Entangling symmetry and topology in correlated electrons

by

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Abstract

In this thesis, I study a class of exotic quantum matter named Symmetry-Protected Topological (SPT) phases. These are short-range-entangled quantum phases hosting non-trivial states on their boundaries. In the free-fermion limit, they are famously known as Topological Insulators (TI). Huge progress has been made recently in understanding SPT phases beyond free fermions. Here I will discuss three aspects of SPT phases in interacting systems, mostly in three dimensions: (1) Novel SPT phases could emerge in strongly correlated systems, with no non-interacting counterpart. In particular, I will discuss interaction-enabled electron topological insulators, including their classification, construction, characterization and realization. (2) When strong interactions are present, the surface of many SPT phases (including the familiar free fermion topological insulator) can be gapped without breaking any symmetry, at the expense of having intrinsic topological order on the surface. (3) Some topological phases that are non-trivial in the free fermion theory become trivial once strong interactions are introduced. The material of this thesis closely parallels that of Refs. [1, 2, 3, 4, 5, 6].

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Chapter 1

Introduction

1.1 Entangled quantum phases of matter

A focus of modern quantum condensed matter physics is the study of phases of matter whose characterization is not captured by the concepts of broken symmetry and associated Landau order parameters[7]. Such phases often come with emergent low energy degrees of freedom giving rise to interesting field theories. A striking example is the fractional quantum hall states[8, 9]: these states host quasi-particle excitations that carry fractional quantum statistics that are neither bosonic nor fermionic, and such particles are called anyons. The anyon statistics cannot be understood within the framework of broken symmetries, and a new paradigm is clearly required to understand them systematically.

It was realized later that the fractional quantum hall states belong to a larger class of gapped quantum matter called **topological ordered** phases, with their low energy properties described by topological quantum field theories (TQFT). These are characterized by emergent excitations with unusual quantum statistics and ground state degeneracies that depend on the topology of the underlying manifold[10]. Other examples of topological order include gapped spin liquids – an exotic class of quantum magnets, in which the magnetic order is destroyed by quantum fluctuations[11, 12]. The simplest kind of gapped quantum spin liquid – often dubbed $\mathbb{Z}_2$ spin liquid – is potentially realized in a mineral called Herbertsmithite[13, 14].

Other examples of quantum phases beyond the Landau paradigm of symmetry breaking are gapless phases of matter, where the gaplessness is protected but is not associated with broken symmetries and Goldstone modes. The most familiar example of such a phase is the Landau Fermi liquid but gapless spin liquids and various non-Fermi liquid phases provide other examples.

A common characterization of all these exotic phases is the presence of non-local many body quantum entanglement in their ground state wave function, in the sense that different parts of the system are necessarily entangled. More precisely, the ground state wavefunction cannot be tuned to an unentangled product state by continuously changing the Hamiltonian without encountering any singularity in its properties (so called adiabatic evolution). Here a "product state" is a quantum state of the form $|\alpha_1\rangle_1 \otimes |\alpha_2\rangle_2 ... \otimes |\alpha_N\rangle_N$, where $|\alpha_i\rangle_i$ is a local state on site i of the underlying lattice. Entanglement has thus become a new organization principle in classifying quantum phases. Those exotic phases that cannot be "disentangled" without encountering a phase transition have come to be known as **highly entangled phases** of matter.

On the contrary, **short-range entangled phases** are those that can be adiabatically evolved into a simple product state. All phases described within the symmetry-breaking paradigm fall into this category. For example, an ising ferromagnet, in the ordered phase, can be adiabatically tuned into a product state with no entanglement: $|\uparrow\rangle \otimes |\uparrow\rangle ... \otimes |\uparrow\rangle$. Most of the band insulators described by free fermions can also be tuned into a product state, with an integer number of fermions occupying each local orbital – sometimes called the atomic insulator limit. Since a product state cannot host any exotic excitation (such as anyons), a short-range entangled phase, which is adiabatically connected to a product state, must also host no exotic excitation.

There is another class of phases that will not be discussed extensively in this thesis but is nevertheless worth mentioning: these are phases that can be adiabatically disentangled into a product state only if the system is combined with another "mirror" state – sometimes referred to as "invertible" phases. In some sense they fall in between the two categories discussed above. In this thesis I will count them as short-range entangled, even though they are sometimes classified as long-range entangled phases in the literature depending on the taste of the authors. These phases have nothing exotic in the bulk of the system: the excitations are gapped and carry regular statistics. However, they host non-trivial boundaries (often gapless) that cannot be eliminated unless the bulk goes through a phase transition into a trivial state. Classic examples include integer quantum hall states and chiral superconductors in two (spatial) dimensions, and the Kitaev Majorana chain in one dimension[15]. The former host chiral fermions on their one dimensional boundaries, and the latter host Majorana zero-modes (zero-dimensional Majorana fermions) at the ends of the system. The boundary theories of these invertible phases cannot be realized alone without a nontrivial bulk. In field theory language, they have gravitational anomalies.

1.2 The interplay between symmetry and topology

The organization of quantum phases according to their entanglement patterns does not require the presence of symmetry in the picture. However, it has been known since its discovery that symmetry can largely enrich the story of highly entangled phases such as topological orders. The oldest example is again the fractional quantum hall states, where the quasi-particle excitations carry not only fractional statistics, but also fractional charge: for example, electrons filling up $1/3$ of the first Landau level (often called the Laughlin $1/3$ state) has anyon excitations that carry charge-$e/3$. In modern language, symmetry (here the $U(1)$ charge conservation) is acting projectively on the quasi-particle excitations. Projective symmetry groups (PSG) also play a key role in understanding other exotic phases including various kinds of spin liquids[10]. Topological orders equipped with different realizations of symmetries are dubbed symmetry-enriched topological orders (SET), and the phenomenon of symmetries acting projectively is often called **symmetry fractionalization**.

It was also realized that symmetry can even enrich the story of short-range entangled phases: there are states that are short-range entangled, but cannot be adiabatically disentangled into product states while preserving the symmetry of the system[16, 17]. In another word, these phases are distinct from a trivial product state if and only if the symmetries are preserved. They are now dubbed **Symmetry-Protected Topological (SPT)** phases. Similar to invertible phases, SPT phases have no exotic excitations in the bulk – all the excitations are gapped and carry trivial statistics. But the boundaries of SPT phases are necessarily nontrivial as long as the symmetries of the system are not broken – the precise meaning of the term "nontrivial" here shall be explained later, and is in fact one of the main themes of this thesis. The crucial point is that the boundaries of SPT phases realize symmetries in an **anomalous** way: the $(d-1)$-dimensional boundary state of an d-dimensional SPT cannot be realized in a strictly $(d-1)$-dimensional system while preserving the symmetries.

1.2.1 Topological insulators: SPT in free fermions

The interest in studying SPT phases was revived recently due to the theoretical and experimental discovery of electronic topological insulators in two and three spatial dimensions[18, 19, 20]. These are band insulators hosting gapless boundary modes as long as time-reversal symmetry and charge conservation are preserved. If these two symmetries are broken – either spontaneously or explicitly – the boundary of topological insulators will be gapped, and the distinction between topological and trivial (atomic) insulators will vanish. This makes topological insulators perfect examples of SPT phases.

The topological insulators realized experimentally so far can be modeled using free lattice fermions, which makes them more accessible theoretically than general interacting SPT phases. Indeed, topological insulators in free fermion systems have been fully classified in any dimension and with any internal symmetry (not involving spatial groups)[21, 22], and tremendous progress has been made in understanding the physics of these phases within band theory. The key feature of these phases is the emergence of gapless Dirac/Weyl/Majorana fermions on the boundaries of the gapped bulk, and they are kept gapless as long as the relevant symmetries are unbroken.

The most famous and interesting example is the three dimensional topological insulator[23, 24, 25]. It is an insulator that cannot be adiabatically tuned to a trivial (atomic) insulator as long as time-reversal symmetry ($\mathcal{T}$) and charge-conservation ($U(1)$ symmetry) are preserved. In another word, the non-triviality of the topological insulator is protected by the $U(1) \rtimes \mathcal{T}$ symmetry. The surface hosts an odd number of two component Dirac cones. In the simplest case, there is only one flavor of two-component Dirac fermion:

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi + \mu\psi^\dagger\psi, \tag{1.1}$$

where ψ is the fermion field, $\gamma_0 = i\sigma_y$, $\gamma_1 = \sigma_z$, $\gamma_2 = \sigma_x$, and μ is the chemical potential. The Dirac dispersion spectrum has been observed experimentally using photoemision spectroscopy[26, 27] in materials such as Bi_2Se_3. Time-reversal symmetry acts as $\mathcal{T}\psi\mathcal{T}^{-1} = i\sigma_y\psi$. It is easy to check that any fermion mass term will necessarily break a symmetry: the Dirac mass term $m\bar{\psi}\psi$ will break time-reversal symmetry, while the pairing mass term $\Delta\psi^T(i\sigma_y)\psi + h.c.$ breaks the $U(1)$ charge rotation symmetry. Therefore the surface is gapless as long as the symmetries are preserved, at least within the free fermion theory. Importantly, such a theory as in Eq. (1.1) cannot be realized in strict two dimensions while preserving the $U(1) \rtimes \mathcal{T}$ symmetry. This is known in the field theory literature[28, 29, 30] as the "parity anomaly".

The free fermion description is clearly the appropriate starting point to discuss the possibility of topological insulators/superconductors (or SPT in general) in weakly correlated materials. In recent years however attention has turned toward materials with strong electron correlations as possible platforms for similar phenomena. These include the mixed valence compound[31, 32, 33] SmB_6, and iridium oxides on pyrochlore lattices[34]. The exploration of topological phenomena in correlated materials brings with it a number of questions. Are the free fermion topological phases stable to the inclusion of strong interactions? Even if they are, could they behave very differently in strongly interacting systems? Are there intrinsically interacting SPT phases that have no free fermion analog? Clearly in addressing these questions there is a need to go beyond the concept of topological band structure and think more generally about the physics of SPT phases.

1.2.2 SPT in one dimension

The first understood SPT phase in interacting system was the Haldane spin-1 chain[35, 36] in one dimension (well before the discovery of topological insulators!). It is a chain of localized spins, each carrying spin-1. The typical Hamiltonian has the form

$$H = \sum_i J\mathbf{S}_i \cdot \mathbf{S}_{i+1} + ..., \tag{1.2}$$

where $\mathbf{S}_i$ is a spin-1 operator living on site i, J is a positive coupling constant, and "..." represents longer range terms that do not affect the qualitative feature of the system. What is surprising about this system is that a spin-1/2 degree of freedom emerges at the end of the chain – clearly not possible for a spin-1 system in strictly zero-dimension (a point). This gives a two-fold degeneracy at each end, and four-fold ground state degeneracy for an open chain. It is also clear that the non-triviality exists only if the relevant symmetry (e.g., $SO(3)$ spin-rotation or time-reversal symmetry) is preserved.

More recently, the systematic study of SPT phases in interacting systems was carried out succesfully in one dimension for both bosonic[37, 38, 39] and fermionic[40, 41, 42] systems. Both the mathematical classification of phases and physical understanding of their properties were achieved. In bosonic systems, where the microscopic degrees of freedom are lattice bosons/spins, the physics of Haldane chain was generalized to all internal symmetries, and many new states were discovered. In fermionic systems, where the microscopic degrees of freedom are interacting lattice fermions, it was realized that some phases that were non-trivial in the free fermion models became trivial once interactions were introduced. In both cases, the common feature in terms of physical properties is that there is always a ground state degeneracy protected by the symmetry associated with the boundary (the end of the one dimensional chain). In other words, if the system is put on an open chain, there will be degenerate ground states as long as the symmetries are preserved; while the system on a closed chain is guaranteed to have a unique ground state. The degenerate states associated with each end realizes symmetry projectively (generalization of spin-1/2 to other symmetries), and cannot be realized alone in a strictly zero dimensional system (a point).

1.3 SPT in interacting bosons

The complete understanding in one dimension was possible largely due to special features of one dimensional systems and many exact results related to them. In higher dimensions the story becomes much more complicated. For bosonic systems a breakthrough was achieved using the mathematical device of group-cohomology[16, 17]: a large number of new SPT states were discovered and classified in all dimensions with all internal symmetries. The group cohomology classification was later realized to be incomplete in three and higher dimensions[43, 1, 44, 45], but it represents a huge progress in classifying SPT phases. However, unlike the SPT phases in one dimensions, it is harder to infer the physical properties of those nontrivial SPT phases in higher dimensions directly from the mathematical classification. For two dimensional SPT phases, the physics of the one dimensional boundary were analyzed in Ref. [46, 47, 48, 49, 50, 51]. The main features are similar to that in one dimension: the boundary (edge) states are guaranteed to be gapless as long as the symmetries are not broken (either spontaneously or explicitly). The symmetry implementations on the boundary theories are anomalous, in the sense that they cannot be realized in a strictly one dimensional system. If the symmetries involved are unitary (i.e. not including time-reversal), these anomalies are related to gauge anomalies in field theory literatures.

On the other hand, the theory of three dimensional boson SPT was relatively less developed, simply because we have less theoretical tools to study those interacting field theories on the $(2+1)$ dimensional boundaries. It was suspected that the story may be similar to those in lower dimensions: the boundary surface will be either gapless or symmetry-breaking. Surprisingly, it was later realized[43, 52] that there is yet another possibility: the surface could be in a topologically ordered state, in which case it would be both gapped and symmetric! Remember the three dimensional bulk of the SPT phase has no topological order since it is short-range entangled by definition.

However, the surface can develop topological orders, hosting anyon excitations that live only on the surface. Such a possibility is possible only on the two (or higher) dimensional surface because topological order does not exist in lower dimensions. It was later realized (by us and other groups) that even the surface of the famous free fermion topological insulator can be symmetrically gapped and develop an intrinsic topological order[2, 53, 54, 55].

The topological orders appeared on the surfaces of SPT states are very unconventional, in the sense that the symmetries are fractionalized in some strange ways. While symmetry fractionalization can certainly happen even in strict $2D$ topological orders (for example in fractional quantum hall states), the particular kinds of symmetry fractionalization on the SPT surfaces are so strange that they cannot be realized in any strict $2D$ system. This is a manifestation of the symmetry anomaly on the SPT boundary.

The phenomenon of anomalous symmetry fractionalization leads us to a fundamental question: what kinds of symmetry fractionalization are anomaly-free, hence allowed in strict $2D$, and what are not? In Ref. [1] we tackled this question for several symmetry groups of practical interest, including time-reversal symmetry and charge conservation. The general idea is to examine for each pattern of symmetry fractionalization whether the $2D$ system could have an $1D$ boundary (edge) to the vacuum that is consistent with the $2D$ bulk theory. This "edgability" criteria turned out to be quite powerful and was able to determine which theories can happen in $2D$ and which can only happen on the surface of a $3D$ bulk: in the former case there should be an edge to the vacuum, while in the latter a consistent edge is clearly not required since the boundary has no boundary. Indeed, we found that all the topological orders realized on the SPT surfaces cannot be realized in strict $2D$ with their symmetry implementations. This approach also led very naturally to an explicit construction of all the bosonic SPT states in three dimensions, using coupled layers of $2D$-allowed topological orders. In particular, the existence of bosonic SPT states beyond the group-cohomology classification was explicitly confirmed.

The understanding of which kinds of symmetry-fractionalization are possible in strict $2D$ is of great practical interest. For example, if a $\mathbb{Z}_2$ spin liquid is indeed realized in Herbersmithite, one would like to know how the symmetry charges (spin, lattice momentum, etc.) are fractionalized. Solving this directly is very challenging both analytically and numerically. The most practical way is to analyze possible experimental signatures that can arise in $\mathbb{Z}_2$ spin liquids with different symmetry-fractionalization, and compare with experimental observations. However, in doing this one must first figure out what are the possible symmetry-fractionalization. In particular, one should not try to compare experiments with phenomenologies of those topological orders that can only appear on the surface of some $3D$ SPTs.

1.4 SPT in interacting fermions

Since all the strongly interacting systems in solid state are made of electrons, the study of SPT in interacting fermions is of great practical value. In particular, the three most important issues are classification, characterization and realization: How many distinct phases are there? How to characterize a nontrivial phase, especially in the laboratory? How to realize those nontrivial phases in real materials?

The understanding of SPT in interacting fermions in higher dimensions ($D > 1$) is even more incomplete than the bosonic ones. Formal classifications using algebraic tools are available for certain special symmetry groups[56, 57], but not including the physically important case of charge conservation and time-reversal symmetry (as is appropriate for insulators). Moreover, it is even harder to read off the physical properties from the mathematical classifications than it is for bosonic SPT. Significant progress has been made for $2D$ fermion SPT phases through the study of their $(1+1)D$ edge states[48, 58, 59, 60, 61], but it is very hard to generalize such approach to higher dimensions, including the physically important case of $3D$ systems.

Given the difficulties of tackling the main issues (classification, characterization and realization) using abstract mathematics, we took a more elementary approach by coming up with various physical arguments. The physical approach turned out to be surprisingly fruitful. In Ref. [3, 4] we presented a series of physical arguments that led to complete classifications of fermionic SPT phases in $3D$ when the symmetry groups contain a normal $U(1)$ subgroup, which include the physically important case of charge-conservation. The resulting physics was very rich: interacting SPT states that cannot be described with non-interacting fermions were identified. On the other hand some phases that were nontrivial in free fermion theory became trivial once interactions were introduced. We also discussed their physics in terms of surface states, in particular their experimental fingerprints that are measurable. In Ref. [5] I also discussed a very simple toy model of one of those beyond-free-fermion SPT states.

It is much harder to say anything certain about the issue of realizing those exotic interacting states in real materials, simply because most realistic interacting materials are described by Hamiltonians that are extremely hard to solve. However, we were able to give some suggestive arguments in Ref. [6], suggesting that frustrated spin-1 magnets in three dimensions might be a good place to find some of these exotic phases. This class of magnets are largely unexplored so far both theoretically and experimentally. It is will be exciting if one can indeed identify a topological paramagnet in these materials, through, for example, surface measurements suggested by us in Ref. [3].

1.5 From SPT to spin liquids

As mentioned above in Sec. 1.3, the study of SPT phases led to an understanding of what kind of topologically ordered spin liquid is allowed in strict two dimensions, and what is not. Surprisingly enough, the study of SPT phases also shed new light on another aspect of spin liquids: different spin liquids can be obtained by gauging different SPT. Here "gauging" means promoting the global symmetry of the SPT (or a subgroup of it) to a gauge symmetry, by coupling a dynamical gauge field to it via minimal coupling.

Such a line of thinking was first developed for the free fermion topological insulators in both two[62, 63] and three[64, 65, 66, 67] dimensions. The most famous example is the three dimensional topological insulator coupled with a $U(1)$ gauge field. It is well known[64] that when the fermions are in a topological band, the effective action of the gauge field will contain a Θ-term at $\Theta = \pi$:

$$\mathcal{L}_\Theta = \frac{\epsilon^{\mu\nu\lambda\rho}}{8\pi}\partial_\mu A_\nu \partial_\lambda A_\rho. \tag{1.3}$$

Such a gauge theory was proposed as an interesting quantum spin liquid dubbed "topological Mott insulator"[65], where the fermionic charge and the dynamical gauge field are emergent fields at low energy. The most interesting feature of such a spin liquid is that the monopole excitations of the gauge field (which is necessarily compact) becomes a charge-1/2 dyon due to Witten effect[68].

Similar phenomenon was also discovered for bosonic SPT states in both two[47] and three[69] dimensions, where coupling the bosonic matter fields to gauge fields led to various interesting gauge theories.

In Ref. [1], we carried out an analysis along this line specifically for a physically motivated case: $U(1)$ spin liquids in three dimensions with time-reversal symmetry. By putting the matter fields ("charge" and "monopole") into different possible SPT phases, we were able to answer the question: what kinds of time-reversal symmetric $U(1)$ gauge theories (spin liquids) are possible to emerge from a magnet (a spin system)? This is a relevant and important question for real materials including Pyrochlore quantum spin ice.

The result in Ref. [1] is a partial classification, and we will carry out a more complete classification in a future work[70].

1.6 Plan of the Thesis

The rest of the thesis is organized as follows: in Chapter 2 I will discuss bosonic SPT states in three dimensions and their surface topological orders. Along the way (Sec. 2.4) I will give an explicit construction of various SPT states using coupled layers of two dimensional topological orders. The latter half of Chapter 2 discusses time-reversal symmetric $U(1)$ spin liquids, obtained through coupling different boson SPT states with $U(1)$ gauge fields (Sec. 2.8). In Chapter 3 I will discuss gapped symmetric topological order on the surface of the famous free fermion topological insulator. In Chapter 4 I discuss in detail SPT phases in interacting electronic insulators, which is a physically well motivated case: I will discuss both the classification and characterizations of different interacting topological insulators. In Chapter 5 I generalize the work in Chapter 4 to interacting fermionic systems with other symmetries, including topological superconductors with time-reversal symmetry. In Chapter 6 I discuss a very simple toy model for some of the SPT phases in interacting fermions. In Chapter 7 I discuss the possibility of realizing a nontrivial SPT phase in frustrated spin-1 magnets in three dimensions.

The work in the remaining parts of the thesis is a synthesis of Refs. [1, 2, 3, 4, 5, 6].

Chapter 2

Bosonic topological insulators and their surface topological orders

In this chapter we will study several aspects of the realization of symmetry in exotic quantum phases - both gapped and gapless, primarily in two dimensions. Of particular importance to us are the results of Ref. [43] on the protected surface states of three dimensional bosonic SPT phases. The surface phase diagram was argued to admit a phase with surface topological order though the bulk itself has no such order. Furthermore this surface topological order implements the defining global symmetry in a manner not allowed in strictly two dimensional systems.

We are thus lead to consider in detail consistent implementation of global symmetries in several highly entangled quantum phases. First we obtain several new results and insights into both gapped and gapless phases that are allowed to exist in strictly 2d systems. These results have immediate application to theories of quantum spin liquid insulators and of non-fermi liquid metals. Along the way we also obtain an explicit construction of the various 3d symmetry protected topological insulators of bosons studied recently in Ref. [43]. In particular we construct a time reversal symmetric $3d$ SPT phase that was suggested to exist in Ref. [43] but is not currently part of the cohomology classification of Ref. [16, 17].

Second we study symmetry realization in three dimensional gapless quantum spin liquids with an emergent photon. Focusing on time reversal symmetry and on phases that can exist in strictly $3d$ systems we show that different such spin liquids may be distinguished by whether the emergent electric charge excitation is a Kramers singlet/doublet and its statistics. We show that this distinction is nicely captured by viewing these phases as different SPT insulators of the dual 'magnetic' particle (the monopole).

Ref. [71] proposed a formal classification of two dimensional topological order described by a deconfined Z_2 gauge theory in the presence of global symmetries. The topological quasiparticles can in principle carry fractional quantum numbers of the global symmetry. More formally this means that they are allowed to transform projectively under the global symmetry group. The approach of Ref. [71] involves finding all consistent ways of assigning projective representations to the different topological quasiparticles. A different classification has also appeared[72] that considers topological order with unitary symmetry but restricts to phases where one of the bosonic quasiparticles has trivial global quantum numbers. Earlier Refs. [73, 74, 75] classified all two dimensional time reversal invariant gapped abelian insulators using a Chern-Simons/K-matrix approach. It is expected that all such insulators can always be described by a multicomponent Chern-Simons theory. The key idea of Refs. [73, 74, 75] is that the bulk two dimensional theory can be completely characterized by studying the $1 + 1$ dimensional edge theory at the interface with the vacuum (or equivalently a topologically trivial gapped insulator). This is a multi-component Luttinger liquid theory in which operators corresponding to various bulk quasiparticles can be easily identified. In particular constraints coming from global symmetries can be straightforwardly implemented. This approach has recently been used to study other symmetry enriched $2d$ topological order in Refs. [76, 77].

How does the interplay of symmetry and topological order that is only allowed at the surface of 3d systems fit in with the emerging results on the classification of 2d topological order with symmetry? In the first part of this chapter we address this question in detail for a few examples. Specifically we restrict attention to boson systems with a few simple internal global symmetries. We also restrict to topological order described by a deconfined Z_2 gauge theory. We first use the procedure of

Ref. [71] to obtain all distinct allowed implementation of the global internal symmetry. Some of these can be realized at the surface of 3d SPT phases. Then we use elementary arguments and the results of Ref. [75] to determine which ones of the phases are allowed in strictly 2d systems. Interestingly the remaining phases are *all* shown to be realized at the surface of 3d SPT phases.

Why does the Chern-Simons/edge theory approach select out only those phases that can exist in strict 2d while the approach of Ref. [71] does not? The key point is that the former approach assumes that the state in question can have a physical edge with the vacuum (equivalently a topologically trivial gapped insulator) while preserving the symmetry. For topological order realized at the surface of a 3d SPT phase this possibility simply does not exist. A trivial gapped symmetry preserving state to which the surface topological ordered state can have an interface is forbidden at the SPT surface. In contrast the methods of Ref. [71] only worry about consistent assignment of symmetries to the various topological quasiparticles. The requirement that the state allow a physical edge to the vacuum is not part of the considerations of this method.

We will denote states which allow a physical edge to the vacuum as *'edgable'*. The topologically ordered states at the surface of a $3d$ SPT phase are not edgable while those allowed in strictly $2d$ are edgable. The concept of edgability will prove to be a powerful criterion in deciding which topological ordered states with symmetry are allowed in strictly $2d$ systems and which not.

Below we will flesh all this out in several concrete examples. We first study $2d$ gapped Z_2 topological order with a few different symmetries in Sections. 2.1, 2.2 and 2.3. Next we use the insights from these results to provide an explicit construction of SPT phases with the same symmetries in a system of coupled layers in Sec. 2.4. We provide a brief discussion of the relationship between the possible surface topological order in a 3d SPT and its bulk topological field theory in Section 2.5. We turn our attention then to highly entangled gapless phases. In Section. 2.6 we argue that a previously proposed gapless vortex fluid (dubbed the 'Algebraic Vortex Liquid') cannot exist with time reversal symmetry in strictly $2d$ systems, but could arise on the surface of a $3d$ SPT state. In Sec. 2.7 we describe the surface Landau-Ginzburg theories for the $3d$ SPT phases of interest in terms of dual vortices with non-trivial structure and discuss the surface phase structure. We also provide an explicit derivation of this dual surface vortex theory. We conclude in Section 2.9 with a discussion.

2.1 Topological ordered boson insulators: Symmetry $U(1) \rtimes Z_2^T$

We begin by considering a system of bosons with a global $U(1)$ symmetry and time reversal (Z_2^T). The bosons are taken to have charge 1 under the global $U(1)$ symmetry. In this section we assume that the boson destruction operator $b \to b$ under Z_2^T. This means that the global symmetry group is $U(1) \rtimes Z_2^T$. We will assume that the topological order in question has 2 non-trivial bosonic particles (dubbed e and m, in analogy with the familiar three dimensional $U(1)$ gauge theory) and a fermion (dubbed ϵ). Any two of these are mutual semions. Further any one of these may be thought of as a bound state of the other two. This corresponds precisely to the excitation structure of a deconfined Z_2 gauge theory in two space dimensions. What are the allowed topological phases with Z_2 gauge structure according to the analysis of Ref. [71]? The time reversal operation $\mathcal{T}$ when it acts on physical states of the bosons must satisfy $\mathcal{T}^2 = 1$. Let us denote by $\mathcal{T}_{e,m}$ the action of time reversal on the e and m particles. The only restriction on these is that they satisfy

$$\mathcal{T}_e^2 = \mu_e \tag{2.1}$$

$$\mathcal{T}_m^2 = \mu_m \tag{2.2}$$

with $\mu_{e,m} = \pm 1$. A value -1 of either of these means that the corresponding particle forms a Kramers doublet, i.e. a two-fold degeneracy protected by time-reversal due to the fact that $\mathcal{T}^2 = -1$. What about symmetry under global $U(1)$ rotations? Here the distinct possibilities correspond to whether the (e, m) particles carry integer or fractional charge. In the latter case their charge must be shifted from an integer by $\frac{1}{2}$.

These possibilities are nicely distinguished by asking about the action of a 2π global $U(1)$ rotation $R_{2\pi}$. On physical states $R_{2\pi} = 1$. Let us again denote by $R_{2\pi}^{e,m}$ the action on the e and m sectors. We then have

$$R_{2\pi}^{e} = \sigma_e \tag{2.3}$$

$$R_{2\pi}^{m} = \sigma_m \tag{2.4}$$

with $\sigma_{e,m} = \pm 1$. The realization of the symmetry in this topologically ordered state is thus described by the numbers $(\sigma_e, \mu_e, \sigma_m, \mu_m)$. Naively this gives 16 phases but we must remember that interchanging e and m is simply a relabeling of particles and does not produce a new phase. This removes 6 possibilities so we are left with a total of 10 phases for this symmetry.

In Table 2.1 we display the quantum numbers of the e and m excitations of these 10 phases. We label these phases by the excitations that carry non-trivial charge (C) or time reversal (T) quantum numbers. Thus $e0m0$ means both the e and m particles carry trivial quantum numbers, while mT refers to a phase where the m particle is Kramers doublet and neither e nor m cary half-integer charge, etc.

Phase	σ_e	σ_m	μ_e	μ_m	Comments
$e0m0$	1	1	1	1	No fractionalization
eT	1	1	-1	1	No fractional charge but Kramers
eC	-1	1	1	1	$b = \Phi^2$
eCT	-1	1	-1	1	$b = \epsilon_{\alpha\beta} f_\alpha f_\beta$; f_α in trivial band
$eCmT$	-1	1	1	-1	$b = \epsilon_{\alpha\beta} f_\alpha f_\beta$;$f_\alpha$ in topological band
$eTmT$	1	1	-1	-1	3d SPT surface
$eCmC$	-1	-1	1	1	3d SPT surface
$eCTmC$	-1	-1	-1	1	$eCmC \oplus eCmT$
$eCTmT$	-1	1	-1	-1	$eTmT \oplus eCmT$
$eCTmCT$	-1	-1	-1	-1	$eCTmC \oplus eCTmT$

Table 2.1: Symmetry action of $U(1) \rtimes Z_2^T$ (charge is $\mathcal{T}$-even) for Z_2 topological ordered states. The first 5 are allowed in strict 2d while the last 5 can only be realized at surface of 3d SPT phases (or derived from them). Interchanging e and m on each rolls is simply a relabeling of particles and does not lead to a new phase.

Note that the discussion above made no reference to the edgability of the state.

To consider only edgable states, let us now switch gears and consider the possibilities for Z_2 topological order with the same symmetry as above but within the Chern-Simons/edge theory approach of Ref. [75]. This naturally selects out edgable states and hence will enable us to decide which of the 10 states in Table 2.1 can be realized in strictly 2d systems. We will show that only the first 5 of these are captured in the Chern-Simons approach.

In the Chern-Simons description of an abelian two dimensional insulator the effective Lagrangian is given by a multi-component Chern-Simons term

$$L = \frac{K_{IJ}}{4\pi}\epsilon^{\mu\nu\lambda}a^I_\mu\partial_\nu a^J_\lambda + \frac{1}{2\pi}\tau_I\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a^I_\lambda \tag{2.5}$$

where the current density of quasiparticle I is given by $j^I_\mu = \frac{\epsilon^{\mu\nu\lambda}\partial_\nu a^I_\lambda}{2\pi}$. The integer matrix K_{IJ} – usually called the K-matrix – gives the topological information of the system, while the charge vector τ_I is an integer valued charge of each quasi-particle through coupling with the external gauge field A_μ. The allowed quasiparticles carry integer charge under the different gauge fields a^I which can be expressed in terms of an integer valued vector l. The mutual statistics of two quasiparticles labeled by l and l' is $\theta_{ll'} = 2\pi l^T K^{-1} l$ while the self-statistics of a quasiparticle is $\theta_l = \pi l^T K^{-1} l$. To describe Z_2 topological order we begin with a 2×2 K-matrix

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{2.6}$$

which captures the statistics of the e and m particles. We will determine the distinct ways in which the $U(1) \rtimes Z_2^T$ symmetry can be realized. First of all note that the electrical Hall conductivity is given by $\sigma_{xy} = \tau^T K^{-1}\tau = \tau_1\tau_2$. Thus time reversal invariant states must necessarily have at most one of $\tau_{1,2} \neq 0$. Henceforth without loss of generality we will therefore set $\tau_1 = 0$ and $\tau_2 = t$. Next the physical charge of a quasiparticle labeled by l is given by $q_l = l^t K^{-1}\tau = \frac{l_1 t}{2}$. Since we only want to distinguish half-integer physical charge from integer the distinct possibilities correspond to $t = 0, 1$. Let us now demand time reversal invariance of the Chern-Simons

Lagrangian. The symmetry realizations classified by the first approach above assume that the symmetry transformation does not interchange e and m particles. Therefore we restrict attention to that subclass here. For the first term to be time reversal invariant it must be that the spatial components a_{1i}, a_{2i} transform oppositely under time reversal. Further if $\tau_2 = t$ is non-zero, then $\epsilon_{ij}\partial_i a_{2j}$ must be even under time reversal. Thus we choose the action of time reversal on the a_i^I to be $a_i^I \to T_{IJ}a_i^J$ with

$$T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.7}$$

As described in Ref. [75] we also need to describe the transformation of the quasi-particle creation operators. This is conveniently accomplished by using the standard edge theory that corresponds to the bulk Chern-Simons Lagrangian:

$$\mathcal{L} = \frac{1}{4\pi}\left(K_{IJ}\partial_x\phi_I\partial_t\phi_J + ...\right) + \frac{1}{2\pi}\epsilon_{\mu\nu}\tau_I\partial_\mu\phi_I A_\nu \tag{2.8}$$

Quasiparticle creation operators corresponding to $l = (1,0)$ and $l = (0,1)$ are $e^{i\phi_1}$ and $e^{i\phi_2}$ respectively. The time reversal transformation of a_{Ii} fixes the transformation of ϕ_I upto an overall phase. Thus we write

$$e^{i\phi_1} \to e^{i(\phi_1+\alpha_1)} \tag{2.9}$$

$$e^{i\phi_2} \to e^{-i(\phi_2+\alpha_2)} \tag{2.10}$$

However by a shift of ϕ_1 we can always set $\alpha_1 = 0$. This is not possible for α_2. A further constraint comes from requiring that all physical operators transform such that $\mathcal{T}^2 = 1$. In particular $\mathcal{T}^2$ should take $e^{2i\phi_2} \to e^{2i\phi_2}$. This imposes the restriction that $\alpha_2 = \frac{\pi x}{2}$ with $x = 0, 1$. If $x = 0$ then the particle created by $e^{i\phi_2}$ is a Kramers singlet. If $x = 1$ however $\mathcal{T}^2$ takes $e^{i\phi_2} \to -e^{i\phi_2}$ so that the particle is a Kramers doublet.

Thus within this 2×2 K-matrix we have four possible states corresponding to the four possible values of the pair t, x. In terms of Table 2.1 these correspond to the four phases $e0m0, eT, eC, eCmT$. Actually a fifth phase eCT is also allowed in strict 2d but requires a 4×4 K-matrix. To see why this is so it is useful to better understand the physics of the 4 states described so far.

First note that with the K-matrix in Eqn. 2.6 the edge phase fields ϕ_1, ϕ_2 satisfy commutation relations such that the fields $f_\pm = e^{i(\phi_1 \pm \phi_2)}$ satisfy fermion anti commutation relations. Indeed these correspond to $l = (1,1), l = (1,-1)$ and describe the bulk fermionic ϵ particle. $f_\pm$ are the right and left moving fermions of the one dimensional edge Luttinger liquid theory. Under a global $U(1)$ symmetry rotation U_θ by angle θ and time reversal, the $f_\pm$ transform as

$$U_\theta^\dagger f_\pm U_\theta = e^{i\frac{t\theta}{2}} f_\pm \tag{2.11}$$

$$\mathcal{T}^{-1} f_\pm \mathcal{T} = e^{\mp i\frac{\pi x}{2}} f_\mp \tag{2.12}$$

Note that as the ϵ particle may be regarded as a bound state of e and m , it has quantum numbers $\sigma_\epsilon = \sigma_e \sigma_f$ and $\mu_f = \mu_e \mu_f$. For the four cases described above in terms of edge Lagrangians this is consistent with the symmetry transformation of the edge fermion fields.

Further insight is obtained by understanding how the four phases corresponding to the four choices of (t, x) are obtained within a slave particle (parton) construction in the bulk. Consider a slave particle (parton) construction obtained by writing the boson operator $b_r = a_r s_r$ at each site r of the lattice. Here a_r destroys a bosonic parton with charge 1 while s_r is an Ising parton (with charge 0). We may take them to belong to the e sector. Under time reversal a_r, s_r remain invariant. Further as the a_r, s_r carry only integer charge, $\sigma_e = \mu_e = 1$. First we take the a_r to form a simple bosonic Mott insulator and s_r to form a simple Ising paramagnet. Then the vison of the Z_2 gauge field associated with the slave particle construction will have trivial quantum numbers so that $\sigma_m = \mu_m = 1$. This corresponds to Phase $e0m0$ in Table. 2.1.

Phase eT can be likewise constructed if we start with two species of physical bosons $b_{1,2}$ and require $b_1 \leftrightarrow b_2$ under time-reversal (e.g. spin-half bosons). Then we write the boson operators as $b_{1,2} = a_{1,2}s_{1,2}$ and put the system into a state such that time reversal is implemented through $(a_1, a_2) \to (-a_2, a_1)$ and $(s_1, s_2) \to (-s_2, s_1)$. The e particles in this phase ($a_{1,2}$ and $s_{1,2}$) are Kramer's doublets, while the m particle (the vision) transforms trivially under time-reversal. Nothing carries fractional charge in this phase.

Phase eC is a familiar one and can be obtained in a slave particle construction by writing $b_r = \Phi_r^2$. The Φ_r destroys a charge-1/2 bosonic parton (denoted "chargon" in the literature). Under time reversal Φ_r is invariant. Explicit microscopic models for the corresponding phase were studied in Refs. [78, 79, 80] with the standard implementation of time reversal symmetry for bosons ($b \to b$).

Phase eCT is also a familiar one. It can be obtained through a parton construction by writing the boson operator as $b_r = \epsilon_{\alpha\beta} f_{r\alpha} f_{r\beta}$ with $f_{r\alpha}$ a fermion. We will refer to $\alpha = 1, 2$ as a pseudospin index. The fermions carry charge -1/2. Time reversal is implemented through $f_{r\alpha} \to i\left(\sigma_y\right)_{\alpha\beta} f_{r\beta}$. Now consider a mean field ansatz where the fermion $f_{r\alpha}$ forms a (topologically trivial) band insulator that preserves time reversal but does not conserve any component of the fermion pseudospin. The result is a Z_2 topologically ordered state with symmetry implemented as defined for Phase 4.

Phase $eCmT$ is obtained from the same parton construction as for Phase eCT but when the $f_{r\alpha}$ band structure is topologically non-trivial, *i.e* the fermions form a 2d topological insulator. Then a π flux seen by the fermions (which we may take to be the m particle) is known to bind a Kramers doublet[62, 63]. Indeed in the edge theory above if we choose $t = 1, x = 1$ the edge Lagrangian becomes identical to that of a fermionic topological insulator formed by the ϵ particle. Thus this parton construction has the symmetries of Phase 5. Three dimensional analogs of these phases were studied in Refs. [66, 67].

It is clear now that the phase eCT can exist in strictly $2d$ systems but is not captured by a Chern-Simons/edge theory description with a 2×2 K-matrix. This can also be seen by noting that since the physical charge is invariant under time-reversal, one cannot have a particle that's non-trivial under both $U(1)$ and $\mathcal{T}$ symmetries within this 2×2 K-matrix formulation.

Note that for $eCmT$ the edge theory is gapless so long as the global symmetry is preserved. In contrast for the phases $e0m0, eT, eC$ the edge theory can be gapped by adding symmetry allowed perturbations. Similarly from the parton construction we know that though the ϵ particle carries the same quantum numbers for both $eCmT$ and for eCT the edge theory for eCT can be gapped. From the theory of the fermion topological insulator it follows that trivial band structure for the ϵ can be built up from the topological band structure by taking 2 copies and allowing all symmetry allowed perturbations. This suggests that the the minimal description of eCT uses a 4×4 K-matrix. Specifically consider

$$K = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.13}$$

with the charge vector $\tau = (0, 1, 0, 0)$. Time reversal is implemented on the edge boson fields ϕ_I through $\phi_I \to T_{IJ} (\phi_I + \alpha_I)$ with $\alpha = (0, 0, \pi, 0)$. It is readily seen that this describes the eCT phase.

In passing we note that we can easily generate other $2d$ Z_2 topological phases with this symmetry by simply adding a layer of the $2d$ SPT phase allowed with $U(1) \rtimes Z_2^T$ symmetry to one of the 5 examples discussed above. This obvious extension does not affect our subsequent discussion and we will not consider it further.

In Sec. 2.1.1 we will explain in detail why all the other phases are not possible within the K-matrix formulation. In the next section we argue that, independent of the K-matrix formulation, the existence of those phases on SPT surfaces implies their non-existence in strict $2d$ systems.

2.1.1 K-matrix descriptions of Z_2 topological order

In this subsection we consider $2d$ states in detail. In most cases a 2×2 K-matrix is enough to describe the state because we can identify all particles with the same symmetry and topological properties through condensing appropriate combinations of them, and there remains only one species of e and m particle, respectively. For example, consider a Kramer's doublet carrying spin-1/2 $b_{\pm}$, the combination b_+b_- is a singlet under time-reversal and carries no spin, so we can condense it and identify $b_- \sim b_+^\dagger$, and time-reversal could be realized through $b_+ \to ib_+^\dagger$ so that $\mathcal{T}^2 = -1$.

The 2×2 K-matrix was considered thoroughly in the previous discussions, and it was straightforward to get all the possible states within the framework. It is also clear from the analysis above that 2×2 K-matrix is enough to describe every state with Z_2^T and $U(1) \times Z_2^T$ (spin) symmetries. For $U(1) \rtimes Z_2^T$ (charge) symmetry, the 2×2 K-matrix describes most of the states, except when there is at least one particle that carries both half-charge and Kramer's doublet, in which case there is no particle bilinear that preserves all symmetries, and we should really consider two species of such particles. For these states, a 4×4 K-matrix is needed.

The general form of such K-matrices was given in Ref. [75], with slight modifications due to the bosonic nature of our systems here. There are three possible forms of K and T matrices. The simplest one of them

$$K = \begin{pmatrix} 0 & A_{2\times 2} \\ A_{2\times 2} & 0 \end{pmatrix}, T = \begin{pmatrix} -1_{2\times 2} & 0 \\ 0 & 1_{2\times 2} \end{pmatrix} \tag{2.14}$$

does not work because the T matrix does not allow a particle to carry both charge and Kramer's doublet structure. The next possibility

$$\begin{aligned} K &= \begin{pmatrix} K_{2\times 2} & W_{2\times 2} \\ W_{2\times 2}^T & -K_{2\times 2} \end{pmatrix}, \tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_1 \\ \tau_2 \end{pmatrix} \\ T &= \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix} \end{aligned} \tag{2.15}$$

with $W_{2\times 2}$ anti-symmetric, does not work either. To see this, simply look at the charge carried by any particle $q_l = l_I K^{-1}_{IJ} \tau_J$. The entries of K^{-1}_{IJ} are either integers or half-integers. From the structure of the T-matrix and the assumption that time-reversal doesn't interchange e and m particle, we find that the only half-integer entries of K^{-1} are $K^{-1}_{12} = K^{-1}_{21}, K^{-1}_{14} = K^{-1}_{41}, K^{-1}_{23} = K^{-1}_{32}, K^{-1}_{34} = K^{-1}_{43}$. Then from the structure of the τ vector it is easy to see that the charge $q_l = l_I K^{-1}_{IJ} \tau_J$ must be an integer for any integer vector l, so there's no quasi-particle that carries half-charge.

The only possibility left is thus

$$K = \begin{pmatrix} 0 & A & B & B \\ A & 0 & C & -C \\ B & C & D & 0 \\ B & -C & 0 & -D \end{pmatrix}, \tau = \begin{pmatrix} 0 \\ \tau_2 \\ \tau_3 \\ \tau_3 \end{pmatrix} \tag{2.16}$$

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

with $\det K = (AD - 2BC)^2 = 4$. The inverse of the K matrix is thus

$$K^{-1} = \frac{\mathrm{sgn}(AD - 2BC)}{2} \begin{pmatrix} 0 & D & -C & -C \\ D & 0 & -B & B \\ -C & -B & A & 0 \\ -C & B & 0 & -A \end{pmatrix} \tag{2.17}$$

Therefore to have the right self and mutual statistics, we need $A = 4m, D = 2n$, and B, C odd, which makes particle-1 ($l = (1,0,0,0)$) or 2 ($l = (0,1,0,0)$) having π-statistics with particle-3 ($l = (0,0,1,0)$) or 4 ($l = (0,0,0,1)$), and all the other mutual or self statistics trivial.

It is clear from the T matrix that particle-2 is time-reversal trivial. Since the bound state of particle-1 and particle-2 ($l = (1,1,0,0)$) has trivial statistics with any particle from the structure of K^{-1}, it must be physical hence time-reversal trivial, which implies that particle-1 should also be time-reversal trivial. Now consider the charge of these two particles. It is straightforward to see that with the given charge vector τ, the charge carried by particle-1 or 2 $q_{1,2} = \tau_I K^{-1}_{IJ} l_J, (J = 1, 2)$ can only be an integer. Hence particle-1 and 2 carry neither fractional charge nor Kramer's doublet.

Recall that our purpose here is to describe phases with a particle that carries both charge-1/2 and Kramer's doublet. Hence particle-3 and particle-4 must form a Kramer's doublet and carries charge-1/2. So we want the charge vector that makes $q = \tau_I K^{-1}_{IJ} l_J$ half-integer when $l \in \{(0,0,1,0),(0,0,0,1)\}$. It is then straightforward to show that we need τ_2 to be odd and $\tau_3 = \tau_4$ to be any integer.

What we have shown above is that if the e-particle carries both charge-1/2 and Kramer's doublet structure, the m-particle must be trivial under both symmetry transforms, i.e. the phase has to be eCT.

For now we make a few comments. Note that in the first four phases the m particle has trivial quantum numbers. It is only natural that such states where one of the e or m particles have trivial quantum numbers have trivial quantum numbers can always be realized in strictly 2d systems. From such states we can always destroy the Z_2 topological order by condensing the m particle to produce a trivial symmetry preserving insulator. This will not be possible for states that can only be realized at the surface of 3d SPT phases. In Phase $eCmT$ both the e and m carry non-trivial quantum numbers. Despite this as we have seen it can be realized in strict 2d.

2.1.2 Surface Equivalence

Now lets move to the last 5 phases of Table. 2.1. Ref. [43] showed that phases $eTmT$ and $eCmC$ both arise at the surface of 3d SPT phases. To discuss the other phases we first define the concept of "surface equivalence" of topologically ordered phases.

We say that two topologically ordered states at the surface of a 3d SPT phase are "surface equivalent" if one can be obtained from the other by combining with a strictly 2d states with the same symmetry. The notion of combining two states will be described in detail below. Consider two Z_2 topologically ordered states - say states

A and B - with distinct realizations of the global symmetry. This means that at least one of the e, m particles transform differently under the global symmetry for the two states. Assume now that A and B have the same symmetry for the e particle or - in obvious notation - that $(\sigma_{eA}, \mu_{eA}) = (\sigma_{eB}, \mu_{eB}) \equiv (\sigma_e, \mu_e)$. Then we must have $(\sigma_{mA}, \mu_{mA}) \neq (\sigma_{mB}, \mu_{mB})$.

Now consider the composite system $A + B$. We allow A and B to couple through all symmetry allowed short ranged interactions. For weak interaction strengths the 2 states will be decoupled, and the combined system will have deconfined $Z_2 \times Z_2$ topological order. However for stronger interactions e_A can mix with e_B as they have the same symmetry. This partially confines the $Z_2 \times Z_2$ topological order to a simpler topological ordered state with just a single deconfined Z_2 gauge structure. We will denote this new phase $A \oplus B$. In this new state the m particles of A and B will be confined together to produce a new particle $m_{A\oplus B} \sim m_A m_B$. Thus $A \oplus B$ has the quantum numbers $(\sigma_e, \sigma_{mA}\sigma_{mB}, \mu_e, \mu_{mA}\mu_{mB})$.

This concept of combining phases enables us to see several equivalences in Table 2.1. For instance it is clear that Phase eCT can be obtained as $eC \oplus eT$ (by letting the m particles mix). Let us now consider surface equivalence. Phase $eCmC$ and $eCmT$ share the same quantum numbers for the e particle. Thus we may combine them to produce a new Z_2 phase $eCmC \oplus eCmT$ which, by inspection, has the same symmetries as Phase $eCTmC$ (after a relabeling of e and m). This means that Phases $eCmC$ and $eCTmC$ are surface equivalent. Specifically consider the 3d SPT phase with Phase $eCmC$ as its surface topological ordered state. We may then deposit a layer of Phase $eCmT$ (which is allowed in strict 2d) at its surface, and then let the e particles mix. This mixing will induce a surface phase transition where the surface topological order becomes that of Phase $eCTmC$. It follows that Phase $eCTmC$ can also only be realized at the surface of the 3d SPT boson insulator.

Similarly the m particle of Phase $eTmT$ has the same quantum numbers as the m particle of Phase $eCmT$. Letting them mix we get Phase $eCTmT$. Phase $eCTmCT$ is also readily seen to be $eCTmC \oplus eCTmT$.

Thus we see that the last 5 phases of Table 2.1 are all obtained at the surface of 3d SPT phases. All these 5 phases are obtained from two "root" phases (Phase $eTmT$ and $eCmC$) by combining with phases that are allowed in strict 2d or with each other.

It is interesting to notice that the realization of the 5 phases at the SPT surfaces implies their absence in strict $2d$ systems, independent of K-matrix consideration. One can understand this as follows: if a surface state can also be realized in strict $2d$, then one can deposit such a $2d$ system onto the surface. The quasi-particles in the two systems (call them (e_1, e_2) and (m_1, m_2)) will then have exactly the same symmetry properties, and the bound states of two particles of the same kind in the two systems (e_1e_2 and m_1m_2) will be trivial under all symmetries. Moreover, e_1e_2 and m_1m_2 are mutual bosons to each other. Hence one can condense both e_1e_2 and m_1m_2 without breaking any symmetry. However, this will confine all the fractional quasi-particles since any one of them will have mutual π-statistics with either e_1e_2 or m_1m_2, and the surface will become a trivial phase, i.e. symmetric, gapped and confined. By definition, the corresponding bulk cannot be a SPT state. Hence the states at SPT surfaces must not be realizable in strict $2d$. This will have interesting implications for $2d$ systems, and an example will be given toward the end of this chapter.

It is also interesting to view this result from a different point of view which inverts the logic followed above. Consider the problem of identifying 3d boson SPT states with this symmetry. The results of this section show that there are precisely two distinct 'root' Z_2 topological orders that can only occur at the surface of SPT phases. phases . This then gives us two "root" 3d SPT states with this symmetry. This is the same conclusion arrived at by direct consideration of surface theories in Ref. [43], and ties in nicely wth the formal cohomology classification (which also gives 2 root states). Note in particular that of the 2 root states $eTmT$ is simply inherited from the 3d SPT with Z_2^T symmetry alone. Thus the only non-trivial SPT state that is unique to the extra $U(1)$ symmetry is the one with surface topological order $eCmC$ as was suggested in Ref. [43].

2.2 Topological Z_2 spin liquids

Here we repeat the excercise above for symmetries appropriate to quantum spin systems. We consider two cases: symmetry $U(1) \times Z_2^T$ and symmetry Z_2^T. The former describes time reversal symmetric quantum spin Hamiltonians with a conserved component of spin. In the latter we only assume time reversal symmetry. The consistent symmetry assignments for Z_2 topological order with bosonic e and m particles are given in Tables. 2.2 and 2.3.

Let's first consider $U(1) \times Z_2^T$, in which case the $U(1)$ charge goes to minus itself under time-reversal. The analysis of Ref. [71] again gives the same 10 phases as before and we will use the same labels. However a difference appears in the K-matrix classification. For this symmetry class we will see that a 2×2 K-matrix is enough to describe all the $2d$ states. We have again

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.18}$$

but now the charge vector must be taken to be $\tau = (t, 0)$ with $t = 0, 1$ due to the different transformation of the density of the global $U(1)$ charge under time reversal. Time reversal on the edge boson fields continues to be implemented as $\Phi_I \to T_{IJ} (\phi_I + \alpha_I)$ with $\alpha = (0, \frac{\pi x}{2})$ $(x = 0, 1)$. With this symmetry implementation we see that the edge field $e^{i\phi_2}$ creates a particle that can either carry $1/2$ charge or be a Kramers doublet or both. The other edge field $e^{i\phi_1}$ creates a particle with trivial quantum numbers. This leads to four phases corresponding to $e0m0, eT, eC$ and eCT.

The standard slave boson/fermion construction of Z_2 spin liquids - as in the classic work of Refs. [81, 82] - give (when the spin symmetry is $U(1)$) the state eCT. The spinon in these constructions both carries spin-1/2 and is a Kramers doublet. The easy axis Kagome lattice spin model of Ref. [83] provides an explicit microscopic model for a Z_2 spin liquid with $U(1) \times Z_2^T$ symmetry. In the standard interpretation the spin S_z of that model labels the two members of a Kramers doublet states of a microscopic Ising spin. Time reversal is then implemented in terms of the spin operators as usual through $\mathbf{S} \to -\mathbf{S}$. In that case in the Z_2 spin liquid phase the spinons are readily seen to both carry $S_z = \pm\frac{1}{2}$ and be Kramers doublets to realize the eCT class. There is a different implementation of time reversal symmetry in this easy axis Kagome spin model. If the S_z labels two members of a microscopic non-Kramers doublet then we must interpret the S as a 'pseudo spin' $1/2$ operator that acts in this two dimensional Hilbert space at each site. Time reversal takes $S_z \to -S_z, S^+ \to S^-$. In that case the spinons in the Z_2 spin liquid phase will have spin $S_z = \pm\frac{1}{2}$ but will be Kramers singlets. Thus we have a realization of the eC phase in the model of Ref. [83].

Phase $eCmT$ which was allowed earlier in Sec. 2.1 now does not appear. Physically this is because the "topological band" in the $U(1) \rtimes Z_2^T$ symmetry becomes trivial in the $U(1) \times Z_2^T$ case. The easiest way to see this is to consider the edge theory, which has two counter-propagating fermions. With $U(1) \times Z_2^T$ symmetry, one can mix the two fermions (hence gap out the edge) without breaking any symmetry, even if the fermions form Kramer's pairs.

The absence of the $eCmT$ phase in strict $2d$ modifies the equivalence relation established in last section. In particular, the last three phases in Table 2.1 will not be equivalent to either $eTmT$ or $eCmC$. Actually in Ref. [43], three distinct 'root' SPT phases were discussed corresponding to those with surface topological orders$eTmT$, $eCTmT$ and $eCTmCT$. The last three phases in Table 2.2 are thus surface topological orders corresponding to other SPT phases that may be obtained by combining these root phases. This is in perfect agreement with the results of Ref. [43].

Next we consider Z_2^T symmetry alone, which is much simpler. It is straightforward to see that the phases $e0m0$ and $eTm0$ can be realized in strict $2d$, while $eTmT$ can only appear on an SPT surface. The corresponding table is simply a subset of the previous two.

Phase	σ_e	σ_m	μ_e	μ_m	Comments
$e0m0$	1	1	1	1	No fractionalization
eT	1	1	-1	1	No fractional charge but Kramers
eC	-1	1	1	1	$b = \Phi^2$
eCT	-1	1	-1	1	$b = \epsilon_{\alpha\beta} f_\alpha f_\beta$
$eTmT$	1	1	-1	-1	3d SPT surface
$eCTmT$	-1	1	-1	-1	3d SPT surface
$eCTmCT$	-1	-1	-1	-1	3d SPT surface
$eCmT$	-1	1	1	-1	$eTmT \oplus eCTmT$
$eCTmC$	-1	-1	-1	1	$eCTmT \oplus eCTmCT$
$eCmC$	-1	-1	1	1	$eCTmC \oplus eCmT$

Table 2.2: Symmetry action of $U(1) \times Z_2^T$ (charge is $\mathcal{T}$-odd) for Z_2 topological ordered states. The first 4 are allowed in strict 2d while the last 6 can only be realized at surface of 3d SPT phases (or derived from them)

Phase	μ_e	μ_m	Comments
$e0m0$	1	1	No fractionalization
eT	-1	1	Kramers
$eTmT$	-1	-1	3d SPT surface

Table 2.3: Symmetry action (Z_2^T) for Z_2 topological ordered states. The first two are allowed in strict 2d while the last one can only be realized at surface of 3d SPT phases

2.3 All fermion Z_2 liquids

We now extend our analysis to a very interesting topological order where there are three distinct topological quasi-particles, all of which are fermions $f_{1,2,3}$, and there's a mutual π-statistics between any two of them. This can be viewed as a variant of the usual Z_2 liquid, in which both the e and m particles become fermions. Since they have a mutual π-statistics, the bound state $\epsilon = em$ is still a fermion and has π-statistics with both e and m.

The statistics of this phase is perfectly compatible with time-reversal symmetry, but the realization in strict $2d$ turns out to be always chiral and hence breaks time-reversal. One way to understand this is to start from a conventional Z_2 topologically ordered liquid with bosonic e and m particles. Then put the fermionic ϵ particle into a band structure such that the vison also becomes a fermion. This may be fruitfully discussed in terms of the edge Lagrangian for the ϵ field. The vison operator appears as a 'twist' field that creates a π phase shift for ϵ. For a single branch of chiral (complex) fermion on the edge $e^{i\phi_{L,R}}$ the twist operator is $e^{i\phi_{L,R}/2}$. This has conformal spin $\pm 1/8$ so that in this case the vision is an anyon with fractional statistics. Very generally take a theory with n_R right moving and n_L left moving fermions all of which correspond to the same bulk ϵ particle which see a single common vison. This acts as a common twist field for all the edge fermions and hence has conformal spin $\frac{n_R - n_L}{8}$. Therefore to make the vison fermionic one needs $n_L - n_R = 4 \text{ mod } 8$. One such realization is given by the 4×4 K-matrix

$$K = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix} \tag{2.19}$$

which has chiral central charge 4. Since the chiral central charge is non-zero this phase clearly cannot arise in a time reversal invariant strictly 2d system.

However Ref. [43] suggested that such an all fermion Z_2 topological order can arise at the surface of a 3d SPT phase with time reversal symmetry. In this state if the surface is gapped by breaking time reversal symmetry then there is a quantized thermal Hall conductivity $\kappa_{xy} = \pm 4$. However if time reversal symmetry is present and the surface is gapped, there will be surface topological order. Ref. [43] proposed that this is a Z_2 topological order which is a time reversal symmetric all fermion state. To understand why this is reasonable consider starting from the all fermion surface topological ordered state. What should we do to confine all the fermion excitations in the surface? It is clear from the discussion above that if we take one of the fermions and put it in a Chern band such that the surface $\kappa_{xy} = \pm 4$ then the other two topological quasiparticles will become bosons. These bosons can now be condensed to get a confined surface state. However this clearly requires broken time reversal symmetry and will give a $\kappa_{xy} = \pm 4$ which is indeed the right Z_2^T broken surface state for this proposed SPT. This kind of 3+1-D SPT phase with Z_2^T symmetry is not present in the cohomology table of Ref. [16, 17]. Including this 3d SPT surface state and using the language in the last few sections, we have a new table (Table 2.4).

Phase	μ_e	μ_m	Comments
e^f0m^f0	1	1	All fermions, singlets
e^fTm^fT	-1	-1	$e^f0m^f0 \oplus eTmT$

Table 2.4: Symmetry action (Z_2^T) for all- fermionic Z_2 states. Both states can only be realized at surfaces of 3d SPT phases

The second phase in the table is obtained from the first by adding a usual Z_2 liquid in the $eTmT$ phase, then condense the bound state of the $\epsilon^f = e^f m^f$ in the fermionic liquid and the $\epsilon = em$ in the $eTmT$ liquid. Since the $eTmT$ phase cannot be realized in strict $2d$, the two phases e^f0m^f0 and e^fTm^fT should be viewed as inequivalent, hence give rise to two distinct SPT phases with time-reversal symmetry in addition to the one with the $eTmT$ surface Z_2 topological order. Thus in total with Z_2^T symmetry we actually have 3 non-trivial SPT phases corresponding to a Z_2^2 classification. (The cohomology classification of Ref. [16, 17] gives instead a Z_2 classification).

2.4 Constructing SPT with coupled layers of Z_2 liquids

From the considerations in the previous sections, it is clear that to construct a 3+1-D SPT state, we only need to construct the corresponding topological order on the surface but have a confined bulk with gapped excitations. In this section we give one such explicit construction using coupled layers of $2d$ Z_2 liquids. Specifically we consider a system of stacked layers where each layer realizes a Z_2 topological order that is allowed in strictly 2d systems. Then we couple the different layers together in such a way that the bulk is confined and gapped. But we show that the surface layer is left unconfined and further corresponds to the surface Z_2 topological order of an SPT phase. A similar coupled-layer construction of the free fermion topological insulator was proposed[84] to obtain the single Dirac cone on the surface. We first illustrate this by constructing the $eTmT$ with Z_2^T symmetry, and it will be clear later that this can be generalized to all the SPT states mentioned in this chapter.

Consider stacking N layers of Z_2 liquids in the eT state which is allowed in strictly $2d$. Now turn on an inter-layer coupling to make the composite particles $e_i m_{i+1} e_{i+2}$ condensed, where i is the layer index running from 1 to $N-2$. Note that the $e_i m_{i+1} e_{i+2}$ all have bosonic self and bosonic mutual statistics so that they may be simultaneously condensed. As illustrated in Figure 2-1, this procedure confines all the non-trivial quasi-particles in the bulk. However four particles on the surfaces survive as the only deconfined objects: $e_1, m_1 e_2, e_N, m_N e_{N-1}$. Notice that e_1 and $m_1 e_2$ are mutual semions and have self-boson statistics. Thus they form a Z_2 liquid at the top surface. Similarly e_N and $m_N e_{N-1}$ are self bosons, have mutual semion statistics and form a Z_2 liquid at the bottom surface. The key point however is that all these particles have $\mathcal{T}^2 = -1$. Thus either surface is in the $eTmT$ state though the bulk has no exotic excitations. By the analysis above we identify this with the 3d SPT state with Z_2^T symmetry.

This construction can be immediately generalized to other SPT states. For example, to get the $eCmC$ (or $eCTmCT$) state with $U(1) \rtimes Z_2^T$ or $U(1) \times Z_2^T$ symmetry, just stack layers of eC (or eCT) states and condense $e_i m_{i+1} e_{i+2}$.

Most interestingly, the all-fermion Z_2 surface topological state with global Z_2^T symmetry, which is quite hard to construct using other methods, can also be constructed in this way: simply start with stacked 2d Z_2 liquids where all particles e, m, ϵ are invariant under T-reversal. Such a Z_2 state is obviously allowed in strict $2d$. Now condense $\epsilon_i m_{i+1} \epsilon_{i+2}$ instead of $e_i m_{i+1} e_{i+2}$ in the above constructions, where $\epsilon_i = e_i m_i$ is the fermion in the $2d$ Z_2 gauge theory. Again the $\epsilon_i m_{i+1} \epsilon_{i+2}$ have both self and mutual boson statistics so that they may be simultaneously condensed. This confines all bulk topological quasiparticles. The surviving surface quasi-particles will be $\epsilon_1, m_1\epsilon_2$ at the top surface and $\epsilon_N, m_N\epsilon_{N-1}$ at the bottom surface. These particles are all fermions, and the two particles at either surface have mutual semion statistics. It follows that either surface realizes the all-fermion Z_2 topological order but now in the presence of Z_2^T symmetry. We have thus explicitly constructed the SPT phase discussed in Section. 2.3.

This coupled layer construction gives very strong support to the results of Ref. [43] on the various SPT phases. In particular it removes any concerns on the legitimacy of the state of Sec. 2.3 with Z_2^T symmetry not currently present in the cohomology classification.

2.5 Relation with bulk topological field theories

Here we provide an understanding of the results obtained above from topological field theories in the bulk. It was shown in Ref. [43] that bosonic topological insulators in $3d$ with $(U(1))^N$ symmetry has a bulk response to external 'probe' gauge fields A^I characterized by a θ-term with $\theta = \pi$:

$$L_\theta = \frac{\theta}{8\pi^2} K_{IJ} \epsilon^{ijkl} \partial_i A_j^I \partial_k A_l^J. \tag{2.20}$$

If under symmetry transformations (e.g. time-reversal) the θ-angle transforms as $\theta \to -\theta$, then the $\theta = \pi$ term is symmetric in the bulk, but on the boundary it

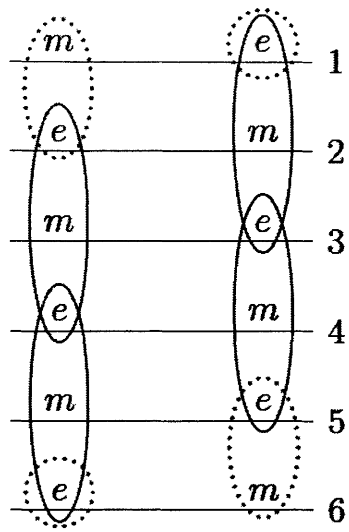

Figure 2-1: Coupled-layer construction of SPT states. The particle composite in the ellipses are condensed, and only the four surface particles in the dotted ellipses survived as deconfined topological quasi-particles.

reduces to a (mutual) Chern-Simons term with symmetry-violating responses. This was a familiar issue in the non-interacting fermionic topological insulator, where a single Dirac cone was introduced on the boundary to cancel the time-reversal violating response through parity anomaly.

In our cases let us understand how this works out when the surface is in a symmetry preserving gapped topological ordered phase of the kind studied in this chapter. We will show that the symmetries of the topological order on the boundary are such as to cancel the Chern-Simons response arising from the θ-term. To illustrate the idea we take $K_{IJ} = \sigma_x$ which applies to a large class of SPT phases in $3d$. This will give a mutual Chern-Simons term on the boundary

$$L_{CS} = \frac{1}{4\pi}\epsilon^{ijk}A_{1,i}\partial_j A_{2,k}. \tag{2.21}$$

This term alone would give a response that breaks time-reversal symmetry. To cure it we put a Z_2 topological liquid on the boundary, with the e and m particles coupling to $A_{1,2}$ respectively. The Lagrangian is given by

$$L_{Z_2} = \frac{1}{\pi}\epsilon^{ijk}a_{1,i}\partial_j a_{2,k} + \frac{1}{2\pi}\left(\epsilon^{ijk}A_{1,i}\partial_j a_{1,k} + \epsilon^{ijk}A_{2,i}\partial_j a_{2,k}\right). \qquad (2.22)$$

Integrating out $a_{1,2}$ induces a mutual Chern-Simons term for $A_{1,2}$ which exactly cancels what arose from the bulk θ-term and hence restores time-reversal symmetry.

The topological ordered states with symmetry that are forbidden in strict $2d$ realize the symmetry in an 'anamolous' way. The corresponding topological field theories cannot be given consistent lattice regularizations which implement the symmetry in a local manner. The discussion in this section illustrates how these theories can nevertheless be given a higher dimensional regularization as the boundary of a non-anomalous field theory. This has the same essence with the anomaly cancellation in free fermion TI. It will be interesting for the future to have criteria to directly identify such 'anamolous' symmetry in a topological field theory.

2.6 Constraints on gapless 2d quantum spin liquids: Absence of Algebraic Vortex Liquids

We now turn to gapless quantum liquids in two space dimensions. Examples are gapless quantum spin liquid phases of frustrated quantum magnets, or non-fermi liquid phases of itinerant fermions or bosons. Symmetry plays a very crucial role in the stability of these phases. The example of the topologically ordered gapped states considered in previous sections lead us to pose the question of what kinds of putative gapless phases/critical points are allowed to exist with a certain symmetry in strictly 2d systems. First of all we note that in contrast with gapped topologically ordered phases global symmetries typically play a much more important role in protecting the gaplessness of a phase. The symmetry may forbid a relevant perturbation to the low energy renormalization group fixed point that, if present, may lead to a flow to a gapped fixed point. Here however we are interested in a more general question. We wish to consider gapless fixed points that can be obtained by tuning any finite number of relevant perturbations. This includes not just bulk 2d phases but also critical or even finitely multi critical quantum systems. We are particularly interested in such gapless 2d fixed points with symmetry that cannot exist in strict 2d but may only exist at the surface of a 3d insulator (SPT or otherwise).

To set the stage consider a simple and familiar example in a free fermion system. The surface of the celebrated time reversal symmetric electron topological insulator (symmetry $U(1) \rtimes Z_2^T$ has an odd number of Dirac cones. Such a gapless state cannot exist in strict 2d fermion systems with the same symmetry even as a multi critical point. However if we give up time reversal symmetry this state is allowed as a critical point in strict 2d. An example is provided by a 2d free fermion model poised right at the integer quantum Hall transition. Thus symmetry provides a strong restriction on what gapless fixed points are allowed in strict 2d.

We focus now on a very interesting gapless state proposed[85, 86, 87] to exist in strict 2d in frustrated XY quantum magnets (symmetry $U(1) \times Z_2^T$) or in boson systems (symmetry $U(1) \rtimes Z_2^T$). This state - dubbed an Algebraic Vortex Liquid (AVL) - was obtained in a dual vortex description by fermionizing the vortices and allowing them to be massless. A suggestive approximation was then used to derive a low energy effective field theory consisting of an even number of massless 2-component Dirac fermions (the vortices) coupled to a non-compact $U(1)$ gauge field. The AVL state has been proposed to describe quantum spin liquid states on the Kagome and triangular lattices. In terms of development of the theory of gapless spin liquids/non-fermi liquids the AVL proposal is extremely important. To date the only known theoretical route to accessing such exotic gapless phases of matter (in $d > 1$) is through a slave particle construction where the spin/electron operator is split into a product of other operators. If some of the resulting slave particles are fermions, they can be gapless. In contrast the AVL presents a new paradigm for a gapless highly entangled state which is likely beyond the standard slave particle approach. It is therefore crucial to explore and understand it thoroughly.

We now argue that the AVL state cannot exist in strictly 2d models with either $U(1) \rtimes Z_2^T$ or $U(1) \times Z_2^T$ symmetry. This is already hinted at by several observations.

First it has never been clear how to implement time reversal in a consistent way in the AVL theory. The AVL is obtained from the usual bosonic dual vortex theory through a flux attachment procedure to fermionize the vortices. This leads to an additional Chern-Simons gauge field that couples to the fermionized vortices. However this new gauge field can be absorbed into the usual dual gauge field to leave behind a three derivative term for a residual gauge field. It was argued that this three derivative interaction is formally irrelevant in the low energy effective theory. This argument is delicate though. In the simplest context [88] where such an approximation was made an alternate description[89] in terms of a sigma model revealed the presence of a topological θ term at $\theta = \pi$. The topological term also has three derivatives but its coefficient is protected by time reversal symmetry and does not flow under the RG. It's presence presumably crucially alters the physics of the model. Thus one may worry about the legitimacy of the approximations invoked to justify the AVL phase.

Note that the fundamental issue that needs to be addressed with the AVL phase is whether it realizes symmetry in a manner that is allowed in 2d spin/boson systems. This is of course the kind of question that is the essence of this chapter. A final hint that the AVL phase may not exist in strict 2d comes from recent work[43] showing that gapless quantum vortex liquids with fermionic vortices can actually arise at the surface of time reversal symmetric 3D SPT phases. This strongly suggest that such phases cannot arise in strict 2d with the same symmetries. Below we will sharpen these arguments.

Consider the proposed effective field theory for the AVL phase:

$$\mathcal{L} = \overline{\psi}_\alpha \left(i\partial\!\!\!/_\mu - i a\!\!\!/_\mu \right) \psi_\alpha + \frac{1}{2e^2} \left(\epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda \right)^2 + \frac{1}{2\pi} a_\mu \epsilon_{\mu\nu\lambda} \partial_\nu A^{ext}_\lambda \tag{2.23}$$

Here ψ_α ($\alpha = 1, ...2N$) are the fermionized vortices, a_μ is a non-compact $U(1)$ gauge field whose curl is 2π times the global $U(1)$ current, and A^{ext}_μ is an external probe $U(1)$ gauge field. Note that the vortices themselves do not carry physical $U(1)$ charge. As mentioned above the realization of time reversal in terms of these fermionized vortex fields has always been a tricky issue for the AVL theory. To sharpen the issue we now consider a phase that is accessed from the AVL phase by pairing and condensing the ψ_α fields. In the original AVL literature[85, 86, 87] a number of phases proximate to the AVL were studied by assuming that four fermion interactions were strong enough to give a mass to the fermions. A number of different such mass terms leading to various symmetry breaking orders were examined. Here instead we imagine a mass term that corresponds to vortex pairing that preserves the global internal symmetry.

The vortex pair condensation will gap out the gauge field a_μ and will give an insulator. However as the ψ_α fields are vortices of the original boson this is a phase with Z_2 topological order. The fermionized vortices survive as 'unpaired' gapped quasiparticles in this topological phase. We may identify them with the ϵ particle which in this case has zero global $U(1)$ charge. In the notation of previous sections $\sigma_\epsilon = 1$. Furthermore the pair condensation will quantize the gauge flux $\nabla \times \mathbf{a}$ in units of π so that one of the topological quasiparticles (which we take to be the e particle) has $1/2$ charge, *i.e* $\sigma_e = -1$.

Now it is clear from Tables 2.1 and 2.2 of the previous sections that in strict 2d such a state can exist only if the ϵ particle also carries $1/2$ charge, *i.e* $\sigma_\epsilon = -1$. However we just argued that the Z_2 topological ordered state realized from the AVL state has $\sigma_\epsilon = 1$, *i.e* it carries zero global $U(1)$ charge. It follows that such a Z_2 topological ordered state cannot exist in strict 2d. Note that we did not explicitly rely on time reversal symmetry in our analysis (though it is implicit in deciding which Z_2 states are allowed in 2d).

Thus the Z_2 topological ordered states that descends from the AVL is not allowed to exist in strict $2d$. This then implies that the AVL itself cannot exist in strictly 2d systems so long as both global $U(1)$ and Z_2^T symmetries are present.

Can gapless quantum vortex liquids ever exist in strictly 2d? One option is to break time reversal symmetry. Then our arguments do not prohibit the formation of fermionic vortices which can then be in a gapless fluid state. Indeed such a gapless magnetic-field induced vortex metal state was proposed to exist in 2d superconducting films in Ref. [90]. A different option - which we elaborated in Ref. [91] - that preserves internal symmetries is obtained by fractionalizing the vortices into fermionic partons which can then be gapless.

2.7 Dual Landau-Ginzburg theory of SPT surface

In this Section we briefly describe the Landau-Ginzburg theory of the surface of 3d SPT states (symmetry $U(1) \rtimes Z_2^T$ or $U(1) \times Z_2^T$) in terms of dual vortex fields. Ref. [43] showed that the difference with a trivial surface is captured very simply in terms of the difference in the structure of the vortex (this should be understood as the point of penetration of the $3d$ vortex line with the surface). Here we elaborate on this dual theory and present an explicit derivation starting from the surface topological ordered phase.

Let us consider $U(1) \times Z_2^T$ and consider the phase labeled by surface topological order $eCmC$ (the symmetry $U(1) \rtimes Z_2^T$ analysis is essentially the same). According to Ref. [43] the boundary vortex is then a Kramers singlet fermion. The corresponding surface Landau-Ginzburg Lagrangian may be written schematically

$$\mathcal{L}_d = \mathcal{L}[c, a_\mu] + \frac{1}{2\pi} A_\mu \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda \tag{2.24}$$

The first term describes a (spineless) fermionic field c coupled minimally to the dual internal gauge field a_μ, and A_μ is an external probe gauge field. The field c describes the fermionic vortex. The global $U(1)$ current is as usual

$$j_\mu = \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda \tag{2.25}$$

If instead the field c were a boson the Lagrangian above would be the standard dual Lagrangian for a system of strictly $2d$ bosons. Let us first describe the phase structure of this dual fermionic vortex theory. We will then provide an explicit derivation that is complementary to the general considerations of Ref. [43].

If the fermionic vortex c is gapped and in a trivial 'band' insulator, then as usual we get a surface superfluid. Note that as c is a fermion it cannot condense. This precludes the usual mechanism of vortex condensation to obtaining a trivial boson insulator as expected for an SPT surface. The surface superfluid order can be killed if pairs of c condense, $i.e < cc > \neq 0$. This leads to a surface topological order described by a Z_2 gauge theory. There is the unpaired fermion that survives as a gapped excitation carrying zero global $U(1)$ charge. We identify this with the neutral ϵ particle. The pair condensation quantizes flux of a_μ in units of π. This carries global $U(1)$ charge $1/2$ and we identify this with the e particle. It follows that this is the $eCmC$ phase.

As described in Ref. [43] if we break time reversal at the surface we can get a gapped phase without topological order. This is obtained by simply letting the c-fermionic vortices completely fill a topological band with Chern number ± 1, $i.e$ the Hall conductivity of the c-fermion is $\sigma^c_{xy} = \pm 1$. To see that this indeed describes the correct T-broken surface state we use a Chern-Simons description of this state. First rewrite the fermion current j^f in terms of a dual gauge field $\tilde{a}$:

$$j^f_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu \tilde{a}_\lambda \tag{2.26}$$

When the fermion has Hall conductivity, say $+1$, the effective Chern-Simons Lagrangian in terms of $(a, \tilde{a})$ becomes

$$\mathcal{L} = \frac{1}{4\pi}\tilde{a}d\tilde{a} + \frac{1}{2\pi}ad\tilde{a} + \frac{1}{2\pi}Ada \tag{2.27}$$

(We have used the compact notation $da \equiv \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda$). This is a 2-component Chern-Simons theory with a K-matrix

$$K = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.28}$$

with charge vector $\tau = (0, 1)$. This state has electrical Hall conductivity $\sigma_{xy} = \tau^T K^{-1} \tau = -1$. Further as this K-matrix has one positive and one negative eigenvalue, it is a non-chiral state with $\kappa_{xy} = 0$. Finally there is no surface topological order as $|detK| = 1$. These are exactly the right properties of the T-broken surface state without topological order at this SPT surface.

Thus the fermionic vortex Landau-Ginzburg theory correctly reproduces the surface phase structure of this SPT phase. We note that if the c-fermion had band structure with an even number of gapless Dirac fermions we get exactly the proposed Lagrangian for the AVL phase, consistent with the claim in Sec. 2.6 that the AVL state can only occur at the surface of an SPT state (with T-reversal) and not in strict $2d$.

A different SPT state with $U(1) \times Z_2^T$ symmetry has a bosonic Kramers doublet vortex z_α, $\alpha = 1, 2$. The corresponding dual Landau-Ginzburg theory takes the form

$$\mathcal{L}_d = \mathcal{L}[z_\alpha, a_\mu] + \frac{1}{2\pi} A da \tag{2.29}$$

Under time reversal $z_\alpha \rightarrow i\sigma^y_{\alpha\beta} z_\beta$. Finally by stacking the two SPT phases described above we obtain a third SPT with a fermionic Kramers doublet vortex field c_α with Lagrangian

$$\mathcal{L}_d = \mathcal{L}[c_\alpha, a_\mu] + \frac{1}{2\pi} A da \tag{2.30}$$

The surface phase structure of these other SPTs can be readily discussed in terms of these dual Landau-Ginzburg theories. In all these cases there is a bulk-edge correspondence that relates the structure of the surface vortex to the properties of the bulk monopole when the global $U(1)$ symmetry is gauged. Including the trivial (i.e non-SPT) insulator, we have four possible SPT phases with four distinct surface vortices (end points of bulk vortex lines). These correspond precisely to the four possible bulk monopoles of the gauged SPT as discussed in Sec. 2.8.2.

We now provide an explicit derivation of Eqn. 2.24 for the corresponding SPT phase. Let us begin with the surface topological order $eCmC$. Under time reversal the e, m particles transform as

$$\mathcal{T}^{-1} e \mathcal{T} = e^\dagger \tag{2.31}$$

$$\mathcal{T}^{-1} m \mathcal{T} = m^\dagger \tag{2.32}$$

while the ϵ particle is left invariant. It is convenient for our purposes to focus on the e (described by a boson field b) and ϵ (described by a fermion f) particles with mutual semion interactions. We will implement this in a lattice model of the surface through two Ising gauge fields σ, μ with a mutual Ising Chern-Simons term[80]. The mutual Chern-Simons term imposes a constraint relating the integer valued lattice 3-current $\mathbf{j}_b$ to the Ising gauge flux of σ that the fermion sees:

$$(-1)^{\mathbf{j}_b} = \prod_P \sigma_{ij} \tag{2.33}$$

Here the plaquette product in the RHS is taken over the space-time plaquette pierced by the link of the dual lattice on which the boson current flows. The lattice space-time Lagrangian may be taken to be

$$\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f \tag{2.34}$$

$$\mathcal{L}_b = \kappa \mathbf{j}_{b\alpha}^2 + i\mathbf{j}_{b\alpha}.\frac{(\mathbf{A}_\alpha)}{2} \tag{2.35}$$

$$\mathcal{L}_f = -\sigma_{ij}\left(t_{ij} f_i^\dagger f_j + h.c + ...\right) \tag{2.36}$$

The fermion Lagrangian will in general also include pairing terms $f_i f_j + h.c.$ As before A is the external probe gauge field. We now implement a standard duality transformation on the b field by first writing $\mathbf{j}_b = \nabla \times \alpha$ with α an integer. As this kind of duality has been explained in detail in Refs. [80, 89] we will be very brief. The mutual semion constraint Eqn. 2.33 can be solved to write

$$(-1)^\alpha = \sigma \tag{2.37}$$

This means that the integer $\alpha = 2\alpha' + \frac{\pi}{2}(1-\sigma)$ with α' an integer. Imposing the integer constraint on α' softly leads to a term

$$-\lambda cos\left(2\pi\alpha'\right) = -\lambda\sigma_{ij} cos\left(\pi\alpha_{ij}\right) \tag{2.38}$$

We may now define $\pi\alpha = a$ and extract a longitudinal piece $a_{ij} \to a_{ij} + \phi_i - \phi_j$ to obtain a dual vortex Lagrangian in Euclidean space-time:

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_f \tag{2.39}$$

$$\mathcal{L}_\phi = -\lambda\sigma_{ij}cos\left(\nabla\phi + \mathbf{a}\right) + \frac{\kappa}{\pi^2}\left(\nabla\times\mathbf{a}\right)^2 \tag{2.40}$$

$$+i\frac{1}{2\pi}\mathbf{A}.\nabla\times\mathbf{a} \tag{2.41}$$

Finally tracing over the Ising gauge fields σ gives a dual vortex theory in terms of two fermionic fields $c_\pm$ defined through

$$c_\pm = fe^{i\pm\phi} \tag{2.42}$$

However generically due to pairing terms in the f Lagrangian c_- can mix with $c_+^\dagger$ so that there is a unique fermion field $c = c_+ \sim c_-^\dagger$. This then gives us the dual fermionic vortex Landau-Ginzburg theory in Eqn. 2.24. For $U(1)\times Z_2^T$ under time reversal we must have $c \to c$. The dual vortex theory for the surface of the $U(1)\rtimes Z_2^T$ SPT (with $eCmC$ surface topological order) can be derived identically, and takes the same form except that under time reversal $c \to c^\dagger$.

2.8 Time reversal symmetric $U(1)$ quantum liquids in $3+1$ dimensions

We now turn our attention to three dimensional highly entangled states with time reversal symmetry. In three dimensions, interesting gapless quantum liquids with an emergent gapless $U(1)$ gauge field are possible[92]. Explicit lattice models for such phases were constructed and their physics studied in Refs. [79, 93, 94, 95, 96, 97, 98]. Interest in such phases has been revived following a recent proposed realization[99] in quantum spin ice materials on three dimensional pyrochlore lattices. It is thus timely to understand the possibilities for the realization of symmetry in such phases with emergent photons. Here we will restrict attention to time reversal symmetry in keeping with the theme of the rest of the chapter.

The excitation spectrum of the $U(1)$ quantum liquid consists, in addition to the gapless photon, point 'electric' charges (the e particle) and point 'magnetic' charges (the m particle or monopole). We will only consider the situation in which both the e and m particles are gapped, and will focus on phases that can be realized in strictly $3d$ systems (as opposed to $U(1)$ phases allowed at the boundary of $4+1$ dimensional SPT phases). Following the discussion of previous sections, a simple restriction that ensures this is to assume that one of the e or m particles has trivial global quantum numbers and is a boson. Without loss of generality we will assume that it is the m particle.

The low energy long wavelength physics of the $U(1)$ liquid state is described by Maxwell's equations. As usual they imply that the emergent electric and magnetic fields transform oppositely under time reversal. We will distinguish two cases depending on whether the electric field is even or odd under time reversal.

2.8.1 Even electric field

First we consider the case $\mathbf{E} \to \mathbf{E}, \mathbf{B} \to -\mathbf{B}$ under time reversal. This is what happens in the usual slave particle constructions of $U(1)$ spin liquids through Schwinger bosons or fermions. The electric field on a bond gets related to the bond energy which is clearly even under time reversal. Consistent with this the magnetic field gets identified with the scalar spin chirality which is odd under time reversal. Then the electric charge $q_e \to q_e$ and magnetic charge $q_m \to -q_m$. Let us introduce creation operators $e^\dagger, m^\dagger$ for the e and m particles. With the assumption that $m^\dagger$ has trivial global quantum numbers and is a boson, it must transform under time reversal as

$$\mathcal{T}^{-1} m^\dagger \mathcal{T} = e^{i\alpha_m} m \tag{2.43}$$

However the phase α_m has no physical significance. It can be removed by combining $\mathcal{T}$ with a (dual) $U(1)$ gauge transformation that rotates the phase of m (more detail follows in Sec. 2.8.3). So we may simply set $\alpha_m = 0$. Let us consider now the e

particle. If there is just a single species of e particle, then we must have

$$\mathcal{T}^{-1}e^{\dagger}\mathcal{T} = e^{i\alpha_e}e^{\dagger} \tag{2.44}$$

Now the phase α_e can be absorbed by redefining the e operator and so we set $\alpha_e = 0$. The e particle transforms trivially under Z_2^T. There are nevertheless two distinct phases depending on whether e is a boson or a fermion. More phases are obtained by considering a 2-component e field: $e = (e_1, e_2)$. The new non-trivial possibility is that this 2-component e field transforms as a Kramers doublet under Z_2^T:

$$\mathcal{T}^{-1}e^{\dagger}\mathcal{T} = i\sigma_y e^{\dagger} \tag{2.45}$$

Clearly we have $\mathcal{T}^{-2}e^{\dagger}\mathcal{T}^2 = -e^{\dagger}$ but the action of $\mathcal{T}^2$ on physical (*gauge invariant* local) operators gives 1. For instance $e_1^{\dagger}e_2$ is a physical operator and we clearly have $\mathcal{T}^{-2}e_1^{\dagger}e_2\mathcal{T}^2 = e_1^{\dagger}e_2$. When e is a Kramers doublet it can again be either a boson or a fermion. The former is obtained in the standard Schwinger boson construction and the latter in the Schwinger fermion construction. Thus we have a total of four possible phases corresponding to e being a Kramers singlet/doublet with bose/fermi statistics and a boson monopole with trivial global quantum numbers.

2.8.2 $U(1)$ quantum liquids as monopole topological insulators

The four $U(1)$ quantum liquids described above were distinguished by the symmetry and statistics of the e particle. We now develop a very interesting alternate view point where we understand these four states as different SPT insulators of the bosonic monopole with trivial global quantum numbers. As the magnetic charge is odd under time reversal, the monopole transforms under $U_g(1) \times Z_2^T$ where $U_g(1)$ is the gauge transformation generated by the monopole charge. It is useful (though not necessary) to perform an electric-magnetic duality transformation: this exchanges the e and m

labels:

$$e \leftrightarrow m_d \tag{2.46}$$

$$m \leftrightarrow e_d \tag{2.47}$$

We included a subscript d on the right side to indicate that these are the dual labels. Now e_d is a gapped boson that transforms under $U_g(1) \times Z_2^T$. Thus we may regard the $U(1)$ quantum liquids as insulating phases of e_d obtained by gauging the $U(1)$ part of a $U(1) \times Z_2^T$ symmetry. Note that e_d transforms under a linear (*i.e* not projective) representation of $U_g(1) \times Z_2^T$. As discussed in previous sections such bosons can be in a number of different SPT phases. We now study their fate when the $U(1)$ symmetry is gauged.

Gauged bosonic SPT phases in 3*d*

In 2*d* Ref. [47] studied the fate of bosonic SPT insulators with discrete global unitary symmetry when that symmetry is gauged. It was shown that the result was a topologically ordered gapped quantum liquid with long range entanglement. A general abstract discussion of such gauged SPT phases for unitary symmetry groups (*i.e* not involving time reversal) has also appeared[100]. Here we are interested in 3*d* SPT phases with $U(1) \times Z_2^T$ symmetry. A gauged 3*d* SPT phase with $U(1) \rtimes Z_2^T$ was also studied very recently in a beautiful paper[69]. Using the known $\theta = 2\pi$ electromagnetic response[43], it was argued that the monopole of this gauged SPT is a fermion, and this was used as a conceptual starting point to discuss the surface of this SPT. Here we will discuss the gauged SPT from a different and more general point of view that will enable us to also discuss SPT phases where the electromagnetic response has no θ term (necessary for the results in this subsection).

Refs. [43, 101] show that a key distinction between different SPT phases with the same symmetry is exposed by considering the end points of vortex lines of the boson at the interface with the vacuum. It will be convenient to label the SPT phases by their possible surface topological order (whether or not such order is actually present in any particular microscopic realization). For one simple example SPT phase (the one whose surface topological order is $eCmC$) these papers argued that a surface Landau-Ginzburg theory is obtained in a dual description in terms of fermionic vortices. In another SPT phase (labeled by surface topological order $eCmT$) the surface vortex is a boson but is a Kramers doublet. By stacking these two phases together we can get a third SPT phase where the surface vortex is a fermionic Kramers doublet. In contrast for topologically trivial insulators the surface vortex is a bosonic Kramers singlet. In Appendix 2.7 we describe these surface dual Landau-Ginzburg theories and their implied surface phase structure. We also provide an explicit derivation that is complementary to the arguments of Ref. [43].

Closely related to this we can also consider external point sources for vortex lines directly in the bulk. In the Hilbert space of the microscopic boson model the vortex lines do not have open ends in the bulk. So these external sources for vortex lines must be thought of as 'probes' that locally modify the Hilbert space. These will behave similarly to the surface end points of vortices. For example in the SPT labelled $eCmT$ Ref. [101] shows that the ground state wave function is a loop gas of vortices where each vortex core is described as a Haldane spin chain. An externally imposed open end for a vortex string will terminate the core Haldane chain so that there is a Kramers doublet localized at this end point. In this case the external vortex source is a bosonic Kramers doublet. In the other example SPT (labelled $eCmC$) the vortices are ribbons with a phase factor (-1) associated with each self-linking of the ribbon. Open end points of such vortex strings are fermionic Kramers singlets. Obviously stacking these two phases together produces an SPT where bulk vortex sources are fermionic Kramers doublets. In contrast in trivial boson insulators such bulk external vortex sources are bosons with trivial quantum numbers under global symmetries.

This understanding of the different SPT phases immediately determines what happens when the $U(1)$ symmetry is gauged. As these phases are gapped insulators (at least in the bulk) there will now be a dynamical photon. More interesting for our purposes is the fate of the magnetic monopole m_d. The monopole serves as a source of 2π magnetic flux for the e_d particle. Thus it should precisely be identified with the source of vortex lines. It follows that m_d can therefore either be a Kramers singlet/doublet and have bose/fermi statistics.

Reversing the duality transformation we see that these are precisely the four distinct $U(1)$ quantum liquids discussed in the previous subsection. We have thus established our promised claim that these different $U(1)$ quantum liquids may be equivalently viewed as different bosonic monopole SPT insulators.

Electric particle	Monopole insulator
$\mathcal{T}^2 = 1$, boson	Trivial
$\mathcal{T}^2 = -1$, boson	SPT -$eCmT$
$\mathcal{T}^2 = 1$, fermion	SPT-$eCmC$
$\mathcal{T}^2 = -1$, fermion	SPT- $eCTmC$=$eCmT \oplus eCmC$

Table 2.5: Phases of $U(1)$ quantum liquids (Z_2^T symmetry and even emergent electric field), labeled by symmetry properties of the electric charge, and the corresponding type of monopole SPT, conveniently labeled by the possible surface topological order.

In Table 2.5 we list all the distinct phases of the $U(1)$ gauge theory with their monopole quantum numbers, and the corresponding SPT states (labeled by the surface topological states) formed by the bosonic matter field. Notice that SPT states descended from that of Z_2^T symmetry (the $eTmT$ and the all-fermion states) didn't appear in Table 2.5. One can understand this by thinking of these states as combinations of trivial insulators and Z_2^T SPT states formed by charge-neutral bosons, hence the $U(1)$ gauge field is decoupled from the SPT states and the vortex source ($i.e$ monopole $m_d = e$) remains trivial.

2.8.3 Odd electric field

We now consider $U(1)$ liquid states where under time reversal the electric field is odd and the magnetic field is even. In the convention of Ref. [94] this includes the case of quantum spin ice. Again we restrict attention to $U(1)$ liquids where the magnetic monopole m is bosonic and transforms trivially under Z_2^T. What then are the possibilities for the e particle?

Based on the insights of the previous subsection let us first see what we can learn by considering different monopole SPT phases. Now the magnetic charge $q_m \to q_m$

under time reversal so that

$$\mathcal{T}^{-1} m^{\dagger} \mathcal{T} = m \tag{2.48}$$

Thus m (or equivalently e_d after the duality transformation) transforms under $U_g(1) \rtimes Z_2^T$.

For bosons with global symmetry $U(1) \rtimes Z_2^T$ there is one non-trivial SPT phase which is again conveniently labeled by its surface topological order $eCmC$. Other SPT phases are inherited from Z_2^T and hence are not pertinent to our present concerns (see the end of Sec. 2.8.2). Thus we have two possible phases - the trivial insulator and the SPT insulator labelled by $eCmC$. In the former case external probes where bulk vortex lines end are bosons while in the latter they are fermions. In both cases the vortex sources are Kramers trivial.

Let us now following the logic of the previous subsection and gauge the $U(1)$ symmetry. The resulting monopole m_d will be identified with the vortex source and will therefore be a Kramers singlet which can be either boson or fermion. Thus this reasoning suggests that for the odd electric field case there are only two possibilities for the e particle ($= m_d$) - it is a Kramers singlet that is either boson or fermion.

Let's understand the above claim directly from the gauge theory point of view, independent of the argument based on SPT. With odd electric field the electric charge at any site q_e is also odd under time reversal. This implies that the e particles transform under $U_g(1) \times Z_2^T$ where $U_g(1)$ is the gauge transformation generated by q_e. Notice that we have $U_\theta \mathcal{T} = \mathcal{T} U_\theta$ for $U(1) \times Z_2^T$ symmetry, where U_θ gives the $U(1)$ rotation.Allowing for the possibility of a multi-component field e_I, time reversal will be implemented by

$$\mathcal{T}^{-1} e_I \mathcal{T} = e^{-i\alpha_e} T_{IJ} e_J^{\dagger} \tag{2.49}$$

We can always change the common phase α_e by defining a new time reversal operator $\tilde{\mathcal{T}} = U(\theta)\mathcal{T}$. As $U(\theta)$ is a gauge transformation $\tilde{\mathcal{T}}$ and $\mathcal{T}$ will have the same action on all physical operators. We therefore can set $\alpha_e = 0$ (or any other value for that matter). In particular under this redefinition $\mathcal{T}^2$ goes to $(U_\theta \mathcal{T})^2 = U_{2\theta}\mathcal{T}^2$ so that the over all phase in the action of $\mathcal{T}^2$ on e can be changed at will, and one can always choose $\mathcal{T}^2 = 1$. The algebraic structure of $U_g(1) \times Z_2^T$ still guarantees a degenerate doublet structure, but the degeneracy here is protected by $U_g(1)\times Z_2^T$ as a whole rather than by Z_2^T alone as in Kramer's theorem. In particular, one can lift the degeneracy by breaking the $U_g(1)$ symmetry but still preserving time-reversal invariance, which is in sharp contrast with the Kramer's case. It is appropriate to regard the electric charge $q_e = \pm 1$ as a non-Kramers doublet. Hence with $U_{gauge}(1) \times Z_2^T$ symmetry, any charged particle should always be viewed as time-reversal trivial. This implies that the e particle is always time reversal trivial for a $U(1)$ gauge theory where the electric field is odd, in full agreement with what we obtained from the SPT point of view.

Before concluding this section let us briefly discuss the putative $U(1)$ spin liquid in quantum spin ice from this point of view. We have just argued that the 'spinons' (in the notation of Ref. [94]) are not Kramers doublets. If the quantum spin ice Hamiltonian has S_z conservation then the spinons will generically carry fractional S_z. However the realistic Hamiltonians currently proposed[99] for quantum spin ice do not have conservation of any component of spin. Thus the "spinons" of the possible $U(1)$ spin liquid in quantum spin ice do not carry any quantum numbers associated with internal symmetries. Their non-Kramers doublet structure is independent of whether or not the microscopic Ising spin is itself Kramers or not. Further microscopically there are two species of electric charge e_1, e_2 (associated with two sub lattices of the diamond lattice formed by the centers of pyrochlore tetrahedra). Time reversal should be implemented by letting $e_1 \to e_1^\dagger, e_2 \to \pm e_2^\dagger$ so that the physical spin operator $e_1^\dagger e_2$ transforms as appropriate with $-$ sign for a microscopic Kramers doublet spin and the $+$ sign for a non-Kramers doublet.

Finally we note that current theoretical work treats the spinons in quantum spin ice as bosons. This is reasonable as the electric strings connecting them are simply made up of the physical spins and do not have the ribbon structure and associated phase factors expected if they were fermions.

2.9 Discussion

In this chapter we studied many aspects of the realization of symmetry in highly entangled quantum phases of matter. We relied heavily on insights obtained from recent work on *short range entangled* symmetry protected topological phases. Despite their short range entanglement the SPT phases provide a remarkable window into the properties of the much more non-trivial highly entangled phases. In turn the connections to the highly entangled phases enhances our understanding of SPT phases themselves. Below we briefly reiterate some of our results and their implications.

The very existence of SPT phases emphasizes the role of symmetry in maintaining distinctions between phases of matter even in the absence of any symmetry breaking. For highly entangled states this leads to the question of whether symmetry is realized consistently in the low energy theory of such a state. We addressed this for the example of $2d$ gapped topological phases described by a Z_2 gauge theory with time reversal symmetry (and possibly a global $U(1)$ symmetry). By combining the methods of two different recent approaches[71, 75] to assigning symmetry to the topological quasiparticles we showed that there are consistent symmetry realizations which nevertheless are not possible in strictly $2d$ systems. Such states were however shown to occur at the surface of $3d$ bosonic SPT phases. Conversely we provided simple arguments that if a Z_2 topological order can occur at the surface of a $3d$ SPT, then it is not allowed to occur in strictly $2d$ systems. Crucial to our discussion was the concept of edgability. A topologically ordered state with (internal) symmetry is allowed in strictly $2d$ if and only if it is edgable, i.e it must admit a physical edge with the vacuum. Topological ordered states that are only allowed at the surface of a $3d$ SPT phase are clearly not edgable.

Thus illustrates how the study of SPT surfaces can provide a very useful "no-go" constraint on what kinds of phases are acceptable in strictly d-dimensional systems. If a phase occurs at the surface of a $d+1$ dimensional SPT phase (for $d > 1$) then it cannot occur with the same realization of symmetry in strictly d dimensions. A nice application of this kind of no-go constraint is to the possibility of gapless vortex fluid phases proposed to exist[85, 86, 87] in two space dimensions in boson/spin systems with both time reversal and global $U(1)$ symmetries. Such phases were argued to exist at the surface of $3d$ SPT phases in Ref. [43] thereby strongly suggesting that they cannot exist in strictly $2d$. We sharpened this conclusion by considering a descendant Z_2 topological ordered phase that is obtained by pairing and condensing vortices of this putative vortex fluid. We showed that the result was a phase that cannot exist in strict $2d$ but can of course exist at the surface of $3d$ SPT.

The study of SPT surfaces thus gives us valuable guidance in writing down legal theories of strictly d-dimensional systems. It thus becomes an interesting exercise to study boundary states of SPT phases in $4+1$ dimensions as a quick route to obtaining some restriction on physically relevant effective field theories of strictly $3+1$ dimensional systems. Quite generally the issue of consistent symmetry realization is likely related to the existence of 'quantum anomalies' in the continuum field theory. For instance the surface field theory of free fermion topological insulator are 'anomalous' and require the higher dimensional bulk for a consistent symmetry-preserving regularization. For other states such as, for example, topological quantum field theories it will be interesting if there is a useful direct diagnostic of whether the theory is anomalous or not.

Finally a very interesting outcome of our results is the explicit construction of coupled layer models for $3d$ SPT phases. For all the symmetry classes discussed in Ref. [43] we provided such a construction. The strategy is to start from a layered $3d$ system where each layer is in a Z_2 topological ordered state that is allowed in strict $2d$. We then coupled the layers together to confine all topological excitations in the bulk but left behind a deconfined Z_2 topological state at the surface. This surface topological order was shown to match the various possible such order at SPT surfaces. In particular this scheme provides an explicit construction of a $3d$ SPT state whose surface is a time reversal symmetric gapped Z_2 topological ordered state with three fermionic excitations that are all mutual semions. This topological order is expected to occur at the surface of a bosonic SPT state with time reversal symmetry proposed in Ref. [43] and is not currently part of the classification of Ref. [16, 17].

In Chapter 7 we propose a possible experimental realization of the $eTmT$ state discussed extensively in this chapter, in frustrated spin-1 magnets.

Chapter 3

Gapped Symmetry Preserving Surface-State for the Electron Topological Insulator

3.1 Introduction

In the last decade, dramatic progress has been made in understanding the topological properties of non-fractional electronic insulators[18, 19, 20]. While the original theoretical constructions were framed in terms of band structures for non-interacting electrons, attention has recently turned towards the interplay of strong correlation and topological insulation. It is now appreciated that the electron topological insulator is part of a larger class of quantum phases of matter known as Symmetry Protected Topological (SPT) phases[16, 17].

The Fu-Kane-Mele electronic TI (eTI)[25] is the first known 3D example of an SPT phase. Its non-trivial surface states are protected by bulk time-reversal symmetry (TRS) and charge conservation ($U(1)_C$) symmetry. If either of these symmetries is broken in the bulk, the eTI can be smoothly deformed into a trivial insulator. It is, by now, well known that the surface can either be 1) a gapless, symmetry-preserving state, or 2) a gapped state that breaks one (or both) of TRS and $U(1)_C$. For some time, it was implicitly assumed that these options exhausted the possible surface phases. Indeed these are the only possibilities accessible in a weakly interacting description of the surface. However in the presence of strong correlations other options for the surface may become available. In particular, we will show that it is possible for the eTI surface to be both fully gapped and preserve all symmetries. The price to pay for having a gapped and symmetric surface is that the surface develops intrinsic topological order (even though the bulk does not). We describe this surface topologically ordered state of the eTI and show that it has non-Abelian quasiparticles. The physical symmetries are realized in this surface topological ordered state in a manner forbidden in a strictly two dimensional insulator with the same topological order.

The prime impetus for our study comes from recent progress in describing bosonic SPT phases in three dimensions, described in detail in Chapter 2. For bosons, interactions are essential to obtain an insulator. Consequently the study of boson SPTs is necessarily non-perturbative in the interaction strength. For such bosonic SPT phases, it was shown that the surface can be both gapped and symmetry preserving[43] if it possesses intrinsic two-dimensional surface topological order (STO). This STO however realizes symmetry in a manner prohibited in strictly two dimensional systems. The STO provides a particularly simple non-perturbative insight into the bulk SPT phase. Indeed targeting such an STO is a useful conceptual tool for constructing SPT phases[1, 44], and can provide very general constraints on lower-dimensional phases[43, 1]. In light of the simplicity and power of the STO as a surface termination of strongly interacting bosons SPTs it is natural to construct the STO appropriate for the fermionic topological insulator.

Our strategy is to start from the TR-symmetric non-Abelian surface superconductor[102], and to restore $U(1)_C$ without destroying the superconducting gap by proliferating vortices in the superconducting phase. The minimal $\frac{hc}{2e}$ superconducting vortices cannot be directly condensed due to their non-Abelian statistics arising from unpaired core Majorana modes. It turns out that, despite being Abelian, the doubled $\frac{hc}{e}$ vortex is a semion and can also not be condensed while preserving TRS. Identifying an appropriate vortex field that can be condensed to disorder the superconductor without breaking TRS requires some care. We find that there are 4-fold ($\frac{2hc}{e}$) vortex fields that can be condensed without breaking any symmetries as a minimal route to producing the STO starting from the surface superconductor.

The resulting phase has identical topological order and charge assignments as the 2D Moore-Read quantum Hall state[103] accompanied by an extra neutral semion. However, in strictly two-dimensions this topological phase cannot be realized in a TR symmetric manner. We will show that the fact that the eTI can realize this TO while preserving TRS provides a complete, non-perturbative definition of the bulk eTI.

3.2 Vortex Condensation in a Conventional Superconductor

As a warm-up for the more-complicated non-Abelian case, we begin by reviewing how insulating states can be produced by quantum disordering a conventional 2D s-wave superconductor through vortex proliferation.

A superconducting state has a charge $2e$ order parameter $\Delta = |\Delta|e^{2i\phi_s}$ that breaks $U(1)$ charge conservation symmetry. Starting from a conventional s-wave superconductor, one can restore $U(1)_C$ symmetry by proliferating vortices in the phase of the order-parameter, ϕ_s. Since the pairing amplitude $|\Delta|$ remains finite (except inside the vortex cores), the resulting state is clearly gapped. Different gapped phases can be obtained by proliferating different types of vortices. For example, proliferating π-vortices in ϕ_s (i.e. superconducting $\frac{hc}{2e}$ vortices) produces a simple band-insulator[104], whereas proliferating 2π vortices produces a gapped phase with $\mathbb{Z}_2$ topological order[104, 80]. These constructions are well known[104, 80], but are useful to review in order to fix notation and to set the stage for the more complicated non-Abelian superconductors that are the subject of this chapter.

All three phases are conveniently described by a parton construction in which the electron annihilation operator with spin-σ, c_σ, is rewritten as $c_\sigma = bf_\sigma$ with b a spinless charge-1 boson (chargon) , and f_σ is a neutral spinful fermion (spinon).

This parton description (often referred to as "slave-boson"), has a $U(1)$ redundancy associated with changing the phase of b and f in opposite ways. Consequently, any field theory description will contain an emergent, compact $U(1)$ gauge-field, whose vector potential we will denote by a^μ.

3.2.1 Superconductor

In the parton description, the s-wave superconductor phase is described by condensing the charged boson, $\langle b\rangle \neq 0$, and introducing an s-wave pairing amplitude for f: $\langle f_\uparrow f_\downarrow\rangle \neq 0$. In this phase, the emergent gauge field is gapped by the Higgs mechanism (or, equivalently confined) due to the charge-1 boson condensate.

The gapped, unpaired f-quasiparticles are neutral fermion excitations. These are ordinary Bogoliubov quasi-particles of the superconductor, that arise from electron states whose charge is screened completely (at long lengthscales) by the pair-condensate.

In addition, there are also π vortices of the f-pair condensate phase. Since f carries internal gauge charge, these vortices carry π "magnetic"-flux of a. The bosons, having internal gauge charge, are also affected by this π flux of a. Writing $b = \sqrt{\rho_b}e^{i\phi_b}$, we see that ϕ_b must wind by π in the vicinity of this vortex in order to avoid an extensive energy penalty. Since b carries the physical electromagnetic charge, this means that a π-vortex in the f-pair-condensate is necessarily accompanied by a physical supercurrent flow in the b-condensate; this object is simply the familiar $\frac{hc}{2e}$ superconducting vortex.

3.2.2 Band-Insulator

In an s-wave SC, $\frac{hc}{2e}$ vortices carry only gapped quasi-particle states in their core. Moreover, the pairing amplitude, Δ, is non-vanishing outside of vortex cores. Consequently, a state with an arbitrary density of non-overlapping vortices has no gapless excitations. Therefore, one can consider starting with a superconductor and creating a quantum superposition of states with various numbers and placements of (well-separated) $\frac{hc}{2e}$ vortices. This state will clearly be gapped.

Moreover, since the spinon excitations of the superconductor see the $\frac{hc}{2e}$ superconducting vortices as π-gauge-magnetic-flux, the spinon and vortex have mutual semionic statistics. This immediately implies that the spinons will be confined in the vortex-proliferated state. The bosonic particles, b, also see the vortices as π-fluxes. Therefore, the physical electron $c = bf$ has trivial mutual statistics with the vortex, remains gapped but deconfined. Therefore, there is no spin-charge separation and the resulting state describes a conventional electron phase.

This phase can be thought of as a Bose-Mott insulator of Cooper pairs. If the electron density is commensurate such that there are an even number of electrons per unit cell, then the Cooper pairs have integer filling and can form a Mott insulating state without further breaking any spatial symmetry. Commensurate Cooper-pair filling is a necessary requirement for forming a band-insulator, and furthermore, the $\frac{hc}{2e}$-vortex-proliferated state has all the properties of an ordinary electronic band-insulator.

Therefore, we see that $\frac{hc}{2e}$ vortex proliferation in a superconductor produces a conventional band-insulator. This description of a band-insulator is clearly more complicated than the usual non-interacting band-structure description. However, this construction provides a complementary "dual" perspective capable of capturing correlated band-insulators, and can be a useful conceptual starting point for constructing more complicated strongly interacting phases.

3.2.3 $\mathbb{Z}_2$ Topological Order

Instead of proliferating $\frac{hc}{2e}$-vortices in the superconductor, one could alternatively proliferate doubled ($\frac{hc}{e}$) vortices. If the electrons are at commensurate filling with the lattice, this proliferation destroys the boson superfluidity ($\langle b\rangle = 0$) without further breaking any other symmetry. Single b-particle excitations are now gapped and the resulting phase is a charge insulator. In this phase a is not confined; rather, the emergent $U(1)$ gauge invariance is broken down to a local $\mathbb{Z}_2$ gauge invariance by the f-pair-condensate. Moreover, since the spinons-f develop a trivial (multiple of 2π)

Berry phase upon encircling an $\frac{hc}{e}$ defect, they remain deconfined.

The excitations of the theory are then b, f, and objects with π-flux of a (visons). The visons are their own antiparticles (since two visons make up the condensed $\frac{hc}{e}$ vortex), having mutual π-statistics with b and f, and the resulting state is fractionalized with $\mathbb{Z}_2$ topological order.

It is worthwhile to pause to reflect on the strategy underlying the vortex condensation route to describing insulators proximate to superconducting phases in two space dimensions. In general a useful effective field theory description of such a system is formulated in terms of degrees of freedom natural in the superconductor - namely the $\frac{hc}{2e}$ vortices and the neutralized Bogoliubov quasiparticles (the f field). The $\frac{hc}{2e}$ vortex field is a mutual semion with the f particle and furthermore is coupled to a non-compact $U(1)$ gauge field. The vortex field of this dual Landau-Ginzburg theory is, in the examples reviewed above, bosonic. Vortex fields with strength $\frac{nhc}{2e}$ can therefore be formally condensed to produce various kinds of insulating states.

Having reviewed the simpler s-wave SC case, we now turn to the problem of producing a topologically ordered phase from the eTI surface-SC.

3.3 Vortices in the eTI Surface Superconductor

Starting from the superconducting surface of the eTI, we know that there should be some obstruction to proliferating superconducting vortices to form an ordinary band-insulator, and indeed the $\frac{hc}{2e}$-vortices in the superconducting TI surface-state are non-Abelian objects that cannot be directly condensed[102]. Since $\frac{hc}{e}$ vortices do not have an unpaired Majorana core state, they are Abelian, and one is tempted to follow the above construction to obtain a $\mathbb{Z}_2$ topologically ordered state by proliferating $\frac{hc}{e}$ vortices.

However, this naive approach fails to produce a symmetric STO state. It turns out that in the eTI surface SC, $\frac{hc}{e}$ vortices have semionic self-statistics[1], and cannot

[1]To precisely define vortex statistics, it is necessary to consider a gauged $U(1)_C$ symmetry (as is the case for real electrons coupled to the physical electromagnetic field, which has very weak fluctuations).

be condensed without breaking TRS. The $\frac{3hc}{2e}$ vortices again have unpaired Majorana cores, and are non-Abelian. We show however that there are $\frac{2hc}{e}$ vortices that are bosonic. Therefore, the minimal route to restoring $U(1)_C$ is to condense such bosonic $\frac{2hc}{e}$ vortices.

We now establish the Abelian statistics of $\frac{hc}{e}$ and $\frac{2hc}{e}$ vortices in the surface-superconductor, by arguing based on the Θ-term electromagnetic response of the bulk.

3.3.1 Bulk Argument for statistics of Abelian vortices

A useful conceptual device for what follows is to modify the problem by coupling the electrons to a weakly fluctuating dynamical compact $U(1)$ gauge field. It is well known that the topological insulating bulk leads to a Θ-term, with $\Theta = \pi$, in the effective action (apart from the usual Maxwell term) for this gauge field obtained by integrating out the electrons. Also well-known is the effect of this Θ term: a unit strength magnetic monopole of this $U(1)$ gauge field acquires electric charge $\frac{1}{2}$ (the Witten effect[68]). Now imagine tunneling such a monopole from the vacuum into the bulk of the (gauged) topological insulator. Such a tunneling process will leave behind at the surface a $\frac{hc}{e}$ vortex. This implies that the $\frac{hc}{e}$ vortex field in the vortex Landau-Ginzburg theory formally also has electric charge $\frac{1}{2}$. As a composite made of charge-1/2 and 2π flux it is natural to expect that this vortex will have semionic statistics.

To demonstrate the semionic statistics of $\frac{hc}{e}$ vortices, consider a slab of bulk eTI with a top and bottom interface with a trivially insulating vacuum. Then create a pair of $\frac{hc}{e}$ vortices on the top surface and a pair of $-\frac{hc}{e}$ vortices on the bottom surface. Since the gauge field A^{μ} is free, except at the superconducting surface, closed magnetic flux lines carrying $\frac{hc}{e}$ flux are condensed in the bulk and in the vacuum. Since the surface is superconducting, a magnetic flux tube can only penetrate the surface at a vortex. For the vortex configuration of Fig. 3-1, there are only two magnetic flux lines that leave the TI bulk. Let us consider just one representative flux line configuration, as shown in Fig. 3-1. Next consider dragging one of the $\frac{hc}{e}$ vortices on the top surface

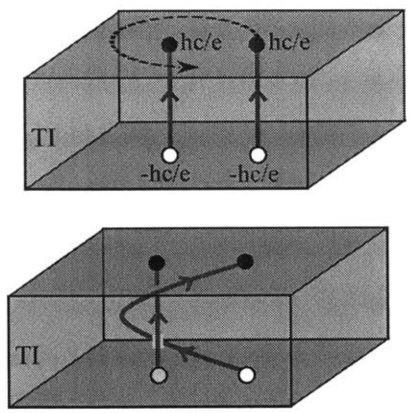

Figure 3-1: Exchanging two $\frac{hc}{e}$ vortices at the superconducting surface of a TI slab (top panel) leads to a linking of their magnetic field lines, which gives a phase of -1, demonstrating that $\frac{hc}{e}$ vortices are semionic.

all the way around the other, as shown in Fig. 3-1 without moving the $-\frac{hc}{e}$ vortices on bottom surface. The new magnetic flux configuration differs from the initial one by a single linking of the magnetic flux lines that thread the vortices.

Due to the bulk topological Θ-term for A:

$$\mathcal{L}_\Theta = i\frac{\varepsilon^{\mu\nu\lambda\rho}}{8\pi}\partial_\mu A_\nu \partial_\lambda A_\rho \tag{3.1}$$

this linking produces a phase of -1 relative to unlinked configurations. This phase can be computed directly from $\mathcal{L}_\Theta$ by considering any convenient choice of A with a linked vortex line. Alternatively, one can imagine creating a linked field line configuration in the bulk by starting with an infinite flux line, creating a monopole anti-monopole pair and dragging the monopole around the flux line before annihilating it with the anti-monopole. Since monopoles in the TI bulk have charge $\frac{e}{2}$, dragging one around a 2π-flux line contributes phase $e^{2\pi i\cdot\frac{1}{2}} = -1$.

We have illustrated this -1 phase for a particular magnetic field line configuration. More generally, the ground-state, $|\Psi_{\text{EM}}\rangle$, of the bulk gauge field, A^μ, is a quantum-superposition of various configurations, $\mathcal{C}$, of magnetic flux lines:

$$|\Psi_{\text{EM}}\rangle = \sum_{\mathcal{C}}(-1)^{L_\mathcal{C}}\Psi_0(\mathcal{C})|\mathcal{C}\rangle \tag{3.2}$$

weighted by phase $(-1)^{L_C}$, where L_C is the number of linked loops in the configuration $\mathcal{C}$, and by amplitude, $\Psi_0(\mathcal{C})$, that is determined by the non-topological dynamical terms for the gauge-field.

This follows directly from computing the wave-function for a given configuration, $\mathcal{A}(\mathbf{r})$, from the (imaginary time) path integral:

$$\begin{aligned}\Psi[\mathcal{A}] &= \langle \mathcal{A}|\Psi\rangle \\ &= \int D[A]\big|_{A_\mu(\mathbf{r},t=0)=\mathcal{A}(\mathbf{r})} e^{-\int_{-\infty}^{0} d\tau \int d^3r \mathcal{L}_\Theta[A]} \\ &\sim e^{i\int d^3r \frac{\epsilon^{\mu\nu\lambda}}{8\pi}\mathcal{A}_\mu \partial_\nu \mathcal{A}_\lambda} = (-1)^{L_{\mathcal{C}[\mathcal{A}]}} \end{aligned} \tag{3.3}$$

we see that the resulting wave-function contains a Chern-Simons (CS) term which just counts the linking number of flux lines of $\mathcal{A}$.

For any configuration of closed bulk field-lines, $\mathcal{C}$, the two-fold exchange of $\frac{hc}{e}$ introduces a single extra linking number. Therefore the two-fold exchange of $\frac{hc}{e}$ vortices produces phase (-1), indicating that a single exchange produces phase $\pm i$; the $\frac{hc}{e}$ vortices are semionic. Let us denote the quantum field that creates a $\frac{nhc}{e}$ vortex with electric charge q by $\Phi_{n,q}$. With this notation $\Phi_{1,\frac{1}{2}}$ is a semionic $\frac{hc}{e}$ vortex with charge $\frac{1}{2}$. The field $f\Phi_{1,\frac{1}{2}}$ produces a neutral fermion bound to this vortex and hence creates an antisemionic $\frac{hc}{e}$ vortex with charge $\frac{1}{2}$. These two $\frac{hc}{e}$ vortices will play an important role below.

Let us now consider strength-4 ($\frac{2hc}{e}$) vortices. A similar argument as above shows that $\frac{2hc}{e}$ vortices are either bosonic or fermionic (fermionic and bosonic $\frac{2hc}{e}$ vortices can be interchanged by binding a neutral f quasi-particle). Note that if we combine two charge-1/2 semionic $\frac{hc}{e}$ vortices, we end up with a charge-1 bosonic $\frac{2hc}{e}$ vortex. *i.e* $(\Phi_{1,\frac{1}{2}})^2 = \Phi_{2,1}$. An electrically *neutral* $\frac{2hc}{e}$ vortex may be obtained by considering the combination $c\Phi_{2,1}$, *i.e* by removing an electron from the charge-1 $\frac{2hc}{e}$ vortex. Clearly this is a fermion.

These strength-4 vortices at the surface correspond in the bulk to strength-2 monopoles. At $\Theta = \pi$, such monopoles always carry integer electric charge. We will denote bulk dyons with magnetic charge n and electric charge q by (n, q). These correspond to surface vortices created by $\Phi_{n,q}$. It is readily seen that the bulk $(2,1)$ dyon (at $\Theta = \pi$) is a boson while the electrically neutral strength 2 monopole (the $(2,0)$ particle) is clearly a fermion (the polarization charge induced by the Θ term does not contribute to the statistics, as explained in Ref.[105]). This is in complete accord with our discussion of surface vortices above. Arguments using bulk monopole properties to constrain surface physics were also recently used for boson topological insulators in Ref. [69].

To disorder the surface superconductor we need to identify bosonic vortices which we can then condense. Though the $\frac{2hc}{e}$ vortex with electric charge-1 seems like a candidate it is problematic. To preserve time reversal we should clearly also condense (with equal amplitude) the $-\frac{2hc}{e}$ vortex with electric charge 1. But then the resulting state also has a condensate of ordinary Cooper pairs so that it is still a superconductor (albeit an exotic one). The neutral $\frac{2hc}{e}$ vortex described above is a fermion and hence cannot condense. Fortunately we also have a different neutral fermion in our theory - the spinon (the f particle). By binding f to the fermionic $\frac{2hc}{e}$ vortex we obtain an electrically neutral bosonic $\frac{2hc}{e}$ vortex. Equivalently this bosonic neutral $\frac{2hc}{e}$ vortex may be viewed as being obtained from the charge-1 bosonic $\frac{2hc}{e}$ vortex by binding to b (*i.e* by removing a chargon). This neutralizes the charge but keeps the statistics as bosonic. We are then free to condense this vortex to destroy the superconducting order.

We emphasize that the bosonic neutral $\frac{2hc}{e}$ vortex is not simply a 4π vortex of the chargon b but requires also binding to the spinon f. An 8π ($\frac{4hc}{e}$) vortex of b, $\Phi_{4,0}$,is an electrically neutral boson. The corresponding bulk monopole is a $(4,0)$ particle which is also a boson. Condensation of the bosonic vortex $f\Phi_{2,0}$ automatically implies condensation of $\Phi_{4,0}$ as the spinon f is paired.

3.3.2 Topological spins of non-Abelian vortices

We now consider non-Abelian vortices, and it is sufficient for our purpose to consider $\pm hc/2e$ vortices, with Majorana core states. Naively, the argument given in Sec.3.3.1 for hc/e vortices implies that the topological spin (see Sec.3.4.2 for its definition) of

$\pm hc/2e$ vortices would be $e^{i\pi/8}$. This can be seen by writing the bulk Θ-term as a boundary Chern-Simons term at level-1/2, which would contribute to the topological spin of $\pm hc/2e$ vortices by $e^{i\pi/8}$. However, the Majorana zero-modes trapped in the vortices contributes another $e^{-i\pi/8}$ to the topological spins[106], hence the total topological spins are

$$\theta_{hc/2e} = \theta_{-hc/2e} = 1. \tag{3.4}$$

The above argument can be made more precise by viewing the surface superconductor as a paired single Dirac cone. One can then add two gapped Dirac cones with opposite masses to the surface without breaking time-reversal symmetry. One can then group one of the massive Dirac cones with the original surface superconductor and rewrite the combination as a $p - ip$ superconductor, and the other massive Dirac cone with the opposite mass gives a half-quantum hall state. The former contributes $e^{-i\pi/8}$ to the topological spin of $\pm hc/2e$ vortices, and the latter gives $e^{i\pi/8}$, hence we have $\theta_{\pm hc/2e} = 1$.

3.4 Surface Topological Order

We are now in a good position to construct a symmetry preserving STO phase from the SC phase. In the parton construction $c_\sigma = bf_\sigma$, we can describe the SC topological insulating surface state by condensing b, $\langle b \rangle \neq 0$, and placing f in the eTI band-structure with a superconducting surface. From the previous section, we saw that the minimal route to restoring the $U(1)_C$ symmetry is to proliferate the electrically neutral bosonic $\frac{2hc}{e}$ vortices.

What topologically distinct classes of particles remain after their proliferation? Since b and f have trivial mutual statistics with the $\frac{2hc}{e}$ vortices, they will clearly survive as gapped quasi-particles with unaltered charge and statistics. Quite generally the condensation of such $\frac{2hc}{e}$ vortices will produce an insulator with gapped bosonic excitations with fractional charge 1/2. We will call this particle β. Clearly two β particles make a chargon: $b = \beta^2$.

Vortices in the superconductor become dressed by the $\frac{2hc}{e}$ condensate. We will see later that they survive as topological quasiparticles but with sharp non-zero electric charge (unlike in the example reviewed above of 2D $\mathbb{Z}_2$ topologically ordered states produced by disordering a proximate superconductor, where the visons are charge neutral). For now, we put aside the charge assignment for these topological particles and focus just on identifying the different particle types.

Going from the superconductor to the STO phase, the non-Abelian $\frac{hc}{2e}$ vortex, v, becomes a new object, τ_v, which is a quantum superposition of odd-strength vortices in the superconductor whose vorticity differs by a multiple of $\frac{2hc}{e}$. Similarly, the $-\frac{hc}{2e}$ anti-vortex, $\bar{v}$, becomes a different object, $\tau_{\bar{v}}$, which is made up of a quantum superposition of $\frac{(4n-1)hc}{2e}$ vortices of the superconductor (with $n \in \mathbb{Z}$).

In the SC, an $\frac{hc}{2e}$ vortex, v, carries a Majorana zero mode in its core[102], and a pair of v's shares a single complex fermion level that can be either occupied or unoccupied. Consequently, there are two possible outcomes from fusing two v's, $v_\pm^2$, both of which have net vorticity $\frac{hc}{e}$ and which differ from each other by adding a neutral Bogoliubov fermion, f. Upon moving into the STO phase by condensing 4-fold vortices, $v_\pm^2$ will turn into distinct objects, $\tau_\pm^2$, which differ by a fermion: $\tau_+^2 = \tau_-^2 \times f$.

Similarly, in the superconductor, a pair of $\bar{v}$'s can fuse to two different $-\frac{hc}{e}$ vortex objects that differ by a fermion, f. Upon condensing $\frac{2hc}{e}$ vortices however, the $\pm\frac{hc}{e}$ vortices become mixed, and fusing two $\tau_{\bar{v}}$ particles should have the same outcome as fusing two τ_v particles: $\tau_{\bar{v}} \times \tau_{\bar{v}} = \tau_v \times \tau_v = \tau_+^2 + \tau_-^2$.

Lastly, in the superconductor, the vortex and anti-vortex pair also share a non-local fermion level due to their Majorana cores. Fusing a v and $\bar{v}$, then produces either the superconducting ground state, $\mathbb{I}$, or the ground-state plus an extra Bogoliubov particle: $v \times \bar{v} = \mathbb{I} + f$. Consequently, in the STO phase, we must have two possible fusion outcomes for $\tau_v \times \tau_{\bar{v}}$, which differ by an f. Naively, one might be tempted to have τ_v and $\tau_{\bar{v}}$ fuse $1 + f$ as in the superconductor. However, more generally we may also have: $\tau_v \times \tau_{\bar{v}} = X \times (1 + f)$ where X is some to-be determined particle that is condensed in the SC. This is consistent with the fusion rules of the surface SC if X is condensed in the SC phase. This requires X to be a boson. Below we will show that X is just the fractional chargon: β.

Finally, we note that $\tau_+^2 \times \tau_+^2 = \beta^2$, and that $\tau_+^2 \times \tau_-^2 = \beta^2 \times f = c$, the physical electron.

A summary of the particle content and fusion rules produced by this line of reasoning is summarized in Tables 3.1 and 3.2 respectively.

3.4.1 Charge Assignments

Having specified the topologically distinct particle classes and fusion rules for the STO phase, we now turn to their symmetry properties under $U(1)_C$. The resulting charge assignments explained below are summarized in Table 3.1.

Since b and f are unaffected by the vortex condensation, b still carries charge e and that f is charge-neutral. What about the excitations that descend from superconducting vortices? $\tau_\pm^2$ particles descend from $\Phi_{1,1/2}$ vortex fields of the superconductor, and hence can be created by dragging a magnetic monopole from the vacuum through the STO surface into the bulk. Since the monopole carries fractional electric charge: $\pm\frac{e}{2}$, its corresponding surface excitations must also have charge $\mp\frac{e}{2}$. Moreover, since τ_+^2 and τ_-^2 differ by a neutral fermion, f they must have the same charge. For concreteness, and without loss of generality, we choose $\tau_\pm^2$ to have charge $+\frac{e}{2}$ and their anti-particles, $\tau_\pm^{-2}$, to have charge $-\frac{e}{2}$. It then immediately follows from the fusion rule: $\tau_v \times \tau_v = \tau_+^2 + \tau_-^2$ that τ_v has charge $\frac{e}{4}$.

It is instructive to understand how these charge assignments come about directly from the surface without recourse to bulk monopoles. To obtain the STO from the SC, we are condensing 4π vortices of the chargon b that are bound to the neutral fermion f. The neutral fermion acquires a π phase when it encircles the $\frac{hc}{2e}$ vortex in the superconductor. Consequently, the $\frac{hc}{2e}$ vortex is a mutual semion with the condensed bosonic $\frac{2hc}{e}$ vortex. As a result, the $\frac{hc}{2e}$ vortex can survive in the STO phase only by binding with some other particle to produce trivial mutual statistics with the condensed bosonic $\frac{2hc}{e}$ vortex. The only possibility is for the $\frac{hc}{2e}$ vortex to bind a fractional charge, $\frac{e}{4}$, which also obtains π-phase upon encircling an $\frac{2hc}{e}$ vortex. Thus we conclude that the particles $\tau_v, \tau_{\bar{v}}$ in the STO phase are the remnants of the $\frac{hc}{2e}$ vortices of the SC phase which have been dressed by charge $e/4$.

Since τ_v and $\tau_{\bar{v}}$ descend from $\pm\frac{hc}{2e}$ vortices in the superconductor, they are related by time-reversal and must have the same charge. Above, we saw that the $v \times \bar{v} = 1 + f$ fusion rule for the surface-SC generalized to: $\tau_v \times \tau_{\bar{v}} = X \times (1 + f)$ in the STO phase, with X to-be-determined particle. The above arguments show that X must have charge $\frac{e}{2}$. Since X is a $\frac{1}{2}$-charge boson that must be condensed in the SC phase, the only possibility is: $X = \beta$.

3.4.2 Topological Spins

The topological spin, θ_a, of a particle in sector a is defined as the phase factor accumulated when an a-particle is adiabatically rotated by 2π in the counter-clockwise (CCW) sense. For Abelian particles, the topological spin coincides with the phase obtained through CCW exchange of a pair of a-particles.

Clearly $\theta_b = 1$ and $\theta_f = -1$. The argument in Sec.3.3.1 established the semionic/anti-semionic statistics of hc/e surface vortices (the semion and anti-semion differ by an f fermion). In the topologically ordered phase the hc/e vortex acquires an additional charge $q_{\tau_\pm^2} = 1/2$. The charge-flux relation thus gives an additional $e^{iq\phi} = -1$ to its topological spin. This shifts a semion to an antisemion and vise versa. But since we have both semionic and anti-semionic vortices already, the shift is just a relabeling of the two different vortices. Hence we establish that $\tau_\pm^2$ have topological spin $\pm i$.

It was also established in Eq.3.4 that the $\pm hc/2e$ vortices have trivial topological spins. In the topologically ordered phase, the $\pm hc/2e$ vortices acquire additional charge-1/4 and becomes $\{\tau_v, \tau_{\bar{v}}\}$. Hence an additional contribution of $e^{iq\phi} = e^{\pm i\pi/4}$ is introduced to the topological spin. Hence we have $\theta_{\tau_v} = e^{i\pi/4}$ and $\theta_{\tau_{\bar{v}}} = e^{-i\pi/4}$.

3.4.3 Exchange Statistics

In a system with non-Abelian particles that have multiple possible fusion outcomes, the phase obtained by the CCW exchange of two particles, a and b, will depend on the fusion channel. When a and b fuse to c, the phase factor obtained by adiabatic CCW exchange of a and b is denoted by R_c^{ab} (for a pedagogical review see Ref. [107]).

The R matrices are related to the topological spin of the underlying particles[107] by $(R^{ab}_c)^2 = \theta_c/\theta_a\theta_b$. This identity just encodes the fact that dragging b around a is nearly the same as rotating the entire a-b composite system CCW by 2π, or equivalently to fusing to c and rotating CCW by 2π giving: θ_c. However, rotating the entire system also rotates a and b individually, which is not part of the exchange process. The factor of $\theta_a\theta_b$ in the denominator compensates for this unwanted rotation of a and b. The proper branch of the square-root can be identified by writing $\theta_{a,b,c} \equiv e^{i\phi_{a,b,c}}$, and choosing an exchange protocol such that the phase is accumulated monotonically over the course of time T: $R^{ab}_c = \lim_{t\to T^-} e^{i(\phi_c-\phi_a-\phi_b)t/2T}$.

For Abelian particles a and b, there is a unique fusion channel, and the lower-index on R is redundant. Therefore, it is common to just specify the mutual statistics of a and b by: $\theta_{a,b} = (R^{ab}_{a\times b})^2$, which is the phase factor obtained by adiabatically dragging b CCW around a. Consequently, the braiding statistics for all particles follows straightforwardly from the previously obtained fusion rules and topological spins tabulated in Tables. 3.1 and 3.2 respectively.

For example, consider the mutual statistics of τ_v and $\tau^2_\pm$. The composite $\tau_v \times \tau^2_\pm = \tau^3_v$ has topological spin: $\theta_{\tau^3_v} = -e^{i\pi/4}$, indicating:

$$\theta_{\tau_v,\tau^2_\pm} = \frac{\theta_{\tau_v\times\tau^2_\pm}}{\theta_{\tau_v}\theta_{\tau^2_\pm}} = \frac{-e^{i\pi/4}}{e^{i\pi/4}e^{\pm i\pi/2}} = -e^{\mp i\pi/2} \tag{3.5}$$

3.4.4 Time-Reversal Properties

We have already identified appropriate charge assignments, which encode the transformation properties of various particles under the $U(1)_C$ symmetry. In this section, we address how TR is implemented in the proposed STO phase. The results of this section are summarized in Table. 3.1.

The first task for implementing TRS is to specify how topological equivalence classes of particles are exchanged under TR. This is relatively straightforward since we have constructed the STO state from the well-understood TR-symmetric superconductor phase. The τ_v descends from an $\frac{hc}{e}$ vortex in the superconductor, which becomes a $-\frac{hc}{e}$ vortex under TR; in turn the $-\frac{hc}{e}$ vortex becomes $\tau_{\bar{v}}$ in the STO phase. Therefore under TR:

$$\tau_v \overset{\mathcal{T}}{\longleftrightarrow} \tau_{\bar{v}} \tag{3.6}$$

Similarly, by going to the superconductor it is clear that f, and $\beta^2 \cong b$ are preserved under TR. It is also clear that the β sector is preserved under TR.

Under TR, counter-clock-wise and clock-wise exchange are interchanged, and hence topological classes of particles that are related by TR must have conjugate topological spin. We see that this is true for all of the above TR transformation rules.

Since $\tau_\pm^2$ descend from both $\pm\frac{hc}{e}$ vortices, we cannot determine their TR properties directly from the superconductor. However, since τ_+^2 and τ_-^2 have conjugate topological spins, they must be exchanged by $\mathcal{T}$:

$$\tau_+^2 \overset{\mathcal{T}}{\longleftrightarrow} \tau_-^2 \tag{3.7}$$

In addition to the action of $\mathcal{T}$ on topological superselection sectors, for sectors that are not interchanged by $\mathcal{T}$, it is meaningful to ask about their eigenvalues under the unitary operation of double-time-reversal, $\mathcal{T}^2$. For particles that reside in TR-invariant superselection sectors, $\mathcal{T}^2 = -1$ has definite physical interpretation as a TRS-protected Kramers degeneracy. Our STO state arises naturally from the superconductor where b has $\mathcal{T}^2 = 1$ and f has $\mathcal{T}^2 = -1$ respectively; hence β^2 and f also have $\mathcal{T}^2 = 1$ and $\mathcal{T}^2 = -1$ in the STO phase. Similarly, β^2 has $\mathcal{T}^2 = 1$ since it is a fraction of b, and since β can be condensed to obtain the SC from the STO phase.

However, for particles, like $\tau_\pm^2$, whose superselection sectors are changed by $\mathcal{T}$, the $\mathcal{T}^2$ eigenvalue does not imply a further degeneracy within that particle sector. Furthermore, for such particles, it turns out that it is not even possible to assign a local gauge-invariant representation of $\mathcal{T}^2$. In the next two sections we further describe the issue of symmetry localization on gauge non-invariant particles.

Gauge (non)-invariance TR Properties for Fractionalized Particles

Fractionalized particles (i.e. particles with non-trivial self- or mutual-statistics) cannot be individually created from the ground-state. Rather, one can only create groups of excitations that fuse to $\mathbb{I}$. For example, to isolate a fractionalized particle X, one can create a particle anti-particle pair, X and X^{-1}, from the ground-state, and pull them far apart from one another. The operator that implements this sequence consists of a string of electron operators connecting the final locations, R_1 and R_2, of X and X^{-1} respectively. This string of operators can be divided into two local operators $\Psi_X^\dagger(R_1)$ and $\Psi_X(R_2)$, that create X and X^{-1} respectively, and a non-local gauge-string, $W_{1,2} = \prod_\Gamma e^{iq_X a_{ij}}$, where i and j label sites on the lattice where Ψ_X is defined, Γ is directed path of links $\langle ij \rangle$ connecting sites R_1 to R_2, q_X is the internal gauge-charge of the particle X, and a_{ij} is a discrete-valued emergent gauge field. This division into particles and strings is inherently arbitrary, which is reflected by the local gauge invariance under the transformations $\Psi_{X,i} \to e^{2\pi i n_i q_x}\Psi_{X,i}$, and $a_{ij} \to a_{ij} - (n_i - n_j)$ (with $n_{i,j} \in \mathbb{Z}$).

Due to the non-local gauge structure there is not always a well-defined gauge invariant way to assign symmetry-transformation properties locally to the particle creation operators $\Psi_X^\dagger$. Rather, one must generically keep track of the transformation property of both the particles, and their gauge-strings, W. However, in special cases it is possible to associate a well-defined action of a symmetry locally on $\Psi_X^\dagger$ even for gauge non-invariant objects. For simplicity, in what follows, we will not distinguish between the label X for a topological class of particles and the corresponding (gauge-non-invariant) annihilation operator Ψ_X.

Since $f \times f = \mathbb{I}$, the phase of f has a sign ambivalence, indicating that f's have $\frac{1}{2}$-gauge charge (i.e. change sign under $e^{2\pi i q_f} = -1$) and are connected pairwise by (unobservable) unoriented $\mathbb{Z}_2$ gauge strings. Similarly $\beta^2 \times f$ is a physical electron c, and so β^2 also has a $\mathbb{Z}_2$ gauge charge. It then follows from $\tau_\pm^2 \times \tau_\pm^2 = \beta^2$, that $\tau_\pm^2$ has internal $\frac{1}{4}$-gauge charge, and that oriented $\mathbb{Z}_4$ gauge strings emanate from $\tau_\pm^2$ particles. We know that $\tau_\pm^2$ have opposite internal gauge charge, since $\tau_+^2 \times \tau_-^2 = c$, and c is a physical (gauge-invariant) local electron. Therefore, we can choose the orientation convention that $\mathbb{Z}_4$ lines emanate from τ_+^2 and terminate on τ_-^2 particles.

$\mathcal{T}^2$ Properties For Sectors that are Exchanged by $\mathcal{T}$

With this gauge-string picture in mind, we now turn to the task of determining to what extent $\mathcal{T}^2$ is defined on particles whose topological classes are interchanged by $\mathcal{T}$. To see why it is important to consider the effects of $\mathcal{T}$ on the gauge string, consider a τ_+^2-τ_-^2 pair. Suppose that we represent $\mathcal{T}$ locally on the particle operators as: $\mathcal{T}^{-1}\tau_+^2\mathcal{T} = e^{i\alpha}\tau_-^2$ and $\mathcal{T}^{-1}\tau_-^2\mathcal{T} = e^{i\beta}\tau_+^2$ where α and β are unknown phases. Then one has: $\mathcal{T}^{-2}\tau_\pm^2\mathcal{T}^2 = e^{\pm i(\beta-\alpha)}$, and naively it appears that $\mathcal{T}^{-2}\tau_+^2\tau_-^2T^2 = |e^{i(\beta-\alpha)}|^2\tau_+^2\tau_-^2$. However, this cannot be the whole story, since $\tau_\pm^2$ fuse to the physical electron, c, which is a Kramers doublet with $\mathcal{T}^2 = -1$.

This puzzle is resolved by noting that $\tau_+^2 \overset{\mathcal{T}}{\leftrightarrow} \tau_-^2$, implies that $\mathcal{T}$ reverses the direction of the gauge string connecting a given τ_+^2-τ_-^2 pair. Then acting twice with $\mathcal{T}^2$ doubly flips the orientation of the connecting gauge-string. A two-fold re-orientation of the gauge-string can also be accomplished by dragging τ_+^2 around τ_-^2. Due to their semionic mutual statistics, this observation dictates that the gauge string contributes an additional factor of -1 to the overall $\mathcal{T}^2$. Therefore, the action of $\mathcal{T}^2$ cannot be consistently implemented in a purely local fashion for the gauge-non-invariant particles $\tau_\pm^2$, which interchange under $\mathcal{T}$.

Note that a nearly identical argument can be applied to monopoles in the bulk of the electron TI to formally establish the intimate connection between the $\theta = \pi$ electromagnetic response of the TI and the Kramers degeneracy of the electron[3] (see also Ref. [53]). This is indeed appropriate, since the $\tau_\pm^2$ particles are the surface-avatars of these bulk dyons.

The issue of non-locality is even more pronounced for the non-Abelian excitations τ_v and $\tau_{\bar{v}}$, since a collection of these particles share a degenerate Hilbert space of non-local fermion modes, and the action of $\mathcal{T}^2$ depends on the total fermion parity of this non-local Hilbert space, which is a global property of the system.

3.5 2D TR Breaking Analog

For bosonic SPT bulk phases, the topological properties of the STO phase can always be realized by a strictly 2D system that does not preserve the underlying symmetries of the 3D SPT. In this section, we provide an analogous construction for the electron TI. Specifically, we show that the STO phase has the same topological order as the Moore-Read QH phase[103] supplemented by an extra neutral semion. We begin by reviewing the Moore-Read and related 2D phases in the language of the parton construction $c_\sigma = bf_\sigma$ used above.

3.5.1 $p + ip$ Superconductor and Kitaev Spin-Liquid

We begin with the $p + ip$ superconductor, and its topologically ordered analog, which are in some sense the simplest "roots" of the non-Abelian Ising topological order for the STO phase. A TR-breaking superconductor with $p+ip$ pairing symmetry, and the TR-broken B-phase of Kitaev's Honeycomb Model (henceforth denoted Kitaev Spin-Liquid, or KSL) are closely related states with non-Abelian Ising anyon excitations. The latter is obtained from the former by condensing $\frac{hc}{e}$-vortices. In the language of the parton construction, this is equivalent to placing b in a Mott insulator, and f into a $p + ip$ superconductor. The resulting phase contains topological particle classes: $\mathbb{I}$ (vacuum), b, f, and a non-Abelian vison, σ that descends from the $\pm\pi$-vortices of the $p + ip$ superconductor.

In the resulting KSL phase, b has charge e, and all other particles are neutral. The edge of this phase contains a single chiral Majorana fermion that contributes $\sigma_H = 0$

and $\kappa_H = \frac{1}{2}$. The fusion rules are:

$$\begin{aligned} b \times f &= c \\ b \times b &= c^2 \cong \mathbb{I} \\ f \times f &= \mathbb{I} \\ \sigma \times f &= \sigma \\ \sigma \times b &= \sigma \\ \sigma \times \sigma &= 1 + f \end{aligned} \tag{3.8}$$

and the topological spins are:

$$\begin{aligned} \theta_b &= 1 \\ \theta_f &= -1 \\ \theta_\sigma &= e^{i\pi/8} \end{aligned} \tag{3.9}$$

3.5.2 Moore-Read Quantum Hall State

The Moore-Read state[103] can be obtained from the KSL phase by placing b in a $\nu = 1/2$ bosonic-Laughlin quantum Hall phase rather instead of a Bose-Mott insulator. This phase is characterized by the idealized wave-function:

$$\Psi_{\mathrm{MR}} \sim \prod_{i<j} (z_i - z_j)^2 \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \tag{3.10}$$

where $z_j = x_j + iy_j$ is the complexified coordinate of the j^{th} electron. The factors of $(z_i - z_j)^2$ stem from the b sector, and the Pf denotes the Pfaffian of the anti-symmetric matrix with entries $\frac{1}{z_i - z_j}$, which describes the BCS wave-function with $p + ip$-pairing[108].

In this phase, the vison, σ of the f-sector is bound to a π-flux of the Bosonic QH fluid which we denote v (similarly, denote a $-\pi$ flux of the Bosonic QH fluid by $\bar{v}$). A π-flux in a $\sigma_H = \frac{1}{2}$ system has charge $\frac{e}{4}$ and hence v has topological spin

$e^{i\pi/8}$. Denoting the non-Abelian vison/charge-$\frac{e}{4}$ vortex composite as σ_v, we have: $\theta_{\sigma_v} = e^{i\pi/4}$. Since $\bar{v}$ is a $-\pi$-vortex bound to charge $-\frac{e}{4}$ it also contributes an extra $e^{i\pi/8}$ to the vison topological spin, indicating that the composite, $\sigma \times \bar{v} \equiv \sigma_v^{-1}$, has $\theta_{\sigma_v^{-1}} = \theta_{\sigma_v} = e^{i\pi/4}$.

3.5.3 2D TR-Breaking Analog

The MR state looks somewhat similar to the STO phase constructed above: there Ising non-Abelions attached to charged Abelian vortices. However, unlike in the TI STO phase, $\sigma_v \times \sigma_v^{-1} = 1 + f$ is charge-neutral. More generally, since σ_v and $\sigma_{\bar{v}}$ have opposite charge, and the same topological spin, it is hard to see how TR-invariance could be implemented in the MR phase, even at the surface of the STO.

We can cure this problem by introducing an extra counter-propagating anti-semion particle, s, with topological spin $\theta_s = e^{-i\pi/2}$ to the boson sector (in the parton language this corresponds to further fractionalizing $b \to b_1 b_2$, with b_1 carrying charge e in a bosonic $\nu = 1/2$ QH phase, and b_2 a charge-neutral in a $\nu = -1/2$ bosonic QH phase). Making the following identifications:

$$\begin{aligned} \beta^{-1} \times \tau_+^2 &= s \\ \tau_v &= \sigma_v \\ \tau_{\bar{v}} &= \sigma_v \times s \\ \tau_v^{-1} &= \sigma_v^{-1} \\ \tau_{\bar{v}}^{-1} &= \sigma_v^{-1} \times s \end{aligned} \tag{3.11}$$

we see that this 2D TR-breaking phase has the same topological order and charge assignments as the STO phase described above. For brevity we denote the 2D TR-breaking phase: MR×AS.

3.6 Connection Between STO and Familiar Non-Fractionalized Surface Phases

In the previous section, we have constructed an STO phase by quantum disordering the TRS surface superconductor state. The fact that a TI can realize this topological order with both $U(1)_C$ and TR symmetries intact actually serves as a non-perturbative definition of the $U(1)_C \ltimes \mathbb{Z}_2^T$ fermion topological insulator. To see this, we need to show that we can obtain all of the usual symmetry broken non-topologically ordered surface phases of the familiar fermion TI through a sequence of surface-phase transitions that do not affect the TI bulk.

3.6.1 STO to TR-Symmetric Non-Abelian Surface SC

Since we have constructed the STO phase from the TR-invariant surface SC, it is straightforward to recover the familiar surface SC. We have already argued that the superconducting surface can be obtained from the STO phase by condensing β. Here we provide some further details.

Since $\beta^2 = b = \sqrt{\rho_b}e^{i\phi_b}$ we may write $\beta = (\rho_b)^{1/4}e^{i\phi_\beta}$. Then 2π vortices of ϕ_β are 4π vortices of ϕ_b, which are condensed in the STO phase. In other words, the STO phase can be viewed as a Mott insulator of β. Then, to recover the TRS surface superconductor from the STO phase, one can simply condense β. Since β has non-trivial mutual statistics with all other particles besides f, the particles $\tau_v, \tau_{\bar{v}}, \tau_\pm^2$ etc... will all be confined in the $\langle\beta\rangle \neq 0$ phase. However, these confined objects do not completely dissapear from the theory, rather they are bound to vortices of ϕ_β (which are now-gapped) to form composites that have trivial mutual statistics with the β-condensate.

Since, β has the same mutual statistics with τ_v as with a $\pi/2$-vortex of ϕ_β they are bound-together in the superconductor. Since β is charged a $+\pi/2$-vortex of ϕ_β has physical circulating charge current. and the τ_v object becomes the superconducting $\frac{hc}{2e}$ vortex (or, more generally, a $\frac{(4n+1)hc}{2e}$ vortex with $n \in \mathbb{Z}$). Similarly, $\tau_{\bar{v}}$ becomes a

$\frac{(4n-1)hc}{2e}$ vortex, and $\tau_\pm^2$ become a $\frac{nhc}{e}$ vortices (with n odd).

3.6.2 STO to 1/2-integer quantum Hall

Next, we connect the STO to the $U(1)_C$ preserving but TR-breaking $\frac{1}{2}$-integer surface quantum Hall insulator (SQHI). In the previous section, we showed that the topological order and charge assignments of the STO can be realized in strict 2D at the expense of breaking TRS. The analogous TR breaking phase was equivalent to the Moore-Read QH phase with an extra neutral semion, denoted MR×AS. Importantly, the MR×AS has $\sigma_H = \kappa_H = \frac{1}{2}$. There is a closely related phase, which we denote $\overline{\text{MR} \times \text{AS}}$, obtained from MR×AS by switching all of the particles of MR×AS with their anti-particles, which has $\sigma_H = \kappa_H = -\frac{1}{2}$.

Starting with the STO phase of the TI, let us "deposit" a layer of $\overline{\text{MR} \times \text{AS}}$ on the TI surface (or alternatively, imagine adjusting the interactions and other parameters of a layer of the bulk near the surface to drive that layer into the $\overline{\text{MR} \times \text{S}}$ phase). Then, suppose we allow the f particle of the $\overline{\text{MR} \times \text{AS}}$ to hybridize with (i.e. tunnel into) the f particle of the STO phase. This confines each non-Abelian τ_v of the STO is bound to a similar non-Abelian τ_v^{-1} of the deposited layer, thereby neutralizing the non-Abelian statistics of the composite object. The resulting composites are all Abelian and have trivial self-statistics, and hence can be straightforwardly condensed (since TR symmetry is already broken). In particular, if we condense the particles containing a τ_v of the STO layer and a τ_v^{-1} of the deposited layer, all other particles are trivially confined, and no excitations with fractional statistics remain.

We have thereby eliminated the surface-topological order, at the expense of breaking TR-symmetry on the surface. What is the quantum Hall response of this non-fractionalized insulating state?

To answer this question we note that we could have equally well followed a time-reversed version of the above procedure, by depositing a different surface layer related to $\overline{\text{MR} \times \text{AS}}$ by TR, which we denote $\overline{\text{MR}^* \times \text{AS}^*}$ and has $\sigma_H = \kappa_H = \frac{1}{2}$. Consider a spherical TI, depicted in Fig. 3-2, and imagine depositing a layer of $\overline{\text{MR} \times \text{AS}}$ on the bottom hemisphere of the TI surface and a layer of $\overline{\text{MR}^* \times \text{AS}^*}$ on the top

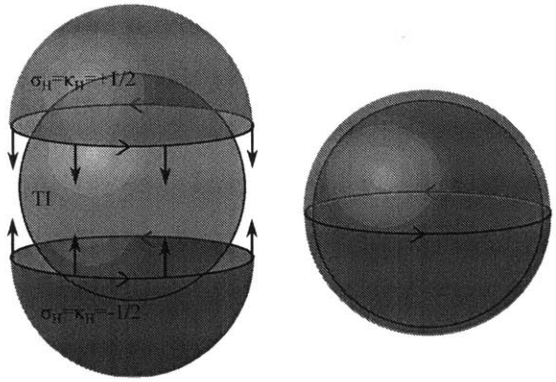

Figure 3-2: The non-fractionalized TR-breaking quantum Hall insulator (QHI) with coating the TI surface with a 2D TR-breaking topologically ordered state with $\sigma_H = \kappa_H = \pm\frac{1}{2}$ (depicted in orange and purple respectively), as explained in the text. The half-integer quantum Hall conductance can be seen by considering a domain between these two coatings as shown in the above figure for a spherical TI,

hemisphere. The edges of the deposited 2D layers meet at the equator, and each contributes a chiral Majorana fermion, a co-propagating charged boson mode and a counter-propagating neutral boson mode. The chiral Majorana fermions from the top and bottom hemisphere propagate in the same direction, and when coupled, combine into a complex (neutral) chiral fermion. The combined edge has overall chirality with a single chiral charged mode, and hence has $\sigma_H = \kappa_H = 1$. This σ_H and κ_H is not effected by condensing τ_v composites in order to remove the topological order.

This line of reasoning shows that, even after destroying the surface-topological order, the interface at the equator possesses a single 1D chiral charge fermion. The non-fractionalized phases that we have produced on the top and bottom hemisphere therefore differ by an electron $\nu = 1$ quantum Hall layer. Since these two phases are related by TR-symmetry, we must democratically assign them $\sigma_H = \kappa_H = \pm\frac{1}{2}$ respectively. We have therefore succeeded in recovering the familiar non-fractionalized surface QH insulating phases from the STO phase.

3.6.3 STO to Gapless Dirac Fermion Surface

In the previous section, we showed how to obtain the surface QH insulator from the STO phase by breaking TR. The resulting phase can have either $\sigma_H = \kappa_H = \pm\frac{1}{2}$.

From here, it is straightforward to produce the symmetry preserving gapless Dirac cone phase by proliferating domain walls between the $\sigma_H = \pm\frac{1}{2}$ surface phases. Such domain walls carry a single chiral (complex) fermion, and it is well known (for example from network models[43]) that their proliferation results in a single gapless Dirac cone.

3.6.4 $\mathbb{Z}_2$ Nature of Surface Order

It is well known that two copies of the ordinary electron topological insulator can be smoothly deformed into the trivial insulator without a bulk phase transition. Therefore, as a final consistency check for the proposed STO, we demonstrate that two coupled STO phases can be deformed to a trivial insulator by surface phase-transitions that leave the bulk gap untouched.

Consider starting with two layers of the STO phase, labeled 1 and 2 respectively, coupled such that electrons can tunnel between them: $\langle c_1^\dagger c_2\rangle$, $\langle c_2^\dagger c_1\rangle \neq 0$. It is straightforward to check that the following set of composite particles are charge-neutral self-bosons with trivial mutual-statistics, which can be simultaneously condensed without breaking either $U(1)_C$ or TRS:

$$\{\beta_1^\dagger \beta_2, \quad \tau_{v1}\tau_{v2}\beta^\dagger, \quad \tau_{\bar{v}1}\tau_{\bar{v}2}\beta^\dagger, \quad \text{and h.c.'s}\} \tag{3.12}$$

with h.c.'s indicating that all operators related by Hermitian conjugation to those listed are also condensed. In order to preserve TRS, we must condense TR conjugate particles with equal amplitude: $\langle \tau_{v1}\tau_{v2}\beta^\dagger\rangle = \langle \tau_{\bar{v}1}\tau_{\bar{v}2}\beta^\dagger\rangle \neq 0$.

It is also straightforward to verify that after condensing these objects, all nontrivial particles in the theory are either confined or condensed, and there are no fractionalized excitations. In particular, f and β^2 both have mutual (-1) statistics with the condensed $\tau_{v1}\tau_{v2}\beta^\dagger$ particles, and are confined together to form the physical electron: $c = \beta^2 f$. The resulting phase has only gapped, physical electron excitations, c, and hence is a trivial band-insulator. Therefore, we have verified that the bulk phase described by our proposed STO indeed has a $\mathbb{Z}_2$ group structure (i.e. that combining two copies of our phase produces a trivial phase) as required for the electron TI.

This set of particles in Eq. 3.12 has a natural physical interpretation: starting with two coupled layers of the TRS surface-SC phase, we know that we can obtain a trivial bulk insulator by condensing the (now Abelian) $\pm\frac{hc}{2e}$ vortices, which now occur in the same location in both layers due to the interlayer tunneling. The set of particles condensed here to trivialize the double-layer STO phase are simply the descendants of these vortices.

3.7 Discussion

We have shown that, in addition to the familiar gapless Dirac surface state, and gapped symmetry-broken states, the electronic topological insulator (TI) can support a gapped and fully symmetric phase with surface topological order (STO). This STO phase provides a complete, non-perturbative definition of the electron TI. Like STO phases of analogous bosonic TIs, the electron TI STO phase has the same topological-order as a 2D phase, but with symmetry implemented in a way that is not allowed in strict 2D.

For boson TIs, the lens of STO provides a useful perspective into 3+1D strongly correlated boson TIs as well as 2+1D gauge theories[1]. The hope is then that understanding of the electron TI STO will enable similar progress for strongly-correlated electronic phases. An essential component for boson TIs was a systematic understanding of symmetry implementation for strictly-2D Abelian bosonic systems[48, 75]. One potentially complicating factor in adapting this approach to fermions is that the electron TI STO is inherently non-Abelian. Consequently an important outstanding task for making progress along these lines is to develop a systematic understanding of symmetry implementation in 2D non-Abelian theories. These theories are not amenable to the simple K-matrix methods that have so successfully utilized for boson systems[48, 75]. However, methods of similar spirit based on using the bulk-boundary correspondence to reduce the problem to symmetry implementation in 1+1D conformal field theories of the edge may still prove fruitful. Such a pursuit would go far beyond the scope of the present chapter and is left as a challenge for future work.

Using a different method, based on Walker-Wang type models, X. Chen, L. Fidkowski, and A. Vishwanath have also constructed a candidate STO phase for a $\Theta = \pi$ electron TI[54]. The relationship between this STO and the one described above is not completely clear, however, in light of the general arguments of Ref. [3] this phase can at most differ from the conventional eTI by an SPT phase of neutral bosons.

This chapter focused on the Fu-Kane-Mele topological insulator, which is the only nontrivial topological insulator in $3D$ free fermion systems. In the next chapter we discuss the possibility of other (bulk) electronic topological insulators in correlated systems.

Topological Superselection Sector ("Particle Type"):	$\mathbb{I}$	β	f	τ_v	$\tau_{\bar{v}}$	τ_+^2	τ_-^2	τ_v^3	$\tau_{\bar{v}}^3$
Conjugate Sector (anti-particle):	$\mathbb{I}$	$\beta^3 \equiv \beta^{-1}$	f	$\beta^{-1}\tau_{\bar{v}}$	$\beta^{-1}\tau_v$	$\beta^{-2}\tau_+^2$	$\beta^{-2}\tau_-^2$	$\beta^{-2}\tau_v$	$\beta^{-2}\tau_{\bar{v}}$
Quantum Dimension (d):	1	1	1	$\sqrt{2}$	$\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
Topological Spin (θ):	1	1	-1	$e^{i\pi/4}$	$e^{-i\pi/4}$	$e^{i\pi/2}$	$e^{-i\pi/2}$	$-e^{i\pi/4}$	$-e^{-i\pi/4}$
Charge (q_e):	$2ne$ ($n \in \mathbb{Z}$)	$\frac{e}{2}$	0	$\frac{e}{4}$	$\frac{e}{4}$	$\frac{e}{2}$	$\frac{e}{2}$	$\frac{3e}{2}$	$\frac{3e}{2}$
Time-Reverse Partner:	$\mathbb{I}$	β	f	$\tau_{\bar{v}}$	τ_v	τ_-^2	τ_+^2	$\tau_{\bar{v}}^3$	τ_v^3
$\mathcal{T}^2$ value (if meaningful):	1	1	-1						

Table 3.1: Summary of the topological content of the surface-topological order phase and the implementation of charge-conservation and TR symmetries. Topological superselection sectors are topological equivalence classes of particle types. The anti-particle of a particle in sector a resides a's conjugate sector. A particle has the same quantum-dimension and $\mathcal{T}^2$ value as its anti-particle, but opposite electrical charge and conjugate topological spin. Other distinct topological particles such as β^2, $\beta\tau_v$, etc... can be obtained by combining the above listed objects. The properties of these composites and anti-particles follows straightforwardly from the information listed above. Superselection sectors have the same quantum dimension, opposite charge, and same topological spin compared to their conjugate sectors (anti-particles). Empty entries in the $\mathcal{T}^2$ row indicate that there is no gauge invariant meaning to the value of $\mathcal{T}^2$ for that type of particle. In addition, there is the physical electron, c, which has $d = 1$, $\theta_c = -1$, $\mathcal{T}_c^2 = -1$. This could be regarded as part of the vacuum sector $\mathbb{I}$ since it has trivial mutual statistics with all other particles. However, since fusing c to another particle changes that particle's topological spin factor of -1 it is convenient to distinguish c from $\mathbb{I}$.

$\beta \times \beta = \beta^2$
$\beta^2 \times \beta = \beta^3 = \beta^{-1}$
$\beta^3 \times \beta = \mathbb{1}$
$\beta^n \times a = \beta^n a$
(for any sector $a \neq \beta$ and $n = 1, 2, 3$)
$f \times f = 1$
$f \times \tau_v = \tau_v$
$f \times \tau_{\bar{v}} = \tau_{\bar{v}}$
$\tau_v \times \tau_v = \tau_+^2 + \tau_-^2$
$\tau_{\bar{v}} \times \tau_{\bar{v}} = \tau_+^2 + \tau_-^2$
$\tau_v \times \tau_{\bar{v}} = \beta + \beta f$
$\tau_\pm^2 \times f = \tau_\mp^2$
$\tau_\pm^2 \times \tau_\pm^2 = \beta^2$
$\tau_+^2 \times \tau_-^2 = \beta^2 \times f = c$
$\tau_v^3 = \tau_v \times \tau_\pm^2$
$\tau_{\bar{v}}^3 = \tau_{\bar{v}} \times \tau_\pm^2$

Table 3.2: Fusion rules for the surface-topological order phase.

Chapter 4

Classification of interacting electronic topological insulators in three dimensions

The last few years have seen tremendous progress[18, 19, 20, 21, 22] in our understanding of electronic topological insulators modeled by band theory. Despite this there is currently very little understanding of the interplay between strong electron interactions and the phenomenon of topological insulation. Can interaction dominated phases be in a topological insulating state? Are there new kinds of topological insulators that might exist in interacting electron systems that have no non-interacting counterpart? These questions acquire particular importance in light of the ongoing experimental search for topological phenomena in strongly correlated materials with strong spin-orbit coupling.

We focus here on the all-important example of time reversal symmetric insulating phases of electrons with a conserved global charge (corresponding to a global $U(1)$ symmetry). Non-interacting insulators with this symmetry in $3D$ have a well known distinction[25, 18, 19, 20] between the topological and trivial band insulators (corresponding to a $\mathbb{Z}_2$ classification).

We show that with interactions there are 6 other non-trivial topological insulating states corresponding to a classification by the group $\mathbb{Z}_2^3$. This group structure means

Topological Insulator	**Representative surface state**	**$\mathcal{T}$-breaking transport signature**	**$\mathcal{T}$-invariant gapless superconductor**
Free fermion TI	Single Dirac cone	$\sigma_{xy} = \frac{\kappa_{xy}}{\kappa_0} = \pm 1/2$	None
Topological paramagnet I ($eTmT$)	$\mathbb{Z}_2$ spin liquid with Kramers doublet spinon(e) and vison(m)	$\sigma_{xy} = \kappa_{xy} = 0$	$N = 8$ Majorana cones
Topological paramagnet II ($e_f m_f$)	$\mathbb{Z}_2$ spin liquid with Fermionic spinon(e) and vison(m)	$\sigma_{xy} = 0; \frac{\kappa_{xy}}{\kappa_0} = \pm 4$	$N = 8$ Majorana cones

Table 4.1: Brief descriptions of the three fundamental non-trivial topological insulators, with their representative symmetry-preserving surface states, and surface signatures when either time-reversal or charge conservation is broken on the surface (with topological orders confined). σ_{xy} is the surface electrical Hall conductivity in units of $\frac{e^2}{h}$. κ_{xy} is the surface thermal Hall conductivity and $\kappa_0 = \frac{\pi^2}{3}\frac{k_B^2}{h}T$ (T is the temperature). N is the number of gapless Majorana cones protected by time-reversal symmetry when the surface becomes a superconductor. A combination of these measurements could uniquely determine the TI.

that all these interacting topological insulators can be obtained from 3 'root' states and taking combinations. One of the 3 root states is the standard topological band insulator. The other two require interactions. They can be understood as Mott insulating states of the electrons where the resulting quantum spins have themselves formed an SPT phase. Such SPT phases of quantum spins were dubbed 'topological paramagnets' in Ref. [43] and their properties in 3D elucidated. The three root states and their properties are briefly described in Table. 4.1.

Previous progress in understanding interacting electronic SPT phases is restricted to one[40, 41, 42, 109] and two[48, 58, 59, 60, 61] space dimensions. A formal abstract classification for some symmetries (which includes neither charge conservation, nor spin-1/2 electrons) in 3d has been attempted[56] but leaves many physical questions unanswered. Our strategy - which sidesteps the difficulties of this prior approach - is to first constrain the symmetries and statistics of monopole sources of external electromagnetic fields. We then incorporate these constraints into a theory of the surface, and determine the resulting allowed distinct states.

In general it is natural to attempt to construct possible SPT phases of fermion system by first forming bosons as composites out of the fermions and putting the bosons in a bosonic SPT state. However not all these boson SPTs remain distinct states in an electronic system. We determine that the distinct such states (see Sec. 4.5.1) can all be viewed as topological paramagnets as described above.

While such spin-SPT phases can clearly exist, we give very general arguments that the only other electronic root state is the original topological band insulator.

We also clarify a number of other questions about interacting topological insulators (see end of the chapter and Sec. 4.6.1, 4.6.2). For instance we explain the fundamental connection between topological insulation and Kramers structure of the electron.

Our results set the stage for a number of future studies including identification of the new topological insulators in microscopic models and in real materials. Strongly correlated materials with strong spin orbit interactions are natural platforms for the various topological insulator phases we described. We expect that our results, especially the knowledge of what exotic phases are possible and what experimental signatures to look for, will inform the many ongoing searches (e.g., in rare earth insulators, or in iridium oxides) for topological phenomena in such materials.

4.1 Generalities

For any fully gapped insulator in 3D, the effective Lagrangian for an external electromagnetic field obtained by integrating out all the matter fields will take the form

$$\mathcal{L}_{eff} = \mathcal{L}_{Max} + \mathcal{L}_{\theta} \tag{4.1}$$

The first term is the usual Maxwell term and the second is the 'theta' term:

$$\mathcal{L}_{\theta} = \frac{\theta}{4\pi^2}\mathbf{E}\cdot\mathbf{B} \tag{4.2}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the external electric and magnetic fields respectively.

Under time reversal, $\theta \to -\theta$ and in a fermionic system the physics is periodic under $\theta \to \theta + 2\pi$. Time reversal symmetric insulators thus have $\theta = n\pi$ with n an integer. Trivial time-reversal symmetric insulators have $\theta = 0$ while free fermion topological insulators have $\theta = \pi$[64]. Any new interacting TI that also has $\theta = \pi$ can be combined with the usual one to produce a TI with $\theta = 0$. Thus it suffices to restrict attention to the possibility of new TIs which have $\theta = 0$.

Consider the symmetry properties of monopole sources of the external magnetic field. At a non-zero θ, this elementary monopole carries electric charge $\frac{\theta}{2\pi}$ so that it is neutral when $\theta = 0$. Under time reversal the monopole becomes an anti-monopole as the magnetic field is odd. Formally if we gauge the global $U(1)$ symmetry to introduce a dynamical monopole field m it must transform under time reversal as

$$\mathcal{T}^{-1} m \mathcal{T} = e^{i\alpha} m^\dagger \tag{4.3}$$

$$\mathcal{T}^{-1} m^\dagger \mathcal{T} = e^{-i\alpha} m \tag{4.4}$$

However[1] by combining with a gauge transformation we can set the phase $\alpha = 0$. Physically this is because the time reversed partner of a monopole lives in a different topological sector with opposite magnetic charge and hence is not simply a Kramers partner. To see this explicitly we observe that the $\mathcal{T}$ operator can be combined with a (magnetic) gauge transformation to define a new time reversal operator:

$$\tilde{\mathcal{T}} = U(\alpha)\mathcal{T} \tag{4.5}$$

where $U(\alpha) = e^{-i\alpha q_m}$ where q_m is the total magnetic charge. Since q_m is odd under time-reversal, we have $U(\alpha)\mathcal{T} = \mathcal{T}U(\alpha)$, hence the order of product in Eq. (4.5) does not matter. When acting on physical gauge invariant states $\tilde{\mathcal{T}}$ has the same effect as $\mathcal{T}$ but the monopole fields $m, m^\dagger$ transform with $\alpha = 0$.

This fixes the symmetry properties of the bulk monopole. There are still in principle two distinct choices corresponding to the statistics of the monopole: it may be either bosonic or fermionic. We will consider them in turn below. Bosonic monopoles will be shown to allow for the topological paramagnets mentioned above and nothing else. Fermionic monopoles will be shown to not occur in electronic SPT phases.

4.2 Topological insulators at $\theta = 0$ - bosonic monopoles

Consider the surface of any insulator with $\theta = 0$ and a bosonic monopole. This is conveniently incorporated into an effective theory of the surface formulated in terms of degrees of freedom natural when the surface is superconducting, *i.e*, it spontaneously breaks the global $U(1)$ but not time reversal symmetry. The suitable degrees of freedom then are $\frac{hc}{2e}$ vortices and (neutralized) Bogoliubov quasiparticles[80] (spinons) which have mutual semion interactions. In general we can also allow for co-existing topological order, i.e. other fractionalized quasi-particles, in the surface superconductor.[1] This gives a dual description of $2D$ electronic systems that is particularly convenient to studying not just the superconducting phase but also some topologically ordered insulating phases.

Imagine tunneling a monopole from the vacuum to the system bulk. Since the monopole is trivial in both regions, the tunneling event - which leaves a $\frac{hc}{e}$ vortex on the surface - also carries no non-trivial quantum number. Hence the surface dual effective field theory has a bosonic $\frac{hc}{e}$-vortex that carries no non-trivial quantum number. We can therefore proliferate (condense) the $\frac{hc}{e}$-vortex on the surface which disorders the superconductor and yields an insulator with the full symmetry $U(1) \ltimes \mathcal{T}$ unbroken. However as is well known from dual vortex descriptions[104, 80] of spin-charge separation in $2D$, the resulting state has intrinsic topological order.

In this surface topologically-ordered symmetry-preserving insulator, a quasi-particle of charge-q sees the $\frac{hc}{e}$-vortex as a $2\pi q/e$ flux. Hence, the $\frac{hc}{e}$-vortex condensate confines all particles with fractional charge and quantizes the charge to $q = ne$ for all the remaining particles in the theory (for a more detailed discussion see Sec. 4.3).

[1]Such a phase with coexistence of topological order and superconductivity was denoted SC* in Ref. [80].

However, we can always remove integer charge from a particle without changing its topological sector by binding physical electrons. Hence the particle content of the surface topological order is $\{1, \epsilon, ...\} \times \{1, c\}$, where only the physical electron c in the theory is charged, and all the non-trivial fractional quasi-particles in $\{1, \epsilon, ...\}$ are neutral. Since time-reversal operation preserves the $U(1)$ charge, its action has to be closed within the neutral sector $\{1, \epsilon, ...\}$. We can therefore describe the surface topological order as a purely charge-neutral quantum spin liquid with topological order $\{1, \epsilon, ...\}$, supplemented by the presence of a trivial electron band insulator, $\{1, c\}$. In particular, any gauge-invariant local operator made out of the topological theory must be neutral (up to binding electrons), but in an electron system a local neutral object has to be bosonic. Hence the theory should be viewed as emerging purely from a neutral boson system. This implies that the bulk SPT order should also be attributed to the neutral boson (spin) sector, *i.e* it should be a SPT of spins in a Mott insulating phase of the electrons (a topological paramagnet).

The SPT states of neutral bosons with time-reversal symmetry are classified[43, 1, 44] by $\mathbb{Z}_2^2$, with two fundamental root non-trivial phases. These can both be understood as Mott insulators in topological paramagnet phases. Adding to this the usual $\theta = \pi$ TI captured by band theory we have 3 root states corresponding to a $\mathbb{Z}_2^3$ classification. To establish that there are no other states we need to still consider the other possibility left open for the bulk response: a fermionic monopole.

4.3 Vortex condensate and charge quantization

Here we provide more details of the argument establishing that electronic topological insulators with $\theta = 0$ and a bosonic monopole can be reduced to bosonic topological paramagnets. It is convenient to start with a symmetry preserving surface termination that has intrinsic topological order. Such a surface state is characterized by a set of anyons $\{1, c, X, \bar{X}, Y_I\}$ where I is a discrete label, and their corresponding braiding and fusion rules. Each anyon will be characterized by a sharply quantized charge q under the global $U(1)$ symmetry. Let us denote this topological information and symmetry assignments as the initial surface anyon theory: $\mathcal{T}_{\text{initial}}$.

A useful theoretical device[69] is to consider creating a monopole source of an external (non-dynamical) magnetic field, and dragging that monopole through the topologically ordered surface at position $\mathbf{R}$. Such a monopole insertion event changes the external magnetic flux, Φ_B, piercing the surface by $\frac{2\pi}{e}$ (in units where $\hbar = c = e = 1$). When the monopole sits close to the under-side of the surface, this extra flux, $\delta\Phi_B$, is concentrated in the vicinity of $\mathbf{R}$. Suppose we take a surface excitation, Y, with fractional charge q_Y, and drag it around a sufficiently large loop that encloses (nearly all) the additional magnetic-flux from the monopole insertion. This process accumulates Berry phase $e^{2\pi i q_Y} \neq 1$ because of Y's fractional charge. However, the total monopole insertion event is a local physical process, and since there are no gapless excitations in the system it cannot have non-trivial action on distant events (clearly if Y is arbitrarily far from the $\mathbf{R}$, it should not be able to discern whether the monopole is infinitesimally above or infinitesimally below the surface). Therefore, if $\mathcal{T}_{\text{initial}}$ contains quasi-particles Y_I with fractional charge, q_I, the monopole insertion event must also create a quasi-particle of type X in the surface theory which has mutual statistics $\theta_{X,Y_I} = e^{-2\pi i q_I}$. This mutual statistics then exactly compensates the non-trivial Berry phase from encircling the additional flux from the monopole insertion, and ensures that the overall monopole insertion event does not have unphysical non-local consequences. Furthermore, since the bulk monopole is chargeless and bosonic, X, is a neutral boson.

We can similarly consider the time-reversed version of this process by inserting an anti-monopole from the vacuum into the bulk. Let us denote by $\bar{X}$ the particle nucleated at the surface. Clearly X and $\bar{X}$ are exchanged by $\mathcal{T}$, indicating that, like X, $\bar{X}$ is a charge-neutral boson. The mutual statistics of an anyon Y with $\bar{X}$ is then $e^{2\pi i q_Y}$. Further as the monopole and antimonopole can annihilate each other to give back the ground state $\bar{X}$ must be the antiparticle of X.

These mutual statistics indicate that driving a phase transition in which $X, \bar{X}$ condense will confine all fractionally charged particles. However, in general it is not guaranteed that the condensation of $X, \bar{X}$ preserves $\mathcal{T}$. To avoid this issue, we take a detour through an intermediate superconducting phase in which descendants of $X, \bar{X}$ can be safely condensed while preserving $\mathcal{T}$. This results in a topologically ordered state, T_{final}, which has the desired structure of a neutral boson theory.

Our strategy is to first enter a superconducting phase obtained by condensing the physical Cooper pair, $b \equiv c_\uparrow c_\downarrow$, from T_{initial} and then to exit it through a different phase transition to reach the final topological order T_{final}. In the theory, T_{initial}, the Cooper pair is local with respect to all nontrivial anyons. Thus its condensation preserves the topological order T_{initial}. The resulting topologically ordered superconductor is conventionally denoted SC* (see Ref. [80]) to distinguish it from the ordinary non-fractionalized BCS superconductor, SC.

Let us denote the Cooper pair field by $b = \sqrt{\rho_b} e^{i\phi}$. A long-wavelength effective Lagrangian density for the theory can be written:

$$\begin{aligned}\mathcal{L}[b, X, \bar{X}, \cdots] = \frac{\rho_b}{2} \left(\partial_\mu \phi\right)^2 + \mathcal{L}_{T_{\text{initial}}}[X, \bar{X}, Y_I, \ldots] \\ + \mathcal{L}_{\text{mixed}}[b, Y_I, \ldots]\end{aligned} \tag{4.6}$$

where $\mathcal{L}_{T_{\text{initial}}}[X, \bar{X}, Y_I, \ldots]$ is the Lagrangian density encoding the topological content of the topologically ordered phase, and $\mathcal{L}_{\text{mixed}} = \sum \lambda_{\{N_I\}} \prod_I \left(e^{iq_I\phi/2} Y_I\right)^{N_I}$ encodes all charge-conserving interaction terms between b and gauge-invariant combinations of operators in the topologically ordered theory. When b condenses to obtain a superconducting phase, apart from the original topological quasiparticles, there will also be quantized vortex excitations where the phase ϕ of b winds by $2n\pi$ with n an integer. Following the terminology of Ref. [80] we will call these vortons (to distinguish from the vortices of conventional superconductors without topological order).

We wish to disorder the superconducting order by condensing a suitable vortex to obtain the desired insulating surface theory T_{final}. This may be done in a dual effective field theory in terms of the vorton degrees of freedom. To formulate such a dual field-theory, it is very convenient to introduce "neutralized" fields: $\tilde{Y}_I = e^{iq_I\phi/2e} Y_I$, obtained by binding a fraction of the Cooper pair to Y_I. In terms of these neutralized variables:

$$\mathcal{L} = \frac{\rho_b}{2}\left(\partial_\mu \phi\right)^2 + \tilde{\mathcal{L}}[X, \bar{X}, \tilde{Y}_I] \tag{4.7}$$

The advantage of this choice of variables is now manifest, as the Cooper-pair phase ϕ is no longer directly coupled to the neutralized fields $\tilde{Y}_I$. The $\tilde{Y}_I$ however now acquire a phase $e^{\pi i q_I}$ on encircling an elementary vorton. Following the standard duality transformation, we can re-write the boson current $j_b^\mu = \rho_b \partial_\mu \phi$ as the flux of a gauge-field α_μ: $j_b^\mu = \frac{\varepsilon^{\mu\nu\lambda}}{2\pi}\partial_\nu \alpha_\lambda$. In the dual theory, the vorton field, denoted by v, is a bosonic field that couples minimally to this gauge field, and in addition has statistical interactions with the $\tilde{Y}$ particles:

$$\begin{aligned}\mathcal{L}_{\text{dual}} =& \frac{1}{8\pi^2 \rho_b}\left(\varepsilon^{\mu\nu\lambda}\partial_\nu \alpha_\lambda\right)^2 + \frac{1}{2}|\left(\partial_\mu - i\alpha_\mu - i a_\mu^I\right) v|^2 \\ &+ V(|v|^2) + \tilde{\mathcal{L}}[X, \bar{X}, \tilde{Y}_I] + \frac{\varepsilon^{\mu\nu\lambda}}{4\pi} a_\mu^I K_{IJ} \partial_\nu a_\lambda^J \\ &+ \ell_J^{(I)} a_\mu^J j_{Y_I}^\mu \end{aligned} \tag{4.8}$$

where the gauge fields, a^I, integer vectors $\ell^{(I)}$, and multi-component Chern-Simons term with K-matrix K_{IJ} capture the mutual statistics between the vortons and the fields Y_I. Here, $j_{Y_I}^\mu$ is the current of the Y_I particles, and $V(|v|^2))$ is a potential for the vorton field.

Now consider the particles $v^2 X, \left(v^\dagger\right)^2 \bar{X}$. These carry vorticity ± 2 and are interchanged under time reversal. These are the relics of a monopole tunneling event in this superconducting state discussed in the main text. Due to the coupling of v to the dual gauge field, α_I, we may always choose a gauge such that time reversal is implemented as:

$$\mathcal{T}^{-1} v^2 X \mathcal{T} = \left(v^\dagger\right)^2 \bar{X} \tag{4.9}$$

$$\mathcal{T}^{-1}\left(v^\dagger\right)^2 \bar{X} \mathcal{T} = v^2 X \tag{4.10}$$

We may now condense $v^2 X, \left(v^\dagger\right)^2 \bar{X}$ and preserve time reversal symmetry. The condensation also destroys the superconducting order and produces the desired new topological order T_{final}. Note that the neutralized particles $\tilde{Y}_I$ have no non-trivial mutual statistics with v^2X as the phase around the v^2 exactly cancels the phase around X. Hence they survive in T_{final} as quasiparticles. The vortex condensate however quantizes electric charge to be an integer. In particular the charge q bosons obtained by fractionalizing the Cooper pair $b_q = e^{\frac{iq\phi}{2}}$ are confined unless q is an integer. In effect the original electrically charged Y_I particles are confined to the fractional bosons to produce the neutral $\tilde{Y}_I$ particles. The vortons v also survive as particles in final but they are electrically neutral.

The detour through the superconductor essentially implements a 'charge-anyon' separation of the original topological theory T_{initial}. This is completely analogous to the conceptual utility of superconducting degrees of freedom in implementing 'spin-charge' separation in $2d$ insulators[80]. Though we will not elaborate this here an alternately route from T_{initial} to T_{final} is through a parton construction where we fractionalize the charged anyons into a charged boson and a neutral anyon.

This proves that T_{final} only has integer charged quasi-particles. Without loss of generality, we may relabel the quasi-particle content of T_{final} by binding an appropriate number of electrons to each quasi-particle to remove the remaining integer charge. The resulting theory has quasi-particle content $\{1, v, \tilde{Y}_I\} \times \{1, c\}$, that can be decomposed into the direct product of a neutral boson sector $\{1, v, \tilde{Y}_I\}$ trivially accompanied by a gapped electron. This completes the desired proof that the $\theta = 0$ classification reduces to the classification of neutral bosonic phases.

4.4 Topological insulators at $\theta = 0$ - fermionic monopoles?

The possibility that the monopole may be fermionic in a system which also has fermionic charges is naively consistent with time-reversal symmetry. However we can show that such a state cannot occur in any electronic $3D$ SPT phase. Crucial to our argument is the requirement of 'edgability' defined in Ref. [1]. Any theory that can occur in strictly d-dimensions (as opposed to the surface of an SPT in $(d+1)$

dimensions) must admit a physical edge to the vacuum. We show that electronic systems with a fermionic monopole are not edgable.

To illustrate the difficulty consider a Bose-Fermi mixture, with both the boson b and the electron c carrying charge-1. Now put the electron into a trivial band insulator, and the boson into a bosonic SPT state. Then the charge-neutral external monopole source becomes a fermion[1, 69]. We may attempt to get rid of the bosons in the bulk by taking their charge gap to infinity (*i.e* projecting them out of the Hilbert space). However they will make their presence felt at the boundary and the theory is not edgable as a purely electronic system. Indeed we show in Sec. 4.4.1 by a direct and general argument that fermionic statistics of the monopole in an SPT phase implies the existence of physical charge-1 bosons at the boundary. This is not possible in a purely electronic system.

4.4.1 Impossibility of a Fermionic Monopole

In this section we provide a general argument against the possibility of fermionic monopoles in a purely electronic SPT insulator. We will show that fermionic monopoles in the bulk necessarily leads to inconsistencies in the boundary theory, as long as the charge $U(1)$ symmetry is preserved. When the charge $U(1)$ is gauged, apart from monopoles we may also consider in the bulk dyons parametrized by (q_m, q_e) where q_e is the electric charge and q_m the magnetic charge. If the neutral monopole $(1, 0)$ is fermionic in a purely electronic system (where the $(0, 1)$ particle is identified with the electron) all dyons with $q_m = 1$ are also fermions. If time reversal is broken in the bulk the θ value may change from 0 leading to these dyons acquiring non-zero charge. However their statistics stays fermionic. It follows that if any putative time reversal symmetric electronic topological insulator phase with a fermionic monopole exists then it will stay a non-trivial topological insulator even in the absence of time reversal symmetry. Thus it suffices to show that fermionic monopoles are forbidden in the absence of time reversal symmetry to rule out such putative topological insulators.

We will show that SPT states of electrons with a global $U(1)$ symmetry admit unphysical boundary excitations if the monopole is fermionic. Suppose we could construct a state with fermionic monopoles. By the arguments of the previous section, we may describe this phase in terms of the surface topological order with particle content:

$$\{1, c, f, , Y_1, Y_2, \cdots\} \tag{4.11}$$

Here, f is the surface excitation corresponding to the bulk monopole, and hence is a neutral fermion having mutual statistics $e^{-2\pi i q_I/e}$, with particles Y_I of charge q_I. (Even if time reversal is not present we imagine tuning to a point where the monopole is neutral).

Following an analogous line of reasoning in Sec. 4.3, we can now pair condense the remnant of the fermionic monopole $\langle ff\rangle \neq 0$, which immediately confines all the fractionally charged particles Y_I unless $q_I = ne/2$ for some integer n, due to their mutual statistics with f. By attaching enough physical electrons (c), we can always take the charge of the particles Y_I to be either 0 or $e/2$. The resulting theory can thus be written as:

$$\{1, c, f, C_I, N_I\} \tag{4.12}$$

where C_I have charge $e/2$, and N_I are neutral quasi-particles. Note that f is local with respect to N_I and is a mutual semion with C_I.

The neutral sector of the theory $\{1, f, N_I\}$ is closed under fusion and braiding due to charge conservation. Moreover they form a consistent topological field theory. To see this, let us momentarily dispense also with charge-conservation symmetry (for example by explicitly breaking it), and then condense $\langle cf\rangle \neq 0$, which confines all 1/2-charged particles C_I while keeping all the neutral particles N_I unaffected. Furthermore, as f is local with respect to all the N_I's, the theory $\{1, f, N_I\}$ can be viewed as a topological field theory of a system with physical fermion f in the *absence* of any symmetry. Such a theory can then be confined to $\{1, f\}$ without obstruction.

Returning to the original theory in Eqn 4.12 this implies that we may get rid of the neutral particles N_I and be left with

$$\{1, c, f, C_i\}, \tag{4.13}$$

where $\{C_i\}$ is a subset of the original charge-$e/2$ particles $\{C_I\}$.

Without loss of generality, we can restrict our attention to a single species of fractional charge particle C_1, and its anti-particle. The only possible fusion outcomes consistent with charge conservation are: $C_1 \times C_1 \in \{c, cf\}$. If two copies of C_1 fuse to c then $c^\dagger C_1$ is the anti-particle of C_1. However, this is not possible, since the topological spin (self-statistics) of $c^\dagger C_1$ and C_1 differs by -1, whereas anti-particles must have the same topological spin. A similar argument rules out the possibility that two copies of C_1 fuses to cf.

This line of reasoning shows that the topological order of Eq. 4.13 is internally inconsistent, unless there are no C_i particles, i.e. unless the topological order contains only the following particles:

$$\{1, c, f\}. \tag{4.14}$$

Since f has trivial mutual statistics with c, it must be a physical object that is microscopically present in the system (i.e. is not an emergent particle). However, there is no such neutral fermion degree of freedom in an electronic system. It follows that in a purely electronic system the monopole cannot be fermionic in an SPT phase with global $U(1)$ symmetry.

We note that the Bose-Fermi example constructed at the beginning of Sec. 4.4 has a neutral fermion excitation (a bound state of the boson and fermion) and hence is allowed to have a fermionic monopole. Let us examine this more closely. We put the electron into a trivial band insulator, and the boson into a boson topological insulator. Then the charge-neutral external monopole source becomes a fermion[1, 69]. We initially consider such a system in a geometry with no boundaries. We then tune the boson charge gap to infinity, so that the charged bosons disappear from the spectrum, and we are left with a purely electronic theory. But since the fermionic monopole does not carry any boson charge, it survives as the only charge-neutral monopole. Now the bulk theory is exactly what we were looking for, but we need to examine its boundary and see if it is consistent with a time-reversal invariant electronic system.

As the electrons are in a trivial insulator they do not contribute anything special on the boundary, so we only have to worry about surface states of the $eCmC$ boson SPT. We first consider a symmetric surface state with topological order. It is known[43] that one of the possible surface states of the bosonic TI is described by a $\mathbb{Z}_2$ gauge theory with both e and m carrying charge-1/2 and the ϵ fermion being charge-neutral (the state denoted $eCmC$ in Ref. [1]). By setting the boson charge-gap to infinity, the e and m particles disappear from the spectrum, but the neutral ϵ fermion survives as a gauge-invariant local object, which is not allowed in a system purely made of charged fermions. Another way to see the inconsistency of the surface is to look at the surface state without topological order in which time-reversal symmetry is broken. The boson topological insulator leads to a surface electrical quantum hall conductance $\sigma_{xy} = \pm 1$ and thermal hall conductance $\kappa_{xy} = 0$.[43] The difference of σ_{xy}, κ_{xy} between the two time-reversal broken states should correspond to an electronic state in two dimensions without topological order. Here we have $\Delta\sigma_{xy} = 2$ and $\Delta\kappa_{xy} = 0$, which cannot be realized from a purely electronic system without topological order. Indeed adding integer quantum Hall states of electrons increases σ_{xy}, κ_{xy} by the same amount. It is possible to add a neutral boson integer quantum Hall state without topological order but that requires $\sigma_{xy=0}, \kappa_{xy} = 0(mod8)$. Hence the boundary as a purely electronic theory is not consistent with time-reversal symmetry, and the bulk theory cannot be realized in strict three dimensions, although it may be realizable at the surface of a four dimensional system. We also note that if we allow topological (or other exotic long range entanglement) in the bulk then the monopole may be fermionic.

4.5 Physical characterization of interacting topological insulators

We now describe phenomena which in principle can be used to completely experimentally identify the various TIs. We consider breaking symmetry at the surface to obtain states with no intrinsic topological order. The results are summarized in Table.4.1. A different, less practical, but conceptually powerful characterization is in terms of a gapped topologically ordered surface state which we describe in Sec. 4.5.1.

First consider surface states breaking time-reversal symmetry (and no intrinsic topological order). Of the 8 insulating phases we obtained, four have electromagnetic response $\theta = \pi$ (of which one is the topological band insulator) and four have $\theta = 0$ (of which one is the trivial insulator). The θ term in the response means that such a surface state will have quantized electrical Hall conductivity $\frac{e^2}{h}\nu$ with $\nu = \frac{\theta}{2\pi} + n$, where n can be any integer signifying conventional integer quantum hall effect on the surface. A further distinction is obtained by considering the thermal Hall effect κ_{xy} in this surface state. In general in a quantum Hall state $\kappa_{xy} = \nu_Q \frac{\pi^2}{3}\frac{k_B^2}{h} T$ where k_B, T are Boltzmann's constant and the temperature respectively. The number ν_Q is a universal property of the quantum Hall state.

Two of the $\theta = \pi$ insulators have $\nu_Q = \nu = 1/2 + n$ (including the topological band insulator) while the other two have $\nu_Q = \nu \pm 4$. Similarly two of the $\theta = 0$ insulators (including the trivial one) have $\nu_Q = \nu = n$ while the other two have $\nu_Q = \nu \pm 4$.[2] Thus a combined measurement of electrical and thermal Hall transport when $\mathcal{T}$ is broken at the surface can provide a very useful practical (albeit partial) characterization of these distinct topological insulators.

Next we consider surface superconducting states (again without topological order) obtained by depositing an s-wave superconductor on top. It was noticed in Ref. [110] that the surface of the topological paramagnets I and II become identical to that of a topological superconductor (see also Sec. 4.5.2 for a simpler derivation). The corre-

[2] A note for experts: $\nu_Q - \nu$ is determined only up to 8, so we have $\nu_Q - \nu = 0(\text{mod}8)$ for half of the insulators and $\nu_Q - \nu = 4(\text{mod}8)$ for the other half.

sponding free fermion superconductor has $N = 8(\text{mod}16)$ gapless Majorana cones at the surface protected by time-reversal symmetry. Thus inducing superconductivity on the surface of either Topological Paramagnet I or II leads to 8 gapless Majorana cones which should be observable through photoemission measurements. Taken together with the T-breaking surface transport we have a unique fingerprint for each of the 8 TIs.

4.5.1 Topologically ordered surface states

A powerful and complete characterization of the different three dimensional interacting topological insulators is in terms of a gapped symmetry preserving surface with intrinsic topological order. The physical symmetries are realized in this surface topological order in a manner which cannot be realized in strict two dimensions. The surface topological order of the topological paramagnets was studied in Refs. [43, 1, 44]. The simplest such surface states have $\mathbb{Z}_2$ topological order, with two particles e and m having a mutual π-statistics. The Topological Paramagnet I supports a surface theory in which both e and m particles are Kramers bosons (denoted as $eTmT$), while Topological Paramagnet II has a surface in which both e and m are non-Kramers fermions ($e_f m_f$). The third state, being a composite of the previous two, has e and m both being Kramers fermions ($e_f T m_f T$).

The topological band insulator can also be characterized in terms of its surface topological order. In contrast to the topological paramagnets the surface topological order in this case is non-abelian and such states have recently been studied in Ref. [2, 53, 54, 55]. The resulting state are variants of the familiar Moore-Read state describing the $\nu = 5/2$ quantum hall system, modified to accommodate time-reversal symmetry. In Table 4.2 we list the representative surface topological orders of the three root states described in the main text.

In hindsight, in interacting electron systems the descendants of neutral boson SPT states are naturally expected to arise. However, one could also have naively included the descendants of boson SPT states made out of Cooper pairs (charge-2 objects). The non-trivial boson SPT made out of physical bosons with charge $q = 2$ supports

Topological Insulator	**Representative surface topological order**
Free fermion TI	Variants of Moore-Read state
Topological paramagnet I ($eTmT$)	$\mathbb{Z}_2$ gauge theory with Kramers doublet e and m, $\epsilon = em$ is singlet
Topological paramagnet II ($e_f m_f$)	$\mathbb{Z}_2$ gauge theory with Fermionic e and m, $\epsilon = em$ is also Fermionic

Table 4.2: Brief descriptions of the three fundamental non-trivial topological insulators, with their representative surface topological orders.

a surface theory[43, 1, 69] in which both e and m are non-Kramers bosons carrying charge $q/2 = 1$ (denoted as $eCmC$). However, since we have physical Kramers fermions with charge-1 in the system (the electrons), we can bind them with the e and m particles. This converts them to neutral Kramers fermions, which becomes exactly one of the SPT surface states (e_fTm_fT) of neutral bosons. Hence the SPT state made out of charge-2 bosons does not add any non-trivial fermion topological insulator.

Apart from its conceptual value the study of the surface topological order also provides a very useful theoretical tool to access the topological paramagnets. It allowed Ref. [1] to construct the root states of the two time reversal symmetric topological paramagnets (as well as other bosonic SPT phases) in a system of coupled layers where each layer forms a state that is allowed in strictly $2d$ systems. Ref. [44] used the surface topological order $e_f m_f$ (Topological Paramagnet II) to construct an exactly solvable model. While the constructions of Refs. [43, 1, 44] establish the existence of Topological Paramagnet II it is absent in the cohomology classification of Ref. [16, 17]. Understanding how to generalize the cohomology classification to include this state is a challenge for the future.

4.5.2 Equivalence between $N = 8$ Majorana cones and the $eTmT$ topological order

In this section, we provide a physical construction of the $eTmT$ topological order from the $N = 8$ Majorana-cone surface state of a time-reversal invariant topological superconductor phase. We start from the free theory

$$\mathcal{L}_{free} = \sum_{i=1}^{8} \chi_{i,a}^T (p_x \sigma^x + p_y \sigma^z)_{a,b} \chi_{i,b} \tag{4.15}$$

where $i \in \{1, \ldots, 4\}$ and $a \in \{\uparrow, \downarrow\}$, and with time reversal acting on the real (Majorana) fermions as

$$\mathcal{T} \chi_i \mathcal{T}^{-1} = i\sigma_{ab}^y \chi_{i,b}. \tag{4.16}$$

We can group the theory into four complex (Dirac) fermions by writing

$$\psi_{i,a} = \chi_{2i,a} + i\chi_{2i-1,a}, \tag{4.17}$$

the Lagrangian then simply describes four gapless Dirac cones

$$\mathcal{L}_{free} = \sum_{i=1}^{4} \psi_i^\dagger (p_x \sigma^x + p_y \sigma^z) \psi_i, \tag{4.18}$$

in which time-reversal acts as

$$\mathcal{T} \psi_i \mathcal{T}^{-1} = i\sigma_y \psi_i^\dagger. \tag{4.19}$$

It is easy to see that the theory is protected from gap-opening at the free (quadratic) level. We can then ask, could a non-perturbative gap be opened when interaction is introduced? The way to tackle this problem is to first introduce a symmetry-breaking mass term into the fermion theory, viewing the mass term as an fluctuating order parameter, and ask if one can recover the symmetry by disordering the phase of the mass field.

For this purpose it is convenient to first introduce an auxiliary global $U(1)$ symmetry

$$U_\theta \psi U_\theta^{-1} = e^{i\theta}\psi \tag{4.20}$$

as a microscopic symmetry in the model (rather than a subgroup of the emergent $SO(8)$ flavor symmetry). This auxiliary symmetry will be removed at the end of the argument, so the final result does not depend on the existence of this $U(1)$ symmetry.

The total symmetry is now enlarged to $U(1)\times\mathcal{T}$, with $U_\theta\mathcal{T} = \mathcal{T}U_\theta$ (i.e. the conserved quantity associated with the auxiliary $U(1)$ symmetry changes sign under $\mathcal{T}$ like a component of spin rather than an electrical charge). One can now write down a pairing-gap term into the theory

$$\mathcal{L}_{gap} = i\Delta\sum_{i=1}^{4}\psi_i\sigma_y\psi_i + h.c. \tag{4.21}$$

which breaks both $U(1)$ and $\mathcal{T}$ ($\Delta\to-\Delta^*$ under time-reversal because $\mathcal{T}^2=-1$ on physical fermions). The task for us now is then to disorder the field Δ and restore time-reversal symmetry. The virtue of the auxiliary $U(1)$ symmetry shows up here: the field Δ is XY-like, so to disorder it we can follow the familiar and well-understood route of proliferating vortices of the order parameter

It is important here to notice that although the gap in Eq. (6.9) breaks both $U(1)$ and $\mathcal{T}$, it does preserve a time-reversal-like subgroup generated by $\tilde{\mathcal{T}} = \mathcal{T}U_{\pi/2}$. Since we want to restore $\mathcal{T}$ by disordering Δ (which surely will restore $U(1)$), we must do it while preserving $\tilde{\mathcal{T}}$. This "modified time-reversal" looks almost like the original one, but there is a crucial difference: $\tilde{\mathcal{T}}^2=1$ when acting on the fermion field ψ.

Now we are ready to disorder the field Δ. At first glance it seems sufficient just to proliferate the fundamental vortex ($hc/2e$-vortex) and obtain a trivial gapped insulator. However, as we will see below, $\tilde{\mathcal{T}}^2=-1$ on these fundamental vortices, hence proliferating them could not restore time-reversal symmetry.

The vortex here is subtle because of the fermion zero-modes associated with it. It is well-known that a superconducting Dirac cone gives a Majorana zero-mode in the vortex core[102]. So the four Dirac cones in total gives two complex fermion zero-modes $f_{1,2}$. We then define different vortex operators as

$$v_{nm}|GS\rangle = \left(f_1^\dagger\right)^n \left(f_2^\dagger\right)^m |FN\rangle, \tag{4.22}$$

where $|FN\rangle$ denotes the state with all the negative-energy levels filled in a vortex background. The $U(1)$ being spin-like under $\mathcal{T}$ (hence $\tilde{\mathcal{T}}$ also) means that a vortex configuration is time-reversal invariant. The only non-trivial action of $\tilde{\mathcal{T}}$ is thus on the zero-modes:

$$\tilde{\mathcal{T}} f_{1,2} \tilde{\mathcal{T}}^{-1} = f_{1,2}^\dagger, \tag{4.23}$$

and by choosing a proper phase definition:

$$\tilde{\mathcal{T}}|FN\rangle = f_1^\dagger f_2^\dagger |FN\rangle. \tag{4.24}$$

It then follows straightforwardly that $\{v_{00}, v_{11}\}$ and $\{v_{01}, v_{10}\}$ form two "Kramers" pairs under $\tilde{\mathcal{T}}$. Moreover, since the two pairs carry opposite fermion parity, they actually see each other as mutual semions.

We thus conclude that to preserve the symmetry, the "minimal" construction is to proliferate double vortices. The resulting insulating state has $\mathbb{Z}_2$ topological order $\{1, e, m, \epsilon\}$ with the e being the remnant of $\{v_{00}, v_{11}\}$, m being the remnant of $\{v_{01}, v_{10}\}$, and ϵ is the neutralized fermion $\tilde{\psi}$.

Now the full $U(1) \times \mathcal{T}$ is restored, we can ask how are they implemented on $\{1, e, m, \epsilon\}$. Obviously these particles are charge-neutral, so the question is then about the implementation of $\mathcal{T}$ alone. However, since the particles are neutral the extra auxiliary $U(1)$ rotation in $\tilde{\mathcal{T}}$ is irrelevant and they transform identically under $\tilde{\mathcal{T}}$ and $\mathcal{T}$. Hence we have $\mathcal{T}^2 = \tilde{\mathcal{T}}^2 = -1$ on e and m, and $\mathcal{T}^2 = \tilde{\mathcal{T}}^2 = 1$ on ϵ, which is exactly the topological order $eTmT$. The charged physical fermion ψ is now trivially gapped and plays no role in the topological theory, one can thus introduce explicit pairing to break the auxiliary $U(1)$ symmetry. Since topological order stems from the charge-neutral sector, pair-condensation of ψ does not alter the topological order, and the resulting state is just the $eTmT$ state with only $\mathcal{T}$ symmetry.

4.6 Other symmetries, Kramers fermions, and $\theta = \pi$ topological insulators

As a by-product of our considerations we are able to address a number of other fundamental questions about interacting topological insulators. For the free fermion systems the Kramers structure is what allows a topological insulator with $\theta = \pi$. What precise role, beyond free fermion band theory, does the Kramers structure of the electron play in enabling $\theta = \pi$? We show non-perturbatively that any gapped insulator with a $\theta = \pi$ response and no intrinsic topological order necessarily has charge carriers that are Kramers doublet fermions. We also use a similar insight to show the necessity of magnetic ordering when the exotic bulk excitations of the topological Mott insulator phase of Ref. [65] are confined. Finally we show that time reversal breaking electronic systems with global charge $U(1)$ symmetry have no interacting topological insulator phase in three dimensions. These results are described in Sec. 4.6.1 and Sec. 4.6.2.

4.6.1 Spinless fermions and other symmetries

We first provide the proof that a $\theta = \pi$ electromagnetic response in a time reversal invariant insulator implies that the charge carriers are Kramers fermions.

When the global $U(1)$ symmetry is gauged, $\theta = \pi$ implies that the monopoles of the resulting $U(1)$ gauge field are 'dyons' (in the Witten sense) with electric charge shifted from integer by $\frac{1}{2}$. Label particles by (q_m, q_e), where q_m is the magnetic charge (monopole strength) and q_e is the electric charge. A strength-1 monopole (dyon) carries charge-1/2, labeled as $(1, 1/2)$, which under time-reversal transforms to the $(-1, 1/2)$ dyon, since electric charge is even while magnetic charge is odd under time-reversal.

Introduce fields d_{q_m,q_e} for dyons with magnetic charge q_m and electric charge q_e.

Under time reversal these transform as

$$\begin{aligned} \mathcal{T}^{-1} d_{(1,1/2)} \mathcal{T} &= e^{i\alpha} d_{(-1,1/2)} \\ \mathcal{T}^{-1} d_{(-1,1/2)} \mathcal{T} &= e^{i\beta} d_{(1,1/2)} \end{aligned} \tag{4.25}$$

where $d_{(q_m,q_e)}$ denotes the corresponding dyon operator. We then have for $\mathcal{T}^2$

$$\begin{aligned} \mathcal{T}^{-2} d_{(1,1/2)} \mathcal{T}^2 &= e^{i(\beta-\alpha)} d_{(1,1/2)} \\ \mathcal{T}^{-2} d_{(-1,1/2)} \mathcal{T}^2 &= e^{i(\alpha-\beta)} d_{(-1,1/2)} \end{aligned} \tag{4.26}$$

The exact value of the phase factor $e^{i(\alpha-\beta)}$ is not meaningful since it is not gauge-invariant.

Now let's consider the bound state of $d_{(1,1/2)}$ and $d_{(-1,1/2)}$, it has $q_m = 0$ and $q_e = 1$, which is nothing but the fundamental charge of the system. These two dyons see each other as an effective monopole. To see this view the $(-1, 1/2)$ dyon as the bound state of the electric charge $(0, 1)$ and $(-1, -1/2)$ which is the anti-particle of $(1, 1/2)$. The Berry phase seen by the $(-1, 1/2)$ dyon is the same as that seen by a charge from a monopole. Hence their bound state will carry half-integer orbital angular momentum and fermionic statistics The angular momentum of the gauge field[68] in this bound state is given by

$$L = \frac{q_{e,1} q_{m,2} - q_{e,2} q_{m,1}}{2} = 1/2. \tag{4.27}$$

The half integer angular momentum goes hand in hand with fermi statistics of the bound state. To determine whether or not the fermion is a Kramers doublet, we need to consider contributions from the internal and orbital degrees of freedom separately. The internal contribution follows readily from Eq. (4.26), which contributes to $\mathcal{T}^2$ by $e^{i(\beta-\alpha)} e^{i(\alpha-\beta)} = 1$. The orbital part contributes to $\mathcal{T}^2$ by -1 due to the half-integer angular momentum. More precisely, since time-reversal exchanges the two dyons, it is generated by a π-rotation along a great circle, hence $\mathcal{T}^2$ is generated by a 2π-rotation along a great circle, which picks up a Berry phase of π due to the mutual-monopole structure of the two dyons. Therefore we conclude that the fundamental charge must be a Kramers fermion, and there's no fermion SPT with $\theta = \pi$ made out of non-Kramers fermions. We emphasize that this argument is non-perturbative, and does not rely on results from free fermion theories.[3]

In the absence of the $\theta = \pi$ TI for non-Kramers fermions ($T^2 = 1$) what are the possible TIs? The arguments advanced earlier go through as before and we again inherit the boson SPTs with symmetry $\mathbb{Z}_2^T$. However, there is a subtle phenomenon that is unique to the non-Kramers fermions: the $eTmT$ topological paramagnet actually becomes trivial when non-Kramers fermions (even though charged) are present in the system. The argument is simple: one can combine the non-Kramers fermion with the e and ϵ particle in the $eTmT$ topological order. This is essentially a relabeling of the same phase. The resulting topological order is $eCmT$, which means the e particle has charge-1 but is non-Kramers, while the m particle is Kramers but charge-neutral. But as discussed in Chapter 2, this topological order is realizable even in strictly two dimensional systems. Hence it is anomaly-free. One way to realize this state is to start from the $eCT\epsilon CT$ state, which is anomaly-free since the m particle is trivial, and then put the ϵ particle into a $2D$ topological insulating band. The resulting state is nothing but the $eCmT$.

Therefore for non-Kramers fermion, the classification should be $\mathbb{Z}_2$, where the only nontrivial state is the $efmf$ topological paramagnet.

Finally we note that the methods of this chapter imply the absence of any topological insulator states of electrons when time reversal is absent (*i.e* when only charge $U(1)$ is present). Progress toward the classification of interacting time reversal symmetric electronic topological superconductors (the charge $U(1)$ is absent) is made in Ref. [110] which proposes a Z_{16} classification.

[3]After this work was completed we learnt of Ref. [53] which also pointed out the relation between Kramers fermion and $\theta = \pi$ TI.

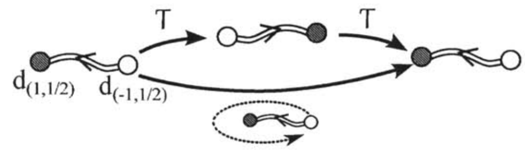

Figure 4-1: For $\theta = \pi$, a monopole and anti-monopole become charge-$\frac{e}{2}$ dyons. Acting twice with $\mathcal{T}$ is equivalent to rotating the pair by 2π, which gives Berry-phase -1 due to the half-angular momentum of the EM field of the dyon-pair.

4.6.2 Implications for topological Mott insulators

Let us now briefly consider the question of confined phases obtained by condensing the dyons of the topological Mott insulator phase[65] whose low energy theory is precisely the gauged TI. Since the $(1, 1/2)$ and $(-1, 1/2)$ dyons see each other as effective monopoles, they cannot condense simultaneously. Condensing one of them should confine the other, just as condensing monopoles will confine electric charges. Since time-reversal relates these two dyons, this implies that the dyons cannot condense (hence confine the gauge theory) without breaking time-reversal symmetry. That the condensation of either of the $(1, \pm 1/2)$ dyons breaks T-reversal was previously pointed[111]. Here we see that it is not possible to simultaneously condense both dyons. Thus the confined phase obtained from the topological Mott insulator necessarily breaks T-reversal and hence is an antiferromagnet.

Chapter 5

Interacting fermionic topological insulators/superconductors in three dimensions

5.1 Introduction

In contrast to bosonic systems, our understanding of fermionic SPT phases beyond band theory is rather limited, particularly in the physically important case of three space dimensions. In dimension $d = 2$ however a simpler Chern-Simons approach provides many definitive results for fermionic SPT states[48]. The effect of strong interaction was also examined for certain kinds of $2D$ fermionic SPT states described by band theory[58, 59, 60, 61], where it was found that some of the topological bands became trivial in the presence of strong interactions. These approaches, however, are difficult to generalize to higher dimensions.

In Chapter 4 (Ref. [3]) we classified and described the physical properties of such interacting three dimensional electronic topological insulators in the physically important situation where both charge conservation and time reversal symmetries are present. The $\mathbb{Z}_2$ classification of such insulators within band theory was shown to be modified to a $\mathbb{Z}_2^3$ classification in interacting systems, resulting in a total of 8 distinct phases. These are generated by 3 'root' states of which one is the topological band insulator and the other two are Mott insulators where the spins form a spin-SPT phase (various models of such "topological paramagnets" were discussed in Ref. [16, 17, 1, 44]). The physics-based methods used in Ref. [3] enabled us to obtain a very clear picture of the physical properties of the various states and determine their experimental fingerprints. It was also shown there that insulators without time-reversal symmetry ($U(1)$ only) have no non-trivial SPT phase.

In this chapter we generalize the ideas of Ref. [3] to discuss $3d$ electronic topological insulators/superconductors with many other symmetries. Free fermion topological phases with various symmetries fall into one of 10 distinct classes. This is known as the 10-fold way[21, 22]. With interactions there is no guarantee that systems with two different symmetries that fall in the same class in the 10-fold way still have the same possible SPT phases. Therefore it is important to specify the symmetry group directly. For symmetry groups represented in each of the famous 10-fold way we are able to ascertain the stability of the free fermion classification to interactions. If the symmetry group has a normal $U(1)$ subgroup we obtain a complete classification of the interacting electronic SPT states. The results are summarized in Table. 5.1.

For time reversal invariant superconductors in three dimensions (class $DIII$) a recent paper[110] showed that the Z classification of band theory reduces to a Z_{16} classification with interactions. For this symmetry we provide a simpler derivation of the same result. For other symmetry classes our results have not been described in the literature as far as we know.

5.2 Generalities

It is useful to first describe a few general ideas that will form the basis of the physical arguments used to establish our results.

Symmetry class	Reduction of free fermion states	Distinct boson SPT	Complete classification
$U(1)$ only (A)	0	0	0
$U(1) \rtimes \mathbb{Z}_2^T$ with $\mathcal{T}^2 = -1$ (AII)	$\mathbb{Z}_2 \to \mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$
$U(1) \rtimes \mathbb{Z}_2^T$ with $\mathcal{T}^2 = 1$ (AI)	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$U(1) \times \mathbb{Z}_2^T$ (AIII)	$\mathbb{Z} \to \mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
$U(1) \rtimes (\mathbb{Z}_2^T \times \mathbb{Z}_2^C)$ (CII)	$\mathbb{Z}_2 \to \mathbb{Z}_2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5$
$(U(1) \rtimes \mathbb{Z}_2^T) \times SU(2)$	0	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$
$\mathbb{Z}_2^T$ with $\mathcal{T}^2 = -1$ (DIII)	$\mathbb{Z} \to \mathbb{Z}_{16}$	0	$\mathbb{Z}_{16}$ (?)
$SU(2) \times \mathbb{Z}_2^T$ (CI)	$\mathbb{Z} \to \mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$ (?)

Table 5.1: Summary of results on classifications of electronic SPT states in three dimensions. The second column gives free fermion states that remain nontrivial after introducing interactions. The third column gives SPT states that are absent in the free fermion picture, but are equivalent to those emerged from bosonic objects such as electron spins and cooper pairs. For symmetries containing a normal $U(1)$ subgroup, we can find the complete classification. In all such examples the complete classifications are simple products of those descending from free fermions and those obtained from bosons. For symmetry class CI, we give suggestive arguments but not a proof that the classification in the last column is complete.

5.2.1 Surface terminations

A crucial property of an SPT phase is the presence of non-trivial surface states protected by the global symmetry. It is thus no surprise that powerful constraints are obtained by thinking about the possible surface terminations of the bulk SPT phase, *i.e* different possible surface phases that correspond to the same bulk phase. The surface either spontaneously breaks the symmetry, or if gapped, has intrinsic topological order. A gapless symmetry preserving surface is also in principle possible. More fundamentally any effective theory for the surface implements symmetry in a manner not possible in a strictly two dimensional theory.

Symmetry broken surface

A particularly useful surface termination is one where the defining global symmetry is either partially or completely broken. In the latter case the surface can be fully gapped without introducing intrinsic topological order. This follows from the assumption that the phase is symmetry protected. The non-triviality of the symmetry broken surface manifests itself in the topological defects of the symmetry breaking order parameter. This ensures that we cannot produce a trivial symmetry preserving surface by proliferating topological defects.

We mention two particularly interesting examples of broken symmetries here. The first one is the breaking of time-reversal symmetry, which can be realized explicitly by depositing a ferromagnet on the surface. Very often (but not always) the domain walls between opposite $\mathcal{T}$-breaking regions hosts chiral modes, which prohibits the domain walls to proliferate and restore $\mathcal{T}$. The chiral modes in the domain wall are related to quantized Hall conductance (say, of charge, spin or heat) in each of the domains. We will discuss Hall transport in more detail in Sec.5.2.2.

The second example is a surface that breaks $U(1)$ symmetry. If the $U(1)$ symmetry corresponds to charge-conservation, this can be realized by depositing a superconductor on the surface. Below we will use the terminology of superconductivity to describe the $U(1)$ symmetry breaking more generally (even if the $U(1)$ symmetry does not actually correspond to charge conservation). It is well known that the $U(1)$ symmetry can be restored by proliferating (condensing) vortices. Therefore if the "superconductor" is gapped (and has no intrinsic topological order), the fundamental ($hc/2e$) vortex must be non-trivial. Otherwise it can be proliferated to restore a trivial insulator on the surface. However, there always exist some higher vortices that are trivial in terms of statistics and symmetry representation, and thus can be condensed. In this case a topologically ordered surface arises, which will be discussed further in Sec.5.2.1 and throughout the chapter.

Symmetry preserving surface topological order

Powerful insights into the SPT phase are provided by a surface termination which is fully gapped and preserves the symmetry at the price of having intrinsic topological order just at the surface. This was first demonstrated in the context of bosonic SPT phases[43]. Conceptually such a topologically ordered surface state provides a nice and non-perturbative characterization of the bulk SPT order[43, 1, 69, 44, 110, 2, 53, 54, 55]. We point out here that it is not always guaranteed that such a symmetry preserving surface topological ordered phase will exist. Indeed later in the chapter we will discuss an example where a symmetry preserving surface is necessarily gapless. When a symmetric surface topologically ordered state exists, it too must realize symmetry in a manner forbidden in strictly two dimensional systems.

5.2.2 Gauging the symmetry: θ terms

Another useful theoretical device is to formally gauge all or part of the defining global symmetry to produce a new physical system. This can be done for all unitary symmetries or for unitary subgroups of the full symmetry group. Two cases will be of particular interest to us. In the first case the full symmetry group G has a normal $U(1)$ subgroup which we can then consistently gauge while retaining the quotient group $G/U(1)$ as an unbroken global symmetry. In the second case the continuous part of the full global symmetry is $SU(2)$. In this case there is no normal $U(1)$ subgroup and instead we gauge the full continuous $SU(2)$ symmetry.

Let us first discuss the case where there is a normal $U(1)$ subgroup to which we couple a gauge field. As the bulk is gapped we may formally integrate out the electrons and obtain an effective long wavelength Lagrangian for the gauge field.

$$\mathcal{L}_{eff} = \mathcal{L}_{Max} + \mathcal{L}_{\theta} \tag{5.1}$$

The first term is the usual Maxwell term and the second is the 'theta' term:

$$\mathcal{L}_{\theta} = \frac{\theta}{4\pi^2}\mathbf{E}\cdot\mathbf{B} \tag{5.2}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields respectively of the $U(1)$ gauge field. The allowed value of θ will be constrained by the unbroken global symmetry $G/U(1)$. In the familiar example of the topological band insulator (with $G = U(1) \rtimes Z_2^T$, where Z_2^T is time reversal), it is well known that $\theta = 0$ or π by time reversal symmetry. This is true as well for other symmetry groups in Table. 5.1 that include time reversal. The $\mathbf{E}$ and $\mathbf{B}$ fields transform oppositely under Z_2^T so that $\theta \to -\theta$. Further on a closed manifold there is periodicity under $\theta \to \theta + 2\pi$ so that the only distinct possibilities are $\theta = 0, \pi$.

The θ term provides very useful constraints on the surface physics. It can be written as the derivative of a Chern-Simons term. Hence at a surface where the Z_2^T symmetry is broken, it leads to a Hall conductivity (associated with transport of the $U(1)$ charge) of

$$\sigma_{xy} \equiv \nu = \frac{\theta}{2\pi} \tag{5.3}$$

(We use units in which the $U(1)$ charge of the fermions is 1 and $\hbar = 1$). Furthermore in all the examples studied in this chapter, such a Z_2^T broken surface termination exists with a gap and without any surface topological order. In that case we can safely say that when ν is fractional , the surface state cannot exist in strictly two dimensional systems, and requires the $3d$ bulk. Thus fractional $\nu = \frac{\theta}{2\pi}$ for the response to a $U(1)$ gauge field implies non-trivial bulk SPT order even in the presence of interactions.

Returning to the Z_2^T broken gapped surface without any topological order, we can further argue that the *difference* in the Hall conductivity of the surface and its time reverse must be a state allowed in $2d$ systems of electrons without topological order. This forces $\frac{\theta}{\pi} = n$ with n an integer.

Another important characterization of such a Z_2^T broken surface state is the thermal Hall conductivity κ_{xy}. Formally this is related to gravitational responses in the bulk and the notion of gravitational anomaly[112] though we will not need to use such a description. For any gapped two dimensional state $\nu_Q = \frac{\kappa_{xy}}{\kappa_0}$ is a universal number with $\kappa_0 = \frac{\pi^2}{3}\frac{k_B^2}{h}T$ (T is the temperature). For a *strictly* two dimensional system if further there is no topological order then $\nu_Q - \nu = 0 \pmod 8$ (see Sec. 5.2.2). Thus a gapped Z_2^T broken surface that either has fractional ν or has $\nu_Q - \nu \neq 0 \pmod 8$ implies non-trivial bulk SPT order even with interactions.

In the case where the continuous symmetry is $SU(2)$ (e.g., associated with spin conservation) we can again gauge this $SU(2)$ symmetry and study the effective Lagrangian of the corresponding matrix-valued $SU(2)$ gauge fields A_μ which again takes the form

$$\mathcal{L}_{eff} = \mathcal{L}_{Max} + \mathcal{L}_\theta \tag{5.4}$$

The first Maxwell term is the usual Lagrangian for the $SU(2)$ gauge field (g is a non-universal coupling constant).

$$\mathcal{L}_{Max} = \frac{1}{2g} Tr\left(F_{\mu\nu}F_{\mu\nu}\right) \tag{5.5}$$

The field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The second 'theta' term takes the form

$$\mathcal{L}_\theta = \frac{\theta}{32\pi^2} Tr\left(\epsilon_{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right) \tag{5.6}$$

On a closed manifold there is periodicity under $\theta \to \theta + 2\pi$. Time reversal if present takes $\theta \to -\theta$, and thus the potentially time reversal invariant possibilities are $\theta = n\pi$. This θ term can once again be written as the derivative of a Chern-Simons term for the $SU(2)$ gauge field:

$$\mathcal{L}_\theta = \frac{\theta}{8\pi^2}\partial_\mu K_\mu \tag{5.7}$$

where

$$K_\mu = \epsilon_{\mu\nu\alpha\beta} Tr\left(A_\nu \partial_\alpha A_\beta + \frac{2}{3}A_\nu A_\alpha A_\beta\right) \tag{5.8}$$

Similarly to the discussion of the $U(1)$ case above this implies that a Z_2^T broken gapped surface without topological order will have a *spin quantum Hall effect* [113]. This is characterized by the spin current induced in the transverse direction in response to a spatially varying Zeeman field. (The spin quantum Hall effect should not be confused with the quantum spin Hall effect - the latter describes the transverse spin current induced by an electrical voltage). The corresponding spin Hall conductivity $\sigma^s_{xy} = \frac{\theta}{\pi}$. Note the factor of 2 difference between the corresponding formula for the $U(1)$ case. In a strictly $2d$ system we must have $\sigma^s_{xy} = 2n$ with n an integer. Therefore an odd $\frac{\theta}{\pi}$ implies bulk SPT order even with interactions.

In both $U(1)$ and $SU(2)$ cases, if the θ term is such that the Z_2^T broken surface state has Hall transport that is allowed in $2d$ we cannot directly conclude anything about whether an SPT state exists or not. In the following subsection we obtain some additional constraints in the $U(1)$ case by thinking about monopole defects of the gauge field.

Electric and thermal hall conductance mismatch

Here we discuss the constraints on quantum hall and thermal hall effect in a two-dimensional charged fermion system in the absence of intrinsic topological order and fractionalization. It is well known that in such cases the electric hall conductance σ_{xy} is quantized in unit of $\sigma_0 = e^2/h$, and the thermal hall conductance κ_{xy} is also quantized in units of $\kappa_0 = \frac{\pi^2}{3}\frac{k_B^2}{h}T$. For free fermions, the two should agree $\sigma_{xy}/\sigma_0 = \kappa_{xy}/\kappa_0 = n$ since the fermions transport both electricity and heat.

With interactions, however, the two integers could differ. A simple example[5] is the following: imagine and odd number (say $2n+1$) of fermions form a bound state $F \sim f^{2n+1}$, which is also a fermion but with charge $e^* = 2n+1$. Now put the bound state fermion F into a Chern band with Chern number ν. The quantum hall conductance is then $\nu(e^*)^2 = (2n+1)^2\nu$, but the thermal hall conductance is simply ν since it does not distinguish the charge carried by the fermion. The two quantized quantities thus have a mismatch

$$\begin{aligned} \frac{\sigma_{xy}}{\sigma_0} - \frac{\kappa_{xy}}{\kappa_0} &= ((2n+1)^2 - 1)\nu \\ &= 8\left(\frac{n(n+1)}{2}\nu\right) = 0(\text{mod}8). \end{aligned} \tag{5.9}$$

In general, it can be shown that the identity Eq. (5.9) is true as long as the system does not develop intrinsic topological order (see Chapter 6 or Ref. [5]). We outline part of the proof here: for a system with any σ_{xy} and κ_{xy}, we can stack it with certain integer quantum hall system made of free fermions so that the net σ_{xy} becomes zero. If the remaining thermal hall conductance $\kappa'_{xy}/\kappa_0 = \kappa_{xy}/\kappa_0 - \sigma_{xy}/\sigma_0$ is non-zero, there must be a chiral edge mode that carries no charge (with chiral central charge κ'_{xy}/κ_0). But since the fermions are charged, the neutral charge mode must be bosonic. Hence it can be viewed as a boson state with a chiral edge. It is known that for a boson system with no topological order, $\kappa'_{xy}/\kappa_0 = 0(\text{mod}8)$ (for a proof, see for example Ref. [48]).

5.2.3 Gauging the symmetry: bulk monopoles and surface states

An important lesson from the work in Ref. [47] is that when the global symmetry in an SPT phase is gauged the defects of the gauge field could become nontrivial. Let us now consider the situation discussed above where the global symmetry has a normal $U(1)$ subgroup which we then gauge. Then the gauge defect is simply the magnetic monopole: a 2π-source of the gauge flux. In three dimensions the monopole statistics can only be bosonic or fermionic. It was shown in Ref. [3] that in any (short-range entangled) system where all the charge-1 particles are fermions, the monopole must be a boson. The monopole may then carry non-trivial quantum numbers under the symmetry group.

The 'electric' charge of the monopole under the $U(1)$ symmetry is determined directly by the θ term in the effective gauge action through the well-known Witten effect: there is a a $U(1)$ charge of $\frac{\theta}{2\pi}$ on the monopole. For $\theta = \pi$ this is fractional. The remaining question is about the symmetry transformation of the monopole under the quotient group $G/U(1)$. In particular it will be important to ask of the monopole transforms under a projective representation of this quotient group. In the familiar case of electrons with $U(1) \rtimes Z_2^T$, the symmetry properties of the monopole under Z_2^T are severely constrained[3]. The monopole goes to an antimonopole under Z_2^T and this makes it meaningless to ask about whether time reversal acts projectively on it or not. In the $U(1) \times \mathbb{Z}_2^T$ case studied in detail below, the gauge magnetic flux is even under time-reversal. Hence it is also possible to have monopoles forming non-trivial (projective) representations under time-reversal, *i.e* it could become a Kramers doublet, with $\mathcal{T}^2 = -1$. It is possible to enumerate all possible nontrivial quantum numbers that can be carried by the monopole. For example, with $U(1) \times \mathbb{Z}_2^T$, the monopole can either carry half-integer $U(1)$ charge (corresponding to $\theta = \pi$), or be charge-neutral while having $\mathcal{T}^2 = -1$, or both.

Understanding the allowed structure of the bulk monopole leads to important constraints on the possible surface terminations of the SPT. Such a point of view was nicely elaborated in Ref. [69] to discuss the physics of the bosonic topological insulator. We emphasize that this procedure of gauging the symmetry and studying the monopole is a purely theoretical device. It is however very powerful.

It is convenient for our purposes in this chapter to consider a surface termination which breaks the $U(1)$ symmetry. Imagine tunneling a monopole from the vacuum into the system bulk, which leaves behind a two-fold (hc/e using the 'superconducting' terminology) vortex on the surface. As the monopole is trivial in the vacuum, if it carries nontrivial quantum numbers in the bulk, the corresponding hc/e vortex on the surface must also carry the same nontrivial quantum numbers, and vice versa. Therefore we could either use the known monopole property to infer the properties of the surface 'superconductor' (as was done in Ref. [2]), or use the knowledge of the surface vortex to infer the quantum numbers carried by the bulk monopole (as was done in Ref. [1], and will be done in Sec.5.3 and 5.6 below).

It is important to emphasize that not all the seemingly consistent projective symmetry representations of monopoles can actually be realized. For example, in a spinless fermion system where $\mathcal{T}^2 = 1$ on the fermions, a theory with half-charged monopole ($\theta = \pi$) is naively consistent, even though there is no free fermion band structure that realizes such a theory. One may wonder if there is an intrinsically interacting SPT state with no free fermion realization that gives $\theta = \pi$. However, it was noticed recently[3, 53] that it is internally inconsistent in a subtle way, and hence cannot be realized even with an interacting SPT. Therefore if a symmetry assignment of the monopole is not realized in any free fermion system, one needs to examine its consistency more carefully.

The next question is then, if a symmetry assignment to the monopole is realized by a representative state (for example, a free fermion model), how many other states exist with the same monopole properties? For such a state with certain monopole quantum numbers, there is always a representative state with monopoles carrying the "opposite" quantum numbers, so that stacking the two states together produces another state with monopoles carrying trivial quantum numbers. Therefore the question can be posed equivalently as: how many nontrivial states exist with monopoles being completely trivial? This was analysed in detail in Chapter 4 (Ref. [3]). The conclusion is that if an SPT state has trivial monopole, it must be equivalent to a SPT state constructed from bosonic particles carrying no $U(1)$ charge (e.g. spins in an electron system). For example, if a fermionic SPT state with $U(1) \rtimes \mathbb{Z}_2^T$ symmetry has a trivial monopole, it is equivalent to a bosonic SPT with $\mathbb{Z}_2^T$ symmetry only.

The above conclusion can be summarized compactly as follows:

> If the group G contains a normal $U(1)$ subgroup, then any 3D SPT state with the symmetry group G must either have a nontrivial $U(1)$ monopole (one that transforms projectively under G), or be equivalent to a bosonic SPT with the symmetry $G/U(1)$.

In the rest of the chapter we utilize these different ways of diagnosing and differentiating SPT phases to classify and understand electronic SPT phases with many different symmetries.

5.3 $U(1) \times \mathbb{Z}_2^T$: AIII class

In this section we study fermions with the symmetry group $U(1) \times \mathbb{Z}_2^T$ (the AIII class), which can be interpreted physically as superconductors with S_z-spin conservation and time-reversal symmetry. The $U(1)$ rotation $U(\theta)$ and time-reversal $\mathcal{T}$ commutes: $U(\theta)\mathcal{T} = \mathcal{T}U(\theta)$, or equivalently, the $U(1)$ charge is odd under time-reversal action, unlike the electric charge. Physically, the action of time-reversal on the fermions has two distinct possibilities: $\mathcal{T}^2 c \mathcal{T}^{-2} = \pm c$, where c is the physical fermion annihilation operator. However, the two symmetries lead to very similar physics, including the classification of SPT states. This is because one can always define a new time-reversal-like operation $\tilde{\mathcal{T}} = \mathcal{T}U(\pi/2)$, and it is easy to see that $\tilde{\mathcal{T}}^2 = -\mathcal{T}^2$ on the fermion. Hence the problem with $U(1) \times \mathbb{Z}_2^T$ with $\mathcal{T}^2 = -1$ on the fermion can be mapped to that with the same symmetry group $U(1) \times \mathbb{Z}_2^{\tilde{T}}$ but with $\tilde{\mathcal{T}}^2 = 1$. We will take $\mathcal{T}^2 = -1$ below in order to be able to connect to other interesting symmetry groups, but the modified time-reversal $\tilde{\mathcal{T}}$ will still be useful as a tool in our argument.

The free fermion band theory gives a $\mathbb{Z}$ classification for this symmetry group. Each state is labeled by an integer n signifying the number of protected gapless Dirac cones on the surface:

$$H = \sum_{i=1}^{n} \psi_i^\dagger (p_x \sigma_x + p_y \sigma_z) \psi_i, \tag{5.10}$$

with the symmetries acting as $U(\theta) : \psi \to e^{i\theta}\psi$ and $\mathcal{T} : \psi \to i\sigma_y \psi^\dagger$.

We will show in the following that the $\mathbb{Z}$ classification from band theory reduces to $\mathbb{Z}_8$ in the presence of interaction: the $n = 1$ state has a bulk $\theta = \pi$ term, the $n = 2$ state, which has $\theta = 2\pi$, has a neutral Kramers monopole ($\mathcal{T}^2 = -1$), the $n = 4$ state is equivalent to an SPT state formed by bosons carrying no $U(1)$ charge, hence the $n = 8$ state, formed by taking two copies of the $n = 4$ state, is trivial. Following the arguments in Sec.5.2, we can also show that taking another bosonic SPT (which cannot be realized using free fermions) into account, we obtain the complete classification given by $\mathbb{Z}_8 \times \mathbb{Z}_2$.

5.3.1 8 Dirac cones: triviality

We first look at the $n = 8$ state, which has eight protected surface Dirac cones in the free fermion theory. We will show explicitly that, with interaction, such surface state can open up a gap and become a trivial state. We use an argument very similar to that in Ref. [3].

We first introduce a singlet pairing term into the theory

$$H_\Delta = \sum_{i=1}^{n} i\Delta\psi\sigma_y\psi + h.c., \tag{5.11}$$

which breaks both the $U(1)$ and $\mathcal{T}$ symmetries (under time-reversal we have $\Delta \to -\Delta^*$). The surface theory is now gapped, with the physical symmetries broken. With interactions, the gap Δ becomes a fluctuating field, hence it is possible to disorder it (have $\langle\Delta\rangle = 0$) and restore the symmetries. To disorder the XY-like field Δ, we can follow the familiar and well-understood route of proliferating vortices of the order parameter.

It is important here to notice that although the gap in Eq. (5.11) breaks both $U(1)$ and $\mathcal{T}$, it does preserve a time-reversal-like subgroup generated by $\tilde{\mathcal{T}} = \mathcal{T}U(\pi/2)$. Since we want to restore $\mathcal{T}$ by disordering Δ (which surely will restore $U(1)$), we must do it while preserving $\tilde{\mathcal{T}}$.

The vortex needs to be examined carefully because of the fermion zero-modes associated with it. It is well-known that a superconducting Dirac cone gives a Majorana zero-mode in the vortex core[102]. Under $\tilde{\mathcal{T}}$, the vortex background is invariant (unlike the case of $U(1) \rtimes \mathbb{Z}_2^T$, where a vortex goes to an anti-vortex), and the Majorana zero-modes transform trivially $\gamma_i \to \gamma_i$. At free fermion level the degeneracy from the zero-modes are robust, since any quadratic term $iA_{ij}\gamma_i\gamma_j$ would break $\tilde{\mathcal{T}}$. However, it is known[40] that with $n = 8$ Majorana zero-modes, the degeneracy can be lifted by a quartic term. The remaining vortex is then completely trivial and can thus be condensed without breaking $\tilde{\mathcal{T}}$, producing a trivial insulator on the surface.

We can also examine the time-reversal properties of the vortices more directly, which will be useful in the following sections. We first pair up the 8 Majorana zero-modes to 4 complex fermion zero-modes $f_i = \gamma_{2i-1} + i\gamma_{2i}$. We then define different vortex operators as

$$v_{nmkl}|GS\rangle = \left(f_1^\dagger\right)^n \left(f_2^\dagger\right)^m \left(f_3^\dagger\right)^k \left(f_4^\dagger\right)^l |FN\rangle, \tag{5.12}$$

where $|FN\rangle$ denotes the state with all the negative-energy levels filled in a vortex background. The $U(1)$ being spin-like under $\mathcal{T}$ (hence $\tilde{\mathcal{T}}$ also) means that a vortex configuration is time-reversal invariant. The only non-trivial action of $\tilde{\mathcal{T}}$ is thus on the zero-modes:

$$\tilde{\mathcal{T}} f_i \tilde{\mathcal{T}}^{-1} = f_i^\dagger, \tag{5.13}$$

and by choosing a proper phase definition:

$$\tilde{\mathcal{T}}|FN\rangle = f_1^\dagger f_2^\dagger f_3^\dagger f_4^\dagger |FN\rangle. \tag{5.14}$$

It is then straightforward to check the modified time-reversal $\tilde{\mathcal{T}}$ only relates vortex operators with the same fermion parity $(-1)^{n+m+k+l}$, and $\tilde{\mathcal{T}}^2 = 1$ on all the v_{nmkl} operators. Moreover, vortices with the same fermion parity are mutually local with each other, and thus can be condensed simultaneously while keeping the $\tilde{\mathcal{T}}$ symmetry. The remaining surface is then a trivial gapped symmetric state.

5.3.2 4 Dirac cones: boson SPT

We now look at the state with $n = 4$ Dirac cones on the surface and try to do the same exercise as in Sec.5.3.1. Again we start from a paired gapped state and try to disorder the pairing gap by proliferating vortices. We have now $n = 4$ Majorana zero-modes in the vortex core, and even with interaction the degeneracy cannot be lifted. The two-fold vortex (hc/e), on the other hand, hosts 8 Majorana zero-modes and hence is trivial. Condensing the two-fold vortex will then give a symmetric gapped state, with an intrinsic $\mathbb{Z}_2$ topological order[80], *i.e* described by a deconfined Z_2 gauge theory. To study this $\mathbb{Z}_2$ topologically ordered state, we need to examine the fundamental vortex in the superconducting state with more care.

Again we group the 4 Majorana zero-modes into two complex fermion zero-modes $f_{1,2}$, and define vortices through

$$v_{nm}|GS\rangle = \left(f_1^\dagger\right)^n \left(f_2^\dagger\right)^m |FN\rangle, \tag{5.15}$$

where $|FN\rangle$ denotes the state with all the negative-energy levels filled in a vortex background. The modified time-reversal again acts as

$$\tilde{\mathcal{T}} f_{1,2} \tilde{\mathcal{T}}^{-1} = f_{1,2}^{\dagger}, \tag{5.16}$$

and by choosing a proper phase definition:

$$\tilde{\mathcal{T}} |FN\rangle = f_1^{\dagger} f_2^{\dagger} |FN\rangle. \tag{5.17}$$

It then follows straightforwardly that $\{v_{00}, v_{11}\}$ and $\{v_{01}, v_{10}\}$ form two "Kramers" pairs under $\tilde{\mathcal{T}}$ (namely $\tilde{\mathcal{T}}^2 = -1$). Moreover, since the two pairs carry opposite fermion parity, they actually see each other as mutual semions.

We thus conclude that to preserve the symmetry, the "minimal" construction is to proliferate double vortices. The resulting insulating state has $\mathbb{Z}_2$ topological order $\{1, e, m, \epsilon\}$ with the e being the remnant of $\{v_{00}, v_{11}\}$, m being the remnant of $\{v_{01}, v_{10}\}$, and ϵ is the neutralized fermion $\tilde{\psi}$.

Now the full $U(1) \times \mathcal{T}$ is restored, and we can ask how they are implemented on $\{1, e, m, \epsilon\}$. Obviously these particles are charge-neutral, so the question is then about the implementation of $\mathcal{T}$ alone. However, since the particles are neutral the extra $U(1)$ rotation in $\tilde{\mathcal{T}}$ is irrelevant and they transform identically under $\tilde{\mathcal{T}}$ and $\mathcal{T}$. Hence we have $\mathcal{T}^2 = \tilde{\mathcal{T}}^2 = -1$ on e and m, and $\mathcal{T}^2 = \tilde{\mathcal{T}}^2 = 1$ on ϵ. This state is denoted as $eTmT$ in Ref. [1], and is a characteristic surface state of a bosonic SPT. We thus conclude that the $n = 4$ free fermion state is equivalent to the $eTmT$ bosonic SPT in the presence of interaction.

5.3.3 2 Dirac cones: Kramers monopole

The $n = 2$ state, being a "square root" of the $n = 4$ state which is equivalent to a bosonic SPT, must involve the $U(1)$ symmetry in a non-trivial manner as argued in Sec.5.2. It must thus have monopoles that are non-trivial under the symmetries. It turns out that the charge-neutral monopole behaves as a Kramers pair under time-reversal ($\mathcal{T}^2 = -1$). Such monopole behavior was also realized in a boson SPT state[1] with $U(1) \times \mathbb{Z}_2^T$ symmetry, where charge-1 carriers are bosons instead of fermions. So in contrast with charge-1/2 monopoles ($\theta = \pi$), the Kramers monopole can be realized in two different systems, one with fermionic charge carriers and one with bosonic one.

We show this by studying the monopole tunneling event on the surface: if the monopole has $\mathcal{T}^2 = -1$ inside the insulator bulk and $\mathcal{T}^2 = 1$ in the vacuum outside, the tunneling event on the surface must leave behind another excitation with $\mathcal{T}^2 = -1$. We can work this out directly from the free fermion surface state, by showing that a monopole insertion operator in a $(2+1)-d$ theory with two Dirac cones has $\mathcal{T}^2 = -1$ due to the fermion zero-modes from the Dirac cones. An alternative route, which we will follow, is to study the paired state described in Sec.5.3.1 and 5.3.2, in which a monopole tunneling event leaves behind a two-fold (hc/e) vortex that now traps four Majorana zero-modes. The argument in Sec.5.3.2 immediately shows that this two-fold vortex has $\tilde{\mathcal{T}}^2 = -1$, which means the monopole inside the bulk also has $\tilde{\mathcal{T}}^2 = -1$. But since the monopole is charge-neutral, it has $\mathcal{T}^2 = \tilde{\mathcal{T}}^2 = -1$.

Surface topological order

The $n = 2$ state can also be analyzed in a similar fashion as for $n = 8$ and $n = 4$ states. As we have noticed, the two-fold vortex in the paired state has $\tilde{\mathcal{T}}^2 = -1$ and hence cannot be condensed to restore time-reversal symmetry. Hence the minimal construction is to condense the four-fold vortex, which traps eight Majorana zero-modes and can be trivially condensed. A charge-1/2 boson (denoted as β) emerges from this non-trivial vortex condensate, which under time-reversal goes to $\beta \to \beta^{-1} \sim \beta^3$ (the last identification comes from the topological triviality of β^4). The particle content of the remaining theory can be represented as $\{1, \beta, \beta^2, \beta^3, v, v,^2, v^3, \beta^n v^m\} \times \{1, c\}$, where v is the remnant of the fundamental vortex with the complex fermion zero-mode un-occupied, and c is the physical fermion, while the remnant of the ψ fermion is denoted as $\epsilon = c^\dagger \beta^2$.

The resulting gapped surface state has $\mathbb{Z}_4$ topological order, with the symmetries implemented in a peculiar way. The remnant of the fundamental vortex with the complex fermion zero-mode un-occupied is v, and that with the zero-mode occupied is ϵv. The two go to each other under time-reversal, and their squares (either v^2 or ϵv^2) have $\mathcal{T}^2 = \tilde{\mathcal{T}}^2 = -1$, with ϵ having $\mathcal{T}^2 = 1$. The topological sector of β^2 does not change under time-reversal, but since it carries charge-1, we have $\mathcal{T}^2 = -\tilde{\mathcal{T}}^2 = -1$ for it, which is consistent with $\mathcal{T}^2 = 1$ on ϵ, since $\beta^2\epsilon \sim c$. The charge-vortex relation gives the obvious mutual statistics $\theta_{\beta v} = e^{i\pi/2}$.

5.3.4 1 Dirac cone: $\theta = \pi$

The $n = 1$ state is the $U(1)\times\mathbb{Z}_2^T$ counterpart of the familiar electronic topological band insulator[25]. The surface single Dirac cone implies a θ-term in the gauge response[64] at $\theta = \pi$. The monopole then carries charge-1/2.

Non-Abelian Surface topological order

Following the reasoning from previous sections, we know that in the paired surface state, the four-fold ($2hc/e$) vortex is Kramers under $\tilde{\mathcal{T}}$ and we need to condense eight-fold vortex to recover the full symmetry. A charge-1/4 boson α emerges out of this condensate, and as in the $n = 2$ case, the charge-1 boson α^4 has $\mathcal{T}^2 = -\tilde{\mathcal{T}}^2 = -1$.

The story about lower vortices, however, is made more complicated due to the structure of the zero-modes. In particular, the fundamental vortex carries only one Majorana zero-mode and is thus non-abelian. The detailed analysis of the fusion and statistics of the vortices was carried out in Ref. [2, 53], which showed that the fundamental vortex has topological spin 1 while the two-fold vortices have topological spin $\pm i$, depending on whether the complex fermion zero-mode is filled or not. Fusing the vortex with an ϵ fermion gives back the vortex: $v\times\epsilon \sim v$, while fusing two vortices gives either the semionic or anti-semionic two-fold vortex: $v\times v \sim v^2 + \epsilon v^2$. The mutual statistics between the fundamental and two-fold vortices are $\pm i$, hence the three-fold vortex has topological spin -1. It also follows that the fundamental vortex and the four-fold vortex (which is Kramers) are mutual semions (mutual statistics -1).

Again the particle content can be written as $\{1, \alpha, ...\alpha^7, v, ...v^7, \alpha^n v^m\} \times \{1, c\}$.

Vortex zero-modes	**Properties**
8 Majorana	Trivial
4 Majorana	$\mathcal{T}^2 = -1$
2 Majorana, 2-fold (hc/e) vortex	semion/anti-semion
2 Majorana, fundamental $(hc/2e)$ vortex	bosonic, $\mathcal{T} : v \to \epsilon v$
1 Majorana	Non-abelian

Table 5.2: Summary of vortex properties, according to the number of Majorana zero-modes trapped. Most of the properties do not depend on the vortex strength (as long as the vortex exists), except when there are two Majorana zero-modes. In $n = 1$ phase such a vortex has strength-2 while in $n = 2$ phase it has strength-1, and the vortex statistics turns out to be different in the two cases.

We summarize the properties of different vortices in different states in Table.5.2.

5.3.5 $\mathbb{Z}_8 \times \mathbb{Z}_2$ classification

We have shown that the $\mathbb{Z}$ classification from free fermion band theory reduced to $\mathbb{Z}_8$ under interaction. The argument outlined in Sec.5.2 makes it possible to further classify all the SPT states, including those not realizable using free fermions.

For any putative new SPT phase that cannot be realized using free fermions, there is always a free fermion state such that the combination of the two has a trivial monopole. This is because every possible nontrivial symmetry implementation of the monopole is realized by a free fermion model. Following the reasonings in Sec.5.2, a phase with trivial monopole can at most be a SPT made of charge-neutral bosons (with $\mathbb{Z}_2^T$ symmetry only). Bosonic SPT states with $\mathbb{Z}_2^T$ symmetry in three dimensions are classified by $\mathbb{Z}_2^2$, with two root states[43]. One of the two root states becomes identical to the $n = 4$ free fermion state. Hence it does not give rise to any new state. But the other root state is independent of all the free fermion states. Hence it provides a new state in the full classification. The final result is thus a $\mathbb{Z}_8 \times \mathbb{Z}_2$

classification of three dimensional fermions with $U(1) \times \mathbb{Z}_2^T$ symmetry.

5.4 $\mathbb{Z}_2^T$ with $\mathcal{T}^2 = -1$: DIII Class

In this section we apply the results obtained in Sec.5.3 to superconductors with only time-reversal symmetry (the DIII class). This was recently discussed in Ref. [110] using powerful Walker-Wang methods. We reproduce part of the results there in a physically simpler and constructive approach following the ideas of Ref. [3] and the previous section. A similar argument has also been independently developed in Ref. [114].

At free fermion level, the DIII class superconductors in 3D are classified by $\mathbb{Z}$, with an integer index ν signifying the number of gapless Majorana cones on the surface protected by time-reversal symmetry:

$$H = \sum_{i=1}^{\nu} \chi_i^\dagger (p_x \sigma_x + p_y \sigma_z) \chi_i. \tag{5.18}$$

If ν is even ($\nu = 2n$), one can group the Majorana cones into n Dirac cones $\psi_i = \chi_{2i-1} + i\chi_{2i}$, and the theory looks exactly the same as Eq. (5.10). The $U(1)$ symmetry $\psi \to e^{i\theta}\psi$ is now an emergent symmetry at low energy. We can instead consider the $U(1)$ as a microscopic symmetry, apply the results in Sec.5.3 to obtain interacting gapped surface states, and then break the $U(1)$ symmetry explicitly by adding fermion pairing term. A similar strategy was useful in the Walker-Wang approach[110]. For the $n = 8$ ($\nu = 16$) state, the resulting surface is trivially gapped, and further breaking the $U(1)$ symmetry does not introduce anything nontrivial. Hence the $\mathbb{Z}$ classification from band theory reduces to $\mathbb{Z}_{16}$ with interaction. For the $n = 4$ ($\nu = 8$) state, the resulting surface is topologically ordered, but all the quasi-particles are charge-neutral under the $U(1)$, hence breaking $U(1)$ symmetry does not affect anything either. These establish the $\nu = 16$ state as a trivial one, and the $\nu = 8$ state as equivalent to a boson SPT, which are consistent with the results in Ref. [110]. The $n = 2$ ($\nu = 4$) and $n = 1$ ($\nu = 2$) states, however, have surface topological orders involving the $U(1)$

symmetry non-trivially, hence need more careful examination.

5.4.1 4 Majorana cones: doubled semion-fermion surface state

We now take the surface topological order in Sec.5.3.3, break the $U(1)$ symmetry but keep time-reversal. Notice that $\mathcal{T}^2 = -1$ on both β^2 and v^2, hence the simplest particle to condense is the charge-1 object $v^2\beta^2$. It can be checked straightforwardly that the remaining theory contains the following deconfined particles (and their combinations):

$$\{1, s_1 = v\beta\} \times \{1, s_2 = cv^{-1}\beta\} \times \{1, c\}, \tag{5.19}$$

where c is now the charge-neutral physical fermion. The mutual statistics between β and v in the original theory $\theta_{\beta,v} = i$ makes the composite $s_1 = v\beta$ a semion with self-statistics i, likewise the particle $s_2 = cv^{-1}\beta$ is also a semion. Under time reversal, $s_1 = v\beta \rightarrow (v\epsilon)\beta^{-1} = s_1\epsilon\beta^2 = s_1c$ which is an anti-semion, likewise $s_2 \rightarrow s_2c$ which is again an anti-semion. The two semions s_1, s_2 are local with respect to each other, and their bound state $s_1s_2 = c\beta^2 = \epsilon$ is a fermion with $\mathcal{T}^2 = 1$. These are in agreement with the result in Ref. [110].

5.4.2 2 Majorana cones: semion-fermion surface state

The fate of the surface topological order in Sec.5.3.4 is more complicated. Again $\mathcal{T}^2 = -1$ on both α^4 and v^4, and the simplest particle to condense is the charge-1 object $v^4\alpha^4$. It can be checked that the only effect of this condensate is to confine odd powers of α: α^{2n+1}. The remaining theory can then be written in the following way:

$$\{1, v, ...v^7\} \times \{1, s = \epsilon\alpha^2v^2\} \times \{1, c\}, \tag{5.20}$$

where $\epsilon = c\alpha^4 = cv^4$. It can be checked that particles in the two sectors $\{1, v, ...v^7\}$ and $\{1, s\}$ are mutually local with respect to each other. The sector $\{1, v...v^7\} \times \{1, c\}$, with its time-reversal implementation, is exactly what was named T-Pfaffian state in Ref. [54] and was proposed to be a possible surface state of the electronic band topological insulator[54, 55]. The only difference here is that there is no charge assignment. The T-Pfaffian state being a surface state of the band TI implies that without charge assignment (i.e. when charge $U(1)$ is broken), it should be possible to completely confine it down to $\{1, c\}$. This is a highly non-trivial statement, since there is no trivial boson in the theory for one to condense, and one need a series of unknown phase transitions to confine it. Now taking the statement as true, we can eliminate the $\{1, v...v^7\}$ sector from Eq. (5.20) and get $\{1, s\} \times \{1, c\}$. Recall that v^2 is a semion, it also has -1 mutual statistics with α^2 from the charge-vortex relation. Hence the composite $s = \epsilon\alpha^2 v^2$ is a semion, and under time-reversal it goes to $s = \epsilon\alpha^2 v^2 \rightarrow \epsilon\alpha^{-2}\epsilon v^2 = \alpha^{-2}v^2 = (\alpha^{-4}\epsilon)s = cs$ which is an anti-semion. These are in agreement with Ref. [110].

5.5 $SU(2) \times \mathbb{Z}_2^T$: CI Class

The results in Sec.5.3 can also be applied to systems with $SU(2) \times \mathbb{Z}_2^T$ symmetry, i.e. superconductors with full spin rotation and time-reversal symmetry: the CI class. Again the free fermion bands are classified by $\mathbb{Z}$. For a state indexed by k, there are $2k$ Dirac cones on the surface, giving k flavors of $SU(2)$-fundamental fermions:

$$H = \sum_{i=1}^{k} \psi_i^\dagger (p_x\sigma_x + p_y\sigma_z) \otimes \tau_0 \psi_i, \tag{5.21}$$

where τ_μ is the $SU(2)$ spin, so that the $SU(2)$ rotation $\mathcal{U}$ acts as

$$\mathcal{U} : \psi_i \rightarrow \sigma_0 \otimes \mathcal{U}\psi_i, \tag{5.22}$$

and time-reversal acts as

$$\mathcal{T} : \psi_i \rightarrow i\sigma_y \otimes \tau_0 \psi_i^\dagger. \tag{5.23}$$

At $k = 1$ when the surface is gapped by breaking time reversal, there is a spin quantum Hall effect of $\sigma^s_{xy} = 1$. This is half of what is allowed in $d = 2$. Correspondingly if we gauge the global $SU(2)$ symmetry the bulk response has a θ term[115] for the corresponding $SU(2)$ gauge field at $\theta = \pi$. As we argued earlier the $k = 1$ state is therefore stable to interactions.

As in previous sections there is an emergent $U(1)$ symmetry in the surface Dirac theory:

$$U(\theta) : \psi_i \to e^{i\theta}\psi_i, \tag{5.24}$$

which we can promote to a physical symmetry, apply the arguments in Sec.5.3 and get a gapped state, then break the $U(1)$ by an explicit pairing. One should, however, be careful in the procedure not to break the $SU(2)$ symmetry. It turns out for even k, it is possible to have an intermediate $U(1)$-breaking phase preserving the $SU(2)$ symmetry, while for odd k this is impossible. Hence the results from Sec.5.3 can be applied to $k = 4$ (8 Dirac cones) to show that it is trivial, and to $k = 2$ (4 Dirac cones) to show that it is equivalent to the $eTmT$ boson SPT. We show the latter in Sec.5.5.1 since it directly implies the former due to the $\mathbb{Z}_2$ nature of the corresponding boson SPT states. For the $k = 1$ (2 Dirac cones) state, we argue in Sec.5.5.2 that it is impossible, even with interactions, to gap out the surface state while keeping the full $SU(2) \times \mathbb{Z}_2^T$ symmetry. Interestingly, it is so far the only known example in 3D with a symmetry protected gapless surface robust even under strong interaction.

The above results lead to a partial classification given by $\mathbb{Z}_4 \times \mathbb{Z}_2$, where the $\mathbb{Z}_4$ subgroup was deduced from the $\mathbb{Z}$ classification in free fermions, and the $\mathbb{Z}_2$ subgroup comes from boson SPT states with $\mathbb{Z}_2^T$ symmetry, as discussed in Sec.5.3.5. Unlike the symmetries with a normal $U(1)$ subgroup, it is not clear in this case if other SPT phases exist with no analog in either free fermion or boson systems. The analysis below in Sec. 5.5.2 suggests (but does not prove) that non-trivial surface states beyond boson SPT can be described by a Hopf term in the non-linear-sigma model, which prevents the surface from opening up a trivial gap. Since the Hopf term is realized in a free-fermion model, this suggests that states beyond boson SPT are either free fermion phases, or the combination of boson SPT and free fermion phases, hence the above $\mathbb{Z}_4 \times \mathbb{Z}_2$ classification may be complete. Likewise, superconductors with only $SU(2)$ symmetry may not support any nontrivial SPT state, since the surface Hopf-angle can always be tuned to zero in the absence of time-reversal symmetry. It is desirable to make the above arguments precise.

5.5.1 4 Dirac cones: boson SPT

We first re-write the $k = 2$ surface Dirac state as

$$H = \psi^\dagger(p_x\sigma_x + p_y\sigma_z) \otimes \tau_0 \otimes \mu_0\psi, \tag{5.25}$$

where μ denotes the flavor index. We now write down the pairing gap term:

$$H_\Delta = i\Delta\psi\sigma_y \otimes \tau_y \otimes \mu_y\psi + h.c., \tag{5.26}$$

which obviously opens up a gap and preserves $SU(2)$ invariance. As in Sec.5.3, time-reversal and the $U(1)$ symmetries are broken, but the modified time-reversal $\tilde{\mathcal{T}} = \mathcal{T}U(\pi/2)$ is kept invariant.

The vortex of Δ field carries four Majorana zero-modes, or two complex fermion zero-modes $f_{1,2}$. Since $SU(2)$ symmetry is kept and the ψ fermion is an $SU(2)$-fundamental, the two complex fermion zero-modes must also form an $SU(2)$ doublet $(f_1, f_2)^T$. Again we define vortices through Eq. (5.15), and time reversal acts as in Eq. (5.16) and (5.17). It is then clear that $\{v_{00}, v_{11}\}$ are $SU(2)$ singlets and $\{v_{01}, v_{10}\}$ form an $SU(2)$ doublet. Both pairs are Kramers under $\tilde{\mathcal{T}}$ ($\tilde{\mathcal{T}}^2 = -1$). Moreover, since the two pairs carry opposite fermion parity, they actually see each other as mutual semions. Condensing two-fold vortices then gives the $\mathbb{Z}_2$ topological order $\{1, e, m, \epsilon\}$, where $e \sim \{v_{00}, v_{11}\}$, $m \sim \{v_{01}, v_{10}\}$ and $\epsilon \sim \psi$. All the particles are neutral under the $U(1)$, hence further breaking the $U(1)$ symmetry does not change anything in the topological order. Now both the e and ϵ particles are $SU(2)$ doublets, so we can bind them with a physical fermion c to produce $SU(2)$ singlets. The topological order can thus be re-written as $\{1, \tilde{e}, m, \tilde{\epsilon}\}$, where $\tilde{e} = c\epsilon$ and m have $\mathcal{T}^2 = -1$, and $\tilde{\epsilon} = ce$ has $\mathcal{T}^2 = 1$, and all the particles are $SU(2)$ trivial. This is indeed the $eTmT$ state promised.

5.5.2 2 Dirac cones: symmetry-enforced gaplessness

With two Dirac cones one cannot write down a gap term that breaks $U(1)$ but not $SU(2)$, hence the previous trick does not apply. In fact, as we will now argue on very general grounds, it is impossible to have a gapped (topologically-ordered) symmetric surface state for the $k = 1$ topological superconductor. Hence the two Dirac cones on the surface are robust even with strong interaction, as long as the full $SU(2) \times \mathbb{Z}_2^T$ symmetry is preserved.

If the surface can be symmetrically gapped by introducing a topological order, then the $SU(2)$ group has to be represented non-projectively for all the particles in the theory, since there is no projective representation for $SU(2)$. One can then always bind a non-trivial quasi-particle with certain number of physical fermions to form an $SU(2)$ singlet. Therefore the theory can always be re-written as $\{1, \epsilon, ...\} \times \{1, c\}$ where all the particles are $SU(2)$ singlets except c. The first sector is also closed under time-reversal, since time-reversal action cannot mix an $SU(2)$ doublet with a singlet. Any local object in the topological order $\{1, \epsilon, ...\}$ must then be bosonic since it is $SU(2)$ trivial. Hence the topological order can be viewed as one emergent from bosonic objects in the theory, and the bulk state can at most be a bosonic SPT state with $\mathbb{Z}_2^T$ symmetry only.

For the $k = 2$ state the above analysis is consistent with what we obtained in Sec.5.5.1. The $k = 1$ state, on the other hand, cannot fit into the above framework: putting two copies of the $k = 1$ state together forms a $k = 2$ state, which is a bosonic SPT. The bosonic SPT's in this case are classified by $\mathbb{Z}_2^3$, so none of them admits a "square root". So the $k = 1$ state cannot be a bosonic SPT, and according to the above analysis we are forced to conclude that a symmetric gapped surface topological order does not exist for the this state. This provide the first known example of "strictly" symmetry-protected gapless surface, since all the other 3D SPT states studied so far admit a gapped symmetric surface with topological order. This also implies that the $k = 1$ (and the combination of the state with other boson SPTs) cannot be constructed using the Walker-Wang approach[44, 110, 54], which relies on the existence of a gapped surface.

$O(3)$ non-linear sigma model: Hopf term

The surface Dirac theory in Eq. (5.21) can also be gapped by introducing a Neel-like order. For $k = 1$ we write down the Dirac fermion coupled to the Neel unit vector $\mathbf{n}$:

$$H = \psi^\dagger (p_x \sigma_x + p_y \sigma_z) \otimes \tau_0 \psi + m \psi^\dagger \sigma_y \otimes \mathbf{n} \cdot \tau \psi. \tag{5.27}$$

Since the fermion is gapped now, one can integrate it out and obtain an effective theory of the Neel vector. The result[116] is a non-linear sigma model with a topological term known as the Hopf term, at $\theta = \pi$:

$$S = \frac{1}{g} \int d^2x dt (\partial_\mu \mathbf{n})^2 + i\pi H_2[\mathbf{n}], \tag{5.28}$$

where H_2 is the integer characterizing $\pi_3(S^2) = \mathbb{Z}$.

The Hopf term changes the statistics of the skyrmions of the $O(3)$ model[117]. Continuum field theory arguments suggest that time reversal (and parity) are preserved so long as the coefficient of the Hopf term is 0 or π. If it is 0 the skyrmions are bosons while if it is π they are fermions. This field theory was once proposed[118] to describe the parent antiferromagnets of the cuprate materials. In the specific context of the square lattice Heisenberg antiferromagnet this proposal was killed by microscopic derivations of the sigma model[119, 120, 121, 122] which revealed a Hopf coefficient of zero. With our modern understanding we can see that a Hopf coefficient of π is not allowed in the presence of time reversal symmetry in any strictly $2d$ quantum magnet. Indeed this theory arises at the surface of the $3D$ topological superconductor.

Our analysis of the $k = 1$ topological superconductor implies that the non-linear sigma model with Hopf term at $\theta = \pi$ does not have a gapped phase that preserves the full $SO(3) \times \mathbb{Z}_2^T$ symmetry, even with topological order. This is an interesting conclusion that is not entirely obvious from other approaches.

As was seen in Sec.5.5.1, the $k = 2$ topological superconductor is also nontrivial under interaction. In particular, a gapped symmetric surface must necessarily develop topological order. Since the $k = 2$ state can also be described using 4 Dirac cones on the surface, the effective theory of the Neel order parameter $\mathbf{n}$ can be described using a non-linear sigma model with a Hopf term at $\theta = 2\pi$. We therefore reach the surprising conclusion that, even a Hopf term at $\theta = 2\pi$ cannot arise in a purely 2D system if time-reversal acts as $\mathbf{n} \to -\mathbf{n}$. Moreover, since the $k = 4$ superconductor is trivial under interaction, a Hopf term with $\theta = 4\pi$ is allowed in strict 2D with time-reversal.

5.6 $U(1) \rtimes (\mathbb{Z}_2^T \times \mathbb{Z}_2^C)$: CII Class

Now we turn to fermions with charge $U(1)$, time-reversal and charge-conjugation symmetries that both square to $\mathcal{T}^2 = \mathcal{C}^2 = -1$ on physical fermions (the CII class). At free fermion level, the insulators are classified by $\mathbb{Z}_2$ in three dimensions. The non-trivial surface state has two Dirac cones:

$$H = \psi^\dagger (p_x \sigma_x + p_y \sigma_z) \otimes \tau_0 \psi, \tag{5.29}$$

where the $U(1)$ symmetry acts in the obvious way, time reversal acts as

$$\mathcal{T} : \psi \to i\sigma_y \otimes \tau_0 \psi, \tag{5.30}$$

and charge conjugation acts as

$$\mathcal{C} : \psi \to \sigma_0 \otimes \tau_y \psi^\dagger. \tag{5.31}$$

The natural question to ask is how stable this phase is when interaction is included.

Again we answer this question by looking at the $U(1)$ monopole. Notice that the composite operation $\mathcal{S} = \mathcal{T}\mathcal{C}$ is an anti-unitary operator that commutes with $U(1)$ rotation. Hence it plays the role of time-reversal in Sec.5.3, where it was shown that the surface state with two Dirac cones gives a "Kramers" monopole. Therefore the monopole in the present case transforms as a Kramers pair under $\mathcal{S}$, which establishes the state as a nontrivial interacting SPT.

One may also ask that whether a (presumably strongly interacting) SPT exist for this symmetry group that gives a $\theta = \pi$ term in the $U(1)$ gauge response, since it looks consistent with symmetries but yet cannot be realized using free fermions. An analysis parallel to that in Ref. [3] and [53] shows that, however, such a state cannot exist. The basic idea is the following: if such a state exist, then combining the $(1, 1/2)$ dyon (monopole carrying charge-1/2) with the $(-1, 1/2)$ dyon gives the fundamental charge-1 fermion. A careful analysis then shows that $\mathcal{C}^2 = 1$ on such a composite, hence requiring the fundamental fermion to have $\mathcal{C}^2 = 1$ as well, which is inconsistent with the microscopic symmetry structure. Indeed, for microscopic symmetry such that $\mathcal{C}^2 = 1$, the state does exist, which is just the descendent of the electronic band TI with the additional symmetry $\mathcal{C} : \psi \to \psi^\dagger$.

Therefore the only non-trivial monopole structure is realized by the free fermion state Eq. (5.29), which contributes a $\mathbb{Z}_2$ subgroup in the classification. The other SPT states, according to Sec.5.2, are those from bosons with symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2^T$, which are classified[16, 17, 43] by $\mathbb{Z}_2^4$. The complete classification is thus given by $\mathbb{Z}_2^5$.

5.7 $(U(1) \rtimes \mathbb{Z}_2^T) \times SU(2)$: $\mathbb{Z}_2^4$ classification from boson SPT

Let us now turn to another physically relevant symmetry: charge conservation ($U(1)$), spin rotation ($SU(2)$) and time-reversal ($\mathcal{T}$). Free fermion band theory gives no non-trivial state, and we would like to examine it more carefully when interaction is included. Obviously one can always have SPTs coming from the charge-neutral bosonic sector, which have $SO(3) \times \mathbb{Z}_2^T$ symmetry, and are classified[16, 17, 43] by $\mathbb{Z}_2^4$. The real question is whether there is a strongly interacting SPT state not descending from bosonic sectors. According to Sec.5.2, such states would necessarily have monopoles carrying nontrivial quantum numbers.

It is easy to first rule out a $\theta = \pi$ state[3, 53], where monopoles become charge-1/2 dyons: the bound state of the $(1, 1/2)$ dyon and the $(-1, 1/2)$ dyon, which are time-reversal partners, is the charge-1 physical fermion, which is an $SU(2)$-fundamental. It is then impossible to assign $SU(2)$ quantum numbers to either of the two dyons that is consistent with time-reversal symmetry.

Now we consider monopoles that are charge-neutral. The only nontrivial quantum number a monopole can carry is then an $SU(2)$-fundamental, since the $SU(2)$ group does not admit a projective representation. It turns out such a state does not exist as well, and the full classification is given simply by $\mathbb{Z}_2^4$ from bosonic SPT. We outline the argument briefly as follows:

We know that the monopole is bosonic and does not carry electric charge, so let's take advantage of that: instead of asking "could fermions give rise to spin-1/2 monopoles", let's ask the dual question instead: could spin-1/2 bosons give rise to fermionic monopoles? Now this becomes a question about boson SPT which is tractable, albeit with a less familiar symmetry. Specifically the appropriate symmetry for these bosons is $U(1) \times SU(2) \times \mathbb{Z}_2^T$. Note that the contrast with the electrons (which are dual monopoles as seen by these bosons).

The question can be further reduced to the following: does a boson SPT that gives a fermionic monopole survive if we further impose $SU(2)$ symmetry on the bosons, and require the charge-1 bosons transform as $SU(2)$ fundamental?

We then argue that for bosons with $U(1) \times \mathbb{Z}_2^T$, the SPT does not survive upon adding $SU(2)$ symmetry: for this symmetry group, $b_\alpha \to b_\alpha^\dagger$ under time-reversal, so the spin-up and down bosons do not get mixed under time-reversal. Therefore each spin-sector gives a time-reversal invariant boson insulator. More precisely, we can integrate out up-spin boson field since they are gapped anyway, the theory left behind contains only down-spin bosons, but it is still time-reversal invariant. Hence the two sectors should contribute equally to the θ-angle in the $U(1)$ gauge response, which must be either 0 or 2π due to time-reversal invariance in each sector. So the total θ must be 0 or 4π. It was shown in Ref. [43, 1, 69] that for boson systems $\theta = 0$ and 4π correspond to a trivial insulator, while $\theta = 2\pi$ gives an SPT state with fermionic monopoles (this is named as "statistical Witten effect" in Ref. [69]). Therefore it is impossible for the $U(1) \times SU(2) \times \mathbb{Z}_2^T$ bosons to induce a fermionic monopole.

The above argument does not work for bosons with $U(1) \rtimes \mathbb{Z}_2^T$, since a theory with only one species cannot be time-reversal invariant ($b_\alpha \to i\sigma_y^{\alpha\beta} b_\beta$ under $\mathcal{T}$), so each sector does not have to contribute to the θ-angle in a time-reversal invariant way. For example, each sector can contribute a π to θ, so the total θ could be 2π.

In the original (un-dual) problem, the above argument shows that it is impossible for fermions with $(U(1) \rtimes \mathbb{Z}_2^T) \times SU(2)$ symmetry to induce a monopole that transforms as $SU(2)$-fundamental. For fermions with $U(1) \times SU(2) \times \mathbb{Z}_2^T$ symmetry, on the other hand, it is possible for the monopole to carry spin-1/2 under $SU(2)$. In fact, it can be shown that the $k = 1$ state discussed in Sec. 5.5.2 survives upon imposing an extra $U(1)$ symmetry that commutes with $\mathcal{T}$, and the monopole of this $U(1)$ symmetry carries precisely spin-1/2 under $SU(2)$.

5.8 Summary and discussion

In this chapter we studied the classification and physical properties of three dimensional interacting electronic topological insulators and superconductors. Free fermion systems in $3d$ fall into different symmetry classes described by the "10-fold way". For all these symmetry classes we were able to determine the stability to interactions, and further to determine if there are any new interacting phases that have no free fermion counterpart. If the symmetry group has a normal $U(1)$ subgroup we obtained the full classification in the presence of interactions. Our methods are physics-based and enable us to describe the physical properties of these various electronic SPT phases in three dimensions.

We now discuss some open questions and some applications of our results. In the cases without a normal $U(1)$ subgroup it will be interesting to establish the completeness or lack thereof of our classification. For the symmetry groups $SU(2) \times Z_2^T$ or just $SU(2)$ in Section. 5.5 we gave arguments why our classification may be complete. It is desirable to have a sharper version of these arguments.

Perhaps the biggest open question about SPT phases is their possible occurrence in specific materials. For the 3d SPT phases with no free fermion counterpart for the most part we do not currently have simple theoretical models which may be useful guides on the kinds of physical systems that are likely platforms for these phases. We hope that the enhanced understanding of these phases that our work provides will help answer such questions. For example, the understanding of the connection between certain free fermion topological phases and some bosonic SPT states enables us to construct slave particle mean field theories for the boson SPT states, which might be a useful guide in searching for material realizations of the SPT phases. We elaborate on this point in Chapter 7.

An interesting application of our work, which we will elaborate in a future work[70], is to the classification of three dimensional time reversal symmetric quantum spin liquids with an emergent photon (known as $U(1)$ spin liquids). These phases may be relevant to quantum spin ice materials. Though such quantum spin liquids are "long range entangled" they may nevertheless be fruitfully understood as gauged versions of SPT phases. The understanding of SPT phases provides a very insightful perspective on these time reversal symmetric quantum spin liquids.

A different application of the results of this chapter is widen the range of two dimensional quantum field theories which have anomalous implementation of symmetry. We showed that strictly two dimensions the non-linear sigma model description of collinear quantum antiferromagnets cannot have a Hopf term with a coefficient $\theta = \pi$ or 2π. The former was proposed[118] as a possibility and discarded[119, 120, 121, 122] on microscopic grounds. Our results show that Hopf terms with $\theta = \pi, 2\pi$ are consistent with time reversal only if the two dimensional magnet is the boundary of a three dimensional SPT phase.

Chapter 6

Bound states of three fermions forming symmetry-protected topological phases

Topological insulators (TIs) are gapped phases of matter hosting non-trivial boundary states protected by symmetries. Most of our current knowledge of topological insulators come from free fermion models[18, 19, 20], which have been fully classified[21, 22] in all dimensions and with different global symmetries. Recently an interesting generalization of the free fermion topological insulators to interacting systems - known as Symmetry-protected topological (SPT) phases - has been pursued theoretically (see Ref. [123, 124] for simple review articles).

It was realized[16, 17] that SPT states can also exist in systems of interacting bosons. The boson SPT states can also be realized in interacting fermion systems, since one can always bind two fermions to form a boson, such as the electron spin $\frac{1}{2}c^{\dagger}\sigma c$ or the Cooper pair $c_{\uparrow}c_{\downarrow}$ - a process which clearly requires fermionic interactions. One can then imagine putting the bound bosons into a boson SPT state.

It should be noted that this approach does not always give new states. Some of the boson SPT states become equivalent to certain free fermion states once physical fermions are introduced into the system[1]. The corresponding free fermion states

[1]By "free fermion state", what we really mean is an interacting fermion state that is adiabatically

could be trivial[125, 110, 3] or topological[48, 110]. A simple example is that with only time-reversal symmetry, the Haldane chain becomes equivalent to four copies of the Kitaev chain[40, 41, 42, 109] with spinless fermions.

There are, however, many other boson SPT states that are distinct from free fermion models. Abundant examples have been found in both 2D[48] and 3D[3, 4]. So far, these states have been exclusively understood within the bosonic approach: in all the models the non-triviality comes entirely from the boson sector (spins, Cooper pairs, etc.), and the existence of fermions does not seem to contribute anything.

In this chapter, we show that some of these non-trivial boson SPT states can also be understood in an intrinsically fermionic approach, even though they are distinct from any free fermion state. Specifically, these states can be viewed as topological insulators of certain fermions - not the free ones, but the bound states of three fermions (or some other odd number) which we refer to as *clustons*. This observation not only provides new insights into the interacting SPT states, but also suggests realizations in cold atom systems: three-body bound state can be achieved in cold atom systems through Efimov effect[126], thus if one can control these Effimov states efficiently and put them in a topological band (which is certainly quite challenging), it will be possible to realize the novel states we propose.

We study specifically two examples in this chapter. The first one is the boson integer quantum hall state (BIQHE) in 2D[50], the second one is the boson topological insulator (BTI) in 3D[43, 1, 69]. In both examples the charged bosons are viewed as Cooper pairs of electrons. One can also view the fermions as slave-particles (partons), and our result gives another way to write down wavefunctions of these states in purely bosonic systems.

6.1 2D: clustons in Chern band

We consider a fermion system with charge $U(1)$ symmetry. Now imagine a situation in which fermions prefer to form three-body bound states (clustons), which clearly

connected to a state realized by a free fermion model.

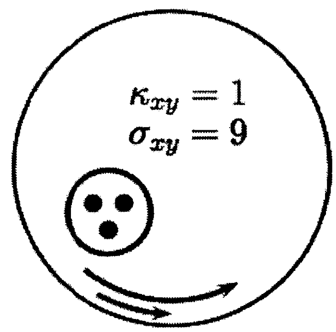

Figure 6-1: Construction of the cluston Chern insulator: putting the three-fermion clustons into a Chern band.

requires strong interaction. We then put the clustered fermions into a Chern band with Chern number $C = 1$. This state is not fractionalized, so one could naturally ask if it is equivalent to a free fermion state. We answer this question by looking at its transport properties.

The quantum hall conductance is given by $\sigma_{xy} = \tilde{e}^2 C = 9$ in units of e^2/h, where $\tilde{e} = 3$ is the charge carried by the clustered fermion. However, the thermal hall conductance κ_{xy} in units of $\frac{\pi^2}{3}\frac{k_B^2}{h}T$, also known as the chiral central charge, is given by $\kappa_{xy} = C = 1$, since the thermal transport is independent with the amount of charge carried by the fermions. Therefore this state is distinct from all the integer quantum hall states made of free fermions, which always have $\sigma_{xy} = \kappa_{xy}$ since heat carriers are also charge carriers in a free fermion systems. For interacting systems made of electrons (fermions with charge e), it is known[48] that as long as the bulk excitation spectrum does not contain fractionalized anyons (i.e. the excitations only include fermions with odd-integer charge and bosons with even-integer charge), the difference between the hall conductance and the thermal hall conductance is always an integer multiple of eight in proper units:

$$\frac{3}{\pi^2}\frac{h}{k_B^2 T}\kappa_{xy} - \frac{h}{e^2}\sigma_{xy} = 8n. \tag{6.1}$$

For completeness we reproduce the derivation of this result in Sec. 6.1.1. Our cluston Chern insulator is thus a minimal state with unequal charge and thermal hall conductance.

To make the structure clearer, we combine the above state with a free fermion IQHE with Chern number $\bar{C} = -1$. The total system now has $\sigma_{xy} = 8$ but $\kappa_{xy} = 0$. We now consider another state with the same transport properties: imagine the fermions form Cooper pairs (charge $e^* = 2$ bosons), and the Cooper pairs form a boson IQHE state. It was shown in Ref. [48, 50, 49, 51] that such a state would be non-chiral ($\kappa_{xy} = 0$), but with quantum hall conductance $\sigma_{xy} = 2(e^*)^2 = 8$. The transport properties of the Cooper pair BIQHE state therefore matches perfectly with the state we constructed above. In Sec. 6.1.2 we use the edge theory to show explicitly that the two states are indeed equivalent, even though they appear to be very different in the way they are constructed. More explicitly, the Cooper pair BIQHE state can be described by the simple fermionic Hamiltonian:

$$H[f, F] = H_{C=1}[F] + H_{C=-1}[f] + \sum_{ijkl} \lambda F_i^\dagger f_j f_k f_l + h.c., \tag{6.2}$$

where f denotes the charge-1 fermion, F denotes the charge-3 clustered fermion, i, j, k, l represent indices such as spins and sub-lattices, H_C is a quadratic Hamiltonian that puts the fermions into a band with Chern number C, and the last term reveals F as the bound state of three fundamental fermions.

One can also consider a different state, where F is in a band with $C = 1$, while f is in a band with $\bar{C} = -9$. The total Hall conductance is then $\sigma_{xy} = C(e^*)^2 + \bar{C}e^2 = 3^2 - 9 = 0$, but the thermal hall conductance is $\kappa_{xy} = C + \bar{C} = -8$. The transport signature is identical to that of the E_8 state[48], which is the minimal chiral state of charge-neutral bosons. In systems of charge-neutral fermions, the E_8 state can be understood as 16 copies of $p + ip$ superconductors[106]. Our work provides another way to understand the state in terms of charged fermions.

6.1.1 Hall conductance of fermion SPT states

Consider a $2D$ system of charged fermions, and assume that there is no fractionalized anyon excitations in the bulk. It is then well known that such a state must have integer hall conductance in unit of e^2/h: this can be shown easily by examining the statistics of a 2π magnetic flux. We can then combine the system with some integer quantum hall state to produce a new state with zero hall conductance. Since integer quantum hall states of fermions have equal charge and thermal hall conductance, the combined state will have a new thermal hall conductance

$$\tilde{\kappa}_{xy} = \kappa_{xy} - \left(\frac{\pi^2 k_B^2 T}{3e^2}\right)\sigma_{xy}, \tag{6.3}$$

where κ_{xy} and σ_{xy} are the thermal and charge hall conductance of the original state before combining with any integer quantum hall state. Therefore in order to prove Eq. (6.1), it is sufficient to prove that the thermal hall conductance must be an integer multiple of eight if the system is non-fractionalized and has zero charge hall conductance.

We now consider the edge state of this system, which in general is a multi-component Luttinger liquid

$$\mathcal{L} = \frac{1}{4\pi}\left(K_{IJ}\partial_x\phi_I\partial_t\phi_J + ...\right) + \frac{1}{2\pi}\epsilon_{\mu\nu}\tau_I\partial_\mu\phi_I A_\nu \tag{6.4}$$

described by a symmetric integer K-matrix with an integer charge vector τ. A local object will carry odd charge *iff* it is a fermion, therefore the parity of the nth diagonal element of K must agree with the parity of the nth entry of τ. For non-fractionalized bulk, we have $|\det(K)| = 1$.

To make our discussion self-contained, we summarize some key facts known about the edge theory: the operator $e^{il_I\phi_I}$ defined by the integer vector l carries spin $S = \frac{1}{2}l^T K^{-1} l$, and charge $Q = \tau^T K^{-1} l$. It could condense on the edge only if $S = 0$. When it condenses, another mode (defined by another integer vector l') can stay gapless on the edge only if the operator $e^{il'_I\phi_I}$ commutes with $e^{il_I\phi_I}$, which means $l^T K^{-1} l' = 0$.

The fact that the state has zero charge hall conductance means that

$$\sigma_{xy} = \tau^T K^{-1}\tau = 0. \tag{6.5}$$

Therefore the operator $e^{i\tau_I\phi_I}$ carries zero spin and charge. We can then introduce a charge-conserving term

$$\Delta\mathcal{L} = U\cos(\tau_I \phi_I), \tag{6.6}$$

which at sufficiently large U will gap out all the modes that do not commute with it. The remaining gapless modes, denoted as $\tilde{\phi}_\alpha = l_{\alpha,I}\phi_I$, must satisfy $\tau^T K^{-1} l = 0$ in order to commute with the condensed operator. But this precisely means that the remaining modes are charge-neutral, since $\tau^T K^{-1} l$ gives the charge carried by the operator $e^{il_I\phi_I}$.

Therefore the remaining edge state can be described using another $\tilde{K}$-matrix after proper field redefinition, with zero charge-vector $\tilde{\tau} = 0$ since all modes are charge-neutral. Since a local object can be charge-neutral only if it is bosonic, the $\tilde{K}$-matrix must describe a bosonic topological state. In particular, the diagonal elements of $\tilde{K}$ must be even integers. It is known[48] that for bosonic states, the minimal non-fractional chiral phase (with a non-zero chiral central charge or thermal hall conductance) is the so-called E_8 state, which has $\kappa = 8$. The corresponding $\tilde{K}$-matrix of such a state is the Cartan matrix for the exceptional Lie group E_8. This proves our assertion under Eq. (6.3), hence also proves Eq. (6.1).

As a side note, the above derivation is valid for general abelian topological orders described by a K-matrix with $|\det| \geq 1$, with the only modification that the charge gap in Eq. (6.6) should be replaced by $\Delta\mathcal{L} = U\cos(N\tau_I\phi_I)$, where N is some integer that makes the operator local. The conclusion is unchanged: if $\sigma_{xy} = 0$ and there is no other symmetry than the charge $U(1)$, then the charge modes on the edge can be gapped, and the remaining modes can be described by a charge-neutral bosonic topological order.

6.1.2 The 2D equivalence from edge theories

We show here the equivalence between the fermion model in Eq. (6.2) and the BIQHE state. It is sufficient to show that the boundary between the two states can be fully gapped while preserving charge conservation. The boundary Luttinger liquid

$$\mathcal{L} = \frac{1}{4\pi}\left(K_{IJ}\partial_x\phi_I\partial_t\phi_J + ...\right) + \frac{1}{2\pi}\epsilon_{\mu\nu}\tau_I\partial_\mu\phi_I A_\nu \tag{6.7}$$

is described by the K-matrix with the charge vector τ:

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tau = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 2 \end{pmatrix}. \tag{6.8}$$

Now consider two possible mass terms:

$$\begin{aligned} \Delta\mathcal{L} &= U_1\cos\Phi_1 + U_2\cos\Phi_2 \\ &= U_1\cos\left(\phi_1 - \phi_2 + \phi_3\right) \\ &\quad + U_2\cos\left(\phi_1 + \phi_2 - 2\phi_4\right), \end{aligned} \tag{6.9}$$

which obviously preserve the $U(1)$ charge conservation. The null vector criteria[127] $\Phi_i K^{-1}\Phi_j = 0$ is easily satisfied. Therefore Eq. (6.9) fully gaps out the edge theory while preserving the $U(1)$ symmetry when the couplings $U_{1,2}$ are large.

6.2 3D: clustons in topological band

We now consider a three dimensional fermion system with $U(1)$ charge conservation and time-reversal symmetry $\mathcal{T}$ with $\mathcal{T}^2 = -1$ on the physical fermions. Again imagine a situation in which fermions prefer to form three-body bound states (clustons). We then put the charge-3 clustons into a Fu-Kane-Mele topological band[25]. The state is obviously interacting and not fractionalized, so one can again ask what phase the state belongs to. It is easy to see that the state should be nontrivial, for example,

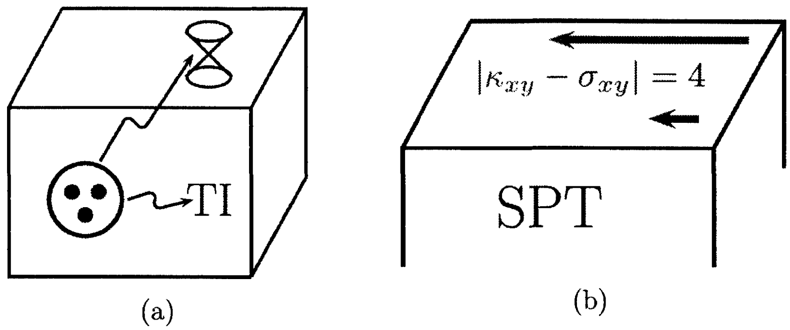

Figure 6-2: (a) Construction of the cluston TI: putting the three-fermion clustons into a topological band. (b) Transport signature on a $\mathcal{T}$-breaking surface. $\sigma_{xy} - \kappa_{xy} = 4(\text{mod}8)$ signifies a nontrivial SPT.

through the magneto-electric response[64] described by a θ-term with

$$\theta = \pi(e^*)^2 = 9\pi = \pi \ (\text{mod}2\pi). \tag{6.10}$$

A naive answer could then be that the state is equivalent (connected) to the free fermion topological insulator. However, we will show that this is not true.

We answer the question by looking at the surface. The simplest symmetric surface state is a single Dirac cone of fermions carrying charge $e^* = 3$. However, for our purpose it is easier to reveal the nontriviality (and fully determine the topological state) from symmetry-breaking surface states. If we break the $U(1)$ charge conservation on the surface (e.g. by depositing a superconductor on top), the Dirac cone can be gapped out. If we keep the $U(1)$ symmetry but break time-reversal $\mathcal{T}$ instead, we can also gap out the Dirac cone, but the surface will have nontrivial transport signatures. It is well known that the Dirac mass gap leads to "half" quantum hall and thermal hall conductance. In our case we have $\sigma_{xy} = \frac{1}{2}(e^*)^2 = \frac{9}{2}$, and $\kappa_{xy} = \frac{1}{2}$ since the carrier charge does not affect thermal transport. We therefore have $\sigma_{xy} - \kappa_{xy} = 4$, which is half of what is allowed in strictly two dimensions without fractionalization (for example, the state considered in Eq. (6.2) has $\sigma_{xy} - \kappa_{xy} = \pm 8$).

The $\sigma_{xy} - \kappa_{xy}$ mismatch is zero on the surface of the free fermion topological insulator, since heat carriers are also charge carriers in free fermion systems. When interactions are introduced, the mismatch can shift by integer multiples of eight, by depositing either the BIQHE state of Cooper pairs, or the E_8 state of spins, *i.e.* $\sigma_{xy} - \kappa_{xy} = 8n$ if the state is equivalent to the free fermion TI. Therefore the state we described above cannot be equivalent to the free fermion TI.

Non-fractional insulators (SPT states) with $U(1)$ and $\mathcal{T}$ symmetries in three dimensions are classified[3] by $\mathbb{Z}_2^3$. There are six nontrivial SPT states distinct from the free fermion TI. Three out the six states have $\sigma_{xy} = \frac{1}{2}(\text{mod}1)$ when $\mathcal{T}$ is broken on the surface. Among the three states, only one of them can be completely gapped out by breaking $U(1)$ charge-conservation (while keeping $\mathcal{T}$). This state also has $\sigma_{xy} - \kappa_{xy} = 4(\text{mod}8)$ when $\mathcal{T}$ is broken on the surface. Thus the clustered TI state described above is precisely this state. Ref. [3] provided two equivalent ways to understand the state: one can think of it either as the combination of the free fermion TI and a spin SPT (dubbed topological paramagnet e_fTm_fT in Ref. [1, 3]), or as the combination of the free fermion TI and the bosonic topological insulator (BTI)[43, 1, 69] of Cooper pairs. Our clustered TI provides another way to view the state, and it differs from the free fermion TI by a bosonic SPT state of either the spin or the Cooper pair.

Our result also provide a simple way to (at least theoretically) construct the BTI state with fermions, in the same spirit as Eq. (6.2):

$$H[f,F] = H_{FKM}[F] + H_{FKM}[f] + \sum_{ijkl} \lambda F_i^\dagger f_j f_k f_l + h.c., \tag{6.11}$$

where f denotes the charge-1 fermion, F denotes the charge-3 clustered fermion, i,j,k,l represent indices such as spins and sub-lattices, H_{FKM} is a quadratic Hamiltonian that puts the fermion into the Fu-Kane-Mele band, and the last term reveals F as the bound state of three fundamental fermions.

Previously, the simplest symmetry-preserving surface state of the topological paramagnet e_fTm_fT was given by a gapped topologically ordered $\mathbb{Z}_2$ gauge theory $\{1, e, m, \epsilon\}$, with both the electric-like particle e and magnetic-like particle m being fermions and Kramers' ($\mathcal{T}^2 = -1$). Likewise, the simplest symmetry-preserving surface state of the boson TI (BTI) was given by another gapped topologically ordered $\mathbb{Z}_2$ gauge theory

$\{1, e, m, \epsilon\}$, with both the e and m particles carrying half-charge (in our case the Cooper pair boson carries charge-2, so e and m carry charge-1). The two states are distinct as purely bosonic states, but in the presence of electrons (charge-1 fermions), the two states become equivalent since the two surface gauge theories can be transformed to each other by attaching an electron to the e and m particles. Eq. (6.11) leads to another simple symmetric surface state without topological order (but is gapless instead), namely two Dirac cones carrying charge $e = 1$ and $e^* = 3$ respectively:

$$\mathcal{L} = \bar{\psi}\sigma^\mu(-i\partial_\mu + A_\mu)\psi + \bar{\Psi}\sigma^\mu(-i\partial_\mu + 3A_\mu)\Psi, \tag{6.12}$$

where A_μ is the external probe gauge field. This also implies that the Dirac theory in Eq. (6.12), even though is free from the famous parity anomaly[28, 29], suffers from another anomaly first proposed by Vishwanath and Senthil[43] in the context of topological quantum field theories. The Vishwanath-Senthil anomaly can be viewed as a gravitational analogue of the parity anomaly: if the theory in Eq. (6.12) is coupled to gravity, then a gravitational Chern-Simons term at level $c = 4(\text{mod}8)$ must be introduced to regularize the theory[2], thus time-reversal symmetry must be broken. In terms of the bulk theory, this corresponds to a gravitational θ-term[112] at $\theta = 8\pi$.

One can also show the equivalence between the fermionic state in Eq. (6.11) and the Cooper pair BTI directly from the symmetric surface states. The idea is to show that the surface Dirac theory in Eq. (6.12) can be gapped by introducing the corresponding $\mathbb{Z}_2$ topological order. However, the argument, which we briefly outline in Sec. 6.2.1, is considerably more technical. The fact that the equivalence was easily established using the result in Ref. [3] is another illustration of the usefulness of the $\mathbb{Z}_2^3$ classification.

One can also imagine similar states with five-fermion clustons, or any other odd number for e^*. By repeating the previous argument, it is easy to show that the cluston TI differs from the free fermion TI by the Cooper pair BTI if $e^* = \pm 3(\text{mod}\, 8)$, and

[2]More precisely, we have $c - k = 4(\text{mod}8)$ where k is the level of the Chern-Simons term of the $U(1)$ gauge field.

equivalent to the free fermion TI if $e^* = \pm 1 (\mathrm{mod}\, 8)$.

6.2.1 The 3D equivalence from symmetric surface states

We show here the equivalence between the fermion model in Eq. (6.11) and the Cooper pair boson TI (BTI) state. One of the defining features[43, 1, 69] of the BTI state is that when the surface breaks $U(1)$ but not $\mathcal{T}$, it is gapped without topological order, but the vortex of the surface superconductor has fermion statistics. To access a fully symmetric surface state, one can imagine driving a surface phase transition and condensed double-vortex (which is a boson). It is well known that double-vortex condensates produce $\mathbb{Z}_2$ topological orders[80, 104], and in our cSCtoBandInsulatorase we get precisely the surface topological orders studied in[43, 1, 69].

It is easy to see that when $U(1)$ symmetry is broken on the surface, the surface Dirac theory Eq. (6.12) (for general odd $e^*/e = n$) can be fully gapped by introducing the pairing term:

$$\Delta\mathcal{L} = i\Delta\psi\sigma_y\psi + i\xi\Delta^n\Psi\sigma_y\Psi + h.c., \tag{6.13}$$

where we wrote the second pairing amplitude as proportional to Δ^n to keep the pairing field Δ formally charge-2, and ξ is a non-universal coupling constant. We then have to show that the vortex in Δ field has fermion statistics for $n = \pm 3(\mathrm{m}od\, 8)$ and boson statistics for $n = \pm 1(\mathrm{m}od\, 8)$.

Since there are even numbers of Dirac cones in total, the vortex[102] does not trap any Majorana zero-mode and is thus abelian. The abelian part of the statistics is then given by the topological spin $e^{i\theta}$, which receives nontrivial contribution from both Dirac cones. It is important to notice here that while the charge-1 fermion ψ sees the fundamental vortex as a π-flux, the charge-n fermion Ψ sees it as a $n\pi$-flux. Therefore the topological spin of the fundamental vortex is given by $e^{i\theta} = e^{i\theta_1}e^{i\theta_n}$, where $e^{i\theta_n}$ is the topological spin of a $n\pi$-flux seen by a paired single Dirac cone. Fortunately this topological spin $e^{i\theta_n}$ has been computed already in Ref. [2, 53], and is given by

$$e^{i\theta_n} = 1 \quad \text{if} \quad n = \pm 1 \,(\mathrm{mod}\, 8), \qquad (6.14)$$
$$e^{i\theta_n} = -1 \quad \text{if} \quad n = \pm 3 \,(\mathrm{mod}\, 8).$$

Therefore, when $n = \pm 1\,(\mathrm{mod}\,8)$the total topological spin is $e^{i\theta} = 1$ and the fundamental vortex is a boson, and when $n = \pm 3\,(\mathrm{mod}\,8)$ the total topological spin is $e^{i\theta} = -1$ and the fundamental vortex is a fermion. When the vortex is boson, it can be condensed and produce a trivial insulator on the surface, and the corresponding bulk state is also trivial. But if the vortex is a fermion, one can no longer condense it to produce a trivial surface insulator. One can instead condense double-vortex to produce an insulator which has intrinsic $\mathbb{Z}_2$ topological order. Such a topological order contains the remnant of the uncondensed vortex ϵ which is a non-Kramers fermion, and the remnant of the Bogoliubov quasi-particle e_f which is a Kramers fermion. The two particles see each other as π-flux, and the bound state of the two (denoted as m_f) is another fermion which is also Kramers. Therefore the $\mathbb{Z}_2$ gauge theory contains three distinct fermions, two of which are Kramers. This is exactly the surface topological order of the topological paramagnet e_fTm_fT. Binding a physical fermion (charge-1, Kramers) to e_f and m_f converts them to charge-1 bosons. Thus the topological order can also be viewed as one with two bosons and one fermions, with both bosons carrying charge-1. This is exactly the surface topological order of the Cooper pair BTI.

6.3 Bosonic states: parton constructions

So far we have discussed various states in fermionic systems (i.e. systems with fermions in the microscopic Hilbert spaces). In this somewhat more technical section we consider purely bosonic systems (without fermions in the microscopic Hilbert space) by gauging the fermions. This leads us to some new parton constructions of various bosonic states.

2D: For the E_8 state, we start from nine copies of free fermion Chern insulator and one copy of charge-3 cluston Chern insulator with the opposite chirality. We then gauge the $U(1)$ symmetry. Since the state has no net hall conductance, the dynamics of the compact $U(1)$ gauge theory does not contain Chern-Simons term. It is well known that in $2+1$ dimensions a compact $U(1)$ gauge theory without Chern-Simons term is always confined. Therefore we automatically obtain a confined (unfractionalized) bosonic state with $\kappa_{xy} = 8$, which has a chiral edge state with chiral central charge $c_+ - c_- = 8$.

For the BIQHE state, we start from Eq. (6.2) and then gauge the fermion parity ($f \to (-1)f$) which is a $\mathbb{Z}_2$ symmetry (constructions using higher gauge symmetries were proposed earlier in Ref. [128, 129, 130, 131]). A simple way to realize this in a parton construction is to start with two flavors of charge-2 bosons $B_{1,2}$ and decompose them as

$$\begin{aligned} B_1 &= f_1 f_2, \\ B_2 &= f_3^\dagger F, \end{aligned} \tag{6.15}$$

where $f_{1,2,3}$ are charge-1 fermions and F is a charge-3 fermion. It is possible to arrange a mean field ansatz for the fermions such that the gauge symmetry reduces to a simple $\mathbb{Z}_2$ which is the fermion parity, and F form a band with Chern number $C = 1$ while $f_{1,2,3}$ together form a band with Chern number $C^* = -1$. Since the state is non-chiral, the $\mathbb{Z}_2$ gauge flux (also dubbed as "vison") carries bosonic statistics. The total quantum hall conductance is $\sigma_{xy} = 8$, so the vison carries integer charge and can always be neutralized by binding certain number of f fermions (which does not change the vison statistics due to the mutual statistics). The system can thus go through a confinement transition by condensing the gauge flux, and the resulting state is a confined bosonic state (with bosons carrying charge $e^* = 2$), with $\sigma_{xy} = 8 = 2(e^*)^2$ and $\kappa_{xy} = 0$.

3D: We start from Eq. (6.11). If we gauge the $U(1)$ symmetry, the $U(1)$ gauge theory does not contain nontrivial θ-angle ($\theta = 9\pi - \pi = 0(\mathrm{mod}\, 2\pi)$), therefore the gauge theory can be confined while preserving $\mathcal{T}$. The resulting system has only charge-neutral bosons (spins), and we obtain the topological paramagnet dubbed e_fTm_fT in Ref. [1, 3].

Instead of gauging the $U(1)$ symmetry, we can also choose to gauge the fermion parity ($f \to (-1)f$) which is a $\mathbb{Z}_2$ symmetry (for example using the parton decomposition in Eq. (6.15)). The $\mathbb{Z}_2$ gauge theory can again be confined, and we get a system of charge-2 bosons. The resulting state is then the BTI of these charge-2 bosons. Constructions with higher gauge symmetries were proposed earlier in Ref. [132, 133].

There are, however, two subtle issues on this construction. The first issue is that whether the $\mathbb{Z}_2$ flux loops coupled to the fermions in Eq. (6.11) can indeed proliferate and produce a confined gapped bulk. This is nontrivial because naively the loop hosts gapless fermion modes[134]. We show in Appendix 6.3.1 that the flux core can indeed be gapped, hence the flux loops can proliferate and confine the fermions. Since the gapless mode in a $\mathbb{Z}_2$ flux core in a $3D$ TI is identical to the edge mode of the $2D$ TI[135], our result also shows that putting charge-3 clustons into a $2D$ TI (quantum spin hall state) does not produce any new state, instead it gives the conventional $2D$ TI. Interestingly, this is related to the absence of Cooper-pair boson SPT state in $2D$ fermion systems.

The second issue is the nature of the confined (bosonic) state. The boson system after confinement has $U(1) \rtimes \mathcal{T}$ symmetry. In such systems the e_fTm_fT topological paramagnet and the bosonic topological insulator (BTI) are two distinct states, unlike in fermion systems where the two become equivalent. We therefore have to determine which boson SPT state one get by confining the fermion state. The construction of symmetric gapped surface state (with topological order) outlined in Sec. 6.2.1 shows that the Kramers' fermions in the e_fTm_fT topological order couples to the $\mathbb{Z}_2$ gauge field, hence they must be confined with the parton fermion and form charge-1 non-Kramers bosons. The deconfined surface state is thus the $eCmC$ topological order, with both e and m being charge-1 non-Kramers bosons, which is exactly the surface state of the boson TI.

However, the above result leaves one question unsolved: since the two states

e_fTm_fT (topological paramagnet) and $eCmC$ (BTI) are equivalent in fermion systems, why would confinement prefer one state over the other? To answer this question, we need to examine the dynamics of the $\mathbb{Z}_2$ gauge field coupled to the fermions more carefully. We show in Sec. 6.3.2 that there are two distinct confinement transitions one can drive the system through: a conventional one resulting from a trivial dynamics of the gauge field (which was implicitly assumed above), and a "twisted" one which requires nontrivial dynamics on the gauge field. In our case, the conventional confinement results in the BTI state, while the twisted confinement results in the e_fTm_fT topological paramagnet. Therefore the BTI state seems to be the more natural confined phase, since it only requires a trivial dynamics on the $\mathbb{Z}_2$ gauge field.

6.3.1 Gapping out cluston helical modes

Here we show that two copies of helical modes in $1D$, one carrying charge-1 and the other one carrying charge-3, can be fully gapped without breaking the $U(1) \rtimes \mathcal{T}$ symmetry. Such helical theory arises both in the $\mathbb{Z}_2$ flux core of the fermion system described by Eq. (6.11), and on the edge of a $2D$ state, which is a combination of the free fermion and the cluston $2D$ TI.

The Luttinger liquid

$$\mathcal{L} = \frac{1}{4\pi}\left(K_{IJ}\partial_x\phi_I\partial_t\phi_J + ...\right) + \frac{1}{2\pi}\epsilon_{\mu\nu}\tau_I\partial_\mu\phi_I A_\nu, \tag{6.16}$$

is described by the 8×8 K-matrix:

$$K = \begin{pmatrix} K_1 & 0 & 0 & 0 \\ 0 & K_2 & 0 & 0 \\ 0 & 0 & K_3 & 0 \\ 0 & 0 & 0 & K_4 \end{pmatrix} = \begin{pmatrix} \sigma_z & 0 & 0 & 0 \\ 0 & \sigma_z & 0 & 0 \\ 0 & 0 & \sigma_z & 0 \\ 0 & 0 & 0 & \sigma_z \end{pmatrix}, \tag{6.17}$$

charge vector τ:

$$\tau = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \tag{6.18}$$

and time-reversal implementation:

$$\mathcal{T}: \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \end{pmatrix} \to \begin{pmatrix} -\sigma_x & 0 & 0 & 0 \\ 0 & -\sigma_x & 0 & 0 \\ 0 & 0 & -\sigma_x & 0 \\ 0 & 0 & 0 & -\sigma_x \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \end{pmatrix} + \pi \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \tag{6.19}$$

The helical modes $K_1 = \sigma_z$ comes from the charge-3 cluston TI, while $K_{2,3,4} = \sigma_z$ come from the charge-1 TI. We choose to work with three charge-1 helical modes instead of only one to avoid subtleties from one-band picture.

Now consider the following term:

$$\begin{aligned}
\Delta\mathcal{L} &= U\cos\Phi_1 + U\cos\Phi_2 + U'\cos\Phi_3 + U''\cos\Phi_4 \\
&= U\cos\left(-\phi_1+\phi_3+\phi_6+\phi_8\right) \qquad (6.20) \\
&\quad +U\cos\left(-\phi_2+\phi_4+\phi_5+\phi_7\right) \\
&\quad +U'\cos\left(\phi_1-\phi_2+\phi_3-\phi_4\right) \qquad (6.21) \\
&\quad +U''\cos\left(\phi_1-\phi_2+\phi_5-\phi_6\right).
\end{aligned}$$

It is straightforward to check that Eq. (6.20) preserves both $U(1)$ and $\mathcal{T}$ symmetry. The null vector criteria[127] $\Phi_i K^{-1}\Phi_j = 0$ is easily satisfied, so Eq. (6.20) fully gaps out the flux core. To ensure that the theory does not break $\mathcal{T}$ spontaneously, we should also check the primitivity condition proposed in Ref. [75], by checking the mutual primitivity of all the 4×4 minors of $\{\Phi_i\}$. This can be done straightforwardly, and indeed the primitivity condition is satisfied.

6.3.2 Conventional and "twisted" confinement

The question can be simplified by considering the combined state of the e_fTm_fT (topological paramagnet) and $eCmC$ (BTI), which has a surface $\mathbb{Z}_2$ topological order with both e and m being electron-like (Kramers fermion carrying charge-1) and is therefore dubbed e_fCTm_fCT. The equivalence of the two states in fermion systems implies that their combined state is equivalent to a trivial fermion insulator, as can be seen directly from the surface topological order: one can condense the composite of the e particle and the microscopic fermion (or parton in gauged systems), which confines the surface completely without breaking any symmetry.

Our question now becomes: can one get the e_fCTm_fCT state in a boson system, by confining fermions in a trivial insulator coupling to a $\mathbb{Z}_2$ gauge field? Naively the confinement should simply lead to a trivial boson state since the underlying fermion state is trivial. However, we will show below that the nontrivial state can indeed be obtained if the dynamics of the gauge field is sufficiently nontrivial (which, crucially, does not require non-triviality of the underlying fermions state). We will focus our discussion on the specific example, though generalizations to other symmetries/systems are straightforward.

In the presence of time-reversal symmetry $\mathcal{T}$, the $\mathbb{Z}_2$ gauge flux loops can proliferate in different ways, hence giving rise to distinct confined phases. There is always a trivial confinement, namely the flux loops proliferate with a positive-definite amplitude for all configurations, written schematically as

$$\langle W_C \rangle \sim 1, \tag{6.22}$$

where W_C is the loop creation operator for configuration C. There is however another way of loop proliferation, namely the amplitude acquires a minus sign whenever the loops self-link (one needs to frame the loops into ribbons to make the self-linking well-defined):

$$\langle W_C \rangle \sim (-1)^{L_C}, \tag{6.23}$$

where L_C is the self-linking number of configuration C. It was shown in Ref. [101] that the end point of such loops on the surface become fermions. If the surface was trivial before confinement, after the "twisted" confinement there will be a $\mathbb{Z}_2$ gauge theory emerging on the surface. The excitations of the surface $\mathbb{Z}_2$ gauge theory include the un-condensed flux which is now a fermion, and the deconfined $\mathbb{Z}_2$ gauge charge which only lives on the surface. In our example, the gauge charge is electron-like (fermion carrying $U(1)$ charge and Kramers' degeneracy), and the gauge flux is a non-Kramers fermion carrying no charge (call it ϵ). The surface $\mathbb{Z}_2$ gauge theory therefore has both e and m particles being electron-like. The confined state is thus nothing but the e_fCTm_fCT boson SPT state.

Chapter 7

Topological Paramagnetism in Frustrated Spin-One Mott Insulators

Frustrated quantum magnets display a rich variety of many–body phenomena. Some such magnets show long–range magnetic order at low temperature, often selected out of a manifold of degenerate classical ground states by quantum fluctuations. A very interesting alternative possibility — known as quantum paramagnetism — is the avoidance of such ordering even at zero temperature. Quantum paramagnets may be of various types. A fascinating and intensely–studied class is the quantum spin liquids: these display many novel phenomena, for instance fractionalization of quantum numbers and topological order, or gapless excitations that are robust despite the absence of broken symmetries [10, 11, 12].

Recently there has been much progress in understanding a different type of remarkable quantum paramagnet. These are phases which have a bulk gap and no fractional quantum numbers or topological order. Despite this, they have nontrivial surface states that are protected by global symmetries. These properties are reminiscent of the celebrated electronic topological band insulators. Hence they have been called topological paramagnets [43]. Topological paramagnets and topological band insulators are both examples of what are known as Symmetry Protected Topological (SPT) phases [37, 38, 39, 40, 41, 42, 16, 17]. In the last few years tremendous progress has been made in understanding such SPT phases and their physical properties in diverse dimensions (for reviews, see Refs. [123, 124]).

The main focus of the present chapter is on three-dimensional topological paramagnets that are protected by time reversal (we also briefly discuss topological paramagnets protected by other symmetries, notably conservation of at least one spin component). These are interesting for a number of reasons. First, time reversal is a robust symmetry of typical physical spin Hamiltonians. In 1D the familiar Haldane/AKLT(Affleck-Kennedy-Lieb-Tasaki) chain is the only time reversal protected topological paramagnet[35, 36, 139, 140] while in 2D there are no time reversal protected topological paramagnets. In 3D however there are three distinct non-trivial phases [43, 1, 44] (corresponding to a classification by the group $\mathbb{Z}_2^2$). Second, regarded as an *electronic* insulator, unlike the 1D Haldane chain [125], these 3D topological paramagnets survive as distinct interacting SPT insulators [3]. The properties and experimental fingerprints of such topological paramagnets were described in Refs. [43, 1, 44, 3]. However there is currently very little understanding of where such phases might actually be found. In this chapter we propose that frustrated spin-1 Mott insulators may be good places to look for an example of such phases.

Already in the familiar 1D example it is the spin-1 antiferromagnetic chain, rather than the spin-1/2 chain, that naturally becomes a topological paramagnet. In 3D for one of the topological paramagnets we provide a physical picture and a parton construction which are both very natural for the spin-1 case. We hope that our observations inspire experimental and numerical studies of frustrated spin-1 quantum magnetism in the future. Towards the end of the chapter we remark on materials that may form such interesting frustrated magnets.

The three 3D topological paramagnets that are protected by time reversal symmetry alone [43, 1, 44] all allow for a gapped surface with $\mathbb{Z}_2$ topological order (*i.e.* a gapped surface $\mathbb{Z}_2$ quantum spin liquid) even though the bulk itself is not topologically ordered. The properties of this *surface* theory give a useful way to label the bulk phases. The surface has gapped quasiparticle excitations — labelled 'e' and 'm' — which are mutual semions. These may be thought of as the electric charge and magnetic flux of a deconfined $\mathbb{Z}_2$ gauge theory (like the vertex and plaquette defects of Kitaev's toric code [136]). At the SPT surfaces these particles have properties — self-statistics or time reversal transformation properties — that are impossible in a strictly 2D system, and which encode the topology of the bulk wavefunction. The three nontrivial bulk states are denoted:

$$eTmT, \qquad efTmfT, \qquad efmf.$$

In the first and second, the surface e and m excitations are each Kramers doublets under time reversal, denoted by T. In the second and third they are fermions (f), while in the first they are bosons. This chapter focuses primarily on the '$eTmT$' state.

We begin by explaining a physical picture of a suitable ground state wave function for the $eTmT$ topological paramagnet. This is most easily visualized on a diamond lattice. We first close–pack each interpenetrating fcc sublattice of the diamond lattice with closed loops. On each loop we place all the spin-1 moments (located at the diamond sites) in the ground state of the 1D AKLT chain. We then superpose all such loop configurations with a crucial (-1) sign factor whenever loops from the two different fcc sublattices link. We argue that this construction yields the topological paramagnet.

To understand the topological properties of such a wave function we describe a simple exactly solvable loop gas Hamiltonian — equivalent to two coupled Ising gauge theories — that clarifies the role of the '$(-1)^{\text{linking}}$' sign structure. The same model has also been independently studied in Ref. [137]. In this solvable model the loops do not have AKLT cores but there are two species of loops on different sublattices with the mutual (-1) linking sign. It demonstrates very simply how this sign leads to a state without intrinsic topological order. (This loop gas is not in the $eTmT$ state, because of the absence of AKLT cores, but we show it to be nontrivial in a different sense.)

Next we use the two–orbital fermionic parton representation developed for spin-1 magnets [138] to construct possible ground states. When the fermionic partons have the mean–field dispersion of a certain topological superconductor, we show that the gauge fluctuations associated with the parton description convert the system into a topological paramagnet. In this construction the mean field state is *unstable* toward confinement by gauge fluctuations, as a result of a continuous nonabelian gauge symmetry. Despite this the bulk gap survives, leaving behind a non-trivial surface that we are able to identify as that of the $eTmT$ topological paramagnet. As a warm up exercise to illustrate some of the ideas of this 3D construction, we also describe how to access the 1D Haldane phase by confining a topological superconductor of parton fermions. The 3D construction naturally suggests alternative bulk wave functions for topological paramagnets, in the form of Gutzwiller–projected topological superconductors. This may be fruitful for future numerical work on the energetics of microscopic models.

This parton construction also gives access to other SPT states for quantum magnets in 3D. For instance we show how to naturally obtain an SPT paramagnet (dubbed $eCmT$ in Ref. [1]) protected by $\mathrm{U}(1) \times \mathbb{Z}_2^T$, where the U(1) describes rotation about one spin axis, say S_z, and $\mathbb{Z}_2^T$ is time reversal.

Finally we show how to access a bulk U(1) quantum spin liquid with non-trivial implementation of time reversal symmetry. Interestingly simply condensing the magnetic monopole of this U(1) spin liquid leads to an SPT state dubbed $eCTmT$ in the presence of both spin rotation and time reversal symmetries. If only time reversal is present this becomes the $eTmT$ state.

7.1 Loop gas states

In this section we describe a loop gas wavefunction that is naturally adapted to spin one magnets and gives an intuitive picture for the $eTmT$ state. The wavefunction is a superposition of loop configurations, with each loop representing an AKLT state [139, 140] for the spins lying on it. A given configuration enters the superposition with a sign factor determined by its topology: specifically, the loops come in two species, A and B (one associated with each sublattice of the bipartite diamond lattice) and the sign depends on the linking number of A loops with B loops. This geometrical picture makes the relationship between the bulk wavefunction and the surface excitations particularly simple. The surface e and m excitations are endpoints of the two species of AKLT chains, and are Kramers doublets since an AKLT chain has dangling spin-1/2s at its ends.

In Sec. 7.2 we describe a similar wavefunction for 'pure loops', i.e. loops that do not carry an internal AKLT structure. This may be regarded as a state of two coupled Ising gauge theories. It is *not* in the $eTmT$ phase, but it illustrates the basic features of the loop gases in a simple model with an exactly solvable Hamiltonian. This 'pure loop' model is also interesting in its own right: when open strands (as opposed to closed loops) are banished from the Hilbert space, i.e. when charge is absent, it is in a nontrivial phase despite the absence of topological order. Therefore it may be viewed as a 'constraint–protected' state. It would be interesting to relate this to the recent ideas of Ref. [141]. We note that the constrained models discussed in Ref. [142] are also believed to be separated from the trivial phase by a phase transition, despite the absence of topological order.

The wavefunctions discussed here are in a similar spirit to the Walker Wang models, which are formulated in terms of string nets with a nontrivial sign structure, and show bulk confinement and surface topological order [143, 144, 44]. Constructions of SPTs using Walker Wang models were given in Refs. [44, 145]. 2D 'symmetry-enriched' topological states [146, 147, 148] and SPT states [149] have also been constructed by attaching AKLT chains to loop-like degrees of freedom (see also [150]).

7.1.1 Fluctuating AKLT chains

The diamond lattice is made up of two fcc sublattices, A and B. If $\mathcal{C}_A$ is a configuration of fully packed loops on A (with every A site visited by exactly one loop), we define $|\mathcal{C}_A\rangle$ to be a product of AKLT states $|\mathcal{L}\rangle$ for each of the loops $\mathcal{L}$ in $\mathcal{C}_A$,

$$|\mathcal{C}_A\rangle = \prod_{\mathcal{L}\in\mathcal{C}_A} |\mathcal{L}\rangle\,. \tag{7.1}$$

Figure 7-1: Two species of AKLT loops, one on each sublattice of the diamond lattice (blue and red). Note that loops live on the links of the fcc sublattices, i.e. on *next-nearest*-neighbour bonds of diamond.

Similarly $|\mathcal{C}_B\rangle$ is the state corresponding to a loop configuration $\mathcal{C}_B$ on B. To define the AKLT states $|\mathcal{L}\rangle$ fully we must choose an orientation for the fcc links, as discussed below (Sec. 7.1.2).

Let $X(\mathcal{C}_A, \mathcal{C}_B)$ be the mutual linking number of the two species of loops. Since the loops are unoriented, this is defined modulo two: $X(\mathcal{C}_A, \mathcal{C}_B) = 0, 1$. A schematic wavefunction for the $eTmT$ phase may be written in terms of $X(\mathcal{C}_A, \mathcal{C}_B)$:

$$|\Phi\rangle = \sum_{\mathcal{C}_A, \mathcal{C}_A} (-1)^{X(\mathcal{C}_A, \mathcal{C}_B)} |\mathcal{C}_A\rangle |\mathcal{C}_B\rangle . \tag{7.2}$$

For concreteness, we take periodic boundary conditions. The sums over $\mathcal{C}_A$ and $\mathcal{C}_B$ are then each restricted to loop configurations with an *even* number of strands winding around the 3D torus in each direction, for reasons discussed below. This global constraint, together with the geometrical fact that the links of A never intersect those of B, ensures that $X(\mathcal{C}_A, \mathcal{C}_B)$ is well defined.

The entanglement between the two sublattices in Eq. 7.2 is entirely due to the sign factor. First consider what happens in the *absence* of this sign factor. Each sublattice then hosts a superposition of loop configurations with positive amplitude, e.g. $\sum_{\mathcal{C}_A} |\mathcal{C}_A\rangle$. By analogy with the usual picture of deconfined $\mathbb{Z}_2$ gauge theory as a superposition of electric flux loop configurations [151], we would expect such a state to show $\mathbb{Z}_2$ topological order. (It is a 3D version of the 'resonating AKLT' states studied in 2D [146, 147, 148].) The endpoint of an open AKLT chain is the deconfined $\mathbb{Z}_2$ charge in this state. Associated with the topological order is ground state degeneracy — different ground states are distinguished by the parity of the winding number in each spatial direction.

In contrast, $|\Phi\rangle$ is *not* expected to show topological order, despite the proliferation of long loops in Eq. 7.2. Instead it describes a phase in which the endpoints of open chains are confined in the bulk. Furthermore there is no ground state degeneracy: states with odd winding numbers are not ground states (i.e. are not locally indistinguishable from $|\Phi\rangle$).

More detailed discussion of this is deferred for the solvable model of Sec. 7.2, but the basic idea is the following. While the amplitude $(-1)^{X(\mathcal{C}_A,\mathcal{C}_B)}$ depends on the *global* topology of the loop configurations, it amounts to the simple *local* rule that the amplitude changes sign if an A strand is passed through a B strand. It is useful to imagine a hypothetical parent Hamiltonian that imposes this sign rule. But the sign rule cannot be consistently imposed if the wavefunction includes open strands or configurations with odd winding numbers (see below). Similar phenomena occur in the confined Walker–Wang models [143, 144, 44].

However, open endpoints are deconfined at the boundary, for appropriate boundary conditions. The minus sign associated with passing an A strand through a B strand in the bulk means that the endpoints are mutual semions [101] — see Fig. 7-2. They are also Kramers doublets. These surface properties are the defining features of the $eTmT$ state. The wavefunction $|\Phi\rangle$ has more symmetry than simply time reversal (e.g. separate spin rotation symmetries for each sublattice) but if it is indeed in the $eTmT$ phase then these symmetries could be weakly broken without leaving the phase.

7.1.2 Further details on fluctuating AKLT state

To write the AKLT-based state explicitly it is convenient to represent the spin-one at each site i in terms of auxiliary spin-1/2 bosons [139, 140, 150]. If the boson creation

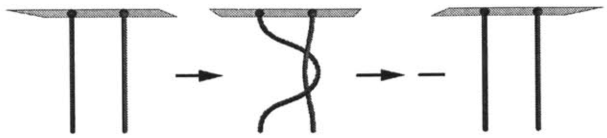

Figure 7-2: For appropriate boundary conditions, endpoints of A and B chains (red and blue respectively) give surface excitations with mutual semionic statistics. Braiding the anyons on the surface (first arrow) changes the sign of the wavefunction, for consistency with the rule that configurations related by passing an A strand through a B strand in the bulk (second arrow) appear in the wavefunction with opposite sign.

operators are $b^\dagger_{i\alpha}$ ($\alpha =\uparrow,\downarrow$), then $\mathbf{S}_i = \frac{1}{2} b^\dagger_{i\alpha}\sigma_{\alpha\beta}b_{i\beta}$. The occupation number $b^\dagger_{\alpha i}b_{i\alpha}$ is equal to two to ensure spin one at each site. The AKLT state $|\mathcal{L}\rangle$ is then created by acting on the boson vacuum with operators $S^\dagger_{ij}$ that create singlet pairs on the links of the loop, which we normalize as $S^\dagger_{ij} = \frac{1}{\sqrt{3}}(b^\dagger_{i\uparrow}b^\dagger_{j\downarrow} - b^\dagger_{i\downarrow}b^\dagger_{j\uparrow})$. This operator is antisymmetric in (i,j), so to define $|\Phi\rangle$ we must fix an orientation for the links of each fcc sublattice. (The fcc lattice has four sublattices, a, b, c, d, so for example we could orient the links from $a\to b$, $a \to c$, $a \to d$, $b\to c\to d\to b$, with the orientations on each sublattice related by inversion symmetry.) Then for each sublattice

$$|\mathcal{C}\rangle = \prod_{\langle ij\rangle\in\mathcal{C}} S^\dagger_{ij}\,|\text{vac}\rangle\,, \tag{7.3}$$

where i is the site at the tail of the oriented link $\langle ij\rangle$. These states satisfy $\langle\mathcal{C}|\mathcal{C}\rangle = \prod_{\text{loops}}(1+(-1)^\ell/3^{\ell-1})$, where ℓ is the length of a given loop [139, 140].

It should be noted that that expectation values in the state $|\Phi\rangle$ are nontrivial, in particular because overlaps $\langle\mathcal{C}|\mathcal{C}'\rangle$ for distinct $\mathcal{C}$, $\mathcal{C}'$ are nonzero. So while it is plausible that $|\Phi\rangle$ is in the $eTmT$ phase, this cannot be established purely analytically. For example, the state could in principle break spatial or spin rotation symmetry spontaneously. A cautionary example is given by the uniform-amplitude resonating valence bond state for spin-1/2s on the cubic lattice: this has weak Néel order [152], despite being a superposition of singlet configurations which individually have trivial spin correlations. In the present model, the entanglement between sublattices supresses off-diagonal elements of the reduced density matrix when written in the AKLT-chain

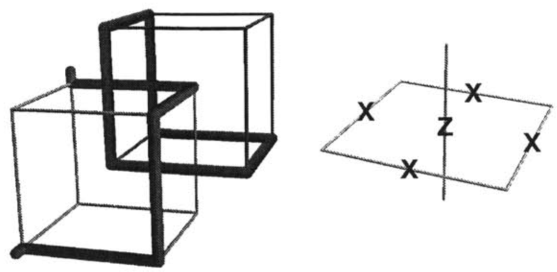

Figure 7-3: Left: Loops on interpenetrating cubic lattices A and B. The state $|\Psi\rangle$ is a superposition of such configurations with signs determined by linking of A and B loops. Right: the product of Pauli matrices defining the flip term $\mathcal{F}$ on a plaquette (see Eq. 7.6).

basis[1]. Together with the non-bipartiteness of the fcc lattice, this makes spin order seem less likely. But since $|\Phi\rangle$ is intended to illustrate the topological structure of the phase, and not as a ground state of a realistic Hamiltonian, it may not be crucial whether it is in the desired phase as written or whether further tuning of the amplitudes is required.

7.2 'Pure loop' state

It is enlightening to look at the simplest model[137] that captures the $(-1)^{\text{linking}}$ sign structure. To this end we take a system of spin-1/2s on the *links* of two interpenetrating cubic lattices A and B, as shown in Fig. 7-3. We think of a down spin (in the 'z' basis) as an occupied link, and an up spin as an unoccupied one. The number of occupied links at each vertex is always even in the state we consider, so the configurations of occupied links, $\mathcal{C}_A$ and $\mathcal{C}_B$, can be decomposed into closed loops.[2] We refer to $\mathcal{C}_A$ and $\mathcal{C}_B$ as loop configurations. Other solvable loop gas/string net models have been considered in Refs. [144, 44], using the Walker Wang construction [143].

The 'pure loop' state analogous to $|\Phi\rangle$ above is (again we sum only over loop

[1]In the 'pure loop' state of Sec. 7.2, the reduced density matrix for a single sublattice is diagonal: $\rho_A \propto \sum_{\mathcal{C}_A} |\mathcal{C}_A\rangle \langle \mathcal{C}_A|$. The reduced density matrix for the AKLT based state does not have a simple form, but by analogy we expect a suppression of off-diagonal elements. (In the artificial limit where they are completely suppressed, the spins are trivially short-range correlated.)

[2]With a harmless ambiguity when the number of occupied links at a vertex exceeds two.

configurations with even winding numbers on each sublattice):

$$|\Psi\rangle = \sum_{\mathcal{C}_A,\mathcal{C}_B} (-1)^{X(\mathcal{C}_A,\mathcal{C}_B)} |\mathcal{C}_A\rangle |\mathcal{C}_B\rangle . \tag{7.4}$$

We may view $\mathcal{C}_A$ and $\mathcal{C}_B$ as the electric flux line configurations for a pair of coupled $\mathbb{Z}_2$ gauge fields, with one $\mathbb{Z}_2$ gauge field living on each cubic lattice. Imposing the above sign structure for the two sets of electric flux lines is equivalent to binding the electric flux line of each gauge field to the magnetic flux line of the other, as will be clear shortly.

It is straightforward to write down a gapped parent Hamiltonian $\mathcal{H}_{\text{linking}}$ for $|\Psi\rangle$, using the fact that flipping the occupancy of all the links on the plaquette changes the linking number $X(\mathcal{C}_A, \mathcal{C}_B)$ if and only if the link piercing the plaquette is occupied. $\mathcal{H}_{\text{linking}}$ is a sum of terms for the plaquettes p of each cubic lattice:

$$\mathcal{H}_{\text{linking}} = -\left(J \sum_{p\in A} \mathcal{F}_{Ap} + J \sum_{p \in B} \mathcal{F}_{Bp} \right) . \tag{7.5}$$

The operators $\mathcal{F}_A$ and $\mathcal{F}_B$ flip the occupancy of the links on a plaquette, with a sign that depends on whether the link piercing it is occupied. Allowing p to denote both a plaquette and the link piercing it, and denoting the Pauli operators on A and B by σ and τ respectively,

$$\mathcal{F}_{Ap} = \tau_p^z \prod_{l\in p} \sigma_l^x, \qquad \mathcal{F}_{Bp} = \sigma_p^z \prod_{l \in p} \tau_l^x . \tag{7.6}$$

These operators all commute, so the Hamiltonian is trivially solvable. $|\Psi\rangle$ is the unique ground state and minimises each term of $\mathcal{H}_{\text{linking}}$ since $\mathcal{F}|\Psi\rangle = |\Psi\rangle$ for each plaquette operator.

The state $|\Psi\rangle$ contains only closed loops, i.e. it satisfies

$$\prod_{l\in v} \sigma_l^z = 1 \qquad \text{for } v \in A, \qquad \prod_{l\in v} \tau_l^z = 1 \qquad \text{for } v \in B \tag{7.7}$$

where v denotes a vertex and $l \in v$ the links touching v. Any state satisfying $\mathcal{F}\left|\Psi\right\rangle = \left|\Psi\right\rangle$ for all the plaquette operators must also satisfy these vertex conditions, because $\prod_{l\in v}\sigma_l^z$ and $\prod_{l\in v}\tau_l^z$ can be written as products of $\mathcal{F}$s.

We may regard Eqs. 7.7 as the gauge constraints for a pair of pure $\mathbb{Z}_2$ gauge theories (the $\mathbb{Z}_2$ versions of $\nabla.\mathbf{E} = 0$). The two electric fields are given by σ^z and τ^z and live on the links of A and B respectively. The magnetic field of each gauge field lives on the links of the *opposite* lattice to its electric field. For example the magnetic field of σ is given by $\prod_{l\in p}\sigma^x$, where p is a plaquette of A, or equivalently a link of B.

In this language, $\mathcal{H}_{\text{linking}}$ simply glues the electric flux line of each species to the magnetic flux line of the other. The σ–magnetic flux and the τ–electric flux are equal since $\mathcal{F}_A = 1$, and the σ–electric and τ–magnetic fluxes are equal via $\mathcal{F}_B = 1$.

The state $\left|\Psi\right\rangle$ is not topologically ordered. Neither is it a time-reversal protected SPT: it can be adiabatically transformed to a product state without breaking time reversal symmetry. However it *is* protected if impose Eqs. 7.7 as constraints: i.e. if we forbid open strands, as opposed to closed loops. In the gauge theory language, this means forbidding charge. With this constraint it is impossible to reach a trivial state without going through a phase transition, as follows from the self–duality of the state described in Sec. 7.2.1.

We will explain these features from several points of view below. One convenient approach which leads to a geometric picture is to switch from the (σ^z, τ^z) basis used in Eq. 7.4 to the (σ^z, τ^x) basis. The σ^z configuration is a loop configuration on the A lattice, as above. We represent the τ^x configuration by a configuration of *membranes* made up of plaquettes on the A lattice: $\tau_p^x = -1$ represents an occupied plaquette, and $\tau_p^x = 1$ an unoccupied one.

The $\mathcal{F}_B$ terms in $\mathcal{H}_{\text{linking}}$ act on a link of the A lattice together with the four plaquettes touching it. $\mathcal{F}_B = 1$ imposes the rule that the σ^z loops are glued to the boundaries of the τ^x membranes, i.e. to the links where an odd number of occupied plaquettes meet. This is the gluing of σ–electric flux lines (where $\sigma^z = -1$) to τ–magnetic flux lines (where $\prod \tau^x = -1$) mentioned above.

Let $\mathcal{M}$ denote a membrane configuration, and $\left|\mathcal{M}\right\rangle$ the corresponding state with

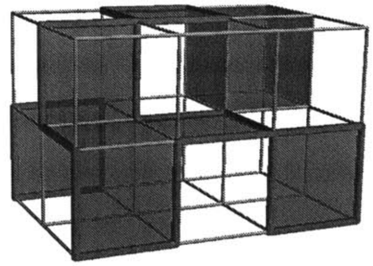

Figure 7-4: After a basis change, $|\Psi\rangle$ is a superposition of membrane configurations ($\tau^x = -1$ on shaded plaquettes) with red loops (where $\sigma^z = -1$) glued to membrane boundaries. (The red loops are σ–electric lines and the membrane boundaries are τ–magnetic lines.)

$\tau^x = -1$ on the occupied plaquettes. Let $\partial\mathcal{M}$ be the loop configuration given by the boundaries of the membranes in $\mathcal{M}$. Then $|\Psi\rangle$ can be written (neglecting an overall constant)

$$|\Psi\rangle = \sum_{\mathcal{C}_A} \sum_{\substack{\mathcal{M} \\ \partial\mathcal{M}=\mathcal{C}_A}} |\mathcal{C}_A\rangle\,|\mathcal{M}\rangle\,. \tag{7.8}$$

Fig. 7-4 shows the geometrical interpretation of this state. It is a soup of τ^x membranes, with σ^z loops glued to their boundaries.

Confinement of string endpoints is easy to see in this basis. A pair of vertex excitations at which $\prod_{l\in v}\sigma^z_l = -1$ are connected by an open string. Since the boundary of $\mathcal{M}$ contains only closed loops, the open string makes it impossible to satisfy the gluing of strings to membrane boundaries demanded by the $\mathcal{F}_B$ terms in $\mathcal{H}_{\text{linking}}$. If the separation of the vertex defects is D, there must be at least D unsatisfied links, giving a linear confining potential for such defects. For similar reasons, a configuration with an odd number of winding σ^z strands in some direction costs an energy proportional to the spatial extent of the system in this direction. By symmetry, this applies equally to the τ^z strings that are present in the original basis.

We can also understand the confinement of string endpoints algebraically (Refs. [144, 44] give analogous arguments for bulk confinement and surface topological order in the Walker Wang models). The Hamiltonian in Eq. 7.5 is clearly exactly soluble not just for the ground state but for all excited states. An 'elementary' excitation is given by a 'defect' in some square plaquette, say on the B lattice, with

$$\mathcal{F}_{Bp} = -1 \tag{7.9}$$

while $\mathcal{F} = +1$ on all other plaquettes of either sublattice. Such a defect plaquette costs energy $2J$. It leads to a violation of the closed loop vertex constraint for σ^z on the two vertices of the A sublattice connected by the A-link that penetrates the defect plaquette. Thus the excitation we have created has two string end-points on nearest neighbor A-sites. To move these string endpoints apart by a distance D we must create $O(D)$ such defect plaquettes. Consequently the energy cost is also $O(D)$ and we have linear confinement of string endpoints.

In the gauge theory language, the reason for the absence of deconfined excitations is that the tensionless lines in this state are not lines of pure electric flux, but rather of electric flux together with magnetic flux of the other species. If such lines could end, their endpoints would be deconfined excitations. But the Hilbert space does not allow for such excitations: a magnetic flux line cannot terminate in the bulk (by virtue of its definition in terms of e.g. $\prod \tau^x$).

Despite the lack of deconfined endpoints in the bulk, A and B strings that terminate on a boundary can give deconfined e and m particles in a surface $\mathbb{Z}_2$ topologically ordered state. To see this, we terminate the system as in Fig. 7-5, including in the Hamiltonian the natural plaquette and vertex terms at the surface. The surface string operators that create pairs of e or pairs of m excitations can then be written explicitly (see Fig. 7-5). They satisfy the same algebra as the string operators in the toric code [136], confirming that e and m are mutual semions as expected from the heuristic argument of Fig. 7-2.

We can adiabatically transform $|\Psi\rangle$ to a product state so long as we allow the intermediate states to violate the closed–loop constraints on at least one sublattice. The membrane picture gives an obvious way to do this, by giving the membranes in $\mathcal{M}$ a surface tension. If 'Area' denotes the number of occupied plaquettes in $\mathcal{M}$, the interpolating state is

$$|\Psi\rangle_\gamma = \sum_{\mathcal{C}_\mathcal{A}} \sum_{\substack{\mathcal{M} \\ \partial\mathcal{M}=\mathcal{C}_A}} e^{-\gamma\times\text{Area}} |\mathcal{C}_A\rangle |\mathcal{M}\rangle . \tag{7.10}$$

When $\gamma = 0$ this is the initial state, and when $\gamma \to \infty$ only the term with zero area survives. This is the state with no loops and no membranes, i.e. the product state $|\sigma^z = 1\rangle |\tau^x = 1\rangle$. To get a gapped parent Hamiltonian for $|\Psi\rangle_\gamma$, we modify the plaquette flip term $\mathcal{F}_A$ in $\mathcal{H}_{\text{linking}}$ to $\mathcal{F}_{Ap} = (\cosh\gamma)^{-1}\left[\tau_p^z \prod_{l\in p} \sigma_l^x + (\sinh\gamma)\, \tau_p^x\right]$. This preserves the simple algebraic properties of the plaquette terms. From the fact that the modified $\mathcal{F}_{Ap}$ does not commute with the closed-loop constraint on the B lattice (or by directly transforming to the τ^z basis) we see that $|\Psi\rangle_\gamma$ violates this constraint when $\gamma > 0$.

7.2.1 Self-duality of $|\Psi\rangle$ and protection by constraints

When the interpolating state above is rewritten in the original (σ^z, τ^z) basis, it includes configurations with open strands, as well as closed loops, on the B lattice. What if we impose the constraint that both lattices have only closed loops? In this case it is impossible to go from $|\Psi\rangle$ to a trivial state without a phase transition. (We will take the reference trivial state to be that with no loops, $|\text{trivial}\rangle = |\sigma^z = 1\rangle |\tau^z = 1\rangle$.)

This follows from a simple duality transformation which exchanges the electric

Figure 7-5: String operators creating surface excitations. Left: acting with a chain of σ^x operators on the links of the upper layer (A lattice surface) gives a pair of e excitations (i.e. endpoints of bulk A strings). Right: a pair of m excitations (i.e. endpoints of B strings) are created by a chain of τ^x operators (thick green strand) on the lower layer (B surface), together with σ^z operators on the corresponding links in the upper layer (thick purple links).

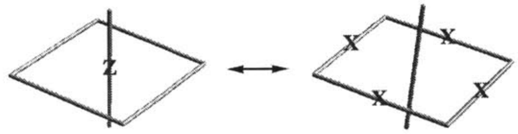

Figure 7-6: Under the mapping (7.11), a σ^z (or τ^z) operator on a link is exchanged with a product of τ^x (resp. σ^x) operators on the surrounding links of the other lattice. (Links of one lattice can equally be thought of as plaquettes of the other.)

flux of each species with the magnetic flux of the other species. The duality maps $|\Psi\rangle$ to itself, but exchanges the trivial state with a topologically ordered one. Thus there is no adiabatic path from $|\Psi\rangle$ to the trivial state. If there were, duality would yield an adiabatic path from $|\Psi\rangle$ to the topologically ordered state, and this is impossible since $|\Psi\rangle$ is not topologically ordered.

The duality transformation makes sense for states obeying the closed loop constraint. (To be precise, we must also impose the global constraint that the loop configurations have even winding in each direction.) As shown in Fig. 7-6, its action is:

$$\sigma_l^z \longleftrightarrow \prod_{p \in l} \tau_p^x, \qquad \tau_p^z \longleftrightarrow \prod_{l \in p} \sigma_l^x \tag{7.11}$$

Here $p \in l$ denotes the four plaquettes p surrounding link l. We have labelled the σs by l for link and the τs by p for plaquette, but the duality acts on the two sets of degrees of freedom symmetrically. It preserves the locality of any Hamiltonian acting in the constrained Hilbert space.

For completeness, we write the action of the duality on states explicitly. Return to the picture of loops + membranes on the A lattice, i.e. the (σ^z, τ^x) basis. One may check that any state satisfying the constraints can be written as a sum over two loop configurations on the *same* lattice,

$$|f\rangle = \sum_{\mathcal{C}_A, \mathcal{C}'_A} f(\mathcal{C}_A, \mathcal{C}'_A) \, |\mathcal{C}_A\rangle_\sigma \, \widetilde{|\mathcal{C}'_A\rangle}_\tau, \tag{7.12}$$

where $\widetilde{|\mathcal{C}'_A\rangle}_\tau$ is defined as the uniform superposition of all membrane configurations

$|\mathcal{M}\rangle_\tau$ with boundary $\partial\mathcal{M} = \mathcal{C}'_A$. We have added subscripts to the kets as a reminder of the degrees of freedom involved. ($\mathcal{C}_A$ is the σ–electric flux configuration, and $\mathcal{C}'_A$ the τ–magnetic flux configuration; the fact that the wavefunction depends on $\mathcal{M}$ only through $\partial\mathcal{M}$ is simply a statement of gauge invariance.) The duality then simply exchanges the two kinds of loops,

$$f(\mathcal{C}_A, \mathcal{C}'_A) \longleftrightarrow f(\mathcal{C}'_A, \mathcal{C}_A). \tag{7.13}$$

The flip operators $\mathcal{F}_A$ and $\mathcal{F}_B$ (Eq. 7.6) are clearly invariant under the duality in Eq. 7.11 and therefore so is $\mathcal{H}_{\text{linking}}$. (We can also see that $|\Psi\rangle$ is invariant from Eq. 7.13 and Eq. 7.8.) On the other hand, the trivial Hamiltonian

$$\mathcal{H}_{\text{trivial}} = -\left(J \sum_{l \in A} \sigma^z_l + J \sum_{l \in B} \tau^z_l \right) \tag{7.14}$$

is exchanged with

$$\mathcal{H}_{\text{deconfined}} = -\left(J \sum_{p \in A} \prod_{l \in p} \sigma^x_l - J \sum_{p \in B} \prod_{l \in p} \tau^x_l \right), \tag{7.15}$$

which describes a pair of deconfined $\mathbb{Z}_2$ gauge theories. This establishes the claim at the beginning of this subsection: while the linking state is invariant, the trivial state is exchanged with a topologically ordered state. It follows that the linking state is in a distinct phase from the trivial state if we do not allow open endpoints in the Hilbert space. (We know from Eq. 7.10 that they are in the same phase if we *do* allow endpoints.)

7.2.2 Heuristic relation between symmetry protection of $eTmT$ and closed–loop constraint

The proposed wave function for the $eTmT$ phase has the two loop species 'stuffed' with Haldane/AKLT chains. The linking sign factor ensures that the ground state is not topologically ordered as required for a topological paramagnet. In particular the open end-points of the loops — which now harbor a Kramers doublet — are confined. However as described in Sec. 7.1.1 the surface implements time-reversal 'anamolously' exactly characteristic of the $eTmT$ state.

We now briefly consider whether the results in the previous subsection for the 'pure loop' state yield a heuristic 'bulk' understanding of why the $eTmT$ state is protected by time reversal. So let us imagine perturbing the schematic $eTmT$ wavefunction of Sec. 7.1.1, and ask why we cannot reach a trivial state without a phase transition.

We make use of the heuristic analogy between the AKLT loops of the spin-1 system and the 'pure loops' of the coupled gauge theory.[3] The result for the pure loop state then indicates that if we only have closed AKLT loops on each sublattice, we cannot get to a trivial state without a phase transition. So, we must consider proliferating open strands on at least one sublattice. But in the spin-1 system, unlike the pure-loop system, open strands introduce bulk spin-1/2 Kramers doublet degrees of freedom. (Binding these emergent spin–1/2s into singlets with others on the same sublattice merely heals the AKLT chains, taking us back to the original situation with separate closed loops on each sublattice.) When time reversal is broken, these spin–1/2s are innocuous — for example we can gap them out using a magnetic field. But it is natural to expect that when time reversal is preserved they prevent us reaching a trivial state without closing the gap.

However, the above argument is incomplete as it does not rule out the possibility of getting to a trivial state by proliferating nearby pairs of open strands on *opposite* sublattices. Such a pair gives two spin–1/2s which can be bound into a singlet to avoid a gapless degree of freedom. In the gauge theory, such pairs correspond to bound pairs of electric charges, one from each $\mathbb{Z}_2$ gauge field. The stability of the $eTmT$ state suggests that the pure loop state remains protected even when such double charges are allowed. We note that at the surface these double charges correspond to the bound state of the e and m particle (in the surface topological order). This is a Kramers singlet spin-0 fermion (conventionally denoted ϵ). The surface Fermi

[3]Recall that the singlet basis allows us to represent any spin-zero state of the spin-1 system in terms of loops of spin-1/2 singlet bonds; these may form minimal length 'loops' which backtrack on a single link, i.e. spin-1 singlet bonds, or longer AKLT loops.

statistics suggests a potential obstruction to 'trivializing' the bulk by proliferating the double charges. We leave an explicit demonstration of this for the future.

7.3 Parton constructions

Though the description of the $eTmT$ topological paramagnet in terms of a loop gas wave function is physically appealing it is desirable to have alternate descriptions which enhance our understanding and which may help with evaluating the energetic stability of this phase in microscopic models. To that end, in this section we propose explicit parton constructions for some topological paramagnets in spin-1 systems.

Historically the parton approach has provided variational wave functions and effective field theories both for spin liquids [10] and non-fractionalized symmetry-breaking states [153]. The parton construction inevitably introduces a gauge symmetry. It describes a fractionalized spin liquid phase whenever it yields an emergent deconfined gauge field. To obtain a non-fractionalized phase such as conventional antiferromagnet or valence bond solid paramagnet, the gauge field should either be Higgsed or confined.

Recently the parton construction has been used to construct SPT states in two [128, 129, 130, 131] and three [132, 133, 154] dimensions. The general idea is to construct a gauge theory (with matter fields) that is confined, but with certain nontrivial features surviving in the confined state that make it an SPT state. However, the currently known constructions in three dimensions use either $\mathbb{Z}_2$ or U(1) gauge theories, which do not confine automatically: strong gauge coupling is needed to reach the confined phase. Furthermore, the constructions using U(1) gauge theories [132, 133] require highly nontrivial dynamics of the gauge fields to condense composite dyon-like objects.

In three dimensions, a continuous non-abelian gauge symmetry is needed to guarantee confinement. We propose two parton constructions in three dimensions with SU(2) gauge symmetry, which confine even if the bare gauge coupling is small, giving rise to topological paramagnets. A similar construction was used previously [131] in

2D to describe an SPT phase of a spin-1 magnet protected by spin SU(2) symmetry and time reversal. We also propose a construction with U(1) gauge symmetry, which confines at sufficiently strong coupling. Crucially, this U(1) construction differs from previous ones in that we only condense simple monopoles to confine the gauge theory, which can be achieved at strong coupling without exotic form of gauge field dynamics.

The spin-1 operators are re-written using the two-orbital fermionic parton representation proposed in Ref. [138],

$$\mathbf{S} = \frac{1}{2} \sum_{a=1,2} f_{a\alpha}^\dagger \sigma_{\alpha\beta} f_{a\beta}. \tag{7.16}$$

where $a = 1, 2$ is the orbital index. As will be discussed below, the two-orbital structure is natural for topological bands corresponding to topological paramagnets. This gives another reason for favoring spin-1 systems.

The physical spin states are represented in the parton description as

$$|S_z = 0\rangle = \tfrac{1}{\sqrt{2}} \left(f_{1\downarrow}^\dagger f_{2\uparrow}^\dagger + f_{1\uparrow}^\dagger f_{2\downarrow}^\dagger \right) |\text{vac}\rangle,$$
$$|S_z = +1\rangle = f_{1\uparrow}^\dagger f_{2\uparrow}^\dagger |\text{vac}\rangle, \qquad |S_z = -1\rangle = f_{1\downarrow}^\dagger f_{2\downarrow}^\dagger |\text{vac}\rangle.$$

where $|\text{vac}\rangle$ is the state with no fermions. States in the physical spin Hilbert space thus have two fermions at each site, $\sum_{a\alpha} f_{a\alpha}^\dagger f_{a\alpha} = 2$, and the two fermions form a singlet in orbital space: denoting the Pauli matrices in orbital space by $\tau^{x,y,z}$, this is $\sum_\alpha f_{a\alpha}^\dagger \tau_{ab} f_{b\alpha} = 0$.

The representation in Eq. 7.16 actually has an Sp(4) gauge redundancy [138] which becomes apparent when we represent the fermions using Majoranas, $f = \frac{1}{2}(\eta_1 - i\eta_2)$. Here $\eta_{1,2}$ are Hermitian operators satisfying $\{\eta_{sI}, \eta_{s'J}\} = 2\delta_{ss'}\delta_{IJ}$, where $s, s' = 1, 2$ are the new indices associated with the Majoranas and I, J represent all other indices (site, spin, orbital). The Majorana representation of the spin is

$$\mathbf{S} = \frac{1}{8} \eta^T \mathbf{\Sigma} \eta, \qquad \mathbf{\Sigma} = \left(\rho^y \sigma^x, \sigma^y, \rho^y \sigma^z \right), \tag{7.17}$$

where $\rho^{x,y,z}$ are Pauli matrices acting on the Majorana index. The generators of the gauge symmetry are ten anti-symmetric imaginary matrices that commute with the physical spin operators:

$$\Gamma = \{\rho^y, \rho^y\tau^{x,z}, \rho^{x,z}\sigma^y, \rho^{x,z}\sigma^y\tau^{x,z}, \tau^y\}, \tag{7.18}$$

where τ_i are Pauli matrices acting on the orbital index. The spin in Eq. 7.17 is invariant under the Sp(4) gauge transformation $\eta \to e^{ia_i\Gamma_i}\eta$.

The effective field theory associated with the parton construction is a gauge theory. The gauge symmetry is determined by the mean field band structure of the partons, and is in general a subgroup of the full Sp(4) group due to some generators being Higgsed. The gauge structure allows symmetry to act projectively on the η fermion [10]. In particular, time-reversal could be either Kramers ($\mathcal{T}^2 = -1$) or non-Kramers ($\mathcal{T}^2 = 1$).

In 3D, band structures of Kramers fermions with $\mathcal{T}$ symmetry are classified by an integer index [21, 22] ν which counts the number of Majorana cones on the surface. It was realized [110, 4, 114] that in the presence of interactions the state with $\nu = 16$ is trivial, while that with $\nu = 8$ is equivalent to a topological paramagnet. More specifically, for $\nu = 8$ the surface state with four Dirac cones (eight Majorana cones) can be gapped without breaking any symmetry via strong interactions, and the resulting gapped surface state must have intrinsic topological order. The simplest such topological order is a $\mathbb{Z}_2$ gauge theory in which the e and m particles are bosons, but transform under time-reversal as Kramers doublets ($\mathcal{T}^2 = -1$). Therefore we can put the slave fermions into a band with $\nu = 8$, and let the gauge fields confine the fermions (either automatically through an SU(2) gauge field or at strong coupling through a U(1) gauge field). Crucially, the topological quasi-particles (e and m) on the surface do not carry the gauge charge, and they survive on the surface as deconfined objects. The resulting phases are therefore confined paramagnets with nontrivial surface states protected by time-reversal symmetry.

Non-Kramers fermions, by contrast, cannot host non-trivial band structure with time-reversal symmetry alone. However, if spin–S_z conservation is present, the band structures can again be assigned an integer topological invariant ν' [21, 22] which is the number of Dirac cones on the surface (or half the number of Majorana cones). It is known [110, 4, 114, 155] that with interactions the state with $\nu' = 8$ is trivial, while that with $\nu' = 4$ is equivalent to a topological paramagnet. We can then put the slave-fermions into a band with $\nu' = 4$ and let the gauge fields confine the fermions, which produces a topological paramagnet with time-reversal and spin-S_z conservation.

In both cases we need to put the slave fermions into band structures with four Dirac cones on the surface. Band structures with two Dirac cones ($\nu = 4$) have been studied on the cubic [84] and diamond [115] lattices. Therefore we can obtain the desired structure simply by putting the partons into two copies of the $\nu = 4$ band. This can be easily done by taking advantage of the two orbitals in Eq. 7.16, making the topological paramagnets very natural in spin-1 systems.

In the next section we outline a similar construction for the one–dimensional Haldane chain, by confining slave fermions which form four copies of the Kitaev chain. This illustrates the essential idea of our constructions in a simpler and more familiar context.

7.3.1 Parton construction for Haldane/AKLT chain

The Haldane phase is an SPT phase with gapless boundary degrees of freedom that are protected by time reversal. As a warm–up exercise, we outline how this phase can be constructed from a topological superconductor of slave fermions. This illustrates some features we will meet again in 3D. A different parton construction for the Haldane phase was considered in Ref. [156].

The fermions are taken to be non-Kramers ($\mathcal{T}^2 = 1$). In 1D, superconducting band structures for *free* non-Kramers fermions are labelled by a $\mathbb{Z}$–valued index [21, 22], ν, which is the number of protected Majorana zero modes at the boundary. The state with a given ν can be viewed as ν copies of Kitaev's p–wave superconducting chain [15]. Interactions reduce this classification to $\mathbb{Z}_8$, i.e. the $\nu = 8$ state becomes trivial [40, 41, 42]. Further, the state with $\nu = 4$ is topologically equivalent to the Haldane chain, modulo the presence of gapped fermions in a trivial band.

Here we therefore put the slave fermions into four copies of the Kitaev bandstructure, in an SU(2)–symmetric manner. Gauge fluctuations (or Gutzwiller projection) will then remove the unwanted degrees of freedom, leaving a topological paramagnet in the Haldane phase.

Starting with an antiferromagnetic spin–1 chain,

$$\mathcal{H} = J\sum_i \mathbf{S}_i.\mathbf{S}_{i+1} + \dots, \tag{7.19}$$

we represent the spins with slave fermions as in Eq. 7.16 or equivalently Eq. 7.17. The valence bond picture of the AKLT state suggests using a mean-field Hamiltonian for the partons with hopping t and spin–singlet, orbital–singlet pairing Δ,

$$H_{\text{MF}} = -\sum_i \left[t\left[f_i^\dagger f_{i+1} + \text{h.c.}\right] + \Delta\left[f_i^\dagger \sigma^y \tau^y f_{i+1}^{\dagger T} + \text{h.c.}\right]\right].$$

In terms of the Majoranas, this is

$$H_{\text{MF}} = -\frac{1}{2}\sum_i \eta_i^T M \eta_{i+1}, \qquad\qquad M = t\rho^y + i\Delta\rho^x\sigma^y\tau^y. \tag{7.20}$$

We first consider this as a free fermion problem, then include the gauge fluctuations.

For simplicity take $\Delta = t$, which makes the terms in H_{MF} for different links commute. The Hamiltonian is simply four copies of the Kitaev chain, as can be seen immediately by going to a basis where $\sigma^y\tau^y$ is diagonal. To be more explicit, it is useful to define the matrix

$$X = \rho^z\sigma^y\tau^y. \tag{7.21}$$

Firstly, we use this to define the action of time reversal $\mathcal{T}$ on the fermions:

$$\mathcal{T}: \qquad \eta \longrightarrow X\eta. \tag{7.22}$$

This definition ensures that the spin changes sign under $\mathcal{T}$ and that H_{MF} is invariant. The fermions are non-Kramers ($\mathcal{T}^2 = 1$ on η).

Secondly let us define matrices that project onto a given value of X, and corresponding fermion modes:

$$\mathcal{P}_{\pm} = \frac{1}{2}(1 \pm X), \qquad \eta^{(\pm)} = \mathcal{P}_{\pm}\eta. \tag{7.23}$$

In an appropriate basis, $\eta^{(+)}$ has four nonzero components. Next, note that

$$M = \mathcal{P}_{-} M \mathcal{P}_{+}, \tag{7.24}$$

since $M = t\rho^y(1 + \rho^z\sigma^y\tau^y) = (2t\rho^y)\mathcal{P}_{+}$, etc. So we may rewrite H_{MF} as

$$H_{\mathrm{MF}} = -\frac{1}{2}\sum_i \eta_i^{(-)T} M \eta_{i+1}^{(+)}. \tag{7.25}$$

Taking open boundary conditions, and denoting the leftmost site of the chain by L, we see that the four modes in $\eta_L^{(+)}$ do not appear in the Hamiltonian.

These four Majoranas correspond to two complex fermion modes that can be occupied or unoccupied, i.e to a degenerate four-dimensional boundary Hilbert space. At the level of free fermions, this degeneracy is protected by time reversal symmetry $\mathcal{T}$, under which $\eta_L^{(+)}$ is invariant (since by definition $X\eta^{(+)} = \eta^{(+)}$).[4]

Once we go beyond mean field theory, the fermions are coupled to confining gauge fluctuations. We will see below that two of the four boundary states are not gauge invariant — i.e. they can be thought of as having an unscreened gauge charge sitting at the end of the chain. Confinement removes these states from the low energy Hilbert space, leaving a single boundary spin–1/2 whose gaplessness is protected by time reversal.

H_{MF} treats spin and orbital degrees of freedom symmetrically, and preserves $\mathrm{SU}(2)_{\mathrm{spin}} \times \mathrm{SU}(2)_{\mathrm{orbital}}$ symmetry. The four boundary states can be labeled by the occupation numbers of two complex fermions $c_{1,2}$. Since the partons transform as

[4]Any quadratic term $i\eta^{(+)T}A\eta^{(+)}$ (where A is real antisymmetric) is forbidden as it is odd under $\mathcal{T}$. However in the presence of interactions a four fermion term $\gamma_1\gamma_2\gamma_3\gamma_4$ — where γ_i are the components of $\eta^{(+)}$ in some basis — is allowed by time reversal, and lifts the boundary degeneracy to a single doublet as in the Haldane chain [109].

doublets under each SU(2), the fermions $c_{1,2}$ should also form doublets under each SU(2). In an appropriate basis the transformations are

$$\begin{aligned} \mathrm{SU}(2)_{\mathrm{spin}}: &\qquad (c_1, c_2)^T \longrightarrow \mathcal{U}_s(c_1, c_2)^T, \\ \mathrm{SU}(2)_{\mathrm{orbital}}: &\qquad (c_1, c_2^\dagger)^T \longrightarrow \mathcal{U}_o(c_1, c_2^\dagger)^T. \end{aligned} \tag{7.26}$$

where $\mathcal{U}_{s,o}$ are SU(2) matrices. It follows that states which are singlets under $\mathrm{SU}(2)_{\mathrm{spin}}$ are doublets under $\mathrm{SU}(2)_{\mathrm{orbital}}$ and vice versa. We denote the spin doublet $|\uparrow\rangle, |\downarrow\rangle$ and the orbital doublet $|1\rangle$, $|2\rangle$. The spin operator for the boundary spin-1 can be split into contributions from the dangling boundary modes $\eta_L^{(+)}$ and from $\eta_L^{(-)}$: $\mathbf{S}_L = \mathbf{S}_L^{(+)} + \mathbf{S}_L^{(-)}$, with

$$\mathbf{S}^{(\pm)} = \frac{1}{8}\eta^{(\pm)T}\mathbf{\Sigma}\,\eta^{(\pm)}, \qquad \mathbf{\Sigma} = (\rho^y\sigma^x, \sigma^y, \rho^y\sigma^z). \tag{7.27}$$

We can make a similar splitting for the orbital spin $\mathbf{T}$, which is related to $\mathbf{S}$ by swapping the σs for τs. We denote the matrices appearing in $\mathbf{T}$ by $\mathbf{\Omega}$:

$$\mathbf{T}^{(\pm)} = \frac{1}{8}\eta^{(\pm)T}\mathbf{\Omega}\,\eta^{(\pm)}, \qquad \mathbf{\Omega} = (\rho^y\tau^x, \tau^y, \rho^y\tau^z). \tag{7.28}$$

The pairs $(|\uparrow\rangle, |\downarrow\rangle)$ and $(|1\rangle, |2\rangle)$ are both Kramers doublets, since the spin and orbital operators for the boundary modes, $\mathbf{S}_L^{(+)}$ and $\mathbf{T}_L^{(+)}$, change sign under $\mathcal{T}$. This can also be checked explicitly by considering the transformation of the boundary states (labeled by fermion occupation numbers) under $\mathcal{T}$, with the fermions transforming as $\mathcal{T}: c_{1,2} \to c_{1,2}^\dagger$.

Now we consider the effect of gauge fluctuations or Gutzwiller projection. We have listed the generators for the Sp(4) gauge group in Eq. 7.18. However, some gauge generators are Higgsed in the above mean field state. In general, to determine the unbroken gauge group we must examine Wilson loops of the form $W = \widehat{u}_{i_1 i_2}\widehat{u}_{i_2 i_3}..\widehat{u}_{i_n i_1}$, where $H_{\mathrm{MF}} = \sum_{ij}\eta_i^T\widehat{u}_{ij}\eta_j$ [10]. The unbroken gauge generators are those that commute with the Wilson loops. Here, the only nontrivial Wilson loop is the matrix X defined in Eq. 7.21. This leaves a subset of six unbroken generators, which may be written in terms of the matrices Ω appearing in the orbital spin (Eq. 7.28):

$$\Gamma_{\mathrm{1D}} = \{\boldsymbol{\Omega}, X\boldsymbol{\Omega}\}. \tag{7.29}$$

Taking linear combinations, we can use instead[5]

$$\Gamma_{\mathrm{1D}} = \{\mathcal{P}_+\boldsymbol{\Omega}\mathcal{P}_+, \mathcal{P}_-\boldsymbol{\Omega}\mathcal{P}_-\}. \tag{7.30}$$

We denote the unbroken gauge group $\mathrm{SU}(2)^{(+)}_{\mathrm{orbital}} \times \mathrm{SU}(2)^{(-)}_{\mathrm{orbital}}$.

To make the Hamiltonian in Eq. (7.20) a reasonable ansatz, we must check that the Sp(4) gauge charges are all zero on average: $\langle \Gamma_i \rangle = 0$ for all i. Fortunately the unbroken gauge symmetry Γ_{1D} guarantees this.

The boundary modes involve only $\eta^{(+)}$, so are invariant under $\mathrm{SU}(2)^{(-)}_{\mathrm{orbital}}$. However, $|1\rangle$ and $|2\rangle$ are not invariant under $SU(2)^{(+)}_{\mathrm{orbital}}$. Therefore after confinement only the doublet $|\uparrow\rangle$, $|\downarrow\rangle$ survives, with corresponding spin $\mathbf{S}^{(+)}_L$. This is the boundary spin-1/2 of the Haldane phase.

In this 1D example we can confirm explicitly that Gutzwiller–projecting the mean-field wavefunction gives the desired SPT phase. In fact the Gutzwiller–projected state for $\Delta = t$, denoted $|\Psi_{\mathrm{spin}}\rangle$, is precisely the AKLT state. To see this we adopt a trick from Ref. [156]. Using the fact that the terms in H_{MF} commute, we can check that $|\Psi_{\mathrm{spin}}\rangle$ has zero amplitude for a pair of adjacent sites to be in a spin–two state. $|\Psi_{\mathrm{spin}}\rangle$ is therefore the ground state of the AKLT Hamiltonian, since this can be written as a sum of projectors onto the spin-two subspace for each link.[6]

It is interesting to consider inversion symmetry here. In the free fermion problem,

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[5]To be more precise, the two types of generators in Eq. 7.29 correspond to elements of the invariant gauge group (IGG) [10] at different momenta, $k = 0$ and $k = \pi$, so when we take linear combinations the two types of generators in Eq. 7.30 alternate on even and odd sites. This is not crucial here. Another subtlety is that the IGG is enlarged at the special point $\Delta = t$.

[6]This correspondence with the AKLT state is less obvious if we simply Gutzwiller–project the BCS ground state of H_{MF}. The ground state of the Kitaev chain involves a long-range Cooper pair wavefunction, $C(r) = (L-2r)/L$ [157], so in the present case $|\Psi_{\mathrm{spin}}\rangle$ is obtained by acting on the vacuum with an exponented sum of long-range singlet creation operators, $\exp(\sum_i \sum_{r>0} C(r) f_i^\dagger \sigma^y \tau^y f_{i+r}^{\dagger T})$, and projecting. The AKLT state may of course be written using only short–range singlet creation operators, $|\mathrm{AKLT}\rangle \propto \mathcal{P} \prod_i (f_i^\dagger \sigma^y \tau^y f_{i+1}^{\dagger T}) |\mathrm{vac}\rangle$. By the previous argument, the two states must be equivalent.

$\nu \to -\nu$ under inversion, so that a nonzero value of ν can only be realised with a Hamiltonian which breaks inversion symmetry. With interactions, $\nu \simeq \nu + 8$, suggesting that $\nu = 4$ can be realised in inversion–symmetric interacting system [158]. The present example is a nice realisation of this. The mean field Hamiltonian H_{MF} appears to break inversion symmetry. However, the symmetry can be restored by combining it with a gauge rotation. So the projected wavefunction is actually inversion symmetric.

We now move on to 3D states.

7.3.2 Cubic lattice

Making use of the cubic band structure studied in Ref. [84], we construct an SU(2) gauge theory which confines to a topological paramagnet. We choose the mean field Hamiltonian

$$H_{\text{MF}} = \sum_{\langle ij\rangle} t_{ij}\eta_i^T \rho^y \eta_j + \sum_{\langle\langle ij\rangle\rangle} i\chi'_{ij}\eta_i^T \rho^x\sigma^y\tau^y\eta_j + \sum_{\langle\langle\langle ij\rangle\rangle\rangle} \chi_{ij}\eta_i^T \rho^x\sigma^y\eta_j, \tag{7.31}$$

where the nearest-neighbor hopping t_{ij} gives a π-flux on every square plaquette, the body-diagonal pairing χ_{ij} follows the pattern studied in Ref. [84], and the next-nearest-neighbor pairing χ'_{ij} is a small perturbation introduced to reduce the gauge group to SU(2) and is not responsible for the gap or the band topology.

To determine the unbroken gauge group, we examine the Wilson loops as above. The fundamental nontrivial ones are proportional to $\rho^z\sigma^y$ and $\rho^x\sigma^y\tau^y$. The unbroken gauge group is generated by those of the Sp(4) generators that commute with the Wilson loops. It is then straightforward to see that the unbroken gauge group is an SU(2) generated by

$$\Gamma_{\text{cubic}} = \{\rho^z\sigma^y\tau^x, \tau^y, \rho^z\sigma^y\tau^z\}. \tag{7.32}$$

One can choose to implement time-reversal $\mathcal{T}$ as $\mathcal{T} : \eta \to i\rho^z\sigma^y\eta$, and it is straightforward to see that $\mathcal{T} : H_{\text{MF}} \to H_{\text{MF}}, \mathbf{S} \to -\mathbf{S}$, and $\mathbf{\Gamma}_{\text{cubic}} \to -\mathbf{\Gamma}_{\text{cubic}}$. The band structure in Eq. (7.31) preserves time-reversal symmetry, and the SU(2) gauge rotation commutes with $\mathcal{T}$. Notice also that $\mathcal{T}^2 = -1$ on the η fermions.

We must check that the Sp(4) gauge charges are all zero on average, $\langle \Gamma_i \rangle = 0$. The unbroken gauge symmetry Γ_{cubic} guarantees that $\langle \Gamma_i \rangle = 0$ for all i except for $\Gamma_5 = \rho^z \sigma^y$. Furthermore, time-reversal invariance $\mathcal{T}$ guarantees that $\langle \Gamma_5 \rangle = 0$. Hence the condition is indeed satisfied for any i.

To determine the band topology, it is sufficient to consider the Hamiltonian H'_{MF} with only the nearest-neighbor and body-diagonal terms in Eq. (7.31). In H'_{MF}, fermions with different orbital indices are decoupled and form two identical bands. Each band is the same as that studied in Ref. [84], with $\nu = 4$ (two Dirac cones on the surface). So the band has $\nu = 8$ in total (four Dirac cones). So Eq. (7.31) indeed gives rise to a topological paramagnet.

In order to understand the role played by spin-rotation symmetry, we examine the surface state in more detail. We start from the surface Dirac theory with $\text{SU}(2)_{\text{gauge}} \times \text{SU}(2)_{\text{spin}} \times \mathcal{T}$ symmetry, with four Dirac cones in total:

$$H = \psi^\dagger (p_x \mu_x + p_y \mu_z) \otimes \tau_0 \otimes \sigma_0 \psi, \tag{7.33}$$

with time-reversal

$$\mathcal{T} : \psi \to i\mu_y \otimes \tau_0 \otimes \sigma_0 \psi^\dagger, \tag{7.34}$$

gauge SU(2)

$$\mathcal{U}_g : \psi \to \mu_0 \otimes \mathcal{U}_g \otimes \sigma_0 \psi, \tag{7.35}$$

and spin SU(2)

$$\mathcal{U}_s : \psi \to \mu_0 \otimes \tau_0 \otimes \mathcal{U}_s \psi. \tag{7.36}$$

We have denoted the $\text{SU}(2)_{\text{gauge}}$ Pauli matrices by τ, but they should not be confused with the Pauli matrices for the orbital spin.

Next we will consider driving this surface theory into a $\mathbb{Z}_2$ topologically ordered state by first introducing an order parameter Δ which gaps out the Dirac fermions, but breaks time reversal symmetry, and then restoring time-reversal symmetry by proliferating double vortices in Δ. The single vortex remains gapped, and gives rise to anyonic surface excitations with nontrivial time reversal properties.

To analyse the symmetry properties it is useful to consider the auxiliary $\mathrm{U}(1)_a$ transformation

$$U_a(\theta) : \psi \to e^{i\theta}\psi \tag{7.37}$$

(which is an emergent symmetry of Eq. 7.33, but not a microscopic symmetry). The gap term of interest is

$$H_\Delta = i\Delta\, \psi\mu_y \otimes \tau_y \otimes \sigma_y \psi + \text{h.c.} \tag{7.38}$$

This is invariant under the $\mathrm{SU}(2)_{\text{gauge}} \times \mathrm{SU}(2)_{\text{spin}}$ symmetry. It is not invariant under time reversal $\mathcal{T}$ or under $\mathrm{U}(1)_a$ separately, but it is invariant under the modified time-reversal transformation $\tilde{\mathcal{T}} \equiv U_a(\pi/2)\mathcal{T}$. Notice that $\tilde{\mathcal{T}}^2 = 1$ on the parton fermions ψ, in contrast to the original $\mathcal{T}$ under which they are Kramers.

As shown in Refs. [3, 4, 114], the fundamental vortex in Δ transforms projectively under $\tilde{\mathcal{T}}$, i.e. $\tilde{\mathcal{T}}^2 = -1$. We now examine the $\mathrm{SU}(2)_{\text{gauge}} \times \mathrm{SU}(2)_{\text{spin}}$ spins carried by the vortex. A key point is that there are four Majorana zero modes trapped in the vortex core. One can label the internal Hilbert space with two complex fermions $c_{1,2}$. Since both $\mathrm{SU}(2)$ groups are preserved in the intermediate gapped phase and the partons transform as doublets under both $\mathrm{SU}(2)$, the two complex fermions $c_{1,2}$ should also be doublets under both $\mathrm{SU}(2)$. In an appropriate basis the transformations are

$$\begin{aligned} \mathcal{U}_g : &\quad (c_1, c_2)^T \to \mathcal{U}_g (c_1, c_2)^T, \\ \mathcal{U}_s : &\quad (c_1, c_2^\dagger)^T \to \mathcal{U}_s (c_1, c_2^\dagger)^T. \end{aligned} \tag{7.39}$$

It follows that states which are singlets under $\mathrm{SU}(2)_{\text{gauge}}$ are doublets under $\mathrm{SU}(2)_{\text{spin}}$ and vice versa. Specifically, there are two distinct kinds of vortices, labeled by the fermion parity $(-1)^{c_1^\dagger c_1 + c_2^\dagger c_2}$: both have $\tilde{\mathcal{T}}^2 = -1$, but one transforms as $(0, 1/2)$ under $\mathrm{SU}(2)_{\text{gauge}} \times \mathrm{SU}(2)_{\text{spin}}$, and the other as $(1/2, 0)$.

We now restore time-reversal symmetry by condensing double-vortices that transform trivially under both $SU(2)_{gauge} \times SU(2)_{spin}$ and $\tilde{\mathcal{T}}$, giving $\mathbb{Z}_2$ topological order on the surface [80, 104]. Single vortices with even and odd fermion parity yield mutual semions which we denote e and $\tilde{m}$ respectively. Both are Kramers bosons ($\mathcal{T}^2 = -1$), and e transforms as $(0, 1/2)$ under $SU(2)_{gauge} \times SU(2)_{spin}$ while $\tilde{m}$ transforms as $(1/2, 0)$. Their bound state, $\tilde{\epsilon}$, is non–Kramers, fermionic, and transforms as $(1/2, 1/2)$.

So far, our treatment of the surface has neglected the confining gauge field.[7] When we take it into account, only excitations that are neutral under $SU(2)_{gauge}$ survive. In addition to e, these include bound states $m = \psi\tilde{\epsilon}$ and $\epsilon = \psi\tilde{m}$ got by attaching a ψ fermion to $\tilde{m}$ and $\tilde{\epsilon}$. This shifts the self-statistics, so m is bosonic while ϵ is fermionic (all three particles are mutual semions). Since ϵ is the bound state of e and m (and its properties follow from this) we do not discuss it further. Note that $m = \psi\tilde{\epsilon}$ is Kramers since ψ is.

The upshot is that the surface topological order surviving after 'gauge neutralization' has an e particle that is Kramers and spin-doublet, and an m particle that is Kramers but spin-singlet. Since both e and m are Kramers bosons, this state is indeed the $eTmT$ phase, like the wavefunction discussed in Sec. 7.1.

However if spin-rotation symmetry is preserved, a finer classification is possible, under which the present state is dubbed $eCTmT$, where the 'C' indicates that e is a spin doublet [1].[8] This finer classification emphasises a difference between the $eTmT$ state constructed here, in which e is a spin doublet and m is not, and that constructed in Sec. 7.1, where both e and m are spin doublets.

Like the 1D example of the previous section, the cubic lattice construction violates inversion symmetry at the free fermion level (this is inevitable if ν is nontrivial [159]) but the resulting spin state is inversion symmetric as a result of gauge invariance.

[7] The fermions in the bulk are confined, giving a non-fractionalized bulk state. It is known [115] that the SU(2) gauge theory has a θ-term at $\theta = (\nu'/2)\pi$. For $\nu' = 4$, as here, we have $\theta = 2\pi$ which has the same physics as at $\theta = 0$. The confined state then can preserve time reversal symmetry. In contrast if $\nu' = 2$, we will have $\theta = \pi$ and the resulting confined phase of the $SU(2)$ gauge theory must be non-trivial in some way. It either breaks time reversal or becomes a quantum spin liquid with long range entanglement.

[8] The $eCTmT$ state is topologically equivalent to the combination of a generic $eTmT$ state and the state $eCmT$; the analogue of the latter for U(1) spin symmetry is discussed in Sec. 7.3.3.

Here, the hopping term in H_{MF} is invariant under inversion, while for an appropriate choice of χ' the pairing terms change sign under inversion. Therefore inversion can be restored by combining it with the gauge transformation $f \to if$, i.e. $\eta \to i\rho^y\eta$. (With the arrow conventions of Sec. 7.1.2, the fluctuating AKLT state is also inversion symmetric.)

7.3.3 Diamond lattice

Next we consider parton theories on the diamond lattice, making use of the band structure of Ref. [115]. First we construct a theory with an SU(2) gauge field which naturally confines (Sec. 7.3.3). The resulting state is a topological paramagnet which requires both time-reversal and XY-spin rotation symmetry to be protected. Then in Sec. 7.3.3 we construct a U(1) gauge theory, which confines at strong coupling. The confined state is a topological paramagnet which only requires time-reversal symmetry.

Topological XY paramagnet from $SU(2)$ gauge theory

The mean field Hamiltonian is

$$H_{\mathrm{MF}} = \sum_{\langle ij\rangle} t\eta_i^T\rho^y\eta_j + \sum_{\langle\langle ij\rangle\rangle} t'_{ij}\eta_i^T\rho^y\eta_j + \sum_{\langle\langle ij\rangle\rangle} \Delta_{ij}\eta_i^T\rho^x\tau^y\eta_j, \tag{7.40}$$

where the nearest-neighbor hopping t is isotropic, while the next-nearest-neighbor hopping t'_{ij} and pairing Δ_{ij} follow the patterns discussed in Ref. [115]. Notice that the pairing term is a singlet in orbital space, but is a triplet in spin space. Hence the spin rotation symmetry is reduced from SO(3) down to O(2) rotations about the S_y axis, corresponding to XY anisotropy in the spin model.

We again calculate the nontrivial Wison loops: the simplest nontrivial ones consist of three links and are proportional to ρ^y and $\rho^x\tau^y$. The unbroken gauge group is generated by

$$\Gamma_{\text{diamond}} = \{\rho^y\tau^x, \tau^y, \rho^y\tau^z\}. \tag{7.41}$$

These are precisely the orbital SU(2) generators $\mathbf{\Omega}$.

One can implement time-reversal symmetry $\mathcal{T}$ as $\eta \to \rho^z\sigma^y\tau^y\eta$, under which η is non-Kramers ($\mathcal{T}^2 = 1$) and $\mathbf{S} \to -\mathbf{S}$ and of course $H_{\text{MF}} \to H_{\text{MF}}$.

As above we must check that the Sp(4) gauge charges are all zero on average: $\langle\Gamma_i\rangle = 0$. The unbroken gauge symmetry Γ_{diamond} guarantees that $\langle\Gamma_i\rangle = 0$ for all i except for $\Gamma_1 = \rho^y$, which is nothing but the total fermion occupation number (minus two). Fortunately the mean field Hamiltonian Eq. (7.40) has a special lattice symmetry[9] that sets $\langle\Gamma_1\rangle = 0$.

To determine the topology of the mean field band structure, it is convenient to consider the modified time-reversal symmetry $\mathcal{T}' : \eta \to i\rho^z\tau^y\eta$ (with $\mathcal{T}'^2 = -1$), which is the combination of time-reversal and spin rotation $i\sigma_y$. Fermions with different physical spins ($\eta_\uparrow$ and $\eta_\downarrow$) do not mix under the modified time-reversal. Furthermore, they are decoupled in the mean field Hamiltonian H_{MF} and form two copies of an identical band. Therefore the topological index ν' is defined for each band separately. Now each band is identical to that studied in Ref. [115], with $\nu' = 4$. The total band therefore has $\nu' = 8$, with four Dirac cones in total on the surface.

We now consider the surface Dirac theory with $\text{SU}(2)_{\text{gauge}} \times \text{U}(1)_{\text{spin}} \times \mathcal{T}$ symmetry, with four Dirac cones in total:

$$H = \psi^\dagger(p_x\mu_x + p_y\mu_z) \otimes \tau_0 \otimes \sigma_0\psi, \tag{7.42}$$

with modified time-reversal

$$\mathcal{T}' : \psi \to i\mu_y \otimes \tau_0 \otimes \sigma_0\psi^\dagger, \tag{7.43}$$

gauge SU(2)

$$\mathcal{U}_g : \psi \to \mu_0 \otimes \mathcal{U}_g \otimes \sigma_0\psi, \tag{7.44}$$

[9]The symmetry is the combination of a $\pi/2$ rotation along $\hat{z}$, a reflection $z \to -z$, then followed by a particle-hole transformation $c_{\alpha,\delta,a} = \sigma^z_{\alpha\beta}\tau^z_{\delta\lambda}\gamma^z_{ab}c^\dagger_{\beta,\lambda,b}$, where $\{\alpha, \beta\}$ denote the spin, $\{\delta, \lambda\}$ denote the orbital, and $\{a, b\}$ denote the sub-lattice index (γ^z is a Pauli matrix acting on the sub-lattice indicies).

and spin U(1)

$$U_s(\theta) : \psi \to e^{i\theta}\psi. \tag{7.45}$$

The actual time-reversal is $\mathcal{T} = U_s(\pi/2)\mathcal{T}'$:

$$\mathcal{T} : \psi \to \mu_y \otimes \tau_0 \otimes \sigma_0 \psi^\dagger. \tag{7.46}$$

Now consider the gap term

$$H_\Delta = i\Delta\, \psi\mu_y \otimes \tau_y \otimes \sigma_y \psi + h.c., \tag{7.47}$$

which preserves both $\mathrm{SU}(2)_{\mathrm{gauge}}$ and $\mathcal{T}$, but breaks $\mathrm{U}(1)_{\mathrm{spin}}$. To restore the $\mathrm{U}(1)_{\mathrm{spin}}$ symmetry and preserve the gap, we need to proliferate vortices in the order parameter field Δ. It was shown in Ref. [3, 4, 114] that the fundamental vortices have $\mathcal{T}^2 = -1$, so condensing double-vortices gives a $\mathbb{Z}_2$ gauge theory, with e being Kramers, $\tilde{m}$ being Kramers and $\mathrm{SU}(2)_{\mathrm{gauge}}$-doublet, and $\tilde{\epsilon}$ being non-Kramers and $\mathrm{SU}(2)_{\mathrm{gauge}}$-doublet. We can then gauge-neutralize the particles by binding ψ fermions to $\tilde{m}$ and $\tilde{\epsilon}$. The neutralized theory then has e being Kramers, and $m = \tilde{\epsilon}\psi$ being non-Kramers (recall that $\mathcal{T}^2 = 1$ on ψ) but carrying spin-1/2 under $\mathrm{U}(1)_{\mathrm{spin}}$ due to the S_y spin carried by ψ. This state is dubbed $eCmT$ in Ref. [1].

The fermions will be confined once the fluctuation of the SU(2) gauge field is introduced, and we obtain a non-fractionalized bulk state. On the surface, the $eCmT$ topological order survives the confinement, since all the non-trivial quasi-particles in the theory are gauge-neutral and are hence decoupled from the gauge field. We have thus obtained the $eCmT$ topological paramagnet.

As a side note, if the spin-1 operators are pseudo-spins such that $\mathcal{T} : \{S_x, S_y, S_z\} \to \{S_x, -S_y, S_z\}$, then the modified time-reversal $\mathcal{T}' : \eta \to i\rho^z\tau^y\eta$ (with $\mathcal{T}'^2 = -1$) could represent the physical time-reversal symmetry. In this case we obtain a topological paramagnet that requires time-reversal only, as will be shown in Sec 7.3.3.

Stable $U(1)$ quantum spin liquids and topological paramagnets

The parton construction of course also gives access to stable quantum spin liquid phases. Of particular interest to us is a time reversal symmetric U(1) quantum spin liquid phase on the diamond lattice. For greater generality we allow for full SU(2) spin symmetry. As usual such a phase has a gapless emergent photon. In addition it has a gapped fermionic spin-1/2 Kramers doublet spinon which has internal 'electric' charge[10] and a gapped bosonic spin-0 magnetic monopole that transforms to an antimonopole under time reversal. We will give the spinons the band structure of a topological superconductor (as in previous sections). The resulting quantum spin liquid phase then inherits the non-trivial surface states of the topological superconductor. The relevance to the present chapter comes from asking about the confined phase that results when the magnetic monopole is condensed. We show below that this is the $eCTmT$ topological paramagnet.

SPT phases in 3D have been accessed previously through confinement of emergent U(1) gauge fields [132, 133]. However in these previous studies the confinement was achieved in a highly non-trivial way involving the condensation of dyons (bound states of magnetic and electric charges). The novel aspect of our construction is that the confinement is achieved directly by simply condensing the magnetic monopole, which will result from the usual dynamics of the gauge field at strong coupling.

Consider the following mean field ansatz:

$$\begin{aligned} H_{\mathrm{MF}} = \quad & \sum_{\langle ij\rangle} t\eta_i^T \rho^y \eta_j + \sum_{\langle\langle ij\rangle\rangle} t'_{ij}\eta_i^T\rho^y\eta_j + \sum_{\langle\langle ij\rangle\rangle} \Delta_{ij}\eta_i^T\rho^x\sigma^y\eta_j \\ & + \sum_i \Delta'\eta_i^T\rho^x\sigma^y\eta_i + \sum_{\langle ij\rangle, i\in A} it''\eta_i^T\rho^y\tau^y\eta_j, \end{aligned} \tag{7.48}$$

where the nearest-neighbor hopping t and on-site pairing Δ' are uniform and isotropic, while the next-nearest-neighbor hopping t'_{ij} and pairing Δ_{ij} follow the patterns discussed in Ref. [115]. Note that the first two terms are the same as in Eq. (7.40),

[10]This 'electric' charge couples to the emergent photon in this spin liquid, and not to physical external electromagnetic fields.

and the third is got by exchanging the role of orbital and physical spin. Contrary to Eq. (7.40), the pairing term Δ is a singlet in physical spin and a triplet in orbital space, so the full spin rotation symmetry is preserved. The nearest-neighbor antisymmetric hopping term t'' is introduced to reduce the gauge symmetry, and does not affect the other arguments in this section as long as it is kept small.

The simplest nontrivial Wilson loops are proportional to ρ^y, $\rho^x\sigma^y$ and $\rho^y\tau^y$. The resulting unbroken gauge group is a U(1) generated by τ^y.

We implement time-reversal symmetry $\mathcal{T}$ through $\eta \to i\rho^z\sigma^y\eta$ (which has $\mathcal{T}^2 = -1$). It is straightforward to check that $\mathbf{S} \to -\mathbf{S}$ and $H_{\text{MF}} \to H_{\text{MF}}$ under the chosen time-reversal symmetry. Moreover, the U(1) gauge charge τ^y is also odd under $\mathcal{T}$, which allows for topologically non-trivial band structures for the partons.

We now check that $\langle \Gamma_i \rangle = 0$. The unbroken U(1) gauge symmetry and time-reversal guarantee that $\langle \Gamma_i \rangle = 0$ for all i except for ρ^y and $\rho^x\sigma^y$, which are nothing but the total fermion occupation number (minus two) and the real part of the on-site pairing. The lattice symmetry again sets $\langle \rho^y \rangle = 0$. For the on-site pairing amplitude, there is no symmetry to set it to zero automatically. We must therefore adjust the on-site pairing term Δ' in Eq. (7.48) to make it zero on average[11].

To determine the topology of the mean field band structure, notice that fermions with different orbital indices (τ indices) do not mix under time-reversal $\mathcal{T} : \eta \to i\rho^z\sigma^y\eta$. They are also decoupled in the mean field Hamiltonian H_{MF}, forming two copies of an identical band. Therefore the topological index ν' is defined for each band separately. Now each band is almost identical to that studied in Ref. [115], with $\nu' = 4$. The total band therefore has $\nu' = 8$, with four Dirac cones in total on the surface.

We now consider fluctuations of the U(1) gauge field. In the weak coupling regime the gauge theory is deconfined, and we have a stable U(1) quantum spin liquid phase. The spinon band structure has time reversal protected surface states that provide a

[11]We must check that this does not close the gap. The total pairing term in momentum space $\Delta_{\mathbf{k}} + \Delta'_{\mathbf{k}}$ must not be positive (or negative) definite, since the onsite pairing vanishes: $\langle f_{i\uparrow} f_{i\downarrow} \rangle = 0$. It is easy to show that this requires $|\Delta'| < 12|\Delta|$. One can then show that such a value of Δ' can never close the gap opened by Δ. Therefore the mean field Hamiltonian Eq. (7.48) can be smoothly connected to a Hamiltonian with no Δ' term without closing the gap.

distinction between this spin liquid and more conventional ones. For a compact U(1) gauge theory, there are always gapped magnetic monopole excitations in the theory. In Ref. [3, 4] it was shown that for the spinon band structure we have here, this magnetic monopole is a spin-0 boson that simply transforms into an antimonopole under time reversal.

As the gauge coupling strength increases, the monopole mass gap decreases and eventually becomes zero. The monopoles will then condense and confine the gauge theory. The trivial symmetry properties of the monopole implies that this condensate does not break $\mathcal{T}$ or the physical spin SU(2) (if present). The confined state is thus a non-fractionalized symmetry preserving paramagnet. To determine which SPT phase the paramagnet belongs to, we need to examine the surface state in more detail. The argument is largely parallel to that in Sec. 7.3.2, with the simple modification that the SU(2) gauge symmetry discussed in Sec. 7.3.2 is reduced to U(1). The conclusion remains the same: the paramagnet is the nontrivial SPT dubbed $eCTmT$ in Ref. [1]. The representative surface state is a gapped $\mathbb{Z}_2$ topological order, with e being Kramers and spin-doublet, and m Kramers but spin-singlet. (If the spin-rotation symmetry is broken, this becomes a generic $eTmT$ state.)

7.3.4 Spin wavefunctions

The parton constructions suggest spin wave functions that may be useful as variational states in future work on specific microscopic models. Following the standard procedure [10] we construct a spin wave function from the mean-field fermion wave function $|\Psi_{\mathrm{MF}}\rangle$ by projecting onto the subspace obeying the constraints $\sum_{a\alpha} f^\dagger_{a\alpha} f_{a\alpha} = 2$ and $\sum_{ab\alpha} f^\dagger_{a\alpha} \tau_{ab} f_{b\alpha} = 0$:

$$|\Psi_{\mathrm{spin}}\rangle = \mathcal{P}|\Psi_{\mathrm{MF}}\rangle. \tag{7.49}$$

Such a projection is expected to roughly mimic the effect of gauge fluctuations. For the states constructed in Sec. 7.3.2 and 7.3.3, the SU(2) gauge fluctuations automatically confine the states. We therefore expect the projected wave functions to represent the confined spin SPT states. For the state in Sec. 7.3.3, the U(1) gauge field is deconfined at weak coupling, and confines to an SPT state at strong coupling. So it is not clear a priori whether the projected wave function will give the $U(1)$ quantum spin liquid state or the confined SPT state.

These spin wave functions are alternate possibilities to the loop gas wave functions described in the first part of the chapter. While the loop gas wave functions are physically appealing they are likely not very tractable numerically due to the linking signs. The parton wave functions, on the other hand, may be studied through variational Monte Carlo calculations though the physical connection to SPT physics is less directly obvious. This situation is similar to existing descriptions of quantum spin liquid phases through either loop gases (string-nets) or through partons which each have their advantages and disadvantages.

For the topological paramagnets, at present we do not have a direct connection between the parton and loop gas wavefunctions. Establishing such a connection is a target for future work, and will confirm the general correctness of the projected wave functions as faithfully capturing the state accessed through the parton description.

7.4 Discussion: Towards models and materials

We have emphasized that frustrated spin-1 magnets in 3D may be fruitful in the search for spin SPT phases.

In the ongoing search for quantum paramagnetism in frustrated systems, the bulk of the attention has focused on spin-1/2 systems. This is guided by the intuition that increasing the spin only leads to more 'classical' physics and hence to a greater tendency to order. Caution however is required in taking this intuition too seriously. In one dimension the spin-1/2 chain is almost antiferromagnetically ordered (power law correlations) while the spin-1 chain is a good paramagnet with a spin gap. This has the following amusing consequence. Consider a two–dimensional rectangular lattice with nearest neighbor antiferromagnetic interactions:

$$H_{\text{rect}} = J_{\parallel} \sum_{\mathbf{r}} \mathbf{S_r} \cdot \mathbf{S_{r+x}} + J_{\perp} \sum_{\mathbf{r}} \mathbf{S_r} \cdot \mathbf{S_{r+y}} \tag{7.50}$$

For $J_{\parallel} = J_{\perp}$ the model is antiferromagnetically ordered for all spin S. When $\frac{J_{\perp}}{J_{\parallel}}$ is decreased from 1 the spin-1/2 model stays ordered unless $J_{\perp} = 0$. The spin-1 model on the other hand becomes a spin gapped paramagnet below a non-zero critical value of $\frac{J_{\perp}}{J_{\parallel}}$. So there is a range of parameters in this 2D model where the spin-1 system is a quantum paramagnet although the spin-1/2 system has long range Neel order.

There are some interesting examples of frustrated spin-1 magnets — most notably $NiGa_2S_4$ and $Ba_3NiSb_2O_9$, in both of which the spin-1 Ni ion forms a triangular lattice [160, 161]. Apart from new and interesting kinds of quantum spin liquids, spin-1 magnets may also harbor novel broken symmetry states (such as spin nematics [162, 163, 164, 165]) more naturally than their spin-1/2 counterparts. To this we add the SPT phase discussed in this chapter as a possible fate for a frustrated 3D spin-1 magnet.

Our results suggest a route to guessing possible microscopic models that might harbor an SPT phase. Starting from the parton mean field Hamiltonian we can write down a lattice gauge theory that captures fluctuations. A strong coupling expansion of this lattice gauge theory will result in a spin Hamiltonian which may then be in the same phase as the same lattice gauge theory at weaker coupling. Such an approach has previously been successfully used to write down lattice models for various spin liquid phases. Given that we are interested here in confined phases we may be cautiously optimistic that a similar approach has an even better chance of resulting in spin models for the SPT phases. As an application let us consider the diamond lattice parton construction. With full SU(2) spin symmetry, the mean field state of Section 7.3.3 suggests (at leading order of the strong coupling expansion in the resulting U(1) gauge theory) an interesting frustrated spin-1 model: the "J_1–J_2" antiferromagnet on the diamond lattice[12]:

$$H = J_1 \sum_{\langle rr' \rangle} \mathbf{S}_r \cdot \mathbf{S}_{r'} + J_2 \sum_{\langle\langle rr' \rangle\rangle} \mathbf{S}_r \cdot \mathbf{S}_{r'} \tag{7.51}$$

[12]Strictly speaking, the J_2-coupling obtained from the previous mean-field ansatz should be anisotropic. It is not clear whether this anisotropy is in reality essential for realizing the topological paramagnet.

The next-nearest neighbour coupling J_2 introduces frustration. Indeed classically once $J_2 > \frac{J_1}{8}$ there are an infinite number of degenerate ground states [166] that are not related by global spin rotation. For large spin, it has been argued that the ground state is magnetically ordered as a result of quantum order by disorder [167]. The ground state for $S = 1$ (or $S = 1/2$) is not known. The SPT paramagnet discussed in this chapter is a candidate. The various descriptions we have provided should be a useful guide in future numerical studies should a paramagnetic ground state be found for this model.

It is interesting to note that since the diamond lattice is 4–fold coordinated classical 2-sublattice Neel order is likely to be more easily destabilized by frustration/quantum fluctuations than in the cubic lattice. Thus the J_1–J_2 diamond magnet for low spin ($S = 1/2$ or 1) may be an excellent candidate to find an interesting quantum paramagnetic ground state.

The frustrated diamond lattice model appears to describe well [166] the physics of the spinel oxide materials $MnAl_2O_4$ and $CoAl_2O_4$ [168] which belong to a general family of materials of the form AB_2O_4. The A site forms the diamond lattice and is magnetic. The Mn and Co compounds have $S = \frac{5}{2}$ and $S = \frac{3}{2}$ respectively. In searching for a material that realizes the $S = 1$ model it is natural then to consider $NiAl_2O_4$. However this is an inverse spinel, in which the A site is instead occupied by Al and the octahedrally coordinated B site is shared randomly between Ni and Al [169]. This randomness will presumably lead to different physics in this compound.

If the regular spinel compound could be synthesized the Ni is expected to be in a d^8 Ni^{2+} configuration and have spin-1. However the A site is tetrahedrally coordinated, and in the resulting crystal field, the Ni^{2+} ion will have orbital degeneracy in addition to spin-1. Further spin-orbit coupling will split the resulting spin-orbital Hilbert space and the physics of the lattice will be determined by its competition with inter-site spin/orbital exchange[170]. Thus spinels with Ni atoms at the A-site, even of they exist, will not be simply described by a spin-1 diamond lattice model.

Nevertheless we hope that our considerations motivate an experimental search for and study of other frustrated spin-1 magnets.

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