Representations of Vertex Operator Algebras and Superalgebras

by

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Abstract

Representations of vertex operator algebras and superalgebras are studied. In Chapter 1, we find explicit formulas for the "top" singular vectors in any Verma module of an affine Kac-Moody Lie superalgebra(or called a superloop algebra) whose irreducible quotient is a unitary integrable representation. Then we prove that these singular vectors give the defining relations of the corresponding representation. In Chapter 2, we develop the theory of vetex operator superalgebras (SVOA). For any SVOA V, we construct an associative algebra $A(V)$. We establish a 1-1 correspondence between the irreducible representations of V and those of $A(V)$. We also define an $A(V)$-module $A(M)$ for any V-module M and then describe the fusion rules in terms of the modules $A(M)$. This is a generalization of Zhu's construction for the vertex operator algebras (VOA) case. In Chapter 3 we consider three classes of SVOAs, corresponding to the affine Kac-Moody superalgebras, Neveu-Schwarz algebras, and free fermions respectively. We apply the machinery developed in Chapter 2 to them to classify irreducible representations, to prove the rationality and to compute fusion rules of these SVOAs. Here the formulas for singular vectors found in Chapter 1 plays an important role for the theory of SVOAs associated to the affine Lie superalgebras. In Chapter 4 we solve a long standing conjecture. We prove that the VOA associated to the Virasoro algebra, denoted by $\overline{V}_c$, is rational iff $c = c_{p,q} = 1 - 6(p-q)^2/pq$, where $p, q \in \{2, 3, 4, ...\}$, and $(p, q) = 1$; furthermore, the irreducible representations of the VOA $\overline{V}_{c_{p,q}}$ are precisely the minimal modules of the Virasoro algebra. Then we compute the fusion rules for the minimal series.

Thesis Supervisor: Victor G. Kac
Title: Professor

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Contents

Introduction

In the study of two-dimensional conformal quantum field theory, physicists introduced the notion of a chiral algebra [BPZ, G, MS]. Independently the axioms of a vertex operator algebra (VOA) were introduced by Borcherds [B] to realize the largest sporadic simple finite group, called the Monster, as a symmetric group of a certain algebraic structure i.e. the Moonshine module [FLM]. The notion of VOAs can be regarded as a mathematical axiomization of the chiral algebras. It is further developed in [FLM], [FHL] and many others.

The extended symmetry which plays an important role in two-dimensional conformal field theories leads one to study the vertex operator superalgebras (SVOA). For instance, a powerful tool in studying conformal field theories is the free field realization. Roughly Speaking, one can study a complicated object by embedding it into a simple object (here the simple object is free fields). There are two basic types of free fields, namely free bosonic fields (i.e. Heisenberg algebras) and free fermionic fields (i.e. Clifford algebras). The theory of SVOAs provides a natural framework for the above consideration. SVOA structures also arise from the study of affine superalgebras, superconformal algebras and many other important subjects.

An important class of conformal field theories is the so-called rational ones. Accordingly, one can also introduce the notion of rationality of a VOA (or SVOA). Recall that a rational SVOA is a SVOA with finitely many irreducible representations and completely reducibility property as well. Among the most important questions in the theory of vertex operator (super)algebras are the classification of rational VOAs (or SVOAs), classification of the irreducible representations of a given (rational) VOA (or SVOA), and the fusion rules.

In his dissertation [Z], Zhu constructed an associative algebra $A(V)$ for any vertex operator algebra V. He established the 1-1 correspondence between the irreducible representations of V and those of $A(V)$. The construction of $A(V)$ plays an important role in the proof of modular invariance of certain classes of vertex operator algebras [Z]. Furthermore, Frenkel-Zhu [FZ] constructed an $A(V)$-bimodule $A(M)$ for any V-module M, such that the fusion rules among the representations of V can be computed in terms of $A(M)$. A remarkable feature of these constructions is that $A(V)$ and $A(V)$-bimodule $A(M)$ can usually be calculated explicitly. Using these machinery, Frenkel-Zhu [FZ] proved the rationality of the vertex operator algebra associated to the affine Kac-Moody algebras with positive integral central charge, classified its irreducible representations, and computed the fusion rules.

In this thesis, we formulate the notion of vertex operator superalgebras, in particular of Neveu-Schwarz(NS) type. We generalize Zhu's $A(V)$ theory and Frenkel-Zhu's $A(M)$ construction to the vertex operator superalgebras cases (see Chapter 2). We study in detail three classes of SVOAs, namely, the SVOAs associated to the affine Kac-Moody superalgebras (or called superloop algebras), the Neveu-Schwarz algebras, and the free fermions. We apply the machinery developed in Chapter 2 to these three classes of SVOAs to prove the rationality, to classify the irreducible representations, and to compute the fusion rules. In the study of the SVOAs associated to the affine superalgebras, explicit formulas for the "top" singular vectors of the unitary representations of the affine superalgebras are crucial. These singular vectors are constructed in Chapter 1. In Chapter 4, We solve a long standing conjecture (cf. [FZ]) on the theory of vertex operator algebras associated to the Virasoro algebra.

This thesis is organized as follows. In Chapter 1, we study the representations of the affine superalgebra $\hat{\hat{\mathfrak{g}}}$. Kac-Todorov [KT] proved that any unitary highest weight representation of $\hat{\hat{\mathfrak{g}}}$ is the irreducible quotient of the Verma module $M(\Lambda + h^\vee d)$, where Λ is an integral dominant weight, and $h^\vee$ is the dual Coxeter number of the corresponding finite dimensional simple Lie algebra $\mathfrak{g}$. The formulas for all but one of the "top" singular vectors in $M(\Lambda+h^\vee d)$ are easily found. We obtain explicit formulas in two different forms for the distinguished "top" singular vector of the Verma module

$M(\Lambda + h^\vee d)$ (See Theorem 1.2.1 and 1.3.1). We then show that these "top" singular vectors generate the maximal proper submodule of $M(\Lambda + h^\vee d)$ and thus give the defining relations of the irreducible quotient $L(\Lambda + h^\vee d)$.

In Chapter 2, we first give the definitions of SVOAs, modules of a SVOA, fusion rules and some other concepts. Then for any SVOA V, we construct an associative algebra $A(V)$. We establish the 1-1 correspondence between the irreducible representations of $A(V)$ and those of V. For any V-module M, we construct a $A(V)$-bimodule $A(M)$ such that one can express the fusion rules in terms of $A(M)$.

In Chapter 3, we apply the machinery developed in Chapter 2 to the SVOAs associated to the affine superalgebras, the Neveu-Schwarz algebras and the free fermions. In Sect. 3.1, we show that the irreducible vacuum module $L_{k,0}$ carries a natural (NS type) SVOA structure. We prove that the SVOA $L_{k,0}$ is rational if the central charge k is a positive integer. We classify all the irreducible representations of such a SVOA and compute the fusion rules. In Sect. 3.2 we show that the irreducible Vacuum module V_c of the Neveu-Schwarz algebra carries a natural SVOA structure. We prove that the rationality of V_c for the unitary series and that its irreducible representations are exactly the unitary minimal modules. We conjecture this is also true for the non-unitary minimal series. Indeed we reduce this conjecture to a calculation of a coinvariant of a nilpotent subalgebra of the Neveu-Schwarz algebra, following the idea in [W]. In Sect. 3.3 we construct the SVOAs corresponding to the free charged or neutral fermions. We show that such a SVOA is always rational and has itself as a unique irreducible representation.

In Chapter 4, we study the irreducible Vacuum module $\overline{V}_c$ of the Virasoro algebra which carries a natural VOA structure. We prove the rationality of $\overline{V}_c$ if and only if $c = c_{p,q} = 1 - 6(p-q)^2/pq$, where $p, q \in \{2, 3, 4, \ldots\}$ and p, q are relatively prime; the irreducible representations of the VOA $\overline{V}_c$ are precisely those which correspond to minimal modules of the Virasoro algebra. Then we also present formulas for the fusion rules in the minimal series cases. As a byproduct, the equivalence is established between the fusion rules defined by Frenkel-Huang-Lepowsky in terms of intertwiners among modules over a vertex operator algebra and those defined by Feigin-Fuchs in terms of the coinvariant of a certain nilpotent infinite dimensional Lie algebra.

Most of the results in this thesis have already appeared in [KWa] and [W].

Chapter 1

Representations of affine Kac-Moody superalgebras

In this chapter we consider the Verma module $M(\Lambda)$ and its irreducible quotient $L(\Lambda)$ of an affine Kac-Moody Lie superalgebra (or called a superloop algebra)

$$\hat{\hat{\mathfrak{g}}} = \mathfrak{g} \bigotimes \mathbb{C}\left[t, t^{-1}, \xi\right] \bigoplus \mathbb{C}\mathbf{k} \bigoplus \mathbb{C}d.$$

It is known [KT] that $L(\Lambda)$ is unitary with respect to a certain canonical Hermitian form if and only if $\Lambda - h^{\vee} d$ is an integral dominant weight, where $h^{\vee}$ is the dual Coxeter number in the corresponding simple Lie algebra $\mathfrak{g}$. We obtain explicit formulas for the "top" singular vectors and then prove that these singular vectors generate the maximal proper submodule of the Verma module $M(\Lambda)$.

1.1 Basic notions

Given a finite-dimensional simple Lie algebra $\mathfrak{g}$ of rank l over $\mathbb{C}$, we fix a Cartan subalgebra $\mathfrak{h}$, a root system $\Delta \subset \mathfrak{h}^*$ and a set of positive roots $\Delta_+ \subset \Delta$. Let ρ be half the sum of positive roots. Let $\mathfrak{g} = \mathfrak{h} \bigoplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$ be the root space decomposition of $\mathfrak{g}$. Let e_i, f_i, $h_i\,(i = 1, \cdots, l)$ be the corresponding Chevalley generators. Denote by θ the highest root and normalize the Killing form

$$(\,,\,) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

by the condition $(\theta, \theta) = 2$. Let σ be the antilinear anti-involution of $\mathfrak{g}$. We choose $f_\theta \in \mathfrak{g}_{-\theta}$ so that $(f_\theta, \sigma(f_\theta)) = 1$, and set $e_\theta = \sigma(f_\theta)$. We denote by r_α the reflection with respect to $\alpha \in \Delta$ in the Weyl group W of $\mathfrak{g}$. The affine Kac-Moody superalgebra (or called superloop algebra) is then defined by

$$\hat{\hat{\mathfrak{g}}} = \mathfrak{g} \bigotimes \mathbb{C}\left[t, t^{-1}, \xi\right] \bigoplus \mathbb{C}\mathbf{k} \bigoplus \mathbb{C}d,$$

where t is a polynomial variable and ξ is a Grassmann variable, i.e. $\xi^2 = 0$, with the following commutation relations

$$\begin{aligned}
[a(m), b(n)] &= [a,b](m+n) + m\delta_{m+n,0}(a,b)\mathbf{k}, && (1.1.1)\\
\left[\bar{a}(m), \bar{b}(n)\right]_+ &= \delta_{m+n+1,0}(a,b)\mathbf{k}, && (1.1.2)\\
\left[a(m), \bar{b}(n)\right] &= \overline{[a,b]}(m+n), && (1.1.3)\\
[\mathbf{k}, a(m)] &= 0, && (1.1.4)\\
[d, a(m)] &= ma(m), && (1.1.5)\\
[d, \bar{a}(m)] &= (m + \frac{1}{2})\bar{a}(m), && (1.1.6)
\end{aligned}$$

where $a, b \in \mathfrak{g}$, $m, n \in \mathbb{Z}$, $a(m) := a \otimes t^m$, $\bar{a}(m) := a \otimes \xi t^m$.

Let

$$\begin{aligned}
\bar{\mathfrak{g}} &= \mathfrak{g} \bigotimes \xi,\\
\hat{\hat{\mathfrak{g}}}_+ &= \mathfrak{g} \bigotimes t\mathbb{C}[t] \bigoplus \mathfrak{g} \bigotimes \xi\mathbb{C}[t],\\
\hat{\hat{\mathfrak{g}}}_- &= \mathfrak{g} \bigotimes t^{-1}\mathbb{C}\left[t^{-1}\right] \bigoplus \mathfrak{g} \bigotimes \xi t^{-1}\mathbb{C}\left[t^{-1}\right].
\end{aligned}$$

$\hat{\hat{\mathfrak{g}}}_+$ and $\hat{\hat{\mathfrak{g}}}_-$ are subalgebras of $\hat{\hat{\mathfrak{g}}}$ and

$$\hat{\hat{\mathfrak{g}}} = \hat{\hat{\mathfrak{g}}}_+ \bigoplus \hat{\hat{\mathfrak{g}}}_- \bigoplus \mathfrak{g} \bigoplus \mathbb{C}\mathbf{k} \bigoplus \mathbb{C}d.$$

Here we identify $\mathfrak{g} \otimes 1$ with $\mathfrak{g}$. We let

$$\hat{\mathfrak{h}} = \mathfrak{h} \bigoplus \mathbb{C}\mathbf{k} \bigoplus \mathbb{C}d,$$

and extend the Killing form on $\mathfrak{h}$ to $\hat{\mathfrak{h}}$ by letting $(\mathbf{k}, d) = 1$, $(\mathbf{k}, \mathbf{k}) = 0$, $(d, d) = 0$, $(\mathbb{C}\mathbf{k} + \mathbb{C}d, \mathfrak{h}) = 0$. We identify $\hat{\mathfrak{h}}^*$ with $\hat{\mathfrak{h}}$ using this bilinear form on $\hat{\mathfrak{h}}$.

Of course the Lie subalgebra

$$\hat{\mathfrak{g}} = \mathfrak{g} \bigotimes \mathbb{C}\left[t, t^{-1}\right] \bigoplus \mathbb{C}\mathbf{k} \bigoplus \mathbb{C}d$$

of $\hat{\hat{\mathfrak{g}}}$ is the affine Kac-Moody algebra associated to $\mathfrak{g}$. As usual, we let

$$\begin{aligned} \hat{\mathfrak{g}}_+ &= \mathfrak{g} \bigotimes t\mathbb{C}[t], \\ \hat{\mathfrak{g}}_- &= \mathfrak{g} \bigotimes t^{-1}\mathbb{C}\left[t^{-1}\right], \\ \hat{\mathfrak{n}}_\pm &= \hat{\mathfrak{g}}_\pm \bigoplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_{\pm\alpha}. \end{aligned}$$

We have the triangular decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \hat{\mathfrak{n}}_+$. For $\Lambda \in \hat{\mathfrak{h}}^*$ we have the Verma module $\bar{M}(\Lambda) = \mathfrak{U}(\hat{\mathfrak{g}}) \otimes_{\mathfrak{U}(\hat{\mathfrak{h}} + \hat{\mathfrak{n}}_+)} \mathbb{C}_\Lambda$ over $\hat{\mathfrak{g}}$, where $\mathbb{C}_\Lambda$ is the 1-dimensional $\mathfrak{U}\left(\hat{\mathfrak{h}} + \hat{\mathfrak{n}}_+\right)$-module defined by $h \longmapsto \Lambda(h)$, $\hat{\mathfrak{n}}_+ \longmapsto 0$. Denote by $\bar{L}(\Lambda)$ the unique irreducible quotient of $\bar{M}(\Lambda)$.

We also have a triangular decomposition

$$\hat{\hat{\mathfrak{g}}} = \hat{\hat{\mathfrak{n}}}_- \bigoplus \mathfrak{h} \bigoplus \hat{\hat{\mathfrak{n}}}_+,$$

where $\hat{\hat{\mathfrak{n}}}_\pm = \hat{\hat{\mathfrak{g}}}_\pm \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_{\pm\alpha}$. Similarly we can define the Verma module $M(\Lambda)$ over $\hat{\hat{\mathfrak{g}}}$ and its irreducible quotient $L(\Lambda)$.

1.2 Formulas for singular vectors

Set $e_0 = e_{-\theta}(1)$, $f_0 = e_\theta(-1)$, and $h_0 = \alpha_0 = \mathbf{k} - \theta$. Given $\Lambda \in \hat{\mathfrak{h}}^*$, let $\lambda_i = \Lambda(h_i)$ and $P_+ = \left\{\Lambda \in \hat{\mathfrak{h}}^* \mid \Lambda(h_i) \in \mathbb{Z}_+\right\}$.

It is well known (cf. [K]) that if $\Lambda \in P_+$, then $\{f_i^{\lambda_i+1}1, i = 0, 1, \ldots, l\}$ are the singular vectors of the Verma module $\bar{M}(\Lambda)$ of the affine algebra $\hat{\mathfrak{g}}$ which generate the maximal proper submodule of $\bar{M}(\Lambda)$, denoted by $\overline{\langle f_i^{\lambda_i+1}1, i = 0, 1, \ldots, l\rangle}$. In other words, we have

$$\bar{L}(\Lambda) \cong \bar{M}(\Lambda)/\overline{\langle f_i^{\lambda_i+1}1, i = 0, 1, \ldots, l\rangle}$$

and that the $\hat{\mathfrak{g}}$-modules $\bar{L}(\Lambda)$, $\Lambda \in P_+$, are all the unitary highest weight modules. Now let us consider the affine superalgebra $\hat{\hat{\mathfrak{g}}}$. There exists a unique hermitian form $H(\cdot, \cdot)$ on the Verma module $M(\Lambda)$ satisfying

$$H(1,1) = 1,$$

$$a(n)^* = \sigma(a)(-n), \tag{1.2.7}$$

$$\bar{a}(n)^* = \overline{\sigma(a)}(-n-1), \tag{1.2.8}$$

where $*$ defines the adjoint operator with respect to the hermitian form $H(\cdot, \cdot)$. Then $L(\Lambda) = M(\Lambda)/Ker\ H$. The $\hat{\hat{\mathfrak{g}}}$-module is called unitary if the form H on $L(\Lambda)$ is positive definite. It is known [KT] that there exists a distinguished unitary highest weight $\hat{\hat{\mathfrak{g}}}$-module of level $h^\vee$, called the minimal representation F which is given by the Fock space realization of the infinite dimensional Clifford algebra (1.1.2) and as a $\hat{\hat{\mathfrak{g}}}$-module is isomorphic to $L(h^\vee d)$. Furthermore, any unitary highest weight representation of $\hat{\hat{\mathfrak{g}}}$ is of the form $L(\Lambda + h^\vee d)$, where $\Lambda \in P_+$, and that one has an isomorphism as $\hat{\hat{\mathfrak{g}}}$-modules:

$$(\Lambda + h^\vee d) \cong F \bigotimes \bar{L}(\Lambda).$$

From the construction of the minimal representation, we also see that as $\hat{\mathfrak{g}}$-modules

$$M(\Lambda + h^\vee d) \cong F \bigotimes \bar{M}(\Lambda). \tag{1.2.9}$$

It is not difficult to see that $\left\{f_i^{\lambda_i+1}1, i = 1, \ldots, l\right\}$ are singular vectors of $M(\Lambda + h^\vee d)$.

By comparing the character formulas of both sides of (1.2.9), we see that there also exists a unique singular vector of weight $\Lambda + h^\vee d - (\lambda_0 + 1)\,\alpha_0$ in $M\,(\Lambda + h^\vee d)$. To get an explicit formula for this singular vector, we need to introduce the following notion of special roots.

Definition 1.2.1 *A root α in Δ_+ is called* special *if $\theta - \alpha$ is also a root.*

Denote by $\mathbb{S}$ the set of all special roots. The following is an equivalent way to define the set $\mathbb{S}$:

Remark 1.2.1 *The set $\mathbb{S}$ is also characterized by the property: $r_\theta(\alpha) = \alpha - \theta$, if $\alpha \in \mathbb{S}$; $r_\theta(\alpha) = \alpha$, if $\alpha \in \Delta - (\mathbb{S} \cup \{\theta\})$. It is clear that*

$$\mathbb{S} \cup \{\theta\} \;=\; \{\alpha \in \Delta_+ | r_\theta(\alpha) \in -\Delta_+\}\,. \tag{1.2.10}$$

Lemma 1.2.1 *The number of special roots is $\#\mathbb{S} = 2(h^\vee - 2)$.*

Proof. Since $(\theta,\theta) = 2$ and $(\rho,\theta) = h^\vee - 1$, we have

$$r_\theta\rho = \rho - \frac{2(\rho,\theta)}{(\theta,\theta)}\theta = \rho - (h^\vee - 1)\theta. \tag{1.2.11}$$

On the other hand, it follows from Remark 1.2.1 that

$$\begin{aligned} r_\theta\rho &= r_\theta\left(\frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha\right) \\ &= \frac{1}{2}\left(\sum_{\alpha\in\Delta_+-\theta} r_\theta(\alpha) - \theta\right) \\ &= \frac{1}{2}\left(\sum_{\alpha\in\Delta_+-\theta}\alpha - (\#\mathbb{S})\theta - \theta\right) \\ &= \rho - \frac{1}{2}\,(\#\mathbb{S} + 2)\,\theta. \end{aligned} \tag{1.2.12}$$

Comparing the right hand sides of (1.2.11) and (1.2.12), we complete the proof of the lemma. □

Remark 1.2.2 *It follows from the above lemma that the length of the reflection r_θ is $l(r_\theta) = 2h^\vee - 3$. One can choose the shortest $w \in W$ such that $w(\alpha_i) = \theta$ for some simple root α_i of $\mathfrak{g}$. $l(w) = h^\vee - 2$. Pick a reduced expression $w = r_{i_1} \cdots r_{i_{h^\vee - 2}}$.*

$$r_\theta = r_{i_1} \cdots r_{i_{h^\vee-2}} r_i r_{i_{h^\vee-2}} \cdots r_{i_1}, \tag{1.2.13}$$

is reduced. Let

$$\beta_1 = \alpha_{i_1}, \beta_2 = r_{i_1}(\alpha_{i_2}), \ldots, \beta_{h^\vee-2} = r_{i_1} \cdots r_{i_{h^\vee-3}} \left(\alpha_{i_{h^\vee-2}}\right)$$

and let

$$\gamma_s := \theta - \beta_s = w r_i r_{i_{h^\vee-2}} \cdots r_{i_{s+1}}(\alpha_{i_s}).$$

We have

$$\mathbb{S} = \{\beta_1, \ldots, \beta_{h^\vee-2}, \gamma_1, \ldots \gamma_{h^\vee-2}\}. \tag{1.2.14}$$

The sum of two elements from $\mathbb{S} \cup \{\theta\}$ is a root if and only if one of them is β_i and the other is γ_i.

Lemma 1.2.2 *Assume that $\delta_i \in \Delta_+ - \{\theta\}$, $i = 1, 2, \ldots, p$ satisfy*

$$\sum_{i=1}^{I} \delta_i = p\theta + \sum_k \eta_k \text{ for some } p \in \mathbb{N} \text{ and } \eta_k \in \Delta_+. \tag{1.2.15}$$

Then $I \geq 2p$, and at least $2p$ δ_i's are contained in $\mathbb{S}$.

Proof. Assume that there are q δ_i's which are contained in $\mathbb{S}$. By applying r_θ to both sides of the equation (1.2.15), it follows from Remark 1.2.1 that

$$\sum_{i=1}^{I} \delta_i - q\theta = -p\theta + \sum_k r_\theta(\eta_k). \tag{1.2.16}$$

By subtracting (1.2.16) from (1.2.15), we have

$$(q-2p)\theta = \sum_k \left(\eta_k - r_\theta\left(\eta_k\right)\right). \tag{1.2.17}$$

By Remark 1.2.1, we see that $\eta_k - r_\theta\left(\eta_k\right) \in Q_+ = \sum_{\alpha\in\Delta_+} \mathbb{Z}_+\alpha$. It follows that $q \geq 2p$.

□

Theorem 1.2.1 *The element $v_{\lambda_0} = \bar{e}_{-\theta}(1)\bar{e}_{-\theta}\Pi_{\alpha\in S}\bar{e}_{-\alpha}\cdot e_\theta(-1)^{\lambda_0+h^\vee+1}1$ is a singular vector in $M\left(\Lambda+h^\vee d\right)$ of weight $\Lambda + h^\vee d - \left(\lambda_0+1\right)\alpha_0$. (Here and further, $\bar{e}$ stands for $\bar{e}(0)$, where $e \in \mathfrak{g}$.)*

A different arrangement of order in $\Pi_{\alpha\in S}\bar{e}_{-\alpha}$ only makes a difference in the sign of v_{λ_0}. In the following proof, for convenience we use $x\left(m+\frac{1}{2}\right)$ to replace $\bar{x}(m)$ in $\hat{\hat{\mathfrak{g}}}$, $m \in \mathbb{Z}$.

Proof. It follows from Lemma 1.2.1 that the weight of v_{λ_0} is $\mu = \Lambda+h^\vee d-\left(\lambda_0+1\right)\alpha_0$. To prove that v_{λ_0} is a singular vector, it suffices to prove that for every homogeneous element $w \in M^s\left(\Lambda+h^\vee d\right)_\mu$, we have $H\left(v_{\lambda_0}, w\right) = 0$ and that v_{λ_0} is nonzero. That v_{λ_0} is nonzero will follow from another formula for v_{λ_0} given in Theorem 1.3.1. Here we prove the former statement.

By (1.2.8), it is enough to show that

$$H\left(e_\theta(-1)^{\lambda_0+h^\vee+1}1, \tilde{w}\right) = 0, \tag{1.2.18}$$

where

$$\tilde{w} = \Pi_{\alpha\in S}e_\alpha\left(-\frac{1}{2}\right)e_\theta\left(-\frac{3}{2}\right)e_\theta\left(-\frac{1}{2}\right)\cdot w. \tag{1.2.19}$$

Any homogeneous element w in $M\left(\Lambda+h^\vee d\right)_\mu$ is of the form

$$\Pi_{i=1}^{I}e_{\gamma_i}(-m_i)\Pi_{j=1}^{J}e_\theta(-n_j)\Pi_{k=1}^{K}e_{-\eta_k}(-l_k)\cdot 1,$$

where $\eta_k \in \Delta_+$, $\gamma_i \in \Delta_+ - \{\theta\}$, $K, I, J \in \mathbb{Z}_+, l_k \in \frac{1}{2}\mathbb{Z}_+$, m_i, $n_j \in \frac{1}{2}\mathbb{N}$, and

$$-\sum_k \eta_k + \sum_i \gamma_i + J\theta = (\lambda_0 + 1)\,\theta, \tag{1.2.20}$$

$$\sum_k l_k + \sum_i m_i + \sum_j n_j = \lambda_0 + 1. \tag{1.2.21}$$

Case 1) $n_j = \frac{1}{2}$ for some j.

Note that $e_\theta\left(-\frac{1}{2}\right)$ commutes up to a sign with elements of the form $e_\theta(-n_j)$ or $e_{\gamma_i}(-m_i)$. Since $e_\theta^2\left(-\frac{1}{2}\right) = 0$, we have $\tilde{w} = 0$ by (1.2.19). (1.2.18) is satisfied automatically.

Case 2) All $n_j \geq 1$.

On one hand, we have the inequality

$$\frac{I}{2} \leq \lambda_0 + 1 - J, \tag{1.2.22}$$

since by (1.2.21) we have

$$\frac{I}{2} = \sum_i \frac{1}{2} \leq \sum_i m_i = (\lambda_0 + 1) - \sum_j n_j - \sum_k l_k \leq d + 1 - \sum_j 1 = d + 1 - J,$$

and the equality in (1.2.22) holds iff

$$m_i = \frac{1}{2}, \quad n_j = 1, \quad l_k = 0, \tag{1.2.23}$$

for all i, j, k.

On the other hand, we rewrite (1.2.20) as

$$\sum_{i=1}^{I} \gamma_i = (\lambda_0 + 1 - J)\,\theta + \sum_k \eta_k.$$

By Lemma 1.2.2,

$$I \geq 2(\lambda_0 + 1 - J). \tag{1.2.24}$$

Comparing (1.2.22) and (1.2.24), we obtain that $I = 2(\lambda_0 + 1 - J)$ and then (1.2.23) holds. Furthermore at least $2(\lambda_0 + 1 - J)$ γ_i's are in $\mathbb{S}$. Now we divide case 2) into two subcases:

Subcase 2.1) $J < \lambda_0 + 1$.

Then at least one γ_{i_0} is in $\mathbb{S}$, i.e. w can be expressed in the form $e_{\gamma_{i_0}}\left(-\frac{1}{2}\right)w'$. Then $\tilde{w} = 0$ since $e^2_{\gamma_{i_0}}\left(-\frac{1}{2}\right) = 0$.

Subcase 2.2) $J = \lambda_0 + 1$.

Under this assumption we have $I = K = 0$. $w = e_\theta(-1)^{\lambda_0+1}1$. Then w' can be expressed in the form $e_\theta(-1)w''$ since $e_\theta(-1)$ commutes with $\Pi_{\alpha\in\mathbb{S}}e_\alpha\left(-\frac{1}{2}\right)$ $e_\theta\left(-\frac{3}{2}\right)e_\theta\left(-\frac{1}{2}\right)$. By (1.2.8), we see that (1.2.18) is equivalent to

$$H\left(e_{-\theta}(1)e_\theta(-1)^{\lambda_0+h^\vee+1}1, w''\right) = 0.$$

However it is well known (cf. formula (3.2.2) in [K]) that $e_{-\theta}(1)e_\theta(-1)^{\lambda_0+h^\vee+1}1 = 0$.

□

1.3 Reformulation for singular vectors in terms of PBW basis

We can choose $e_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}, \alpha \in \mathbb{S} \cup \{\theta\}$, in such a way that

$$[e_\alpha, e_{-\alpha}] = -\alpha, \tag{1.3.25}$$

$$[e_{-\gamma_i}, e_\theta] = e_{\beta_i}, \tag{1.3.26}$$

$$[e_{-\beta_i}, e_\theta] = -e_{\gamma_i}. \tag{1.3.27}$$

Indeed, we can pick $e_{-\gamma_i}$ and e_θ arbitrarily, and define e_{β_i} by the formula (1.3.26). Then (1.3.25) fixes e_{γ_i} and $e_{-\beta_i}$. The formula (1.3.27) holds automatically since

$$([e_{-\beta_i}, e_\theta], e_{-\gamma_i}) = (e_\theta, [e_{-\gamma_i}, e_{-\beta_i}]) = -(e_{\beta_i}, e_{-\beta_i}) = 1.$$

In the above notations, the singular vector v_{λ_0} can be written as (cf. equation (1.2.14))

$$v_{\lambda_0} = \bar{e}_{-\theta}(1)\bar{e}_{-\theta}\prod_{i=1}^{h^\vee-2}(\bar{e}_{-\beta_i}\bar{e}_{-\gamma_i})\cdot e_\theta(-1)^{\lambda_0+h^\vee+1}1.$$

Note that v_{λ_0} is independent of the order of the index i in $\prod_{i=1}^{h^\vee-2}(\bar{e}_{-\beta_i}\bar{e}_{-\gamma_i})$. Now let us rewrite this formula of singular vectors in terms of a PBW basis. We need to introduce a combinatorial symbol $[m]_n = m(m-1)\cdots(m-n+1)$.

Theorem 1.3.1 *The singular vector formula in Theorem 1.2.1 can be rewritten in terms of PBW basis as follows:*

$$\begin{aligned} v_{\lambda_0} = & \sum_{s=0}^{h^\vee-2}\sum_{(i_1,\ldots,i_s)}(k+h^\vee)^{h^\vee-s-2}[\lambda_0+h^\vee+1]_{2(h^\vee-2)-s} \\ & \times\Big((k+h^\vee)[\lambda_0-s+3]_2\, e_\theta(-1)^{\lambda_0-s+1} \\ & + [\lambda_0-s+3]_3\, e_\theta(-1)^{\lambda_0-s}\bar{e}_\theta(-1)\bar{h}_\theta(-1) \\ & + [\lambda_0-s+3]_4\, e_\theta(-1)^{\lambda_0-s-1}\bar{e}_\theta(-1)\bar{e}_\theta(-2)\Big)\, Q_{i_1}\cdots Q_{i_s}\cdot 1, \end{aligned}$$

where $Q_i = \bar{e}_{\beta_i}(-1)\bar{e}_{\gamma_i}(-1)$, and the sum $\sum_{(i_1,\ldots,i_s)}$ is taken over all subsets of the set $\{1,\ldots,h^\vee-2\}$.

Proof. We assume that whenever some negative power of $e_\theta(-1)$ appears in the following, the corresponding monomial term is zero.

It is not hard to prove by induction that

$$\begin{aligned} & \bar{e}_{-\beta_i}\bar{e}_{-\gamma_i}\cdot e_\theta(-1)^n \\ = {} & n\,(k+h^\vee)\,e_\theta(-1)^{n-1} + n(n-1)e_\theta(-1)^{n-2}\bar{e}_{\beta_i}(-1)\bar{e}_{\gamma_i}(-1) \\ & + e_\theta(-1)^n\bar{e}_{-\beta_i}\bar{e}_{-\gamma_i} - ne_\theta(-1)^{n-1}\bar{e}_{\beta_i}(-1)\bar{e}_{-\beta_i} \\ & - ne_\theta(-1)^{n-1}\bar{e}_{\alpha_i}(-1)\bar{e}_{-\alpha_i}. \end{aligned} \tag{1.3.28}$$

It follows that

$$\begin{aligned} & \bar{e}_{-\beta_i}\bar{e}_{-\gamma_i}\cdot e_\theta(-1)^n\cdot 1 \\ = {} & n\,(k+h^\vee)\,e_\theta(-1)^{n-1}\cdot 1 + n(n-1)e_\theta(-1)^{n-2}\bar{e}_{\beta_i}(-1)\bar{e}_{\gamma_i}(-1)\cdot 1. \end{aligned} \tag{1.3.29}$$

Using (1.3.28) and (1.3.29), we get by induction that

$$\begin{aligned} v' &:= \prod_{i=1}^{h^\vee-2} \bar{e}_{-\beta_i}\bar{e}_{-\gamma_i} \cdot e_\theta(-1)^{\lambda_0+h^\vee+1}1 \\ &= \sum_{s=0}^{h^\vee-2} \sum_{(i_1,\dots,i_s)} (k+h^\vee)^{h^\vee-s-2} [\lambda_0+h^\vee+1]_{(2(h^\vee-2)-s)} \\ &\quad \times e_\theta(-1)^{\lambda_0-s+3} Q_{i_1}\cdots Q_{i_s}\cdot 1. \end{aligned}$$

Since

$$[\bar{e}_{-\theta}, Q_i] = 0$$

and

$$[\bar{e}_{-\theta}, e_\theta(-1)^n] = n(n-1)e_\theta(-1)^{n-2}\bar{e}_{-\theta}(-2) + ne_\theta(-1)^{n-1}\bar{h}_\theta(-1),$$

we have

$$\begin{aligned} \bar{e}_{-\theta}v' &= \sum_{s=0}^{h^\vee-2} \sum_{(i_1,\dots,i_s)} (k+h^\vee)^{h^\vee-s-2} [\lambda_0+h^\vee+1]_{(2(h^\vee-2)-s)} \times \\ &\quad \times \Big([\lambda_0-s+3]_2\, e_\theta(-1)^{\lambda_0-s+1}\bar{e}_{-\theta}(-2) \\ &\quad + (\lambda_0-s+3)\, e_\theta(-1)^{\lambda_0-s+2}\bar{h}_\theta(-1)\Big) Q_{i_1}\cdots Q_{i_s}\cdot v_0. \end{aligned}$$

Using another identity

$$[\bar{e}_{-\theta}(1), e_\theta(-1)^n] = n(n-1)e_\theta(-1)^{n-2}\bar{e}_{-\theta}(-1) + ne_\theta(-1)^{n-1}\bar{h}_\theta,$$

we get the desired formula. □

Remark 1.3.1 *It follows from Theorem 1.3.1 that $v_{\lambda_0} \neq 0$. Moreover the only term which does not involve the odd factors is a non-zero multiple of $e_\theta(-1)^{\lambda_0+1}$. It follows that the submodule of the Verma module $M(\Lambda + h^\vee d)$ generated by v_{λ_0} is again a Verma module.*

1.4 Defining relations for the unitary representations

Theorem 1.4.1 *We have the following isomorphism*

$$L(\Lambda + h^\vee d) \cong M(\Lambda + h^\vee d)/\left\langle v_{\lambda_0}, f_i^{\lambda_i+1}1, i = 1, \dots, l \right\rangle,$$

where $\left\langle v_{\lambda_0}, f_i^{\lambda_i+1}1, i = 1, \dots, l \right\rangle$ *denotes the submodule of* $M(\Lambda + h^\vee d)$ *generated by the singular vectors* v_{λ_0}, *and* $f_i^{\lambda_i+1}1, i = 1, \dots, l$.

Proof. Since the weights of $v_{\lambda_0}, f_i^{\lambda_i+1}1, i = 1, \dots, l$ are $\Lambda + h^\vee d - (\lambda_i + 1)\alpha_i, i = 0, 1, \dots, l$ respectively, we have the following isomorphism of $\hat{\mathfrak{g}}$-modules:

$$\begin{aligned}
&\left\langle v_{\lambda_0}, f_i^{\lambda_i+1}1, i = 1, \dots, l \right\rangle \\
&= \sum_{i=0}^{l} M(\Lambda + h^\vee d - (\lambda_i + 1)\alpha_i) \\
&= \sum_{i=0}^{l} F \otimes \bar{M}(\Lambda - (\lambda_i + 1)\alpha_i) \\
&= F \otimes \overline{\left\langle f_i^{\lambda_i+1}1, i = 0, 1, \dots, l \right\rangle}.
\end{aligned}$$

Therefore we have the following isomorphism of $\hat{\mathfrak{g}}$-modules:

$$\begin{aligned}
&M(\Lambda + h^\vee d) \Big/ \left\langle v_{\lambda_0}, f_i^{\lambda_i+1}1, i = 1, \dots, l \right\rangle \\
&= \frac{F \otimes \bar{M}(\Lambda)}{F \otimes \overline{\left\langle f_i^{\lambda_i+1}1, i = 0, 1, \dots, l \right\rangle}} \\
&= F \otimes \bar{L}(\Lambda) \\
&= L(\Lambda + h^\vee d).
\end{aligned}$$

□

Chapter 2

General theory of vertex operator superalgebras

In this chapter we generalize Zhu's $A(V)$ theory [Z] and Frenkel-Zhu's $A(M)$ construction [FZ] to vertex operator superalgebras. We construct an associative algebra $A(V)$ corresponding to a SVOA V and establish a 1-1 correspondence between the irreducible representations of V and those of $A(V)$. We also define an $A(V)$-module $A(M)$ for any V-module M and then describe the fusion rules in terms of the modules $A(M)$. If we view a VOA as a SVOA with zero odd part, then our constructions reduce to the original ones in [Z] and [FZ].

2.1 Definitions and basics

For a rational function $f(z,w)$, with possible poles only at $z=w, z=0$ and $w=0$, we denote by $\iota_{z,w}f(z,w)$ the power series expansion of $f(z,w)$ in the domain $|z|>|w|$. Set $\mathbb{Z}_+=\{0,1,2,\ldots\}$, $\mathbb{N}=\{1,2,3,\ldots\}$.

A superalgebra is an algebra V with a $\mathbb{Z}_2$-gradation $V=V_{\bar{0}}\oplus V_{\bar{1}}$. Elements in $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) are called even (resp. odd). In particular, a $\frac{1}{2}\mathbb{Z}_+$-graded vector space $V=\bigoplus_{n\in\frac{1}{2}\mathbb{Z}_+}V_n$ has a natural $\mathbb{Z}_2$-gradation with $V_{\bar{0}}=\bigoplus_{n\in\mathbb{Z}_+}V_n$ as its even part and $V_{\bar{1}}=\bigoplus_{n\in\frac{1}{2}+\mathbb{Z}_+}V_n$ as its odd part. Let $\tilde{a}$ be 0 if $a\in V_{\bar{0}}$, and 1 if $a\in V_{\bar{1}}$. The general principle to extend identities in VOAs to SVOAs is the usual one: if in certain formulas of VOAs there are some monomials of vertex operators with interchanged terms, then in the corresponding formulas in SVOAs every interchange of neighboring terms, say a and b, is accompanied by multiplication of the monomial by the factor

$(-1)^{\tilde{a}\tilde{b}}$.

The general definition of a VOA was given in [B, FLM]. However, in [FKRW] we gave a somehow different but essentially equivalent definition which has simpler axioms and is easier to check in practice. The definition of SVOA below is an analog of the latter definition.

Definition 2.1.1 *A* vertex operator superalgebra *is a $\frac{1}{2}\mathbb{Z}_+$-graded vector space $V = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}_+} V_n$ with a sequence of linear operators $\{a(n) \mid n \in \mathbb{Z}\} \subset End\ V$ associated to every $a \in V$, whose generating series $Y(a,z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1} \in (End\ V)[[z,z^{-1}]]$, called the* vertex operators *associated to a, satisfy the following axioms:*

(A1) *(vacuum axiom) There exists an element $1 \in V_0$ such that $Y(1,z) = Id$ and $\lim_{z\to 0} Y(a,z)1 = a$.*

(A2) *(translation invariance) There exists an operator $T \in End V$ such that*

$$T1 = 0, \quad \partial_z Y(a,z) = Y(Ta,z) = [T, Y(a,z)].$$

(A3) *(locality) For any $a,b \in V$, there exists a positive integer N depending on a,b such that*

$$(z-w)^N Y(a,z)Y(b,w) = (-1)^{\tilde{a}\tilde{b}}(z-w)^N Y(b,w)Y(a,z).$$

A SVOA V is called *conformal* of *rank* $c \in \mathbb{C}$ if there exists an element $\omega \in V_2$ (called the *Virasoro element*), such that the corresponding vertex operator $Y(\omega,z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ satisfies $L_{-1} = T$, $L_0|_{V_n} = n\cdot \mathrm{Id}$, and $L_2\omega = \frac{1}{2}c|0\rangle$.

One can show that a SVOA automatically satisfies the associativity property:

$$Y(a,z)Y(b,w) = Y(Y(a,z-w)b,w). \tag{2.1.1}$$

Here the left-hand (resp. right-hand) side is the analytic continuation from the domain $|z| > |w|$ (resp. $|w| > |z-w|$). Formula (2.1.1) gives the operator product expansion. In particular, one easily derives that the Fourier coefficients L_n's in $Y(\omega, z)$ form a Virasoro algebra with central charge c.

A SVOA satisfies the Jacobi identity:

$$\begin{aligned}
& \mathrm{Res}_{z-w}(Y(Y(a,z-w)b,w)\iota_{w,z-w}((z-w)^m z^n)) \\
= & \mathrm{Res}_z(Y(a,z)Y(b,w)\iota_{z,w}(z-w)^m z^n) \\
& -(-1)^{\tilde{a}\tilde{b}}\mathrm{Res}_z(Y(b,w)Y(a,z)\iota_{w,z}(z-w)^m z^n)
\end{aligned}$$

for any $m, n \in \mathbb{Z}$.

This implies the equivalence between our definition of VOA and that in [B], [FLM] (cf. e.g. [FHL] [DL] [Li]).

A SVOA is called an $N = 1$ *(NS-type) SVOA* if there exists an element $\tau \in V_{\frac{3}{2}}$ (called the *Neveu-Schwarz element*), such that the corresponding vertex operator

$$Y(\tau, z) = \sum_{n\in\mathbb{Z}} G_{n+\frac{1}{2}} z^{-n-2}$$

satisfies the following two properties:

1. Let $\omega = \frac{1}{2}G_{-\frac{1}{2}}\tau$. Then ω is a Virasoro element of rank c;
2. $L_1\tau = 0, G_{-\frac{1}{2}}\tau = 2L_{-2}, G_{\frac{1}{2}}\tau = 0, G_{\frac{3}{2}}\tau = \frac{2}{3}c.$

It follows that $L_m, G_{m+\frac{1}{2}}, m \in \mathbb{Z}$ satisfy the Neveu-Schwarz algebra commutation relations

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}c, \\
\left[G_{m+\frac{1}{2}}, L_n\right] &= \left(m+\frac{1}{2}-\frac{n}{2}\right)G_{m+n+\frac{1}{2}}, \\
\left[G_{m+\frac{1}{2}}, G_{n-\frac{1}{2}}\right]_+ &= 2L_{m+n} + \frac{1}{3}m(m+1)\delta_{m+n,0}c, \quad m,n \in \mathbb{Z}.
\end{aligned}$$

An element $a \in V$ is called *homogeneous* of degree n if a is in V_n. In this case we write $\deg a = n$.

Given two vertex operators $Y(a, z)$ and $Y(b, z)$, we define the normal ordered product as

$$:Y(a,z)Y(b,z):=Y_-(a,z)Y(b,z)+(-1)^{\tilde{a}\tilde{b}}Y(b,z)Y_+(a,z), \tag{2.1.2}$$

where $Y_-(a,z)=\sum_{n<0}a(n)z^{-n-1}, Y_+(a,z)=\sum_{n\geq 0}a(n)z^{-n-1}$.

We list some properties of $SVOA$s which are anologous to those in the VOA case. For more detail see [FLM].

$$a(n)V_m\subset V_{m+\deg a-n-1}, \tag{2.1.3}$$

$$[a(n),Y(b,z)]_{\mp} = \sum_{i\geq 0}\binom{n}{i}z^{n-i}Y(a(i)b,z), \tag{2.1.4}$$

$$\begin{aligned}
[L_0,Y(a,z)] &= \left(z\frac{d}{dz}+\deg a\right)Y(a,z),\\
[L_{-1},Y(a,z)] &= \frac{d}{dz}Y(a,z), \qquad (2.1.5)\\
Y(a,z)1 &= e^{zL_{-1}}a,\\
Y(a,z)b &= (-1)^{\tilde{a}\tilde{b}}e^{zL_{-1}}Y(b,-z)a,\\
a(n)1 &= 0, \quad \text{for } n\geq 0,\\
a(-n-1)1 &= \frac{1}{n!}L_{-1}^n a \quad \text{for } n\geq 0,\\
Y(a(-1)b,z) &= :Y(a,z)Y(b,z):.
\end{aligned}$$

Moreover, $N=1$ SVOAs have the extra property that:

$$[G_{-\frac{1}{2}},Y(a,z)]_{\mp}=Y(G_{-\frac{1}{2}}a,z).$$

The following proposition allows one to check easily the SVOA axioms.

Proposition 2.1.1 *[FKRW] Let V be a $\frac{1}{2}\mathbb{Z}_+$-graded vector space. Suppose that to some vectors $a^{(0)}=|0\rangle\in V_0, a^{(1)}\in V_{\Delta_1},\ldots$, one associates fields $Y(|0\rangle,z)=Id$, $Y(a^{(1)},z)=\sum_j a_j^{(1)}z^{-j-\Delta_1},\ldots$ of conformal dimensions $0,\Delta_1,\ldots$, such that the following properties hold:*

(1) *all fields $Y(a^{(i)},z)$ are local with respect to each other (the field is even (or odd) $Y(a^{(i)},z)$ if $\Delta_i \in \mathbb{Z}_+ \frac{1}{2}+\mathbb{Z}_+$);*

(2) $\lim_{z\to 0} Y(a^{(i)},z)|0\rangle = a^{(i)}$*;*

(3) *the space V is spanned by the vectors*

$$a^{(k_s)}_{-j_s-\Delta_{k_s}} \cdots a^{(k_1)}_{-j_1-\Delta_{k_1}}|0\rangle,\ \ j_1,\ldots,j_s \in \mathbb{Z}_+; \tag{2.1.6}$$

(4) *there exists an endomorphism T of V such that*

$$\left[T, a^{(k)}_{-j-\Delta_k}\right] = (j+1)a^{(k)}_{-j-\Delta_k-1}, \quad T(|0\rangle) = 0. \tag{2.1.7}$$

Then letting

$$\begin{aligned} &Y(a^{(k_s)}_{-j_s-\Delta_{k_s}} \cdots a^{(k_1)}_{-j_1-\Delta_{k_1}}|0\rangle, z) \\ &= \ (j_1!\cdot\ldots\cdot j_s!)^{-1}\cdot :\partial_z^{j_s}Y(a^{(k_s)},z)\ldots\partial_z^{j_2}Y(a^{(k_2)},z)\partial_z^{j_1}Y(a^{(k_1)},z): \end{aligned} \tag{2.1.8}$$

(where the normal ordering of more than two fields is from right to left as usual), gives a well-defined SVOA structure on V.

We shall say that the SVOA constructed in Proposition 2.1.1 is *generated* by the fields $Y(a^{(i)},z), i>0$.

Now we formulate the definition of a module (or representation) of a conformal SVOA. One can easily modify this to define a module of any SVOA.

Definition 2.1.2 *Given a conformal SVOA V,* a representation of V (or V-module) *is a $\frac{1}{2}\mathbb{Z}_+$-graded vector space $M = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}_+} M_n$ and a linear map*

$$\begin{aligned} V &\longrightarrow (End\ M)\,[[z,z^{-1}]]\,, \\ a &\longmapsto Y_M(a,z) = \textstyle\sum_{n\in\mathbb{Z}} a(n)z^{-n-1}, \end{aligned}$$

satisfying

(R1) $a(n)M_m \subset M_{m+\deg a-n-1}$ *for every homogeneous element a.*

(R2) $Y_M(1,z) = I_M$.

Setting $Y_M(\omega,z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, *we have*

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}c, \\ Y_M(L_{-1}a, z) &= \frac{d}{dz}Y_M(a,z) \text{ for every } a \in V. \end{aligned}$$

(R3) *The* Jacobi identity

$$\begin{aligned} & Res_{z-w}\left(Y_M\left(Y(a,z-w)b,w\right)\iota_{w,z-w}\left((z-w)^m z^n\right)\right) \\ &= Res_z\left(Y_M(a,z)Y_M(b,w)\iota_{z,w}(z-w)^m z^n\right) \\ &\quad -(-1)^{\tilde{a}\tilde{b}}Res_z\left(Y_M(b,w)Y_M(a,z)\iota_{w,z}(z-w)^m z^n\right) \end{aligned}$$

holds for any $m,n \in \mathbb{Z}$.

Definition 2.1.3 *Given an* $N=1$ *SVOA* V, M *is called* a representation of V *if the axiom* **(R2)** *is replaced by the following stronger axiom:*

(R2') *Set* $Y_M(\tau,z) = \sum_{n\in\mathbb{Z}} G_{n+\frac{1}{2}} z^{-n-2}$ *and* $\omega := \frac{1}{2}G_{-\frac{1}{2}}\tau$. *Then* ω *satisfies (R2), and the commutation relations*

$$\begin{aligned} \left[G_{m+\frac{1}{2}}, L_n\right] &= \left(m+\frac{1}{2}-\frac{n}{2}\right)G_{m+n+\frac{1}{2}}, \\ \left[G_{m+\frac{1}{2}}, G_{n-\frac{1}{2}}\right]_+ &= 2L_{m+n} + \frac{1}{3}m(m+1)\delta_{m+n,0}c, \quad m,n\in\mathbb{Z} \end{aligned}$$

also hold.

The notions of submodules, quotient modules, submodules generated by a subset, direct sums, irreducible modules, completely reducible modules, etc., can be introduced in the usual way. As a module over itself, V is called the *adjoint module*. A submodule of the adjoint module is called an *ideal* of V. Given an ideal I in V such that $1 \notin I$, $\omega \notin I$, the quotient V/I admits a natural SVOA structure.

Definition 2.1.4 *A SVOA is called* rational *if it has finitely many irreducible modules and every module is a direct sum of irreducibles.*

We will now extend the definition of intertwining operators and fusion rules of representations of VOAs [FHL] to SVOAs.

For simplicity, we will only define an intertwining operator for V-modules $M^i = \oplus_{n\in\frac{1}{2}\mathbb{Z}_+} M^i(n)$, $i = 1,2,3$, satisfying $L_0 \mid_{M^i(n)} = (h_i + n)\, I \mid_{M^i(n)}$, for some complex numbers h_1, h_2, h_3. We define a $\mathbb{Z}_2$-gradation of M^i by letting $\tilde{v} = 0$ if $v \in M^i(n)$, $n \in \mathbb{Z}$; $\tilde{v} = 1$ if $v \in M^i(n)$, $n \in \frac{1}{2} + \mathbb{Z}$.

Definition 2.1.5 *Under the above assumptions, an intertwining operator of type* $\binom{M^3}{M^1\ M^2}$ *is a linear map*

$$I(\cdot, z) : v \mapsto \sum_{n\in I} v(n) z^{-n-1+(h_3-h_1-h_2)}, \quad v \in M^1, \quad v(n) \in Hom_{\mathbb{C}}\left(M^2, M^3\right)$$

satisfying

(I1) *For homogeneous* $v \in M^1$,

$$I(L_{-1}v, z) = \frac{d}{dz} I(v, z) \ \ \textit{for every } v \in M^1,$$

(I2) *For any* $a \in V, v \in M^1$, *and* $m, n \in \mathbb{Z}$,

$$\begin{aligned} &Res_{z-w}\left(I(Y(a, z-w)v, w)\iota_{w,z-w}\left((z-w)^m z^n\right)\right) \\ = \;& Res_z(Y(a,z)I(v,w)\iota_{z,w}(z-w)^m z^n) \\ &- (-1)^{\tilde{a}\tilde{v}} Res_z\left(I(v,w)Y(a,z)\iota_{w,z}(z-w)^m z^n\right). \end{aligned}$$

We denote by $I\binom{M^3}{M^1\ M^2}$ the vector space of intertwining operators of type $\binom{M^3}{M^1\ M^2}$. An immediate consequence of this definition is that for homogeneous $v \in M^1$,

$$v(n) M^2_m \subset M^3_{m+\deg v-n-1},$$

where $\deg v = k$ means that $v \in M^1_k$.

Assuming that V is a rational SVOA and $\{M^i,\ i \in J\}$ is the complete set of the irreducible modules of V, we denote by N^{ij}_k the dimension of the vector space $I\binom{M^k}{M^i\ M^j}$ and define the fusion rules as the formal product

$$M^i \times M^j = \sum_{k\in J} N^{ij}_k M^k.$$

2.2 Zhu algebra $A(V)$ and related theorems

Definition 2.2.1 *We define bilinear maps $* : V \times V \to V$, $\circ : V \times V \to V$ as follows. For homogeneous a, b, let*

$$a * b = \begin{cases} Res_z\left(Y(a,z)\frac{(z+1)^{\deg a}}{z}b\right), & \text{if } a,b \in V_{\bar{0}},\\ 0, & \text{if } a \text{ or } b \in V_{\bar{1}}.\end{cases}$$

$$a \circ b = \begin{cases} Res_z\left(Y(a,z)\frac{(z+1)^{\deg a}}{z^2}b\right), & \text{for } a \in V_{\bar{0}},\\ Res_z\left(Y(a,z)\frac{(z+1)^{\deg a-\frac{1}{2}}}{z}b\right), & \text{for } a \in V_{\bar{1}}.\end{cases}$$

and then extend to $V \times V$ bilinearly. Letting $O(V) \subset V$ be the linear span of elements of the form $a \circ b$, we define $A(V)$ to be the quotient space $V/O(V)$.

Remark 2.2.1 *1) $O(V)$ is a $\mathbb{Z}_2$-graded subspace of V.*

2) If $a \in V_{\bar{1}}$ homogeneous, then

$$a \circ 1 = Res_z\left(Y(a,z)\frac{(z+1)^{\deg a-\frac{1}{2}}}{z}1\right) = a.$$

Hence $O(V) = O_{\bar{0}}(V) + V_{\bar{1}}$, where $O_{\bar{0}}(V) = O(V) \cap V_{\bar{0}}$. Thus $A(V) = V_{\bar{0}}/O_{\bar{0}}(V)$. Denote by $O_e(V)$ (resp. $O_d(V)$) the linear span of the elements $a \circ b$ for $a, b \in V_{\bar{0}}$ (resp. $V_{\bar{1}}$). The intersection $O_e(V) \cap O_d(V)$ need not be empty.

It is convenient to introduce an equivalence relation $\sim$ as follows. For $a, b \in V$, $a \sim b$ means $a - b \equiv 0 \bmod O(V)$. For $f, g \in End\ V$, $f \sim g$ means $f \cdot c \sim g \cdot c$ for any $c \in V$. Denote by $[a]$ the image of a in V under the projection of V onto $A(V)$.

Lemma 2.2.1 *1)* $L_{-1}a + L_0 a \sim 0$ *if* $a \in V_{\bar{0}}$.

2) For every homogeneous element $a \in V$, *and* $m \geq n \geq 0$, *we have*

$$Res_z \left(Y(a,z) \frac{(z+1)^{\deg a+n}}{z^{2+m}} \right) \sim 0, \quad \textit{if } a \in V_{\bar{0}}.$$
$$Res_z \left(Y(a,z) \frac{(z+1)^{\deg a+n-\frac{1}{2}}}{z^{1+m}} \right) \sim 0, \quad \textit{if } a \in V_{\bar{1}}.$$

3) For any homogeneous element $a, b \in V_{\bar{0}}$, *we have*

$$a * b \sim Res_z \left(Y(b,z) \frac{(z+1)^{\deg b-1}}{z} a \right).$$

Proof. Noting that $V_{\bar{0}}$ is a vertex operator algebra, we see that 1), the first part of 2) and 3) are essentially the same as Lemma 2.1.1, 2.1.2 and 2.1.3 in [Z]. The proof of the second part of 2) is similar to that of the first part. □

Theorem 2.2.1 *1)* $O(V)$ *is a two-sided ideal of* V *under the multiplication* $*$. *Moreover, the quotient algebra* $(A(V), *)$ *is associative.*

2) $[1]$ *is the unit element of the algebra* $A(V)$.

3) $[\omega]$ *is in the center of* $A(V)$.

4) $A(V)$ *has a filtration* $A_0(V) \subset A_1(V) \subset \ldots$, *where* $A_n(V)$ *is the image of* $\oplus_{i \in \frac{1}{2}\mathbb{Z}_+, i \leq n} V_i$.

Sketch of a proof. To prove 1), it is enough to prove the following relations:

$$O_{\bar{0}}(V) * V \subset O(V),$$
$$V_{\bar{0}} * O_{\bar{0}}(V) \subset O(V),$$
$$(a * b) * c - a * (b * c) \in O(V).$$

By the definition of the operation $*$ and Remark 2.2.1, it suffices to prove that for homogeneous elements a, b, c one has

$$(a \circ b) * c \in O(V) \text{ for } a, b, c \in V_{\bar{0}}, \tag{2.2.9}$$

$$a * (b \circ c) \in O(V) \text{ for } a, b, c \in V_{\bar{0}}, \tag{2.2.10}$$

$$(a \circ b) * c \in O(V) \text{ for } a, b \in V_{\bar{1}}, \tag{2.2.11}$$

$$a * (b \circ c) \in O(V) \text{ for } a \in V_{\bar{0}}, b, c \in V_{\bar{1}}, \tag{2.2.12}$$

$$(a * b) * c - a * (b * c) \in O(V) \text{ for } a, b, c \in V_{\bar{0}}. \tag{2.2.13}$$

The proofs of (2.2.9), (2.2.10) and (2.2.13) are essentially the same as in the VOA cases (see the proof of Theorem 2.1.1 in [Z]).

To prove (2.2.11), for $a, b \in V_{\bar{1}}$, $c \in V$ homogeneous, we have

$$\begin{aligned}
&(a \circ b) * c \\
= \;& Res_z \left(Y(a, z) \frac{(z+1)^{\deg a - \frac{1}{2}}}{z} b \right) * c \\
= \;& \sum_{i=0}^{\deg a - \frac{1}{2}} \binom{\deg a - \frac{1}{2}}{i} (a(i-1)b) * c \\
= \;& \sum_{i=0}^{\deg a - \frac{1}{2}} \binom{\deg a - \frac{1}{2}}{i} Res_w \left(Y(a(i-1)b, w) \frac{(w+1)^{\deg a + \deg b - i}}{w} c \right) \\
= \;& \sum_{i=0}^{\deg a - \frac{1}{2}} \binom{\deg a - \frac{1}{2}}{i} Res_w Res_{z-w} \\
& \times \left(Y(Y(a, z-w)b, w)(z-w)^{i-1} \frac{(w+1)^{\deg a + \deg b - i}}{w} c \right) \\
= \;& Res_w Res_{z-w} \left(Y(Y(a, z-w)b, w) \frac{(z+1)^{\deg a - \frac{1}{2}} (w+1)^{\deg b + \frac{1}{2}}}{w(z-w)} c \right) \\
= \;& Res_z Res_w \left(Y(a, z) Y(b, w) \frac{(z+1)^{\deg a - \frac{1}{2}} (w+1)^{\deg b + \frac{1}{2}}}{w(z-w)} c \right) \\
& + Res_w Res_z \left(Y(b, w) Y(a, z) \frac{(z+1)^{\deg a - \frac{1}{2}} (w+1)^{\deg b + \frac{1}{2}}}{w(z-w)} c \right) \\
= \;& \sum_{i \in \mathbb{Z}_+} Res_z Res_w \left(Y(a, z) Y(b, w) z^{-1-i} w^i \frac{(z+1)^{\deg a - \frac{1}{2}} (w+1)^{\deg b + \frac{1}{2}}}{w} c \right)
\end{aligned}$$

$$- \sum_{i\in\mathbb{Z}_+} Res_w Res_z \left(Y(b,w)Y(a,z)w^{-1-i}z^i \frac{(z+1)^{\deg a-\frac{1}{2}}(w+1)^{\deg b+\frac{1}{2}}}{w} c \right).$$

By Lemma 2.2.1 the right hand side of the last identity is in $O(V)$.

To prove (2.2.12), for $a \in V_{\bar{0}}, b, c \in V_{\bar{1}}$ homogeneous, we have

$$\begin{aligned} & a * (b \circ c) - b \circ (a * c) \\ = \; & Res_z \left(Y(a,z)\frac{(z+1)^{\deg a}}{z} \right) Res_w \left(Y(b,w)\frac{(w+1)^{\deg b-\frac{1}{2}}}{w} c \right) \\ & - Res_w \left(Y(b,w)\frac{(w+1)^{\deg b-\frac{1}{2}}}{w} \right) Res_z \left(Y(a,z)\frac{(z+1)^{\deg a}}{z} c \right) \\ = \; & Res_w Res_{z-w} \left(Y(Y(a,z-w)b,w)\frac{(z+1)^{\deg a}}{z}\frac{(w+1)^{\deg b-\frac{1}{2}}}{w} c \right) \\ = \; & \sum_{i=0}^{\deg a}\sum_{j\in\mathbb{Z}_+} \binom{\deg a}{i} Res_w \left(Y(a(i+j)b,w)(-1)^j \frac{(w+1)^{\deg a+\deg b-i-\frac{1}{2}}}{w^{j+2}} c \right). \end{aligned}$$

Since $\deg(a(i+j)b) = \deg a + \deg b - i - j - 1$, and $a(i+j)b \in V_{\bar{1}}$, by Lemma 2.2.1, the right-hand side of the last identity is in $O(V)$. The second term of the left-hand side is also in $O(V)$ by definition. Then so is the first term.

The proof of statements 2), 3) and 4) is the same as in the VOA case. (For details see the proof of Theorem 2.1.1 in [Z]). □

The associative algebra $A(V)$ corresponding to a vertex operator algebra V was constructed in [Z]. From now on, we will refer to $A(V)$ as Zhu algebra.

The following proposition follows from the definition of $A(V)$.

Proposition 2.2.1 *Let I be an ideal of V with the $\mathbb{Z}_2$-gradation $I_{\bar{0}} \oplus I_{\bar{1}}$ consistent with that of V. Assume $1 \notin I, \omega \notin I$. Then the Zhu algebra $A(V/I)$ is isomorphic to $A(V)/[I]$, where $[I]$ is the image of I in $A(V)$.*

For any homogeneous $a \in V_{\bar{0}}$ we define $o(a) = a(\deg a - 1)$ and extend this map linearly to $V_{\bar{0}}$. It follows from (2.1.3) that $o(a)M_n \subset M_n$. In particular, $o(a)$ maps M_0 into itself. We may assume that $M_0 \neq 0$ without loss of generality.

Theorem 2.2.2 *Let $M = \oplus_{n\in\frac{1}{2}\mathbb{Z}_+} M_n$ be a V-module. Then M_0 is an $A(V)$-module* *defined as follows: for* $[a] \in A(V)$*, let* $a \in V_{\bar{0}}$ *be a preimage of* $[a]$*. Then* $[a]$ *acts on* M_0 *as* $o(a)$*.*

Proof. An equivalent way to state this theorem is that for $a, b \in V_{\bar{0}}$, $o(a)o(b) |_{M_0} = o(a * b) |_{M_0}$, and for $c \in O(V) = O_d(V) + O_e(V)$, $o(c) |_{M_0} = 0$. We only need to prove that $o(c) |_{M_0} = 0$ for $c \in O_d(V)$ since $V_{\bar{0}}$ is a vertex operator algebra and so the rest of the statements above holds (For details see Theorem 2.1.2 and its proof in [Z]).

Given $a, b \in V_{\bar{1}}$ homogeneous, we have

$$
\begin{aligned}
& o(a \circ b) \\
&= o\left(Res_z\left(Y(a,z)\frac{(z+1)^{\deg a - \frac{1}{2}}}{z}b\right)\right) \\
&= \sum_{i=0}^{\deg a - \frac{1}{2}} \binom{\deg a - \frac{1}{2}}{i} o\left(a(i-1)b\right) \\
&= \sum_{i=0}^{\deg a - \frac{1}{2}} \binom{\deg a - \frac{1}{2}}{i} \left(a(i-1)b\right)\left(\deg a + \deg b - i - 1\right) \\
&= Res_w Res_{z-w} \sum_{i=0}^{\deg a - \frac{1}{2}} \binom{\deg a - \frac{1}{2}}{i} \times \\
&\quad \times \left(Y((a, z-w)b, w)(z-w)^{i-1} w^{\deg a + \deg b - i - 1}\right) \\
&= Res_w Res_{z-w} \left(Y((a, z-w)b, w)\frac{z^{\deg a - \frac{1}{2}} w^{\deg b - \frac{1}{2}}}{z-w}\right) \\
&= Res_z Res_w \left(Y(a,z)Y(b,w)\frac{z^{\deg a - \frac{1}{2}} w^{\deg b - \frac{1}{2}}}{z-w}\right) \\
&\quad + Res_w Res_z \left(Y(b,w)Y(a,z)\frac{z^{\deg a - \frac{1}{2}} w^{\deg b - \frac{1}{2}}}{z-w}\right) \\
&= \sum_{i \in \mathbb{Z}_+} Res_z Res_w \left(Y(a,z)Y(b,w) z^{\deg a - i - \frac{3}{2}} w^{\deg b - \frac{1}{2} + i}\right) \\
&\quad - \sum_{i \in \mathbb{Z}_+} Res_w Res_z \left(Y(b,w)Y(a,z) z^{\deg a + i - \frac{1}{2}} w^{\deg b - i - \frac{3}{2}}\right) \\
&= \sum_{i \in \mathbb{Z}_+} a(\deg a - i - \frac{3}{2}) b(\deg b + i - \frac{1}{2}) \\
&\quad - \sum_{i \in \mathbb{Z}_+} b(\deg b - i - \frac{3}{2}) a(\deg a + i - \frac{1}{2}).
\end{aligned}
$$

The right-hand side of the above identities acting on M_0 is 0 since

$$a(\deg a + i - \frac{1}{2})\mid_{M_0} = b(\deg b + i - \frac{1}{2})\mid_{M_0} = 0.$$

□

Theorem 2.2.3 *Given an $A(V)$-module (W, π), there exists a V-module $M = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}_+} M_n$ such that the $A(V)$-modules M_0 and W are isomorphic. Moreover, this gives a bijective correspondence between the set of irreducible $A(V)$-modules and the set of irreducible V-modules.*

Sketch of a proof. First we have the following recurrent formula for n-correlation functions on $\langle M_0, (M_0)^*\rangle$ for a given V-module $M = \oplus_{i\in\frac{1}{2}\mathbb{Z}_+} M_i$, where M_0^* is the dual space of M_0. (The proof is similar to that of Lemma 2.2.1 in [Z].) Given $v \in M_0$, $v' \in M_0^*$, and homogeneous $a_1 \in V$, we have

$$\begin{aligned}
&\langle v',\ Y(a_1,z_1)Y(a_2,z_2)\cdots Y(a_m,z_m)v\rangle \\
&= \begin{cases}
\sum_{k=2}^{m}\sum_{i\in\mathbb{Z}_+}(-1)^{(\tilde{a}_2+\cdots+\tilde{a}_{k-1})}F_{\deg a_1-\frac{1}{2},i}(z_1,z_k) \\
\quad \times \langle v', Y(a_2,z_2)\cdots Y(a_1(i)a_k,z_k)\cdots Y(a_m,z_m)v\rangle \text{ if } a_1\in V_{\bar{1}}, \\
\sum_{k=2}^{m}\sum_{i\in\mathbb{Z}_+}F_{\deg a_1,i}(z_1,z_k)\times \\
\quad \times \langle v', Y(a_2,z_2)\cdots Y(a_1(i)a_k,z_k)\cdots Y(a_m,z_m)v\rangle \\
\quad + z_1^{-\deg a_1}\langle a_1(\deg a_1-1)^*v', Y(a_2,z_2)\cdots Y(a_m,z_m)v\rangle \text{ if } a_1\in V_{\bar{0}},
\end{cases}
\end{aligned}$$

where $F_{\deg a,i}$ is defined by

$$\begin{aligned}
F_{n,i}(z,w) &= \sum_{j\in\mathbb{Z}_+}\binom{n+j}{i}z^{-n-j}w^{n+j-i} \\
&= \iota_{z,w}\left(z^{-n}\frac{1}{i!}\left(\frac{d^i}{dw^i}\right)\frac{w^n}{z-w}\right).
\end{aligned}$$

This recurrent formula means that the n-correlation functions on $\langle M_0, (M_0)^*\rangle$ are determined by the $A(V)$-module structure on M_0. The completion of the proof of this theorem is similar to that in Theorem 2.2.1 in [Z]. □

Remark 2.2.2 *Thus we have a functor from the category of V-modules to the category of $A(V)$-modules which is bijective on the sets of irreducibles.*

2.3 Fusion rules

In this subsection, to generalize the construction of [FZ], we define a bimodule $A(M)$ of the Zhu algebra $A(V)$ for every V-module M. We then give a description of the fusion rules in terms of $A(M)$.

Definition 2.3.1 *Given a V-module M, we define bilinear operations $a * v$ and $v * a$ by letting*

$$a * v = Res_z \left(Y(a,z) \frac{(z+1)^{\deg a}}{z} v \right), \text{ for } a \in V_{\bar{0}}, \tag{2.3.14}$$

$$v * a = Res_z \left(Y(a,z) \frac{(z+1)^{\deg a - 1}}{z} v \right), \text{ for } a \in V_{\bar{0}}, \tag{2.3.15}$$

$$a * v = 0, \quad v * a = 0, \text{ for } a \in V_{\bar{1}},$$

for a homogeneous element $a \in V$ and $v \in M$, and then extend linearly to V. We also define $O(M) \subset M$ to be the linear span of elements of the forms

$$Res_z \left(Y(a,z) \frac{(z+1)^{\deg a}}{z^2} v \right), \text{ for } a \in V_{\bar{0}} \text{ and}$$

$$Res_z \left(Y(a,z) \frac{(z+1)^{\deg a - \frac{1}{2}}}{z} v \right), \text{ for } a \in V_{\bar{1}}.$$

Let $A(M)$ be the quotient space $M/O(M)$.

We have the following theorem.

Theorem 2.3.1 *$A(M)$ is an $A(V)$-bimodule with the left action of $A(V)$ defined by (2.3.14) and the right action by (2.3.15). Moreover the left and right action of $A(V)$ commute with each other.*

Sketch of a proof. By a similar argument to Lemma 2.2.1, we see that

$$Res_z\left(Y(a,z)\frac{(z+1)^{\deg a+n}}{z^{2+m}}v\right) \in O(M), \text{ for } a\in V_{\bar{0}},$$
$$Res_z\left(Y(a,z)\frac{(z+1)^{\deg a+n-\frac{1}{2}}}{z^{1+m}}v\right) \in O(M), \text{ for } a\in V_{\bar{1}},$$

for $m\geq n\geq 0$, $v\in M$.

Recall that $O(V)=O_d(V)+O_e(V)$. To prove the theorem, we need to check that

$$O_d(V)*v\subset O(M), \qquad v*O_d(V)\subset O(M), \tag{2.3.16}$$
$$O_e(V)*v\subset O(M), \qquad v*O_e(V)\subset O(M), \tag{2.3.17}$$
$$a*O(M)\subset O(M), \qquad O(M)*a\subset O(M), \tag{2.3.18}$$
$$(a*b)*v-a*(b*v) \in O(M), \tag{2.3.19}$$
$$(v*a)*b-v*(a*b) \in O(M), \tag{2.3.20}$$
$$(a*v)*b-a*(v*b) \in O(M). \tag{2.3.21}$$

The proof of (2.3.16) is similar to that of Theorem 2.2.1. The proofs of (2.3.17), (2.3.18), (2.3.19), (2.3.20), and (2.3.21) are similar to those in [Z]. □

Given left V-modules $M^i=\oplus_{n\in\frac{1}{2}\mathbb{Z}_+}M^i(n)$, $i=1,2,3$. Note that $M^2(0)$ is a left module over $A(V)$, $(M^3(0))^*$ is a right module over $A(V)$, and $A(M^1)$ is a bimodule over $A(V)$. Hence we can consider the tensor product $M^3(0)^*\otimes_{A(V)}A(M^1)\otimes_{A(V)}M^2(0)$ of $A(V)$-modules.

Theorem 2.3.2 *Let V be a rational VOA and $M^i=\sum_{n\in\frac{1}{2}\mathbb{Z}_+}M^i(n)(i=1,2,3)$ be V-modules, satisfying $L_0\mid_{M^i(n)}=(h_i+n)I\mid_{M^i(n)}$, for some complex numbers h_1,h_2,h_3.*

1) *Let $I(\cdot,z)$ be an intertwining operator of type $\binom{M^3}{M^1\ M^2}$. Then $\langle v_3',o(v_1)v_2\rangle$ defines a linear functional f_I on $M^3(0)^*\otimes_{A(V)}A(M^1)\otimes_{A(V)}M^2(0)$, where $v_3'\in M^3(0)^*$, $v_1\in M^1$, $v_2\in M^2$,*

2) *The map $I\mapsto f_I$ given in 1) defines an isomorphism of vector spaces $I\binom{M^3}{M^1\ M^2}$ and $\left(M^3(0)^*\otimes_{A(V)}A(M^1)\otimes_{A(V)}M^2(0)\right)^*$ if M^i $(i=1,2,3)$ are*

irreducible.

Proof. The argument is similar to that in Theorem 2.2.3. □

As a consequence, we obtain the following proposition, which is an anologue of Proposition 1.5.4 in [FZ].

Proposition 2.3.1 *1) Given a V-module M and a submodule M^1 of M, then the image $A(M^1)$ of M^1 in $A(M)$, is a submodule of $A(V)$-bimodule $A(M)$, and the quotient $A(M)/A(M^1)$ is isomorphic to the bimodule $A(M/M^1)$ corresponding to the quotient V-module M/M^1.*

2) If I is an ideal of V, $1 \notin I$, $\omega \notin I$, and $I \cdot M \subset M^1$, then $A(V/I)$-bimodule $A(M)/A(M^1)$ is isomorophic to the $A(M/M^1)$.

Remark 2.3.1 *One can also consider the pre-SVOA (i.e., the SVOA which may not admit a Virasoro element). Similarly to the VOA case, one can still define the associative algebra $A(V)$ and the $A(V)$-module $A(M)$ for any V-module M [L]. Theorems 2.2.3 and 2.3.2 are valid for the pre-SVOAs.*

Chapter 3

Three classes of vertex operator superalgebras

In Sect. 3.1 we construct the SVOAs $M_{k,0}$ and $L_{k,0}$ corresponding to the affine Kac-Moody superalgebras. We prove the rationality, classify the irreducible representations of $L_{k,0}$ for positive integer k and compute the fusion rules. In Sect. 3.1.2 we construct the SVOAs M_c and V_c corresponding to the Neveu-Schwarz algebras. We prove that the rationality of V_c for the unitary series and that its irreducible representations are exactly the unitary minimal modules. We conjecture that a similar statement remains true for the non-unitary minimal series. In Sect. 3.3 we construct the SVOAs corresponding to the free charged or neutral fermions. We show that such a SVOA is rational and has itself as a unique irreducible representation.

3.1 SVOAs associated to affine Kac-Moody superalgebras

3.1.1 SVOA structures on $M_{k,0}$ and $L_{k,0}$

We continue using the notations on the affine (super)algebras in Chap. 1.

Given a $\mathfrak{g}$-module V and a complex number k, we can define the induced module $\tilde{V}_k$ over $\hat{\mathfrak{g}}$ as follows: V can be viewed as a module over $\hat{\mathfrak{g}}_+ + \mathfrak{g} + \mathbb{C}\mathbf{k} + \mathbb{C}d$ by letting $(\hat{\mathfrak{g}}_+ \oplus \mathbb{C}d)V = 0$ and $\mathbf{k} = (k + h^\vee)\ I\ |_V$. Then let

$$\tilde{V}_k = \mathfrak{U}(\hat{\mathfrak{g}}) \bigotimes_{\mathfrak{U}(\hat{\mathfrak{g}}_+ + \mathfrak{g} + \mathbb{C}\mathbf{k} + \mathbb{C}d)} V.$$

Here and further $\mathfrak{U}(\mathfrak{A})$ denotes the universal enveloping algebra of a Lie (super)algebra $\mathfrak{A}$. In particular for any $\lambda \in \mathfrak{h}^*$, we let $L(\lambda)$ be the irreducible highest weight $\mathfrak{g}$-module with highest weight λ, and denote the $\hat{\hat{\mathfrak{g}}}$-module $\widetilde{L(\lambda)}_k$ by $M_{k,\lambda}$. Let $J_{k,\lambda}$ be the maximal proper submodule of the $\hat{\hat{\mathfrak{g}}}$-module $M_{k,\lambda}$. Denote $M_{k,\lambda}/J_{k,\lambda}$ by $L_{k,\lambda}$. Note that $M_{k,\lambda}$ is a quotient module of the Verma module $M\Lambda$, where $\Lambda \mid \mathfrak{h} = \lambda, \Lambda(\mathbf{k}) = k, \Lambda(d) = 0$. If $\lambda = 0$, then $L(0)$ is the trivial $\mathfrak{g}$-module $\mathbb{C}$ and $M_{k,0} \cong \mathfrak{U}\left(\hat{\hat{\mathfrak{g}}}_-\right)$ as $\hat{\hat{\mathfrak{g}}}_-$-modules.

Define a $\frac{1}{2}\mathbb{Z}$-gradation of $\hat{\hat{\mathfrak{g}}}$ by the eigenvalues of $-d$:

$$\deg \mathbf{k} = 0, \quad \deg a(n) = -n, \quad \deg \bar{a}(n) = -n - \frac{1}{2}, \quad \text{for } a \in \mathfrak{g}.$$

This induces $\frac{1}{2}\mathbb{Z}$-gradations of $\mathfrak{U}(\hat{\hat{\mathfrak{g}}}), \mathfrak{U}(\hat{\hat{\mathfrak{g}}}_-)$, and thus of $M_{k,\lambda}$ if we let the degree of the highest weight vector, denoted by 1, of $M_{k,0}$ to be zero. We have the gradation decompositions of $M_{k,0}$

$$M_{k,0} = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_+} M_{k,0}(n).$$

In particular, we have

$$M_{k,0}(0) = \mathbb{C} \cdot 1, \quad M_{k,0}(\frac{1}{2}) = \bar{\mathfrak{g}}(-1) \cdot 1 \cong \bar{\mathfrak{g}}, \quad M_{k,0}(1) = \mathfrak{g}(-1) \cdot 1 \cong \mathfrak{g}.$$

Define $Y(1, z) = Id$. For $a \in \mathfrak{g} \subset M_{k,0}$, $\bar{a} \in \bar{\mathfrak{g}} \subset M_{k,0}$, we define

$$a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}, \bar{a}(z) = \sum_{n \in \mathbb{Z}} \bar{a}(n) z^{-n-1}.$$

It is clear that $a(z)$ is even while $\bar{a}(z)$ is odd.

Choose a basis $\{u_i, i = 1, \cdots, \dim \mathfrak{g}\}$ of $\mathfrak{g}$ satisfying $(u_i, u_j) = \frac{1}{2}\delta_{ij}$, $[u_i, u_j] = \sqrt{-1} f_{ijk} u_k$, where f_{ijk} is anti-symmetric in i, j, k and real valued. Here and below we assume summation over repeated indices as usual .

Theorem 3.1.1 *$(M_{k,0}, 1, \omega, \tau, Y(\ , z))$ is an $N = 1$ SVOA of rank c_k provided that*

$k \neq -h^\vee$, where

$$\begin{aligned} c_k &= \frac{\dim \mathfrak{g}}{2} + \frac{k \dim \mathfrak{g}}{k+h^\vee} \\ \tau &= \frac{2}{k+h^\vee} u_i(-1)\bar{u}_i(-1)1 + \frac{4\sqrt{-1}}{3(k+h^\vee)^2} f_{ijk}\bar{u}_i(-1)\bar{u}_j(-1)\bar{u}_k(-1)1, \\ \omega &= \frac{1}{k+h^\vee}\{u_i(-1)u_i(-1)1 + \bar{u}_i(-2)\bar{u}_i(-1)1\} \\ &\quad + \frac{2\sqrt{-1}}{3(k+h^\vee)^2} f_{ijk}\bar{u}_i(-1)\bar{u}_j(-1)u_k(-1)1, \end{aligned}$$

and the map

$$Y(\ ,z) : M_{k,0} \to End\ (M_{k,0})[[z,z^{-1}]]$$

is defined by letting

$$u_{p_n}(-i_n-1)\dots u_{p_1}(-i_1-1)\bar{u}_{q_m}(-j_m-1)\dots\bar{u}_{q_1}(-j_1-1)1 \longmapsto$$

$$(i_1!\cdot\ldots\cdot i_n! j_1!\cdot\ldots\cdot j_m!)^{-1}\cdot :\partial_z^{i_n}u_{p_n}(z)\dots\partial_z^{i_1}u_{p_1}(z)\partial_z^{j_m}\bar{u}_{q_m}(z)\dots\partial_z^{j_1}\bar{u}_{q_1}(z) :,$$

where $p_1,\dots,p_n,q_1,\dots,q_m$ take values in $1,\dots,\dim\mathfrak{g}$, and $i_1,\dots,i_n,j_1,\dots,j_m \in \mathbb{Z}_+$.

Proof. The fact that the components of the fields

$$Y(\tau,z) = \sum_{n\in\mathbb{Z}} G_{n+\frac{1}{2}} z^{-n-2}, \quad Y(\omega,z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2},$$

satisfy the commutation relations of the Neveu-Schwarz algebra with the central charge c_k is ensured by Theorem 4 in [KT]. One can easily check that

$$\begin{aligned} (z-w)^2[a(z),b(w)] &= 0, \\ (z-w)^2[a(z),\bar{b}(w)] &= 0, \\ (z-w)^2[\bar{a}(z),\bar{b}(w)]_+ &= 0. \end{aligned}$$

Then it follows from Proposition 2.1.1 that $Y(\ ,z)$ defines a SVOA structure on $M_{k,0}$ with $T = L_{-1}$ as the translation operator. □

It follows from [KT] that $[L_0, a(m)] = -ma(m)$, $[L_0, \bar{a}(m)] = -(m+\frac{1}{2})\bar{a}(m)$. Thus $L_0 + d$, which commutes with all $a(m), \bar{a}(m) \in \hat{\hat{\mathfrak{g}}}$ acts as a scalar on a highest weight representation. We call the element $\Omega = 2(k+h^\vee)(L_0+d)$ the generalized Casimir operator. Then we have

$$\Omega(v) = (\mu + 2\rho, \mu)v \tag{3.1.1}$$

if v is a singular vector of weight μ.

Let $J_{k,0}$ be the maximal proper submodule of $M_{k,0}$. It is easy to see that if $k \neq -h^\vee$ then $1 \notin J_{k,0}$, $\tau \notin J_{k,0}$ and hence the quotient $L_{k,0} = M_{k,0}/J_{k,0}$ is also a *SVOA*.

3.1.2 Rationality and the fusion rules of the SVOA $L_{k,0}$

Lemma 3.1.1 *The Zhu algebra $A(M_{k,0})$ is canonically isomorphic to $\mathfrak{U}(\mathfrak{g})$.*

Proof. By the definition of $A(M_{k,0})$ and Lemma 2.2.1 we have

$$[c] * [a(-1)1] = [a(-1)c],$$

where $a \in \mathfrak{g}$, $c \in M_{k,0}$. Hence

$$[a_m(-1)1] * \cdots * [a_1(-1)1] = [a_1(-1) \cdots a_n(-1)1].$$

Therefore we have a homomorphism of associative algebras

$$F : \mathfrak{U}(\mathfrak{g}) \longrightarrow A(M_{k,0}) \tag{3.1.2}$$

given by

$$a_m \cdots a_1 \mapsto [a_1(-1) \cdots a_n(-1)1].$$

It is clear that

$$(a(-n-2) + a(-n-1))c \;=\; Res_z\left(Y(a(-1)1, z)\frac{z+1}{z^{n+2}}c\right),$$

reverse it. Let v_μ be the vacuum vector of $L_{k,\mu}$. Pick a vector $v'_{\mu'} \in M$ of weight μ such that $\pi\left(v'_\mu\right) = v_\mu$. We claim that v'_μ is a singular vector of M, i.e. $e_i v'_\mu = 0$ for any i. Indeed, if $e_i v'_\mu \neq 0$ for some i, then

$$\pi\left(e_i v'_\mu\right) = e_i \pi\left(v'_\mu\right) = e_i v_\mu = 0.$$

So

$$e_i v'_\mu \quad = \quad \iota(u) \tag{3.1.4}$$

for some nonzero $u \in L_{k,\lambda}$, since the short sequence is exact. Comparing the weights of both sides of equation (3.1.4), we have $\lambda - \beta = \mu + \alpha_i$ for nonzero $\alpha_i, \beta \in Q_+$. It follows that $\lambda = \mu + \alpha_i + \beta > \mu$, which is a contradiction.

Denote by M' the submodule of M generated by the singular vector v'_μ. It suffices to show that the module M' is irreducible. But this follows in the same way as in Chapter 11 of [K] by making use of formula (3.1.1). □

Recall that the Lie subalgebra

$$\hat{\mathfrak{g}} = \mathfrak{g} \bigotimes \mathbb{C}\left[t, t^{-1}\right] \bigoplus \mathbb{C}\mathbf{k} \bigoplus \mathbb{C}d$$

is the usual affine Kac-Moody algebra. Given $\lambda \in P_+$ and $k \in \mathbb{Z}_+$, one can regard the irreducible $\mathfrak{g}$-module $\bar{L}(\lambda)$ as a module over $\hat{\mathfrak{g}}_+ \bigoplus \mathfrak{g} \bigoplus \mathbb{C}\mathbf{k} \bigoplus \mathbb{C}d$ by letting $(\hat{\mathfrak{g}}_+ \bigoplus \mathbb{C}d)\, \bar{L}(\lambda) = 0$ and $\mathbf{k} = kId\,|_V$. Let

$$\bar{M}_{k,\lambda} \quad = \quad \mathfrak{U}(\hat{\mathfrak{g}}) \bigotimes_{\mathfrak{U}\left(\hat{\mathfrak{g}}_+ \bigoplus \mathfrak{g} \bigoplus \mathbb{C}\mathbf{k} \bigoplus \mathbb{C}d\right)} \bar{L}(\lambda),$$

and $\bar{L}_{k,\lambda} = \bar{M}_{k,\lambda}/\bar{J}_{k,\lambda}$, where $\bar{J}_{k,\lambda}$ is the unique maximal $\hat{\mathfrak{g}}$-submodule of $\bar{M}_{k,\lambda}$.

The corresponding Zhu algebra $A(\bar{L}_{k,0})$ of the VOA $\bar{L}_{k,0}$ was computed in [FZ]. Comparing with their results, we see that the associative algebras $A\left(L_{k,0}\right)$ and $A\left(\bar{L}_{k,0}\right)$ are the same. And so the irreducible modules of the SVOA $L_{k,0}$ are canonically in 1–1 correspondence with those of the VOA $\bar{L}_{k,0}$. One can calculate the fusion rules using the $A(L_{k,0})$-modules similarly to Section 3.2 in [FZ] and find that the fusion rules for the modules of the SVOA $L_{k,0}$ are canonically in 1–1 correspondence with those for the VOA $\bar{L}_{k,0}$ (see the statements in Theorem 3.2.3 and Corollary 3.2.1 in [FZ]).

3.2 SVOAs associated to the Neveu-Schwarz algebra

3.2.1 SVOA structure on $M_{c,0}$ and $L_{c,0}$

Let us recall first that the Neveu-Schwarz algebra is the Lie superalgebra

$$\mathfrak{NS} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \bigoplus \bigoplus_{m\in\frac{1}{2}+\mathbb{Z}} \mathbb{C}G_m \bigoplus \mathbb{C}C$$

with commutation relations ($m, n \in \mathbb{Z}$):

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}C, \\
\left[G_{m+\frac{1}{2}}, L_n\right] &= \left(m+\frac{1}{2}-\frac{n}{2}\right) G_{m+n+\frac{1}{2}}, \\
\left[G_{m+\frac{1}{2}}, G_{n-\frac{1}{2}}\right]_+ &= 2L_{m+n} + \frac{1}{3}m(m+1)\delta_{m+n,0}C, \\
[L_m, C] = 0, &\qquad \left[G_{m+\frac{1}{2}}, C\right] = 0.
\end{aligned}$$

The $\mathbb{Z}_2$-gradation is given by $\tilde{L_n} = \tilde{C} = \bar{0}$, $\tilde{G_{n+\frac{1}{2}}} = \bar{1}$ (so that the even part is the Virasoro algebra). Set

$$\mathfrak{NS}_\pm = \bigoplus_{n\in\mathbb{N}} \mathbb{C}L_{\pm n} \bigoplus \bigoplus_{m\in\frac{1}{2}+\mathbb{Z}_+} \mathbb{C}G_{\pm m}.$$

Given complex numbers c and h, the Verma module $M_{c,h}$ over $\mathfrak{NS}$ is the free $\mathfrak{U}(\mathfrak{NS}_-)$-module generated by 1, such that $\mathfrak{NS}_+1 = 0$, $L_0 1 = h\cdot 1$ and $C\cdot 1 = c\cdot 1$. There exists a unique maximal proper submodule $J_{c,h}$ of $M_{c,h}$. Denote the quotient $M_{c,h}/J_{c,h}$ by $L_{c,h}$. Recall that $v \in M_{c,h}$ is called a singular vector if $\mathfrak{NS}_+ v = 0$ and v is an

$$\bar{b}(-n-1))c = Res_z\left(Y(\bar{b}(-1)1,z)\frac{1}{z^{n+1}}c\right).$$

By Lemma 2.2.1, we have

$$O'(M_{k,0}) \subset O(M_{k,0}),$$

where

$$O'(M_{k,0}) = \left\{(a(-n-2)+a(-n-1))\,c,\quad \bar{b}(-n-1)c \text{ for } n \geq 0\right\}.$$

Then it follows that

$$[a_1(-i_1-1)\cdots a_m(-i_m-1)] = (-1)^{i_1+\cdots+i_m}[a_1(-1)\cdots a_n(-1)1]$$

for $i_1,\ldots i_m \geq 0$. So F is an epimorphism. To show that F is indeed an isomorphism, we still need to show that

$$O'(M_{k,0}) = O(M_{k,0}). \tag{3.1.3}$$

The proof of (3.1.3) is standard (see the proof of a similar fact in Sect. 4.2). □

Lemma 3.1.2 *If k is a positive integer, then the map (3.1.2) induces an isomorphism from $\mathfrak{U}(\mathfrak{g})/\langle e_\theta^{k+1}\rangle$ onto $A(L_{k,0})$, where $\langle e_\theta^{k+1}\rangle$ is the two-sided ideal of $\mathfrak{U}(\mathfrak{g})$ generated by e_θ^{k+1}.*

Proof. It follows from Theorem 1.4.1 that the SVOA $M_{k,0}$ is isomorphic to

$$M\ (\Lambda_k + h^\vee d)/\langle f_i 1, i=1,\ldots,l\rangle,$$

with Λ_k given by $\lambda_i = \Lambda_k(h_i) = 0$, $i=1,\ldots,l$, $\lambda_0 = \Lambda_k(\mathbf{k}) = k$. Then the SVOA $L_{k,0}$ is isomorphic to $M_{k,0}/\langle v_k\rangle$, where v_k is defined in Theorem 1.3.1. By Theorem 1.3.1, Remark 1.3.1 and the identity (3.1.3), we see that under the isomorphism (3.1.2), v_{λ_0}

maps to $e_\theta^{k+1} \in \mathfrak{U}(\mathfrak{g})$. Hence the lemma follows from Proposition 2.2.1. □

Lemma 3.1.3 *If $x \in \mathfrak{g}$, and $N \in \mathbb{N}$, then the algebra $\mathfrak{U}(\mathfrak{g})/\langle x^N \rangle$ is finite dimensional and semisimple.*

Proof. Let G be the adjoint group of $\mathfrak{g}$. Since G is generated by $exp(ad\, y)$, $y \in \mathfrak{g}$, the ideal $\langle x^N \rangle$ is G-invariant, hence it contains all elements $g(x)^N, g \in G$. Since $\mathfrak{g}$ is simple, it coincides with the linear span of the orbit $G(x)$, hence $u_i^N \in \langle x^N \rangle$ for some basis $\{u_i\}$ of $\mathfrak{g}$. It follows that $\dim \mathfrak{U}(\mathfrak{g})/\langle x^N \rangle \leq N^{\dim \mathfrak{g}}$.

Since any finite-dimensional representation of $\mathfrak{U}(\mathfrak{g})$ is semisimple, it follows that any representation of $\mathfrak{U}(\mathfrak{g})/\langle x^N \rangle$ is semisimple. Hence the latter algebra is semisimple. □

Theorem 3.1.2 *For any positive integral k, the SVOA $L_{k,0}$ is rational. Moreover, $L_{k,\lambda}$, for $\lambda \in \mathfrak{h}^*$ dominant integrable with $\langle \lambda,\, \theta \rangle \leq k$, are precisely all the irreducible $L_{k,0}$-modules.*

Proof. The second part of this theorem follows from Theorem 2.2.3 and Lemma 3.1.2 because by Lemma 3.1.3, $L_{k,\lambda}$, for $\lambda \in \mathfrak{h}^*$ dominant integrable with $\langle \lambda,\, \theta \rangle \leq k$, are all the irreducible modules of $\mathfrak{U}(\mathfrak{g})/\langle e_\theta^{k+1} \rangle$. Any $L_{k,0}$-module M is a restricted module over $\hat{\hat{\mathfrak{g}}}$. Hence any $\hat{\hat{\mathfrak{g}}}$-submodule of M is also an $L_{k,0}$-submodule of M. To prove the complete reducibility of any $L_{k,0}$-module, we only need to prove the following.

Lemma 3.1.4 *Given $\lambda,\ \mu \in P_+$ such that $\langle \lambda, \theta \rangle \leq k, \langle \mu, \theta \rangle \leq k$, any short exact sequence of $\hat{\hat{\mathfrak{g}}}$-modules*

$$0 \longrightarrow L_{k,\lambda} \xrightarrow{\iota} M \xrightarrow{\pi} L_{k,\mu} \longrightarrow 0$$

splits.

Proof. Let $Q_+ = \Sigma_i \mathbb{Z}_+ \alpha_i$. First let us define a partial order in P_+ as follows: $\lambda > \mu$ iff $\lambda - \mu \in Q_+$ and $\lambda \neq \mu$. Without loss of generality, we may assume that $\lambda \not> \mu$. Otherwise we can apply the contragredient functor to the short exact sequence to eigenvector of L_0. For example, $G_{-\frac{1}{2}}1$ is a singular vector of $M_{c,0}$ for any c. Denote $M_{c,0}/\langle G_{-\frac{1}{2}}1\rangle$ by M_c, where $\langle G_{-\frac{1}{2}}1\rangle$ is the submodule of $M_{c,0}$ generated by the singular vector $G_{-\frac{1}{2}}1$. For simplicity we denote $L_{c,0}$ by V_c.

It is well known that

$$L_{-i_1}L_{-i_2}\cdots L_{-i_m}G_{-j_1}G_{-j_2}\cdots G_{-j_n},$$

for $i_1 \geq \cdots \geq i_m \geq 1$, $j_1 > \cdots > j_n \geq \frac{1}{2}$, $i_1 \cdots i_m \in \mathbb{N}$, and $j_1 \cdots j_n \in \frac{1}{2} + \mathbb{Z}_+$ is a basis of $\mathfrak{U}(\mathfrak{NS}_-)$. There is a natural gradation on $M_{c,0}$, M_c and V_c given by the eigenspace decomposition of L_0:

$$\deg L_{-i_1}L_{-i_2}\cdots L_{-i_m}G_{-j_1}G_{-j_2}\cdots G_{-j_n}1 \;=\; i_1 + i_2 + \cdots + i_m + j_1 + \cdots + j_n.$$

Set

$$\begin{aligned} L(z) &= \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \\ G(z) &= \sum_{n\in\mathbb{Z}} G_{n+\frac{1}{2}} z^{-n-2}. \end{aligned}$$

Clearly $L(z)$ is even while $G(z)$ is odd. We have the following theorem.

Theorem 3.2.1 *$(M_c, 1, \omega, \tau, Y(\cdot, z))$ is an $N = 1$ (NS-type) SVOA, where $\tau = G_{-3/2}1, \omega = L_{-2}1$, and the linear map*

$$Y(\ ,z) : M_c \longrightarrow End\ (M_c)[[z, z^{-1}]]$$

is defined by

$$L_{-i_n-2}\ldots L_{-i_1-2}G_{-j_m-\frac{3}{2}}\ldots G_{-j_1-\frac{3}{2}}1 \longmapsto$$

$$(i_1!\cdot\ldots\cdot i_n! j_1!\cdot\ldots\cdot j_m!)^{-1}\cdot :\partial_z^{i_n}L(z)\ldots\partial_z^{i_1}L(z)\partial_z^{j_m}G(z)\ldots\partial_z^{j_1}G(z):,$$

where $0 \leq i_1 \leq \ldots \leq i_n, 0 \leq j_1 < \ldots < j_m$.

Proof. First of all it is easy to see that

$$(z-w)^4[L(z),L(w)] = 0, \tag{3.2.1}$$

$$(z-w)^2[L(z),G(w)] = 0, \tag{3.2.2}$$

$$(z-w)^3[G(z),G(w)]_+ = 0. \tag{3.2.3}$$

Then it follows from Proposition 2.1.1 that $Y(\ ,z)$ defines a SVOA structure on M_c with $T = L_{-1}$ as the translation operator. □

3.2.2 Rationality and the fusion rules of the SVOA $V_{\tilde{c}_{p,q}}$

Lemma 3.2.1 *There exists an isomorphism of associative algebras,* $F : A(M_c) \cong \mathbb{C}[x]$*, given by* $[\omega]^n \mapsto x^n$*,where* $\mathbb{C}[x]$ *is the polynomial algebra on one variable* x*.*

Proof. Set

$$M_c = M_c^0 + M_c^1,$$

where M_c^0 (resp. M_c^1) is the even (resp. odd) part of M_c. By Lemma 2.2.1 we have

$$(L_{-m-3} + 2L_{-m-2} + L_{-m-1})\, b = \mathrm{Res}_z \left(Y(\omega, z) \frac{(z+1)^2}{z^{2+n}} b \right) \in O(M_c), \tag{3.2.4}$$

for every $m \geq 0, b \in M_c^0$.

$$(G_{-n-1} + G_{-n})\, b = \mathrm{Res}_z \left(Y(\tau, z) \frac{(z+1)}{z^{1+n-\frac{1}{2}}} b \right) \in O(M_c), \tag{3.2.5}$$

for every $n \in \frac{1}{2} + \mathbb{Z}_+, b \in M_c^1$. It follows by induction that

$$L_{-m} \sim (-1)^m \left((m-1)\left(L_{-2} + L_{-1}\right) + L_0 \right), \tag{3.2.6}$$

for every $m \geq 1$.

$$G_{-n} \sim (-1)^{n-\frac{1}{2}} G_{-\frac{1}{2}}, \text{ for every } n \in \frac{1}{2} + \mathbb{Z}_+. \tag{3.2.7}$$

By Lemma 2.2.1, we have

$$[b] * [\omega] \;=\; [(L_{-2} + L_{-1})b], b \in M_c^0. \tag{3.2.8}$$

Using (3.2.4) and (3.2.5), it is easy to show by induction on $m+n$ that

$$[L_{-i_1} L_{-i_2} \cdots L_{-i_m} G_{-j_1} G_{-j_2} \cdots G_{-j_{2n}} 1] \;=\; P([\omega])$$

for some $P(x) \in \mathbb{C}[x]$. Since the elements

$$L_{-i_1} L_{-i_2} \cdots L_{-i_m} G_{-j_1} G_{-j_2} \cdots G_{-j_{2n}} 1$$

for $i_1 \geq \cdots \geq i_m \geq 1$, $j_1 > \cdots > j_{2n} \geq \frac{1}{2}$, $i_1 \cdots i_m \in \mathbb{N}$, $j_1 \cdots j_{2n} \in \frac{1}{2} + \mathbb{Z}_+$, span M_c, the homomorphism of associative algebras

$$F : \mathbb{C}[x] \to A(M_c)$$

given by $x^n \mapsto [\omega]^n$ is surjective. (This homomorphism is well defined since $[\omega]$ is in the center of $A(M_c)$.)

To prove that F is also injective, it suffices to show that $O(M_c)$ is the linear span of the elements of the form (3.2.4) and (3.2.5), i.e.

$$O(M_c) = \text{lin. span} \left\{ (L_{-n-3} + 2L_{-n-2} + L_{-n-1})\, b,\; \left(G_{-n-3/2} + G_{-n-3/2}\right) b, n \geq 0, b \in M_c \right\}.$$

This can be proved in a standard way (for a proof of a similar fact see Sec. 4.2). □

Set

$$\begin{aligned} \tilde{c}_{p,q} &= \frac{3}{2}\left(1 - \frac{2(p-q)^2}{pq}\right), \\ \tilde{h}^{r,s}_{p,q} &= \frac{(sp - rq)^2 - (p-q)^2}{8pq}. \end{aligned}$$

Whenever we mention $\tilde{c}_{p,q}$ again, we always assume that $p, q \in \{2, 3, 4, \ldots\}$, $p-q \in 2\mathbb{Z}$, and that $(p-q)/2$ and q are relatively prime to each other. The submodule structure of a Verma module over the Neveu-Schwarz algebra [A] is very similar to that for the Virasoro algebras [FF1]. From the results of [A], we have the following lemma which is an analogue of the results in [FF1] (also see Lemma 4.2 of [W]).

Lemma 3.2.2 *1) $J_{c,0}$ is generated by the singular vector $G_{-\frac{1}{2}}1$ if $c \neq \tilde{c}_{p,q}$.*

2) $J_{c,0}$ is generated by two singular vectors if $c = \tilde{c}_{p,q}$. One of them is $G_{-\frac{1}{2}}1$. The other, denoted by $v_{p,q}$ has degree $\frac{1}{2}(p-1)(q-1)$.

From this lemma we immediately derive an analogue of Corollary 4.1 in [W].

Corollary 3.2.1 *If $c \neq \tilde{c}_{p,q}$, then V_c is not rational.*

Proof. See the proof of Corollary 4.1 in [W]. □

In the remaining part of this section, we shall always assume that $c = \tilde{c}_{p,q}$ and that $\tilde{h}^{r,s} = \tilde{h}^{r,s}_{p,q}$ for the sake of simplicity. It follows from Lemma 3.2.2 that $V_c = M_c/\langle \tilde{v}_{p,q}\rangle$, where $\langle \tilde{v}_{p,q}\rangle$ denotes the submodule of M_c generated by $\tilde{v}_{p,q}$.

Proposition 3.2.1 *The Zhu algebra $A(V_c)$ is isomorphic to $\mathbb{C}[x]/\langle F_{p,q}(x)\rangle$, where* $\deg F_{p,q} = \frac{1}{4}(p-1)(q-1)$ *if p, q are odd;* $\deg F_{p,q} = \frac{1}{4}(p-1)(q-1) + \frac{1}{4}$ *if p, q are even.*

Proof. If p, q are odd, $\tilde{v}_{p,q}$ is an even element of degree $\frac{1}{2}(p-1)(q-1)$ which corresponds to a polynomial $F_{p,q}$ of degree $\frac{1}{4}(p-1)(q-1)$; if p, q are even, $\tilde{v}_{p,q}$ is an odd element of degree $\frac{1}{2}(p-1)(q-1)$. From the definition of the associative algebra $A(V_c)$, it is $G_{-\frac{1}{2}}\tilde{v}_{p,q}$ which corresponds to $F_{p,q}$ of degree $\frac{1}{4}(p-1)(q-1) + \frac{1}{4}$. □

We expect the following conjecture, which is an analogue of Theorem 4.2 in [W], to be true.

Conjecture 3.2.1 *The vertex operator superalgebra $V_{\tilde{c}_{p,q}}$ is rational. Moreover, the minimal series modules $L_{c,h_{r,s}}, 0 < r < p, 0 < s < q, r - s \in 2\mathbb{Z}$ are all the irreducible representations of V_c.*

Remark 3.2.1 *If $p-q=2$, $V_{c,\tilde{h}^{r,s}_{p,q}}$ is unitary. In this case we can prove Conjecture 3.2.1 by using the well-known Goddard-Kent-Olive construction [KW2] and Theorem 3.1.2 (see [DMZ] for a similar proof in the Virasoro algebra case).*

From the argument of Lemma 3.2.1, we see that

$$A(V_c) = H_0(\mathfrak{S}, V_c),$$

where $\mathfrak{S} = \{L_{-n-2}+2L_{-n-1}+L_{-n}, G_{-n-1}+G_{-n}, n>0\}$ is a nilpotent subalgebra of $\mathfrak{NS}$. This conjecture can probably be proved [W] by calculating the coinvariants $H_0(\mathfrak{S}, V_c)$. It is easy to see by Lemma 3.2.1 that

$$A(L_{c,\tilde{h}^{r,s}}) = H_0(\mathfrak{S}, L_{c,\tilde{h}^{r,s}}).$$

Then by applying Theorem 2.3.2, and Proposition 2.3.1, we can obtain the fusion rules for the $V_{\tilde{c}_{p,q}}$-modules $L_{c,\tilde{h}_{r,s}}$, $0<r<p$, $0<s<q$, $r-s\in 2\mathbb{Z}$ if the coinvariants $H_0\left(\mathfrak{S}, L_{c,\tilde{h}^{r,s}}\right)$ are calculated.

To support our conjecture, we present some examples.

Example 1. Consider the case $(p,q)=(5,3)$, $\tilde{c}_{5,3}=7/10$, $\tilde{h}^{1,1}=0$, $\tilde{h}^{2,2}=1/10$. It is easy to check that the singular vector $\tilde{v}_{5,3}$ is given by

$$\tilde{v}_{5,3} = 3L_{-4}1+10L_{-2}^2 1-15G_{-5/2}G_{-3/2}1.$$

Using (3.2.6), (3.2.7) and (3.2.8), we have

$$F_{5,3}(x) = 10\left(x^2-\frac{1}{10}x\right)$$

which gives the values of $\tilde{h}^{1,1}$ and $\tilde{h}^{2,2}$.

Example 2. Let $(p,q)=(8,2)$, $\tilde{c}_{8,2}=-\frac{21}{4}$, $\tilde{h}^{1,1}=0, \tilde{h}^{3,1}=-\frac{1}{4}$. This is a non-unitary case. The singular vector $\tilde{v}_{8,2}$ is given by

$$\tilde{v}_{8,2} = 3G_{-7/2}1-4L_{-2}G_{-3/2}1.$$

Since $\tilde{v}_{8,2}$ is an odd element, we consider $G_{-\frac{1}{2}}\tilde{v}_{8,2}$ in order to get the polynomial $F_{8,2}(x)$. Using (3.2.6), (3.2.7) and (3.2.8), we get

$$G_{-\frac{1}{2}}\tilde{v}_{8,2} \sim -8x\left(x+\frac{1}{4}\right)$$

which gives the values of $\tilde{h}^{1,1}$ and $\tilde{h}^{3,1}$.

3.3 SVOAs generated by free fermionic fields

The free fermionic fields are

$$\begin{aligned} \Phi^a(z) &= \sum_{i\in\frac{1}{2}+\mathbb{Z}} \phi_i^a z^{-i-\frac{1}{2}}, \quad \text{(neutral)} \\ \Psi^{a,\pm}(z) &= \sum_{i\in\frac{1}{2}+\mathbb{Z}} \psi_i^{a,\pm} z^{-i-\frac{1}{2}}, \quad \text{(charged)} \end{aligned}$$

with the following nontrival commutation relations

$$\begin{aligned} \left[\phi_i^a, \phi_j^b\right]_+ &= \delta_{a,b}\delta_{i,j}, \\ \left[\psi_i^{a,+}, \psi_j^{b,-}\right]_+ &= \delta_{a,b}\delta_{i,j}, \end{aligned}$$

where $a, b = 1, \ldots, l$.

It is easy to see that from a pair of charged free fermionic fields $\Psi^{\pm}(z)$ one can construct two free neutral fermionic fields $\Phi(z)$ by letting

$$\begin{aligned} \Phi^1(z) &= \frac{1}{\sqrt{2}}\left(\Psi^+(z)+\Psi^-(z)\right), \\ \Phi^2(z) &= \frac{i}{\sqrt{2}}\left(\Psi^+(z)-\Psi^-(z)\right), \end{aligned}$$

and vise versa. Hence we only need to consider the SVOAs generated by free neutral fermionic fields. Let F be the Fock space defined by $\phi_i^a 1 = 0$, $a = 1, 2, \ldots, l$, $i > 0$, where 1 is the highest weight vector. As a vector space F is isomorphic to the universal envelopping algebra of the Lie superalgebra generated by $\left\{\phi^a_{-i-\frac{1}{2}}, a = 1, \ldots, l, i < 0\right\}$.

F is a SVOA with the Virasoro element $\omega = \frac{1}{2}\sum_{a=1}^{l} \phi^a_{-\frac{3}{2}}\phi^a_{-\frac{1}{2}}1$, and central charge $c = \frac{1}{2}$. F admits a natural gradation by letting $\deg \phi^a_{-i-\frac{1}{2}} = i + \frac{1}{2}$ and $\deg 1 = 0$.

Define $Y\left(\phi^a_{-\frac{1}{2}}1, z\right) = \Phi^a(z)$. One can define (cf. [T]) the vertex operator associated to any $v = \phi^{a_n}_{-i_n-\frac{1}{2}} \cdots \phi^{a_1}_{-i_n-\frac{1}{2}}1 \in F$ by

$$Y(v, z) \;=\; :\partial_z^{i_n}\Phi^{a_n}(z)\ldots\partial_z^{i_1}\Phi^{a_1}(z):,$$

where $a_1, \ldots, a_n \in \{1, \ldots, l\}$, and $i_1, \ldots, i_n \in \mathbb{Z}_+$.

Theorem 3.3.1 *F is a rational SVOA. Moreover, F has a unique irreducible representation, namely F itself.*

Proof. First let us calculate the Zhu algebra $A(F)$. By Lemma 2.2.1, we have

$$\phi^a_{-\frac{1}{2}-n}v \;=\; Res_z\left(Y(\phi^a_{\frac{1}{2}}1, z)v\frac{1}{z^{1+n}}\right) \in O(F), \quad n \geq 0, \quad v \in F.$$

Since

$$\left\{\phi^a_{-\frac{1}{2}-n}v,\ 1 \leq a \leq l, n \geq 0,\ v \in F\right\} \;=\; \bigoplus_{n\in\frac{1}{2}\mathbb{N}} F_n,$$

we have $\bigoplus_{n\in\frac{1}{2}\mathbb{N}} F_n \subset O(F)$.

On the other hand, it is easy to check by definition of $O(F)$ that $O(F) \subset \bigoplus_{n\in\frac{1}{2}\mathbb{N}} F_n$. Thus $O(F) = \bigoplus_{n\in\frac{1}{2}\mathbb{N}} F_n$, and so $A(F) = F/O(F) \cong \mathbb{C}$. Hence there exists a unique representation of the associative algebra $A(F) \cong \mathbb{C}$, i.e., $\mathbb{C}$ itself. By Theorem 2.2.3, there exists a unique representation of F, i.e., F itself.

The complete reducibility of modules of F follows from a similar argument to the proof of Lemma 3.1.4. So F is rational. □

Remark 3.3.1 *The above SVOA F is not an $N = 1$ SVOA. To construct the $N = 1$ SVOAs one needs to add some bosonic fields. For example, one can prove that the Fock space of one free bosonic field and one free neutral fermionic field is an $N = 1$ SVOA of rank $\frac{3}{2}$. Indeed this is just the special case of the SVOA associated to the affine Kac-Moody superalgebra corresponding to the 1-dimensional Lie algebra $\mathfrak{g}$.*

Chapter 4

Vertex operator algebras associated to the Virasoro algebra

One of the most important examples of VOAs (cf. [FZ]) is the VOAs corresponding to the irreducible vacuum representations of the Virasoro algebra $\mathfrak{L}$, denoted by $\overline{V}_c$. It has been conjectured (cf. [FZ]) that $\overline{V}_c$ is rational if and only if $c = c_{p,q} = 1 - 6(p-q)^2/pq$, where $p, q \in \{2, 3, 4, \ldots\}$, and p, q are relatively prime. In this chapter we prove this conjecture, and show that when $c = c_{p,q}$, $p, q \in \{2, 3, 4, \ldots\}$, and $(p, q) = 1$, all the irreducible representations of the VOA $\overline{V}_c$ are precisely those which correspond to irreducible minimal modules of the Virasoro algebra. Then we also present formulas for the fusion rules in the minimal series cases. As a byproduct, the equivalence is established between the fusion rules defined by Frenkel-Huang-Lepowsky in terms of intertwiners among modules over a vertex operator algebra and those defined by Feigin-Fuchs in terms of the coinvariant of a certain nilpotent infinite dimensional Lie algebra.

4.1 Rationality and the fusion rules of the VOA $\overline{V}_{c_{p,q}}$

Let us recall that the Virasoro algebra is the Lie algebra $\mathfrak{L} = \bigoplus_{n=-\infty}^{\infty} \mathbb{C}L_n \oplus \mathbb{C}C$ with commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3 - m}{12}C,$$

$$[L_m, C] = 0.$$

Set

$$L_+ = \bigoplus_{n=1}^{\infty} \mathbb{C}L_n, \ \ L_- = \bigoplus_{n=1}^{\infty} \mathbb{C}L_{-n}.$$

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}.$$

Given complex numbers c and h, the Verma module $\overline{M}_{c,h}$ over $\mathfrak{L}$ is a free $\mathfrak{U}(L_-)$-module generated by 1, such that $L_+1 = 0, L_0 1 = h \cdot 1$ and $C \cdot 1 = c \cdot 1$. There exists a unique maximal proper submodule of $\overline{M}_{c,h}$, say $\overline{J}_{c,h}$. Denote the quotient $\overline{M}_{c,h} \,/\, \overline{J}_{c,h}$ by $\overline{L}_{c,h}$. Recall that $v \in \overline{M}_{c,h}$ is called a singular vector if $L_+v = 0$ and v is an eigenvector of L_0. For example, $L_{-1}1$ is a singular vector of $\overline{M}_{c,0}$ for any c. Denote $\overline{M}_{c,0}/\langle L_{-1}1\rangle$ by $\overline{M}_c$, where $\langle L_{-1}1\rangle$ is the submodule of $\overline{M}_{c,0}$ generated by the singular vector $L_{-1}1$. There is a natural gradation on $\overline{M}_{c,0}$ and $\overline{M}_c$ given by

$$\deg L_{-i_1}L_{-i_2}\cdots L_{-i_n}1 = i_1 + i_2 + \cdots + i_n.$$

It is well known (cf. [FZ]) that $\overline{M}_c$ and $\overline{L}_{c,0}$ admit natural VOA structures, with the Virasoro elment $\omega = L_{-2}1$. The proof of this fact can be much simplified by using our Proposition 2.1.1. We denote the VOA $\overline{L}_{c,0}$ by $\overline{V}_c$ to emphasize the VOA structure on $\overline{L}_{c,0}$. The vertex operator corresponding to $v = L_{-i_n-2}\ldots L_{-i_2-2}\ldots L_{-i_1-2}1$ is given by

$$Y(v,z) = (i_1! \cdot i_2 \cdot \ldots \cdot i_n!)^{-1} \cdot : \partial_z^{i_n} L(z)\ldots\partial_z^{i_2} L(z)\partial_z^{i_1} L(z) :,$$

where $0 \le i_1 \le i_2 \le \ldots \le i_n$.

Lemma 4.1.1 *There exists an isomorphism of associative algebras, $F : A(\overline{M}_c) \cong \mathbb{C}[x]$, given by $[\omega]^n \mapsto x^n$, where $\mathbb{C}[x]$ is the polynomial algebra in one variable x.*

Proof. By Lemma 2.2.1 we have

$$(L_{-n-3} + 2L_{-n-2} + L_{-n-1})b = \mathrm{Res}_z(Y(\omega,z)\frac{(z+1)^2}{z^{2+n}}b) \in O(\overline{M}_c), \qquad (4.1.1)$$

for every $n \geq 0, b \in \overline{M}_c$. It follows by induction that

$$L_{-n} \sim (-1)^n \left((n-1)(L_{-2}+L_{-1})+L_0\right), \text{for every } n \geq 1. \tag{4.1.2}$$

Note that the first step of the induction is based on Lemma 2.2.1.

By Lemma 2.2.1, we have

$$[b] * [\omega] = [(L_{-2}+L_{-1})b]. \tag{4.1.3}$$

Note that

$$L_0 L_{-i_1} L_{-i_2} \cdots L_{-i_n} 1 = (i_1 + i_2 + \cdots + i_n) L_{-i_1} L_{-i_2} \cdots L_{-i_n} 1. \tag{4.1.4}$$

Using (4.1.2) and (4.1.4), it is easy to show by induction that

$$[L_{-i_1} L_{-i_2} \cdots L_{-i_n} 1] = P([\omega]) \text{ for some } P(x) \in \mathbb{C}[x].$$

Since

$$L_{-i_1} L_{-i_2} \cdots L_{-i_n} 1, \ i_1 \geq i_2 \geq \cdots \geq i_n \geq 2,$$

generate $\overline{M}_c$, the homomorphism of associative algebras

$$F : \mathbb{C}[x] \to A(\overline{M}_c)$$

given by $x^n \mapsto [\omega]^n$ is surjective. This homomorphism is well defined since $[\omega]$ is in the center of $A(\overline{M}_c)$.

To prove that F is also injective, it suffices to show that the elements in (4.1.1) generate the whole $O(\overline{M}_c)$. For a proof of this statement see Sec. 4.2. □

Let

$$c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}.$$

Convention 4.1.1 *For $v \in \overline{M}_c$, we also write $v \sim P(x)$ if $[v] = P([\omega])$ in $A(\overline{M}_c)$, by abuse of the symbol $\sim$.*

Whenever we mention $\mathbf{c_{p,q}}$ again in this section, we shall always assume that $\mathbf{p, q \in \{2, 3, 4, ...\}}$, and $\mathbf{p, q}$ are relatively prime.

Lemma 4.1.2 *1) $\overline{J}_{c,0}$ is generated by the singular vector $L_{-1}1$ if $c \neq c_{p,q}$.*

2) $\overline{J}_{c,0}$ is generated by two singular vectors if $c = c_{p,q}$. One of them is $L_{-1}1$. The other is denoted by $v_{p,q}$, where $\deg v_{p,q} = (p-1)(q-1)$.

Proof. It follows from Kac's determinant formula [K1] and the submodule structure theorem of $\overline{M}_{c,h}$ by Feigin-Fuchs [FF1]. □

From this lemma we immediately have the following corollary.

Corollary 4.1.1 *If $c \neq c_{p,q}$, then $\overline{V}_c$ is not a rational VOA.*

Proof. By Lemma 4.1.2, $\overline{V}_c = \overline{M}_c$ if $c \neq c_{p,q}$. It follows from Lemma 4.1.1 that $A(\overline{V}_c) \cong \mathbb{C}[x]$. Since there are infinitely many (1-dimensional) irreducible representations of $\mathbb{C}[x]$, $\overline{V}_c$ is not rational by Theorem 2.2.3. □

In the remaining part of this section, we shall always assume that $\mathbf{c = c_{p,q}}$. It follows from Lemma 4.1.2 that $\overline{V}_c = \overline{M}_c/\langle v_{p,q}\rangle$, where $\langle v_{p,q}\rangle$ denotes the submodule of $\overline{M}_c$ generated by $v_{p,q}$. We rewrite $v_{p,q} = \sigma_{p-1,q-1}1$, where $\sigma_{p-1,q-1} \in \mathfrak{U}(L_-)$.

Lemma 4.1.3 *The coefficient of the term $L_{-2}^{\frac{1}{2}(p-1)(q-1)}1$ in $v_{p,q}$ is nonzero.*

Proof. A projective formula of singular vectors was given in [FF1] as follows:

$$(\pi(\sigma_{p-1,q-1}))^2 = \prod_{i=o}^{p-2}\prod_{j=o}^{q-2}[L_{-1}^2 + ((p-2-2i)\theta^{-1} + (q-2-2j)\theta)^2 L_{-2}],$$

where $\theta^2 = -q/p$, π is the projection from $\mathfrak{U}(L_-)$ to $\mathfrak{U}(L_-)/\mathfrak{U}(L_-)L_{-3}$. Then it is easy to see that the coefficient of $L_{-2}^{\frac{1}{2}(p-1)(q-1)}$ is

$$\prod_{i=o}^{p-2}\prod_{j=o}^{q-2}((p-2-2i)\theta^{-1} + (q-2-2j)\theta)^2 \neq 0.$$

□

$v_{p,q}$ is unique up to a nonzero scalar. Now we can fix $v_{p,q}$ by letting the coefficient of the term $L_{-2}^{\frac{1}{2}(p-1)(q-1)}1$ be 1. We call n the length of the monomial

$$L_{-i_1}L_{-i_2}\cdots L_{-i_n}1, \ i_1 \geq i_2 \geq \cdots \geq i_n \geq 2.$$

Thanks to Lemma 4.1.1 we can assume $v_{p,q} \sim G_{p,q}(x)$ for some $G_{p,q}(x) \in \mathbb{C}[x]$. Then we have the following proposition.

Proposition 4.1.1 *The Zhu algebra $A(\overline{V}_c)$ is isomorphic to $\mathbb{C}[x]/\langle G_{p,q}(x)\rangle$, where* $\deg G_{p,q} = \frac{1}{2}(p-1)(q-1)$.

Proof. From the argument of Lemma 4.1.1, we see that a monomial in $\overline{M}_c$ of length n corresponds to a polynomial of $\deg n$ in $\mathbb{C}[x]$ by means of $F^{-1} : A(\overline{M}_c) \cong \mathbb{C}[x]$. Since $L_{-2}^{\frac{1}{2}(p-1)(q-1)}1$ is the only term in $v_{p,q}$ which has the maximal length $\frac{1}{2}(p-1)(q-1)$ among the monomials in $v_{p,q}$, we have $\deg G_{p,q} = \frac{1}{2}(p-1)(q-1)$.

Using Lemma 2.2.1 and (4.1.2), (4.1.4), it is easy to prove that

$$L_{-i_1}L_{-i_2}...L_{-i_n}v_{p,q} \sim F(x)G_{p,q}(x), \ \ i_1 \geq i_2 \geq \cdots \geq i_n \geq 1$$

for some $F(x) \in \mathbb{C}[x]$. Now this proposition follows from Proposition 2.2.1. □

Now we need to digress on nilpotent subalgebras of $\mathfrak{L}$ and the coinvariants.

Recall that the Virasoro algebra is the central extension of $diff\,(S^1)$, the polynomial vector fields on S^1. Let $\mathfrak{L}_{0,0}$ (*resp.* $\mathfrak{L}_{1,1}$) be the Lie subalgebra of vector fields of the form $z(z+1)p(z)\frac{d}{dz}, p(z) \in \mathbb{C}[z]$ (*resp.* $z^2(z+1)^2p(z)\frac{d}{dz}$). $\mathfrak{L}_{1,1}$ is a two-codimensional ideal of $\mathfrak{L}_{0,0}$. Set

$$e_0' = z(z+1)^2\frac{d}{dz}, \quad e_0'' = z^2(z+1)\frac{d}{dz}\ .$$

Identifying L_n with $z^{-n+1}\frac{d}{dz}$, we have

$$\mathfrak{L}_{1,1} = \bigoplus_{n\in\mathbb{Z}_+} \mathbb{C}\left(L_{-n-3} + 2L_{-n-2} + L_{-n-1}\right),$$

and $e_0' = L_{-2} + 2L_{-1} + L_0$, $e_0'' = L_{-2} + L_{-1}$.

An ordered triple of pairs of integers $((m,n),(m',n'),(m'',n''))$ is *admissible* [1] if $0 < m, m', m'' < p$, $0 < n, n', n'' < q$, $m+m'+m'' < 2p$, $n+n'+n'' < 2q$, $m < m'+m''$, $m' < m+m''$, $m'' < m+m'$, $n < n'+n''$, $n' < n+n''$, $n'' < n+n'$ and the sums $m+m'+m''$, $n+n'+n''$ are odd. We identify the triples $((m,n),(m',n'),(m'',n''))$ and $((m,n),(p-m',q-n'),(p-m'',q-n''))$.

Recall that given a Lie algebra $\mathfrak{g}$, and a $\mathfrak{g}$-module M, the coinvariant (i.e. 0-th homology) of $\mathfrak{g}$ with coefficients in M, denoted by $H_0(\mathfrak{g}, M)$, is defined to be the quotient $M \,/\, \mathfrak{g} \cdot M$.

Let

$$h_{m,n} = \frac{(np-mq)^2 - (p-q)^2}{4pq}.$$

$\overline{L}_{c,h_{m,n}}, 0 < m < p,\ 0 < n < q$ is called the *minimal* module of $\mathfrak{L}$.

The following theorem is from [FF2].

Theorem 4.1.1 *Let $\left\{((m,n),\ (m_i',n_i'),\ (m_i'',n_i'')), i = 1,\ldots,N\right\}$ be the set of all admissible triples with the fixed first pair (m,n). Then $H_0(\mathfrak{L}_{1,1},\ L_{c,h_{m,n}})$ can be decomposed into a sum of N 1-dimensional spaces, such that on the i-th one of these spaces e_0' and e_0'' act as the multiplications by $h_{m_i',n_i'}$ and $h_{m_i'',n_i''}$ respectively, where $N = \frac{1}{2}mn(p-m)(q-n)$.*

Remark 4.1.1 *There is a slight difference between here and [FF2]. In [FF2], $\mathfrak{L}_{0,0}$ (resp. $\mathfrak{L}_{1,1}$) appeared to be the Lie subalgebra of vector fields of the form $z(z-1)p(z)\frac{d}{dz}$ $\left(\text{resp. } z^2(z-1)^2p(z)\frac{d}{dz}\right), p(z) \in \mathbb{C}[z]$, and e_0' was $z(z-1)^2\frac{d}{dz}$, e_0'' was $z^2(z-1)\frac{d}{dz}$. But the conclusion and the argument of the theorem remain the same after the changes of signs.*

Proposition 4.1.2 *$A(\overline{V}_c) = H_0(\mathfrak{L}_{1,1},\ L_{c_{p,q},h_{1,1}}) \cong \mathbb{C}[x]/\langle G_{p,q}(x)\rangle$, where*

$$G_{p,q}^2 = \prod_{m=1}^{p-1}\prod_{n=1}^{q-1}(x - h_{m,n}).$$

[1] The admissibility condition of [FF2] is incorrect although the argument there remains valid. And from their argument we can get the admissibility condition given here.

Note that $G_{p,q}$ is a polynomial in x since $h_{m,n} = h_{p-m,q-n}$.

Proof. Since $h_{1,1} = 0$, we have $\overline{L}_{c_{p,q},h_{1,1}} = \overline{V}_c$. By definition,

$$H_0\left(\mathfrak{L}_{1,1},\, L_{c_{p,q},h_{1,1}}\right) = H_0\left(\mathfrak{L}_{1,1},\, \overline{V}_c\right) = \overline{V}_c/\mathfrak{L}_{1,1}\cdot\overline{V}_c. \tag{4.1.5}$$

On the other hand, we have $A(\overline{V}_c) = \overline{V}_c/O(\overline{V}_c)$ by definition. Since elements of the form (4.1.1) spans $O(\overline{M}_c)$, we see that $O(\overline{V}_c) = \mathfrak{L}_{1,1}\cdot\overline{V}_c$ by Proposition 2.2.1 and the definition of $\mathfrak{L}_{1,1}$. So we have $A(\overline{V}_c) = H_0(\mathfrak{L}_{1,1},\, L_{c_{p,q},h_{1,1}})$. Note that the admissible triples with the first pair $(1,1)$ are exactly

$$\{((1,1),\,(m,n),\,(m,n)),\, 0<m<p,\, 0<n<q\}.$$

It follows that the action of e_0' and e_0'' are the same on $H_0(\mathfrak{L}_{1,1},\, L_{c_{p,q},h_{1,1}})$. The rest of this proposition follows from Theorem 4.1.1 and Proposition 4.1.1. □

For the following lemma see [DGK] or [FF2].

Lemma 4.1.4 *Let S and S' be two minimal $\mathfrak{L}$-modules. Then $Ext^1(S,\, S') = 0$.*

Now we are ready to prove the following theorem.

Theorem 4.1.2 *The vertex operator algebra $\overline{V}_c$ with $c = c_{p,q}$ is rational. And the minimal modules $\overline{L}_{c,h_{m,n}}, 0 < m < p, 0 < n < q$ are all the irreducible representations of $\overline{V}_c$.*

Proof. It is obvious that all the irreducible representations of the Zhu algebra $A(\overline{V}_c) \cong \mathbb{C}[x]/\langle G_{p,q}(x)\rangle$ are one dimensional with $[\omega]$ acting as multiplications by $h_{m,n}$. $[\omega] \in A(\overline{V}_c)$ acts on the top level of a irreducible $\overline{V}_c$-module via $\omega(\deg\omega - 1) = L_0$ by Theorem 2.2.2.

If M is a $\overline{V}_c$-module, then M is a $\mathfrak{L}$-module, and all $\mathfrak{L}$-submodules (resp. quotient modules) of M are $\overline{V}_c$-modules. Thus any irreducible module of $\overline{V}_c$ should be of the form $\overline{L}_{c,h}$ for some $h \in \mathbb{C}$.

Thus by Proposition 4.1.2 and Theorem 2.2.3, we see that the minimal modules $\overline{L}_{c,h_{m,n}}, 0 < m < p, 0 < n < q$, are exactly all the irreducible representations of $\overline{V}_c$.

A highest weight representation $\overline{V}_{c,h}$ of $\mathfrak{L}$ is not a representation of $\overline{V}_c$ unless $\overline{V}_{c,h}$ is one of the minimal modules. Indeed, if $h \neq h_{m,n}$, then obviously $\overline{V}_{c,h}$ is not a $\overline{V}_c$-module. If $h = h_{m,n}$ but $\overline{V}_{c,h}$ is reducible, then it contains some highest weight representation $\overline{V}_{c,h'}$, where $h' \neq h_{m,n}$ for any $0 < m < p, 0 < n < q$, which is not a $\overline{V}_c$-module.

Now assume $M = \bigoplus_{n=0}^{\infty} M_n$ is finitely generated by vectors $w_1, \cdots, w_s \in M$. Let k be the maximal degree of $w_1, \cdots, w_s$. There exist finitely many singular vectors in $\bigoplus_{n=0}^{k} M_n$ (because $\dim(\bigoplus_{n=0}^{k} M_n) < \infty$). Let v be one of them. Then the highest weight submodule $S = \mathfrak{U}(L_-)v$ of M must be isomorphic to some $\overline{L}_{c,h_{m,n}}$, $0 < m < p$, $0 < n < q$. Denote by $\mathbb{P}$ the projection from M to $M^1 = M/S$. $M^1 = \bigoplus_{n=0}^{\infty} M_n^1$ is also a $\overline{V}_c$-module with finitely many generators $\mathbb{P}(w_1), \cdots, \mathbb{P}(w_s)$. (Note that some of $\mathbb{P}(w_i)$ may be 0). Hence the maximal degree of $\mathbb{P}(w_1), \cdots, \mathbb{P}(w_s)$ is at most k. M^1 has also finitely many singular vectors in $\bigoplus_{n=0}^{k} M_n^1$. Furthermore $\dim(\bigoplus_{n=0}^{k} M_n^1) < \dim(\bigoplus_{n=0}^{k} M_n) < \infty$. By induction on $\dim(\bigoplus_{n=0}^{k} M_n)$ we can assume that M^1 is a completely reducible module whose irreducible submodules are the irreducible minimal ones. By Lemma 4.1.4, so is M. □

Now let us turn to the fusion rules between the minimal modules over $\overline{V}_c$. Recall that for any vertex operator (super)algebra V and a V-module M, one can construct a bimodule $A(M)$ over the associative algebra $A(V)$. Relations between the fusion rules and the bimodule $A(M)$ were given in Theorem 2.3.2. Using a similar argument to the one which leads to Proposition 4.1.2, we have

$$A(\overline{L}_{c,h_{m,n}}) = H_0(\mathfrak{L}_{1,1}; \overline{L}_{c,h_{m,n}})$$

with $[\omega] \in A(\overline{V}_c)$ acting on the left as e_0' and on the right as e_0''. Then we get the fusion rules by combining Theorem 4.1.1 and Theorem 2.3.2.

Theorem 4.1.3 *The fusion rules between modules $\overline{L}_{c,h_{m',n'}}$ and $\overline{L}_{c,h_{m'',n''}}$ are*

$$\overline{L}_{c,h_{m',n'}} \times \overline{L}_{c,h_{m'',n''}} = \sum_{(m,n)} N_{(m',n'),(m'',n'')}^{(m,n)} \overline{L}_{c,h_{m,n}},$$

where $N^{(m,n)}_{(m',n'),(m'',n'')}$ *is 1 iff* $\left((m,n),(m',n'),(m'',n'')\right)$ *is an admissible triple of pairs, and 0 otherwise.*

4.2 A proof of an identity in Section 4.1

Let

$$O'(\overline{M}_c) = \left\{(L_{-n-3} + 2L_{-n-2} + L_{-n-1})b, \;\; b \in \overline{M}_c, n \geq 0\right\}.$$

We have shown that $O'(\overline{M}_c) \subset O(\overline{M}_c)$. Now we will prove that $O(\overline{M}_c) \subset O'(\overline{M}_c)$.

We say $a \approx b$ iff $a \equiv b \; \left(\text{mod} O'(\overline{M}_c)\right)$.

We prove by induction on $\deg(L_{-n}a)$ that

$$T = Res_w\left(Y\left(L_{-n}a, w\right)\frac{(w+1)^{n+\deg a}}{w^k}b\right) \in O'\left(\overline{M}_c\right), \quad n,\, k \geq 2.$$

Indeed,

$$\begin{aligned} T &= Res_{z-w}Res_w\left(Y\left(Y\left(\omega, z-w\right)a, w\right)\iota_{w,z-w}(z-w)^{-n+1}\frac{(w+1)^{n+\deg a}}{w^k}b\right) \\ &= T_1 - T_2, \quad \text{by the Jacobi identity} \end{aligned}$$

where

$$T_1 = Res_z Res_w\left(Y(\omega, z)Y(a, w)\iota_{z,w}(z-w)^{-n+1}\frac{(w+1)^{n+\deg a}}{w^k}b\right),$$

and

$$T_2 = Res_z Res_w\left(Y(a, w)Y(\omega, z)\iota_{w,z}(z-w)^{-n+1}\frac{(w+1)^{n+\deg a}}{w^k}b\right).$$

Recall that

$$\begin{aligned} \iota_{w,z}(z-w)^{-n+1} &= \sum_{i\geq 0}(-1)^{-n+1-i}\binom{1-n}{i}z^i w^{-n+1-i} \\ &= \sum_{i\geq 1}(-1)^{-n+1-i}\binom{1-n}{i}z^i w^{-n+1-i} + (-1)^{-n+1}w^{-n+1}. \end{aligned}$$

By applying the induction assumption on a we see that

$$Res_z Res_w \left(Y(a,w)Y(\omega,z)\sum_{i\geq 1}(-1)^{-n+1-i}\binom{1-n}{i} z^i \frac{w^{-n+1-i}(w+1)^{n+\deg a}}{w^k} b \right)$$

is in $O'(\overline{M}_c)$.

So

$$\begin{aligned} T_2 &\approx Res_z Res_w \left(Y(a,w)Y(\omega,z)(-1)^{-n+1}\frac{(w+1)^{n+\deg a}}{w^{n-1+k}} b \right) \\ &= Res_w \left(Y(a,w)(-1)^{-n+1}\frac{(w+1)^{n+\deg a}}{w^{n-1+k}} L_{-1} b \right) \\ &\approx Res_w \left(Y(a,w)(-1)^{-n+1}\frac{w(w+1)^{\deg a}}{w^k} L_{-1} b \right) \qquad (4.2.6) \\ &\approx Res_w \left(Y(a,w)(-1)^{-n+1}\frac{(w+1)^{\deg a+1}}{w^k} L_{-1} b \right) \quad \text{by induction on } a. \\ &= T_{21} - T_{22}, \end{aligned}$$

where

$$T_{21} = L_{-1} Res_w \left(Y(a,w)(-1)^{-n+1}\frac{(w+1)^{\deg a+1}}{w^k} b \right),$$

$$T_{22} = Res_w \left(Y(L_{-1}a,w)(-1)^{-n+1}\frac{(w+1)^{\deg a+1}}{w^k} b \right),$$

and (4.2.6) holds by induction since

$$\frac{(w+1)^{n+\deg a}}{w^{n-1+k}} = \frac{w(w+1)^{\deg a}}{w^k} + \sum_{i=1}^{n}\binom{n}{i}\frac{(w+1)^{\deg a}}{w^{k+i-1}}.$$

We see that $T_{22} \sim 0$ by applying induction assumption to $L_{-1}a$.

$$\begin{aligned} T_1 &= Res_z Res_w \left(Y(\omega,z)Y(a,w)\iota_{z,w}(z-w)^{-n+1}\frac{(w+1)^{n+\deg a}}{w^k} b \right) \\ &= \sum_{i=0}^{\infty}\binom{1-n}{i} Res_z Res_w \left(Y(\omega,z)Y(a,w)(-1)^i z^{-n-i+1}\frac{w^i(w+1)^{n+\deg a}}{w^k} b \right) \\ &= \sum_{i=0}^{\infty}(-1)^i\binom{1-n}{i} Res_w \left(L_{-n-i}Y(a,w)\frac{w^i(w+1)^{n+\deg a}}{w^k} b \right) \end{aligned}$$

$$
\begin{aligned}
&\approx \sum_{i=0}^{\infty}(-1)^n\binom{1-n}{i} Res_w\left(((n+i-1)(L_{-2}+L_{-1})+L_0)\,Y(a,w)\frac{w^i(w+1)^{n+\deg a}}{w^k}b\right)\\
&= T_{11}+T_{12},
\end{aligned}
$$

where

$$
\begin{aligned}
T_{11} &= \sum_{i=0}^{\infty}(-1)^n\binom{1-n}{i} Res_w\left(((n+i-1)(L_{-2}+L_{-1}))\,Y(a,w)\frac{w^i(w+1)^{n+\deg a}}{w^k}b\right)\\
&= (-1)^n(n+1)(L_{-2}+L_{-1})Res_w\left(Y(a,w)\frac{(w+1)^{\deg a}}{w^k}b\right)\approx 0,
\end{aligned}
$$

by (4.1.3) and the induction assumption on a and the identity

$$
\sum_{i=0}^{\infty}\binom{1-n}{i}(n+i-1)w^i=(n-1)(w+1)^{-n},
$$

and

$$
T_{12}=(-1)^n L_0 Res_w\left(Y(a,w)\frac{(w+1)^{\deg a+1}}{w^k}b\right),
$$

by using the identity

$$
\sum_{i=0}^{\infty}\binom{1-n}{i}w^i=(w+1)^{-n+1}.
$$

Then

$$
T=T_{11}+T_{12}-T_{21}-T_{22}\approx(-1)^n(L_{-1}+L_0)Res_w\left(Y(a,w)\frac{(w+1)^{\deg a+1}}{w^k}b\right)\approx 0,
$$

by Lemma 2.2.1.

This completes the proof that $O(\overline{M}_c)\subset O'(\overline{M}_c)$.

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