its component over C_2/C_2 is a functor $F_{C_2/C_2} : C \to D_{C_2/C_2}$ such that*

$$F_{C_2}(X \amalg Y) \cong F_{C_2/C_2}(X) \amalg \coprod_{C_2} F_{C_2/e}(X, Y) \amalg F_{C_2/C_2}(Y)\,.$$

Lemma 3.18. *Let C be an ∞-category with finite products such that the products commute with colimits separately in each variable. Then the product functor*

$$\mu : \prod_f C^{\mathbf{U}} \to C^{\mathbf{U}'}$$

is a distributive functor.

Proof. Thanks to the universal property of the category of $\mathbf{U}'$-objects μ can be identified with a functor

$$\prod_f C^{\mathbf{U}} \to C$$

that, when restricted to the fiber above $[W \to U'] \in \mathbf{U}'$ is the functor

$$\prod_{V \in \mathrm{Orbit}(U \times_{U'} W)} \mathrm{Fun}(\mathbf{V}, C) \xrightarrow{\prod ev_V} \prod_{V \in \mathrm{Orbit}(U \times_{U'} W)} C \xrightarrow{\times} C\,.$$

In the proof of the distributivity we can assume without loss of generality that U' is an orbit and choose an $\mathbf{U}$-colimit diagram

$$p : K \star_{\mathbf{U}} \mathbf{U} \to C$$

Remember that such a diagram is $\mathbf{U}$-colimit if and only if for every $[V \to U] \in \mathbf{U}$ the restriction to the fiber $p_V : K^{\triangleright}_{[V\to U]} \to C$ is a colimit diagram. Then we need to prove that the diagram

$$\left(\prod_f K\right) \star_{\mathbf{U}'} \mathbf{U}' \to \prod_f C^{\mathbf{U}} \to C$$

is a colimit diagram. But the restriction to the fiber above $[W \to U'] \in \mathbf{U}'$ is

$$\left(\prod_{V\in \mathrm{Orbit}(U\times_{U'} W)} K_V\right)^{\triangleright} \to \prod_{V\in \mathrm{Orbit}(U\times_{U'} W)} K_V^{\triangleright} \xrightarrow{\prod p_V} \prod_{V\in \mathrm{Orbit}(U\times_{U'} W)} C \xrightarrow{\times} C$$

which is a colimit diagram since the cartesian product in C commutes with colimits separately in each variable. □

Proposition 3.19. *Let $f : U \to U'$ be a map of finite T-sets, C be a small $\mathbf{U}$-∞-category and D a cocomplete $\mathbf{U}'$-∞-category. Then the precomposition with the normed-up Yoneda embedding $\prod_f j : \prod_f C \to \prod_f P_T(C)$*

$$\mathrm{Fun}^{\delta}_{\mathbf{U}}\left(\prod_f P_{\mathbf{U}}(C), D\right) \to \mathrm{Fun}_{\mathbf{U}'}\left(\prod_f C, D\right)$$

is an equivalence.

Proof. We claim that the left Kan extension along $\prod_f j$ is an inverse. Since proposition 1.21 implies that $\prod_f j$ is fully faithful (since that can be checked fiberwise), the restriction of the left Kan extension is the original functor.

First let us show that if $F : \prod_f P_T(C) \to D$ is a distributive functor, it is the left Kan extension of its own restriction. In fact if $x \in \prod_f C$ we can write, thanks to proposition 1.19 and proposition 1.21

$$\left(\prod_f j\right)_! F(x) = \operatorname*{colim}_{y\in \prod_f C\times_{\prod_f P_{\mathbf{U}}(C)} \left(\prod_f P_{\mathbf{U}}(C)\right)_{/_{\mathbf{U}'} jx}} F(y) \cong$$

$$\cong \operatorname*{colim}_{y\in\prod_f \left(C\times_{P_{\mathbf{U}}(C)} P_{\mathbf{U}}(C)_{/_{\mathbf{U}} jx}\right)} F(y) \cong F\left(\operatorname*{colim}_{y\in C\times_{P_{\mathbf{U}}(C)} P_{\mathbf{U}}(C)_{/_{\mathbf{U}} x}}\right) \cong$$

$$\cong (F \circ j_! j)(x) = F(x)\,.$$

So we are done if we show that for every functor $F : \prod_f C \to D$ the left Kan extension $(\prod j)_! F$ is distributive. We will first show it when F is the Yoneda embedding $j : \prod_f C \to P_T\left(\prod_f C\right)$. Since colimits in presheaf categories are detected by the evaluation functors, pick $X \in \left(\prod_f C\right)_{[W \to U']} \cong \mathrm{Fun}_{\mathbf{U}}(\mathbf{U}_W, C)$, and consider the functor

$$ev_X \circ \left(\prod j\right)_! j \cong \left(\prod j\right)_! \mathrm{Map}(X, -) : \prod_{fw} P_{\mathbf{U_W}}(C|_{\mathbf{U_W}}) \to \mathrm{Top}^{\mathbf{W}} .$$

Since every distributive functor is the left Kan extension of its restriction it is enough to exhibit a distributive functor whose restriction is $\mathrm{Map}(X, -)$. This is the following composition:

$$\prod_{fw} P_T(C)|_{\mathbf{U_W}} \xrightarrow{\prod ev_X} \prod_{fw} \mathrm{Top}^{U_W} \xrightarrow{\mu} \mathrm{Top}^W .$$

The composition is distributive since ev_X is colimit preserving and μ is distributive.

Now let us treat the case of a general functor $F : \prod_f C \to D$. Since colimits are detected by representable functors, it is enough to show that for every $d \in D_{[W \to U']}$ the functor

$$\mathrm{Map}\left(\left(\prod j\right)_! F-, d\right) \cong \left(\prod j\right)_! \mathrm{Map}(F-, d) : \prod_{fw} P_{\mathbf{U_W}}(C|_{\mathbf{U_W}}) \to (\mathrm{Top}^{U'})^{\mathrm{vop}}$$

is distributive. Again it is enough to exhibit a distributive functor whose restriction to $\prod_{fw} C|_{\mathbf{U_W}}$ is $\mathrm{Map}(F-, d)$. This is the composition

$$\prod_{fw} P_T(C)|_{\mathbf{U_W}} \xrightarrow{(\prod j)_! j} P_T\left(\prod_{fw} C|_{\mathbf{U_W}}\right) \xrightarrow{\mathrm{Nat}(-, \mathrm{Map}(F-, d))} (\mathrm{Top}^{U'})^{\mathrm{Vop}} .$$

This is distributive because $(\prod j)_! j$ is distributive by what shown above and $\mathrm{Nat}(-, \mathrm{Map}(F-, d))$ is representable, hence colimit preserving. $\square$

In order to study how distributive functors interact with Bousfield localizations we will need an auxiliary definition.

Definition 3.20. *Let $F : \prod_f C \to D$ be a functor. Then for every $[V \to U] \in \mathbf{U}$ we say that an arrow e in C_V is an F-equivalence if for every diagram*

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ W \arrow[r] & U' \end{tikzcd}$$

such that $V \to U \times_{U'} W$ is a summand inclusion and every $c \in \prod_{V \neq O \in \mathrm{Orbit}(U \times_{U'} W)} C_O$ the arrow $(e, 1_c)$ of $\prod_{O \in \mathrm{Orbit}(U \times_{U'} W)} C_O = \left(\prod_f C\right)_W$ is sent to an equivalence in D_W by F_W.

Example 3.21. *If $S \subseteq C$ is the $\mathbf{U}$-∞-subcategory of C generated by the F-equivalences, then F sends every arrow of $\prod_f S \subseteq \prod_f C$ to a cocartesian arrow of D. In fact every arrow of $\prod_f S$ can be written as the composition of a cocartesian arrow and a sequence of arrows of the form $(e, 1_c)$ where e is an arrow of S_V for some V.*

Lemma 3.22. *If F is a distributive, F-equivalences form a strongly saturated collection in the sense of [24, Df.5.5.4.5].*

Proof. We need to prove that F-equivalences are closed under colimits, pushouts and that they satisfy the 2-out-of-3 property. The latter fact is obvious.

Now let suppose that $e = \mathrm{colim}_{i \in K} e_i$ for some diagram p in $\mathrm{Fun}(\Delta^1, C_V)$, where e_i is F-good for every $i \in K$. Fix W and c as in the definition of F-equivalence. We need to prove that $F(e, 1_c)$ is an equivalence in D_W knowing that $F(e_i, 1_c)$ is one. We can form two $\mathbf{U_W}$-diagrams in

$$C|_{\mathbf{U_W}} \cong \coprod_{O \in \mathrm{Orbit}(U \times_{U'} W)} C|_{\mathbf{O}}$$

indexed by $K_+ = (K \times \mathbf{V}) \amalg (\mathbf{U_W} \smallsetminus \mathbf{V})$ that is a cocartesian lift of $p|_0$ and $p|_1$ over $K \times \mathbf{V}$ and the constant diagram at c on the other component. Then e can be seen as a natural transformation between those two $\mathbf{U_W}$-colimit diagrams such that over every component it is an F-equivalence. Then, by applying the definition of distributive functor, we see that $F(e, 1_c)$ is the component over the cone point of a natural transformation between two $\mathbf{V}$-colimit diagrams such that all other components are equivalenes. Hence $F(e, 1_c)$ is an equivalence.

It is only left to prove that F-equivalences are closed under pushouts. Let $\bar{p} : (\Lambda^2_0)^{\triangleright} \to C_V$ be a pushout square such that $\bar{p}([0,1])$ is an F-equivalence. Fix W and c as in the definition of F-equivalence. We can identify $\bar{p}$ with its cocartesian lift $(\Lambda^2_0 \times \mathbf{V}) \star_{\mathbf{V}} \mathbf{V} \to C$ and let us consider the diagram

$$\bar{p} \amalg c : (\Lambda^2_0 \times \mathbf{V} \amalg (\mathbf{U_W} \smallsetminus \mathbf{W})) \star_{\mathbf{U_W}} \mathbf{U_W} \to C|_{\mathbf{U_W}}.$$

This is an $\mathbf{U_W}$-colimit diagram, so, since F is distributive, the diagram

$$\prod_f \left(\Lambda^2_0 \times \mathbf{V} \amalg (\mathbf{U_W} \smallsetminus \mathbf{W})\right) \star_{\mathbf{W}} \mathbf{W} \to D|_{\mathbf{W}}$$

is a $\mathbf{W}$-colimit diagram. Note that the fiber of the indexing diagram over $[W' \to W]$ is $\prod_{O \in \mathrm{Orbit}(V \times_W W')} \Lambda^2_0$. Moreover all arrows whose components are copies of $\Delta^{[0,1]}$ times identities are sent to equivalences in $D_{\mathbf{W}'}$ by hypothesis. So the inclusion of the vertex $(2)_{O \in \mathrm{Orbit}(V \times_W W')}$ is right marked anodyne. This means that the diagram

$$\mathbf{W} \star_{\mathbf{W}} \mathbf{W} \to \prod_f \left(\Lambda^2_0 \times \mathbf{V} \amalg (\mathbf{U_W} \smallsetminus \mathbf{W})\right) \star_{\mathbf{W}} \mathbf{W} \to D|_{\mathbf{W}}$$

is still a colimit diagram. But this is exactly the arrow that we wanted to prove to be an equivalence.

□

Proposition 3.23. *Let $f : U \to U'$ be a map of finite T-sets and let C and D be presentable $\mathbf{U}$-∞-category and $\mathbf{U}'$-∞-category respectively. Let $S \subseteq C$ be a small T-∞-subcategory and $L_S C$ the associated Bousfield localization of C (that is the T-∞-subcategory spanned by the S-local objects). Then precomposition with $\prod_f L$ where L is the localization functor $L : C \to L_S C$, induces a fully faithful embedding*

$$\mathrm{Fun}^{\delta}_{\mathbf{U}'}\left(\prod_f L_S C, D\right) \to \mathrm{Fun}^{\delta}_{\mathbf{U}'}\left(\prod_f C, D\right)$$

whose essential image are those distributive functors sending all arrows of $\prod_f S$ to cocartesian arrows of D.

Proof. Let $i : L_S C \to C$ be the inclusion. Since i and L form an $\mathbf{U}$-adjunction such that the counit $Li \to \mathbf{1}$ is an equivalence, it can be verified fiberwise that $\prod_f i$ and $\prod_S L$ form a similar $\mathbf{U}'$-adjunction. Then, since $\prod_f i$ is fully faithful, it follows that precomposition by $\prod_f L$ is fully faithful with image those functors that send the unit $1 \to \prod_f (iL)$ to an equivalence. It is immediate to verify that those are exactly the functors F for which over every orbit V L_V-equivalences are F-equivalences. But since the collection F-equivalences is strongly saturated and L_V-equivalences are the strong saturation of S_V this is equivalent to asking that S_V is composed by F-equivalences. □

3.4 Tensor products of presentable T-∞-categories

Definition 3.24. *Let us consider the cartesian T-symmetric monoidal structure on T-∞-categories $(\mathrm{Cat}^T_\infty)^\times$. It is a cartesian fibration over $\mathbf{F}^T_*$ such that the fiber over $[U \to V]$ is the ∞-category of $\mathbf{U}$-∞-categories and such that if $[U \leftarrow gW \to fU']$ is an arrow in $\mathbf{F}^T_*$ the mapping space of arrows above f is*

$$\mathrm{Map}^f(C, D) \cong \mathrm{Map}\left(\prod_f C|_{\mathbf{W}}, D\right)$$

where the mapping space in the right hand side is in the category of $\mathbf{U}'$-categories. We let $\mathrm{Pr}^\otimes_T$ be the subcategory of $(\mathrm{Cat}^T_\infty)^\times$ whose objects are presentable $\mathbf{U}$-T-categories for all $[U \to V] \in \mathbf{F}^T_*$ and whose mapping space is the subspace of those functors $\prod_f C|_{\mathbf{W}} \to D$ that are distributive. We call $\mathrm{Pr}^\otimes_T$ (together with the restriction of the projection to $\mathbf{F}^T_*$) the T-symmetric monoidal T-∞-category of presentable T-∞-categories.

In order to justify the name we have to check that it is in fact a T-symmetric monoidal T-∞-category. Let's start by checking that it is a T-∞-operad. Since it is a subcategory of the source of an inner fibration the projection is still an inner fibration. Moreover the cocartesian arrows above inert arrows are contained in the subcategory (since the identity functor is colimit-preserving). Finally the square

$$\begin{tikzcd} \mathrm{Map}^f_{\mathrm{Pr}}(C,D) \cong \mathrm{Map}^\delta_{\mathbf{U}'}\left(\prod_f C, D\right) \arrow[r] \arrow[d, hook] & \prod_{W\in\mathrm{Orbit}(U')} \mathrm{Map}^\delta_{\mathbf{W}}\left(\prod_{f_W} C|_{\mathbf{U_W}}, D|_{\mathbf{W}}\right) \arrow[d, hook] \\ \mathrm{Map}^f_{\mathrm{Cat}}(C,D) \cong \mathrm{Map}_{\mathbf{U}'}\left(\prod_f C, D\right) \arrow[r] & \prod_{W\in\mathrm{Orbit}(U')} \mathrm{Map}_{\mathbf{W}}\left(\prod_{f_W} C|_{\mathbf{U_W}}, D|_{\mathbf{W}}\right) \end{tikzcd}$$

is cartesian, since a functor is distributive if and only if all its components are. Hence since the bottom arrow is an equivalence, so is the top. Checking that $\mathrm{Pr}^{\otimes}_T \to \mathbf{F}^T_*$ is a cocartesian fibration will be a bit more elaborate.

Proposition 3.25. *The projection $\mathrm{Pr}^{\otimes}_T \to \mathbf{F}^T_*$ is a cocartesian fibration.*

Proof. We will show that it is a locally cocartesian fibration and that composition of locally cocartesian arrows is locally cocartesian. Fix $[U \leftarrow W \xrightarrow{f} U']$ arrow in $\mathbf{F}^T_*$ and a presentable $\mathbf{U}$-∞-category C. Then we need to find a presentable $\mathbf{U}'$-∞-category NC and a distributive functor $\mu : \prod_f C|_{\mathbf{W}} \to NC$ such that for every presentable $\mathbf{U}'$-∞-category D precomposition with μ

$$\mathrm{Fun}^L_{U'}(NC, D) \to \mathrm{Fun}^\delta_{U'}\left(\prod_f C|_{\mathbf{W}}, D\right)$$

is an equivalence. Since C is presentable we can find C_0 small U-∞-category and $S \subseteq P_T(C_0)$ small $\mathbf{U}$-subcategory such that C can be identified with the category of S-local objects in $P_T(C_0)$. Then we can choose NC to be the category of $\prod_f S|_{\mathbf{W}}$-local objects in $P_T\left(\prod_f C_0|_{\mathbf{W}}\right)$. In fact there is a cartesian square

$$\begin{tikzcd} \mathrm{Fun}^L_{\mathbf{U}'}(NC, D) \arrow[r, hook] \arrow[d] & \mathrm{Fun}^L_{\mathbf{U}'}\left(P_T\left(\prod_f C_0|_{\mathbf{W}}\right), D\right) \arrow[d, "\wr"] \\ \mathrm{Fun}^\delta_{\mathbf{U}'}\left(\prod_f C|_{\mathbf{W}}, D\right) \arrow[r, hook] & \mathrm{Fun}^\delta_{\mathbf{U}'}\left(\prod_f P_T C_0|_{\mathbf{W}}, D\right) \end{tikzcd}$$

where both horizontal arrows are the inclusion of those functors sending all arrows of $\prod_f S|_{\mathbf{W}}$ to cocartesian arrows. Since the right vertical arrow is an equivalence, so is the left vertical arrow. Now the transitivity of Bousfield localizations shows that the composition of two locally cocartesian arrows is locally cocartesian and so the $\mathrm{Pr}^{\otimes}_T$ is a T-symmetric monoidal T-∞-category. $\square$

Example 3.26. *Exactly with the same proof as in [25, Pr. 4.8.1.16] the tensor product of presentable T-∞-category has a very concrete description:*

$$C \otimes D = \underline{\mathrm{Fun}}^R_T(C^{\vee\mathrm{op}}, D)\,.$$

However, the norm functor is much more opaque in this context. For example if G is a finite group $N^G(\mathrm{Sp}) = \underline{\mathrm{Sp}}^G$.

3.5 Idempotence of spectra

Proposition 3.27. *Let D be an $\mathbf{U}$-stable $\mathbf{U}$-∞-category. Then the evaluation at the sphere spectrum*

$$\mathrm{Fun}^{\delta}_{\mathbf{U}'}\left(\prod_f \mathrm{Sp}^{\mathbf{U}}, D\right) \to \mathrm{Fun}_{\mathbf{U}}(\mathbf{U}, D)\,.$$

Proof. Since $\mathrm{Sp}^{\mathbf{U}} \cong \underline{\mathrm{Lin}}^U(\mathrm{Top}_*^{U,fin}, \mathrm{Top})$ we can identify it with the Bousfield localization of presheaves on $(\mathrm{Top}_*^{U,fin})^{v\mathrm{op}}$ at the collection of arrows S composed by (j here indicates the Yoneda embedding)

- $\varnothing \to j*$;
- $\Sigma j(\Sigma X) \to jX$ where $\Sigma j(\Sigma X)$ is the pushout in the category of presheaves of the diagram

$$\begin{tikzcd} j(\Sigma X) \arrow[r] \arrow[d] & j* \\ j* & \end{tikzcd} ;$$

- $\coprod_I j(X) \to j\left(\bigvee_I X\right)$ where the map is defined as in the definition of B-additivity.

Hence

$$\mathrm{Fun}^{\delta}_{\mathbf{U}'}\left(\prod_f \mathrm{Sp}^{\mathbf{U}}, D\right)$$

is equivalent to

$$\mathrm{Fun}^{L}_{\mathbf{U}'}\left(L_S P_T\left(\prod_f (\mathrm{Top}_*^{fin,\mathbf{U}})^{v\mathrm{op}}\right), D\right)$$

But since D is $\mathbf{U}'$-stable this category remains the same if we replace the source with the category of spectral presheaves on $\prod_f(\mathrm{Top}_*^{fin,\mathbf{U}})$.

But the explicit description of this Bousfield localization tells us that this is just the category of distributive functors $\prod_f \mathrm{Top}_*^{fin,\mathbf{U}} \to \mathrm{Sp}^{\mathbf{U}'}$, that is equivalent to $\mathrm{Sp}^{\mathbf{U}'}$ by 3.19. ∎

Corollary 3.28. *The T-∞-category $\underline{\mathrm{Sp}}^T$ together with the functor*

$$\Sigma^\infty_+ : \underline{\mathrm{Top}}_T \to \underline{\mathrm{Sp}}^T$$

is a weak idempotent in the T-∞-category of presentable T-∞-categories. In particular the space of T-symmetric monoidal structures such that the sphere spectrum is the unit is contractible.

Proof. We need to prove that for every map $f : U \to U'$ the map

$$\underline{\mathrm{Sp}}^{\mathbf{U}'} \to \underline{\mathrm{Sp}}^{\mathbf{U}'} \otimes N^f(\underline{\mathrm{Sp}}^{\mathbf{U}})$$

is an equivalence. But the right hand side can be identified with

$$\underline{\mathrm{Sp}}^{\mathbf{U}'} \otimes N^f(\underline{\mathrm{Sp}}^{\mathbf{U}}) = \underline{\mathrm{Fun}}^L_{\mathbf{U}'}(N^f(\underline{\mathrm{Sp}}^{\mathbf{U}}), (\underline{\mathrm{Sp}}^{\mathbf{U}'})^{\vee\mathrm{op}})^{\vee\mathrm{op}} = \underline{\mathrm{Fun}}^\delta_{\mathbf{U}'}\left(\prod_f \underline{\mathrm{Sp}}^{\mathbf{U}}, (\underline{\mathrm{Sp}}^{\mathbf{U}'})^{\vee\mathrm{op}}\right)^{\vee\mathrm{op}}.$$

Now the thesis follows immediately from the previous proposition, by noting that $(\underline{\mathrm{Sp}}^{\mathbf{U}'})^{\vee\mathrm{op}}$ is T-stable. $\square$

Appendix A

Equivariant stable homotopy theory via spectral Mackey functors

Spectral Mackey functors as a model for stable equivariant homotopy theory were first introduced in [20], where Guillou and May proved that they can be used as a model for G-spectra. This idea has been further developed in [14] and [12] where they have been used to extend the notion of G-spectrum to profinite G and to construct algebraic K-theory G-spectra for Waldhausen categories with a suitable action of G (for example the category of $\bar{F}$-vector spaces equipped with a Galois action).

It became soon clear that stable equivariant homotopy theory can be developed in the framework of Mackey functors, and this approach makes many results clearer (for example the tom Dieck splitting reveals itself as yet another instance of the Barratt-Priddy-Quillen theorem). This appendix wants to be a demonstration of this fact. None of the results here are original (in fact, one could say that all of them are classical), but hopefully this presentation will show that it is possible and maybe even simpler to develop the whole theory of G-spectra from the perspective of spectral Mackey functors.

A.1 The effective Burnside category of finite G-sets

Let G be a finite group and let $\mathbf{F}_G$ be the category of finite G-sets and $\mathbf{O}_G \subseteq \mathbf{F}_G$ be the full subcategory spanned by transitive G-sets. The objects of $\mathbf{O}_G$ are called **orbits** and $\mathbf{O}_G$ is the **orbit category**.

We let the **effective Burnside category** of G $\mathbf{A}^{\mathrm{eff}}(G)$ be the ∞-category whose n-simplices are functors $X : \tilde{\mathbf{O}}(\Delta^n)^{\mathrm{op}} \to \mathbf{F}_G$ such that, for every pairs

$i < j$ and $k < l$ the square

$$\begin{tikzcd}
X_{ik} \arrow[r] \arrow[d] & X_{jk} \arrow[d] \\
X_{il} \arrow[r] & X_{jl}
\end{tikzcd}$$

is cartesian. That is the objects of $\mathbf{A}^{\mathrm{eff}}(G)$ are finite G-sets, a map from I to J in $\mathbf{A}^{\mathrm{eff}}(G)$ is a span

$$\begin{tikzcd}
& K \arrow[dl] \arrow[dr] & \\
I & & J
\end{tikzcd}$$

and composition is given by pullbacks

$$\begin{tikzcd}
& & L \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\wedge" very near start] & & \\
& K \arrow[dl] \arrow[dr] & & M \arrow[dl] \arrow[dr] & \\
I & & J & & N
\end{tikzcd}.$$

The mapping spaces in $\mathbf{A}^{\mathrm{eff}}(G)$ are easy to describe

Lemma A.1. *Let I, J be finite G-sets. Then $\mathrm{Map}_{\mathbf{A}^{\mathrm{eff}}(G)}(I, J)$ is equivalent to the groupoid of finite G-sets above $I \times J$.*

Proof. Let us use the model for the mapping space whose n-simplices are functors $\sigma : \Delta^{n+1} \to \mathbf{A}^{\mathrm{eff}}(G)$ such that $\sigma(0) = I$ and $\sigma|_{[1,\dots,n+1]}$ is the degenerate simplex at J. Then in this case n-simplices are diagrams of the form

$$\begin{tikzcd}[column sep=small]
& & & & K_n \arrow[dl, "\sim"'] \arrow[dr] & & & & \\
& & & \cdots \arrow[dl, "\sim"'] & & J \arrow[dl, equal] \arrow[dr, equal] & & & \\
& & K_1 \arrow[dl, "\sim"'] \arrow[dr] & & \cdots \arrow[dl, equal] & & \cdots \arrow[dr, equal] & & \\
& K_0 \arrow[dl] \arrow[dr] & & J \arrow[dl, equal] \arrow[dr, equal] & & \cdots & & J \arrow[dl, equal] \arrow[dr, equal] & \\
I & & J & & J \arrow[ur, equal] & & \cdots & & J
\end{tikzcd}$$

where the maps $K_i \to K_{i-1}$ are equivalences because the underlying square needs to be cartesian. Then it is immediate to see that this simplicial set is isomorphic to the nerve of the groupoid of finite sets above $I \times J$. $\square$

The ∞-category $\mathbf{A}^{\text{eff}}(G)$ is self dual via the obvious isomorphism of simplicial sets that reflects the n-simplices across the vertical axis. Moreover this isomorphism fixes the 0-simplices and acts as the obvious isomorphism $\iota(\mathbf{F}_G)_{/I\times J} \cong \iota(\mathbf{F}_G)_{/J\times J}$ on mapping spaces.

Proposition A.2. *The ∞-category $\mathbf{A}^{\text{eff}}(G)$ has direct sums, given by the disjoint union of G-sets.*

Proof. From the description of the mapping spaces we can see that the empty set $\varnothing$ is a zero object. Let I, J, K are finite G-sets

$$\begin{aligned}&\mathrm{Map}_{\mathbf{A}^{\text{eff}}(G)}(I \amalg J, K) \cong \iota(\mathbf{F}_G)_{/(I\amalg J)\times K} \cong \\ &\quad \cong \iota(\mathbf{F}_G)_{/I\times K} \times \iota(\mathbf{F}_G)_{/J\times K} \cong \mathrm{Map}_{\mathbf{A}^{\text{eff}}(G)}(I, K) \times \mathrm{Map}_{\mathbf{A}^{\text{eff}}(G)}(J, K)\,,\end{aligned}$$

and, in the same way,

$$\begin{aligned}&\mathrm{Map}_{\mathbf{A}^{\text{eff}}(G)}(K, I \amalg J) \cong \iota(\mathbf{F}_G)_{/K\times(I\amalg J)} \cong \\ &\quad \cong \iota(\mathbf{F}_G)_{/K\times I} \times \iota(\mathbf{F}_G)_{/K\times J} \cong \mathrm{Map}_{\mathbf{A}^{\text{eff}}(G)}(K, I) \times \mathrm{Map}_{\mathbf{A}^{\text{eff}}(G)}(K, J)\,.\end{aligned}$$

So $I \amalg J$ is both the coproduct and the product of I and J in $\mathbf{A}^{\text{eff}}(G)$ (as it should be, since $\mathbf{A}^{\text{eff}}(G)$ is self dual by a duality that fixes the objects). It is now an easy exercise to see that the comparison map $I \amalg J \to I \amalg J$ from the coproduct to the product is the identity. $\square$

Warning A.3. *The ∞-category $\mathbf{A}^{\text{eff}}(G)$ is not additive. In fact the set of maps $[I, J]$ in the homotopy category is just the set of isomorphism classes of finite G-sets above $I \times J$. This is a commutative monoid under disjoint union but it is not a group. The ∞-category obtained by group-completing all mapping spaces is called the* ***Burnside category*** *of G and denoted $\mathbf{A}(G)$. As we will see, it is equivalent to the subcategory of G-spectra spanned by the suspension spectra of finite G-sets.*

Proposition A.4. *The ∞-category $\mathbf{A}^{\text{eff}}(G)$ has a symmetric monoidal structure such that the functor $\mathbf{F}_G \to \mathbf{A}^{\text{eff}}(G)$ is symmetric monoidal when the source has the cartesian monoidal structure. That is, $I \otimes J$ is the cartesian product $I \times J$ in the category of finite G-sets. In particular the one point set $*$ is the unit for the symmetric monoidal structure.*

Proof. This is a special case of [12, Pr. 2.14]. $\square$

Corollary A.5. *The unit and counit maps in $\mathbf{A}^{\text{eff}}(G)$*

$$\begin{tikzcd}[column sep=small] & I \arrow[dl] \arrow[dr, "\Delta"] & & & I \arrow[dl, "\Delta"'] \arrow[dr] & \\ I \times I & & * & * & & I \times I \end{tikzcd} \quad , ,$$

exhibit every element $I \in \mathbf{A}^{\mathrm{eff}}(G)$ as dualizable with dual equal to itself.

Proof. The verification of the identities in the definition of dualizable object is left as an exercise to the reader. □

Remark A.6. *This is not true if the category $\mathbf{O}_G$ does not have a terminal object, as it is the case, for example, for the category of cofinite orbits of a compact Lie group.*

A.2 G-spectra

Definition A.7. *Let G be a finite group. Then a **G-spectrum**, or **spectral Mackey functor**, is a product-preserving functor from $\mathbf{A}^{\mathrm{eff}}(G)$ to the ∞-category of spectra* Sp. *We will denote the ∞-category of G-spectra as* Sp^G.

Since Σ and Ω are product-preserving functors on Sp, they are computed pointwise in Sp^G. In particular Sp^G is a stable ∞-category by [25, 1.4.2.27]. Moreover it clearly supports a t-structure, where the connective objects are those G-spectra E such that $E(I)$ is connective for every finite G-set I. The heart of that category is the classical category of Mackey functors ([34, 2.9], [35]). Additionally, thanks to [12, Lm. 3.7], we can equip it with the localization of the Day convolution symmetric monoidal structure.

There is a functor $\Omega^\infty : \mathrm{Sp}^G \to \mathrm{Top}_{G,*}$ sending a G-spectrum E to $\Omega^\infty \circ E \circ j$, where $j : \mathbf{O}_G^{\mathrm{op}} \to \mathbf{A}^{\mathrm{eff}}(G)$ is the obvious functor. Since limits are computed pointwise on Sp^G, Ω^∞ preserves all limits and so it has a left adjoint

$$\Sigma^\infty : \mathrm{Top}_{G,*} \to \mathrm{Sp}^G .$$

Lemma A.8. *For every finite G-set I there is an equivalence*

$$\mathrm{MapSp}(\Sigma^\infty_+ I, E) = E(I)$$

Proof. Since both sides are product preserving, it suffices to prove the thesis for I an orbit. Then

$$\mathrm{Map}(\Sigma^\infty_+ I, E) = Map(I, \Omega^\infty E) = \Omega^\infty E(I)$$

Since the G-space I is the presheaf represented by I. But then both sides of

$$\mathrm{MapSp}(\Sigma^\infty_+ I, E) = E(I)$$

exact functors from G-spectra to spectra that coincide after applying Ω^∞. So the thesis follows from [25, 1.4.2.22]. □

Notation A.9. *Let H be a subgroup of G and $E \in \mathrm{Sp}^G$ a G-spectrum. Then we will denote with E^H the spectrum*

$$E(G/H) = \mathrm{MapSp}(\Sigma^\infty_+ G/H, E)$$

and call it the ***Lewis-May fixed points at*** H *(or fixed points for short) of* E. *Note that*

$$(\Omega^\infty E)^H = \Omega^\infty(E^H)\,,$$

but $(\Sigma^\infty X)^H \neq \Sigma^\infty(X^H)$ *(see corollary A.14).*

Since the category $\mathbf{A}^{\mathrm{eff}}(G)$ has direct sums, there is a canonical homomorphism

$$\mathbf{A}^{\mathrm{eff}}(G) \to \mathrm{Fun}^\times(\mathbf{A}^{\mathrm{eff}}(G), \mathrm{Top}) \cong \mathrm{Fun}^\times(\mathbf{A}^{\mathrm{eff}}(G), \mathrm{CMon}(\mathrm{Top}))$$

where the last equivalence is given by arguing as in remark C.1.5.3 of [26]. Since the group-completion functor CMon(Top) $\to$ Sp preserves finite products, we can use it to get a functor

$$L : \mathbf{A}^{\mathrm{eff}}(G) \to \mathrm{Fun}^\times(\mathbf{A}^{\mathrm{eff}}(G), \mathrm{Sp}) = \mathrm{Sp}^G\,.$$

Moreover this functor preserves the direct sums (since it is a composition of functors that do so). The functor H is not fully faithful, but we want to prove that it is very close to be so. First let us notice that L is essentially a version of Σ^∞_+.

Lemma A.10. *There is a commutative diagram*

$$\begin{array}{ccc} \mathbf{F}_{G,*} & \xrightarrow{i} & \mathrm{Top}_{G,*} \\ \downarrow & & \downarrow{\scriptstyle \Sigma^\infty} \\ \mathbf{A}^{\mathrm{eff}}(G) & \xrightarrow{L} & \mathrm{Sp}^G \end{array}$$

where the left vertical map is given by the identification of $\mathbf{F}_{G,*}$ *with the subcategory of* $\mathbf{A}^{\mathrm{eff}}(G)$ *given by the morphisms where the "wrong way leg" of the span is an injection.*

Proof. Recall that Σ^∞ is characterized by the equivalence

$$\mathrm{Map}_{\mathrm{Sp}^G}(\Sigma^\infty(I_+), E) = \mathrm{Map}_{\mathrm{Top}_{G,*}}(I_+, \Omega^\infty E) = \Omega^\infty E(I)$$

as a functor $\mathbf{F}^{\mathrm{op}}_{G,*} \times \mathrm{Sp}^G \to \mathrm{Top}$. It suffices to show that Li satisfies the same property. In fact

$$\mathrm{Map}_{\mathrm{Sp}^G}(LI, E) = \mathrm{Map}_{\mathrm{Fun}^\times(\mathbf{A}^{\mathrm{eff}}(G),\mathrm{Top})}(LI, \Omega^\infty E) = \Omega^\infty E(I)\,.$$

where the last equivalence is just the Yoneda lemma. □

So from now on we will denote the functor L by Σ^∞_+ without fear of ambiguity.

Corollary A.11. *For any finite* G*-set* I*, the* G*-spectrum* $\Sigma^\infty_+ I$ *is canonically self dual.*

Proof. Since I is self dual in $\mathbf{A}^{\text{eff}}(G)$ and the functor $\Sigma^\infty_+ : \mathbf{A}^{\text{eff}}(G) \to \mathrm{Sp}^G$ is symmetric monoidal by [12, Pr. 4.5], the thesis follows. □

Remark A.12. *Note that $\Sigma^\infty : \mathrm{Top}_{G,*} \to \mathrm{Sp}^G$ is symmetric monoidal since it is just the left Kan extension of a symmetric monoidal functor $\mathbf{F}_{G,*} \to \mathrm{Sp}^G$.*

Our description allows us to completely understand the full subcategory of Sp^G spanned by suspension spectra of finite G-sets in terms of suspension spectra of finite sets.

Lemma A.13. *Let I, J be finite G-sets. Then the map*

$$\mathrm{Map}_{\mathbf{A}^{\text{eff}}(G)}(I, J) \to \mathrm{Map}_{\mathrm{Sp}^G}(\Sigma^\infty_+ I, \Sigma^\infty_+ J)$$

is a group completion map, when the source and target have the canonical E_∞ structure given to mapping spaces in a category with direct sums.

Proof. Since

$$\mathrm{Map}_{\mathrm{Sp}^G}(\Sigma^\infty_+ I, \Sigma^\infty_+ J) = (\Sigma^\infty_+ J)(I)$$

this follows immediately from our description of $\Sigma^\infty_+ J$ as the group completion of the functor corepresented by J. □

Corollary A.14 (tom Dieck splitting). *Let X be a G-space. Then*

$$(\Sigma^\infty_+ X)^G = \bigoplus_{(H)} \Sigma^\infty_+ (X^H)_{hWH}\,,$$

where the sum is over the conjugacy classes of subgroups of G and $WH = NH/H = \mathrm{Aut}_G(G/H)$ is the Weyl group of H.

Proof. First let us prove it in the case of a finite G-seet I. By lemma A.13 and A.1 the spectrum

$$(\Sigma^\infty_+ I)^G = \mathrm{MapSp}(\Sigma^\infty_+ *, \Sigma^\infty_+ I)$$

is the group completion of the space $\iota(\mathbf{F}_G)_{/I}$. Let $(\mathbf{F}_G)^H_{/I}$ be the subcategory of $(\mathbf{F}_G)_{/I}$ consisting of those G-sets for which all orbits are isomorphic to G/H. Since we can write every finite G-set over I is a disjoint union of orbits, we can decompose

$$\iota(\mathbf{F}_G)_{/I} = \prod_{(H)} \iota(\mathbf{F}_G)^H_{/I}$$

as E_∞-spaces. We claim that the category $(\mathbf{F}_G)^H_{/I}$ is equivalent to the action groupoid of WH on $\mathbf{F}_{/I^H}$. In fact we can identify both with the category whose objects are G-equivariant maps

$$G/H \times S \to I$$

where S is a finite set, and maps are G-equivariant maps $G/H \times S \to G/H \times S'$ making the obvious triangle commute. So it follows that

$$(\Sigma^\infty_+ I)^G = (\iota(\mathbf{F}_G)_{/I})^+ = \prod_{(H)} (\iota \mathbf{F}_{/I^H})^+_{hWH}$$

and the thesis follows from the fact that, for every finite set J, the group completion of $(\iota\mathbf{F}_{/J})$ is $\Sigma^\infty_+ J$, which is proven in [31].

Finally, the general case follows from the fact that every G-space is a filtered colimit of finite G-sets and both sides of the identity commute with colimits as functors from G-spaces to spectra. $\square$

A.3 Change of groups and the Wirthmüller isomorphism

Let H be a subgroup of G. Then there is a restriction map $\operatorname{res}^G_H : \mathbf{F}_G \to \mathbf{F}_H$, that comes with a left adjoint $\operatorname{ind}^G_H : \mathbf{F}_H \to \mathbf{F}_G$ given by

$$\operatorname{ind}^G_H(I) = G \times_H I =: (G \times I)/H\,,$$

where in the last formula H acts by $h(g,i) = (gh^{-1}, hi)$.

Since both res^G_H and ind^G_H preserve pullbacks, they induce maps

$$\operatorname{res}^G_H : \mathbf{A}^{\text{eff}}(G) \to \mathbf{A}^{\text{eff}}(H) \quad \text{and} \quad \operatorname{ind}^G_H : \mathbf{A}^{\text{eff}}(H) \to \mathbf{A}^{\text{eff}}(G)\,.$$

Moreover, since for every maps $I \to I'$ of H-sets and $J \to J'$ of G-sets the squares

$$\begin{tikzcd} I \arrow[r] \arrow[d] & \operatorname{res}^G_H \operatorname{ind}^G_H I \arrow[d] \\ I' \arrow[r] & \operatorname{res}^G_H \operatorname{ind}^G_H I' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \operatorname{ind}^G_H \operatorname{res}^G_H J \arrow[r] \arrow[d] & J \arrow[d] \\ \operatorname{ind}^G_H \operatorname{res}^G_H J' \arrow[r] & J' \end{tikzcd}$$

are cartesian, it follows that the unit and counit are natural transformations of functors between the Burnside categories. That is, there is an adjunction

$$\operatorname{ind}^G_H \dashv \operatorname{res}^G_H : \mathbf{A}^{\text{eff}}(G) \leftrightarrows \mathbf{A}^{\text{eff}}(H)\,.$$

Moreover, under the canonical identification of $\mathbf{A}^{\text{eff}}(G)$ and $\mathbf{A}^{\text{eff}}(H)$ with their opposites, we have $(\operatorname{res}^G_H)^{\text{op}} = \operatorname{res}^G_H$ and $(\operatorname{ind}^G_H)^{\text{op}} = \operatorname{ind}^G_H$. So ind^G_H is also the right adjoint of res^G_H, when seen as functors between the Burnside categories.

Let then now define res^G_H as precomposition by ind^G_H and ind^G_H as precomposition by res^G_H so that

$$(\operatorname{res}^G_H E)(H/K) = E(G/K) \qquad \left(\text{i.e. } (\operatorname{res}^G_H E)^K = E^K\right)$$

(this is defined so that for every $K < H$, the K-fixed points of E and $\operatorname{res}^G_H E$ are the same). From this explicit description we can extract the following important result

Theorem A.15 (Wirthmüller isomorphism). *The left and right adjoint to* res^G_H *are canonically equivalent*

Remark A.16. *From [9], res^G_H is a symmetric monoidal functor. Since corollary A.5 implies that Sp^G is rigidly-compactly generated in the sense of [2, Hp. 1.2] we can apply all the results of that paper to the adjunction $\mathrm{ind}^G_H \dashv \mathrm{res}^G_H$.*

A.4 Isotropy separation and geometric fixed point

Let F be a family of subgroups of G that is closed under conjugation and taking subgroups or, equivalently, a sieve in the orbit category $\mathbf{O}_G$. Then we can define a G-space EF such that

$$(EF)^K = \begin{cases} * & \text{if } K \in F \\ \varnothing & \text{if } K \notin F \end{cases}.$$

In fact EF is the presheaf on $\mathbf{O}_G$ corresponding to the sieve F under the equivalence between sieves and -1-truncated presheaves. Then we can form a cofiber sequence in $\mathrm{Top}_{G,*}$

$$EF_+ \to S^0 \to \widetilde{EF}$$

where the first map is just obtained by applying $(-)_+$ to $EF \to *$. That is $\widetilde{EF}$ is the unreduced suspension of EF. It follows that

$$(\widetilde{EF})^K = \begin{cases} * & \text{if } K \in F \\ S^0 & \text{if } K \notin F \end{cases}.$$

When F is the family consisting only of the trivial subgroup, the G-spaces EF and $\widetilde{EF}$ are denoted EG and $\widetilde{EG}$.

Example A.17. *Let $G = C_p$ the cyclic group of prime order p. Then it is easy to see that EG is just $S(\infty\bar{\rho})$, the unit sphere in the infinite sum of copies of the reduced regular representation*[1]*. So $\widetilde{EG} = S^{\infty\bar{\rho}}$.*

Lemma A.18. *Let E be any G-spectrum then, if $K \in F$*

$$(E \wedge \Sigma^\infty \widetilde{EF})^K = 0\,.$$

Viceversa, if $K \notin F$, the canonical map

$$E^K \to (E \wedge \Sigma^\infty \widetilde{EF})^K$$

induced by $S^0 \to \widetilde{EF}$ is an equivalnece.

Proof. We have

$$(E \wedge \Sigma^\infty \widetilde{EF})^K = \mathrm{MapSp}(\Sigma^\infty_+ G/K, E \wedge \Sigma^\infty \widetilde{EF}) = (E \wedge \Sigma^\infty(\widetilde{EF} \wedge G/K_+))^G$$

[1]In fact, instead of $\bar{\rho}$ we could have used any real representation that does not contain copies of the trivial one

So it suffices to prove that

$$\widetilde{EF} \wedge G/K_+ = \begin{cases} * & \text{if } K \in F \\ G/K_+ & \text{if } K \notin F \end{cases},$$

as a pointed G-space. But that can be easily checked on the fixed points. $\square$

We say that a G-spectrum E is concentrated away from F if its fixed points for any subgroup in F are contractible. From the lemma it is clear that this is equivalent to say that the map

$$E \to E \wedge \Sigma^\infty \widetilde{EF}$$

is an equivalence. The fiber sequence

$$E \wedge \Sigma^\infty_+ EF \to E \to E \wedge \Sigma^\infty \widetilde{EF}$$

is called the **isotropy separation sequence**.

The following result shows that for every family of subgroups we get a smashing localization (and so a recollement). In [9] this is generalized to every left orbital functor between orbital ∞-categories.

Theorem A.19. *Let F be a family of subgroups of G closed under conjugation and subgroups. Then there is a smashing localization such that the local objects are the G-spectra concentrated away from F.*

Moreover if $\mathbf{F}_{F^c}$ denotes the subcategory of $\mathbf{F}_G$ consisting of those finite G-sets whose stabilizers are not in F, then the category of G-spectra concentrated away from F is equivalent to

$$\mathrm{Fun}^\times(\mathbf{A}^{\mathrm{eff}}(\mathbf{F}_{F^c}), \mathrm{Sp}),$$

and the inclusion is precomposition with the map $\psi : \mathbf{A}^{\mathrm{eff}}(\mathbf{F}_G) \to \mathbf{A}^{\mathrm{eff}}(\mathbf{F}_{F^c})$ that sends every finite G-set I to the subset of points with stabilizers in F.

Proof. The fact that it is a smashing localization is obvious from the previous discussion and the fact that $\widetilde{EF} \wedge \widetilde{EF} = \widetilde{EF}$. Since $\mathbf{F}_{F^c}$ is closed under pullbacks, we have a product-preserving functor

$$\phi : \mathbf{A}^{\mathrm{eff}}(G) \to \mathbf{A}^{\mathrm{eff}}(\mathbf{F}_{F^c})$$

and it is clear that the image of

$$\phi^* : \mathrm{Fun}^\times\left(\mathbf{A}^{\mathrm{eff}}(\mathbf{F}_{F^c}), \mathrm{Sp}\right) \to \mathrm{Sp}^G$$

consists exactly of those spectral Mackey functors concentrated away from F. So it is enough to provide an inverse functor from the category of G-spectra concentrated away from F. But such an inverse is the left Kan extension along ϕ. $\square$

Note that the localization functor

$$\psi_! : \mathrm{Sp}^G \to \mathrm{Fun}^\times(\mathbf{A}^{\mathrm{eff}}(\mathbf{F}_{F^c}), \mathrm{Sp})$$

is given by the left Kan extension along ψ, which sends product-preserving functors to product-preserving functors by [18, Lm. 2.18].

Example A.20. *Let H be a subgroup of G and F be the family of subgroups that do not contain a conjugate of H. Then the functor*

$$\psi_! : \mathrm{Sp}^G \to \mathrm{Fun}^\times(\mathbf{A}^{\mathrm{eff}}(\mathbf{F}_{F^c}), \mathrm{Sp})$$

*is called the **H-geometric fixed point functor** and denoted Φ^H or $(-)^{\phi H}$. When H is a normal subgroup the category $\mathbf{F}_{F^c}$ is simply the subcategory of finite G/H-sets and so $(-)^{\phi H}$ has values in $\mathrm{Sp}^{G/H}$.*

When $E = \Sigma^\infty X$ is a suspension spectrum the geometric fixed point have a particular nice form: they are equivalent to $\Sigma^\infty(X^H)$ as a G/H-spectrum. This is because both $(\Sigma^\infty -)^{\phi H}$ and $\Sigma^\infty(-^H)$ are colimit-preserving functors from Top_G that coincide on $\mathbf{O}_G$ by the definition of ψ.

A.5 Borel-complete G-spectra

In this section we will describe the completion corresponding to the smashing localization given by EG. This will turn out to have a very pleasant answer.

Lemma A.21. *Let H be a subgroup of G, and let it act on the finite G-set G/e by right multiplication. Then the projection map*

$$G/e \to G/H \times EG$$

is H-equivariant, where H acts trivially on the right hand side, and identifies the homotopy orbits of G/e with $G/H \times EG$.

Proof. The fact that the map is equivariant is obvious, so it is enough to see that it induces an equivalence of G-spaces $(G/e)_{hH} \cong G/H \times EG$. To check that this is an equivalence, it suffices to check it for all fixed points. But the fixed points of both sides for nontrivial subgroups are empty, so it is enough to check that the projection

$$(G/e)_{hH} \to G/H$$

is an equivalence of spaces. Since we can choose a splitting of H-sets $G/e = G/H \times H$, this is equivalent to prove that $H_{hH} = *$, but this is a well known fact. ∎

Corollary A.22. *For any G-spectrum E and every subgroup H there is an equivalence*

$$F(\Sigma^\infty_+ EG, E)^H = (uE)^{hH}$$

where $uE = \mathrm{MapSp}(\Sigma^\infty_+ G/e, E)$ is the underlying spectrum of E.

Proof.

$$F(\Sigma^\infty_+ EG, E)^H = \mathrm{MapSp}(\Sigma^\infty_+ G/H, F(\Sigma^\infty_+ EG, E)) = \mathrm{MapSp}\, \Sigma^\infty_+(G/H \times EG), E) = \\ = \mathrm{MapSp}(\Sigma^\infty_+(G/e)_{hH}, E) = \mathrm{MapSp}(\Sigma^\infty_+ G/e, E)^{hH} = (uE)^{hH}\,.$$

□

We say that a G-spectrum E is **Borel-complete** if the map

$$E \to F(\Sigma^\infty_+ EG, E)$$

is an equivalence. Note that this is equivalent to be complete with respect to the smashing localization associated with the family of subgroups $F = \{e\}$. The category of Borel-complete G-spectra has a very pleasant description:

Theorem A.23. *The functor sending every G-spectrum to its underlying spectrum induced an equivalence of the category of Borel-complete G-spectra with the category Sp^{hG} of spectral presheaves over BG*

Proof. It suffices to exhibit an inverse $i : \mathrm{Sp}^{hG} \to \mathrm{Sp}^G$. Let us consider the composition

$$j : \mathbf{A}^{\mathrm{eff}}(G) \to \mathrm{Sp}^G \xrightarrow{u} \mathrm{Sp}^{hG}$$

and choose i to be

$$iE(I) = \mathrm{Sp}^{hG}(jI, E) = \mathrm{Sp}^{hG}(u\Sigma^\infty_+ I, E)$$

(so that i is the right adjoint of $u : \mathrm{Sp}^G \to \mathrm{Sp}^{hG}$). Since $u\Sigma^\infty_+ G/e$ is the object of Sp^{hG} corepresenting the identity it follows that $ui = 1_{\mathrm{Sp}^{hG}}$. Moreover since $u\Sigma^\infty_+ G/H$ corepresents the functor $E \mapsto E^{hH}$ (since the presheaf of spaces uG/H on BG is equivalent to $(uG/e)_{hH}$), it follows from the previous corollary that $iuE = F(\Sigma^\infty_+ EG, iuE) = F(\Sigma^\infty_+ EG, E)$, so i is the inverse we were looking for. □

Lemma A.24. *Let E be a G-spectrum and H be a subgroup of G. Then there is a canonical equivalence*

$$(\Sigma^\infty_+ EG \wedge E)^H = (uE)_{hH}$$

Proof.

$$(\Sigma^\infty_+ EG \wedge E)^H = \mathrm{MapSp}(\Sigma^\infty_+ G/H, \Sigma^\infty_+ EG \wedge E) = \mathrm{MapSp}(\Sigma^\infty_+ *, \Sigma^\infty_+(G/H \times EG) \wedge E) = \\ = \mathrm{MapSp}(\Sigma^\infty_+ *, \Sigma^\infty_+(G/e)_{hH} \wedge E) = \mathrm{MapSp}(\Sigma^\infty_+ G/e, E)_{hH} = (uE)_{hH}\,.$$

where we have used that finite G-sets are canonically self dual and compact objects (in particular fixed points commute with colimits). □

Example A.25. *Let us fix $E \in \mathrm{Sp}^{hG}$ and let $iE \in \mathrm{Sp}^{G}$ be the corresponding Borel G-spectrum. Let us consider the isotropy-separation sequence*

$$iE \wedge \Sigma^\infty_+ EG \to iE \to iE \wedge \Sigma^\infty_+ \widetilde{EG}$$

and let us take fixed points. The sequence becomes

$$E_{hG} \to E^{hG} \to (iE \wedge \Sigma^\infty_+ \widetilde{EG})^G$$

and the first map is just the norm map from homotopy orbits to homotopy fixed points. So the last term

$$(iE \wedge \Sigma^\infty_+ \widetilde{EG})^G = E^{tG}$$

coincides with the Tate construction. This is sometimes taken as the definition of the Tate construction. If $G = C_p$ is a cyclic group of prime order moreover we can identify the left hand side with the geometric fixed points $(iE)^{\phi C_p}$. So

$$(iE)^{\phi C_p} = E^{tC_p}\,.$$

Applying the previous result to the case of the underlying spectrum uE of a G-spectrum E, the map $E \to F(EG_+, E)$ induces a pullback diagram

$$\begin{tikzcd}
E^G \arrow[r] \arrow[d] & E^{\phi G} \arrow[d] \\
(uE)^{hG} \arrow[r] & (uE)^{tG}
\end{tikzcd}$$

since the fibers of both rows are equivalent to $(uE)_{hG}$.

Bibliography

[1] V. Angeltveit, A. Blumberg, T. Gerhardt, M. Hill, T. Lawson, and M. Mandell. Topological cyclic homology via the norm. preprint `arXiv:1401.5001`, 2014.

[2] Paul Balmer, Ivo Dell'Ambrogio, and Beren Sanders. Grothendieck–neeman duality and the wirthmüller isomorphism. *Compositio Mathematica*, 152(08):1740–1776, 2016.

[3] C. Barwick. Algebraic K-theory of parametrized ∞-categories. To appear.

[4] C. Barwick. Multiplicative structures on algebraic K-theory. Preprint `arXiv:1304.4867.`, April 2013.

[5] C. Barwick, E. Dotto, S. Glasman, D. Nardin, and J. Shah. Elements of parametrized higher category theory. Preprint `arXiv:1608.03657`.

[6] C. Barwick, E. Dotto, S. Glasman, D. Nardin, and J. Shah. Parametrized ∞-operads and parametrized symmetric monoidal ∞-categories. To appear.

[7] C. Barwick, E. Dotto, S. Glasman, D. Nardin, and J. Shah. Parametrized higher category theory and higher algebra: A general introduction. Preprint `arXiv:1608.03654`, Aug 2016.

[8] C. Barwick and S. Glasman. On the fibrewise effective Burnside ∞-category. Preprint available from *arXiv:1607.02786.*

[9] C. Barwick and S. Glasman. Orbital ∞-categories. To appear.

[10] C. Barwick and S. Glasman. Cyclonic spectra, cyclotomic spectra, and a conjecture of Kaledin. Preprint `arXiv:1602.02163`, February 2016.

[11] C. Barwick, S. Glasman, and D. Nardin. Dualizing cartesian and cocartesian fibrations. Preprint from the web pages of the authors.

[12] C. Barwick, S. Glasman, and J. Shah. Spectral Mackey functors and equivariant algebraic K-theory (II). Preprint `arXiv:1505.03098`, May 2015.

[13] C. Barwick, D. Nardin, and J. Shah. Parametrized presentable ∞-categories and distributive functors. To appear.

[14] Clark Barwick. Spectral mackey functors and equivariant algebraic k-theory (i). *Advances in Mathematics*, 304:646–727, 2017.

[15] A. J. Blumberg. Continuous functors as a model for the equivariant stable homotopy category. *Algebr. Geom. Topol.*, 6:2257–2295, 2006.

[16] Andrew J Blumberg and Michael A Hill. Operadic multiplications in equivariant spectra, norms, and transfers. *Advances in Mathematics*, 285:658–708, 2015.

[17] E. Dotto and K. Moi. Homotopy theory of G-diagrams and equivariant excision. *Algebr. Geom. Topol.*, 16(1):325–395, 2016.

[18] S. Glasman. Stratified categories, geometric fixed points and a generalized Arone-Ching theorem. *ArXiv e-prints*, July 2015.

[19] S. Glasman. Goodwillie calculus and mackey functors. Preprint `arXiv:1610.03127`, 2016.

[20] B. Guillou and J. P. May. Models of G-spectra as presheaves of spectra. Preprint `arXiv:1110.3571v3`, July 2013.

[21] Michael A Hill and Michael J Hopkins. Equivariant symmetric monoidal structures. preprint `arXiv:1610.03114`, 2016.

[22] Michael A Hill, MJ Hopkins, and DC Ravenel. On the nonexistence of elements of kervaire invariant one. *Annals of Mathematics*, 184(1):1–262, 2016.

[23] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[24] J. Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[25] J. Lurie. Higher algebra. Preprint from the web page of the author, March 2016.

[26] J. Lurie. Spectral algebraic geometry. Preprint from the web page of the author, October 2016.

[27] M. A. Mandell and J. P. May. Equivariant orthogonal spectra and S-modules. *Mem. Amer. Math. Soc.*, 159(755):x+108, 2002.

[28] J Peter May and Johann Sigurdsson. *Parametrized homotopy theory*. Number 132. American Mathematical Soc., 2006.

[29] D. Nardin. Stability with respect to an orbital ∞-category. arXiv preprint `arXiv:1608.07704`.

[30] S. Schwede. Global homotopy theory. Preprint from the web page of the author.

[31] Graeme Segal. Categories and cohomology theories. *Topology*, 13(3):293–312, 1974.

[32] J. Shah. Parametrized homotopy limits and colimits. To appear.

[33] Neil Strickland. Tambara functors. *arXiv preprint arXiv:1205.2516*, 2012.

[34] Tammo tom Dieck. *Transformation groups*, volume 8. Walter de Gruyter, 1987.

[35] Peter Webb. A guide to mackey functors. *Handbook of algebra*, 2:805–836, 2000.